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“An Istanbul Meeting for World Mathematicians”

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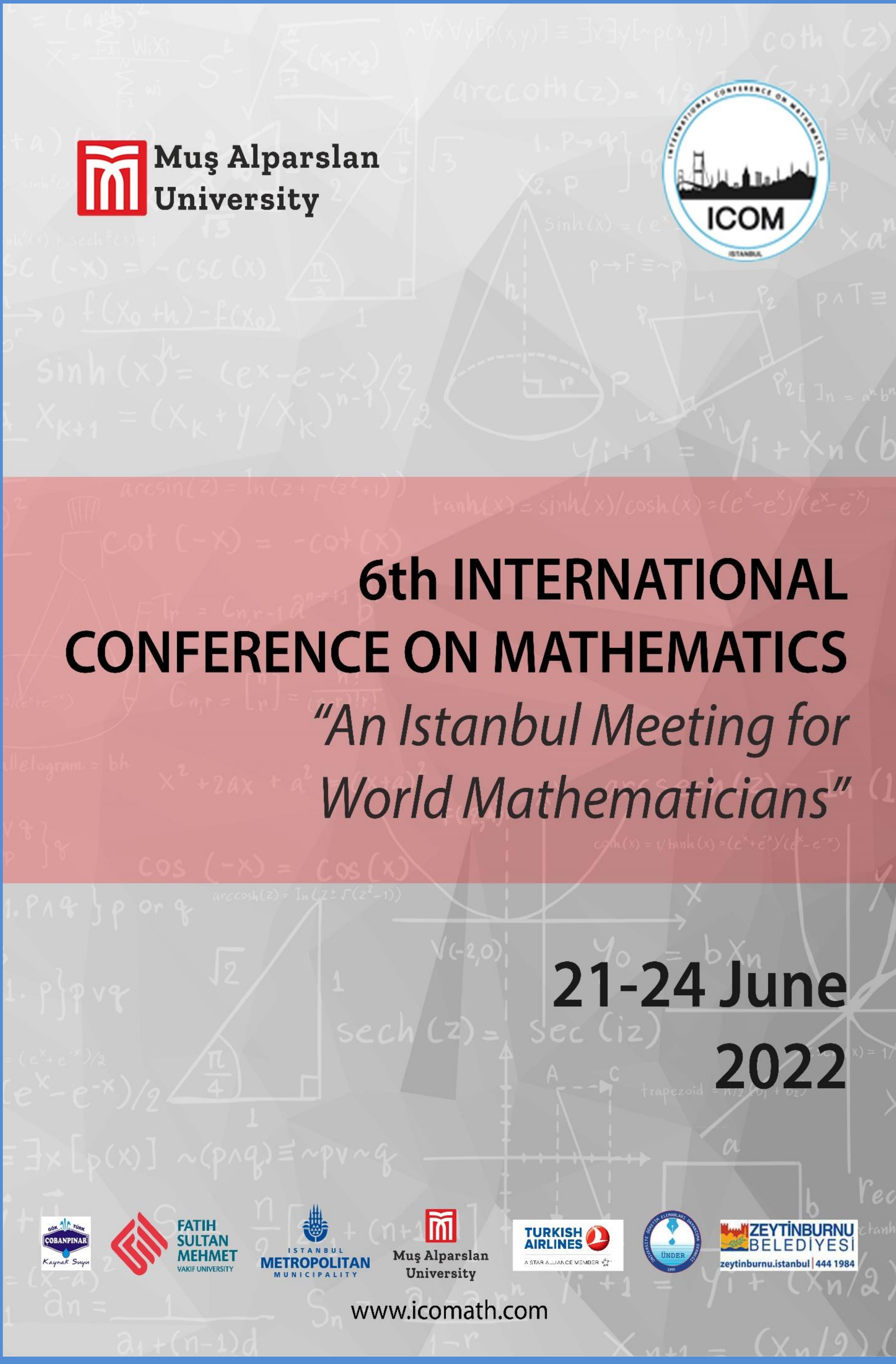

ICOM 2022
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Dear Colleagues and Dear Guests,

On behalf of the organizing committee, welcome to International Conference on Mathematics: An Istanbul Meeting for World Mathematicians, 21-24 June 2022, Istanbul, Turkey.

First of all, we present our deepest thanks to our supporters Fatih Sultan Mehmet Vakif University, Muş Alparslan University, Zeytinburnu Municipality, Çobanpınar water company, Istanbul Metropolitan Municipality, Turkish Airlines and UNDER for their supports and efforts during the conference days.

The conference aims to bring together leading academic scientists, researchers and research scholars to exchange and share their experiences and research results about mathematical sciences.

Besides these academic aims, we also have some social programs for introducing our culture and Istanbul to you. We hope that you will have nice memories in Istanbul for conference days.

We wish to all participants efficient conference and nice memories in Istanbul.

Thank you very much for your interest in International Conference on Mathematics: An Istanbul Meeting for World Mathematicians.

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ICOM 2022 CONFERENCE PROCEEDINGS BOOK

Contents

ICOM 2022 CONFERENCE PROCEEDINGS BOOK

A Comparative Examination Of High School Transition System (LGS) Mathematics Questions And 8th Grade Mathematics Textbooks Unit Evaluation Questions Within The Framework Of "Connection Skills"

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Abstract

The aim of this study is to compare the mathematics questions in the Transition System to High Schools (LGS), which is one of the central exams for transition to secondary education institutions, with the questions in the 8th grade mathematics textbook distributed by the Ministry of National Education and the alternative textbook of a private publishing house, which is frequently used by teachers, revealing their similarities and differences. For this purpose, the questions discussed were analyzed within the framework of 'connection skills' in order to determine what type and how often they contain connections. Document analysis method was used as the research model. The data source of the research is LGS questions, the 8th grade textbook of the Ministry of National Education (MEB) publications, and the unit evaluation questions in the alternative textbook of a private publishing house. According to the data obtained from the study, the types of connection skills were included at a higher rate in the questions belonging to the private publishing house, and it was seen that they were similar to the LGS questions. Less correlations were found in the textbook of MEB publications compared to the others. In all three cases, it was determined that the most connection was 'connection with real life', while the least connection was 'connection with different disciplines'. While the category of 'connection with different disciplines' was not found in LGS, connections were included in all categories in both textbooks. While connection skills were observed in each question in the textbook of the private publishing house, no connection skill was found in only 1 of 47 questions in LGS and in 43 of 104 questions in the textbook of MEB publications. As a result, the findings reveal that LGS and the textbook belonging to the private publisher (B) are at a similar level in terms of connection frequency, but the textbook (A) of the MEB publisher is insufficient.

Keywords:Mathematical connection, connection skills, high school transition system (LGS), mathematics textbooks.

1. Introduction

The ability to make connections plays an important role in students' permanent and meaningful learning, applying what they have learned, and increasing the quality of teaching, which affects the academic success of students in terms of mathematics learning and teaching. For this reason, the ability to make connections for effective mathematics learning and use is among the basic skills that are aimed to be developed in students (MEB, 2009a; 2009b and 2013). The knowledge of mathematics itself is used to make connections or bridges between mathematical ideas (Eli, 2009). Connection in the standards of the National Board of Mathematics Teachers (NCTM, 2000); express the relationships within and between mathematical ideas. It includes associating mathematics with life and other disciplines. With the adoption of the constructivist approach in our country, the linking skill has started to be conceptually included in the secondary school mathematics teaching programs implemented since 2005. Especially in the 2005 program, more detailed explanations are included along with sample linking activities on the basis of topics related to linking. In the literature, it is seen that connection skills are generally examined in four categories (Bingölbali & Coşkun, 2016; Mumcu, 2018; Leikin & Levav-Waynberg, 2007; Eli, 2009): i) associating with real life (Kurtuluş Kayan, 2019; Özgeldi & Osmanoğlu, 2017; Koyunkaya). et al., 2018), ii) connection between different representations of the concept (Duval, 1998; Kaput, 1987; Lesh, Post, & Behr, 1987; Janvier, 1987; Vil1egas, 2009; Mesquita, 1998), iii) connection between concepts (Businskas, 2008; Lockwood, 2011; Leikin & Levav-Waynberg, 2007) and iv) linking with different disciplines (Özgen, 2017; Yıldırım, 1996; Bodner, 2007; Matthews, Adams, & Goos, 2009). These four connection frameworks are also emphasized in the primary school mathematics curriculum (MSLC). According to Özsoy and İkikardeş (2004), the main source of classroom practices is the curriculum, and the resources used in classroom environments are shaped within the framework of the curriculum. Mathematics textbooks are the primary sources used in teaching mathematics. The course book and the lecture notes kept in the course are the only source of many students' experiences in mathematics (Özsoy & İkikardeş, 2004). Unit evaluation questions in the textbooks allow students to evaluate as a result of what they have learned. However, the use of textbooks among teachers occurs in two different ways: the textbook distributed by the Ministry of National Education (MEB) and a special resource. In addition to the textbook distributed by the Ministry of National Education, teachers often use additional special resources that are preferred. Özmantar et al. (2017) determined that the use of source books other than textbooks by mathematics teachers is 80%.

Using the questions in the textbooks, the students also prepare for the national exams with the experience they have gained from these questions. In our country, since 2018, changes have been made in the questions of the transition system to high schools (LGS), and it is called new generation questions or skill-based questions in the exam, which can not be solved only with the use of information, where beyond the knowledge, knowing the essence of the subject and reasoning about the subject and between the subjects, as much as possible, with examples from real life. created questions began to take place . In the Ministry of National Education Transition to Secondary Education directive prepared by the Ministry of National Education, it is stated that the exam questions are prepared in a way to measure the student's reading comprehension, problem solving, interpretation, analysis, critical thinking, inference, scientific process skills and similar skills (MEB, 2018a). However, while questions containing high-level thinking skills are asked in the exam, it is understood that textbooks are not very sufficient to prepare students to answer these questions (Kızkapan & Nacaroğlu, 2019).

Comparing the LGS questions with the questions in the textbooks in terms of associating skills was deemed worthy of research in terms of whether the questions were compatible or not. In this context, it is thought that this study, in which the unit evaluation questions in the mathematics textbooks and the high school transition system (LGS) mathematics questions of the last 3 years will be classified according to the connection skills, will contribute to the literature.

Bingölbali and Coşkun (2016), Dilegelen (2018); In these two studies, the conceptual framework was examined, and some arrangements suitable for our study were deemed necessary. The conceptual framework used in this context consists of four categories: 1) associating between concepts, 2) associating between different representations of the concept, 3) associating with real life, and 4) associating with different disciplines.

In line with the purpose of the research, the following three main research questions and sub-questions will be answered.

1. What kind of "connection skills" are included in the question roots of the 2018-2019-2020 LGS mathematics questions?

1.1. *How often was the connection with real life included in LGS math questions for the years 2018-2019-2020?*
1.2. *How often was the connection between different representations of the concept included in the 2018-2019-2020 LGS math questions?*
1.3. *How often was the connection between concepts included in LGS math questions for the years 2018-2019-2020?*
1.4. How often was the connection with different disciplines included in LGS mathematics questions for the years 2018-2019-2020?

2. What kind of "connection skills" are included in the question roots of the evaluation questions of the subjects in the 8th grade A and B mathematics textbooks?
2.1. How often is the *connection with real life* included in the evaluation questions of the subjects in the 8th grade A and B mathematics textbooks?
2.2. How often is the *connection between different representations of the concept* included in the evaluation questions of the subjects in the 8th grade A and B mathematics textbooks?
2.3. How often is the *connection between concepts* included in the evaluation questions of the subjects in the 8th grade A and B mathematics textbooks?
2.4. How often is the *relationship between different disciplines* included in the evaluation questions of the subjects in the 8th grade A and B mathematics textbooks?

3. What are the similarities and differences between the LGS mathematics questions and the evaluation questions of the subjects in the 8th grade A and B mathematics textbooks in terms of the types of connections made?

2. Method

Model of the Research

In this study, the document analysis method, which is included in the qualitative research design, was used. Qualitative research is the process of revealing events and perceptions in their natural environment in a holistic way without going beyond their reality, in which methods such as interview, document analysis and observation are used to collect data. In qualitative research, data is generally collected through observation, interview and document analysis (Yıldırım & Şimşek, 2016, p. 189).

Document analysis is the process of coding and examining according to a certain norm or system by bringing together the existing records and documents related to a work to be done (Çepni, 2010). In this study, document analysis was used as a method. Mathematics questions in LGS exams and 8th grade mathematics textbooks unit evaluation questions were analyzed according to sub-learning areas, and the classification was finalized by taking expert opinions. The research data analyzed as a result of the examination are presented in tables and the data obtained are given numerically. Thus, in this study, which has a qualitative structure, quantitative methods were also used.

Universe-sample (Reviewed Documents)

The universe of the study; The mathematics questions in the 2018-2019-2020 LGS exams, the 8th grade mathematics textbook distributed by the Ministry of National Education for use in public schools, and alternatively, the mathematics textbook questions of a private publishing house that are frequently used by teachers. The mathematics questions of the central exam held under the name of 2018-2019-2020 LGS were accessed from the relevant website of the Education Information Network (EBA). The examined book A is the 8th grade mathematics textbook published by the Ministry of National Education (MEB, 2019). The examined book B is an 8th grade mathematics textbook belonging to a private publishing house (Varışlı & Demir, 2020).

Data Collection Tools

In this study, two different mathematics textbooks used in 8th grade and LGS mathematics questions were used as data collection tools. 104 questions in book A were taken from the unit evaluation questions, and 80 questions in book B were taken from the skill-based questions section. A total of 231 questions, together with 47 questions in LGS, were examined according to the components of associating skills. 4 of the 5 learning areas in the secondary school mathematics curriculum were selected and the content of the questions belonging to 8 sub-learning areas was used as a data collection tool. 5 learning areas in mathematics; numbers and operations, data manipulation, probability, algebra, geometry, and measurement. Geometry, by its nature, has a separate structure in itself, apart from other learning areas. In this learning area, there may be connections with the sub-concepts of the concept. Therefore, geometry and measurement learning area is excluded because it is more limited and does not provide rich data compared to the types of connection seen in other learning areas. It should also be noted that the expression "sub-learning area" is also used as the word "subject" in other parts of the study.

Table 1. *Learning areas and sub-learning areas examined in LGS and 8th grade mathematics textbooks*

A. Numbers and Operations	B. Data processing	C. Possibility	D. Algebra
Factors and Multiples	Data analysis	Probability of Simple Events	Algebraic Expressions and Identities
Exponential Expressions			Linear Equations
Square Root Expressions			Inequalities

Table 2. *The number of examined questions belonging to sub-learning domains*

Sub-Learning Areas	Book A Number of Questions	Book B Number of Questions	Number of LGS Questions
Factors and Multiples	13	10	5
Exponential Expressions	13	10	7
Square Root Expressions	20	10	10
Data analysis	6	10	4
Probability of Simple Events	11	10	6
Algebraic Expressions and Identities	15	10	6
Linear Equations	14	10	6
Inequalities	12	10	3

The distribution of LGS questions belonging to sub-learning domains by years is given in the table below. The units were examined in the order included in the secondary school mathematics curriculum. The contents of the units in the A and B textbooks are the same.

Analysis of Data

In this section, the stages in the analysis process of the data obtained in the research will be mentioned. The connections analyzed in LGS and textbook questions were evaluated within the framework of four categories. These are: i.) associating between concepts, ii.) associating between different representations of the concept, iii.) associating with real life and iv.) associating with different disciplines. These categories were determined by considering the study of Bingölbali and Coşkun (2016). Within the scope of Dilegelen's (2018) study, a new sub-category was added to the findings obtained within the framework of these categories and sub-categories: real-life object use. The subcategories *of connection between different representations of the concept were arranged in accordance with the content of our study.* A new category has been opened for questions that are not associated.

Linking Concepts

In this study axis, interconceptual connection takes place as a component of the connection skill and there are two sub-categories of this component. (Bingölbali and Coşkun, 2016).

- Establishing a relationship between the concept and other concepts
- Establishing a relationship between the concept and its sub-concepts and the sub-concepts themselves

However, the component of 'establishing a relationship between the concept and its sub-concepts and between the sub-concepts' is not included in this study since it naturally exists in the content of the questions. Therefore, only the sub-category of 'relationship between the concept and other concepts' was discussed.

Establishing a relationship between the concept and other concepts

It is the connection of a mathematical expression or concept with different expressions/concepts (Bingölbali & Coşkun, 2016). In this component, it may be possible to make connections between the same learning areas or different learning areas.

In the question given below, the solution of decimal representations of numbers using integer powers of 10 is handled with the help of exponential expressions. A correlation has been made between exponential expressions and decimal notation. In this respect, it has been determined that the question has the ability to establish a relationship between the concept, which is a sub-category of the category of *connection between concepts, and other concepts.*

Writing a decimal notation as a sum of place values is called decimal notation analysis. The analyzed figure of the heights of five players in a basketball team is given in the table below.

Table:Players' Heights

Name	Length (cm)
Ayça	$2 \cdot 10^2 + 1 \cdot 10^0 + 1 \cdot 10^{-1}$
Beyza	$1 \cdot 10^2 + 7 \cdot 10^1 + 5 \cdot 10^0 + 5 \cdot 10^{-1}$
Ceyda	$1 \cdot 10^2 + 8 \cdot 10^1 + 4 \cdot 10^0$
Derya	$1 \cdot 10^2 + 8 \cdot 10^1 + 7 \cdot 10^0 + 2 \cdot 10^{-1}$
Esra	$1 \cdot 10^2 + 8 \cdot 10^1 + 5 \cdot 10^0 + 6 \cdot 10^{-1}$

The coach of the team will play one of the players shorter than 185 centimeters tall as the quarterback. How many players can play as playmaker among the given players?

A)4 B)3 C)2 D)1

Figure 1. *Example of a question belonging to the 2020 LGS exponential expressions sub-learning domain (Booklet A, P.3)*

Associating Different Representations of the Concept

Representation is a mathematical expression/concept or the presentation of a mathematical relationship in a certain way (NCTM, 2000). The concept of mathematical notation is generally expressed in the form of representations in the international literature.

In this study, the notation styles that are frequently used in the literature and the connections between them are taken as basis. The main titles are taken from the study of Bingölbali and Coşkun (2016). The main headings that shape the subcategories are as follows, and the transformations between the display formats will be given in detail in the conceptual framework table.

- ***Verbal expression*:** It is the connection of a given verbal expression with other forms of representation.
- ***Concrete object*:** Number stamps, fraction bars, real models, mockups, etc. establishing a relationship between objects and other representations.
- ***Picture/diagram*:** Number line, area model, etc. using images and establishing relationships with other display formats.

- ***Written symbols***: It is the connection between symbols such as fraction notation, algebraic expression, and other representations.
- ***Table*****:** Establishing a relationship between a frequency table or a table containing any mathematical idea and other display formats.
- ***Graph:*** It is the establishment of a relationship between display formats such as column graph, line graph, circle graph and other display formats.
- ***Equation:*** Establishing a relationship between the structure of the equation and other forms of representation.
- ***Figure*****:** It is the establishment of a relationship between figures containing mathematical ideas (triangle, quadrilateral, circle, etc.) and other representations.

The sub-categories, LGS questions and textbooks unit evaluation questions that emerged with the help of the above titles and the study of Dilegelen (2018) were reviewed and revised in accordance with the content of this study.

In the question given below, the net numbers according to the courses are given as a column chart. In order to determine the information about the graphic, questions containing verbal expressions are given. Therefore, the question was evaluated as a *graphic-verbal expression as* a sub-category of the category of *connection between different representations of the concept.*

Column showing the nets that Beste made in Turkish, mathematics and science courses in a practice exam graph is given on the side. According to this:

a) Which course did he make the most clear?

b) How many nets did he make in all three courses?

c) By how many times is the net number in the science lesson more than the net number in the mathematics lesson?

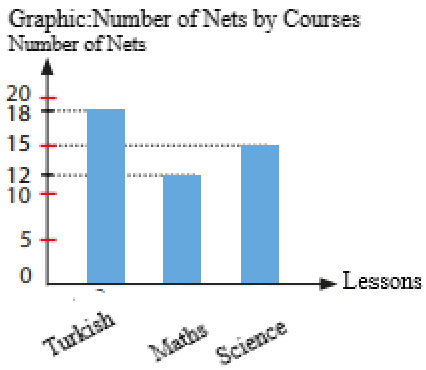


Figure 2. *Example of a question belonging to the data analysis sub-learning domain of book A (p.98, question 12)*

Connection with real life

Today's world, with its changing and developing structure, also changes its expectations from people and the world of education. While education is a part of life, its connection with daily life has been inevitable. It is expected that the problem situations encountered will be associated with daily life and awareness will be raised in this direction. The connection between real life and mathematics, which has an important place in the learning-teaching process of mathematics, is frequently emphasized in the relevant literature.

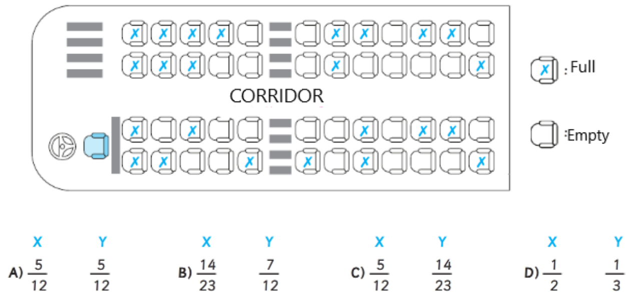
This type of connection is covered under three sub-components in this study:

- Considering the concept in a context
- Giving real-life verbal examples
- Use of real-life objects

Considering the concept in a context

The Turkish Language Connection (TDK) defines the concept of context as situations, events, relationships, or connection in any case. Context, with its educational aspect, is the connection of the phenomenon, event or technology that an individual encounters in real life with the course content (MEB, 2012).

In the question given below, it is seen that the concept of probability is built on the choice of bus seats and associated with social and social space from daily life. Therefore, the question was evaluated in the subcategory *of taking the concept in a context*, which is the subcategory of the category of *associating with real life.*



In the figure above, the seats of a bus are shown with occupied and empty seats.

• The probability that the ticket Ahmet bought is by the window is x.

• The probability that Fatih's ticket is on the aisle side is y.

• Fatih bought his ticket after Ahmet bought it.

Based on this, which of the following is true?

Figure 3. *Example of question belonging to book B probability sub-learning domain (p.181,1st question)*

Verbal example from daily life

Verbal expression of real-life contexts is the only verbal use of real-life relationships when teaching a mathematical concept. (Bingölbali and Coşkun, 2016).

In the statements given in the question below, it is desired to distinguish between true and false. For example, in the 3rd expression, the inequality is given in accordance with the verbal expression that is associated with real life. Since the concept's use of verbal expression is in question, the question was evaluated in the subcategory of *giving verbal examples from real life.*

Write (T) next to the statements in the table that are true and (F) next to the ones that are incorrect. Write the correct ones for the incorrect statements.

Expressions	T/F	Correct if wrong
The line y=20 is parallel to y-axis.		
The line y=x passes through the origin.		
"Photographs of those who are 15 years old applying for an identity card Inequality of the sentence "must" x≥ 15.		
Same negative on both sides of the inequality. inequality direction when number is added changes.		

Figure 4. *Example of a question belonging to the inequality sub-learning domain of book A (p.186,1st question)*

Use of real-life objects

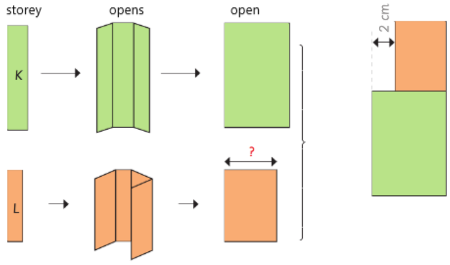
It is the construction of mathematical expressions and concepts over an object that will make it easier to understand that concept from any daily life. The use of the surfaces of medicine boxes can be evaluated within the framework of this sub-category while exploring the surface areas of the rectangular prism.

Apart from the study of Bingölbali and Coşkun (2016), this component was found appropriate to be included as a sub-category by examining Özgen's (2018) study.

The following question explains the steps of setting up and solving equations with the help of folding traces using cardboard material. Thus, the question was evaluated in the *real-life object use* sub-category.

Orange and green colored K and L folded rectangular papers are fully unfolded as shown below.

The short side of L when folded is 1 cm more than half of the short side of K.



When fully opened rectangles are placed along their short sides without any gaps as above, a distance equal to the specified length is formed. What is the short side of the orange rectangle that is fully extended?

A) 4 B) 8 C) 16 D) 24

Figure 5. *Example of a question belonging to the linear equations sub-learning field in book B (p.230,5.question)*

Linking with different disciplines

According to the Turkish Language Institution, discipline; It is defined as the whole of knowledge that is or may be the subject of teaching, that is, a branch of science. Within the framework of this component, it is seen that studies are mostly carried out on the interdisciplinary teaching (disciplinary) approach in the literature. According to Yıldırım (1996), if it is thought that disciplinary teaching is teaching within the framework of a certain subject area (Science, Mathematics, History, etc.), interdisciplinary teaching can be expressed as bringing together subject areas in a meaningful way around certain concepts.

The category of connection with different disciplines is examined in two subcomponents (Bingölbali & Coşkun, 2016).

- Considering the concept in a different disciplinary context
- Expressing the connection with different disciplines with verbal examples

Considering the concept in a different disciplinary context.

Teaching a mathematical concept or expression through the context of a different discipline is evaluated within the framework of this sub-component.

a 12-unit string length into different lengths, Pythagoras obtained notes of different thickness and fineness. 16/15 of the length of a string that makes the sound of the note Don gives the sound of Si, and 6/5 makes the sound La, 4/3 the Sol sound, 3/2 the Fa sound, 8/5 the Fa sound. 8/5 of them give the Mi sound and 16/9 of them give the Re sound (Orhan, 1995). Pythagoras' studies in this direction formed the basis of the relationship between mathematics and music. Thus, the use of fractions in the discipline of music can be shown as an example of this component.

In the question given below, the concept of inequality in mathematics was tried to be expressed with a subject belonging to science by using temperature values. Thus, the question was evaluated in the subcategory of considering the *concept in a different disciplinary context.*

A broken thermometer can show the temperature in the environment up to 3 °C higher or up to 4 °C less than the actual temperature. The actual temperature of an environment where this thermometer shows 18 °C is a degree Celsius. Which of the following inequalities is the largest interval that a can take?

A) $15 < a < 22$ B) $14 \leq a \leq 21$ C) $15 \leq a \leq 22$ D) $14 < a < 21$

Figure 6. *Example of a question belonging to the data analysis sub-learning domain of book A (p.187,12.question)*

Expressing the connection with different disciplines with verbal examples.

In this component, it includes only verbal connection of a mathematical concept/expression or subject with other disciplines (Bingölbali & Coşkun, 2016). In the teaching of negative numbers included in integers, verbal use of the concepts of temperature and thermometer belonging to science while giving examples of numbers is a situation evaluated in this sub-category.

For example; In the question given below, "In the census in a province, it would be more appropriate to show the number of people according to the districts with a circle graph." In the statement, the use of a circle graph belonging to the data analysis sub-learning domain has been associated with social sciences, which is a different subject area. Thus, the question was evaluated in the subcategory *of expressing the connection with different disciplines with verbal examples*.

Write (T) next to the statements in the table that are true and (F) next to the ones that are incorrect. Write the correct ones for the incorrect statements.

Expressions	T/F	Correct if wrong
The order of √21<9<3√11 is correct.		
The result of the operation 5√4+√2 is a rational number.		
It would be more appropriate to show the grades of the students in a class in the Turkish course with a circle graph.		
In the census in a province, it would be more appropriate to show the number of people according to the districts with a circle graph.		

Figure 7. *Example of a question belonging to the data analysis sub-learning domain of book A (p.96, 1st question)*

No connection

If any correlation could not be established between mathematical concepts and expressions, they were evaluated in this category.

For example; In the question on the subject of EBOB (25, 125) = 25 (greatest common divisor) multipliers and multiples given below, only the greatest common divisor of two numbers, which is a mathematical term, is given. It is expected that the statement is false/true to be determined numerically. Therefore, the question was not evaluated in any connection category.

Write (T) next to the statements in the table that are true and (F) next to the ones that are incorrect. Write the correct ones for the incorrect statements.

Expressions	T/F	Correct if wrong
$5^0 + 5^{-1} = 5^{-1}$		
$96 \cdot 10^{15} = 0{,}96 \cdot 10^{17}$		
EBOB $(25,125) = 25$		
The prime factors of 256 are 2 and 3.		

Figure 8. *Example of a question belonging to the factor and multiples sub-learning domain of book A (p.53,1st question)*

3. Results

General evaluation of the connection between concepts

In this category, 33 connections (70%) were found in 47 questions in LGS, 23 (22%) connections were found in 104 questions in book A, and 49 (61%) connection skills were found in 80 questions in book B.

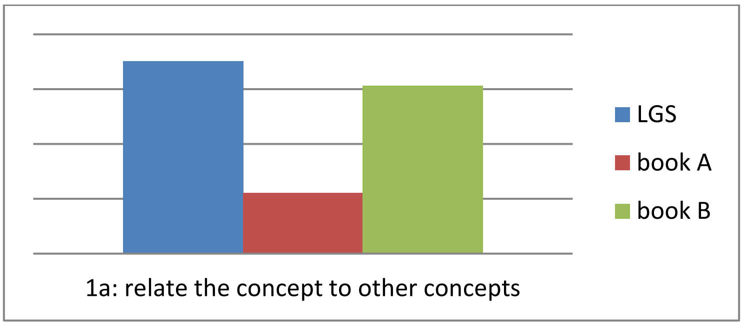


Figure 9. *General assessment of the connection between concepts*

In the chart above, the connections related to the sub-categories are given collectively in order to provide an overview. For example; While 'relationship between the concept and other concepts' was seen in LGS questions and book B at a similar rate, it was determined as almost half in book A.

General assessment of the connection between different representations of the concept

In this category, 24 connections (51%) were found in 47 questions in LGS, 21 connections were found in 104 questions in book A (20%), and 23 connection skills were found in 80 questions in book B (28%).

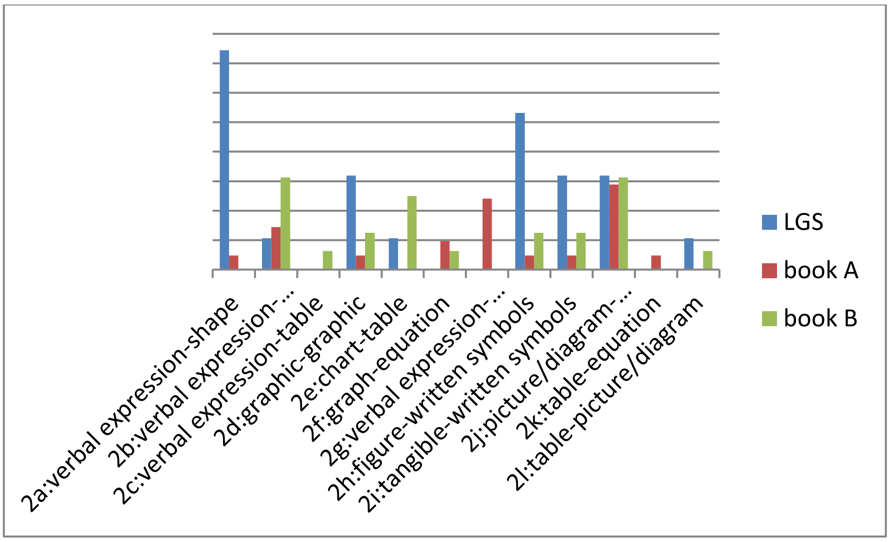


Figure 10. *General assessment of the connection between different representations of the concept*

In the chart above, the connections related to the sub-categories are given collectively in order to provide an overview. When we look at the sub-categories of 'Connection between different representations of the concept', it is seen more in most of the LGS exams than in the others. The most common sub-category was picture/diagram-written symbols (2j). 2c, 2g and 2k categories were found only in one of the three.

General assessment of real-life connection

In this category, 44 connections (93%) were found in 47 questions in LGS, 28 connections (27%) in 104 questions in book A, and 79 (98%) connection skills in 80 questions in book B.

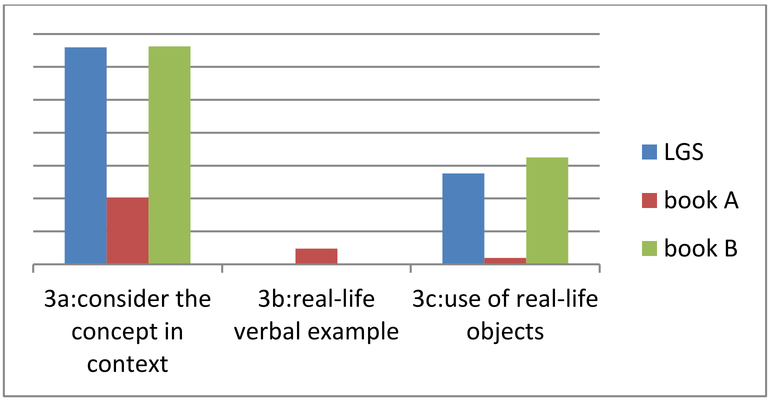


Figure 11. *General assessment of real-life connection*

In the chart above, the connections related to the sub-categories are given collectively in order to provide an overview. Considering the sub-categories of 'relationship with real life', it was seen that the concept was mostly handled in a context (3a). While LGS and book B are at a similar level, book A lags behind them.

General assessment of linking with different disciplines

In this category, 5 connections (5%) were found in 104 questions in book A, and 1 (1%) in 80 questions in book B.

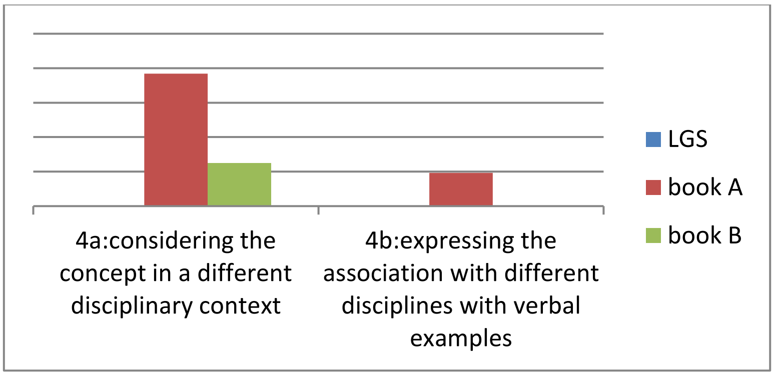


Figure 12. *General assessment of linking with different disciplines*

In the chart above, the connections related to the sub-categories are given collectively in order to provide an overview. It is seen that there are no sub-categories of 'connection with different disciplines' in LGS. It was found in book A at a higher rate than the others.

General assessment of unrelated questions

In this category, one (2%) out of 47 questions in LGS and 43 (41%) out of 104 questions in book A were found to be unrelated questions.

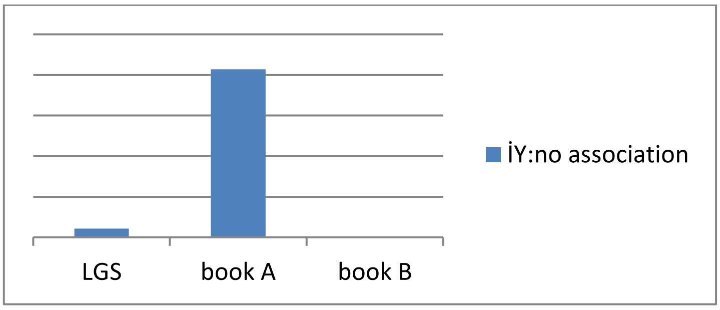


Figure 13. *General assessment of unrelated questions*

In the chart above, the connections related to the sub-categories are given collectively in order to provide an overview. While it was seen that there were a higher rate of unrelated questions in book A than in the others, this situation was never encountered in book B.

4. Discussion, Conclusion

Of connection between concepts was the second most common connection skill for LGS, book A and book B for all three. For the connection between concepts, 8 sub-learning areas belonging to 4 learning areas (numbers and operations, data processing, probability and algebra) were examined. When viewed proportionally, the highest number of connections are seen in LGS questions in the category of 'connection between concepts', followed by book B and then book A. In this respect, while LGS and the B book belonging to a private publishing house are similar in terms of connection frequency, it is seen that half of them have connections in the A book belonging to the Ministry of National Education publications. In addition to how often the connection is given, the quality of the connection is also important. In this respect, the connections made in LGS and book B are more complex and usually made between more than two concepts, while the connections made in book A are simpler and usually between two concepts at most. For example, with square root expressions; While the concept of area and length can be associated at the same time, square root expressions in book A are only associated with the concept of length.

In general, in the literature and in this study, the correlation of mathematics within itself has the result that it is at a sufficient level compared to other types of connection. In Dilegelen's (2018) analysis of the activities in the textbooks in terms of the types of connection skills, the most connection was seen as the connection between concepts, and it can be said that although it is not the most common connection in our study, it is at a high level. In Yorulmaz and Çokçalışkan's (2017) study of pre-service teachers' views on mathematical connection, it was seen that the relation within mathematics itself came to the fore. Many concepts in mathematics are related to each other and the previous concept facilitates the learning of the next concept. For example; during the analysis of decimal representations, exponential expressions are used by using integer powers of 10 and both concepts are associated with each other.

Therefore, due to the nature of mathematics, it is inevitable to include connections within themselves while writing or explaining the concepts.

The connection category LGS between different representations of the concept was the 3rd most common skill out of the 4 categories examined for both book A and book B. 8 sub-learning domains belonging to 4 learning domains (numbers and operations, data processing, probability and algebra) were examined to make connections between different representations of the concept. When viewed proportionally, the highest number of connections were found in LGS in the category of 'connection between different representations of the concept'. It is seen that there is a similar ratio in book A and book B and at half the level of LGS. While LGS and B book are similar in the categories of associating between concepts and associating with real life, books A and B are similar here and lag behind LGS. But it should be noted that; While the notation style used in book A is seen with 1 connection in 1 question, other types of connections have been encountered in addition to 1 connection seen in 1 question in this category in book B. The situation stated for book B is similarly valid for LGS questions. Students can focus on the forms of representation they frequently encounter in books and expect in that direction. In this respect, different forms of representation should be frequently included in the books and in practice in accordance with the structure of the subject.

In the study of Gürbüz and Birgin (2008), the comparison of the ability of students at different education levels to perform operations with different representations of rational numbers was examined. As the education level of the students increased, their ability to operate with rational numbers using geometric models, algebraic expressions and number line representations increased; However, it was determined that the skills of making operations using algebraic expressions increased more than other notation formats. In our study, algebraic expressions were evaluated in the 'written symbol' category, for example, connection of algebraic expressions using the area model (picture/diagram) was frequently seen. The fact that the subject of algebraic expressions can be used frequently in many different subjects, especially in problem posing and solving, has made it inevitable to develop and use this skill. In addition, algebraic representations are mostly included in mathematics textbooks in different studies (Baştürk, 2007, 2010).

Associating with real life LGS, book A and book B was the most frequently encountered skill out of the 4 categories examined for all three. For the ability to relate to real life, 8 sub-learning domains belonging to 4 learning domains (numbers and operations, data processing, probability and algebra) were examined.

When viewed proportionally, the highest number of connections in the 'relationship with real life' category were found in book B. It is seen that there are similar correlations in LGS and book B. While LGS and B book are similar in the categories of associating between concepts and associating with real life, book A lags behind them here. Students preparing for the exam using book A may have difficulties if they take the exam by being familiar with the questions that are not associated.

It is seen that similar connection processes take place in advanced mathematics subjects as well. Mumcu (2018), in his study, investigated the extent to which pre-service teachers can use their connection skills for the concept of derivative, and found that pre-service teachers use real-life connection the most. The high level of connection used supports our research.

In addition, Lee (2012) investigated the real-life perspectives of pre-service classroom teachers. In one of the research findings, approximately 47% of pre-service teachers believed that almost any mathematical concept could be connected to something in the real world. The remaining 53% said that there are more appropriate concepts for real-life connection. The explanation of one participant, "For all kinds of mathematical concepts, if there is a mathematical concept that we cannot associate with the real world, why do we need to learn it?" is remarkable. From this point of view, it can be thought that most of the subjects, concepts and mathematical expressions can be associated with daily life. In our study, it was inevitable that 'connection with daily life' was the most common component in connection skill.

Associating with different disciplines category LGS, book A and book B were the least common linking skills for all three. 8 sub-learning domains belonging to 4 learning domains (numbers and operations, data processing, probability and algebra) were examined for associating with different disciplines. While no connection was found in this category in LGS questions, the interesting thing is that while the least connection was seen in book A in other categories, the highest number of connections were found in book A here. Associating with different disciplines is frequently emphasized in the related literature and it is generally recommended to teach with an interdisciplinary approach (Drake & Burns, 2004; Klein, 1990; Lattuca, 2001; MEB, 2009; Özgen, 2016; Yıldırım, 1996; Bodner, 2007). However, this approach reveals that although it is an important point in teaching, it is not given enough importance in question writing.

Dilegelen (2018) has never encountered the ability to associate with different disciplines in the activities included in the 5th grade textbooks. Özdiner (2021), in his research on the connection skills of activities in primary and secondary school textbooks, found the ability to associate with different disciplines in 2% of the activities. Coşkun (2013)'s research findings show that mathematics and classroom teachers hardly establish any relationship between mathematics and other disciplines in the classroom. Similarly, in our study, this component was found very rarely in the questions in the textbooks, and surprisingly, the component of associating with different disciplines was not found in LGS. This shows that both the questions in the textbooks and the LGS questions do not attach importance to associating them with different disciplines. On the contrary, the importance of establishing relationships with different disciplines in mathematics teaching programs in MEB (2013 and 2018) mathematics curriculum was emphasized.

General evaluation

As a result of this research, book A of the Ministry of National Education publications distributed in public schools falls far behind the LGS exam in terms of connection frequency. In this respect, the frequency of associating skills in a book B belonging to a private publishing house, which is frequently used by teachers, was found to be worth investigating, and it was seen that it was similar or even higher than the connections in the questions of the LGS exam. The difference between the questions in book A and the questions in the LGS exam in terms of the frequency of connection skills may affect students' perceptions and increase their anxiety about the exam. Güler et al. (2019) received the opinions of teachers regarding the LGS exam, the students' achievements in LGS were generally defined as insufficient, and another result reached is that while the teachers found the new exam system questions positive in terms of quality, they stated that the existing infrastructure was insufficient, reducing the difficulty level of the questions. They suggested that the exam time should be increased.

When the connection rates are examined, it is seen that in all three of the LGS, book A and book B, 'connection with real life', then 'connection between concepts', then 'connection between different representations of the concept' and 'connection with different disciplines' at least. Although they are similar in this direction, LGS and B book have closer results in terms of connection frequencies. In book A, the connection frequencies are insufficient compared to the others. In the 'Connection between different representations of the concept' component, this situation differs, and both book A and book B lag behind the frequency of connection in LGS questions.

There are no connections with different disciplines in LGS questions and it is seen that this component is not taken into account sufficiently in the textbooks. It is a shortcoming that this component is not found in LGS questions. In the literature, making connections with different disciplines in the MEB (2009a; 2009b; 2013) and NCTM (2000) standards is among the competencies that should be acquired by students.

Although it is seen that more connections are made in 'relationship with real life' and 'connection between concepts', it is necessary to determine whether these relations are really qualified. For example; The real-life context in book A The expressions in the real-life context in book B are not qualitatively similar. While simpler and shorter expressions are used in the questions in the book A, long expressions with more than one connection are used in the questions in the LGS and book B. As a result of the results of this study, it is thought that it will make important contributions to the literature in terms of showing how and to what extent correlation is included in the exam questions, in the textbook (A) of the Ministry of National Education publications and in the alternative textbook (B) of a private publisher.

5. Suggestions

In this study, the type and frequency of connections in the questions in the LGS exam and in the textbooks were investigated. Studies on LGS generally focus on teacher and student opinions (Kızkapan & Nacaroğlu, 2019; Güler et al., 2019; Kablan & Bozkus, 2021; Şıvkın et al., 2020; Karakaya et al., 2020). There are also studies that include comparisons with international exams (Küçükgençay et al., 2021; Aktaş, 2022; Batur & Beyret, 2019). It is important for the transition to secondary education institutions that LGS is primarily adopted by the students in the national framework and that they achieve success in this exam. In addition to examining written sources, it may be suggested to conduct studies that examine classroom practices in terms of connection types of LGS questions. The main source of classroom practices is textbooks. There are researches within the framework of associating skills in the activities in the textbooks (Dilegelen, 2018; Özdiner, 2021). The use and benefit of the developed conceptual framework in the writing of textbooks and the development of curricula can be subject to further research. Most importantly, it is a necessity to revise the inadequacy in the textbooks and raise them to the current LGS level. In this regard, cooperation can be made with the Ministry of Education Board of Education and Discipline, which examines the textbooks distributed for use in public schools. If the types and frequencies of connections in the questions in the textbooks are increased, both the relational understanding and the working principle for the exam will be provided. It should be noted here that although the ability to relate is emphasized in the curriculum, it is understood that the question content of the textbook distributed by the Ministry of National Education is insufficient in this regard.

In addition to all connection skills, individual connection types can be examined in LGS questions. For example, only the real-life component can be focused on.

For the connection component between the different representations of the few concepts, the content of the questions should be presented, which includes bidirectional conversion between the representation formats. For example; There should be a transition not only from the verbal expression representation to the equation representation, but also from the equation representation form to the verbal expression. Question contents that adopt an interdisciplinary approach should be produced for associating with a small number of different disciplines. In the literature, it is common to associate mathematics with the disciplines of science and technology (Karakuş et al., 2017; Çelikler et al., 2018; Bülbül et al., 2019; Kızılay and Kırmızıgül, 2019; Güder and Gürbüz, 2018; Bakırcı and Kutlu, 2018). In addition to these, studies can be carried out in which mathematics is associated with other different disciplines (music, painting, art, social sciences, history, etc.). Question contents can be produced in this direction.

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A differential search algorithm combined with support vector machine to predict the risk of mortality in patients with STEMI-CS

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Abstract

The ST-segment elevation myocardial infarction- cardiogenic shock (STEMI-CS) is one of the strongest factors in patient mortality within hospitals. This paper presents a hybrid machine learning based approach for predicting the risk of mortality in patients with STEMI-CS. The proposed method combines an efficient evolutionary differential search algorithm (DSA) with support vector machine (SVM) in risk prediction phase. The incentive mechanism of using DSA is to optimally tune the parameters of SVM to improve its prediction ability. With a test on a real-world benchmark dataset, the proposed DSA-SVM is confirmed to have significant improvement compared with multiple machine learning models.

Keywords: Myocardial infarction, cardiogenic shock, STEMI-CS, risk prediction, SVM, DSA.

1. Introduction

Cardiogenic shock (CS) is the leading cause of within-hospital morality in patients with ST-segment elevation myocardial infarction (STEMI), and occurs in about 5-10% of patients [1]. Studies indicate that the admission rates for cases of STEMI has increased about 4-fold and even more in recent years [2].
Machine learning (ML) is a multidisciplinary data analysis technique that automates the construction of analytical models. ML is an important branch of artificial intelligence developed according to the idea that computers can learn from data, recognize patterns, and make decisions with minimal human intervention.

In recent years, researchers have been used various machine learning methods to predict the risk of morality in STEMI-CS patients. Bai et al. [1] used machine learning algorithms to establish an accurate and easy method for predicting the occurrence of STEMI. They showed that least absolute shrinkage and selection operator (LASSO) model has better predictive performance compared with other methods. The limitation of their work is that all possible factors that influence STEMI-CS are not considered in risk prediction phase. Deng et al. [3] used machine learning algorithms to establish an optimal model to predict the within-hospital death that occurred in STEMI-CS patients who underwent primary percutaneous coronary intervention. They proved that the random forest algorithm outperformed other ML algorithms in morality risk prediction process. Wu et al. [4] developed three deep learning models for enhancing the effectiveness of the STEMI diagnosis. Their prediction models are convolutional neural network (CNN), long short-term memory (LSTM), and hybrid CNN-LSTM. With a test on 883 STEMI patients, the CNN-LSTM model performed better than LSTM and CNN, and even doctors in predicting STEMI. Shetty el al. [5] show the effectiveness of the ML models in the morality detection of STEMI patients. They proved that the ML models overcome the limitations of the traditional logistic regression based models. Their results showed that the random forest and multiple perceptron models outperformed counterparts in terms of accuracy measure. Lee et al. [6] used logistic regression with regularization, random forest, extreme gradient boosting (EGB), and SVM models to predict the short- and long-term mortality of STEMI patients. Their comparison showed that the ML-based models outperformed other algorithms in terms of solution quality. Al-Zaiti et al. [7] developed some classifiers for the prediction of underlying acute myocardial ischemia in patients with chest pain. Their proposed method outperformed the doctors and commercial interpretation software. Liu et al. [8] developed a deep learning model as a diagnostic support tool based on a 12-lead electrocardiogram. The objective was to improve the diagnosis of STEMI disease.

A review of related work shows that promising results have been made in diagnosing and determining the risk of mortality in STEMI-CS patients. However, the performance of existing methods is not ideal and more effort is needed in this area. In this research, we proposed DSA-SVM algorithm to predict the risk of morality in STEMI-CS patients. The proposed model takes as input the admission features of patients, analysis them and groups them in one of the high-risk or low-risk classes. The reason for using the DSA algorithm together with the SVM is to optimally adjust the parameters of the SVM to improve its grouping performance. The proposed model is compared with standard support vector machine (SVM), least absolute shrinkage and selection operator (LASSO), and adaptive neuro fuzzy inference system (ANFIS) model. The models were evaluated on 300 and 100 STEMI-CS patients in the training and test datasets, respectively. Models were compared based on the precision, recall, and F1-measure criteria. The results justify that the proposed DSA-SVM model showed best predictive performance and outperformed other models in terms of performance measures.

2. Support vector regression

SVM is a supervised machine learning method equipped with association learning algorithms [9], [10], [11]. For a dataset $D=\{O_i, y_i\}_{i=1}^{N}$, where N indicates the number of data objects, $O_i = \{o_{i1}, o_{i2}, ..., o_{im}\}$ is a m-dimensional data object defined, and y_i is the label that assigned to O_i. In the SVM algorithm, each data object $O_i \in D$ is considered as a point in m-dimensional space. The goal is to create a prediction model using some training data to separate data objects through finding a hyperplane that differentiates the data objects into some separate groups. This hyperplane is calculated based on a few data points, known as support vectors. In other words, SVM aims to maximize the minimum distance of data points from a separator hyperplane by solving the following equation:

$$f(o) = \sum_{i=1}^{N} (\alpha_i - \alpha_i^*) K(o, o_i) + b \tag{1}$$

where $K(o,o_i)$ denotes the kernel function, that is computed by multiplying the two inner vectors o and o_i in the feature space $\varphi(o)$ and $\varphi(o_i)$, respectively. Three well-known kernel functions are radial basis function (RBF), sigmoid, and polynomial basis function. We use the RBF kernel function in SVM due to its high performance and easy configuration compared to other kernel functions. The precise tuning of C, γ and ε is an important task to increase the prediction performance of the SVM with RBF kernel. To optimize these parameters, we used DSA algorithm, which is described in the next section.

3. Differential search algorithm

Figure 1 shows the working principle of differential search algorithm (DSA). The DSA begins its work with a randomly distributed population P, known as superorganism. Each member, P_i in the superorganism is defined as follows:

$$P_{i,j} = rand.(up_j - low_j) + low_j \qquad \begin{array}{l} i = 1,\ 2,\ \ldots,\ N \\ j = 1,\ 2,\ \ldots,\ D \end{array} \tag{1}$$

where N indicates the population size and D indicates the problem dimension. up_j and low_j are the upper and lower bond of the value that each variable of P_i can posses. After initialization, stopover vectors are generated between the organisms. Searching for a stopover site at the areas can be described by a Brownian-like random walk model. The following equation is defined to calculate the stopover vector S_i

$$S_i = P_i + \omega.(O_i - P_i) \tag{2}$$

$O_i \in O$ is the ith individual in the historical superorganism O, and ω is the scale factor, which controls the amplitude of the search-direction matrix $O_i - P_i$. Historical superorganism helps the DSA to uses its experiences from previous generations. O is defined as follows:

$$O = F(P) \tag{3}$$

The function F is a transformation function that converts superorganism P into superorganism O according to a transformation strategy. Standard DSA has four different transformation strategy including bijective, surjective, elitist-1 and elitist-2. The philosophy behind bijective method is to evolve the superorganism towards to "permutated superorganism", i.e. random directions. In surjective method, the superorganism go to some of the random top-best solutions. The elitist-1 method evolves the superorganism towards to "one of the random top-best" solution. In elitist-2 method, the superorganism go towards to "the best" solution. The value of scale factor W is generated by a gamma random number generator controlled by a uniform distribution random number between 0 and 1. W is defined as follows:

$$\omega = g[2.r_1] \cdot (r_2 - r_3) \tag{4}$$

where g is a random number produced using a gamma random number generator, r_1, r_2 and r_3 are random numbers generated in the range of [0, 1]. The formula used for computation of ω allows the superorganism to radically change direction in the habitat. The stopover vector S_i that will participate in producing the population at next generation is calculated as follows:

$$S_i^{'} = \begin{cases} S_i & \text{if } r_i=0 \\ P_i & \text{if } r_i=1 \end{cases} \tag{5}$$

r_i is an integer number either 0 or 1. After computation of stopover vectors, selection operator is used to choose the next population between the stopovers and the population. For each member P_i the selection operator operates as follows:

$$P_i^{t+1} = \begin{cases} S_i^{'} & \text{if } f\left(S_i^{'}\right) \le f\left(P_i\right) \\ P_i & \text{if } f\left(S_i^{'}\right) > f\left(P_i\right) \end{cases} \tag{6}$$

P_i^{t+1} is the member at generation t+1, $f\left(S_i^{'}\right)$ and $f\left(P_i\right)$ are the fitness of stopover $S_i^{'}$ and member P_i, respectively. Some members of the superorganism obtained at the end of DSA's process can overflow the allowed search space limits. The individuals beyond the search-space limits are regenerated using Eq. (1). At the end, the best member is introduced as a final solution to the problem.

1. Initialization
    Set the value of limits, and the maximum iteration number
repeat
    2. Compute the stopover vectors
    3. Compute the historical organisms
    4. Compute the next generation population
    5. Control the limit of stopover site
until stopping conditions are met
Return the best solution

Figure 1. The working principle of the DSA algorithm

4. DSA-SVM

Figure 2 shows the working principle of the DSA-SVM algorithm for predicting the morality risk of STEMI-CS patients. The proposed approach includes two phase: model construction and model usage. In the former phase, the system creates a classification model using the training data, and identifies the most important features that improve the classification performance. In the latter phase, the system applies the constructed model on the unseen test data to classify the data objects. The main tasks of model construction are tuning SVM parameters and feature selection. The DSA algorithm is used to tune optimal values for three SVM's parameters C, γ, and ε. The algorithm also identifies an optimal subset of demographic characteristics, medical history, risk factors, clinical symptoms, and treatment strategies that have the maximum influence on the morality risk prediction. In the model construction phase, the algorithm creates a population of solutions. Each solution X_i is composed of several patient features permutation and SVM's parameters combination, as follows:

$$X_i = \{p_1, p_2, p_3, f_1, f_2, \ldots, f_6\} \tag{7}$$

where p_1, p_2, p_3 are float numbers, which are candidate values for the three parameters C, γ and ε. These values are generated randomly. The boundary of values for p_1 is [0, 100], and for p_2 and p_3 is [0, 1]. Each feature f_j is a binary variable with value 1 when the candidate feature f_j is considered for model construction, and 0 when the feature is ignored.

The fitness of individuals is measured using the mean squared error (MSE) of 10-fold cross-validation for SVM. The fitness function is defined as

$$MSE = \frac{1}{n}\sum_{i=1}^{n}(\widehat{X}_i - X_i)^2 \tag{8}$$

where $\widehat{X}_i$ indicates the predicted value, X_i indicates the observed value, and n indicates the number of all data objects in the dataset. The individual with a smaller value of MSE is more preferable.

Until termination conditions are met, the algorithm iteratively updates the population. Finally, the best solution is identified and returned, which is composed of the most important patient's features and the optimum values for SVM parameters.

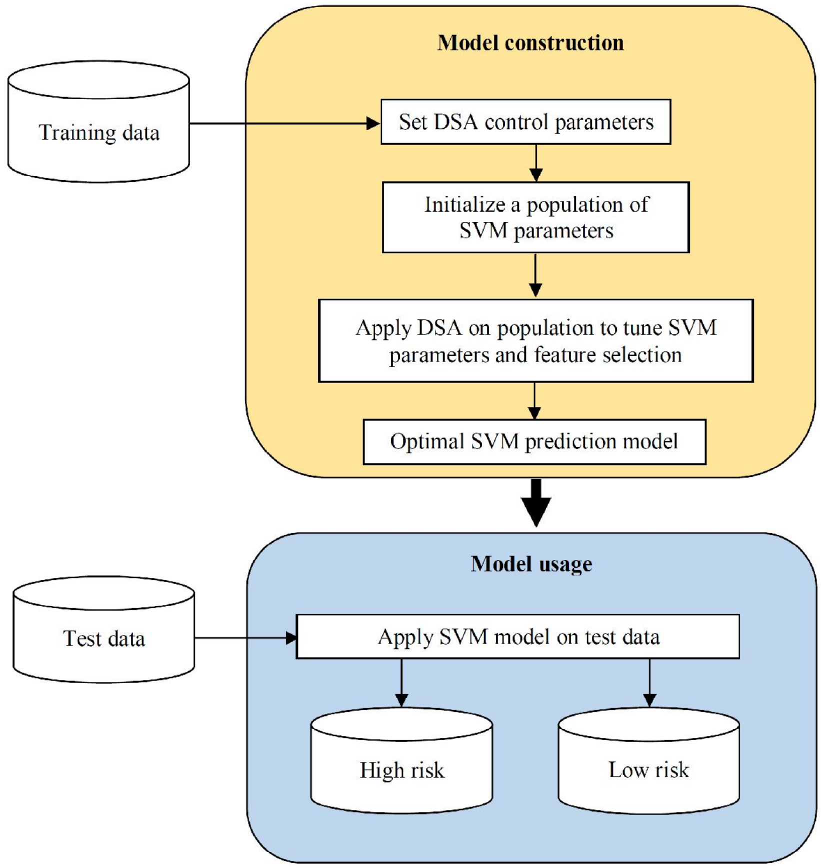

Figure 2. Flowchart of the proposed DSA-SVM algorithm

5. Numerical results and discussion

To evaluate the proposed model a benchmark data set including 400 records of STEMI-CS patients is used. The dataset is about patients admitted due to STEMI-CS complication in Shahid Madani Hospital of Tabriz University of Medical Sciences. The collected data are related to a 10-year period from 2009 to 2018. This dataset includes five main features: demographic characteristics, type of myocardial infarction, risk factors, clinical symptoms, and type of treatment used. The patient's demographic characteristics include age and gender. The age of patients is in the range of 20 to 99 years. Risk factors include diabetes, high blood pressure, kidney failure, smoking, history of myocardial infarction, and history of coronary artery bypass surgery. Chest pain, symptoms of shortness of breath and disturbance of consciousness are among the clinical symptoms of the patient. Therapeutic strategies used include thrombolytics, mechanical ventilation, non-thrombolytic drug therapy, glycoprotein IIb/IIIa inhibitors, inotropes, percutaneous vascular interventions, coronary artery bypass graft surgery, balloon pump and complete vein revascularization. It should be noted that 80% of the data set records are considered as training set, and 20% of the records are considered as test set.

The proposed DSA-SVM method is compared with three well-known methods including the standard SVM model, LASSO regression, and ANFIS. The parameters of these algorithms are configured based on the values specified by the authors in the original publications. In order to determine the most effective factors in predicting the morality risk of STEMI-CS patients, the proposed DSA-SVM model is trained with different combinations of patient admission features and treatment strategies. Then the best combination of features that provide the highest prediction performance is considered as the best combination. To select the most effective features, first all the features are considered to train the SVM model, then the remaining features are ignored one by one and the model is trained with the same structure.

Figures 3 and 4 show the results generated by the predictive models on the training and test datasets, respectively. The results show that the proposed DSA-SVM model is more efficient than other models. The DSA-SVM model has reached 83.47% precision on the training dataset and 81.67% precision on the test dataset. The LASSO and ANFIS models have almost the same performance. Comparing SVM and DSA-SVM models, it can be easily seen that the using DSA in optimizing the SVM parameters has worked well and has improved its performance.

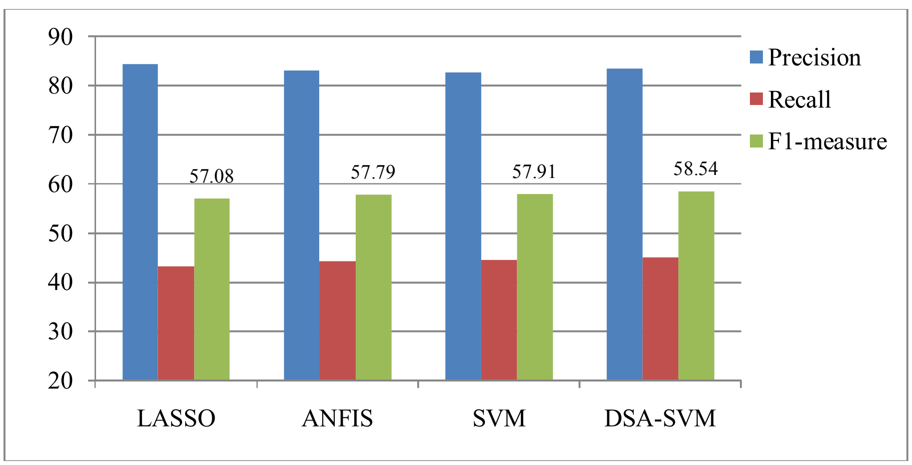

Figure 3. Comparison of the performance of predictive models on training dataset

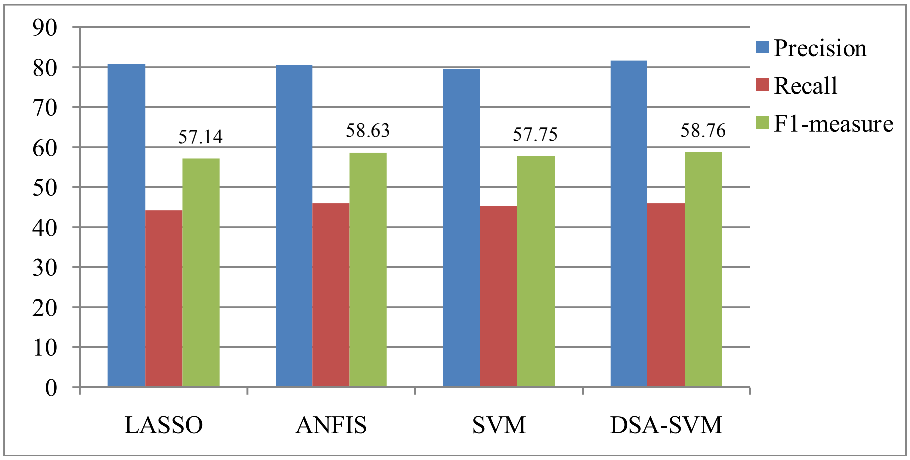

Figure 4. Comparison of the performance of predictive models on test dataset

Figures 5 and 6 respectively show the improvement rate of the proposed DSA-SVM algorithm compared with its counterpart methods on training and test datasets in terms of *F1*-measure. The improvement rate of DSA-SVM model in comparison with LASSO, ANFIS, and SVM models on the training dataset is equal to 2.49%, 1.29%, and 1.07%, respectively. On the test dataset, the improvement rate of DSA-SVM model compared to LASSO, ANFIS and SVM models is 2.76%, 0.22%, and 1.73%.

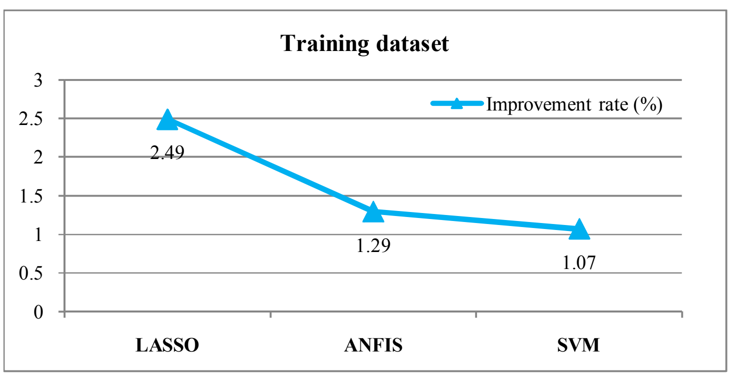

Figure 5. Improvement rate of DSA-SVM compared with its counterparts on training dataset

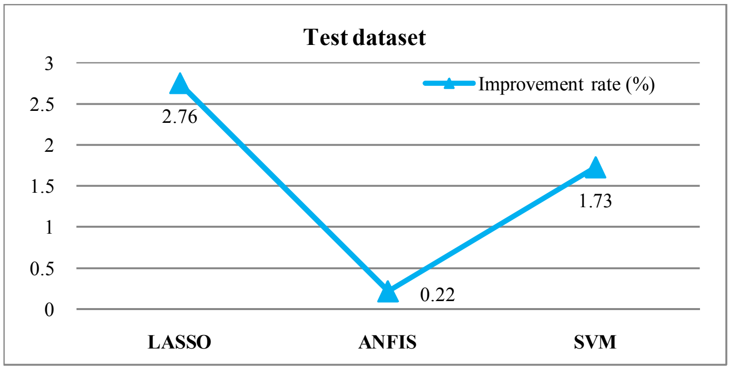

Figure 6. Improvement rate of DSA-SVM compared with its counterparts on test dataset

6. Conclusion

In this study, we introduce a hybrid data mining method in order to predict the morality risk of STEMI-CS patients. The proposed method is based on differntial search algorithm and support vecttor regression. The DSA algoithm is used to find the optimal subset of patient features that have the highest effect on risk prediction and also configure the parameters of SVM. The present method are applied on a collection of STEMI-CS ptients. the results are shown by figures and table. The method has significant perfrmance compared with counterpart methods. One of the interesting work is to apply the proposed method for further problems and assess its application at interdiciplinary area.

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A Markov Chain Monte Carlo Simulated Annealing Algorithm for Path Traveling Salesman Problem

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Abstract

The path traveling salesman problem is one the most well-known NP-complete problem, where it is restricted to traverse all the nodes exactly ones between two prespecified the source and the destination nodes. So, one the most discussed challenges for this kind of a hard problem is to find a good solution by some. There are some approximate algorithms with tight approximation ratio in the version of the problem that the arc cost values satisfy the triangle inequality. The simulated annealing algorithm is a local search method, which is based on the Markov chain formulation and it discovers a nearby optimal solution. The simulated annealing algorithm produces a solution state space and the Markov chain Monte Carlo search method improves the produced state space. The Metropolis algorithm and Boltzmann distribution function have the critical role to accept or reject the produced solution. The simulated annealing algorithm starts with a relatively high temperature and then the most of the solutions are accepted; however, it is cooled gradually, and at end of the annealing process the temperature is so low that the most of the energy increasing solutions (for a minimizing objective function) are rejected. The Markov chain Monte Carlo method produces some samples around the accepted and the rejected states for the possibly improving directions in the solution state space.

Keywords: Simulated annealing, Markov chain Monte Carlo, Metropolis algorithm, Traveling salesman problem, Sampling methods, Local search methods.

1. Introduction

In the stochastic network optimization problems, decisions are made over some statistical information of the parameters of the network. Kalaia and Vempala [1] picked a path from a given source toward a given destination, and then the times on the all arcs are revealed. Polychronopoulos and Tsitsiklis [2] assumed the deterministic unknown arc costs. Provan [3] considered the stochastic traversal costs of arcs those become known upon arrival at the tail of arcs. Ardakani and Sun [4] assumed the stochastic realization of the arc costs. György et al. [5] considered adversarial changes of arc weights in a directed acyclic graph. Awerbuch and Kleinberg [6] assumed a network with unknown arc delays varying adversarially over time. Also, there are some online versions of TSP applying some advanced statistical information [7]-[12].

The symmetric travelling salesman problem (S-TSP) is known as NP-hard problem [13]. However, there are some approximation algorithms where the arc costs satisfy the triangle inequality (see [14] and [15]). In our considered model of the S-TSP the costs of arcs are not revealed until the end of the optimization process. Instead, we use the expected values as the given advanced information. We establish a discrete time Markov chain (DTMC) in undirected networks according to the uniform distribution of the transition probabilities. The states of DTMC are feasible tours and they traverse every node of the network exactly once. Then, the simulated annealing (SA) is applied to obtain a good improvement of the initial approximated solution. However, the local optimality is one of the critical challenges for SA, and we apply the Markov chain Monte Carlo (MCMC) sampling method for the rejected solutions (states) by SA.

2. Preliminaries

We assume a complete undirected network G=(N,A) with node set N and arc set A. For any (i,j)∈A, the cost from node i toward j, c_ij, is equal to the cost from node j toward i, c_ji (symmetry), also $c_{ij} \leq c_{ik} + c_{kj}$ for any (i,j)∈A and k∈N (triangle inequality). In our model there is some advanced statistical information according to the expected values of the costs. The triangle inequality is an essential assumption to approximate the solution of TSP (see [13] and [16]).

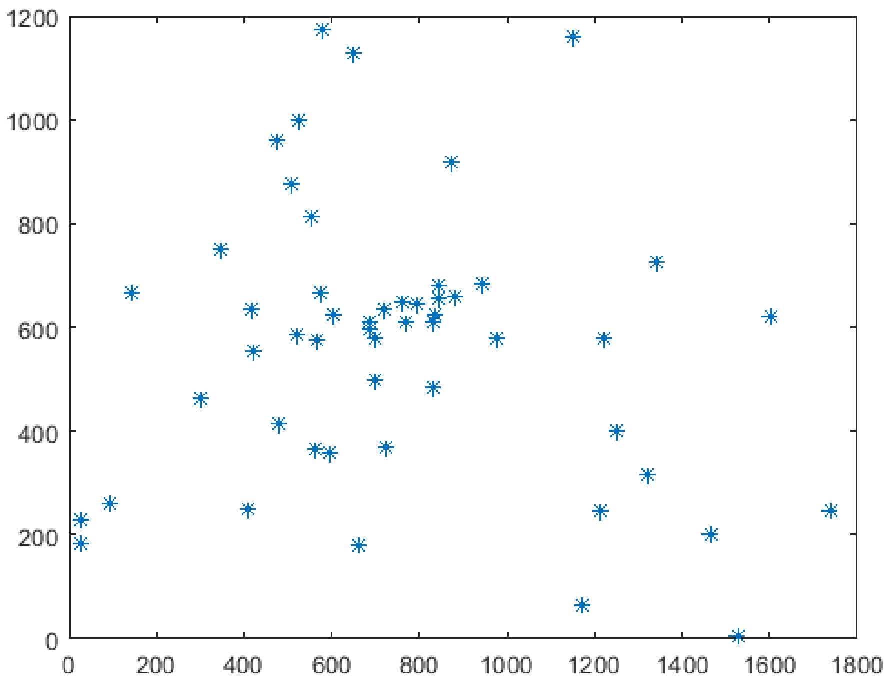

Figure 1. The instance network berlin52 with Euclidean distances.

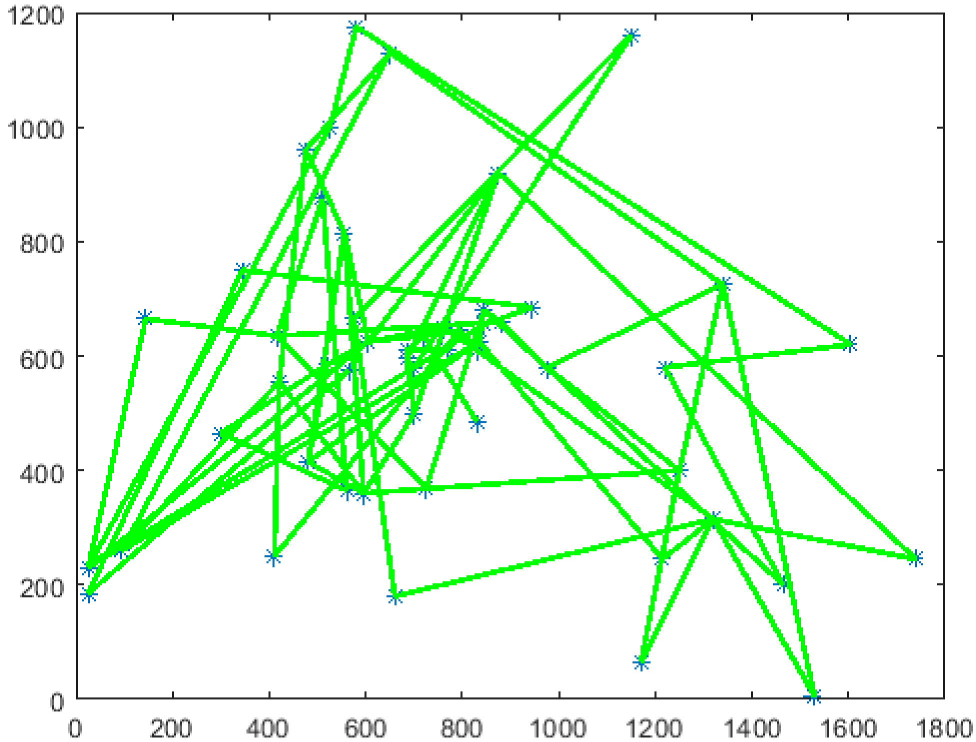

Figure 2. The minimum spanning tree for the instance network berlin52.

For example, the instance network berlin52 is shown in figure 1 according to the node coordination set and two-dimensional Euclidean distance function presented by TSPLIB [24].

Initially, an approximated solution is created by Christofides algorithm [14] which is applied to the TSP with triangle inequality assumption on the arc costs. It must be decided to move from the current node toward the next node starting from the given source node and return to it finally after traversing all the nodes exactly once. The decisions are made according to the obtained limiting probabilities and the expected costs. Figure 2 shows the minimum spanning tree of the instance network berlin52, thus it is transformed into an 1.5-approximate solution as shown in figure 3.

Kalaia and Vempala [1] defined some periods for online made decisions and in each period they chose the decision which has done best so far. Polychronopoulos and Tsitsiklis [2] followed a policy that leads to a path with minimum expected cost. Ausiello et al. [13] and Wen et al. [12] assumed to traverse a number of nodes and not all the nodes. Ausiello et al. [9] and Zhang et al. [11] considered it is possible to visit a node more than once. Jaillet and Lu [8] considered some penalties for not served online made requests and tried to minimize the time of accepted requests and the penalties of rejected ones. In our proposed model, it is restricted to traverse any node exactly once and to return to the source node finally.
The initial ρ-approximated solution is used to establish a DTMC on the network. The minimum spanning tree implies a 2-approximation solution (see [16]) and Christofides [14] produced a 3/2-approximation solution. After producing an expected ρ-approximation of the optimal solution, next possible online decisions are considered as the created state by the current state with the uniformly distributed transition probabilities.

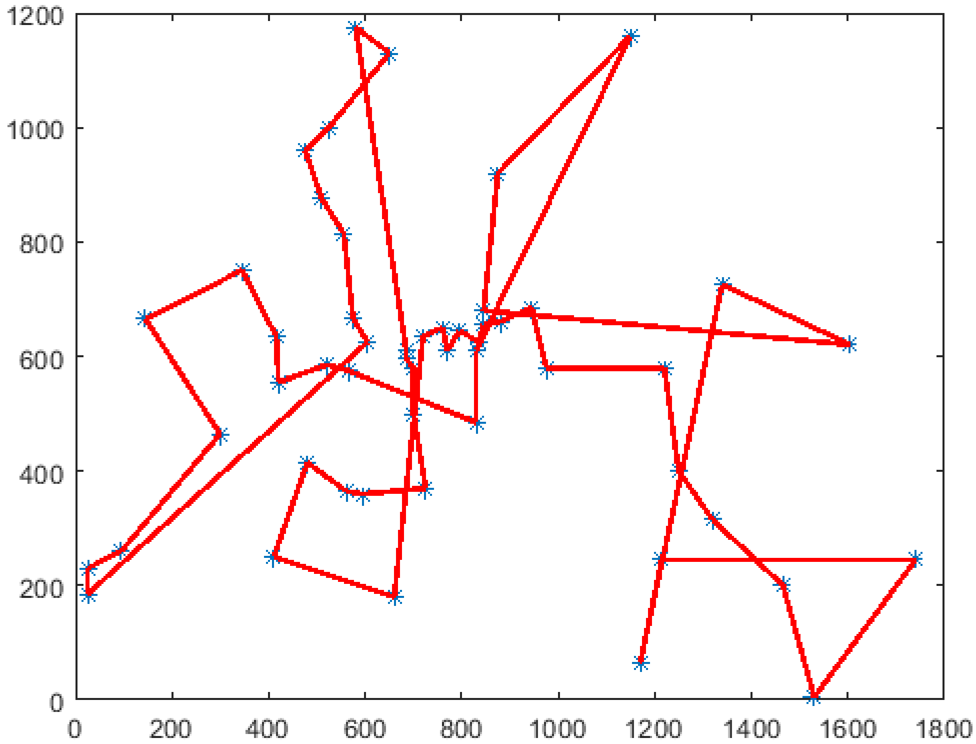

Figure 3. The obtained 1.5-approximate solution for the instance network berlin52.

3. The established discrete time Markov chain

A DTMC is established at any iteration as introduced by Shirdel and Abdolhosseinzadeh [27]; so, by transition from the current state toward a new state exactly one arc is traversed. The process is started and ended in the given source node when all the nodes are traversed exactly once. State $S_{t,k}$, t=1,…,n-2 and k=1,2,…,n-(t+1) contains a set of nodes those are created some tour. The initial state contains the pre-obtained approximated solution and its first node is fixed (the source node), and a fixed node is defined as the node which was traversed previously. We suppose one node is allowed to be fixed at any time t$. The next state S_{t+1,j}$ is created only by single permutation of the unfixed nodes of the current decided state $S^*_{t,i}$. So, if v_{k-1} is the last fixed node of state $S^*_{t,i}=\{1=\bar{v}_1,\bar{v}_2,\ldots,\bar{v}_{k-1},v_k,v_{k+1},\ldots,v_n,1\}$ then state S_{t+1j} is created by permutation node v_k with unfixed nodes v_{k+1}, v_{k+2},…,v_n (|N|=n). So, the total size of the general search space for network G with n nodes is (n-2)(n-1)/2+1. The initial state $S_{0,1}$ contains the pre-obtained approximated solution and its first node is fixed. There is not any repeated state among those are created by an unfixed index. Clearly, any state could be accounted in its opposite direction and both are the same. The source node is the first and the last node of the created states, so the permutation is a circular permutation. Where, there are at least two fixed indices, the permutation does not cause a repeated state because it is not possible double (or more) permutation and just a single node allowed to permute. To determine the repeated states consider the created states starting from the initial state. The repeated state cannot occur after index 2n-5 (double permutation is not possible), however, it may be occurred before in following way: the initial state could be repeated in index n-2, so $v_{n-1}=v_3$ then [27]

$$\{1=\bar{v}\_1,v_2,v_3,\ldots,v_{n-1},v_n,1\}\equiv\{1=\bar{v}_1,v_n,v_3,\ldots,v_{n-1},v_2,1\}\Rightarrow n\text{-}1=3\Rightarrow n=4.$$

The state of index n-2 (where v_{k2} is unfixed) could be repeated through the states from n-1 to 2n-5 (where $\bar{v}_{k2}$ is fixed). Suppose index r is a repeated state

$$\{1=v_1,v_n,v_3,\ldots,v_{n-1},v_2,1\}\equiv\{1=\bar{v}_{l\,1},\bar{v}_{l2},v_{l\,3},\ldots,v_{l\,n-2},v_{l\,n-1}\},v_{l\,n},1\}$$

then $\bar{v}_{l\,1}=v_1=1,v_{l\,n-1}=v_3$ and $v_{l\,n}=v_2$, so if $v_4=\varnothing$ then $v_{l\,n-2}=\varnothing$ and n=5, otherwise if $v_4\neq\varnothing$ then $v_{l\,n-2}=v_4$ and n=6.

The arc costs realizations were done by Provan [3], and then the states of their considered Markov process are created. Polychronopoulos and Tsitsiklis [2] considered pairs of the location and the information sets for the states of a Markov chain. In this paper the established DTMC applies the expected improvement of the initial approximate solution.

4. The simulated annealing heuristic

Markov decision problems are solved in polynomial time [17], however the computations for the state space creation grow exponentially in practice [18]. So, for the large size networks, we apply a Markov chain form of the SA over the created states by DTMC

The Markov chain formulation of the SA presented by [19]. Suppose $\bar{C}_{S_{t,j}}(T)$ is the expected cost of state $S_{t,j}$ when the temperature is T, then $\Delta\bar{C}_{i,j}(T)=\bar{C}_{S_{t+1,j}}(T)-\bar{C}_{S^*_{t,i}}(T)$ is the expected cost difference of the transition from state $S^*_{t,i}$ toward state $S_{t+1,j}$. The acceptance probability $A_{i,j}(T)$ is the probability of the accepting $S_{t+1,j}$, when the online policy determined to be in $S^*_{t,i}$ previously. By Metropolis rule the acceptance probabilities are defined as following

$$A_{i,j}(T)=\begin{cases} e^{\frac{-\Delta\bar{C}_{i,j}(T)}{T}} & \text{, if } \Delta\bar{C}_{i,j}(T)>0 \\ 1 & \text{, if } \Delta\bar{C}_{i,j}(T)\leq 0. \end{cases}$$

In the case $\Delta\bar{C}_{i,j}(T)>0$, state $S_{t+1,j}$ is accepted if by producing a random number r, then it satisfies $r\leq e^{-\Delta\bar{C}_{i,j}(T)/T}$.

For the SA the transition probability $M_{i,j}(T)$ form state $S^*_{t,i}$ toward state $S_{t+1,j}$ is obtained as following

$$M_{i,j}(T)=\begin{cases} p_{i,j}A_{i,j}(T) & \text{, if } i\neq j \\ 1-\sum_{k\neq i}p_{i,k}A_{k,j}(T) & \text{, if } i=j. \end{cases}$$

If the online policy decided to be in $S^*_{t,i}$ previously, then the static probability $q_{i,j}(T)$ represents the probability that the online policy decides to be in state $S_{t+1,j}$, when the static condition is reached (the static probabilities are equivalent to the limiting-state probabilities that show the steady state analysis for the established DTMC, for example see [20]).

So, for the instance network berlin52 the improved solution by the SA algorithm is shown in figure 4.

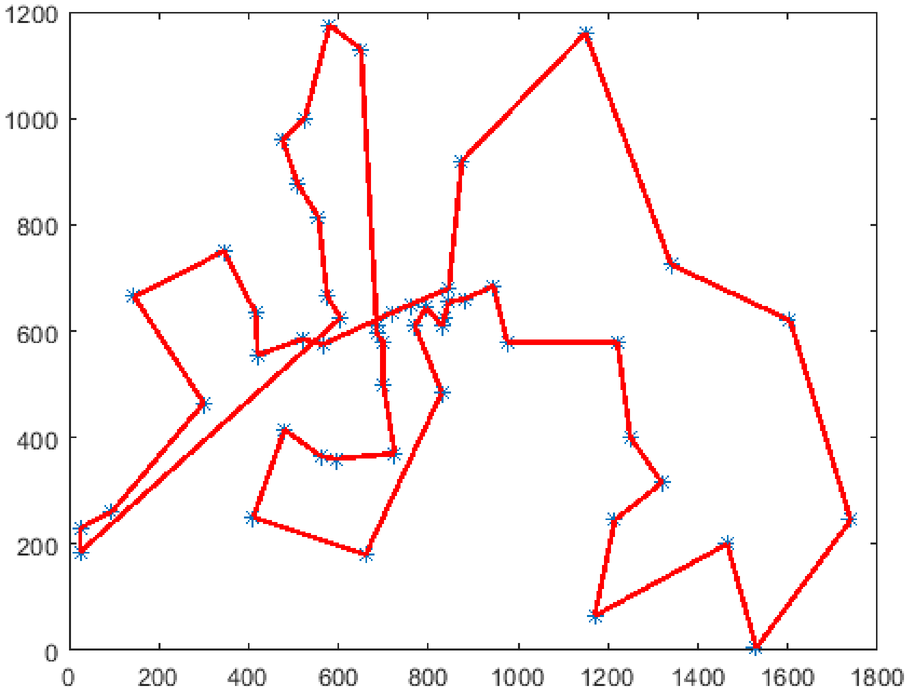

Figure 4. The improved solution from the initial 1.5-approximate solution for the instance network berlin52.

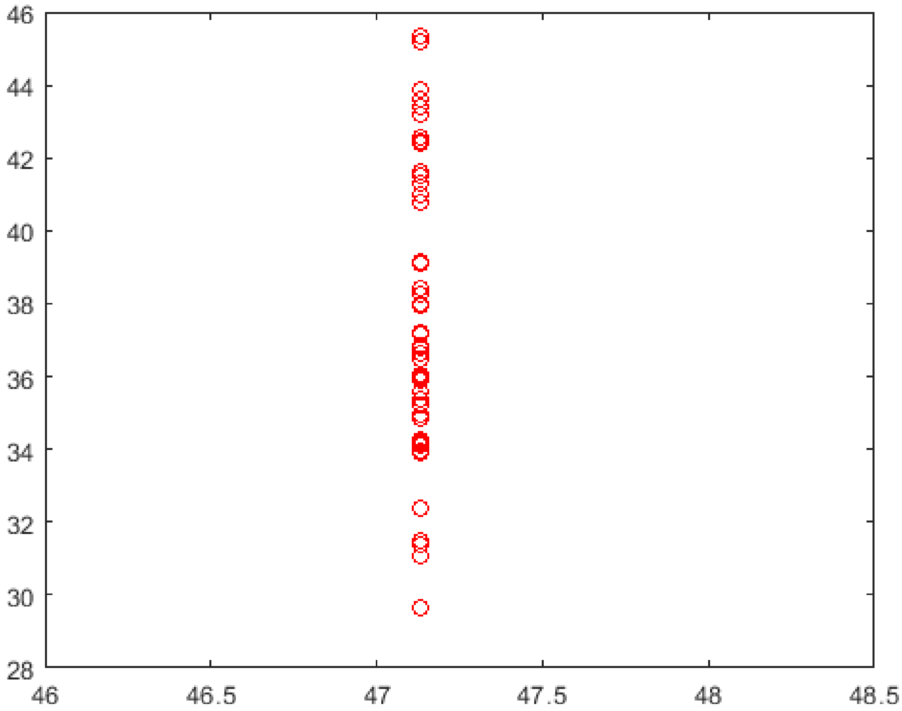

Figure 4. The improved solution from the initial 1.5-approximate solution for the instance network berlin52.

5. The Markov chain Monte Carlo sampling method

In the SA, it is not explored around the rejected solutions to avoid possibly the local optimality. So, we apply the sampling method based on the limiting state distribution, and some states are produced for a rejected state so the state space is explored for the nearby optimal solution. The goal distribution for producing the sample by MCMC is Boltzmann distribution function $A_{ij}(T)$ [28]. One of the produced samples according to a rejected solution by the SA is shown in figure 5; the axis values determine the absolute differences of the produced sample costs from the rejected solution cost. So, figure 5 shows the explored solutions from the rejected states by the SA for the instance network berlin52.

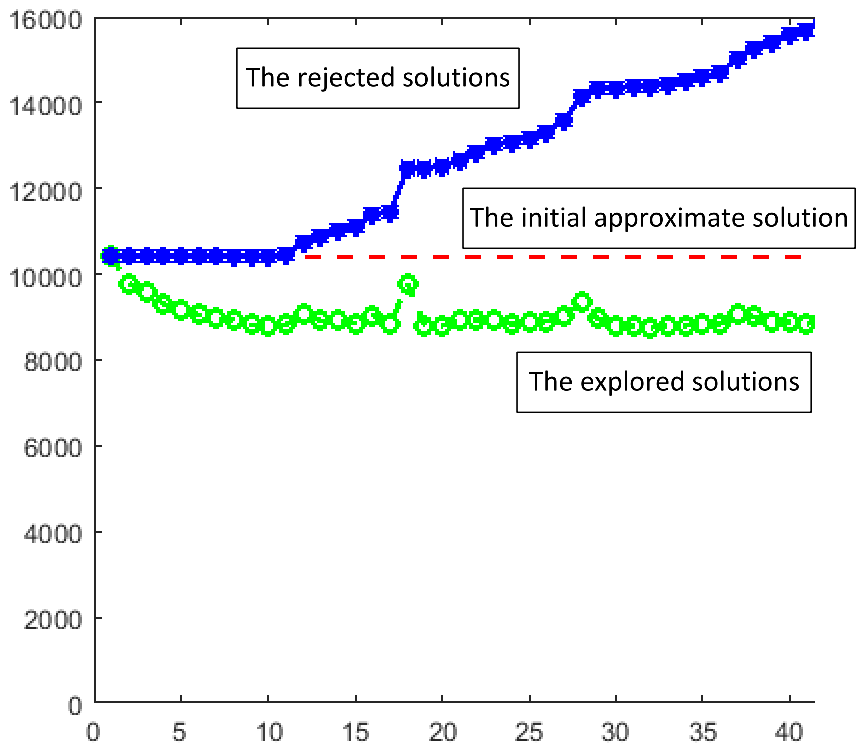

Figure 5. The explored solutions from the rejected solutions by the SA for the instance network berlin52.

By improvement of the initial approximate solution and the exploration around the rejected solutions the optimal solution of the instance network berlin52 is shown in figure 6.

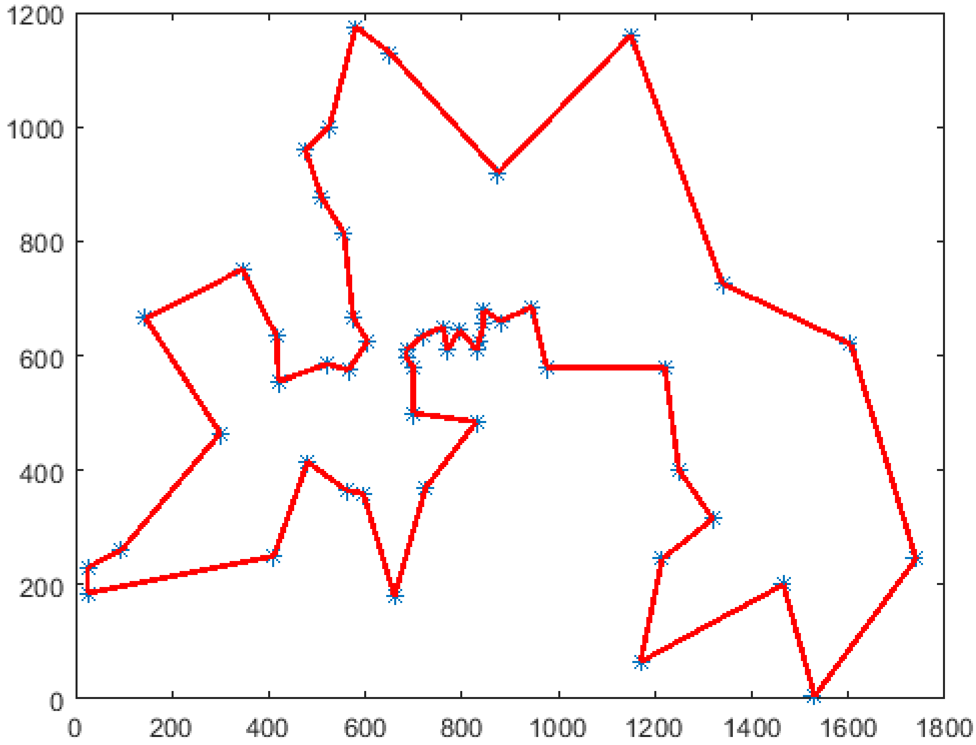

Figure 6. The optimal solution for the instance network berlin52.

6. Conclusion

The Christofides' approximation algorithm is applied to obtain a good 1.5-approximate solution, while the arc cost values satisfy the triangle inequality. So, the simulated annealing algorithm is implemented based on the Markov chain formulation on the local optimal solution and it is improved as 1.3-approximate solution. The simulated annealing algorithm produces a solution state space and the Markov chain Monte Carlo sampling method explores the state space around the rejected states. Then, the proposed method could improve the initial local optimal solution into a nearby optimal solution.

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A Meta-Heuristic Solution for a Chance Constrained Mathematical Model of the Vehicle Routing Problem with Stochastic Demand

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Abstract

In the globalizing economy, supply chain and logistics management has an important place in the transportation of the processed raw material from the production center to the enterprises and from the enterprises to the end customers. Vehicle routing problems (VRP), which is the last stage of supply chain and logistics management, stand out as increasing demands, changing roads, different optimization problems in a limited time. Vehicle routing problem is defined as providing the necessary service to customers located in different geographical locations from the center, defined as the warehouse, with more than one vehicle at the shortest distance and returning the vehicles to the warehouse after providing the service. Mathematically, vehicle routing problems are modeled stochastically due to some uncertainties arising from parameters such as time, route, service and demand. Stochastic demand vehicle routing problems (SDVRP) are modeled as problems where customer demands are not known precisely beforehand and the service vehicle is known exactly after reaching the customer location.

In this study, the chance-constrained SDVRP model was considered and the near-optimal solutions of the problem were calculated with the meta-heuristic algorithm, Simulation Annealing (SA).

Keywords: Optimization, Stochastic Vehicle Routing, Simulation Annealing

1. Introduction

Vehicle routing problem (VRP) is one of the problems that requires finding the most suitable routes with minimum cost in order to serve customers in different geographical locations with one or more vehicles from one or more warehouses. VRP is a combinatorial (discrete) optimization problem used to design an optimum route for a fleet of vehicles to serve a range of customers, given a number of constraints [1]. Combinatorial vehicle routing problem is one of the NP-Hard (Nondeterministic polynomial) problems in its simplest form [2]. In VRP, it is generally defined as minimizing the total route distance, keeping transportation costs to a minimum, minimizing the auxiliary action (penalty) costs and the number of vehicles that will meet the demand, as a result of the distribution of the demands piece by piece, taking into account the purpose functions, vehicle capacity and service time. [3] and [4]. In VRP, the problems in which the parameters are known beforehand are known as deterministic. Problems involving probabilistic information where the parameters are not known beforehand are called stochastic vehicle routing problems (SVRP). The mathematical model for SVRP is problems in which some or all parameters of the routing problem are random. Typically, these problems are modeled when customer demands, travel times, customers, and service times are stochastic. Such situations occur in real life, in problems where precise determination of parameters is difficult. The basic idea in the solution of stochastic VRP is to transform the probabilistic structure of the problem into its equivalent, deterministic model. SVRP is modeled in two ways in the literature: chance-constrained stochastic programming and auxiliary-action (recourse) stochastic programming [5].

The first study in VRP started in 1959 when Dantzig and Ramser created an optimum route between a fleet of gasoline delivery trucks and service stations [6]. In 1964, Clarke and Wright conducted the study that required choosing the most appropriate possible route for a fleet of trucks from a central warehouse to multiple delivery points located at different locations [7].

In 1992, Bertsimas proposed a heuristic method for the stochastic model of customer demands in capacity VRP and showed that probabilistic analysis techniques and results are a powerful and useful alternative to the re-optimization strategy [8]. In 1992, Teodorovic and Pavkovic developed a stochastic programming model with auxiliary action for SDVRP. In the problem, they assumed that the customer demands came from a uniform distribution and used the annealing simulation algorithm to solve the problem [9]. In 2003, Hu et al. discussed SDVRP and extended it with real-time information for the dynamic vehicle routing problem. They modeled the problem with chance constraint and solved it in CPLEX with branch-and-bound techniques [10].

In 2006 Tavakkoli-Moghaddam et al solved the model in their study with hybrid SA based on nearest neighbor. The proposed model enabled the creation of routes that will serve all customers with the minimum number of vehicles and maximum capacity [11]. In 2020, İlhan used the SA algorithm for capacity VRP in his study. He used three different route development operators of the algorithm, namely the change, addition and inversion operator [12].

2. Material and Method

In this study, a data set consisting of 1 warehouse, 20 customers, customer coordinates and 5 daily requests was randomly generated in Matlab. A SDVRP model has been created, which ensures that the stochastic demands with %95 probability obtained from the data set are met. The model was modeled as chance constrained and converted to its equivalent, deterministic integer mathematical model. The deterministic model is solved with a meta-heuristic annealing simulation algorithm.

Branches	X	Y	Demand1	Demand2	Demand3	Demand4	Demand5	Average	Variance
1	9	130	40	35	40	25	25	33	57,5
2	70	36	50	45	65	30	90	56	517,5
3	94	44	40	50	50	20	55	43	195
4	20	47	35	40	25	20	50	34	142,5
5	56	15	45	35	45	35	45	41	30
6	84	97	60	50	50	35	50	49	80
7	9	163	55	50	55	45	50	51	17,5
8	94	190	65	60	60	55	50	58	32,5
9	39	166	80	75	70	55	50	66	167,5
10	23	20	90	95	90	80	70	85	100
11	60	164	25	30	25	25	35	28	20
12	39	110	65	60	50	45	40	52	107,5
13	37	197	45	50	55	45	50	49	17,5
14	33	117	80	85	70	70	75	76	42,5
15	73	59	60	60	55	50	55	56	17,5
16	64	163	75	70	60	65	60	66	42,5
17	38	156	60	65	60	65	65	63	7,5
18	95	142	120	125	110	100	120	115	100
19	51	44	65	60	60	55	60	60	12,5
20	51	145	50	60	60	50	50	54	30
Total			1205	1200	1155	970	1145	1135	

Table1: Customer coordinates, demands, mean and variance

Table 1 contains the data set consisting of customer coordinates, 5-day customer demands, mean and variance.

In this study, it is assumed that customer demands come from a normal distribution with $\mu_i$mean and σ_i^2variance. Based on this assumption, the stochastic demand quantities obtained from the mean and variances of the 5-day demands for each data group were calculated from the expression $\mu_j + z_{1-\alpha}\,\sigma_j$ with the help of the standard normal table (Z) specified in the model. Here, the $z_{1-\alpha}$value is an upper bound value on the right side of the standard normal table.

No	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Demand	45	93	66	54	50	64	58	67	87	101	35	69	56	87	63	77	68	131	66	63

Table2: Customer demands with 95% probability

With %95 probability, total customer demands are taken as 1400 and α =0.05, $z_{1-\alpha}$ =1.645 and given in Table 2.

Stochastic mathematical modeling tries to find solutions for cases where some parameters of the problem are random variables. These random situations are often seen in real life problems where it is difficult to determine the parameters precisely. The basic idea in stochastic mathematical modeling is to transform the probabilistic nature of the problem into the deterministic state, which is the equivalent of the problem. In this study, chance-constrained mathematical modeling technique will be discussed. The near-optimal solution is obtained by transforming the chance-constrained model into a deterministic model. The name of the chance constraint comes from the fact that the constraint occurs with a minimum probability of $1-\alpha$ [13].

3. Mathematical Model for Chance Constrained ARP with Stochastic Demand

The STARP problem is defined on the undirected graph $G=(V,E)$, $V=\{v_0,\dots,v_n\}$ set of vertices (customers), $E=\{(v_i,v_j): v_i,v_j \in V, i<j\}$ edge (springs) set. Repository is represented by $v_0$and customers are represented by $\{v_1,\dots,v_n\}$. The travel cost (distance) of each $(v_i,v_j) \in E$ edge is c_{ij}. There are k vehicles in the warehouse, each with a capacity of C. It is assumed that customer claims are distributed uniformly and independently. Each customer has a demand for d_i from a stochastic, known probability distribution (normal distribution) with μ_i mean and σ_i^2 variance. Vehicle routes planned in the first place should start and end at the warehouse. Each customer should be visited once with a single vehicle [14].

The closed form of the chance-constrained model for SDVRP is as follows [15].

$$\text{MinZ} = \sum_k \sum_{i,j} c_{ij} x_{ijk} \tag{1}$$

$$P\left[\sum_{i,j} d_i x_{ijk} \le C\right] \ge 1-\alpha, \qquad k=1,\dots,K \tag{2}$$

$$x=\{x_{ijk}\} \in S_k \tag{3}$$

In the closed model, the Eq (1) objective function ensures that the total distance traveled is minimized. Eq (2) is a constraint of chance.

Here,

c_{ij} : the distance from node i to node j,

x_{ijk}:1 if vehicle k is going from i to j, 0 otherwise.

K : Number of vehicles available,

S_k : All possible solution sets of the K-traveling salesman problem,

d_i : Random variable representing the demand of customer i, $d_i \sim N(\mu_i, \sigma_i^2)$,

α : İs the maximum breakage probability allowed for course break.

C : Vehicle capacity,

4. A Chance Constrained Integer Programming Model for Stochastic Demand VRP

The purpose of chance constrained integer programming is to ensure that the total demands of customers on a route do not exceed the vehicle capacity C'yi, and the minimum route length $(Pr \leq \alpha)$ is below the specified limit or probability level (α). It is accepted that the demands expressing the stochastic situation in the model provide a normal distribution [15].

If demand d_i,is $d_i \sim N(\mu_i, \sigma_i)$ while $X = \sum_{i=1}^{n_k} d_i$, is taken as the sum of the demands of customers on a route,

$X \sim N\left(\sum \mu_i, \sqrt{\sum \sigma^2{}_i}\right)$ $\quad z = \frac{X - \sum \mu_i}{\sqrt{\sum \sigma^2{}_i}}$ with the transformation $z \sim N(0,1)$ becomes. (4)

Since the probability that the total demands on the route will exceed the vehicle capacity will be α at most, $P(X > C) \leq \alpha$ can be written.

If necessary adjustments are made, it becomes $1 - P(X \leq C) \leq \alpha$ or $P(X \leq C) \geq 1 - \alpha$.

If z transform is done from here,

$$P\left(z \leq \frac{C - \sum \mu_i}{\sqrt{\sum \sigma^2{}_i}}\right) \geq 1 - \alpha \tag{5}$$

$$P(z \leq z_{1-\alpha}) = 1 - \alpha \tag{6}$$

$$z_{1-\alpha} \leq \frac{C - \sum \mu_i}{\sqrt{\sum {\sigma^2}_i}} \tag{7}$$

$$\sum \mu_i + z_{1-\alpha}\sqrt{\sum {\sigma^2}_i} \leq C \tag{8}$$

Adding the variable x_{ijk}, 0-1 to the above equation results in the following new capacity constraint.

$$\sum \mu_i x_{ijk} + z_{1-\alpha}\sqrt{\sum {\sigma^2}_i x_{ijk}} \leq C, \qquad j = 1,2,\dots,n \tag{9}$$

$$\sum_{j=1}^{N}\sum_{\substack{i=0\\ i\neq j}}^{N} \mu_j x_{ijk} + z_{1-\alpha}\sum_{j=1}^{N}\sum_{\substack{i=0\\ i\neq j}}^{N}\sqrt{{\sigma^2}_j x_{ijk}} \leq C, \qquad \forall_k \text{ için} \tag{10}$$

Eq (10) is a non-linear constraint, so we can obtain the new linear Eq (12) constraint by using Eq (11) [15] and [16].

$$\sqrt{\sum_{i=1}^{n} a_i^2} \leq \sum_{i=1}^{n} a_i, \qquad a_i \in \Re^+ \tag{11}$$

$$\sum_{j=1}^{N}\sum_{\substack{i=0\\ i\neq j}}^{N} \mu_j x_{ijk} + z_{1-\alpha}\sum_{j=1}^{N}\sum_{\substack{i=0\\ i\neq j}}^{N} \sigma_j x_{ijk} \leq C \tag{12}$$

Thus, the nonlinear constraint for the model is linearized.

5. Proposed Linear Approximation Model for Chance Constrained VRP with Stochastic Demand

$$\text{MinZ} = \sum_{i=0}^{N}\sum_{j=0}^{N}\sum_{k=1}^{K} c_{ij}x_{ijk} \tag{13}$$

Constraints:

$$\sum_{j=1}^{N}\sum_{k=1}^{K} x_{ijk} \le K, \qquad i = 0 \quad \text{depodan çıkışların kontrolü için,} \tag{14}$$

$$\sum_{\substack{i \\ i\neq j}} x_{ijk} - \sum_{\substack{j \\ j\neq i}} x_{jik} = 0, \qquad \forall_{j,k} \quad \text{için} \tag{15}$$

$$\sum_{j=0}^{N}\sum_{k=1}^{K} x_{ijk} = 1, \qquad i \neq j \;\; and \;\; i \neq 0\,, \;\; \forall\, i \;\; \text{için} \tag{16}$$

$$\sum_{i=0}^{N}\sum_{k=1}^{K} x_{ijk} = 1, \qquad j \neq i \;\; and \;\; j \neq 0 \tag{17}$$

$$\sum_{j=1}^{N}\sum_{\substack{i=0 \\ i\neq j}}^{N} \mu_j x_{ijk} + z_{1-\alpha}\sum_{j=1}^{N}\sum_{\substack{i=0 \\ i\neq j}}^{N} \sigma_j x_{ijk} \le C \tag{18}$$

$$U_{ik} - U_{jk} + N * x_{ijk} \le N - 1, \qquad i \neq 0, \qquad j \neq 0 \tag{19}$$

$$x_{ijk} = 0 \text{ or } 1, \tag{20}$$

U_{ik} : An arbitrary number greater than 0, $i \neq 0$

Eq (13) objective function in the model minimizes the total distance. Eq (14) constraint shows that the number of vehicles leaving the warehouse should be at most K. Eq (15) ensures that the number of arcs exiting and entering a node is equal. Eq (16) and Eq (17) constraints ensure that a node is visited by only one tool. The Eq (18) constraint represents the capacity constraint with the linear approach. The Eq (19) constraint is the sub-round elimination constraint set [17]. Eq 20) are decision variables.

6. Simulation Annealing Algorithm

Simulation Annealing (SA) algorithm, first developed by Metropolis in 1953, is a stochastic search algorithm that simulates energy changes in a cooling system until it converges to an equilibrium state (freezing state) [18]. SA is a probability-based optimization algorithm generally used for discrete optimization problems, inspired by the slow cooling of solids after heating until crystallization [19]. It is used in the SA algorithm to scan the solution area and to select a better solution than the previous one in each round. According to this simulation, the temperature value is used to determine the probability of accepting solutions worse than the best solution found. In application problems, the SA algorithm is defined as a random local search method in which the change in the current solution is accepted with a probability that causes an increase in the solution cost.

The algorithm is started with a sufficiently high temperature and at each step a certain number of solutions are obtained before the temperature is lowered. New solutions are either accepted or rejected according to established criteria. Each decrease in temperature affects the probability of leaving the obtained solution and switching to a new solution. The algorithm is terminated when the temperature reaches the lowest value or when the TB algorithm runs for the desired number of repetitions [20].

In the context of combinatorial optimization, a solution corresponds to the specific state of the physical system and the solution cost value corresponds to the energy of the system. At each iteration, the current solution is modified by choosing a random move from a certain transform class (which defines the neighbors of the solutions). If the new solution provides an improvement, it is automatically accepted and the new existing solution is considered. Otherwise, the new solution is,

$$\mathrm{P} = e^{\left(-\frac{\Delta}{kT}\right)} \tag{21}$$

The Eq (21) is accepted according to the Metropolis probability criterion. Here; Δ, is the change in the Objective (Energy) function, T is the Temperature parameter, k is the Boltzmann constant.

Based on the stated criteria, a move with a high temperature and low cost increase seems more likely to be accepted. The temperature parameter is gradually reduced according to some predefined cooling program, and a certain number of iterations are performed at each temperature level. At sufficiently low values of temperature, only improvement movements are accepted and the process stops at a local optimum. Unlike most meta-heuristics, this method converges asymptotically to the global optimum (assuming an infinite number of iterations) [21].

The purpose of the SA algorithm; is to find a solution x that will optimize a function f(x) defined in a subset of all possible solution points (S). The SA algorithm starts searching with a randomly chosen initial solution. Then, with a suitable method, it chooses a solution adjacent to this solution and calculates the change in f(x). If the change is in the desired direction, it takes the neighboring solution as the current solution. If there is no change in the desired direction, the SA algorithm accepts the solution according to the Metropolis probability criterion. The acceptance of the solution, which causes a change in the opposite direction in the objective function, with a certain probability value, ensures that the SA algorithm gets rid of the local optimum points. When the T temperature value is higher than the Eq (21) probability value, most of the increases in the objective function will be accepted. As the T temperature decreases, the amount of acceptance will also decrease. For this reason, the initial temperature value should be chosen high enough to avoid getting stuck at local points in the SA algorithm, and it should be reduced gradually [22].

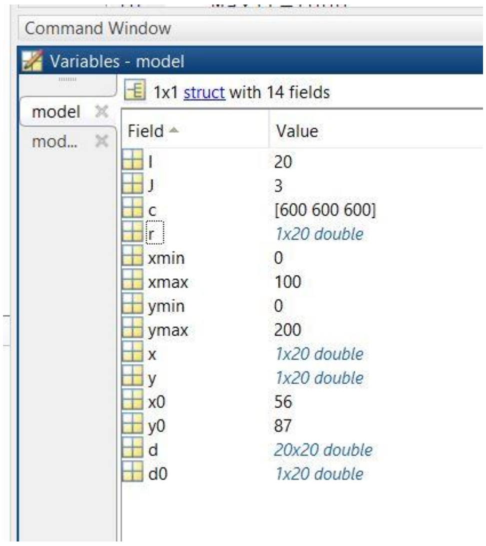

Figure1: Model created in Matlab

In Figure 1, I is the number of customers, j is the number of vehicles, c is the vehicle capacities, r is stochastic demands, x and y is the customer coordinates, x0 and y0 is the warehouse coordinate, d is the distance matrix between customers, d0 is the distance matrix between the warehouse and customers.

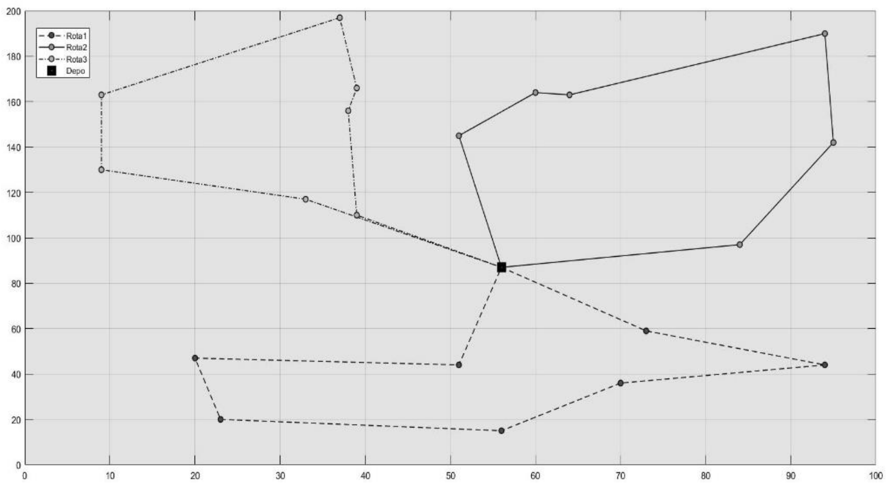

Figure 2: Routes of 3 vehicles calculated in Matlab

Figure 2: It shows the circulation shape of the 3 routes obtained with Matlab in the coordinate axis.

Table1: Routing results where demands are met with % 95 probability

Method	Routes	Route distance	Route request	Total distance
SA	R1 : 0-15-3-2-5-10-4-19-0	244,0781	493	749,7366
	R2 : 0-6-18-8-16-11-20-0	247,7904	437	
	R3 : 0-14-1-7-13-9-17-12-0	257,8681	470	

The table shows the routes and results that start at the warehouse and end again at the warehouse.

7. Conclusion

In this study, a chance-constrained model was created for the stochastic demand vehicle routing problem with 20 customers and 3 vehicles with a capacity of 600. The problem solution of the created model was obtained with the Simulation Annealing algorithm in Matlab, and a near-optimal result was obtained. The demands of each customer were met without exceeding the vehicle capacities from the shortest routes obtained from the solution of the problem. Total customer demands were met with 3 different routes at optimum distance.

8. References

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A New Theorem on Quasi Power Increasing Sequences

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Abstract

The aim of this paper is to generalize a known theorem on absolute Riesz summability for the $\varphi-\left|A;\delta\right|_k$ summability method by using a quasi β-power increasing sequence.

Keywords: Absolute matrix summability, summability factors, quasi β-power increasing sequences, infinite series.

1. Introduction

Let $\sum a_n$ be an infinite series with is partial sums (s_n). Let $A=(a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Let (φ_n) be any sequence of positive real numbers. The series $\sum a_n$ is said to be summable $\varphi-\left|A;\delta\right|_k$, $k\geq 1$, $\delta\geq 0$, if [1]

$$\sum_{n=1}^{\infty}\varphi_n^{\delta k+k-1}\left|A_n(s)-A_{n-1}(s)\right|^k<\infty$$

where

$$A_n(s)=\sum_{v=0}^{n}a_{nv}s_v \;,\quad n=0,1,\ldots$$

Let (p_n) be a sequence of positive numbers such that

$$P_n=\sum_{v=0}^{n}p_v\to\infty \text{ as } (n\to\infty),\ (P_{-i}=p_{-i}=0,\ i\geq 1).$$

For $\varphi_n=\dfrac{P_n}{p_n}$ and $\delta=0$, $\varphi-\left|A;\delta\right|_k$ summability reduces to $\left|A,p_n\right|_k$ summability [2]. Also, for $\varphi_n=\dfrac{P_n}{p_n}$, $\delta=0$ and $a_{nv}=\dfrac{p_v}{P_n}$, we get $\left|\overline{N},p_n\right|_k$ summability [3]. A positive sequence $X=(X_n)$ is said to be a quasi β - power increasing sequence if there exists a constant $K=K(\beta,X)\geq 1$ such that $Kn^{\beta}X_n\geq m^{\beta}X_m$ holds for all $n\geq m\geq 1$.

Every almost increasing sequence is a quasi β - power increasing sequence for any nonnegative β, but the converse need not be true as can be seen by taking an example, say $X_n = n^{-\beta}$ for β >0 [4]. A sequence $(\lambda_n) \in BV$, if $\sum|\Delta\lambda_n| = \sum|\lambda_n - \lambda_{n+1}| < \infty$.

Theorem 1.1 ([5]). Let (X_n) be a quasi β - power increasing sequence for some $0 < \beta < 1$ and let there be sequences (γ_n) and (λ_n) such that

$$|\Delta\lambda_n| \le \gamma_n, \tag{1}$$

$$\gamma_n \to 0 \quad \text{as} \quad n \to \infty, \tag{2}$$

$$\sum_{n=1}^{\infty} n|\Delta\gamma_n| X_n < \infty, \tag{3}$$

$$|\lambda_n| X_n = O(1) \quad \text{as} \quad n \to \infty. \tag{4}$$

If the conditions

$$\sum_{v=1}^{n} \frac{p_v}{P_v} |s_v|^k = O(X_n) \text{ as } n \to \infty, \tag{5}$$

$$\sum_{n=1}^{\infty} P_n X_n |\Delta\gamma_n| < \infty, \tag{6}$$

$$\sum_{n=1}^{m} \frac{|s_n|^k}{P_n} = O(X_m) \quad \text{as } m \to \infty \tag{7}$$

are satisfied, then the series $\sum a_n\lambda_n$ is summable $\left|\bar{N}, p_n\right|_k$, $k \ge 1$.

2. Main Result

There are some papers on quasi β - power increasing sequences, see ([6-15]). Before stating our main theorem, we must first introduce some further notations. Given a normal matrix $A = (a_{nv})$, two lower semimatrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ are defined as follows:

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni} \qquad n, v = 0, 1, 2, \ldots \tag{8}$$

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v} \qquad n = 1, 2, \ldots \tag{9}$$

and

$$A_n(s)=\sum_{v=0}^{n}a_{nv}s_v=\sum_{v=0}^{n}\bar{a}_{nv}a_v \tag{10}$$

$$\bar{\Delta}A_n(s)=\sum_{v=0}^{n}\hat{a}_{nv}a_v. \tag{11}$$

Theorem 2.1. Let $A=(a_{nv})$ be a positive normal matrix which satisfies the following conditions

$$\bar{a}_{n0}=1\,,\quad n=0,1,\ldots \tag{12}$$

$$a_{n-1,v}\geq a_{nv}\quad \text{for}\quad n\geq v+1, \tag{13}$$

$$a_{nn}=O\left(\frac{p_n}{P_n}\right). \tag{14}$$

Let $(\lambda_n)\in BV$. Let (X_n) be a a quasi β -power increasing sequence for some $0<\beta<1$ and $\varphi_n p_n=O(P_n)$. If conditions (1)-(4), (6) of Theorem 1.1 and

$$\sum_{v=1}^{n}\varphi_v^{\delta k-1}\left|s_v\right|^k=O(X_n)\quad \text{as}\quad n\to\infty, \tag{15}$$

$$\sum_{n=v+1}^{\infty}\varphi_n^{\delta k}\left|\Delta_v(\hat{a}_{nv})\right|=O(\varphi_v^{\delta k-1})\,, \tag{16}$$

$$\sum_{n=v+1}^{\infty}\varphi_n^{\delta k}\left|\hat{a}_{n,v+1}\right|=O(\varphi_v^{\delta k})\,, \tag{17}$$

and

$$\sum_{n=1}^{m}\varphi_n^{\delta k}\frac{\left|s_n\right|^k}{P_n}=O(X_m)\quad \text{as}\quad m\to\infty \tag{18}$$

are satisfied, then the series $\sum a_n\lambda_n$ is summable $\varphi-\left|A;\delta\right|_k$ $k\geq 1$ and $0\leq\delta<1/k$.

We need the following lemmas for the proof of Theorem 2.1.

Lemma 2.1 ([4]). Under the conditions on (X_n), (γ_n) and (λ_n) as taken in the statement of Theorem 2.1, we have

$$nX_n\gamma_n = O(1) \quad \text{as } n \to \infty, \tag{19}$$

$$\sum_{n=1}^{\infty} \gamma_n X_n < \infty . \tag{20}$$

Lemma 2.2 ([5]). Under the conditions (2) and (6), we have

$$P_n X_n \gamma_n = O(1) \text{ as } n \to \infty, \tag{21}$$

$$\sum_{n=1}^{\infty} p_n X_n \gamma_n < \infty. \tag{22}$$

where (X_n) is a quasi β - power increasing sequence for some $0 < \beta < 1$.

3. Proof of Theorem 2.1

Let (M_n) denotes A- transform of the series $\sum a_n \lambda_n$. Then, (10) and (11) imply that

$$\bar{\Delta} M_n = \sum_{v=1}^{n} \hat{a}_{nv} \lambda_v a_v$$

By using Abel's transformation, we get

$$\begin{aligned}
\bar{\Delta} M_n &= \sum_{v=1}^{n-1} \Delta_v(\hat{a}_{nv}\lambda_v) \sum_{r=1}^{v} a_r + \hat{a}_{nn}\lambda_n \sum_{v=1}^{n} a_v \\
&= \sum_{v=1}^{n-1} (\hat{a}_{nv}\lambda_v - \hat{a}_{n,v+1}\lambda_{v+1} - \hat{a}_{n,v+1}\lambda_v + \hat{a}_{n,v+1}\lambda_v) s_v + a_{nn}\lambda_n s_n \\
&= a_{nn}\lambda_n s_n + \sum_{v=1}^{n-1} \Delta_v(\hat{a}_{nv})\lambda_v s_v + \sum_{v=1}^{n-1} \hat{a}_{n,v+1}\Delta\lambda_v s_v \\
&= M_{n,1} + M_{n,2} + M_{n,3} .
\end{aligned}$$

For the proof of Theorem 2.1, we have prove

$$\sum_{n=1}^{\infty} \varphi_n^{\delta k+k-1} \left| M_{n,r} \right|^k < \infty \qquad \text{for} \qquad r = 1,2,3.$$

First, by using Abel’s transformation, we have

$$\begin{aligned}\sum_{n=1}^{m}\varphi_n^{\delta k+k-1}\left|M_{n,1}\right|^k &=\sum_{n=1}^{m}\varphi_n^{\delta k+k-1}a_{nn}^k\left|\lambda_n\right|^k\left|s_n\right|^k\\ &=O(1)\sum_{n=1}^{m}\varphi_n^{\delta k-1}\left|\lambda_n\right|\left|s_n\right|^k\\ &=O(1)\sum_{n=1}^{m-1}\Delta\left|\lambda_n\right|\sum_{r=1}^{n}\varphi_r^{\delta k-1}\left|s_r\right|^k+O(1)\left|\lambda_m\right|\sum_{n=1}^{m}\varphi_n^{\delta k-1}\left|s_n\right|^k\\ &=O(1)\sum_{n=1}^{m-1}\gamma_n X_n+O(1)\left|\lambda_m\right|X_m\\ &=O(1) \quad \text{as} \qquad m\to\infty\end{aligned}$$

by virtue of the hypotheses of Theorem 2.1 and Lemma 2.1.

By using Hölder’s inequality, we achieve

$$\begin{aligned}\sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}\left|M_{n,2}\right|^k &\leq\sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}\left(\sum_{v=1}^{n-1}\left|\Delta_v(\hat{a}_{nv})\right|\left|\lambda_v\right|\left|s_v\right|\right)^k\\ &\leq\sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}\left(\sum_{v=1}^{n-1}\left|\Delta_v(\hat{a}_{nv})\right|\left|\lambda_v\right|^k\left|s_v\right|^k\right)\left(\sum_{v=1}^{n-1}\left|\Delta_v(\hat{a}_{nv})\right|\right)^{k-1}.\end{aligned}$$

Here, from (8) and (9), we have

$$\Delta_v(\hat{a}_{nv})=\hat{a}_{nv}-\hat{a}_{n,v+1}=\bar{a}_{nv}-\bar{a}_{n-1,v}-\bar{a}_{n,v+1}+\bar{a}_{n-1,v+1}=a_{nv}-a_{n-1,v}.$$

Also, by using (8), (12) and (13), we get

$$\sum_{v=1}^{n-1}\left|\Delta_v(\hat{a}_{nv})\right|=\sum_{v=1}^{n-1}\left(a_{n-1,v}-a_{nv}\right)\leq a_{nn}.$$

Then, we get

$$\begin{aligned}\sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}\left|M_{n,2}\right|^k &=O(1)\sum_{n=2}^{m+1}\varphi_n^{\delta k}\left(\sum_{v=1}^{n-1}\left|\Delta_v(\hat{a}_{nv})\right|\left|\lambda_v\right|^k\left|s_v\right|^k\right)\\ &=O(1)\sum_{v=1}^{m}\left|\lambda_v\right|\left|s_v\right|^k\sum_{n=v+1}^{m+1}\varphi_n^{\delta k}\left|\Delta_v(\hat{a}_{nv})\right|\\ &=O(1)\sum_{v=1}^{m}\varphi_v^{\delta k-1}\left|\lambda_v\right|\left|s_v\right|^k=O(1) \quad \text{as} \quad m\to\infty\end{aligned}$$

as in $M_{n,1}$.

Now, we have

$$\sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}\left|M_{n,3}\right|^k \le \sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}\left(\sum_{v=1}^{n-1}\left|\hat{a}_{n,v+1}\right|\left|\Delta\lambda_v\right|\left|s_v\right|\right)^k$$

$$\le \sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}\left(\sum_{v=1}^{n-1}\left|\hat{a}_{n,v+1}\right|\left|\Delta\lambda_v\right|\left|s_v\right|^k\right)\left(\sum_{v=1}^{n-1}\left|\hat{a}_{n,v+1}\right|\left|\Delta\lambda_v\right|\right)^{k-1}.$$

Using the fact that $\left|\hat{a}_{n,v+1}\right|\le a_{nn}$, $(\lambda_n)\in BV$, and using the condition (1), we have

$$\sum_{n=3}^{m+1}\varphi_n^{\delta k+k-1}\left|M_{n,3}\right|^k = O(1)\sum_{n=2}^{m+1}\varphi_n^{\delta k}\left(\sum_{v=1}^{n-1}\left|\hat{a}_{n,v+1}\right|\gamma_v\left|s_v\right|^k\right).$$

Thence, we get

$$\sum_{n=3}^{m+1}\varphi_n^{\delta k+k-1}\left|M_{n,3}\right|^k = O(1)\sum_{v=1}^{m}\gamma_v\left|s_v\right|^k\sum_{n=v+1}^{m+1}\varphi_n^{\delta k}\left|\hat{a}_{n,v+1}\right|$$

$$= O(1)\sum_{v=1}^{m}P_v\gamma_v\frac{\left|s_v\right|^k}{P_v}\varphi_v^{\delta k}$$

$$= O(1)\sum_{v=1}^{m-1}\Delta\left(P_v\gamma_v\right)\sum_{r=1}^{v}\varphi_r^{\delta k}\frac{\left|s_r\right|^k}{P_r} + O(1)P_m\gamma_m\sum_{v=1}^{m}\varphi_v^{\delta k}\frac{\left|s_v\right|^k}{P_v}$$

$$= O(1)\sum_{v=1}^{m-1}P_v\left|\Delta\gamma_v\right|X_v + O(1)\sum_{v=1}^{m-1}p_v\gamma_vX_v + O(1)P_m\gamma_mX_m$$

$$= O(1) \quad \text{as} \quad m\to\infty$$

by virtue of the hypotheses of Theorem 2.1 and Lemma 2.2.

If we take $\delta = 0$, $\varphi_n = \dfrac{P_n}{p_n}$ and $a_{nv} = \dfrac{p_v}{P_n}$, then Theorem 2.1 reduces to Theorem 1.1.

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A review on machine learning-based models for mortality risk prediction in STEMI-CS patients

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Abstract

ST-segment elevation myocardial infarction-cardiogenic shock (STEMI-CS) is one of the important cardiovascular diseases, with a high rate of mortality. A timely diagnosis of STEMI is important to guide treatment and reduce sudden cardiac death. Recently, machine learning (ML) methods were developed to establish predictive models to identify the in-hospital mortality risk of STEMI-CS patients. The experimental results reported in the literature showed that the ML methods obtain relatively high performance on benchmark STEMI-CS datasets. To determine how the ML methods were developed in the past years, this paper surveys recent machine learning methods developed for STEMI-CS risk prediction. The existing methods are examined through a comparison framework. After discussing the development of the field in recent years, some open problems and new emerging trends are identified.

Keywords: Artificial intellience, machine learning, data mining, cardiogenic shock, STEMI-CS, risk prediction.

1. Introduction

According to the report released by WHO in 2019, myocardial infarction has been an important cause of death globally in the last two decades [1]. Cardiogenic shock (CS) is the most important cause of in-hospital death in patients with ST-segment elevation myocardial infarction (STEMI) [2], [3], [4]. Despite using different therapeutic approaches, early mortality of STEMI-CS patients is still high, ranging from 30% to 50% [3]. CS and its related complications need a huge financial and medical burden. Some researchers stated that high mortality and complication rates of STEMI-CS patients are associated with the lack of effective early preventive treatments [5]. Given the risk of CS and the different risk factors associated with it, accurate clinical risk prediction tools need to be developed to accurately predict the onset of CS.

Machine learning (ML) is an important branch of artificial intelligence that allows a computational machine to automatically learn and progress without explicit programming. The main objective is the development of computer programs that can access data and use it for their own learning. ML creates automated predictive models to automatically learn and adjust its behavior accordingly, without human intervention. ML method can incorporate more features and analyze more sophisticated mathematical problems compared with traditional statistical models.

In recent years, researchers have suggested ML methods to improve the performance of the CS risk prediction models. ML models can overcome the limitations of regression-based risk prediction systems, such as parametric assumptions, reliance on linearity, and limited capability in evaluating the higher order.

As an element of STEMI-CS research, this paper reviews the development of mortality risk prediction methods in STEMI-CS through a literature survey and the categorization of papers, with the range 2012 to 2022. This survey includes the following objectives:

- Describing the STEMI-CS and its related topics
- Proposing a comparison framework to survey the state-of-the-art methods, from 2012 to 2022 according to the proposed comparison framework
- Giving an overview of new trends and open problems

After this short introduction, Section 2 presents the theoretical background. Section 3 describes the proposed comparison framework for surveying mortality risk prediction in STEMI-CS patients. Section 4 reviews the state-of-the-art methods. Section 5 discusses the methods, open challenges, and some future directions in the field. Finally, Section 6 concludes the paper.

2. Background

2.1. Problem definition

Cardiogenic shock (CS) is defined as a state of critical end-organ hypoxia and hypoperfusion because of primary cardiac disorders [4]. Acute myocardial infarction (AMI) causing left ventricular dysfunction is the most important cause of CS and accounts for 80% of cases. CS associated with STEMI is a potentially life-limiting condition. Figure 1 shows how a coronary artery blockage causes a STEMI heart attack. CS is the most important cause of in-hospital death in STEMI patients. Early diagnosis of STEMI-CS risk is crucial in the proper treatment of the patients. Physicians examine the patient's symptoms at admission and determine whether the patient is at high risk based on the patient's symptoms and treatment history. Determining the most important features that have the greatest impact on the mortality of STEMI-CS patients is one of the major challenges.

2.2. Building blocks of mortality risk prediction system

Figure 2 illustrates the general structure of a typical risk prediction system. The goal of the risk prediction system is to predict the risk score of the STEMI-CS. The system takes as input the characteristics of patients, pre-process the input data, extracts the important features that affect the risk prediction performance, trains the predictive model, and finally uses the constructed model to predict the risk score of unseen patients.

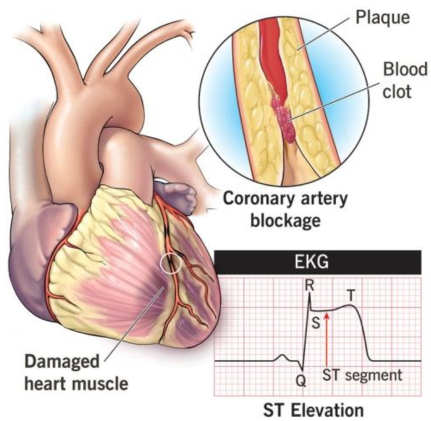

Figure 1. A big picture of STEMI heart attack[1]

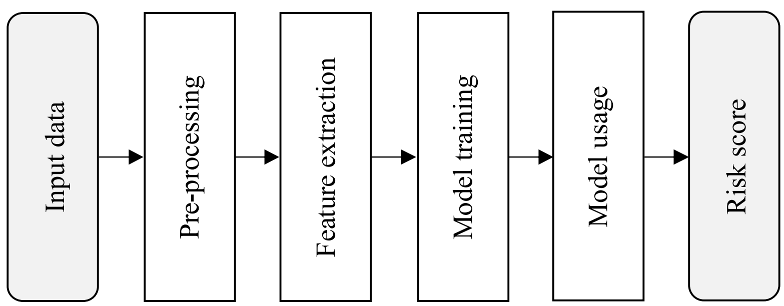

Figure 2. The general structure of a typical ML-based STEMI-CS risk prediction system

3. Comparison framework

Several papers have been published in international scientific journals and conferences to investigate the mortality risk prediction in the STEMI-CS issue. The mortality risk prediction systems differ in several important respects, including the evaluation metrics and datasets, the prediction features, the prediction method, and the scale. These are the key factors that make up the comparison framework. Figure 3 shows the proposed comparison framework.

3.1. Risk prediction technique

Researchers have used different prediction models to predict the mortality risk of STEMI-CS patients. The existing approaches for risk prediction range from those using only statistical methods [6] and regression-based methods at one end of the spectrum, to those that are based on data mining and machine learning methods [5] at the other side of the spectrum. In addition, hybrid methods, which are a combination of approaches, are developed.

[1] [https://my.clevelandclinic.org/health/diseases/22068-stemi-heart-attack, last access 12 Jun 2022]

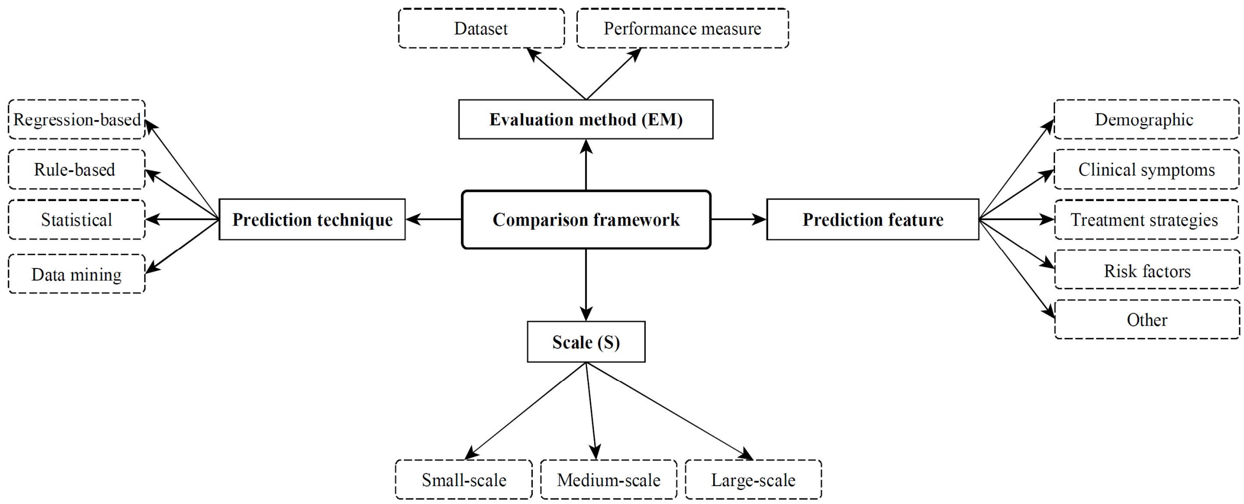

Figure 3. The proposed comparison framework

The main limitation of regression-based methods is low quality and high computational complexity in the risk prediction phase. The risk scores generated by regression-based methods may cause a delay in calculating the risk while awaiting the results. Statistical and regression-based methods mainly focused on using univariate and multivariate regression analysis to identify the independent variables in predicting mortality risk, which leads to low performance [7]. This issue can be easily resolved by data mining and ML methods. Data mining and ML can create automated data-driven predictive models and program complex problems with many factors through statistical tools [7].

ML and data mining methods are considered the best solution for mortality risk prediction in STEMI-CS patients because they can handle information overload, data source noise, data redundancy, and uncertainty. These methods achieve promising results and have high domain adaptability. However, these methods present many challenges that impede applications in the mortality risk prediction scope, such as the need for large datasets and annotated data to train and develop predictive models. Some popular ML and data mining methods used in the literature for the mortality risk prediction in STEMI-CS patients are support vector machine (SVM) [1], [8], logistic regression (LR) [5], Naïve Bayes (NB) [8], least absolute shrinkage and selection operator (LASSO) [5], extreme gradient boosting (XGBoost) [5], light gradient boosting machine (LightGBM) [5], adaptive neuro-fuzzy inference system (ANFIS), random forest algorithm (RF) [7], artificial neural networks (ANNs) [9], [10], [11], convolutional neural network (CNN) [12], recurrent neural networks (RNNs) [12], [13]. Figure 4 shows the timeline of the ML and DM-based mortality risk prediction in STEMI-CS patients.

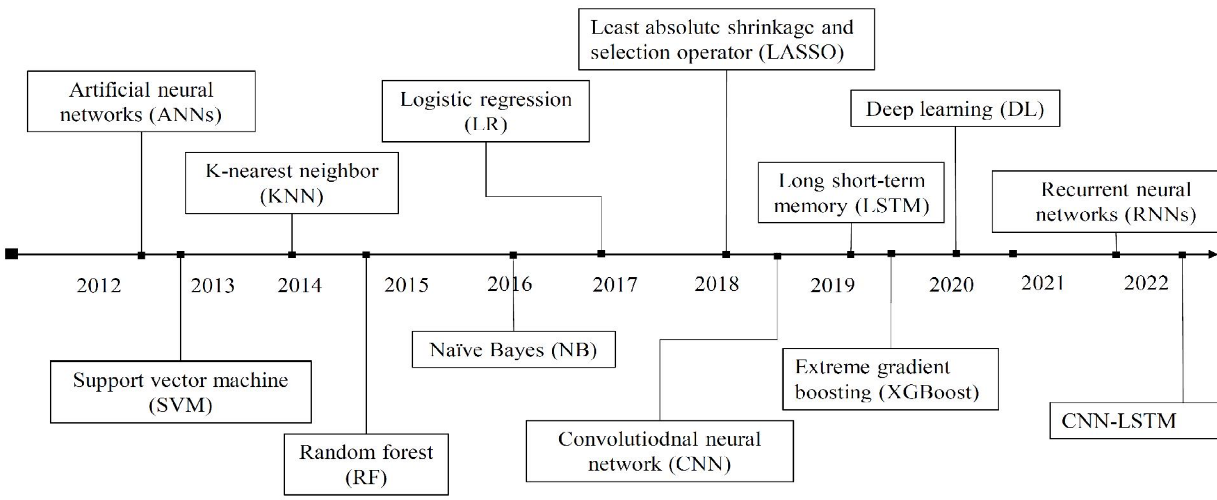

Figure 4. The timeline of the ML and DM based mortality risk prediction in STEMI-CS patients from 2012 to 2022

Green et al. [10] compared the ANN models with LR in predicting acute coronary syndrome in the emergency room. They found that the ANN approach outperforms LR models in terms of performance metrics and calibration assessments. Arif et al. [14] proposed an automatic model for the detection and localization of myocardial infarction using a KNN classifier. The objective is to categorize normal individuals without myocardial infarction and individuals suffering from myocardial infarction. Berikol et al. [8] used SVM algorithm to diagnose acute coronary syndrome and assisting the medical doctor with his decision to discharge or to hospitalize STEMI patients. Their approach relies on age, sex, threat factors, and cardiac enzymes of patients.

Acharya et al. [12] employed deep CNN model for automated detection of myocardial infarction using ECG signals. The main advantage of their work is that no feature extraction is performed, and can accurately identify the unknown ECG signals even with noise. Sharma et al. [15] developed a risk prediction model using SVM with both linear and radial basis function kernel and KNN. They proved that the proposed technique can successfully detect the myocardial infarction. Sharma et al. [16] proposed an automated system for classifying MI and regular ECG signals using a single-channel ECG database. The KNN classifier was used to classify both noisy and clean MI-ECG signals. In other work, Sharma and Sunkaria [17] used SVM and KNN to classify between subjects admitted for health control and patients suffering from inferior myocardial infarction.

Strodthoff et al. [18] proposed an ensemble of fully (CNNs) for the detection of myocardial infarction that operates directly on ECG data without any preprocessing efforts. The proposed classifier achieves 93.3% sensitivity and computed using 10-fold cross-validation with sampling based on patients. Goto et al. [19] proposed a machine learning model to predict needs for urgent revascularization from 12-

leads electrocardiography in emergency patients recorded in the emergency room at Keio University Hospital. Wu et al. [20] presented an ANN model to predict none STEMI patients. They proved that ANN is able to determine the most important features and predict none STEMI and unstable chest pain with higher accuracy. Lui and Chow [13] presented an myocardial infarction classifier that combines both CNN and RNNs. They develop multiclass classification to discriminate the myocardial infarction from those of patients with existing chronic heart conditions and healthy individuals. Baloglu et al. [21] proposed a deep CNN model that provides automatic recognition of myocardial infarction (MI). Their approach provides a powerful classifier without any handcrafted feature extraction. Cosentino et al. [3] discussed the long-term mortality in STEMI-CS patients and developed a risk scoring system based on the logistic regression model to measure the risk of death. Their findings show that the long-term mortality of STEMI-CS patients remains high after hospital discharge.

Bai et al. [5] used five prominent ML algorithms for in-hospital STEMI-CS prediction. The employed ML algorithms are LR, LASSO, SVM, LightGBM, and XGBoost. They showed that LASSO model has better predictive performance compared with other methods. The limitation of their work is that all possible factors that have impact on STEMI-CS are not considered in risk prediction phase. Deng et al. [7] used machine learning algorithms to establish an optimal model to predict the within-hospital death that occurred in STEMI-CS patients who underwent primary percutaneous coronary intervention. With a test on 854 STEMI patients they proved that the random forest algorithm outperformed other ML algorithms in mortality risk prediction process.

Wu et al. [22] developed three deep learning models for enhancing the effectiveness of the STEMI diagnosis. Their prediction models are convolutional neural network (CNN), long short-term memory (LSTM), and hybrid CNN-LSTM. With an evaluation on 883 STEMI patients, the CNN-LSTM model performed better than LSTM and CNN, and even doctors in predicting STEMI. In other work, Cao et al. [23] proposed a multi-scale deep learning model combined with a residual network and attention mechanism for the detection and localization of MI. They showed that the proposed ECGNet model outperforms traditional machine learning methods including CNNs and RNNs in terms of diagnostic performance and performance metrics. Lee et al. [24] used logistic regression with regularization, random forest, extreme gradient boosting (EGB), and SVM models to predict the short- and long-term mortality of STEMI patients. Their comparison showed that the ML-based models outperformed other algorithms in terms of solution quality.

Shetty et al. [25] show the effectiveness of the ML models in the mortality prediction of STEMI patients. They proved that the ML models overcome the limitations of the traditional logistic regression-based models. Their results showed that the random forest and multiple perceptron models outperformed counterpart models in terms of accuracy metric. Al-Zaiti et al. [26] developed some large-scale classifiers for the prediction of underlying acute myocardial ischemia in patients with chest pain. Their considered classifiers are gradient boosting machine (GBM), LR, and ANN. Their proposed method outperformed the doctors and commercial interpretation software. Liu et al. [27] developed a deep learning model as a diagnostic support tool based on a 12-lead electrocardiogram. The objective was to improve the diagnosis of STEMI disease. Kavak et al. [28] examine the application of CNN for the detection and localization of STEMI Using 12-lead ECG images. They employed the gradient-weighted class activation mapping (Grad-CAM) method to localize the STEMI signals in the ECG images. With a test on 537 ECG testing images, the proposed method achieved 96.3% accuracy.

3.2. Scale

This dimension is focused on the operation domain of the risk prediction system. According to the performance and adaptability measures, the risk prediction systems can be divided into three categories, small, medium, and large scales. Systems with high complexity and low adaptability were categorized as small. Medium systems have medium-sized performance and work with medium-sized data. Finally, the risk prediction systems are categorized as large if they are scalable and can work with high dimension data. The majority of data mining and machine learning methods are usually large-scale. Statistical methods usually have a good ability on a medium scale data. Hand-crafted and rule-based methods are considered small scale. Large-scale systems are preferable compared with other methods.

3.3. Evaluation Method

Three observations need to be considered when evaluating a mortality risk prediction system. To facilitate evaluation, the expected output of the system must first be accurately determined. Second, we need to identify the purpose of the evaluation. Third, we need to answer the question, "what are the appropriate datasets and evaluation metrics to evaluate the performance of the risk prediction system?"

Datasets and evaluation measures should appropriate to evaluate the system's soundness, accuracy, completeness, etc. Researchers have been used various benchmark datasets of different size. Table 1 lists the datasets used to evaluate the risk prediction systems. In Table 1, the "Name" column represents the name of the dataset. The "Literature" column provides a reference to the work, including the author and year of publication. A total of 20 datasets have been reviewed in this work. The largest dataset is the D2 dataset, which includes 20,000 ECG beats of normal and different types of myocardial infarction. The smallest dataset is the D9 dataset, which includes only 21 STEMI-CS patients.

Performance measures need for evaluating the performance of the risk prediction systems, should be suitable for evaluating system's accuracy, completeness, soundness, and so on. Each prediction model needs a specific evaluation measures. Some well-known evaluation measures are precision (P), recall (R), F1-measure, accuracy (A), area under precision-recall curve (AUC), area under receiver operating characteristics (ROC) [10]. For more detail about these metrics refer to [1]. These measures indicate the basic metrics for evaluating the performance of risk prediction system. In ideal state, a prediction system should score CS risks with high precision, recall, F1-measure, and AUC. In addition to the mentioned metrics, researchers have been developed other specific metrics to assess the performance of prediction systems.

A risk prediction machine consists of a set of components. Obviously, the performance of each component influences the overall performance of the subsequent components of the predictive system.

We believe, for the sake of comparing risk predictive systems comprehensively, evaluation process must be carried out at two levels: the component level and the system level. At the component level evaluation, the components of system are evaluated one at a time to discover the potential and weakness of them. At the system-level evaluation, the end-to-end overall performance of the system are evaluated. When as compared to the component-level evaluation, system-level evaluation has received much interest for comparing mortality risk prediction systems.

Currently, the state-of-the-art mortality risk prediction systems achieve around 90% of F-1 measure on some datasets. However, the existing systems cannot reach to the ideal performance. This shows that a lot of effort is needed in this area to improve the performance. Because different benchmark datasets and performance metrics have been used to evaluate systems, the generated performances cannot be directly compared. It is clear that after changing the dataset and scope, the performance of predictive systems decreases substantially. An open challenge is to evaluate the different systems under a unified benchmarking approach to identify the potentials and weaknesses of systems and pick out the most powerful structures for a certain application.

Table 1. Overview of the datasets used to evaluate the performance of mortality risk prediction in STEMI-CS patients

Name	Description	Year	Literature
D1	634 patients	2006	[10]
D2	20,000 ECG beats of normal and different types of myocardial infarction	2012	[14]
D3	1074 myocardial infarction ECG frames	2015	[15]
D4	228 patients	2016	[8]
D5	200 patients	2017	[12]
D6	50728 ECG epochs taken from 200 patients	2018	[16]
D7	200 patients	2018	[17]
D8	549 patients	2018	[13]
D9	21 patients	2019	[18]
D10	268 chest pain patients	2019	[20]
D11	52 normal patients and 148 MI patients	2019	[21]
D12	362 patients required urgent revascularization	2019	[19]
D13	737 patients	2021	[27]
D14	2282 STEMI patients	2021	[5]
D15	854 STEMI patients	2022	[7]
D16	506 control and 377 STEMI patients	2022	[22]
D17	3,635 STEMI patients	2022	[25]
D18	1244 patients	2022	[26]
D19	540 ECG images	2022	[28]
D20	52 normal cases and 148 MI patients	2022	[23]

4. Discussions

Literature reviews of current state-of-the-art techniques have been conducted to provide insights into the different methods of mortality risk prediction in STEMI-CS patients. This section discusses 20 risk prediction models. Table 2 lists 20 ML and DM-based mortality risk prediction methods. In Table 2, the methods are categorized based on the dimensions of the comparison framework.

Figure 5 shows the distribution of papers according to the risk prediction model. As shown in the figure, the majority of the methods are deep learning-based strategies, especially the CNN model. The second rank belongs to SVM and ANNs are in third place. This ranking shows the high capability of DM and ML methods in the risk prediction field.

Table 2. The list of STEMI-CS's mortality risk prediction systems addressed in this paper

Literature	**Prediction model**	**Scale**
Green et al. [10]	ANNs	M
Arif et al. [14]	KNN	M
Sharma et al. [15]	SVM	M
Berikol et al. [8]	SVM	M
Acharya et al. [12]	CNN	L
Sharma et al. [16]	KNN	M
Sharma and Sunkaria [17]	SVM	M
Lui and Chow [13]	CNN-LSTM	L
Strodthoff et al. [18]	Resnet	L
Wu et al. [20]	ANN	M
Baloglu et al. [21]	CNN	L
Goto et al. [19]	LSTM	L
Liu et al. [27]	DLM	L
Bai et al. [5]	LASSO	M
Deng et al. [7]	RF	M
Wu et al. [22]	CNN	L
Shetty el al. [25]	Extra Tree ML	L
Al-Zaiti et al. [26]	ANN	M
Kavak et al. [28]	CNN	L
Cao et al. [23]	ECGNet	L

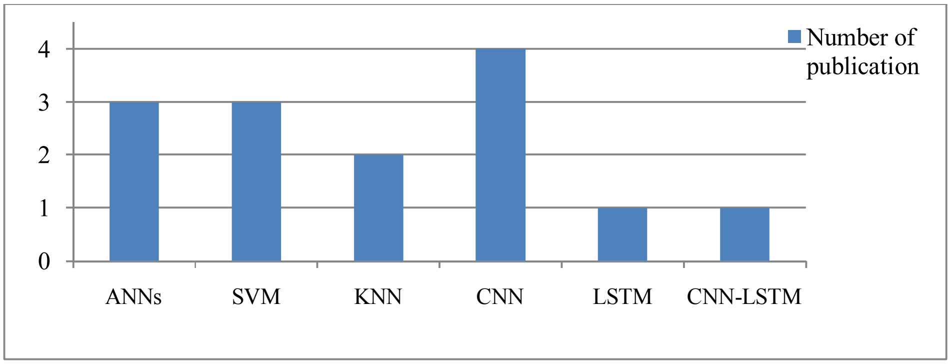

Figure 5. The number of papers reviewd in terms of risk prediction method

Figure 6 shows the distribution of reviewed articles in terms of the risk prediction method and scale dimension. The majority of methods are able to work on large-scale datasets. CNN, LSTM and CNN-LSTM methods are capable to work on large-scale data, because they can cope with noise, data loss, and data update. Since KNN, RF, LASSO and ANN have medium computational complexity, they can often operate on medium-scale data.

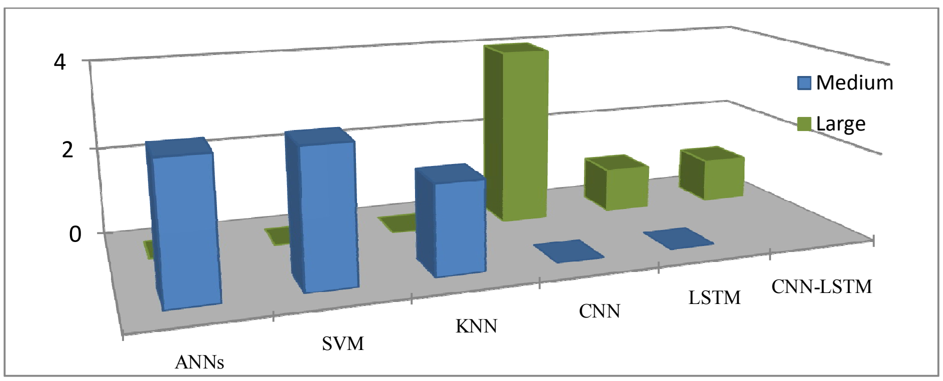

Figure 56 The number of papers reviewd in terms of scale dimension and risk prediction method

Table 3 shows the performance obtained by the risk prediction systems on different benchmark datasets given in Table 1. It is important to notice that because of restrained space, in Table 3, we only provide best results generated by the risk prediction system for every dataset.

Table 3. Performances of risk prediction systems on different datasets listed in Table 1 in terms of evaluation measures

Literature	Prediction model	Dataset	Performance
Green et al. [10]	ANNs	D-2	ROC=0.81
Arif et al. [14]	KNN	D17	A=0.98
Sharma et al. [15]	SVM	D16	A=0.96
Berikol et al. [8]	SVM	D00	A=0.99
Acharya et al. [12]	CNN	D-1	A=0.95
Sharma et al. [16]	KNN	D13	A=0.99
Sharma and Sunkaria [17]	SVM	D14	A=0.99
Lui and Chow [13]	CNN-LSTM	D08	F1=0.95
Strodthoff et al. [18]	Resnet	D15	P=0.94
Wu et al. [20]	ANN	D07	A=0.93
Baloglu et al. [21]	CNN	D09	A=0.99
Goto et al. [19]	LSTM	D10	A=0.83
Liu et al. [27]	DLM	D06	AUC=0.98
Bai et al. [5]	LASSO	D01	AUC=0.82
Deng et al. [7]	RF	D02	AUC= 0.789
Wu et al. [22]	CNN	D03	AUC=0.99
Shetty el al. [25]	Extra Tree ML	D04	AUC=0.80
Al-Zaiti et al. [26]	ANN	D05	AUC=0.78
Kavak et al. [28]	CNN	D11	A=0.96
Cao et al. [23]	ECGNet	D12	A=0.99

A review of performance generated by the risk prediction models shows that promising results have been made in diagnosing and determining the risk of mortality in STEMI-CS patients. However, the performance of existing methods is not ideal and there is a room for more effort in this domain.

One of the main challenges in evaluating mortality risk prediction models is to evaluate existing methods with unified datasets and the same performance criteria to identify the strengths and weaknesses of the methods. Another challenge is to improve the performance of existing models and even combine them with other prominent methods to be able to work in real-world hospital environments. Another challenge is the need for large data sets to train the risk predictive model. Methods need to be considered that have the ability of predicting mortality risk scores without the need for huge training data.

5. Conclusion

Widespread interest in mortality risk prediction of STEMI-CS patients has resulted in the development of different machine learning-based predictive models. To identify how risk prediction models have developed, during the last years, this research surveys risk prediction models through a review of related work and the categorization of publications based on a comparison framework, from 2012 to 2022. Among all machine learning-based risk prediction models, a total of 20 representative models were reviewed and examined. After surveying the risk prediction models, some open issues, new trends, and several promising directions for future research are discussed. This survey confirms that the research in the field is still an open problem and there is room for more efforts.

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A Study on the Earthquake Hazard and Forecasting in the Lake Van and its surroundings, Turkey

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Abstract

In this study, statistical region-time analyses of current earthquake activity in the Lake Van and its adjacent area are performed by considering the Relative Intensity (RI), Pattern Informatics (PI), combined forecast model (RIPI) and Coulomb stress variations. The relations between these variables are used to evaluate the recent earthquake hazard and to make earthquake forecasting in and around the Lake Van. These types of techniques are well-known and have been used in earthquake statistics, especially in statistical seismology. The results show that the areas especially having great stressed distributions at the beginning of 2022 and hotspot locations from combined forecast map between 2022 and 2032 are detected in several same parts of the study area including Erciş and Yeniköşk faults, Van and Saray fault zones between Muradiye, Özalp, Erçek, Van and Gevaş. As a remarkable fact, all anomaly areas of estimated parameters are observed in the same parts of the study region and therefore, these anomaly areas estimated at the beginning of 2022 and also between 2022 and 2032 may be considered as one of the most likely zones for the next strong/large earthquakes. Thus, the interrelationships between these variables may supply more accurate and more reliable interpretations for earthquake forecasting and hazard assessment in this part of Turkey.

Keywords: The Lake Van, Coulomb stress, combined forecast, earthquake hazard.

1. Introduction

Statistical studies on the region-time distributions of earthquake activity are one of the most significant process in the determination of earthquake potential. Many authors have used different variables, scaling laws, measurements and approaches for seismicity analyses of different parts of the world, such as magnitude-frequency *b*-value, fractal dimension *Dc*-value, seismic quiescence Z-value, earthquake probability, recurrence time, Relative Intensity (RI), Pattern Informatics (PI), VAN method, Region-Time-Length (RTL) algorithm, M8 and CN algorithms, Coulomb stress change, moment and energy releases [1]-[5]. Thus, if the earthquake forecasting can be attributed to a statistical basis, statistical behaviors of earthquake occurrences become very important in the earthquake hazard and for the forecasting of possible future earthquakes [6].

The static stress changes caused by a previous earthquake can change the current stress state and trigger the future earthquake occurrence on a fault and its adjacent zones. Literature studies show that earthquakes produce stress perturbations in the crust and this process can significantly advance or delay the earthquakes on nearby faults [7], [8]. It is suggested that tectonic stress throughout the faults may increase during the loading cycle for large events and quickly decrease after a strong rupture in and around the earthquake area. From this point of view, the possible locations of the future earthquakes may be related to the stress situations uploaded by the previous events and current seismotectonic conditions. In recent years, another approach to forecast the earthquakes has been proposed [3], [6]. This method is named as Pattern Informatics (PI) and is based on the strong region-time relations of earthquake activity. Relative intensity (RI) is the other statistics to forecast earthquakes and may specify the locations of the largest seismic activity of earthquakes with the smallest magnitude. Significant developments have been obtained from recent advances in the PI technique, especially after combining the results from RI analyses. The results of these forecasting techniques provide a map of regions in a seismogenic area in which earthquakes are considered to occur in a future time interval, generally five to ten years [9].

The Lake Van region is located in the southeast of Karlıova Triple Junction (~125 km) and ~100 km to the north of the promontory thrust of the Bitlis-Zagros Thrust Zone [10]. The Lake Van consists of three deep sub-basins (Northern basin, Tatvan basin and Deveboynu basin) and basement-ridges (e.g. Northern, Ahlat) in the present time. This region is characterized by oblique-slip boundary faults, N-S shortened and domal morphological structure [11]. Main tectonics in and around the Lake Van are plotted in Fig. 1 [12]. Active faults and fault zones in the study area can be given as Erciş fault, Çaldıran fault, Süphan fault, Nemrut fault, Nazik Gölü fault, Malazgirt fault, Yeniköşk fault, Saray fault zone and Van fault zone. Erciş, Çaldıran, Nazik Gölü faults and Saray fault zone show the right-lateral strike-slip mechanism, whereas Süphan and Malazgirt faults are major left-lateral strike-slip faults. On the contrary, Yeniköşk fault and Van fault zone are the reverse or thrust fault mechanisms typically E-W trending, while Nemrut and Tendürek faults extension fissures show the normal fault mechanisms.

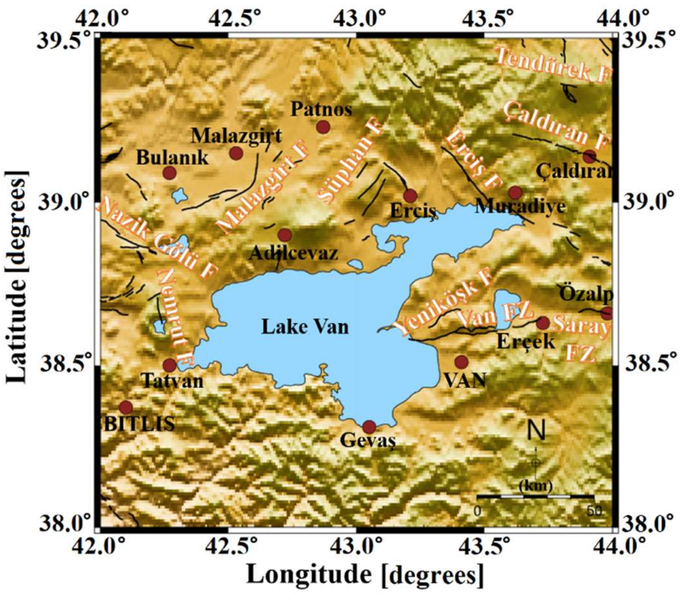

Figure 1. Main tectonic structures from [12] and city locations in the Lake Van and its adjacent area.

Some destructive earthquakes occurred in and around Van in the past and recent years such as; April 29, 1903 Malazgirt (*Ms*=6.7, surface wave magnitude), November 24, 1976 Muradiye-Van (*Ms*=7.5) and October 23, 2011 Van (*Ms*=7.5) (Bogazici University, Kandilli Observatory and Research Institute, KOERI). Although many statistical studies with different contents have been made to reveal the earthquake potential for the Lake Van and its adjacent area, these types of hazard assessments and earthquake forecasting studies are relatively rare for this part of the Turkey. In this context, Coulomb stress changes are realized and the applicability of the RI and PI methods with their combination are firstly performed to forecast the strong/large earthquakes in the intermediate-term in and around the Lake Van region of Turkey.

2. Earthquake Catalog and Analysis Methods

A part of data was taken from Öztürk [13] for the time period from 1970 to 2006. This catalog is homogeneous for duration magnitude, M_d, and contains 392 events. Also, the earthquakes between 2006 and 2022 were provided from KOERI, and there are 13,786 events in this time period. The shallow earthquakes (depth<70 km) were used to achieve the statistical analyses since the seismogenic depth is given between 40 and 45 km for this part of the East Anatolian Region (EAR) [14]. Thus, a database consisting of 14,178 earthquakes from November 28, 1970 to December 31, 2021, with a magnitude range of $1.0 \leq M_d \leq 6.6$ was obtained. Epicenter distributions of the catalog were plotted in Fig. 2. Also, to determine the Coulomb stress changes, 66 events (moment magnitude, $M_w \geq 4.5$, moment magnitude) that occurred in the Lake Van and its surrounding area from 2011 to 2021 were used. Detailed earthquake information (dip, strike, rake, etc.) was taken from the Disaster and Emergency Management Authority (AFAD).

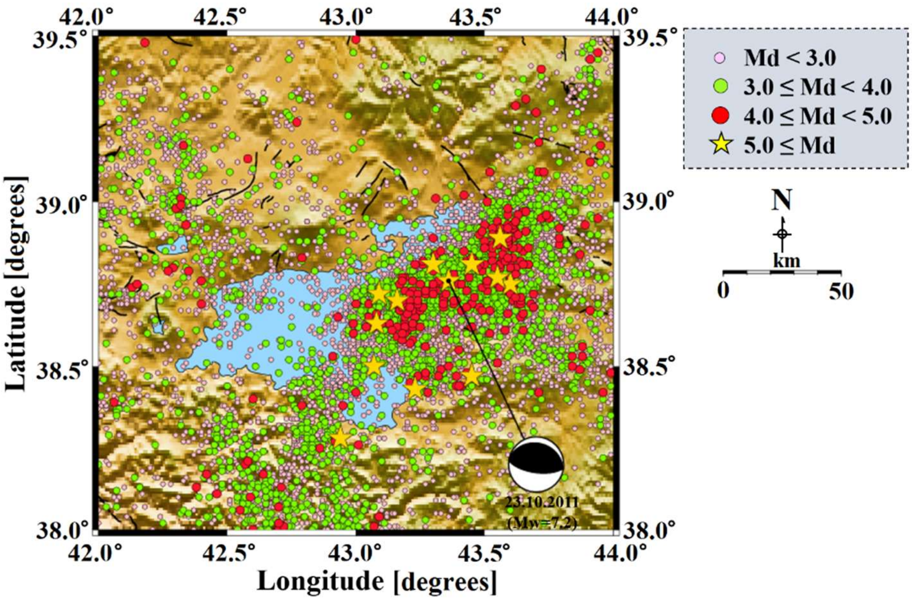

Figure 2. Epicenter distributions of the 14,178 earthquakes with $M_d \geq 1.0$ between 1970 and 2022. The fault plane solution shows the focal mechanism of the October 2011 Van earthquake.

Relative Intensity (RI), Pattern Informatics (PI) and Combined Forecasting Method

RI and PI techniques using the statistical behaviors of earthquake occurrences do not predict the earthquakes, but they forecast the seismogenic regions (hot spots) in which earthquakes are most likely to occur in the future 5 to 10-year time interval. RI and PI algorithms can be found in many studies [9], [15]-[17] and RI forecast can briefly be given in the following steps (for details see [16]):

i. The study region is binned into boxes with a linear dimension Δx.
ii. The number of events having a magnitude M equal to or larger than a smaller Mc-value in box j is estimated for the time period between t_S and t_E, beginning and ending times of catalog, respectively. This number is the average to determine the number of events per day, defined by $n_j(t_S,t_E)$.
iii. The relative value of this number is named as RI score. This score is given in the form $n_j(t_S,t_E)/n_{MAX}$, where n_{MAX} is the highest value of $n_j(t_S,t_E)$. The RI score varies from 0 to 1.
iv. Considering a threshold value w between zero and one ($0\leq w\leq 1$), the next large events are expected only in boxes of RI scores higher than this w-value. The boxes of RI score lower than the threshold w-value are the regions where the next large events are forecasted not to occur.
v. Thus, according to the RI forecast framework, great events are considered most likely to occur in regions with higher earthquake activity.

PI method and its modification consist of the following steps (for details see [16]):

i. As in the RI forecast, the study region is binned into boxes with a linear dimension Δx.
ii. All earthquakes in the region with $M\geq Mc$-value since the time defined by t_0 are included.
iii. Three-time intervals are taken into consideration:
   a) A reference time interval between t_b and t_1.
   b) A second-time interval between t_b and t_2 ($t_2> t_1$). The change period over which seismic activity variations are computed is then from t_1 to t_2. The time t_b is chosen from t_0 to t_1. The aim is to determine anomalous earthquake activity in the change period between t_1 to t_2 relative to the reference interval from t_b and t_1.
   c) The forecast time period from t_2 to t_3 is the period for which the forecast is valid. The change and forecast periods must have the same length.
iv. The earthquake intensity of a box for a time period is the average number of an earthquake with $M\geq Mc$-value that occurred during the time period. The earthquake intensity of box j over the reference period t_b to t_1, $n_j(t_b,t_1)$, is the average earthquakes number between t_b and t_1. The earthquake intensity of box j over the period from t_b to t_2, $n_j(t_b,t_2)$, is the average earthquakes number between t_b to t_2.
v. To compare the intensities from two different time periods, they are required to have the same statistical features. Therefore, the earthquake intensities are normalized by subtracting the mean earthquake activity of all boxes and dividing by the standard deviation of the earthquake activity in all boxes. These normalized intensities are defined by $n_j^*(t_b,t_1)$ and $n_j^*(t_b,t_2)$.
vi. Anomalous seismicity measure in box j is the difference between the two normalized earthquake intensities, $\Delta n_j^*(t_b,t_1,t_2) = n_j^*(t_b,t_2)-n_j^*(t_b,t_1)$.

vii. To reduce the relative importance of random fluctuations in earthquake activity, the average change $\Delta n_j^*(t_b,t_1,t_2)$ is computed over all possible initial times t_b from t_0 to t_1. The result is $\Delta\underline{n}_j^*(t_0,t_1,t_2)$.

viii. The probability of a future earthquake in box j, $P_j(t_0,t_1,t_2)$, is defined as the square of the average intensity change, $P_j(t_0,t_1,t_2) = \{\Delta\underline{n}_j^*(t_0,t_1,t_2)\}^2$.

ix. In order to denote anomalous areas, it is necessary to compute the change in the probability $P_j(t_0,t_1,t_2)$ relative to the background so that the mean probability is subtracted over all boxes $<P_j(t_0,t_1,t_2)>$. This variation in the probability is defined by $P'_j(t_0,t_1,t_2) = P_j(t_0,t_1,t_2) - <P_j(t_0,t_1,t_2)>$.

x. The relative value of the variation in the probability is named as PI score. This score is given in the form $P'_j(t_0,t_1,t_2)/P_{MAX}$, where P_{MAX} is the greatest value of $P'_j(t_0,t_1,t_2)$. Because it is interested in seismic activation/ quiescence relative to the background, if boxes have PI scores smaller than zero, these scores are replaced by zero. PI score is between 0 and 1.

xi. If a threshold w-value is considered in the interval between 0 and 1, next large earthquakes are expected likely in boxes of PI scores larger than this w-value. The boxes of PI scores smaller than the threshold w-value are the regions where future large earthquakes are forecasted not to occur.

xii. Thus, according to the PI forecast framework, great earthquakes are considered likely to occur in regions with higher earthquake activity or quiescence.

In the last step, PI and RI maps are combined to create a forecast map. This map is then renormalized to unit probability over the future 5 to 10-year period. The details of the algorithm can be given in the following steps (for details see [17]):

a. In the first stage, a relative intensity map is created for all areas. Then, relative values larger than 10^{-1} are adjusted to 10^{-1} and nonzero values smaller than 10^{-4} are adjusted to 10^{-4}. Finally, every box with zero historic earthquake activity is set to 10^{-5}.

b. A pattern informatics calculation is performed over the top 10% of most active sites of the study region. The times t_0, t_1, and t_2 are defined for the calculations. Since future earthquakes are expected to occur in hot spots, they are given in a probability value of unity.

c. In the last stage, a combined forecast map is created by superimposing the PI map and its Moore neighborhood (the pixel + its eight adjacent neighbors) on top of the RI map. Entire hot spot pixels have a probability of 1, and all other pixels have probabilities that change between 10^{-5} and 10^{-1}.

Coulomb Stress Analysis

One of the best-known methods to evaluate the stress situations under which a failure occurs in the source fault is the Coulomb failure stress ($\Delta\sigma_{cfs}$) and it is calculated with following formula:

$$\Delta\sigma_{cfs} = \Delta\tau_s + \mu'\Delta\sigma_{n'} \quad (1)$$

where, $\Delta\tau_s$ indicates the shear stress variation associated with the positive direction of receiver fault slip, $\Delta\sigma_{n'}$ is the normal stress change along the fault plane and μ' is the effective friction coefficient on the fault [18]. The effective coefficient of friction μ' in Eq. 1 means to include the effects of pore-pressure changes and varies between 0 and 1. For this analysis, we considered 0.4 in an elastic half-space with uniform isotropic elastic properties. Dimensionless Poisson's ratio (v) is assumed as 0.25 and Young modulus (E) is accepted as 8×10^5 bars. The positive Coulomb stress change indicates the loading stress, pushing the fault toward brittle failure, whereas the negative Coulomb stress change corresponds to the unloading stress, obstructing the earthquake rupture [7], [8], [18].

3. Results and Discussions

For the determination of minimum magnitude in region-time analyses of seismicity, determination of the magnitude completeness, *Mc*-value, must be the first step since *Mc*-value changes in time. Time variations in *Mc*-value can be estimated with a moving time window. In this work, time changes in *Mc*-value were plotted with its standard deviation for every 250 samples per window and all 14,718 earthquakes were used. Time variation of *Mc*-value was plotted in Fig. 3. *Mc*-value is between 2.8 and 3.3, relatively large, until 2011. Then, it decreases to about 2.5 at the beginning of 2012, and it shows a variation from 2.0 to 2.5 after 2012. Thus, temporal changes in *Mc*-value are not stable and there exists a clear variation between 2.0 and 3.3 from 1970 to 2022. It can be stated that *Mc*=2.5 is suitable and consistent with the literature researches covering the study area.

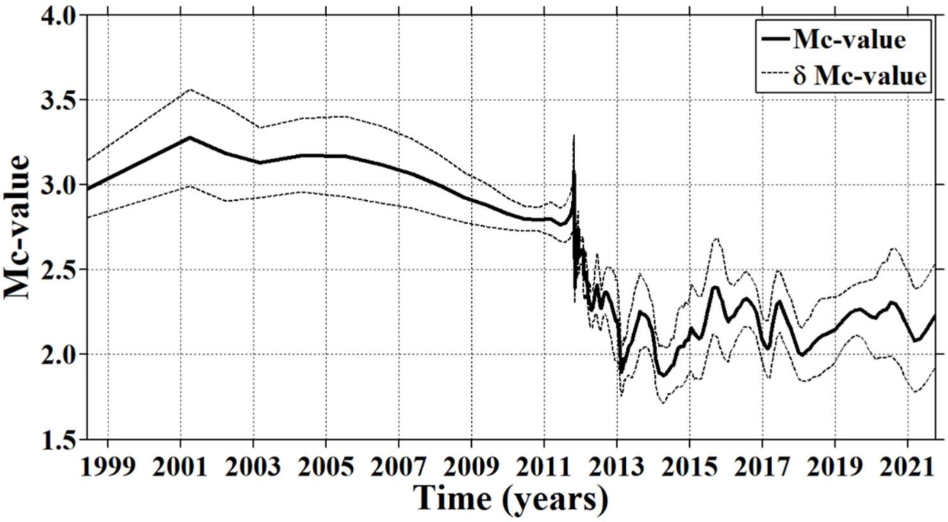

Figure 3. Temporal variation of *Mc*-value. The standard deviation, δ*Mc*-value, was also given.

The forecast map in and around the Lake Van region was created by combining the RI (Fig. 4a) and PI maps (Fig. 4b). The study area was divided into rectangular cells spacing of 0.02°×0.02° in latitude and longitude (Δx = 0.02°), and *Mc*-value was taken as 2.5. Then, the large events during the forecast interval were described by the total number of M_d≥5.0 events expected over the future ten-year period. In the last stage, the times (t_S, t_E, t_0, t_1, t_2, t_3) and time intervals were described. The forecast period is selected between

January 1, 2022 and January 1, 2032. The change interval is considered from January 1, 2012 to January 1, 2022. Thus, the times can be given as t_3 = January 1, 2032, $t_E = t_2$ = January 1, 2022, t_1 = January 1, 2012, and $t_S = t_0$ = November 28, 1970 (the initial time is t_0 = 1970, the change period is from t_1 = 2012 to t_2 = 2022, and the forecast period is from t_2 = 2022 to t_3 = 2032). Previous studies suggest that the length of the change interval must be equal to the length of the forecast interval [15], [17]. Using these input values, region-time forecasting of strong earthquakes expected in the Lake Van and its adjacent region in the intermediate-term (t_3 - t_2 = 10 years) was made for the time period between January 1, 2022 and January 1, 2032. A composite forecast map obtained by combining the RI map with the PI map was plotted in Fig. 4c. As seen in Fig. 4c, hotspot regions were defined more clearly on the composite map created by combining RI and PI maps. In the framework of these methods, great earthquakes are considered most probably to occur with higher seismic activity or quiescence. As seen from seismicity map in Fig. 2, the regions with higher earthquake activity and the areas having strong earthquakes greater than 5.0 have complied with the earthquake zones expected in the future on the RI, PI, and combined forecast maps. Also, there exist some regions that are the forecast hotspots for the occurrence of $M_d \geq 5.0$ earthquakes during the period between 2022 and 2032. These regions were observed in and around Bulanık, in the south of the Lake Van including Gevaş and its surrounding areas, in the east part of the Lake Van including some parts of Erciş fault, among Muradiye-Çaldıran-Van-Erçek in and around Yeniköşk fault, Van fault zone, and Saray fault zone. Thus, these methods aim to limit the times and regions in which earthquakes are most likely to occur in the next and to forecast the earthquakes in the intermediate-term.

The Coulomb stress changes were plotted for the depths of 10 km using a grid size of 0.1 by 0.1 km on the maps (Fig. 5). Increases and decreases in the stress changes were presented with red and blue colors, respectively. As shown in Fig. 5, there are two high-stress lobes along the northwest-southeast direction and two low-stress lobes along the northeast-southwest direction. It is also seen that the Coulomb stress could not be transferred to the western part of the Lake Van. In the Coulomb stress change map, it can be stated that the eastern part of the Lake Van, Erciş fault and Saray fault zone are high stressed. Around the Çaldıran, another important observation is that a mostly low-stress change for all depths appears. Also, it can be observed that small and great earthquakes occurred along the positive stress regions on the NE-SW direction after recent seismic activity since 2011.

As mentioned above, there exist some anomaly regions having hotspot points from created forecast map and high stressed distributions in the same parts of the study area. High-stress distribution and forecasted hotspot locations were estimated between Muradiye, Özalp, Erçek, Van and Gevaş covering Erciş and Yeniköşk faults, Van and Saray fault zones. It is accepted that high-stress distribution may indicte the areas in which the next possible earthquake will occur. Also, the combined forecast map based on analyzing the region-time patterns of past events may indicate the possible locations of future occurrence of $M_d \geq 5.0$ events that can be expected to occur between 2022 and 2032. As a remarkable fact, almost all the anomaly regions of estimated parameters were observed in the same regions and therefore, special emphasis needs to be paid to these anomaly areas.

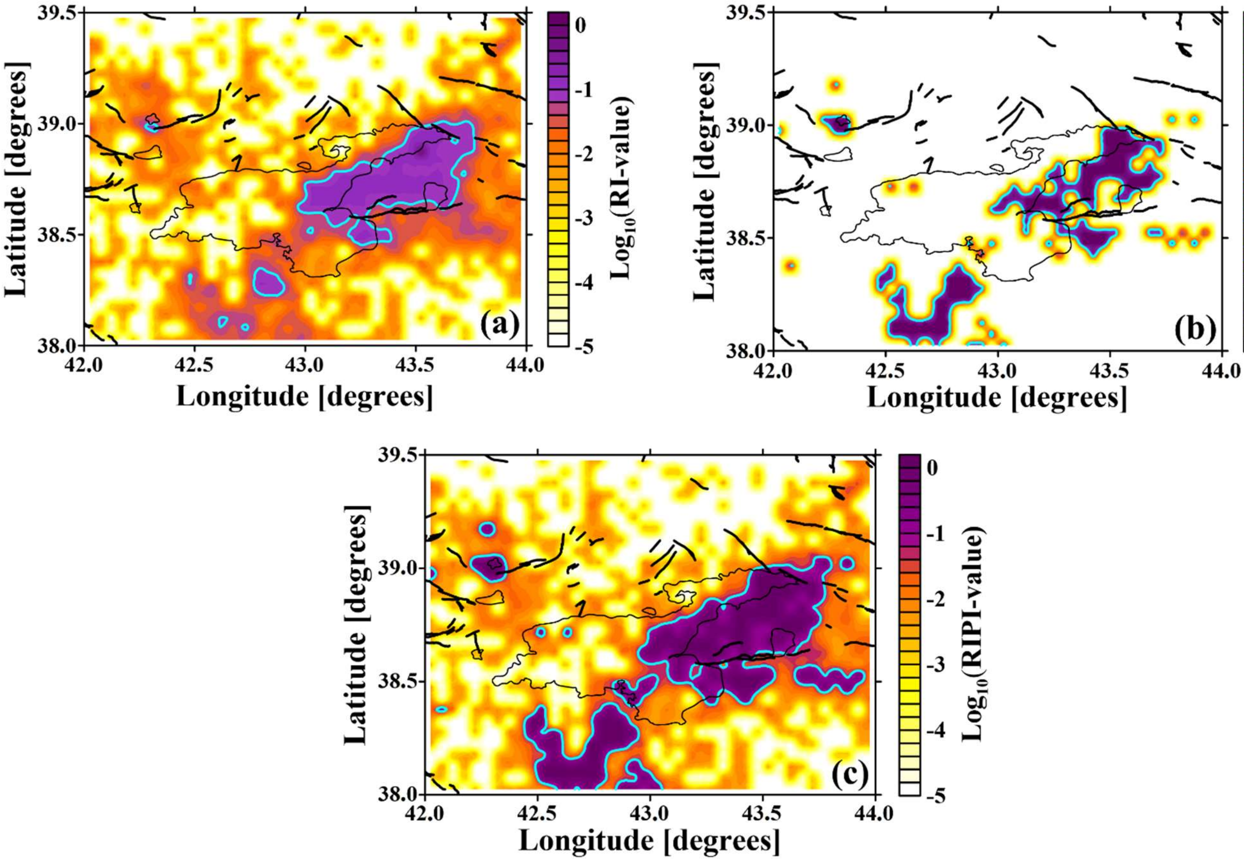

Figure 4. (a) RI map, (b) PI map and (c) Combined forecast map.

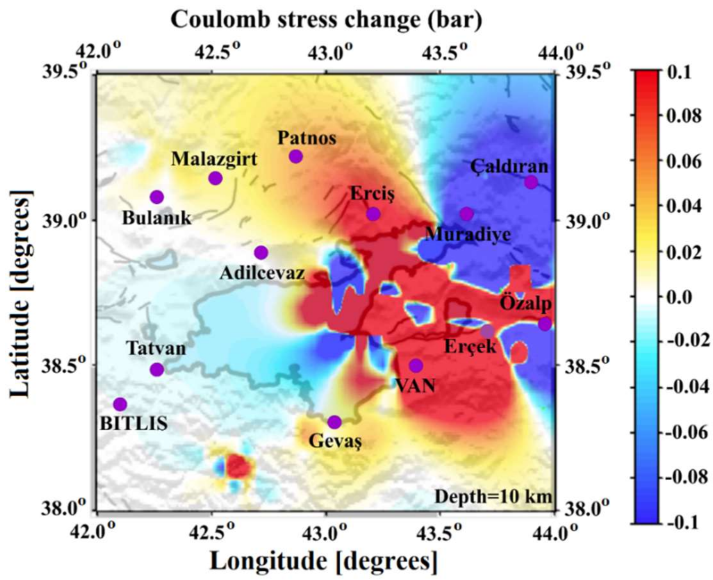

Figure 5. Coulomb stress variation for the Lake Van and its surrounding region.

4. Conclusions

In this study, RI and PI techniques and their combination with the Coulomb stress changes are evaluated for a reliable seismic hazard and more persuasive forecasting of the strong/large earthquakes in and around the Lake Van region of Turkey. For this purpose, we use a homogeneous catalog including 14,178 shallow events (depth<70 km) with $1.0 \leq M_d \leq 6.6$ from November 28, 1970 to December 31, 2021. We perform our analyses in a rectangular area covered by coordinates 38.0°N and 39.5°N in latitude and 42.0°E and 44.0°E in longitude. For the mapping of Coulomb stress changes, 66 events with $M_w \geq 4.5$ that occurred in and around the Lake Van region between 2011 and 2021 are used. The anomaly regions with hotspot points from combined forecast map and high Coulomb stress changes cover Erciş and Yeniköşk faults, Van and Saray fault zones consisting of Muradiye, Özalp, Erçek, Van and Gevaş. It is well known that the regions with high-stress distribution are considered to be the most likely places where the future expected strong/large earthquakes will occur. The forecast of where the next earthquakes with $M_d \geq 5.0$ are expected to occur during a future time interval of ten years between 2022 and 2032 shows that the possible locations of future earthquake occurrences are compatible with the anomaly regions obtained from stress analysis. Thus, almost all the anomaly regions of estimated parameters are found in the same regions and seismic hazard in these regions is high.

Acknowledgement: Some figures were drawn with ZMAP and GMT [19], [20]. The maps of Coulomb stress changes were plotted with the Coulomb 3.4 package [18]. The authors would like to thank Dr. Kazuyoshi Z. Nanjo for helping to prepare the RI, PI and combined codes.

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Amply Cofinitely Weak e-Supplemented Modules

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Abstract

In this work, every ring is associative with identitiy and every module is an unital left module. Let M be an R-module and $U\leq M$. If for every $V\leq M$ such that $M=U+V$, U has a weak supplement X in M with $X\leq V$, then we say U has ample weak supplements in M. If every cofinite essential submodule of M has ample weak supplements in M, then M is called an amply cofinitely weak e-supplemented module. In this work, some properties of these modules are investigated.

Keywords: Essential Submodules, Small Submodules, Cofinite Submodules, Supplemented Modules.

2020 Mathematics Subject Classification: 16D10, 16D70.

1. INTRODUCTION

In this paper all rings are associative with identity and all modules are unital left modules.

Let R be a ring and M be an R-module. It is denoted a submodule N of M by $N\leq M$. Let M be an R-module and $N\leq M$. If $L=M$ for every submodule L of M with $M=N+L$, then N is called a *small* (or *superfluous*) submodule of M and denoted by $N\ll M$. A submodule N of an R-module M is called an *essential* submodule of M and denoted by $N\trianglelefteq M$ if $K\cap N\neq 0$ for every submodule $K\neq 0$, or equvalently, $N\cap L=0$ for $L\leq M$ implies that $L=0$. N is called a *cofinite* submodule of M if M/N is finitely generated. Let M be an R-module and $U,V\leq M$. If $M=U+V$ and V is minimal with respect to this property, or equivalently, $M=U+V$ and $U\cap V\ll V$, then V is called a *supplement* of U in M. M is said to be *supplemented* if every submodule of M has a supplement in M. M is said to be *essential supplemented* (or briefly, *e-supplemented*) if every essential submodule of M has a supplement in M. M is called a *cofinitely supplemented* module if every cofinite submodule of M has a supplement in M. If every cofinite essential submodule of M has a supplement in M, then M is called a *cofinitely essential supplemented* (or briefly, *cofinitely e-supplemented*) module. Let M be an R-module and $U\leq M$. If for every $V\leq M$ such that $M=U+V$, U has a supplement V' with $V'\leq V$, we say U has *ample supplements* in M. If every submodule of M has ample supplements in M, then M is called an *amply supplemented* module. M is said to be *amply essential supplemented* (or briefly, *amply e-supplemented*) if every essential submodule of M has ample supplements in M. If every cofinite submodule of M has ample supplements in M, then M is called an *amply cofinitely supplemented* module. If every cofinite essential submodule of M has ample supplements in M, then M is called an *amply cofinitely essential supplemented* (or briefly, *amply ce-supplemented*) module. Let M be an R-module and $U,V\leq M$. If $M=U+V$ and $U\cap V\ll M$, then V is called a *weak supplement* of U in M. M is said to be *weakly supplemented* if every submodule of M has a weak supplement in M. M is said to be *weakly essential supplemented* (or briefly, *weakly e-supplemented*) if every essential submodule of M has a weak supplement in M. M is called a *cofinitely weak supplemented* module if every cofinite submodule of M has a weak supplement in M. M is said to be *cofinitely weak essential supplemented* (briefly, *cwe-supplemented*) if every cofinite essential submodule of M has a weak supplement in M. Let M be an R-module and $U\leq M$. If for every $V\leq M$ such that $M=U+V$, U has a weak supplement V' with $V'\leq V$, we say U has *ample weak supplements* in M. If every submodule of M has ample weak supplements in M, then M is called an *amply weak supplemented* module. M is said to be *amply weak essential supplemented* (or briefly, *amply weak e-supplemented*) if every essential submodule of M has ample weak supplements in M.The intersection of maximal submodules of an R-module M is called the *radical* of M and denoted by $RadM$. If M have no maximal submodules, then we denote $RadM=M$.

Some properties of (amply) supplemented modules are in [1] and [12]. Some informations about (amply) cofinitely supplemented modules are in [1]. The definition of essential supplemented module and some properties of this module are in [10]. Some properties of amply essential supplemented modules are in [11]. More informations about (amply) cofinitely essential supplemented modules are in [6] and [7]. More informations about weakly supplemented modules are in [3] and [8]. More informations about cofinitely weak supplemented modules are in [2]. More results about weakly (amply weak) essential supplemented modules are in [5] and [9]. More results about cofinitely weak essential supplemented modules are in [4].

2. AMPLY COFINITELY WEAK e-SUPPLEMENTED MODULES

Definition 2.1. Let M be an R-module. If every cofinite essential submodule of M has ample weak supplements in M, then M is called an *amply cofinitely weak e-supplemented* (or briefly, *amply cwe-supplemented*) module.

Proposition 2.2. Let M be an amply weak e-supplemented module. Then M is amply cwe-supplemented.
Proof. Clear from definitions.

Proposition 2.3. Let M be a finitely generated R-module. If M is amply cwe-supplemented, then M is amply weak e-supplemented.
Proof. Let $U\leq M$. Since M is finitely generated, M/U is also finitely generated. Then U is a cofinite submodule of M and since M is amply cwe-supplemented, U has ample weak supplements in M. Hence M is amply weak e-supplemented, as desired.

Proposition 2.4. Let M be an amply cwe-supplemented R-module. Then M is cwe-supplemented.
Proof. Clear from definitions.

Proposition 2.5. Let M be an amply cwe-supplemented R-module. Then $M/RadM$ have no proper cofinite essential submodules.
Proof. Since M is amply cwe-supplemented, by Proposition 2.4, M is cwe-supplemented. Then by [4, Proposition 1], $M/RadM$ have no proper cofinite essential submodules.

Proposition 2.6. Let M be an amply cwe-supplemented R-module and K be a proper cofinite essential submodule of M with $RadM \leq K$. Then $K/RadM$ is not essential in $M/RadM$.
Proof. Since M is amply cwe-supplemented, by Proposition 2.4, M is cwe-supplemented. Then by [4, Proposition 2], $K/RadM$ is not essential in $M/RadM$.

Proposition 2.7. Let M be an R-module, U be a cofinite essential submodule of M and $M_1 \leq M$. If M_1+U has a weak supplement in M and M_1 is amply cwe-supplemented, then U has a weak supplement in M.
Proof. Since M_1 is amply cwe-supplemented, by Proposition 2.4, M_1 is cwe-supplemented. Then by [4, Lemma 2], U has a weak supplement in M.

Proposition 2.8. Let M be an R-module, U be a cofinite essential submodule of M and $M_i \leq M$ for $i=1,2,...,n$. If $M_1+M_2+...+M_n+U$ has a weak supplement in M and M_i is amply cwe-supplemented for every $i=1,2,...,n$, then U has a weak supplement in M.
Proof. Since M_i is amply cwe-supplemented for every $i=1,2,...,n$, by Proposition 2.4, M_i is cwe-supplemented. Then by [4, Corollary 1], U has a weak supplement in M.

Proposition 2.9. Let M be the sum of the family $\{M_i\}_{i\in I}$. If M_i is amply cwe-supplemented for every $i\in I$, then M is cwe-supplemented.
Proof. Since M_i is amply cwe-supplemented for every $i\in I$, by Proposition 2.4, M_i is cwe-supplemented. Then by [4, Lemma 3], M is cwe-supplemented, as desired.

Corollary 2.10. Let M be an amply cwe-supplemented R-module. Then $M^{(\Lambda)}$ is cwe-supplemented for every index set Λ.
Proof. Clear from Proposition 2.9.

Proposition 2.11. Let M be an amply cwe-supplemented R-module. Then every factor module of M is cwe-supplemented.
Proof. Since M is amply cwe-supplemented, by Proposition 2.4, M is cwe-supplemented. Then by [4, Lemma 4], every factor module of M is cwe-supplemented, as desired.

Corollary 2.12. Every homomorphic image of an amply cwe-supplemented module is cwe-supplemented.

Proof. Clear from Proposition 2.11.

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Amply e-Supplemented Lattices

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Abstract

In this work, all lattices are complete modular lattices with the greatest element 1 and the smallest element 0. Let L be a lattice. If every essential element of L has ample supplements in L, then L is called an amply e-supplemented lattice. In this work, some properties of these lattices are investigated.
Keywords: Lattices, Essential Elements, Small Elements, Supplemented Lattices.

2020 Mathematics Subject Classification: 06C05, 06C15.

1. INTRODUCTION

In this paper, every lattice is complete modular lattice with the smallest element 0 and the greatest element 1. Let L be a lattice, $x,y\in L$ and $x\leq y$. A sublattice $\{a\in L|x\leq a\leq y\}$ is called a *quotient sublattice* and denoted by y/x. An element a of a lattice L is called a *complement* of b in L if $a\vee b=1$ and $a\wedge b=0$, in this case we denote $1=a\oplus b$ (here we also call a and b are *direct summands* of L). L is said to be *complemented* if each element has at least one complement in L. An element x of L is said to be *small* or *superfluous* and denoted by $x\ll L$ if $y=1$ for every $y\in L$ such that $x\vee y=1$. Let L be a lattice and $k\in L$. If $t=0$ for every $t\in L$ with $k\wedge t=0$, then k is called an *essential element* of L and denoted by $k\trianglelefteq L$. The meet of all maximal ($\neq 1$) elements of a lattice L is called the *radical* of L and denoted by $r(L)$. If $a\ll L$, then $a\leq r(L)$ holds. An element a of L is called a *supplement* of b in L if it is minimal for $1=b\vee a$. a is a supplement of b in a lattice L if and only if $1=b\vee a$ and $b\wedge a\ll a/0$. A lattice L is said to be *supplemented* if every element of L has a supplement in L. If every essential element of L has a supplement in L, then L is called an *essential supplemented* (briefly, *e-supplemented*) lattice. We say that an element y of L *lies above* an element x of L if $x\leq y$ and $y\ll 1/x$. Let L be a lattice. L is said to be *hollow* if every element distinct from 1 is superfluous in L. If every essential element of L with distinct from 1 is small in L or L have no essential elements with distinct from 1, then L is called an *e-hollow* lattice. L is said to be *local* if L has the greatest element ($\neq 1$). If L has an essential element $c\neq 1$ such that $k\leq c$ for every $1\neq k\trianglelefteq L$, then L is called an *e-local* lattice (here k is called the *greatest essential element* ($\neq 1$)) of L. We say an element $x\in L$ has *ample supplements* in L if for every $y\in L$ with $x\vee y=1$, x has a supplement z in L with $z\leq y$. L is said to be *amply* *supplemented* if every element of L has ample supplements in L. Let L be a lattice. It is defined β^* relation on the elements of L by $a\beta^*b$ with $a,b\in L$ if and only if for each $t\in L$ such that $a\vee t=1$ then $b\vee t=1$ and for each $k\in L$ such that $b\vee k=1$ then $a\vee k=1$.

More informations about (amply) supplemented lattices are in [1], [2], [4] and [5]. More results about (amply) supplemented modules are in [10]. More informations about essential supplemented lattices are in [9]. More informations about (amply) essential supplemented modules are in [7] and [8]. The definition of β^* relation on lattices and some properties of this relation are in [6]. This relation is a generalization of β^* relation on modules. The definition of β^* relation on modules and some properties of this relation are in [3].

Lemma 1.1. Let L be a lattice and $a,b,c,d\in L$. Then the followings are hold.
(i) If $a\leq b$ and $b\ll L$, then $a\ll L$.
(ii) Let $a\leq b$. If $a\ll L$ and $b\ll 1/a$, then $b\ll L$.
(iii) If $a\ll b/0$, then $a\ll t/0$ for every $t\in L$ with $b\leq t$.
(iv) Let $a\leq b$ and b be a direct summand of L. Then $a\ll b/0$ if and only if $a\ll L$.
(v) If $a\ll b/0$, then $a\vee c\ll (b\vee c)/c$.
(vi) If $a\ll L$, then $a\vee b\ll 1/b$.
(vii) If $a\ll b/0$ and $c\ll d/0$, then $a\vee c\ll (b\vee d)/0$.
(viii) If $a\ll L$ and $b\ll L$, then $a\vee b\ll L$.
Proof. See [4].

2. AMPLY e-SUPPLEMENTED LATTICES

Definition 2.1. Let L be a lattice. If every essential element of L has ample supplements in L, then L is called an *amply essential supplemented* (briefly, *amply e-supplemented*) lattice.

Proposition 2.2. Every amply e-supplemented lattice is essential supplemented.
Proof. Clear from definitions.

Proposition 2.3. Let L be an amply e-supplemented lattice. If every element with distinct from 0 is essential in L, then L is amply supplemented.

Proof. Let $a\in L$. If $a=0$, then a has ample supplements in L. Let $a\neq 0$. Then by hypothesis $a\trianglelefteq L$. Since L is amply e-supplemented, a has ample supplements in L. Hence every element of L has ample supplements in L and L is amply supplemented.

Proposition 2.4. Let L be an amply e-supplemented lattice. Then $1/r(L)$ have no essential elements with distinct from 1.
Proof. Since L is amply e-supplemented, by Proposition 2.2, L is essential supplemented. Then by [9, Proposition 4], $1/r(L)$ have no essential elements with distinct from 1.

Proposition 2.5. Let L be a lattice, $x \trianglelefteq L$ and $a \in L$. If $a \vee x$ has a supplement in L and $a/0$ amply e-supplemented, then x has a supplement in L.
Proof. Since $a/0$ is amply e-supplemented, by Proposition 2.2, $a/0$ essential supplemented. Then by [9, Lemma 3], x has a supplement in L.

Corollary 2.6. Let L be a lattice, $x \trianglelefteq L$ and $a_1, a_2, ..., a_n \in L$. If $a_1 \vee a_2 \vee ... \vee a_n \vee x$ has a supplement in L and $a_i/0$ amply e-supplemented for every $i=1,2,...,n$, then x has a supplement in L.
Proof. Clear from Proposition 2.5.

Proposition 2.7. Let L be a lattice and $a,b \in L$. If $a/0$ and $b/0$ are amply e-supplemented, then $(a \vee b)/0$ is essential supplemented.
Proof. Since $a/0$ and $b/0$ are amply e-supplemented, by Proposition 2.2, $a/0$ and $b/0$ is essential supplemented. Then by [9, Lemma 4], $(a \vee b)/0$ is essential supplemented, as desired.

Corollary 2.8. Let L be a lattice, $a,b \in L$ and $1=a \vee b$. If $a/0$ and $b/0$ are amply e-supplemented, then L is essential supplemented.
Proof. Clear from Proposition 2.7.

Corollary 2.9. Let L be a lattice, $a_1, a_2, ..., a_n \in L$ and $1=a_1 \vee a_2 \vee ... \vee a_n$. If $a_i/0$ is amply e-supplemented, then L is essential supplemented.
Proof. Clear from Proposition 2.7.

Proposition 2.10. Let L be an amply e-supplemented lattice and $a \in L$. Then the quotient sublattice $1/a$ is essential supplemented.
Proof. Since L is amply e-supplemented, by Proposition 2.2, L is essential supplemented. Then by [9, Lemma 5], $1/a$ is essential supplemented, as desired.

Proposition 2.11. Let L be an amply e-supplemented lattice. Then $a/0$ is essential supplemented for every direct summand a of L.

Proof. Since a is a direct summand of L, there exists an element b of L such that $1=a\oplus b$. Since L is amply e-supplemented, by Proposition 2.11, $1/b$ is essential supplemented. Then by $1/b=(a\vee b)/b\cong a/(a\wedge b)=a/0$, $a/0$ is essential supplemented.

Proposition 2.12. Every hollow lattice is amply e-supplemented.
Proof. Clear from definitions.

Proposition 2.13. Every e-hollow lattice is amply e-supplemented.
Proof. Clear from definitions.

Proposition 2.14. Every local lattice is amply e-supplemented.
Proof. Clear from definitions.

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An Alternative Analysis of Simple Chaotic Hyperjerk System in Mathematica

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Abstract

MATLAB is a software that is easy to use, gives accurate and reliable results and is the most popular one in literature. Therefore it is usually preferred to solve nonlinear equation sets in literature. In this study, Mathematica, an alternative simulation program, is also used to solve nonlinear differential equations. Mathematica is mostly used to make scientific mathematical computations. In this paper, the chaotic dynamics and the chaotic attractor illustrations of a hyperjerk system with exponential nonlinear equation [1] which are obtained from Mathematica are presented.

Keywords: Mathematica, hyperjerk system, analysis.

1. Introduction

Chaos theorem is a very important area in literature as it allows to study interdisciplinary such as physic, engineering, biology, mathematic, etc [2-3]. If a system is autonomous, this system must have at least three differential equations to exhibit chaotic behavior [2, 4]. Jerk systems which are presented by Sprott in literature are very attractive among autonomous chaotic systems [5-7]. The jerk systems have 3th-order differential equations which are derivative of each other. The hyperjerk systems have 4th-order differential equations in the form $\frac{d^4x}{dt^4} = J\left(\frac{d^3x}{dt^3}, \frac{d^2x}{dt^2}, \frac{dx}{dt}, x\right)$. While the studies in literature are generally about jerk systems, there are very few studies on hyperjerk systems in literature [1, 8-11]. Those studies were investigated while realizing numerical analysis and MATLAB have been generally preferred as a tool in literature.

Mathematica is an alternative analysis tool of MATLAB to solve differential equations. Mathematica was developed by Stephen Wolfram and improved by Wolfram Research [12-14]. Wolfram language is used as the programming language in Mathematica [15]. Mathematica is a computer program that can perform all kinds of symbolic and numerical calculations and plot 2D and 3D graphics.

In this study the hyperjerk system which proposed to literature by Dalkiran and Sprott [1] was analyzed numerically in Mathematica.

2. Analysis of Chaotic Hyperjerk System in Mathematica

The differential equation set of the hyperjerk system which is investigated in this study is given below [1].

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= z \\ \dot{z} &= w \\ \dot{w} &= -aw + f(z) - by - cx \end{aligned} \tag{1}$$

where f(z) is an exponential nonlinear function and plays an important role on the hyperjerk system entering into chaos. The equation of f(z) is given in Eq. (2).

$$f(z) = -exp(z) \tag{2}$$

The parameters a, b and c in Eq. (1) are the system parameters and values of the those are 1, 3 and 1, respectively.

In this study the differential equations described in Eq. (1) and (2) were numerically analyzed in Mathematica. Hence NDSolve command was used in Mathematica to solve those equations [15]. It is seen that how to write the differential equations and the initial conditions of those equations in Fig. 1. The commands in Fig. 2 were used to plot attractor in (x-y) plane.

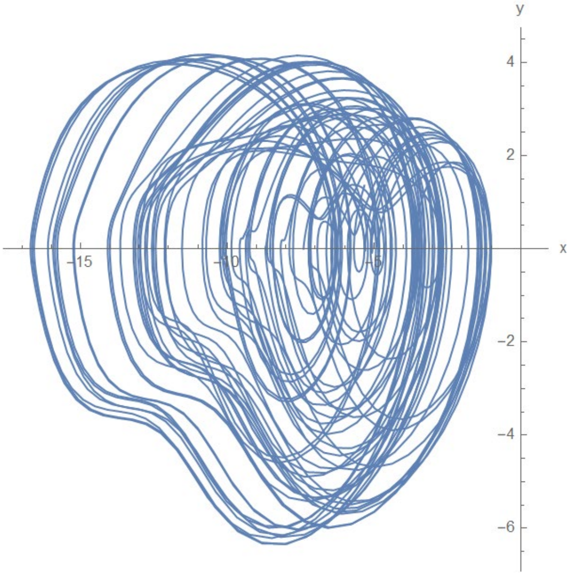
```
In[1]:= simplechaoticsys =
         NDSolve[{x'[t] == y[t], y'[t] == z[t], z'[t] == w[t],
           w'[t] == -w[t] - Exp[z[t]] - 3 y[t] - x[t], x[0] == -7.4,
           y[0] == 0, z[0] == 0, w[0] == 0.1}, {x, y, z, w}, {t, 0, 400},
          MaxSteps → Infinity]
```

Figure 1. NDSolve command was used to solve differential equations in Mathematica.

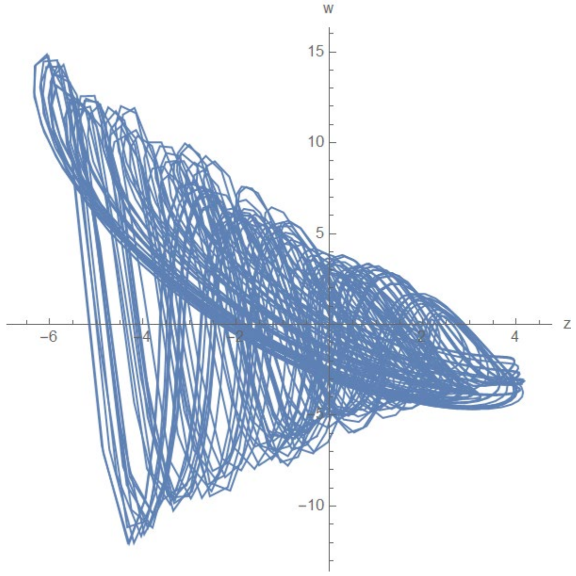
The chaotic attractor illustrations which are obtained by numerical analysis of the chaotic hyperjerk system in Mathematica are shown in Fig. 3.

```
In[2]:= ParametricPlot[Evaluate[{x[t], y[t]} /. simplechaoticsys],
         {t, 0, 400}, AxesLabel → {"x", "y"}, AspectRatio → 1]
```

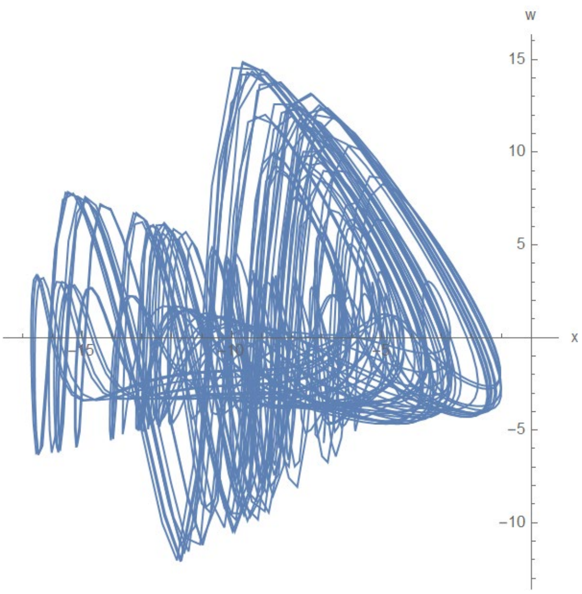
Figure 2. The command in Mathematica was used for plotting attractor.



(a)



(b)



(c)

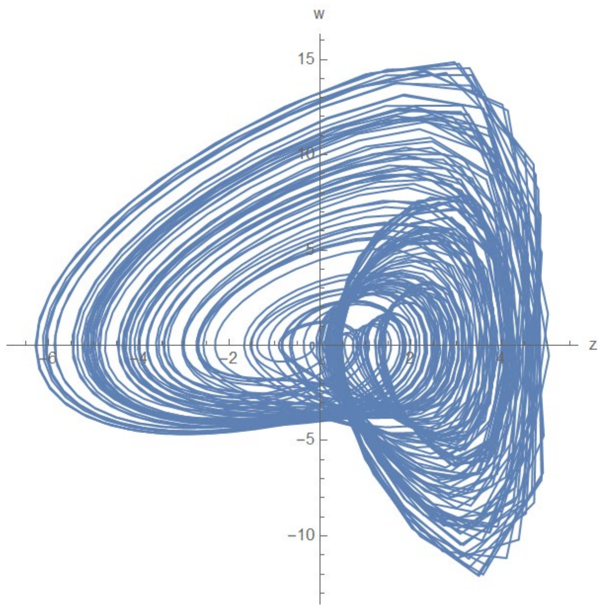


Figure 3. Analysis results of the chaotic hyperjerk system in Mathematica, (a) in plane (x-y), (b) in plane (z-w), (c) in plane (x-w), (d) in plane (z-w).

3D plotting can be realized in Mathematica as well as in MATLAB. While the command which is used to plot the attractor of the hyperjerk system in 3D is given in Fig. 4, three dimensional analysis result is shown in Fig. 5.

```
In[9]:= ParametricPlot3D[
         Evaluate[{x[t], y[t], w[t]} /. simplechaoticsys],
         {t, 0, 400}, AxesLabel → {"x", "y", "w"}, AspectRatio → 1]
```

Figure 4. The command in Mathematica was used for 3D plotting.

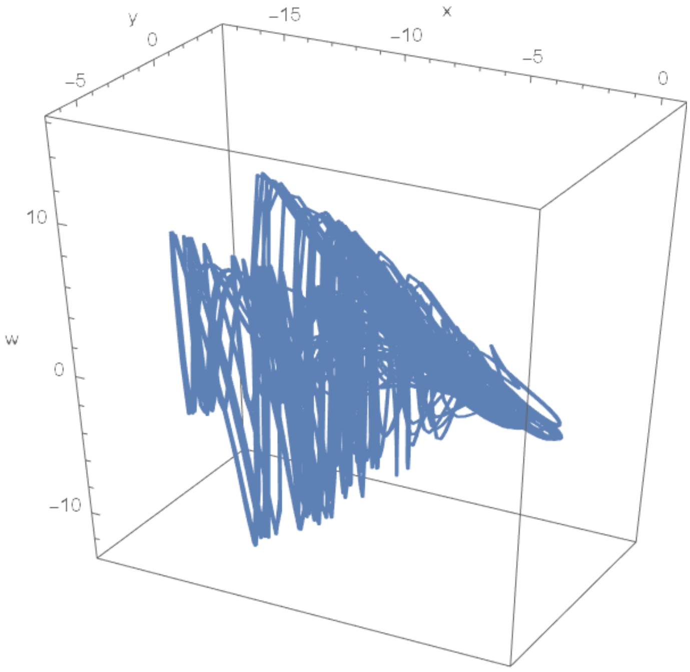


Figure 5. Three dimensional analysis result of the chaotic hyperjerk system.

The chaotic dynamics of the hyperjerk system are drawing using *plot* command in Mathematica given in Fig. 6. It the same with MATLAB. Those dynamics were graphed on the interval [0, 400]. The analysis results of the chaotic dynamics varies according to time domain were shown in Fig. 7.

```
In[10]:= Plot[Evaluate[x[t] /. simplechaoticsys], {t, 0, 400},
          AxesLabel → {"t", "x"}]
```

Figure 6. The command was used to plot the chaotic dynamics in Mathematica.

3. Conclusion

In conclusion, the numerical analysis of the chaotic hyperjerk system is presented using Mathematica. The analysis results of that system obtained using Mathematica have been plotted as chaotic dynamics, chaotic attractors and 3D plotting. In this study it is clearly seen that the analysis results which are obtained using both Mathematica and MATLAB are matched each other. Therefore Mathematica can be considered as an alternative to MATLAB to solve the differential equations.

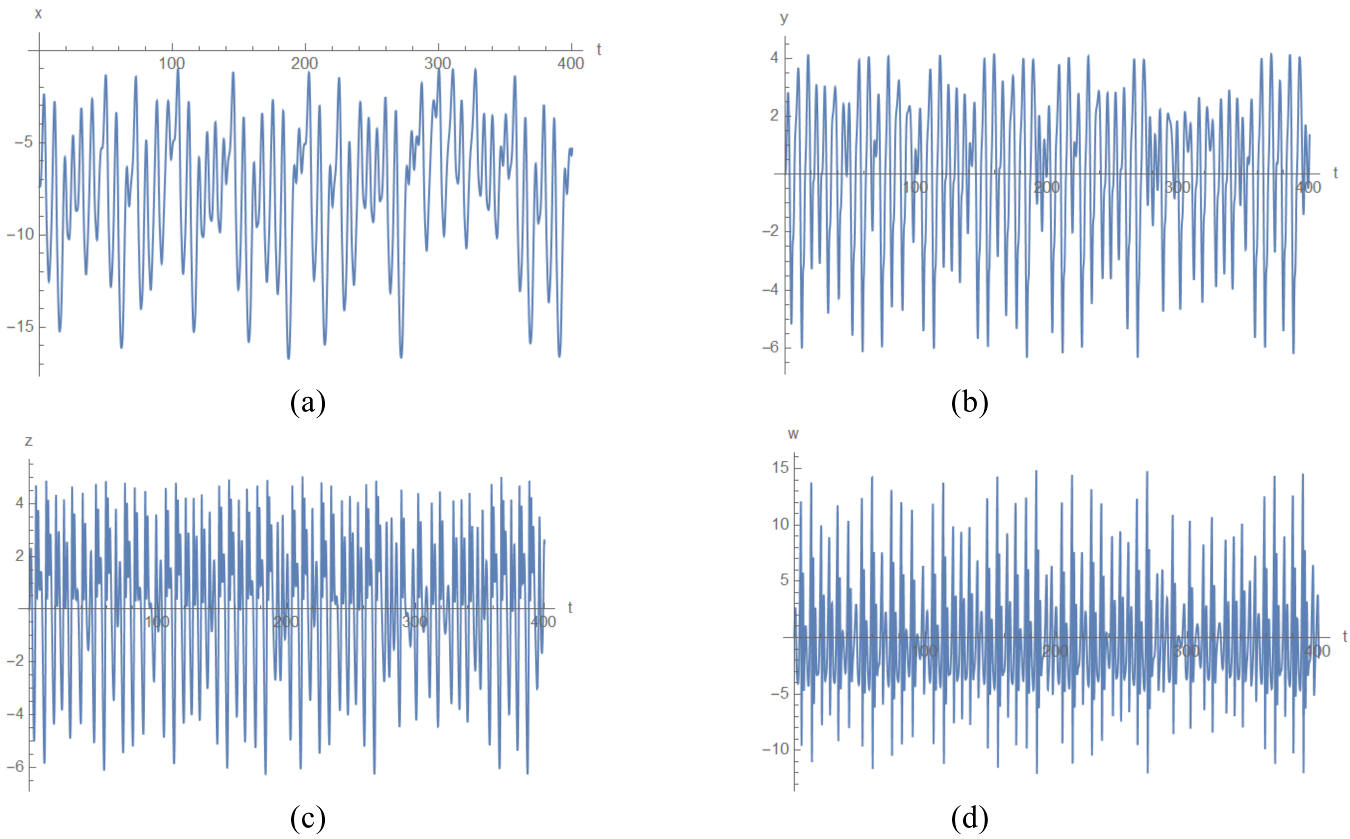


Figure 7. The chaotic dynamics of the hyperjerk system which obtained using Mathematica, (a) in plane (t-x), (b) in plane (t-y), (c) in plane (t-z), (d) in plane (t-w).

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An Application of The Becker's Univalence Criteria

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Abstract

We study univalence properties for certain subclasses of univalent functions. These subclasses are associated with a generalized differential operator. The extended Becker-typed univalence criteria will be studied for these subclasses.

Keywords: Univalence, Becker's Univalence Criteria, Differential operator.

1. Introduction and Preliminaries

Let A denote the class of analytic functions f in the open unit disk $U = \{z : |z| < 1\}$ normalized by $f(0) = f'(0) - 1 = 0$. Thus, each $f \in A$ has a Taylor series represantation

$$f(z) = z + \sum_{k=3}^{\infty} a_k z^k \tag{1}$$

Let A_2 be the subclass of A consisting of functions of the form

$$f(z) = z + \sum_{k=3}^{\infty} a_k z^k \tag{2}$$

Let $\Re$ be the univalent subclass of A which satisfies

$$\left| \frac{z^2 f'(z)}{\left(f(z)\right)^2} - 1 \right| < \mu < 1 \qquad z \in U \tag{3}$$

Let $\Re_2$ be the subclass of $\Re$ for which $f''(0) = 0$. Let $\Re_{2,\mu}$ be the subclass of $\Re_2$ consisting of functions of the form (2) which satisfy

$$\left|\frac{z^2 f'(z)}{\left(f(z)\right)^2}-1\right| < \mu, \qquad 0<\mu\le 1, \quad z\in U. \tag{4}$$

Next, we a subclass $S(p)$ of A consisting of all functions $f(z)$ that satisfy

$$\left|\left(\frac{z}{\left(f(z)\right)}\right)''\right| \le p, \quad 0<p\le 2, \quad z\in U. \tag{5}$$

In [7] (see, also [8]), Deniz and Özkan defined the differential operator D_λ^m (say: Deniz-Özkan differential operator) as follows:

For the parametres $\lambda\ge 0$ and $m\in N_0 = N\cup\{0\}$ the differential operatör D_λ^m on A defined by

$$D_\lambda^0 f(z) = f(z)$$

$$D_\lambda^1 f(z) = \lambda z^3 f'''(z) + (2\lambda+1) z^2 f''(z) + z f'(z)$$

$$D_\lambda^m f(z) = D(D_\lambda^{m-1} f(z)$$

for $z\in U$.

For a function f in A, from the definition of the differential operatör D_λ^m, we can easily see that

$$D_\lambda^m f(z) = z + \sum_{n=2}^{\infty} n^{2m}(\lambda(n-1)+1)^m a_n z^n.$$

Also, $D_\lambda^m f(z)\in A.$ For the special cases of $\lambda = 0{,}1$ we obtain Salagean differential operator (see [15]).

In this study, we consider that the integral operator

$$\Upsilon_{\gamma_i}\left(n,m,\rho,\lambda;z\right) = \left\{\rho\int_0^z t^{\rho-1}\prod_{i=1}^{n}\left(\frac{D_\lambda^m f_i(t)}{t}\right)^{1/\gamma_i} dt\right\}^{1/\rho}, \qquad f_i\in A \tag{6}$$

which generalize many integral operators. In fact, if we choose suitable values of parameters in this type of operator, we get the some interesting operators (see [3,4,9,10])

For our main result,we need the following lemmas.

Lemma 1.1 (see[1,2]). Let c be complex number, $|c|\le 1, c\ne -1$. *If* $f(z)=z+a_2z^2+\ldots$ is analytic function in U and

$$\left|c|z|^2+\left(1-|z|^2\right)\frac{zf''(z)}{f'(z)}\right|\le 1, \qquad \forall z\in U\,, \tag{7}$$

then the function f is univalent in U .

Lemma 1.2 (Schwarz Lemma). Let the function $f(z)$ be analytic in the disk $U_R=\{z\in\mathbb{C}:|z|<R\}$ with $|f(z)|<M$ .If $f(z)$ has one zero with multiply$\ge m$ for $z=0$, then

$$|f(z)|\le \frac{M}{R^m}|z|^m, \qquad \forall z\in U_R\,, \tag{8}$$

and aquality holds only *If* $f(z)=e^{i\theta}\left(M/R^m\right)|z|^m$, where θ is constant.

Lemma 1.3 (see[12]). Let δ be c complex number with $R_e\delta>0$ such that $c\in\mathbb{C},|c|\le 1, c\ne -1.$ If $f\in A$ satisfies the condition

$$\left|c|z|^{2\delta}+\left(1-|z|^{2\delta}\right)\frac{zf''(z)}{\delta f'(z)}\right|\le 1, \quad \forall z\in U \tag{9}$$

then the function

$$F_\delta(z)=\left\{\delta\int_0^z t^{\delta-1}f'(t)\,dt\right\}^{1/\delta} \tag{10}$$

Lemma 1.4 (see[16]). If $f\in S(\rho)$, then

$$\left|\frac{z^2 f'(z)}{\left(f(z)\right)^2}-1\right|\le \rho|z|^2, \qquad \forall z\in U\,. \tag{11}$$

2. Univalence Properties

In this section, we will discuss the univalence properties of the new family of integral operators mentioned above.

Teorem 2.1. Let c be complex number $\left|D_\lambda^m f_i(z)\right| \le M_i, M_i \ge 1$, and $f \in S(p_i)$ for $i = \{1,2,3,\dots\}$ such that

$$\operatorname{Re}(\rho) \ge \sum_{i=1}^{\infty} \frac{\left((M_i - 1)p_i + 2\right)M_i - 1}{|\gamma_i|(M_i - 1)}, \quad M_i \ge 1 \tag{12}$$

where ρ, γ_i are complex numbers. If

$$|c| \le 1 - \frac{1}{R(\rho)} \sum_{i=1}^{\infty} \frac{\left((M_i - 1)p_i + 2\right)M_i - 1}{|\gamma_i|(M_i - 1)}, \quad M_i \ge 1, \tag{13}$$

then the family $\Upsilon_{\gamma_i}(n,m,\rho,\lambda;z)$ is univalent.

Proof. Since $\left|D_\lambda^m f_i(z)\right| \le M_i$, so by Lemma 1.4, we have

$$\left|\frac{z^2\left(D_\lambda^m f_i(z)\right)'}{\left(D_\lambda^m f_i(z)\right)^2} - 1\right| \le p_i|z|^2, \qquad \forall z \in U \ . \tag{14}$$

Now, by using hypotheis, we have

$$\left|D_\lambda^m f_i(z)\right| \le M_i, \tag{15}$$

so by Lemma 1.3, we get

$$\left|D_\lambda^m f_i(z)\right| \le M_i z. \tag{16}$$

Since

$$\frac{D_\lambda^m f(z)}{z} \neq 0 \tag{17}$$

for $z=0$ we have

$$\prod_{i=1}^{m}\left(\frac{D_\lambda^m f_i(z)}{z}\right)^{1/\gamma_i} = \left(\frac{D_\lambda^m f_1(z)}{z}\right)^{1/\gamma_i} \ldots \left(\frac{D_\lambda^m f_m(z)}{z}\right)^{1/\gamma_m} = 1. \tag{18}$$

Let

$$F'(z) = \left(\left(\frac{D_\lambda^m f_1(z)}{z}\right)^{1/\gamma_1} \ldots \left(\frac{D_\lambda^m f_m(z)}{z}\right)^{1/\gamma_m}\right), \tag{19}$$

which implies that

$$\frac{zF''(z)}{F'(z)} = \frac{1}{\gamma_1}\left(\frac{z\left(D_\lambda^m f_1(z)\right)'}{D_\lambda^m f_1(z)} - 1\right) + \ldots + \frac{1}{\gamma_m}\left(\frac{z\left(D_\lambda^m f_m(z)\right)'}{D_\lambda^m f_m(z)} - 1\right), \tag{20}$$

and

$$\left|\frac{zF''(z)}{F'(z)}\right| \le \sum_{i=1}^{\infty}\frac{1}{|\gamma_i|}\left(\left|\frac{z\left(D_\lambda^m f_i(z)\right)'}{\left(D_\lambda^m f_i(z)\right)}\right| + 1\right). \tag{21}$$

This implies that

$$\left|\frac{zF''(z)}{F'(z)}\right| \le \sum_{i=1}^{\infty}\frac{1}{|\gamma_i|}\left(\left|\frac{z\left(D_\lambda^m f_i(z)\right)'}{\left(D_\lambda^m f_i(z)\right)^2}\right|\left|\frac{\left(D_\lambda^m f_i(z)\right)}{z}\right| + 1\right), \tag{22}$$

or

$$\left|\frac{zF''(z)}{F'(z)}\right| \le \sum_{i=1}^{\infty}\frac{1}{|\gamma_i|}\left(\left|\frac{z\left(D_\lambda^m f_i(z)\right)'}{\left(D_\lambda^m f_i(z)\right)^2}\right| M_i + 1\right). \tag{23}$$

Using (16), we get

$$\left|\frac{zF''(z)}{F'(z)}\right| \leq \sum_{i=1}^{\infty}\frac{1}{|\gamma_i|}\left(\left|\frac{z\left(D_\lambda^m f_i(z)\right)'}{\left(D_\lambda^m f_i(z)\right)^2}-1\right|M_i+M_i+1\right). \tag{24}$$

This implies that

$$\left|\frac{zF''(z)}{F'(z)}\right| \leq \sum_{i=1}^{\infty}\frac{1}{|\gamma_i|}\left(\rho_i|z|^2 M_i+M_i+1\right). \tag{25}$$

By using (14), we get

$$\left|\frac{zF''(z)}{F'(z)}\right| \leq \sum_{i=1}^{\infty}\frac{1}{|\gamma_i|}\left(\rho_i M_i+\left(M_i+M_i^{\,2}+M_i^{\,3}\ldots\right)+1\right), \tag{26}$$

which implies that

$$\left|\frac{zF''(z)}{F'(z)}\right| \leq \sum_{i=1}^{\infty}\frac{1}{|\gamma_i|}\left(\rho_i M_i+\left(M_i+M_i^{\,2}+M_i^{\,3}\ldots\right)+1\right), \tag{27}$$

and the condition $M_i, M_i^{\,2}, M_i^3, \ldots \geq 1$ implies that

$$\left|\frac{zF''(z)}{F'(z)}\right| \leq \sum_{i=1}^{\infty}\frac{1}{|\gamma_i|}\left(p_i M_i+\left(\frac{M_i}{M_i-1}\right)+1\right)=\sum_{i;=1}^{\infty}\frac{1}{|\gamma_i|}\left(p_i M_i+\left(\frac{2M_i-1}{M_i-1}\right)\right),$$

$$\left|\frac{zF''(z)}{F'(z)}\right| \leq \sum_{i=1}^{\infty}\frac{1}{|\gamma_i|}\left(\left(\frac{p_i M_i^2-p_i M_i+2M_i-1}{M_i-1}\right)\right) \leq \sum_{i=1}^{\infty}\frac{1}{|\gamma_i|}\left(\left(\frac{\left(p_i M_i-p_i+2\right)M_i-1}{M_i-1}\right)\right),$$

$$\left|\frac{zF''(z)}{F'(z)}\right| \leq \sum_{i=1}^{\infty}\frac{1}{|\gamma_i|}\left(\left(\frac{\left((M_i-1)p_i+2\right)M_i-1}{M_i-1}\right)\right). \tag{28}$$

Now, we calculate

$$\left|c|z|^{2p}+\left(1-|z|^{2p}\right)\frac{zF''(z)}{\rho F'(z)}\right|\le|c|+\frac{1}{|\rho|}\left|\frac{zF''(Z)}{F'(z)}\right|\le|c|+\frac{1}{\operatorname{Re}(\rho)}\left|\frac{zF''(z)}{F'(z)}\right|. \tag{29}$$

This implies that

$$\left|c|z|^{2p}+\left(1-|z|^{2p}\right)\frac{zF''(z)}{\rho F'(z)}\right|<|c|+\frac{1}{\operatorname{Re}(\rho)}\sum_{i=1}^{\infty}\frac{1}{|\gamma_i|}\left(\left(\frac{\left((M_i-1)p_i+2\right)M_i-1}{M_i-1}\right)\right). \tag{30}$$

By using (57), we conlude that

$$\left|c|z|^{2p}+\left(1-|z|^{2p}\right)\frac{zF''(z)}{\rho F'(z)}\right|\le|c|+\frac{1}{|\rho|}\left|\frac{zF''(Z)}{F'(z)}\right|\le 1. \tag{31}$$

Hence, by Lemma 1.3, the family of integral operators $\Upsilon_{\gamma_i}(n,m,\rho,\lambda;z)$ is univalent.

Corollary 2.2 Let c be complex number, $\left|D_\lambda^m f_i(z)\right|\le M$, $M\ge 1$ and $D_\lambda^m f_i(z)\in S(p_i)$, $M_i=M\ge 1$ for all $i=\{1,2,3,\ldots\}$ such that

$$\operatorname{Re}(\rho)\ge\sum_{i=1}^{\infty}\frac{\left((M-1)p_i+2\right)M-1}{|\gamma_i|(M-1)}, \tag{32}$$

where ρ , γ_i are complex numbers, If

$$|c|\le 1-\frac{1}{\operatorname{Re}(\rho)}\sum_{i=1}^{\infty}\frac{\left((M-1)p_i+2\right)M-1}{|\gamma_i|(M-1)},\quad M\ge 1, \tag{33}$$

then the family $\Upsilon_{\gamma_i}(n,m,\rho,\lambda;z)$ is univalent.

Corolary 2.3 Let c be complex number, $\left|D_\lambda^m f_i(z)\right|\le M$, $M\ge 1$ and the family $D_\lambda^m f_i(z)\in S(p_i)$, $M_i=M\ge 1$, $|\gamma_i|=|\gamma|$, for all $i=\{1,2,3,\ldots\}$ such that

$$\operatorname{Re}(\rho)\ge\sum_{i=1}^{\infty}\frac{\left((M-1)p_i+2\right)M-1}{|\gamma|(M-1)}, \tag{34}$$

where ρ , γ_i are complex numbers. If

$$|c| \le 1 - \frac{1}{\operatorname{Re}(\rho)} \sum_{i=1}^{\infty} \frac{\left((M-1)p_i + 2\right)M - 1}{|\gamma|(M-1)}, \quad M \ge 1, \tag{35}$$

then the family $\Upsilon_{\gamma_l}(n, m, \rho, \lambda; z)$ is univalent.

Using the method given in the proof of Theorem 2.1, one can prove the following results.

Theorem 2.4 Let c be complex number, $\left|D_\lambda^m f_i(z)\right| \le M_i, M_i \ge 1$ for all $i = \{1, 2, 3, \ldots\}$ and the family $D_\lambda^m f_i(z) \in S(p_i)$ for $i = \{1, 2, 3, \ldots\}$ and such that

$$\operatorname{Re}(\rho) \ge \sum_{i=1}^{\infty} \frac{(p_i M_i - 1)M_i + 1}{|\gamma_i| p_i M_i}, \tag{36}$$

where ρ, γ_i are complex numbers. If

$$|c| \le 1 - \frac{1}{\operatorname{Re}(\rho)} \sum_{i=1}^{\infty} \frac{(p_i M_i - 1)M_i + 1}{|\gamma_i|(p_i M_i)}, \quad M_i \ge 1, \tag{37}$$

then the family $\Upsilon_{\gamma_l}(n, m, \rho, \lambda; z)$ is univalent.

Theorem 2.5 Let c be complex number, $\left|D_\lambda^m f_i(z)\right| \le M_i, M_i \ge 1$ for all $i = \{1, 2, 3, \ldots\}$ and the family $D_\lambda^m f_i(z) \in S(p_i)$ for $i = \{1, 2, 3, \ldots\}$ and such that

$$\operatorname{Re}(\rho) \ge \sum_{i=1}^{n} \frac{\left(p_i(M_i - 1) + M_i^{\,n} - 2\right)M_i + 1}{|\gamma_i|(M_i - 1)}, \tag{38}$$

where ρ, γ_i are complex numbers. If

$$|c| \le 1 - \frac{1}{\operatorname{Re}(\rho)} \sum_{i=1}^{n} \frac{\left(p_i(M_i - 1) + M_i^n - 2\right)M_i + 1}{|\gamma_i|(M_i - 1)}, \quad M_i \ge 1, \tag{39}$$

then the family $\Upsilon_{\gamma_l}(n, m, \rho, \lambda; z)$ is univalent.

Theorem 2.6 Let c be complex number, $\left|D_\lambda^m f_i(z)\right| \le M_i, M_i \ge 1$ for all $i = \{1,2,3,\dots\}$ and the family $D_\lambda^m f_i(z) \in S(p_i)$ for $i = \{1,2,3,\dots\}$ and such that

$$\operatorname{Re}(\rho) \ge \sum_{i=1}^{n} \frac{\left(p_i + \left(n(n+1)/2\right)\right)M_i - 1}{|\gamma_i|}, \tag{40}$$

where ρ, γ_i are complex numbers. If

$$|c| \le 1 - \frac{1}{\operatorname{Re}(\rho)} \sum_{i=1}^{\infty} \frac{\left(p_i + \left(n(n+1)/2\right)\right)M_i - 1}{|\gamma_i|}, \quad M_i \ge 1 \tag{41}$$

then the family $\Upsilon_{\gamma_l}(n,m,\rho,\lambda;z)$ is univalent.

Theorem 2.7 Let c be complex number, $\left|D_\lambda^m f_i(z)\right| \le M_i, M_i \ge 1$ for all $i = \{1,2,3,\dots\}$ and the family $D_\lambda^m f_i(z) \in \Re_{2,\mu_i}$, for $i = \{1,2,3,\dots\}$ such that

$$\operatorname{Re}(\rho) \ge \sum_{i=1}^{n} \frac{\left(\mu_i + n(n+1)\right)M_i}{|\gamma_i|}, \tag{42}$$

where ρ, γ_i are complex numbers. If

$$|c| \le 1 - \frac{1}{\operatorname{Re}(\rho)} \sum_{i=1}^{n} \frac{\left(\mu_i + n(n+1)\right)M_i}{|\gamma_i|}, \quad M_i \ge 1 \tag{43}$$

then the family $\Upsilon_{\gamma_l}(n,m,\rho,\lambda;z)$ is univalent.

Proof. Using the proof of Theorem 2.1, we have

$$\left|\frac{zF''(z)}{F'(z)}\right| \le \sum_{i=1}^{\infty} \frac{1}{|\gamma_i|}\left(\left|\frac{z\left(D_\lambda^m f_i(z)\right)'}{\left(D_\lambda^m f_i(z)\right)^2} - 1\right| M_i + 1\right). \tag{44}$$

Since $D_\lambda^m f_i(z) \in \Re_{2,\mu_i}$, so by using (4), we get

$$\left|\frac{z^2\left(D_\lambda^m f_i(z)\right)'}{\left(D_\lambda^m f_i(z)\right)^2}-1\right|\le \mu_i, \qquad 0<\mu\le 1,\ \ z\in U. \tag{45}$$

So from (44), we get

$$\left|\frac{zF''(z)}{F'(z)}\right|\le \sum_{i=1}^{\infty}\frac{1}{|\gamma_i|}\left(\left|\frac{z\left(D_\lambda^m f_i(z)\right)'}{\left(D_\lambda^m f_i(z)\right)^2}-1\right|M_i+M_i+1\right), \tag{46}$$

or

$$\left|\frac{zF''(z)}{F'(z)}\right|\le \sum_{i=1}^{\infty}\frac{1}{|\gamma_i|}\left(\mu_i M_i+2M_i\right), \qquad M_i>1,$$

$$\left|\frac{zF''(z)}{F'(z)}\right|\le \sum_{i=1}^{\infty}\frac{1}{|\gamma_i|}\left(\mu_i M_i+2M_i+4M_i+\ldots+n-times\right), \qquad M_i>1 \tag{47}$$

$$\left|\frac{zF''(z)}{F'(z)}\right|\le \sum_{i=1}^{\infty}\frac{1}{|\gamma_i|}\left(\mu_i M_i+n(n+1)M_i\right), \qquad M_i>1.$$

Now, we evaluate the expession

$$\left|c|z|^{2p}+\left(1-|z|^{2p}\right)\frac{zF''(z)}{\rho F'(z)}\right|\le |c|+\frac{1}{|\rho|}\left|\frac{zF''(z)}{F'(z)}\right|\le |c|+\frac{1}{\mathrm{Re}(\rho)}\left|\frac{zF''(z)}{F'(z)}\right|,$$

$$\left|c|z|^{2p}+\left(1-|z|^{2p}\right)\frac{zF''(z)}{\rho F'(z)}\right|< |c|+\frac{1}{\mathrm{Re}(\rho)}\sum_{i=1}^{\infty}\frac{1}{|\gamma_i|}\left(\mu_i M_i+n(n+1)M_i\right). \tag{48}$$

Using (46) and (47), we conclude that

$$\left|c|z|^{2p}+\left(1-|z|^{2p}\right)\frac{zF''(z)}{\rho F'(z)}\right|\le 1. \tag{49}$$

Hence by using Lemma 1.3, the family $\Upsilon_{\gamma_i}(n,m,\rho,\lambda;z)$ is univalent.

Corollary 2.8 Let c be complex number, $\left|D_\lambda^m f_i(z)\right| \le M_i, M_i \ge 1$ for all $i = \{1,2,3,\ldots\}$ and the family $D_\lambda^m f_i(z) \in \Re_{2,\mu_i}$, for $i = \{1,2,3,\ldots\}$ such that

$$\operatorname{Re}(\rho) \ge \sum_{i=1}^{n} \frac{\left(\mu_i + n(n+1)\right)M}{|\gamma_i|}, \tag{50}$$

where ρ, γ_i are complex numbers. If

$$|c| \le 1 - \frac{1}{\operatorname{Re}(\rho)} \sum_{i=1}^{n} \frac{\left(\mu_i + n(n+1)\right)M}{|\gamma_i|}, \qquad M \ge 1 \tag{51}$$

then the family $\Upsilon_{\gamma_i}(n, m, \rho, \lambda; z)$ is univalent.

Corollary 2.9 Let c be complex number, $\left|D_\lambda^m f_i(z)\right| \le M$, $M \ge 1$ for all $i = \{1,2,3,\ldots\}$ and the family $D_\lambda^m f_i(z) \in \Re_{2,\mu_i}$, for $i = \{1,2,3,\ldots\}$ such that

$$\operatorname{Re}(\rho) \ge \sum_{i=1}^{n} \frac{\left(\mu_i + n(n+1)\right)M}{|\gamma|}, \tag{52}$$

where ρ, γ_i are complex numbers. If

$$|c| \le 1 - \frac{1}{\operatorname{Re}(\rho)} \sum_{i=1}^{n} \frac{\left(\mu_i + n(n+1)\right)M}{|\gamma|}, \qquad M \ge 1 \tag{53}$$

then the family $\Upsilon_{\gamma_i}(n, m, \rho, \lambda; z)$ is univalent.

Using a similar method as in the proof of Theorem 2.7, one prove the following results.

Theorem 2.10 Let c be complex number, $\left|D_\lambda^m f_i(z)\right| \le M$, $M \ge 1$ for all $i = \{1,2,3,\ldots\}$ and the family $D_\lambda^m f_i(z) \in \Re_{2,\mu_i}$, for $i = \{1,2,3,\ldots\}$ such that

$$\operatorname{Re}(\rho) \ge \sum_{i=1}^{n} \frac{(\mu_i M_i - 1)M_i + M_i^n M_i}{|\gamma_i|(M_i - 1)}, \tag{54}$$

where ρ, γ_i are complex numbers. If

$$|c| \le 1 - \frac{1}{\operatorname{Re}(\rho)} \sum_{i=1}^{n} \frac{(\mu_i M_i - 1) M_i + M_i^n M_i}{|\gamma_i| (M_i - 1)}, \qquad M \ge 1 \tag{55}$$

then the family $\Upsilon_{\gamma_I}(n, m, \rho, \lambda; z)$ is univalent.

Theorem 2.11 Let c be complex number, $\left|D_\lambda^m f_i(z)\right| \le M$, $M \ge 1$ for all $i = \{1, 2, 3, \ldots\}$ and the family $D_\lambda^m f_i(z) \in \Re_{2,\mu_i}$, for $i = \{1, 2, 3, \ldots\}$ such that

$$\operatorname{Re}(\rho) \ge \sum_{i=1}^{n} \frac{(\mu_i M_i - \mu_i + 2) M_i - 1}{|\gamma_i| (M_i - 1)}, \tag{56}$$

where ρ , γ_i are complex numbers. If

$$|c| \le 1 - \frac{1}{\operatorname{Re}(\rho)} \sum_{i=1}^{n} \frac{(\mu_i M_i - \mu_i + 2) M_i - 1}{|\gamma_i| (M_i - 1)}, \quad M \ge 1 \tag{57}$$

then the family $\Upsilon_{\gamma_I}(n, m, \rho, \lambda; z)$ is univalent.

Note that some other related work involving integral operators regarding univalence criteria can ajso be found in [5,6,11,13,14,17].

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An extension of TOPSIS method to the generalized spherical fuzzy environment

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Abstract

The Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) is a multi-attribute decision approach that is based on choosing an alternative that has the shortest distance from the positive ideal solution (PIS) and the longest distance from the negative ideal solution (NIS). Since this method provides the best and worst alternatives in a more realistic form, the obtained solution is thought as more reliable and effective. In this study, we aim to extend the TOPSIS method to the generalized spherical fuzzy environment. First, we establish a novel method to solve the multi-attribute group decision-making (MAGDM) problems based on TOPSIS by using generalized spherical fuzzy data. Then, we exemplify this approach to provide the steps more understandable, and finally, we compare the results of the same problem by solving it with the proposed and existing methods.

Keywords: Einstein operations, Hamacher operations, Generalized spherical fuzzy sets, multi-criteria group decision, TOPSIS.

1. Introduction

In 1981, TOPSIS was proposed by Hwang and Yoon [1] to determine the best alternative based on the concepts of the compromise solution in the decision-making processes. The mentioned compromise solution can be considered as choosing the solution with the shortest Euclidean distance from the PIS and the farthest Euclidean distance from the NIS. After Zadeh [2] introduced to the fuzzy set theory to handle the uncertainty in the real-life problems, TOPSIS was applied to fuzzy environment by Chen [3]. Also, in literature, there are different types of set theories which are generalizations of fuzzy set theory such as intuitionistic fuzzy set theory (Atanassov [4]), Pythagorean fuzzy set theory (Yager [5]), picture fuzzy set theory (Cuong [6]), spherical fuzzy set theory (Kutlu Gündoğdu and Kahraman [7], Ashraf et al. [8]), spherical fuzzy soft set theory (Perveen et al. [9]) and etc. Until now, various decision-making methods, including TOPSIS, have been developed on the mentioned set theories [10, 11, 12, 13, 14, 15, 16, 17, 18,19]. Recently, Hague et al. [20] initiated the generalized spherical fuzzy set theory as a generalization of the spherical fuzzy set to use when this theory cannot enough to handle the data in the problems consisting of uncertain information. Some recent studies on generalized spherical fuzzy set theories and decision-making approaches have been done can be found in [21, 22, 23, 24].

In this study, we aim to extend the TOPSIS to the generalized spherical fuzzy environment. For this aim, we first construct the linguistic table whose elements are generalized spherical fuzzy to use in evaluating the alternatives or criteria in the decision-making process. Then, we establish a novel MAGDM approach based on TOPSIS with algebraic, Einstein and Hamacher operations. Then, we give an illustrative example to clearly explain the steps of the proposed method by comparing the results.

2. Preliminaries

In this section, we recall some fundamental definitions which will be used in the main sections. Throughout this paper U will refer the set of the discourse.

Definition 2.1. [4, 5] Let $\mu: U \to [0,1]$ and $\nu: U \to [0,1]$ be any two mappings. A set $I = \{< x, \mu(x), \nu(x) > | x \in U\}$ is called a/an

(i) intuitionistic fuzzy set (IFS) if the condition $0 \leq \mu(x) + \nu(x) \leq 1$ hold for all $x \in U$.

(ii) Pythagorean fuzzy set (PyFS) if the condition $0 \leq \mu^2(x) + \nu^2(x) \leq 1$ hold for all $x \in U$.

The values $\mu(x), \nu(x) \in [0,1]$ describe the pm-d and nm-d of x to I, respectively.

The pair $I =< \mu, \nu >$ where $\mu, \nu \in [0,1]$ and $\mu + \nu \leq 1$ (or $\mu^2 + \nu^2 \leq 1$), is called an intuitionistic fuzzy number (IFN) (or Pythagorean fuzzy number (PyFN)).

Remark 2.2. [4, 5] The set of IFNs is the subset of the set of PyFNs.

Definition 2.3. [6, 7, 8, 20] Let $\mu: U \to [0,1], \iota: U \to [0,1]$ and $\nu: U \to [0,1]$ be three mappings. A set $G = \{< x, \mu(x), \iota(x)\ \nu(x) > | x \in U\}$ is called a

(i) picture fuzzy set (PFS) if the condition $0 \leq \mu(x) + \iota(x) + \nu(x) \leq 1$ hold for all $x \in U$.

(ii) spherical fuzzy set (SFS) if the condition $0 \leq \mu^2(x) + \iota^2(x) + \nu^2(x) \leq 1$ hold for all $x \in U$.

(ii) generalized spherical fuzzy set (GSFS) if the condition $0 \leq \mu^2(x) + \iota^2(x) + \nu^2(x) \leq 3$ hold for all $x \in U$.

The values $\mu(x), \iota(x), \nu(x) \in [0,1]$ denote the pm-d, neum-d and nm-d of x to G, respectively.

The triplet $G =< \mu, \iota, \nu >$ where $\mu, \iota, \nu \in [0,1]$ and $\mu^2 + \iota^2 + \nu^2 \leq 3$ (or $\mu + \iota + \nu \leq 1$ and $\mu^2 + \iota^2 + \nu^2 \leq 1$, resp.), is called a generalized spherical fuzzy number (GSFN) (or Picture fuzzy number (PFN) and Spherical fuzzy number (SFN), resp.).

Remark 2.4. [20] (1) The set of SFNs is the subset of the set of GSFNs and the set of PFNs is the subset of the set of SFNs.

(2) In PFN, since the sum of the pm-d, neum-d and nm-d is ≤ 1, this sum is considered as linearly, and this represents a plane in space. But in the theories of SFN and GSFN, we take the nonlinear form of membership degrees which represents a sphere in space.

Definition 2.5. [20] Let $G =< \mu, \iota, \nu >$, $G_1 =< \mu_1, \iota_1, \nu_1 >$, $G_2 =< \mu_2, \iota_2, \nu_2 >$ be three GSFNs and $a \geq 0$. Then the algebraic operations between GSFNs are defined as follows:

(i) $G^c =< \nu, \iota, \mu >$,

(ii) $G_1 \leq G_2$ iff $\mu_1 \leq \mu_2, \iota_1 \geq \iota_2$ and $\nu_1 \geq \nu_2$,

(iii) $G_1 = G_2$ iff $G_1 \leq G_2$ and $G_2 \leq G_1$,

(iv) $G_1 \oplus G_2 =< \sqrt{\mu_1^2 + \mu_2^2 - \mu_1^2\mu_2^2}, \iota_1\iota_2, \nu_1\nu_2 >$,

(v) $G_1 \odot G_2 =< \mu_1\mu_2, \iota_1\iota_2, \sqrt{\nu_1^2 + \nu_2^2 - \nu_1^2\nu_2^2} >$,

(vi) $a \times G =< \sqrt{1-(1-\mu^2)^a}, \iota^a, \nu^a >$,

(vii) $G^a =< \mu^a, \iota^a, \sqrt{1-(1-\nu^2)^a} >$.

Definition 2.6. [21] Let $G =< \mu, \iota, \nu >, G_1 =< \mu_1, \iota_1, \nu_1 >$ and $G_2 =< \mu_2, \iota_2, \nu_2 >$ be three GSFNs and $a \geq 0$. Then the Einstein operations are defined over the GSFNs as follow:

(i) $G_1 \oplus_E G_2 =< \sqrt{\frac{\mu_1^2+\mu_2^2}{1+\mu_1^2.\mu_2^2}}, \sqrt{\frac{\iota_1^2.\iota_2^2}{1+(1-\iota_1^2)(1-\iota_2^2)}}, \sqrt{\frac{\nu_1^2.\nu_2^2}{1+(1-\nu_1^2)(1-\nu_2^2)}} >$,

(ii) $G_1 \odot_E G_2 =< \sqrt{\frac{\mu_1^2.\mu_2^2}{1+(1-\mu_1^2)(1-\mu_2^2)}}, \sqrt{\frac{\iota_1^2.\iota_2^2}{1+(1-\iota_1^2)(1-\iota_2^2)}}, \sqrt{\frac{\nu_1^2+\nu_2^2}{1+\nu_1^2.\nu_2^2}} >$,

(iii) $a \times_E G =< \sqrt{\frac{(1+\mu^2)^a-(1-\mu^2)^a}{(1+\mu^2)^a+(1-\mu^2)^a}}, \sqrt{\frac{2\iota^{2a}}{(2-\iota^2)^a+\iota^{2a}}}, \sqrt{\frac{2\nu^{2a}}{(2-\nu^2)^a+\nu^{2a}}} >$,

(iv) $G^{\wedge_E a} =< \sqrt{\frac{2\mu^{2a}}{(2-\mu^2)^a+\mu^{2a}}}, \sqrt{\frac{2\iota^{2a}}{(2-\iota^2)^a+\iota^{2a}}}, \sqrt{\frac{(1+\nu^2)^a-(1-\nu^2)^a}{(1+\nu^2)^a+(1-\nu^2)^a}} >$.

Definition 2.7. [22] Let $G =< \mu, \iota, \nu >, G_1 =< \mu_1, \iota_1, \nu_1 >$ and $G_2 =< \mu_2, \iota_2, \nu_2 >$ be three GSFNs, $\lambda > 0$ and
$a \geq 0$. Then the Hamacher operations are defined over the GSFNs as follow:

(i) $G_1 \oplus_H G_2 =< \sqrt{\frac{\mu_1^2+\mu_2^2-\mu_1^2\mu_2^2-(1-\lambda)\mu_1^2\mu_2^2}{1-(1-\lambda)\mu_1^2.\mu_2^2}}, \sqrt{\frac{\iota_1^2.\iota_2^2}{\lambda+(1-\lambda)(\iota_1^2+\iota_2^2-\iota_1^2.\iota_2^2)}}, \sqrt{\frac{\nu_1^2.\nu_2^2}{\lambda+(1-\lambda)(\nu_1^2+\nu_2^2-\nu_1^2.\nu_2^2)}} >$,

(ii) $G_1 \odot_H G_2 =< \sqrt{\frac{\mu_1^2.\mu_2^2}{\lambda+(1-\lambda)(\mu_1^2+\mu_2^2-\mu_1^2.\mu_2^2)}}, \sqrt{\frac{\iota_1^2.\iota_2^2}{\lambda+(1-\lambda)(\iota_1^2+\iota_2^2-\iota_1^2.\iota_2^2)}}, \sqrt{\frac{\nu_1^2+\nu_2^2-\nu_1^2\nu_2^2-(1-\lambda)\nu\nu_2^2}{1-(1-\lambda)\nu_1^2.\nu_2^2}} >$,

(iii) $a \times_H G =< \sqrt{\frac{(1+(\lambda-1)\mu^2)^a-(1-\mu^2)^a}{(1+(\lambda-1)\mu^2)^a+(\lambda-1)(1-\mu^2)^a}}, \sqrt{\frac{\lambda\iota^{2a}}{(1+(\lambda-1)(1-\iota^2))^a+(\lambda-1)\iota^{2a}}}, \sqrt{\frac{\lambda\nu^{2a}}{(1+(\lambda-1)(1-\nu^2))^a+(\lambda-1)\nu^{2a}}} >$,

(iv) $G^{\wedge_H a} =< \sqrt{\frac{\lambda\mu^{2a}}{(1+(\lambda-1)(1-\mu^2))^a+(\lambda-1)\mu^{2a}}}, \sqrt{\frac{\lambda\iota^{2a}}{(1+(\lambda-1)(1-\iota^2))^a+(\lambda-1)\iota^{2a}}}, \sqrt{\frac{(1+(\lambda-1)\nu^2)^a-(1-\nu^2)^a}{(1+(\lambda-1)\nu^2)^a+(\lambda-1)(1-\nu^2)^a}} >$.

Remark 2.8. [22] We note that the Hamacher operations are coincident with the algebraic operations and Einstein operations when $\lambda = 1$ and $\lambda = 2$, respectively.

Lemma 2.9. [22] Let $\Delta_1 =< \mu_1, \iota_1, \nu_1 >, \Delta_2 =< \mu_2, \iota_2, \nu_2 >$ be any two GSFNs and $a, a_1, a_2 \geq 0$. Then the following assertions are satisfied:

(i) $G_1 \oplus_H G_2 = G_2 \oplus_H G_1$,

(ii) $\mathrm{a} \times_{\mathrm{H}} (G_1 \oplus_{\mathrm{H}} G_2) = \mathrm{a} \times_{\mathrm{H}} G_1 \oplus_{\mathrm{H}} \mathrm{a} \times_{\mathrm{H}} G_2$,

(iii) $(a_1 + a_2) \times_H G_1 = a_1 \times_H G_1 \oplus_H a_2 \times_H G_2$,

(iv) $G_1 \odot_H G_2 = G_2 \odot_H G_1$,

(v) $(G_1 \odot_H G_2)^{\wedge_H a} = G_1^{\wedge_H a} \odot_H G_2^{\wedge_H a}$,

(vi) $G^{\wedge_H a_1} \odot_H G^{\wedge_H a_2} = G^{\wedge_H (a_1+a_2)}$,

(vii) $\left(G_1^{\wedge_H a_1}\right)^{\wedge_H a_2} = G_1^{\wedge_H a_1 a_2}$.

Definition 2.10. [20, 21, 22] Let $\mathcal{G}$ be a family of all GSFNs and $(G_1, G_2, \dots, G_n) \in \mathcal{G}^n$ where $G_i =< \mu_i, \iota_i, \nu_i >$ for all $i = 1,2,\dots,n$ and $w = (w_1, w_2, \dots, w_n)^T$ be the weight vector corresponding to $(G_i)_{i=1}^n$ such that $w_i \geq 0$ for all i and $\sum_{i=1}^n w_i = 1$. A mapping $GSEWA_w: \mathcal{G}^n \to \mathcal{G}$ is called a

(i) generalized spherical fuzzy weighted averaging (GSWA) operator and defined by

$$GSWA_w(G_1, G_2, \dots, G_n) = w_1 \times G_1 \oplus w_2 \times G_2 \oplus \dots w_n \times G_n = \oplus_{i=1}^n w_i \times G_i.$$

(ii) generalized spherical fuzzy weighted geometric (GSWG) operator and defined by

$$GSWG_w(G_1, G_2, \dots, G_n) = G_1^{w_1} \odot G_2^{w_2} \odot \dots \odot G_n^{w_n} = \odot_{i=1}^n G_i^{w_i}.$$

(iii) generalized spherical fuzzy Einstein weighted averaging (GSEWA) operator and defined by

$$GSEWA_w(G_1, G_2, \dots, G_n) = w_1 \times_E G_1 \oplus_E w_2 \times_E G_2 \oplus_E \dots \oplus_E w_n \times_E G_n = {\oplus_{\mathrm{E}}}_{i=1}^n w_i \times_E G_i.$$

(iv) generalized spherical fuzzy Einstein weighted geometric (GSEWG) operator and defined by

$$GSEWG_w(G_1, G_2, \dots, G_n) = G_1^{\wedge_E w_1} \odot_E G_2^{\wedge_E w_2} \odot_E \dots \odot_E G_n^{\wedge_E w_n} = \odot_{i=1}^n G_i^{\wedge_E w_i}.$$

(v) generalized spherical fuzzy Hamacher weighted averaging (GSHWA) operator and defined by

$$GSHWA_w(G_1, G_2, \dots, G_\mathrm{n}) = w_1 \times_H G_1 \oplus_H w_2 \times_H G_2 \oplus_H \dots w_n \times_H G_n = \oplus_{i=1}^n w_i \times_H G_i.$$

(vi) generalized spherical fuzzy Hamacher weighted geometric (GSHWG) operator and defined by

$$GSHWG_w(G_1, G_2, \dots, G_\mathrm{n}) = G_1^{\wedge_H w_1} \odot_H G_2^{\wedge_H w_2} \odot_H \dots \odot_H G_n^{\wedge_H w_n} = {\odot_{\mathrm{H}}}_{i=1}^n G_i^{\wedge_H w_i}.$$

Definition 2.11. [20] Let $\mathcal{G}$ be the family of all GSFNs and $G \in \mathcal{G}$ where $G =< \mu, \iota, \nu >$.

(i) A score function $SF: \mathcal{G} \to [-1,1]$ is defined as $SF(G) = \frac{3\mu^2 - 2\iota^2 - \nu^2}{3}$.

(ii) An accuracy function $AF: \mathcal{G} \to [0,1]$ is defined as $AF(G) = \frac{1 + 3\mu^2 - \nu^2}{4}$.

Definition 2.12. [20] Let $G_1 =< \mu_1, \iota_1, \nu_1 >$ and $G_2 =< \mu_2, \iota_2, \nu_2 >$ be any two GSFNs. Then the ranking method (comparison technique) as follows:

(i) $SF(G_1) < SF(G_2) \Rightarrow G_1 < G_2$,

(ii) $SF(G_1) > SF(G_2) \Rightarrow G_1 > G_2$,

(iii) If $SF(G_1) = SF(G_2)$, then; (a) $AF(G_1) < AF(G_2) \Rightarrow G_1 < G_2$,

(b) $AF(G_1) > AF(G_2) \Rightarrow G_1 > G_2$,

(c) $AF(G_1) = AF(G_2) \Rightarrow G_1 = G_2$.

3. A method based on TOPSIS with generalized spherical fuzzy information

In this section, we construct a method to solve the MAGDM problems based on TOPSIS under the generalized spherical fuzzy environment. First, we construct the linguistic table whose elements are GSFNs to evaluate alternatives in the decision process. Then, we design an approach consisting of some steps by considering the TOPSIS method when the weights of attributes and DMs are completely unknown. Also, we give a numerical example to clearly explain the steps of the new method.

3.1. Proposed method

Suppose that $A = \{A_1, A_2, \dots, A_m\}$ is the set of m different options and $E = \{E_1, E_2, \dots, E_n\}$ the set of n different attributes. Also assume that $D = \{D_1, D_2, \dots, D_k\}$ is the set of k distinct decision-makers (or experts) in a given real-life problem.

***Table 1**. Linguistic terms for alternatives and the weights of DMs/attributes.*

Lingusitic terms	**GSFNs ($\boldsymbol{\mu}, \boldsymbol{\iota}, \boldsymbol{\nu}$)**
Absolutely more importance (AMI)	$(0.9, 0, 0)$
Very high importance (VHI)	$(0.71, 0.1, 0.15)$
High importance (HI)	$(0.59, 0.36, 0.18)$
Slightly more importance (SMI)	$(0.5, 0.4, 0.4)$
Equally importance (EI)	$(0.375, 0.325, 0.425)$
Slightly low importance (SLI)	$(0.39, 0.39, 0.65\ )$
Low importance (LI)	$(0.33, 0.52, 0.72)$
Very low importance (VLI)	$(0.22, 0.7, 0.8)$
Absolutely low importance (ALI)	$(0.01, 0.85, 0.99)$

Step I: Construct the decision matrices according to the DMs opinions.
In this step, each DM establishes decision matrices in which it evaluates alternatives via attributes by taking linguistic terms from Table 1. Let us denote the decision matrices by $D^{(r)}$ where the elements are given as $D_{ij}^r =< \mu_{ij}^r, \iota_{ij}^r, \nu_{ij}^r > (i = 1,2,\dots,m)$,$(j = 1,2,\dots,n)$, $(r = 1,2,\dots,k)$ and the associated decision matrix is given as follows in Table 1:

***Table 1**. Decision Matrix $D^{(r)}$.*

Alternatives	Attributes			
	E_1	E_2	$\dots$	E_n
A_1	D_{11}^r	D_{12}^r	$\dots$	D_{1n}^r
A_2	D_{21}^r	D_{22}^r	$\dots$	D_{2n}^r
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
A_m	D_{m1}^r	D_{m2}^r	$\dots$	D_{mn}^r

Step II: Determine the weights of DMs D_r for all $r = 1,2,\dots,k$.
In this step, the weights of DMs D_r for all $r = 1,2,\dots,k$ can be given as a real number such that $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k)$ is the weight vector of the attribute D_r $(r = 1,2,\dots,k)$ where $\varepsilon_r \geq 0$ for all $r = 1,2,\dots,k$ and $\sum_{r=1}^k \varepsilon_r = 1$. Or the weights of DMs D_r for all $r = 1,2,\dots,k$ can be chosen as a linguistic term (denote $G_r =< \mu_r, \iota_r, \nu_r >$) given in Table 1 such as "absolutely more importance", "very high importance" and etc. Then, the real value of the weights of DMs (denote by $\varepsilon_r \geq 0$ for all $r = 1,2,\dots,k$) is calculated by using the following formula:

$$\varepsilon_r = \frac{|3\mu_r^2 - 2\iota_r^2 - \nu_r^2|}{\sum_{r=1}^k |3\mu_r^2 - 2\iota_r^2 - \nu_r^2|}.$$

Step III: Aggregate the decision matrices $D^{(r)}$ according to the weights of DMs ε_r for all $r = 1,2,\dots,k$. Here, the individual decision matrices $D^{(r)}$ are merged into the one decision matrix (denote $\widetilde{D} =< \mu_{ij}, \iota_{ij}, \nu_{ij} >$) by using operators GSWA, GSWG, GSEWA, GSEWG, GSHWA or GSHWG given in Definition 2.10. So, if we use to collect the matrices the GSWA operator, then we obtain $\widetilde{D}$ as follows:

$$\widetilde{D} =< \mu_{ij}, \iota_{ij}, \nu_{ij} >= GSWA_{\varepsilon}\left(D^{(1)}, D^{(2)}, \dots, D^{(r)}\right) = \bigoplus_{r=1}^{k} \varepsilon_r \times D^{(r)}.$$

Step IV: Determine the weights of attributes E_i (weight matrix) for all $i = 1,2,\dots,n$.
In this step, the weights of of attributes E_i for all $i = 1,2,\dots,n$ can be given as a real number such that $\omega = (\omega_1, \omega_2, \dots, \omega_k)$ is the weight vector of the attribute E_i $(i = 1,2,\dots,n)$ where $\omega_i \geq 0$ for all $i = 1,2,\dots,n$ and $\sum_{i=1}^{n} \omega_r = 1$. Or the weights of attributes E_i for all $i = 1,2,\dots,n$ can be chosen as a linguistic term (denote $\Omega_i =< \mu_i, \iota_i, \nu_i >$) given in Table 1 such as "absolutely more importance", "very high importance" and etc. Then, the real value of the weights of attributes (denote by $\omega_i \geq 0$ for all $i = 1,2,\dots,n$) is calculated by using the following formula:

$$\omega_i = \frac{|3\mu_i^2 - 2\iota_i^2 - \nu_i^2|}{\sum_{i=1}^{n} |3\mu_i^2 - 2\iota_i^2 - \nu_i^2|}.$$

Step V: Calculate the weighted decision matrix by using the weights of attributes.
Here, we obtain the weighted decision matrix (denote $D^* = \left(d_{ij}^*\right) =< \mu_{ij}^*, \iota_{ij}^*, \nu_{ij}^* >$)) by multiplying the decision matrix $\widetilde{D}$ with the weight matrix of attributes using the scalar multiplication given in Definition 2.5, Definition 2.6 or Definition 2.7. If we use classical scalar multiplication given in Definition 2.5, then we obtain D^* as follows:

$$D^* = \left(d_{ij}^*\right) =< \mu_{ij}^*, \iota_{ij}^*, \nu_{ij}^* >= \omega \times \widetilde{D}.$$

Step VI: Normalize the weighted aggregated decision matrix D^*.
If the attribute is benefit type, then this step is skipped. But, if there are some non-benefit type attributes, then the weighted decision matrix D^* is normalized and obtained the normalized weighted decision matrix $D' = \left(d'_{ij}\right) =< \mu'_{ij}, \iota'_{ij}, \nu'_{ij} >$ as follows:

$$D' = \left(d'_{ij}\right) =< \mu'_{ij}, \iota'_{ij}, \nu'_{ij} >= \begin{cases} < \mu_{ij}^*, \iota_{ij}^*, \nu_{ij}^* >, & if\ the\ attribute\ E_j\ is\ benefit\ type \\ < \nu_{ij}^*, \iota_{ij}^*, \mu_{ij}^* >, & if\ the\ attribute\ E_j\ is\ non-benefit\ type \end{cases}.$$

Step VII: Calculate the positive ideal solution (PIS) and negative ideal solution (NIS) by using score function and accuracy function. Then calculate the distance for each alternative to the PIS and NIS.
Here, denote PIS and NIS by S^+ and S^-, respectively. S^+ and S^- are computed as follows:

$$S^+ = \left(S_j^+\right) = \max\left\{SF\left(d'_{1j}\right), SF\left(d'_{2j}\right), \dots, SF\left(d'_{mj}\right) : j = 1,2 \dots, n\right\}$$
$$S^- = \left(S_j^-\right) = \min\left\{SF\left(d'_{1j}\right), SF\left(d'_{2j}\right), \dots, SF\left(d'_{mj}\right) : j = 1,2 \dots, n\right\}$$

where $SF\left(d'_{1j}\right)$ is the score value of d'_{1j} under the score function. We calculate the for each alternative to the PIS and NIS by using Euclidean distance as follows:

$$d(A_i, S^+) = \sqrt{\sum_{j=1}^{n} \left(SF\left(d'_{ij}\right) - S_j^+\right)^2}, d(A_i, S^-) = \sqrt{\sum_{j=1}^{n} \left(SF\left(d'_{ij}\right) - S_j^-\right)^2}$$

Step VIII: Calculate the relative closeness index $R(A_i)$ for all $i = 1,2,..,m$ and rank the alternatives.
We compute the relative closeness index $R(A_i)$ as follows:

$$R(A_i) = \frac{d(A_i, S^-)}{\max_i d(A_i, S^-)} - \frac{d(A_i, S^+)}{\min_i \ d(A_i, S^+)}$$

for all $i = 1,2,..,m$. Then, the alternatives are ranked according to the relative closeness index $R(A_i)$ on the descending order and the alternative with the biggest value is the best solution.

3.2. An illustrative example

Now, we consider the decision-making problem given in [23] to explain the proposed method step by step. There is a three-shareholder company in which the rates of share are effective at the decisions to be made by shareholders and the sharing of the earnings. Let the shareholders be denoted by D_1, D_2 and D_3. The shareholder D_1 has 35% share rate, the shareholder D_2 has 45% share rate and the shareholder D_3 has 20% share rates. This company is planning to make an investment in an area where the alternatives are A_1: Development of small business, A_2: Information Technology, A_3: Tourism, A_4: Transportation. They are taking into consideration the degree of risk, volume of income and investment recovery period when making an investment in these areas. Let the degree of risk, volume of income and investment recovery period be denoted by E_1, E_2 and E_3, respectively. A prioritization relationship among the attribute E_i $(i = 1,2,3)$ which satisfies $E_2 > E_1 > E_3$ was determined according to the shareholder's preferences. So, assume that $\omega = (0.3, 0.45, 0.25)$ is the weight vector of the attribute $\{E_1, E_2, E_3\}$. In this problem, whereas the attributes E_1 and E_3 are non-benefit types, E_2 is benefit type. To choose the optimum investment, the shareholder's D_1, D_2 and D_3 with the decision-makers weight vector is given $\varepsilon = (0.35, 0.45, 0.2)$.

Step I: The decision matrices are given in Table 3, Table 4 and Table 5 as follows:

Table 3. *Decision Matrix* $D^{(1)}$.

	E_1	E_2	E_3
A_1	<0.6,0.8,0,2>	<0.4, 0.3, 0.7>	<0.2, 0.7, 0.4>
A_2	<0.55,0.2,0.8>	<0.8,0.75,0.65>	<0.9, 0.8, 0.2>
A_3	<0.7, 0.4, 0.4>	<0.55,0.2,0.45>	<0.5, 0.7, 0.8>
A_4	<0.35,0.6,0.5>	<0,7, 0.8,0.55>	<0.8, 0.6, 0.5>

Table 4. *Decision Matrix* $D^{(2)}$.

	E_1	E_2	E_3
A_1	<0.85, 0.7,0.8>	<0.4, 0.75, 0.8>	<0.6, 0.8, 0.5>
A_2	<0.3, 0.4, 0.4>	<0.8, 0.2, 0.45>	<0.5, 0.6, 0.8>
A_3	<0.9, 0, 8, 0,2>	<0.4, 0.8, 0.7>	<0.8, 0.7, 0.4>
A_4	<0.75,0.3, 0.5>	<0.8, 0.5, 0.45>	<0.5, 0.6, 0.8>

Table 5. *Decision Matrix* $D^{(3)}$.

	E_1	E_2	E_3
A_1	<0.75, 0.4, 0.5>	<0.8, 0.8, 0.45>	<0.8, 0.6, 0.8>
A_2	<0.9, 0, 6, 0, 4>	<0.4, 0.6, 0.9>	<0.2, 0.7, 0.4>
A_3	<0.55, 0.5, 0.8>	<0.8,0.75,0.85>	<0.6, 0.8, 0.2>
A_4	<0.75, 0.4, 0.8>	<0.4, 0.8, 0.45>	<0.8, 0.6, 0.6>

Step II: Since the decision-makers weight vector is given as $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3) = (0.35, 0.45, 0.2)$, this step is skipped.

Step III: We obtain the aggregated decision matrices $\widetilde{D}$ according to the weights of DMs ε_r for all $r = 1,2,3$ by using GSWA operator as follows:

***Table 6**. Aggregated Decision Matrix $\widetilde{D}$.*

	$F_{4\times1}^1$	$F_{4\times1}^2$	$F_{4\times1}^3$
	E_1	E_2	E_3
A_1	<0.7699,0.7091,0.7874>	<0.5394,0.8402,0.7710>	<0.5853,0.8166,0.7001>
A_2	<0.6276,0.5978,0.5512>	<0.7573,0.4376,0.6836>	<0.7160,0.7399,0.7530>
A_3	<0.8074,0.7874,0.4635>	<0.5793,0.8539,0.8245>	<0.6912,0.8145,0.4799>
A_4	<0.6647,0.4843,0.7001>	<0.7199,0.7001,0.5951>	<0.7065,0.7175,0.8166>

Step IV: Since the weight of attributes is given as $\omega = (\omega_1, \omega_2, \omega_3) = (0.3, 0.45, 0.25)$, this step is skipped.

Step V: We calculate the weighted aggregated decision matrix by using the weights of attributes and scalar multiplication given in Definition 5.

***Table 7**. Weighted Aggregated Decision Matrix D^*.*

	$F_{4\times1}^1$	$F_{4\times1}^2$	$F_{4\times1}^3$
	E_1	E_2	E_3
A_1	<0.4485,0.9176,0.940>	<0.3786,0.9247,0.8895>	<0.3439,0.9410,0.8986>
A_2	<0.3430,0.8793,0.8617>	<0.5644,0.6894,0.8427>	<0.4404,0.9136,0.9184>
A_3	<0.4815,0.9420,0.8251>	<0.4100,0.9314,0.9168>	<0.4208,0.9403,0.8023>
A_4	<0.3683,0.8342,0.9147>	<0.5292,0.8518,0.7917>	<0.4328,0.9052,0.9410>

Step VI: We obtain the normalized weighted aggregated decision matrix D^* as follows:

***Table 8**. Normalized Weighted Aggregated Decision Matrix D'.*

	$F_{4\times1}^1$	$F_{4\times1}^2$	$F_{4\times1}^3$
	E_1	E_2	E_3
A_1	<0.3439,0.9410,0.8986>	<0.3786,0.9247,0.8895>	<0.4485,0.9176,0.940>
A_2	<0.4404,0.9136,0.9184>	<0.5644,0.6894,0.8427>	<0.3430,0.8793,0.8617>
A_3	<0.4208,0.9403,0.8023>	<0.4100,0.9314,0.9168>	<0.4815,0.9420,0.8251>
A_4	<0.4328,0.9052,0.9410>	<0.5292,0.8518,0.7917>	<0.3683,0.8342,0.9147>

Step VII, VIII: We calculate the positive ideal solution (PIS) and negative ideal solution (NIS) by using score function and accuracy function as follows:

***Table 9**. PIS and NIS Values.*

S^+	0,5382	-0,0766	0,5684
S^-	0,2512	-0,4054	0,3419

Also, we obtain the distance for each alternative to the PIS and NIS, relative closeness index and ranking as follows:

Table 10. *Distance, Relative Closeness and Ranking.*

Alternatives	$d(A_i, S^+)$	$d(A_i, S^-)$	$R(A_i)$	Ranking
A_1	0,3361	0,3099	-0,0074	3
A_2	0,1366	0,4083	0,6942	1
A_3	0,4889	0,0041	-0,9905	4
A_4	0,0941	0,4344	0,0000	2

Therefore, we conclude that the alternative A_4 is the best solution of this problem.

4. Comparative analyses

In this section, we solve the same problem by using the GSWG, GSEWA, GSEWG, GSHWA and GSWG operators in the related steps of the proposed method. We take the value $\lambda = 0.5$ when calculating the steps with GSHWA and GSWG operators. As seen in Table 11, we obtain the same ranking as a result of the solutions.

Table 11. *Ranking by solving this problem with the GSWG, GSEWA, GSEWG, GSHWA and GSWG operators.*

Operators	A_1	A_2	A_3	A_4
GSWG	3	1	4	2
GSEWA	3	1	4	2
GSEWG	3	1	4	2
GSHWA	3	1	4	2
GSHWG	3	1	4	2

Also, when we compare the solution by using the method given in [21], we get the sama result. So, this shows the validity and reliability of the proposed approach.

5. Conclusion

In this study, we establish a novel approach to solving the decision-making problems based on the TOPSIS method by using algebraic, Einstein and Hamacher operations under the generalized spherical fuzzy environment. For future work, we plan to study some traditional methods such as VIKOR, ELECTRE, WASPS, AHP and etc. to use in the decision-making process with the generalized spherical fuzzy data.

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Applications of the Closing Feature to the Limits of Textures

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Abstract

Some applications of the closure operator on inverse limits of the inverse systems in the category, whose objects are ditopological plain texture spaces and morphisms are bicontinuous point functions satisfying a compatibility condition, are discussed in this study. Specifically, we will be interested in the closure operator taken according to the corresponding joint topologies of the ditopologies on the inverse limit spaces.

Keywords : Joint topology, inverse limit, ditopology, plain texture, directed set, category

1. Introduction and Preliminaries

The notion of *texture* was introduced by Lawrence M. Brown as a point-based setting for the study of complement-free mathematical concepts, besides crisp sets, fuzzy sets, L-valued sets and intuitionistic sets. Accordingly, if S is a set then by a *texturing of* S we mean a subset $\mathcal{S} \subseteq \mathcal{P}(S)$ which is a point-separating, complete, completely distributive lattice containing S and $\emptyset$, and for which meet coincides with intersection and finite join with union. The pair $(S, \mathcal{S})$ is then called a *texture*. In particular, if the arbitrary join coincides with union then $(S, \mathcal{S})$ is called *plain texture*.

Since a texturing $\mathcal{S}$ need not be closed under the operation of taking the set-complement, the notion of topology is replaced by that of *dichotomous topology* or *ditopology*, namely a pair (τ, κ) of subsets of $\mathcal{S}$, where the set of *open sets* τ satisfies the conditions $S, \emptyset \in \tau$, $G_1, G_2 \in \tau \implies G_1 \cap G_2 \in \tau$, $G_i \in \tau,\ i \in I \implies \bigvee_i G_i \in \tau$, and the set of *closed sets* κ satisfies the dual conditions. A ditopological texture space with respect to a ditopology (τ, κ) on the texture $(S, \mathcal{S})$ is denoted by $(S, \mathcal{S}, \tau, \kappa)$.

An adequate introduction to the theory of ditopological texture spaces may be obtained from [2,5].

On the other hand, classical theory of inverse systems-limits are important in the extension of homology and cohomology theory. An exhaustive discussion of inverse systems which are in the classical categories of sets, topological spaces, groups and rings, was presented by [3].

The theory of inverse systems - limits in the context of ditopological texture spaces is handled in [6] first-time. Accordingly, it is seen that a method used to construct a new ditopological space is the theory of *ditopological inverse systems* and their limit spaces under the name *ditopological inverse limits* as the subspaces of ditopological product spaces described in [2].

A detailed analysis of the theory of ditopological inverse systems and inverse limits insofar as the category **ifPDitop** whose objects are the ditopological texture spaces which have plain texturing and morphisms are the bicontinuous, special (called w-preserving) point functions, is concerned in [6].

In this study, some applications of the closure operator on inverse limits in **ifPDitop** are discussed. Specifically, as in [4] we will deal with the closure operator taken according to the corresponding joint topologies for the ditopologies on the inverse limit spaces.

Incidentally, the reader is referred to [1] for terms from category theory not defined here.

2. Closing Operator on inverse systems and limits in the category ifPDitop

First of all, we will need some notations in order to present a few helpful lemmas, as follows.

Notation: For the inverse system $\{(S_\alpha, \mathcal{S}_\alpha, \tau_\alpha, \kappa_\alpha), \varphi_{\alpha\beta}\}_{\alpha\geq\beta}$, constructed in **ifPDitop**, over the directed set Λ, the notations $(\tau_\infty, \kappa_\infty)$ and $(S_\infty, \mathcal{S}_\infty, \tau_\infty, \kappa_\infty)$ will be used as *inverse limit ditopology* and *(ditopological) inverse limit space* as in [6, Theorem 4.6], respectively. Here, $S_\infty = \underleftarrow{\lim}\{S_\alpha\}$.

Therefore, with the above notations we have:

Lemma 2.1. $U_\infty = \underleftarrow{\lim}\ \{U_\alpha\} \subseteq \underleftarrow{\lim}\ \{S_\alpha\} = S_\infty$

Proof. By the assumption $U_\infty = \underleftarrow{\lim}\ \{U_\alpha\} \not\subseteq \underleftarrow{\lim}\ \{S_\alpha\} = S_\infty$, there exists $s = \{s_\alpha\} \in \prod\limits_{\alpha\in\Lambda} S_\alpha$ such that $U_\infty \not\subseteq Q_s$ and $P_s \not\subseteq S_\infty$. In this case, $s \in \prod\limits_{\alpha\in\Lambda} U_\alpha$ and $\varphi_{\alpha\beta}|_{U_\alpha}(s_\alpha) = s_\beta$ for every $s_\alpha \in U_\alpha$, $\alpha, \beta \in \Lambda$ such that $\alpha \geq \beta$. Moreover, we have the equality $\varphi_{\alpha\beta}|_{U_\alpha}(s_\alpha) = \varphi_{\alpha\beta}(s_\alpha)$ for $s_\alpha \in U_\alpha$. Thus, because of the facts $s_\alpha \in S_\alpha$, $\alpha \in \Lambda$ and $\varphi_{\alpha\beta}(s_\alpha) = s_\beta$ for $\alpha \geq \beta$, the point $s = \{s_\alpha\}$ becomes an element of S_∞, obviously and this gives a contradiction. □

Now we may associate with the ditopology (τ, κ) on a plain texture $(S, \mathcal{S})$ a topology $\mathfrak{J}_{\tau\kappa}$ on S, called *appropriate joint topology for a ditopology* as given in [6]:

Definition 2.2. Let $(S, \mathcal{S}, \tau, \kappa) \in \mathrm{Ob}\,$**ifPDitop**. We define the joint topology on S in terms of its family $\mathfrak{J}^c_{\tau\kappa}$ of closed sets by the condition

$$W \in \mathfrak{J}^c_{\tau\kappa} \iff (s \in S,\ G \in \eta(s), K \in \mu(s) \implies G \cap W \not\subseteq K) \implies s \in W.$$

Here $\eta(s) = \{N \in \mathcal{S} \mid \quad P_s \subseteq G \subseteq N \not\subseteq Q_s \quad \text{for some } G \in \tau\}$ and $\mu(s) = \{M \in \mathcal{S} \mid \ P_s \not\subseteq M \subseteq K \subseteq Q_s \text{ for some } K \in \kappa\}$. For the details see [5,6].

Clearly $\mathcal{J}^c_{\tau\kappa}$ satisfies the closed-set axioms and on passing to the complement this verifies that

(i) $\{G \subseteq S \mid G \in \tau\} \cup \{S \setminus K \subseteq S \mid K \in \kappa\}$ is a subbase, and
(ii) $\{G \cap (S \setminus K) \subseteq S \mid G \in \tau, K \in \kappa\}$ a base

of open sets for the topology $\mathcal{J}_{\tau\kappa}$ on S.

Because of the appropriate joint topology described for a ditopology, the next lemma arises.

Lemma 2.3. *Take the inverse system* $\{(S_\alpha, \mathcal{S}_\alpha, \tau_\alpha, \kappa_\alpha), \varphi_{\alpha\beta}\}_{\alpha\geq\beta} \in \mathrm{Ob}\,\mathbf{Inv}_{\mathbf{ifPDitop}}$, *over a directed set* Λ *and* $(S_\infty, \mathcal{S}_\infty, \tau_\infty, \kappa_\infty) \in \mathrm{Ob}\,\mathbf{ifPDitop}$ *as the inverse limit of that system.*
If $U_\alpha \in \mathcal{J}^c_{\tau_\alpha\kappa_\alpha}$, $\alpha \in \Lambda$ *and* $\underleftarrow{\lim}\,\{U_\alpha\} = U_\infty$ *for* $\{(U_\alpha, \mathcal{S}_\alpha|_{U_\alpha}, \tau_\alpha|_{U_\alpha}, \kappa_\alpha|_{U_\alpha}), \varphi_{\alpha\beta}|_{U_\alpha}\}_{\alpha\geq\beta} \in \mathrm{Ob}\,\mathbf{Inv}_{\mathbf{ifPDitop}}$, *then* $U_\infty \in \mathcal{J}^c_{\tau_\infty\kappa_\infty}$.

Proof. Because of the equality $\varphi_{\alpha\beta}|_{U_\alpha}(s_\alpha) = \varphi_{\alpha\beta}(s_\alpha)$ for $s_\alpha \in U_\alpha$, $\alpha \geq \beta$, the inclusion $U_\infty = \underleftarrow{\lim}\,\{U_\alpha\} \subseteq \underleftarrow{\lim}\,\{S_\alpha\} = S_\infty$ is immediate, as mentioned in Lemma 2.1 as well.

We can prove now that $U_\infty \in \mathcal{J}^c_{\tau_\infty\kappa_\infty}$: If $P_a \not\subseteq U_\infty$, that is $a \notin U_\infty$ for $a = \{a_\alpha\} \in S_\infty$, then $a \notin \prod_{\alpha\in\Lambda} U_\alpha$ due to the equality $\varphi_{\alpha\beta}|_{U_\alpha}(s_\alpha) = \varphi_{\alpha\beta}(s_\alpha)$ for $s_\alpha \in U_\alpha$, $\alpha \geq \beta$. In this case, there exists $\alpha_0 \in \Lambda$ such that $a_{\alpha_0} \notin U_{\alpha_0}$, that is $P_{a_{\alpha_0}} \not\subseteq U_{\alpha_0}$. Additionally, the subset $\mu^{-1}_{\alpha_0}[U_{\alpha_0}] \subseteq S_\infty$ is an element of $\mathcal{J}^c_{\tau_\infty\kappa_\infty}$ since the limiting projection map $\mu_{\alpha_0} : S_\infty \to S_{\alpha_0}$ is continuous between the corresponding joint topological spaces and $U_{\alpha_0} \in \mathcal{J}^c_{\tau_{\alpha_0}\kappa_{\alpha_0}}$.

Also the statements $P_a \not\subseteq \mu^{-1}_{\alpha_0}[U_{\alpha_0}]$ and $U_\infty \subseteq \mu^{-1}_{\alpha_0}[U_{\alpha_0}]$ are trivial.

Now assume that $P_a \subseteq \mu^{-1}_{\alpha_0}[U_{\alpha_0}]$. In this case we have $\mu_{\alpha_0}(a) = a_{\alpha_0} \in U_{\alpha_0}$ which is a contradiction.

In addition, if $U_\infty \not\subseteq \mu^{-1}_{\alpha_0}[U_{\alpha_0}]$ then there exists a point $z \in S_\infty$ such that $U_\infty \not\subseteq Q_z$ and $P_z \not\subseteq \mu^{-1}_{\alpha_0}[U_{\alpha_0}]$. Hence, $\mu_{\alpha_0}(z) = z_{\alpha_0} \notin U_{\alpha_0}$ and so $z = \{z_\alpha\} \notin \prod_{\alpha\in\Lambda} U_\alpha$ gives the fact that $z \notin U_\infty$ a contradiction. □

By virtue of the last lemma, we have the following theorem:

Theorem 2.4. *If* $\overline{U}$ *denotes the closure of the subset* $U \subseteq S_\infty$ *with respect to the joint topology of the limit ditopology* $(\tau_\infty, \kappa_\infty)$ *then*

(1) $\underleftarrow{\lim}\{\overline{U}_\alpha\}$ *is jointly closed subspace of* S_∞
(2) $\underleftarrow{\lim}\{\overline{U}_\alpha\} = \overline{U} \subseteq S$

Proof. (1) If the inclusion $\underleftarrow{\lim}\{\overline{U}_\alpha\} \subseteq S_\infty$ ($S_\infty = \underleftarrow{\lim}\{S_\alpha\}$) is not true then there exists a point $s = \{s_\alpha\} \in \prod_{\alpha\in\Lambda} S_\alpha$ such that $\underleftarrow{\lim}\{\overline{U}_\alpha\} \not\subseteq Q_s$ and $P_s \not\subseteq S_\infty$. Hence, $\overline{U}_\sigma \not\subseteq Q_{s_\sigma}$ and $s_\sigma \in \overline{U}_\sigma$ for every $\sigma \in \Lambda$, so we have $\overline{\varphi}_{\alpha\beta}(s_\alpha) = s_\beta$ for $\alpha \geq \beta$.

On the other hand, it is easy to see that $P_{s_\sigma} \subseteq S_\sigma$ since the set $\overline{U}_\sigma$ is a subset of S_σ for every $\sigma \in \Lambda$, and so $P_s = \prod_{\sigma\in\Lambda} P_{s_\sigma} \subseteq \prod_{\sigma\in\Lambda} S_\sigma$. Also, if recall the equality $\overline{\varphi}_{\alpha\beta}(s_\alpha) = \varphi_{\alpha\beta}(s_\alpha)$ for $s_\alpha \in \overline{U}_\alpha$ and $\alpha \geq \beta$, then we have $\varphi_{\alpha\beta}(s_\alpha) = s_\beta$ due to the fact that $\overline{\varphi}_{\alpha\beta}(s_\alpha) = s_\beta$ for $s_\alpha \in \overline{U}_\alpha$ and $\alpha \geq \beta$. Thus, by the definition of inverse limit, $s = \{s_\alpha\} \in S_\infty$ and it is a contradiction.

In addition, it is easy to verify that $\varprojlim\{\overline{U}_\alpha\}$ is a jointly closed subspace of S_∞ with the help of limiting projection map.
Also, the fact that $\overline{U} = \varprojlim\{\overline{U}_\alpha\}$ is clear.

(2) Firstly, let us show that $U \subseteq \varprojlim\{\overline{U}_\alpha\}$. Conversely, if $U \not\subseteq \varprojlim\{\overline{U}_\alpha\}$, then there exists $b \in S_\infty = \varprojlim\{S_\alpha\}$ such that $U \not\subseteq Q_b$ and $P_b \not\subseteq \varprojlim\{\overline{U}_\alpha\}$. Thus $P_{b_\alpha} \subseteq \overline{U}_\alpha$ as $\mu_\alpha(b) \in \mu_\alpha(U)$. Hence $P_b = \prod_{\alpha\in\Lambda} P_{b_\alpha} \subseteq \prod_\alpha \overline{U}_{\alpha\in\Lambda}$.

On the other hand, note that $b \in \prod_\alpha S_{\alpha\in\Lambda}$ and $\varphi_{\alpha\beta}(b_\alpha) = b_\beta$ for $\alpha \geq \beta$, $\alpha, \beta \in \Lambda$. Also, by the definition of $\overline{\varphi}_{\alpha\beta}$ for $\alpha \geq \beta$ and $b_\alpha \in U_\alpha$ for every $\alpha \in \Lambda$, the equality $\overline{\varphi}_{\alpha\beta}(b_\alpha) = \varphi_{\alpha\beta}(b_\alpha)$ is satisfied. Hence, $\overline{\varphi}_{\alpha\beta}(b_\alpha) = b_\beta$ for $\alpha \geq \beta$. That is, we obtained that $b \in \varprojlim\{\overline{U}_\alpha\}$ which is a contradiction.

Therefore, from (1) if recall the fact that the space $\varprojlim\{\overline{U}_\alpha\}$ is jointly closed with respect to the limit ditopology $(\tau_\infty, \kappa_\infty)$ on $(S_\infty, \mathcal{S}_\infty)$, then the inclusion $\overline{U} \subseteq \varprojlim\{\overline{U}_\alpha\}$ is immediate.

For the other direction, assume $\varprojlim\{\overline{U}_\alpha\} \not\subseteq \overline{U}$. Thus, there exists a point $a = \{a_\alpha\} \in S_\infty$ such that $\varprojlim\{\overline{U}_\alpha\} \not\subseteq Q_a$ and $P_a \not\subseteq \overline{U}$. By the definition of joint topology, there exist $M \in \mu(a)$ and $N \in \eta(a)$ such that $\overline{U} \subseteq N \cap (S_\infty \setminus M)$ and so we have the sets $G \in \tau_\infty$ and $K \in \kappa_\infty$ such that $P_a \subseteq G \subseteq M$, $N \subseteq K \subseteq Q_a$ and $\overline{U} \subseteq K \cap (S_\infty \setminus G)$. Hence, there exist $\alpha_0, \alpha_1 \in \Lambda$ and $A_{\alpha_0} \in \tau_{\alpha_0}$, $B_{\alpha_1} \in \kappa_{\alpha_1}$ such that the conditions $P_a \subseteq \mu_{\alpha_0}^{-1}[A_{\alpha_0}] \subseteq G$ and $K \subseteq \mu_{\alpha_1}^{-1}[B_{\alpha_1}] \subseteq Q_a$ are satisfied. In this case, the inclusion $\overline{U} \subseteq (S_\infty \setminus \mu_{\alpha_0}^{-1}[A_{\alpha_0}]) \cap \mu_{\alpha_1}^{-1}[B_{\alpha_1}]$ is trivial. Finally, we obtained $\alpha_1 \in \Lambda$ satisfying the conditions $U \subseteq \overline{U} \subseteq \mu_{\alpha_1}^{-1}[B_{\alpha_1}]$ and $P_a \not\subseteq \mu_{\alpha_1}^{-1}[B_{\alpha_1}]$. Thus $U_{\alpha_1} \subseteq B_{\alpha_1}$ for $\alpha_1 \in \Lambda$, because of the inclusions $\mu_{\alpha_1}(U) \subseteq \mu_{\alpha_1}(\mu_{\alpha_1}{}^{-1}[B_{\alpha_1}]) \subseteq B_{\alpha_1}$. If we consider the closure operator on these sets, it is clear that $\overline{U}_{\alpha_1} \subseteq B_{\alpha_1}$ and so $\mu_{\alpha_1}(P_a) \not\subseteq \overline{U}_{\alpha_1}$ by the fact that $\mu_{\alpha_1}(a) \notin B_{\alpha_1}$. Moreover, it is easy to verify that $\mu_{\alpha_1}(P_a) = P_{a_{\alpha_1}}$:

$\mu_{\alpha_1}(P_a) = \{\mu_{\alpha_1}(x) \mid x \in P_a\} = \{x_{\alpha_1} \mid x \in P_a\} = \{x_{\alpha_1} \mid x \in \prod_{\alpha\in\Lambda} P_{a_\alpha}\} = \{x_{\alpha_1} \mid x_\alpha \in P_{a_\alpha}, \forall\alpha\} = P_{a_{\alpha_1}}$. As a result of these facts, we have $P_{a_{\alpha_1}} \not\subseteq \overline{U}_{\alpha_1}$ and so $a_{\alpha_1} \notin \overline{U}_{\alpha_1}$ for $\alpha_1 \in \Lambda$. This argument gives the fact that $a \notin \prod_\alpha \overline{U}_{\alpha\in\Lambda}$, clearly. It means that $a \notin \varprojlim\{\overline{U}_\alpha\}$ and it is a contradiction. □

According to all the considerations presented above, the next result will be obvious.

Corollary 2.5. i) $U \subseteq \varprojlim\{U_\alpha\}$

ii) $\varprojlim\{U_\alpha\} \subseteq \varprojlim\{\overline{U}_\alpha\}$

3. Conclusion

As a further aspect of the inverse systems and inverse limits constructed in the category **ifPDitop** consisting of ditopological plain texture spaces, in this study we investigated the effect of closure operator on inverse systems and limits in **ifPDitop**, with respect to the joint topologies correspond to the ditopologies located on those inverse limits. Particularly, we confined our attention to inverse systems - limits in the context of the category **ifPDitop**. It is clear that there are considerable difficulties involved in establishing a suitable theory of inverse systems and their limits for the general categories of ditopological spaces.

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Asymptotical Deferred statistical and Cesàro Equivalence of Order β for Double Sequences of Sets

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Abstract

In this study, it was intend to present the notions of asymptotical deferred statistical equivalence of order β and asymptotical deferred Cesàro equivalence of order β $(0 < \beta \leq 1)$ in the Wijsman sense for double set sequences, to investigate some properties of these notions and to examine the relationship between them.

Keywords: Asymptotical equivalence, deferred statistical convergence, deferred Cesàro summability, order β, Wijsman convergence, double sequences of sets.

1. Introduction

In Agnew [1] first introduced the notion of deferred Cesàro mean for real (or complex) sequences. Long after this, the notion of deferred statistical convergence is presented by Küçükaslan and Yılmaztürk [2] and the authors showed the relationship of this notion with the strongly deferred Cesàro summability. Using order α, similar notions were also studied by Et et al. [3] in metric spaces. Also, for double sequences, the notions of deferred statistical convergence and deferred Cesàro summability were introduced and studied by Dağadur and Sezgek [4]. Furthermore, using the concept of asymptotical equivalence, the notions of asymptotical deferred statistical and asymptotical deferred Cesàro equivalence were studied by Koşar et al. [5] for non-negative real sequences.

For sequences of sets, on the notions of strongly deferred Cesàro summability and deferred statistical convergence in the Wijsman sense were studied by Altınok et al. [6]. Using order α, similar notions were also studied by Yılmazer et al. [7]. Also, for double sequences of sets, the notions of deferred Cesàro summability and deferred statistical convergence were introduced and studied by Ulusu and Gülle [8]. Furthermore, using the concept of asymptotical equivalence, the notions of asymptotical deferred equivalence were studied by Altınok et al. [9] for sequences of sets. The similar concepts were also carried out by Et et al. [10] using order α.

The aim of this work is to introduce some asymptotical deferred equivalence types of order β in the Wijsman sense for double set sequences and to study on these notions.

More information on the notions in this study can be found in [11-32].

2. Basic Notions

Let's start by recalling some fundamental definitions and notations firstly (See, [8, 33-38]).

For a metric space $(\mathcal{X},\mu)$, $d(x,E)$ indicates the distance from x to E where

$$d(x,E) = \inf_{e\in E}\mu(x,e) := d_x(E)$$

for any $x \in \mathcal{X}$ and any non-empty $E \subseteq \mathcal{X}$.

For a non-empty set $\mathcal{X}$, let a function $h: \mathbb{N} \to P(\mathcal{X})$ is defined by $h(j) = E_j \in P(\mathcal{X})$ for each $j \in \mathbb{N}$. Then, the sequence $\{E_j\} = \{E_1, E_2, \dots\}$, which is the codomain elements of h, is called sequences of sets.

Throughout the study, $(\mathcal{X},\mu)$ will be considered as a metric space and E, E_{ij}, F_{ij} $(i,j \in \mathbb{N})$ will be considered as any non-empty closed subsets of $\mathcal{X}$.

The double sequence $\{E_{ij}\}$ is said to be Wijsman convergent to the set E if

$$\lim_{i,j\to\infty} d_x(E_{ij}) = d_x(E),$$

for each $x \in \mathcal{X}$ and it is denoted by $E_{ij} \stackrel{W_2}{\longrightarrow} E$.

The double sequence $\{E_{ij}\}$ is said to be Wijsman Cesàro summable of order β $(0 < \beta \leq 1)$ to the set E if

$$\lim_{m,n\to\infty} \frac{1}{(mn)^\beta} \sum_{i=1}^{m} \sum_{j=1}^{n} d_x(E_{ij}) = d_x(E),$$

for each $x \in \mathcal{X}$ and it is denoted by $E_{ij} \stackrel{W_2(C)^\beta}{\longrightarrow} E$.

The double sequence $\{E_{ij}\}$ is said to be Wijsman strong Cesàro summable of order β $(0 < \beta \leq 1)$ to the set E if

$$\lim_{m,n\to\infty} \frac{1}{(mn)^\beta} \sum_{i=1}^{m} \sum_{j=1}^{n} \left|d_x(E_{ij}) - d_x(E)\right| = 0,$$

for each $x \in \mathcal{X}$ and it is denoted by $E_{ij} \stackrel{W_2[C]^\beta}{\longrightarrow} E$.

The double sequence $\{E_{ij}\}$ is said to be Wijsman statistically convergent of order β $(0 < \beta \leq 1)$ to the set E if for every $\varepsilon > 0$

$$\lim_{m,n\to\infty} \frac{1}{(mn)^\beta} \left|\left\{(i,j): i \leq m, j \leq n: |d_x(E_{ij}) - d_x(E)| \geq \varepsilon\right\}\right| = 0,$$

for each $x \in \mathcal{X}$ and and it is denoted by $E_{ij} \overset{W_2(S)^\beta}{\longrightarrow} E$.

The double sequence $\{E_{ij}\}$ is said to be bounded if $\sup_{i,j}\{d_x(E_{ij})\} < \infty$ for each $x \in \mathcal{X}$. Also, L_∞^2 denotes the class of all bounded double set sequences.

The deferred Cesàro mean $D_{\psi,\phi}$ of a double sequence $\mathcal{E} = \{E_{ij}\}$ is defined by

$$(D_{\psi,\phi}\mathcal{E})_{uv} = \frac{1}{\psi_u \phi_v} \sum_{i=p_u+1}^{r_u} \sum_{j=q_v+1}^{s_v} d_x(E_{ij}),$$

where $[p_u]$, $[r_u]$, $[q_v]$ and $[s_v]$ are sequences of non-negative integers satisfying following conditions:

$$p_u < r_u, \ \lim_{u\to\infty} r_u = \infty; \quad q_v < s_v, \ \lim_{v\to\infty} s_v = \infty \tag{2.1}$$

$$r_u - p_u = \psi_u; \quad s_v - q_v = \phi_v. \tag{2.2}$$

Throughout the paper, unless otherwise specified, $[p_u]$, $[r_u]$, $[q_v]$ and $[s_v]$ are considered as sequences of non-negative integers satisfying (2.1) and (2.2).

For any non-empty closed subsets $\{E_{ij}\}, \{F_{ij}\} \in \mathcal{X}$ such that $d_x(E_{ij}) > 0$ and $d_x(F_{ij}) > 0$ for each $x \in \mathcal{X}$, the double sequences $\{E_{ij}\}$ and $\{F_{ij}\}$ are said to be asymptotically Wijsman equivalent to multiple η if

$$\lim_{i,j\to\infty} \frac{d_x(E_{ij})}{d_x(F_{ij})} := \lim_{i,j\to\infty} d_x\left(\frac{E_{ij}}{F_{ij}}\right) = \eta,$$

for each $x \in \mathcal{X}$ and it is denoted by $E_{ij} \overset{W_2^\eta}{\sim} F_{ij}$.

As an example to this concept, the following sequences of circles in $\mathbb{R}^2$ can be given.

Let $\mathcal{X} = \mathbb{R}^2$ and double sequences $\{E_{ij}\}$ and $\{F_{ij}\}$ be defined as following:

$$E_{ij} := \{(a,b) \in \mathbb{R}^2 : a^2 + b^2 + 4ija = 0\},$$

$$F_{ij} := \{(a,b) \in \mathbb{R}^2 : a^2 + b^2 - 4ija = 0\}.$$

Then, we have

$$\lim_{i,j\to\infty} d_x\left(\frac{E_{ij}}{F_{ij}}\right) = 1$$

for each $x \in \mathcal{X}$, i.e. $E_{ij} \overset{W_2^1}{\sim} F_{ij}$ $(\eta = 1)$.

3. New Concepts

In this section, we have presented the notions of asymptotical deferred statistical equivalence of order β and asymptotical deferred Cesàro equivalence of order β $(0 < \beta \leq 1)$ in the Wijsman sense for double set sequences.

In the following definitions, we will consider that $d_x(E_{ij}) > 0$ and $d_x(F_{ij}) > 0$, for each $x \in \mathcal{X}$ and any non-empty closed subsets $\{E_{ij}\}, \{F_{ij}\} \in \mathcal{X}$.

Definition 3.1 The double sequence $\{E_{ij}\}$ and $\{F_{ij}\}$ are said to be asymptotically Wijsman deferred statistical equivalent of order β $(0 < \beta \leq 1)$ to multiple η if for every $\varepsilon > 0$

$$\lim_{u,v\to\infty} \frac{1}{(\psi_u \phi_v)^\beta} \left| \left\{ (i,j) : i \in (p_u, r_u], j \in (q_v, s_v], \left| d_x\left(\frac{E_{ij}}{F_{ij}}\right) - \eta \right| \geq \varepsilon \right\} \right| = 0,$$

for each $x \in \mathcal{X}$. In this case, the notation $E_{ij} \overset{W_2^\eta (DS)^\beta}{\sim} F_{ij}$ is used.

Remark 3.1 The notion of asymptotical Wijsman deferred statistical equivalence of order β for double set sequences is reduced to;

- the notion of asymptotical Wijsman deferred statistical equivalence in [39], for $\beta = 1$.
- the notion of asymptotical Wijsman statistical equivalence of order β in [40], for $p_u = 0, r_u = u$ and $q_v = 0, s_v = v$.
- the notion of asymptotical Wijsman statistical equivalence in [36], for $\beta = 1$, and $p_u = 0, r_u = u$ and $q_v = 0, s_v = v$.

Definition 3.2 The double sequences $\{E_{ij}\}$ and $\{F_{ij}\}$ are said to be asymptotically Wijsman deferred Cesàro equivalent of order β $(0 < \beta \leq 1)$ to multiple η if

$$\lim_{u,v\to\infty} \frac{1}{(\psi_u \phi_v)^\beta} \sum_{i=p_u+1}^{r_u} \sum_{j=q_v+1}^{s_v} d_x\left(\frac{E_{ij}}{F_{ij}}\right) = \eta,$$

for each $x \in \mathcal{X}$. In this case, the notation $E_{ij} \overset{W_2^\eta (D)^\beta}{\sim} F_{ij}$ is used.

Definition 3.3 The double sequences $\{E_{ij}\}$ and $\{F_{ij}\}$ are said to be asymptotically Wijsman strongly deferred Cesàro equivalent of order β $(0 < \beta \leq 1)$ to multiple η if

$$\lim_{u,v\to\infty} \frac{1}{(\psi_u \phi_v)^\beta} \sum_{i=p_u+1}^{r_u} \sum_{j=q_v+1}^{s_v} \left| d_x\left(\frac{E_{ij}}{F_{ij}}\right) - \eta \right| = 0,$$

for each $x \in \mathcal{X}$. In this case, the notation $E_{ij} \overset{W_2^\eta[D]^\beta}{\sim} F_{ij}$ is used.

Remark 3.2 The notion of asymptotical Wijsman strong deferred Cesàro equivalence of order β for double set sequences is reduced to;

- the notion of asymptotical Wijsman strong deferred Cesàro equivalence in [39], for $\beta = 1$.
- the notion of asymptotical Wijsman strong Cesàro equivalence of order β in [40], for $p_u = 0$, $r_u = u$ and $q_v = 0, s_v = v$.
- the notion of asymptotical Wijsman strong Cesàro equivalence in [36], for $\beta = 1$, and $p_u = 0, r_u = u$ and $q_v = 0, s_v = v$.

4. Main Results

In this section, we have investigated some properties of the notions of asymptotical Wijsman deferred statistical equivalence of order β and asymptotical Wijsman deferred Cesàro equivalence of order β $(0 < \beta \leq 1)$ for double set sequences, and have examined the relationship between these notions.

Theorem 4.1 If $0 < \beta < \gamma \leq 1$, then

$$E_{ij} \overset{W_2^\eta(DS)^\beta}{\sim} F_{ij} \Rightarrow E_{ij} \overset{W_2^\eta(DS)^\gamma}{\sim} F_{ij}.$$

Proof. Let $0 < \beta < \gamma \leq 1$ and assume that $E_{ij} \overset{W_2^\eta(DS)^\beta}{\sim} F_{ij}$. For every $\varepsilon > 0$ and each $x \in \mathcal{X}$, we can write

$$\frac{1}{(\psi_u \phi_v)^\gamma} \left| \left\{ (i,j): i \in (p_u, r_u], j \in (q_v, s_v], \left| d_x\left(\frac{E_{ij}}{F_{ij}}\right) - \eta \right| \geq \varepsilon \right\} \right|$$

$$\leq \frac{1}{(\psi_u \phi_v)^\beta} \left| \left\{ (i,j): i \in (p_u, r_u], j \in (q_v, s_v], \left| d_x\left(\frac{E_{ij}}{F_{ij}}\right) - \eta \right| \geq \varepsilon \right\} \right|.$$

Therefore, by our assumption, we get $E_{ij} \overset{W_2^\eta(DS)^\gamma}{\sim} F_{ij}$.

If $\gamma = 1$ is taken in Theorem 4.1, then the following corollary is obtained.

Corollary 4.1 If double sequences $\{E_{ij}\}$ and $\{F_{ij}\}$ are $W_2^\eta(DS)^\beta$-equivalence $(0 < \beta \leq 1)$, then the sequences are $W_2^\eta(DS)$-equivalence.

Theorem 4.2 If $0 < \beta \leq \gamma \leq 1$, then

$$E_{ij} \overset{W_2^\eta[D]^\beta}{\sim} F_{ij} \Rightarrow E_{ij} \overset{W_2^\eta[D]^\gamma}{\sim} F_{ij}.$$

Proof. Let $0 < \beta < \gamma \leq 1$ and assume that $E_{ij} \overset{W_2^\eta[D]^\beta}{\sim} F_{ij}$. For each $x \in \mathcal{X}$, we can write

$$\frac{1}{(\psi_u \phi_v)^\gamma} \sum_{i=p_u+1}^{r_u} \sum_{j=q_v+1}^{s_v} \left| d_x\left(\frac{E_{ij}}{F_{ij}}\right) - \eta \right| \leq \frac{1}{(\psi_k \phi_j)^\beta} \sum_{i=p_u+1}^{r_u} \sum_{j=q_v+1}^{s_v} \left| d_x\left(\frac{E_{ij}}{F_{ij}}\right) - \eta \right|.$$

Therefore, by our assumption, we get $E_{ij} \overset{W_2^\eta[D]^\gamma}{\sim} F_{ij}$.

If $\gamma = 1$ is taken in Theorem 4.2, then the following corollary is obtained.

Corollary 4.2 If double sequences $\{E_{ij}\}$ and $\{F_{ij}\}$ are $W_2^\eta[D]^\beta$-equivalence $(0 < \beta \leq 1)$, then the sequences are $W_2^\eta[D]$-equivalence.

Theorem 4.3 Let $0 < \beta \leq 1$. If double sequences $\{E_{ij}\}$ and $\{F_{ij}\}$ are $W_2^\eta[D]^\beta$-equivalence, then the sequences are $W_2^\eta(DS)^\beta$-equivalence.

Proof. Let $0 < \beta \leq 1$ and assume that $E_{ij} \overset{W_2^\eta[D]^\beta}{\sim} F_{ij}$. For every $\varepsilon > 0$ and each $x \in \mathcal{X}$, we can write

$$\sum_{i=p_u+1}^{r_u} \sum_{j=q_v+1}^{s_v} \left| d_x\left(\frac{E_{ij}}{F_{ij}}\right) - \eta \right| \geq \sum_{\substack{i=p_u+1 \\ \left| d_x\left(\frac{E_{ij}}{F_{ij}}\right) - \eta \right| \geq \varepsilon}}^{r_u} \sum_{j=q_v+1}^{s_v} \left| d_x\left(\frac{E_{ij}}{F_{ij}}\right) - \eta \right|$$

$$\geq \varepsilon \left| \left\{ (i,j) : i \in (p_u, r_u], j \in (q_v, s_v], \left| d_x\left(\frac{E_{ij}}{F_{ij}}\right) - \eta \right| \geq \varepsilon \right\} \right|$$

and so

$$\frac{1}{\varepsilon} \frac{1}{(\psi_u \phi_v)^\beta} \sum_{i=p_u+1}^{r_u} \sum_{j=q_v+1}^{s_v} \left| d_x\left(\frac{E_{ij}}{F_{ij}}\right) - \eta \right|$$

$$\geq \frac{1}{(\psi_u \phi_v)^\beta} \left| \left\{ (i,j) : i \in (p_u, r_u], j \in (q_v, s_v], \left| d_x\left(\frac{E_{ij}}{F_{ij}}\right) - \eta \right| \geq \varepsilon \right\} \right|.$$

Therefore, by our assumption, we get $E_{ij} \overset{W_2^{\eta}(DS)^{\beta}}{\sim} F_{ij}$.

Remark 4.1 The converse of Theorem 4.3 is true only in the case $\beta = 1$ and $\{E_{ij}\}, \{F_{ij}\} \in L_{\infty}^2$, which has already been shown in [39].

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Bootstrap in Gaussian Mixture models and Performance assesement

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Abstract

The Gaussian mixture model is one of the well-known clustering approaches while representing the data from different normal distributions with distinct parameters. In inference of the model parameters, various methods can be used. Among alternatives, we select the expectation- maximization algorithm as the number of parameters is large to perform maximum likelihood estimators with explicit forms. The underlying estimation problem becomes serious when the numbers of parameter (p) exceeds the number of observations (n). Hence, in this study, we propose to apply the bootstrapping strategy so that the difference between n and p can be smaller. In bootstrapping procedure, we implement the Efron's approach which is non-parametric and computationally faster than its alternatives. Then, we evaluate the performance of this proposal model with well-known model selection criteria like the Bayesian Information Criterion (BIC) and the ones which are designed specifically for high dimensional datasets, namely, Consistent Akaike information criterion (CAIC) and Information and Complexity Selection (ICOMP) aproach. In numerical examples, we investigate the performance of suggested model with real datasets. By means of this study, we aim to improve the accuracy of the Gaussian mixture model in clustering big datasets and to generate an alternative model selection criterion in comparison of the results.

Keywords: Clustering, Bootstrap methods, Gaussian mixture models, Model selection methods

1. Introduction

Gaussian mixture model (GMM) is a popular machine learning algorithm for unsupervised learning purpose. Expected maximization (EM) algorithm is a standard procedure to estimate GMM. But, when the number of parameters (p) is more than the number of observations (n), EM algorithm has some limitations in convergence. To unreval this problem, the bootstrapping method is suggested and is already inserted in the popular R programming package "mclust" by Scrucca et al. (2015). The package can perform both the Efron`s bootstrapping idea which is nonparametric (Efron, 1979) and the weighted likelihood bootstrapping idea which a generalized version of nonparametric bootstrap (Newton and Raftey,1994). From previous analyses, it has been shown that the bootstrap can be useful for the decision of the number of clustering when the sample size of variables is limited to get convergent estimates from EM algorithm. The mclust package supports Bayesian Information Criterion (BIC), also known as Schwartz criterion (Schwartz, 1978; Fraly and Raftery, 1998) and integrated complete data likelihood (ICL) criterion (Biernacki et al., 2000) in order to select the best partition of the complete data. But it is known that these criteria can give inconsistent

results, particularly, under $n<p$ conditions. In this study, as the novelty, we insert the Consistent Akaike Information Criterion (CAIC) and Information and Complexity selection (ICOMP) in the calculation of GMM in order to choose the optimal clustering of the data. CAIC and ICOMP have developed by Bozdoğan (1987, 2010) and from previous studies, it has been found that these criteria are successful in the selection of the best model under various linear and nonlinear models. Hereby, in this study, we perform both CAIC and ICOMP for bootstrapped datasets and compare the findings under BIC, CAIC and ICOMP criteria and evaluate which model selection criterion have better performance in GMM. We use real bench-mark datasets in our analyses.
Accordingly, in the organization of the study we present GMM, CAIC and ICOMP in Section 2, we present the analysis in Section 3, and finally, in Section 4 we conclude our findings and discuss the future works.

2. Gaussian mixture model (GMM)

Traditionally, the clustering analysis is one of the most important tasks in statistical theory. Moreover, Gaussian mixture model (GMM) is a probabilistic model and uses the soft clustering approach for distributing the points in different clusters while representing normally distributed subpopulations within an overall population. Mixture models, in general, do not require the knowledge of subpopulation which a data point belongs to, instead, they allow the model to learn the subpopulations automatically similar to K-means clustering by Hartigan (1979).

Hence, in GMM we assume a (multivariate) Gaussian distribution for each component k whose parameter θ , i.e., f_k $(\boldsymbol{x}; \theta_k) \sim \mathrm{N}(\boldsymbol{\mu}_k, \Sigma_k)$, while clusters are centered at the mean vector $\boldsymbol{\mu}_k$, and with scale parameter determined by the covariance matrix Σ_k. Paramater estimation of the covariances matrices can be obtained by means of an eigen-values of the form, $\sum_k \lambda_k D_k A_k D_k^T$ where λ_k is a scalar value controlling the volume of the ellipsoid, A_k is a diagonal matrix specifying the shape of the density contours with $\det(A_k) = 1$, and D_k is an orthogonal matrix which determines the orientation of the corresponding ellipsoid. Here, $(\,.\,)^T$ denotes the transpose of the given matrix.

GMM is a popular unsupervised learning algorithm whose cluster parameters are estimated by the expectation-maximization algorithm via mixtures of Gaussian distributions with iterative approach. In calculation, it is assumed that $\boldsymbol{x} = \{x_1, x_2, ..., x_i, ..., x_n\}$ is a sample of n independent identically distributed observations. The distribution of every observation is specified by a probability density function through a finite mixture model with G number of mixture components, which takes the following form

$$f(x_i, \Psi) = \sum_{k=1}^{G} \pi_k f_k\left(x_i; \theta_k\right), \tag{1}$$

where $\Psi = \{\pi_1, \ldots, \pi_{G-1}, \theta_1, \ldots, \theta_G\}$ are the parameters of the mixture model, $f_k(x_i; \theta_k)$ is the kth component density for observations x_i with parameter vector θ_k, and $(\pi_1, \ldots, \pi_{G-1})$ are the mixing weights or probabilities such that $\pi_k > 0$, and $\sum_{k=1}^{G} \pi_k = 1$.

In the computation, it is accepted that G is fixed and the parametersof the mixture model are unknown in general. Therefore, we need to estimate it with the following likelihood equation

$$(\Psi; x_1, ..., x_n) = \sum_{i=1}^{n} \log f_k(x_i; \Psi). \quad (2)$$

Direct estimation of the log-likelihood function is very complex, so, the maximum likelihood estimator (MLE) of a finite mixture model is usually obtained via the EM algorithm (Dempster et al., 1977; McLachlan and Peel, 2000). Expected-maximization (EM) algorithm is a standard procedure to estimate GMM. However, when the number of parameter is more than the number of observation, this task can be challenging. Therefore, the bootstrap procedure is preferred to infer GMMs with higher accuracy.

2.2. Bootstrap assessment for Gaussian mixture model (GMM)

EM algorithm can have convergent problem in inference of GMM, in particular, when the number of parameter (p) exceeds the number of observation (n). For this reason, we propose bootstrap procedure when GMM is used to find the best partition of the data as stated beforehand. Because the bootstrap is a powerful statistical tool that can be used to quantify the uncertainty in associated parameter with a given estimator or statistical learning method via resampling approach. Thus, the the idea of bootstrapping for GMM can be defined as follows:

Step1 : Calculate F_n which is the empirical distribution function of observations $(x_1, \ldots, x_n)$ from replacement via $(x_1^*, \ldots, x_n^*)$,
Step2 : Fit GMM to obtained bootstrap samples $\widehat{\Psi^*}$,
Step3 : Repeat step1-2 , say B=50,100 times, to get $\widehat{\Psi_1^*}, \widehat{\Psi_2^*}, \ldots, \widehat{\Psi^*_B}$estimators frombBootstrap samples. Then, bootstrap covariance matrix can be obtained as $cov_{boot}(\widehat{\Psi}) = \frac{1}{B-1}\sum_{jb=1}^{B} (\widehat{\Psi_{J,}^*} - \Psi^*)(\widehat{\Psi_{J,}^*} - \Psi^*)^T$ and the mean vector is found via $\widehat{\Psi^*} = \frac{1}{B-1}\sum_{j=1}^{B} \widehat{\Psi_J^*}$.

2.3 CAIC and ICOMP Selection Criteria

There are two classical model selection criteria, namely, Akaike‘s information criterion (AIC) and Schwarz‘s Bayesian information criterion that are applicable in different fields. Moreover, different classification methods for finite mixture models were suggested by Göğebakan and Erol (2018, 2019) and different selection methods for GMM models have been compared by Akoğul and Erisoğlu (2016). On the other hand, in order to improve the performance of accuracy, Bozdoğan (1987,2010) proposed two alternative approaches. The first method is called the consistent AIC selections criterion (CAIC) which makes the distance between the true model and the real value as small as possible. In the study of Bozdoğan (1987), the smallest distance is computed by the Kullback-Leibler divergence and this criterion has the following form:

$$CAIC(k) = -2\log L(\hat{\theta}_k) + k[logn + 1] \quad (3)$$

in which the likelihood of θ is shown by logL($\hat{\theta}_k$) and k denotes the degrees of freedom of the distribution. It is seen that there is a similarity between CAIC(k) and BIC of ($k \log n$) and $k[\log n + 1]$ terms that have a stronger penalty term.

As an extension of this method, Bozdoğan (2010) also proposed the Information and COMPlexity (ICOMP) measure. Basically, ICOMP can penalize the free parameters and the covariance matrix directly with a third term. This third term in the loss function has a capability to calculate the distance when the parameter estimates are correlated in the model fitting stage. Hence, the expression for ICOMP can be shown as below.

$$ICOMP = -2\log L\left(\hat{\theta}_k\right) + 2C(\hat{\Sigma}\ ), \tag{4}$$

where $\log L(\hat{\theta}_k)$ is the log-likelihood, $\hat{\theta}_k$ denotes the maximum likelihood estimate of the parameter vector of θ_k, C expresses a real-valued complexity measure and finally, $\hat{\Sigma} = \widehat{cov}(\hat{\theta}_k)$ refers to the estimated covariance matrix of the parameter vector of the candidate model. This covariance matrix can be obtained in different ways. Bozdoğan (2010)` s choice is the computation of the inverse of the Cramer-Rao lower bound matrix that is obtained from the estimated inverse Fisher information matrix with the following equation.

$$\hat{F}^{-1} = \left\{-E\left(\frac{\partial^2 logL(\theta)}{\partial\theta\partial\theta'}\right)\right\}^{-1}. \tag{5}$$

In this expression, the (s × s)-dimensional second-order partial derivatives of the log-likelihood function of the estimated model is denoted by $\hat{F}^{-1}$. As a result, a more general form of ICOMP can be expressed via

$$ICOMP = -2\log L\left(\hat{\theta}_k\right) + 2C(\hat{F}^{-1}) \tag{5}$$

when

$$C(\hat{F}^{-1}) = \frac{s}{2}\log\left[\frac{\widehat{tr}F^{-1}}{s}\right] - \frac{1}{2}\log\left|\hat{F}^{-1}\right|.$$

In this expression, the second term shows the information complexity of the estimated inverse Fisher information matrix of the model and $s = \dim(\hat{F}^{-1}) = rank\left(\hat{F}^{-1}\right)$ while dim(.) shows the dimension of the given matrix.

3. Data analyses

We have two datasets to show the efficiency of proposed model selection procedure. The first data set is The Hemophilia data set which contains two measured variables on 75 women, belonging to two groups: Here, the sample size of groups are denoted by n_1 and n_2 where n_1=30 of them are non-carriers (normal group) and n_2=45 are known hemophilia carriers (obligatory carriers), respectively.

This data set is originally analized in the context of discriminant analysis by Habemma and Hermans (1974). The objective is to find a procedure for detecting potential hemophilia A carriers on the basis of two measured variables: X1=log10(AHV activity) and X2=log10(AHV-like antigen). The first group of n_1=30 women consists of known non-carriers (normal group) and the second group of n_2=45 women is selected from known hemophilia A carriers (obligatory carriers). This data set was also analyzed by Johnson and Wichern (1998), as well as, in the context of robust Linear Discriminant Analysis by Hawkins and McLachlan (1997) and Hubert and Van Driessen (2004).

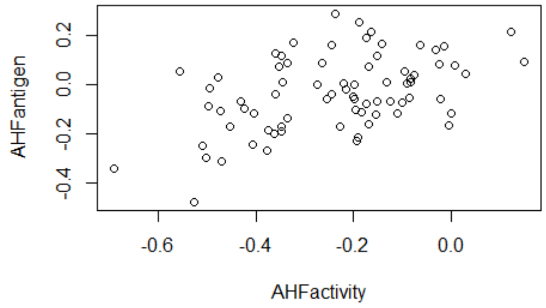


Figure 1: The scatter plot of hemophilia dataset.

In the analysis, first of all, the Gaussian finite mixture model is fitted by EM algorithm and the following mean terms in each group are computed.

Means	**Group1**	**Group2**
AHFactivity	-0.116	-0.366
AHFantigen	-0.025	-0.045

Table 1: Means for AHFactivity and AHFantigen for hemophilia dataset.

Furthermoe, the associated estimated covariance-variance matrix of AHFactivity and AHFantigen for each group is shown as below:

$$\hat{\Sigma}_1 = \begin{bmatrix} 0.011 & 0.007 \\ 0.007 & 0.012 \end{bmatrix} \text{ and } \hat{\Sigma}_2 = \begin{bmatrix} 0.016 & 0.015 \\ 0.015 & 0.032 \end{bmatrix}.$$

Then, in order to increase the number of observations in each group and improve the accuracy of the eastimates, we perfrom nonparametric bootstrapping procedure. Accordingly, we use 38 replications in each group and calculate the following confidence intervals for estimated means and variances with a 0.05 significance level:

Means	**Group1**	**Group2**
AHFactivity	(-0.245, -0.078)	(-0.444, -0.276)
AHFantigen	(-0.105, 0.032)	(-0.169, 0.094)

Table 2: Means of 95% bootstrap confidence intervals for hemophilia dataset.

Variance	Group1	Group2
AHFactivity	(0.006, 0.034)	(0.004, 0.024)
AHFantigen	(0.006, 0.019)	(0.010, 0.043)

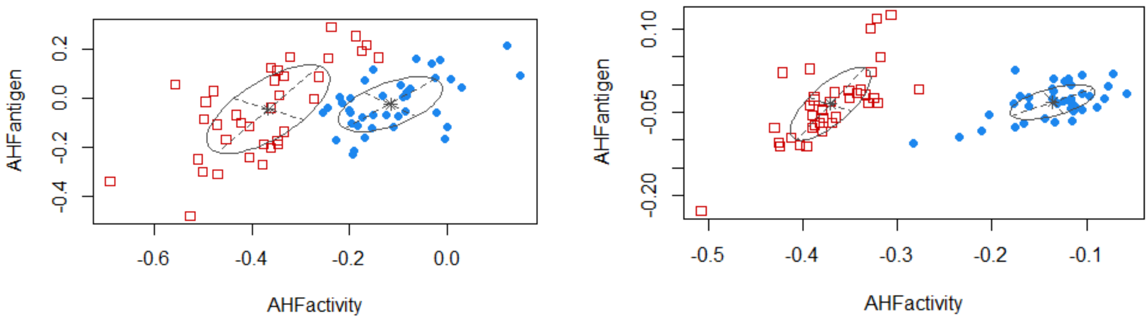
Table 3: Variances of 95% bootstrap confidence intervals for hemophilia dataset.

The values of CAIC and ICOMP with BIC selection criteria for hemophilia data set are listed in Table 4.

Selection criterion	GMM without bootstrap	GMM with bootstrap
BIC	106.565	407.285
CAIC	95.5647	396.430
ICOMP	133.097	422.011

Table 4. Results of selection criteria for hemophilia data set w/o and w/ bootstrap

From Table 4, it is seen that CAIC decreases the log-likelihood regarding BIC, but, ICOMP is the worse. On the other hand, from the comparison of values without/with bootstrap, it is seen that the bootstrap method increases the variance between groups which enables us to better generate the clustering. Figure 2a and 2b validate our findings in such a way that the bootstarp data have better separated cluster than without bootstrap data set.



(a) (b)

Figure 2 Gaussian finite mixture model for classification task (a) without and (b) with bootstrap for hemophilia data set

Our second data set is the diabates data which examine the relationship among blood chemistry measures of glucose tolerance and insulin in 145 nonobese adults. Reaven and Miller (1979) used the PRIM9 system at the Stanford Linear Accelerator Center to visualize the data in 3D, and discovered a peculiar pattern that looked like a large blob with two wings in different directions.

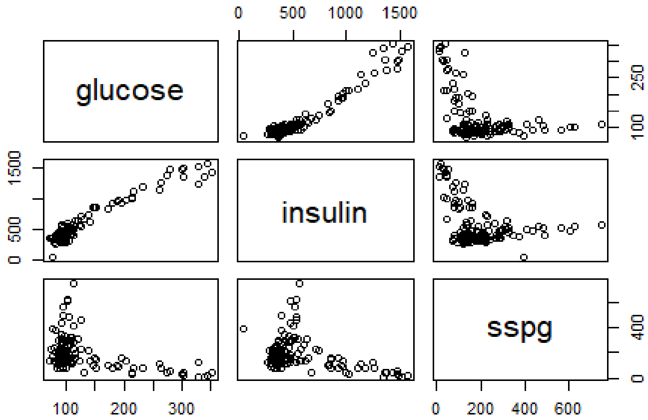


Figure3. The scatter plot of diabetes dataset.

The scatter plots of 3 groups, namely, glucose, insülin and sspg, are shown in Figure 3. In the analyses of this data via GMM whose inference is conducted by the EM algorithm, the mixing probabilities of each group are found as 0.537, 0.265, 0.198, in order, and the estimated means as well as covariance-variance matrices are presented as below:

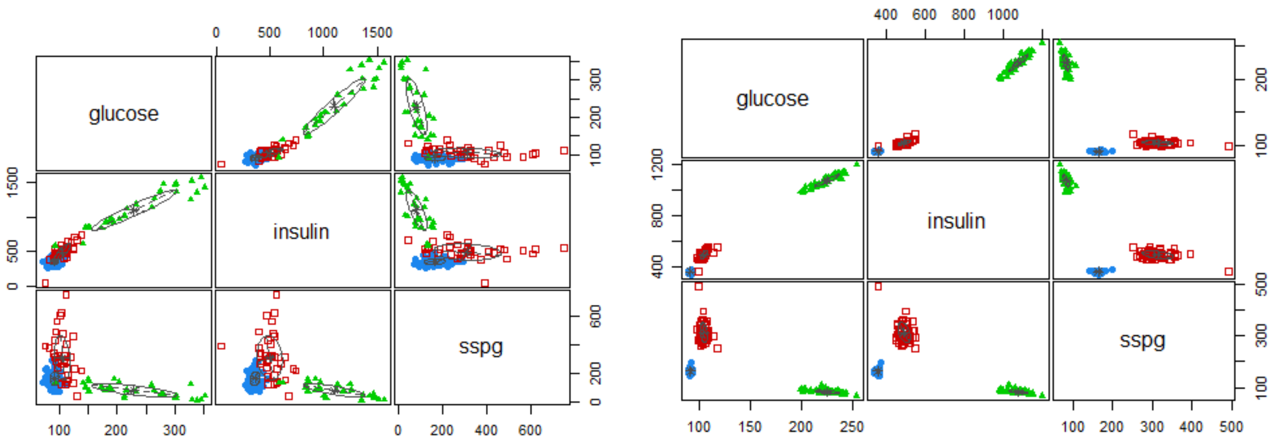
Means	Group1	Group2	Group3
glucose	90.962	104.534	229.421
insulin	357.791	494.826	1098.260
sspg	163.749	309.558	81.600

Table5: Estimated mean values for each group for diabates dataset.

$$\hat{\Sigma}_1 = \begin{bmatrix} 57.180 & 75.832 & 14.732 \\ 75.832 & 2101.766 & 322.823 \\ 14.732 & 322.823 & 2416.991 \end{bmatrix}, \hat{\Sigma}_2 = \begin{bmatrix} 185.029 & 1282.340 & -509.731 \\ 1282.340 & 14039.283 & -2559.025 \\ -509.732 & -2559.025 & 23835.728 \end{bmatrix} \text{ and}$$

$$\hat{\Sigma}_3 = \begin{bmatrix} 5529.250 & 20389.09 & -2486.208 \\ 20389.088 & 83132.48 & -10393.004 \\ -2486.208 & -10393.00 & 2217.533 \end{bmatrix}.$$

In Figure 4, the GMM classifications of diabates datasets without and with bootstrap can be seen visually.



(a) (b)

Figure 4. GMM classifications of diabates datasets (a) without and (b) with bootstrap.

On the other hand, the variance of each group after bootstrap and their 95% confidence intervals are shown as follows:

Means	Group1	Group2	Group3
glucose	(89.627, 92.865)	(98.973, 113.150)	(201.371, 243.072)
insulin	(342.1126, 370.092)	(449.721, 548.226)	(983.205, 1146.30)
sspg	(146.036, 181.321)	(264.278, 389.978)	(71.220, 96.781)

Table 6: 95% confidence interval for means of bootstrapped diabates dataset.

Variance	Group1	Group2	Group3
glucose	(45.765, 76.171)	(82.954, 367.743)	(4377.933, 7409.077)
insulin	(1293.432, 3327.144)	(2985.338, 44335,695)	(58913.5, 115432.000)
sspg	(1287.704, 4071.764)	(13030.360, 33893.19)	(1646.896, 2889.926)

Table7: 95% confidence interval for variances of bootstrapped diabates dataset.

Selection criterion	GMM without bootstrap	GMM with bootstrap
BIC	-4751.316	-3515.975
CAIC	-4780.316	-3545.175
ICOMP	-4818.154	-3537.380

Table 8: Results of selection criteria for diabates dataset without and with bootstrap.

Finally, the performances of all model selection criteria are listed in Table 8. The tabulated results indicate that similar to hemophilia data set, the bootstrap increases the variances in the groups, but, helps better partition of the augmented data as seen in Figure 4. Moreover it is seen that under without and with bootstrap conditions, CAIC and ICOMP have lower loss of information with respect to BIC results. This supports our previous findings in the first dataset.

4. Conclusion

We have proposed alternative model selection criteria for Gaussian mixture models with Bozdogan`s CAIC and ICOMP selection methods in order to find more consistent selection procedure. When the number of paramater is more than the number of observations, EM algorithm can be difficult to compute. For this reason, we have used the nonparametric bootstrap method for Gaussian mixture model (GMM) which improves the accuracy of the clustering. From the analyses, we have shown that the bootstrap raises the variances within each group of data sets and this increase becomes helpful in the partition of the data with GMM. Furthermore, we have seen that CAIC and ICOMP can decrease the loss of information with respect to BIC values. As a future study, we consider to make more comprehensive analyses based on simulated data via distinct scenarios and data types so that we can get more precise conclusion about suitability of CAIC and ICOMP in GMM.

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Calculation of the Shortest Distance and the Lowest Fuel Cost by Simulating Annealing in the Heterogeneous Fleet Vehicle Routing Problem with Capacity Constraints

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Abstract

The vehicle routing problem (VRP), which is the last stage of the logistics systems and supply chain in the growing economy market, has gained great importance. Vehicle routing problem is a mathematical optimization problem in which minimum cost, shortest distance routes are determined for customer demands located in different locations from a central warehouse. In this study, VRP consisting of capacity-constrained, heterogeneous fleet vehicles has been handled, and the optimization problem in which the optimum route set is calculated and fuel costs are minimized has been examined. The approximate solutions of the problem are calculated with the meta-heuristic Simulated Annealing algorithm in Matlab.

Keywords: Optimization, Vehicle Routing, Simulated Annealing

1. Introduction

Vehicle routing problem is one of the combinatorial optimization problems that deals with finding the optimum route as much as the number of vehicles in order to meet the demands of a group of customers located in certain places with a vehicle fleet from a central warehouse in a minimum distance and time and return to the warehouse (or not return). Combinatorial vehicle routing problem is one of the NP-Hard (Nondeterministic polynomial) problems in its simplest form [1]. The work that can be considered the first in VRP started in 1959 when Dantzig and Ramser created an optimal route between a fleet of gasoline delivery trucks and multiple service stations [2]. In 1964, Clarke and Wright conducted the study that requires optimal routing of a fleet of trucks of varying capacities from a central warehouse to a number of delivery points, requiring selection from multiple possible routes if the number of delivery points is large [3]. In 1953, Metropolis et al. first developed the Simulated Annealing (SA) algorithm [4]. In 1983, Kirkpatrick et al. used the algorithm for solving optimization problems [5]. Cerny suggested its use in combinatorial problems in 1985 [6]. Osman in 1993 used the SA algorithm in VRP for the first time in its hybridization with tabu search to solve the capacity VRP (CVRP) [7]. In 1995, Breedam proposed an advanced heuristic based on SA to solve a standard VRP [8]. In their study, Çetin et al. proposed a new heuristic algorithm for the solution of simultaneous distribution and aggregation vehicle routing problems with heterogeneous vehicle fleets and created a decision support system based on the proposed algorithm [9]. In their study, Çetin and Gencer describe the problem of simultaneous distribution-collection vehicle routing with heterogeneous vehicle fleet and precise time window. They proposed a mathematical model for the defined problem and Solomon test problems were arranged in terms of demand; They have tried for samples with 5,10,15,20 clients [10]. In the study of Alağaş et al., the capacity constraint in the transportation of hazardous materials was considered both in terms of weight and volume. More than one vehicle with different features is used in the shipments made from a central warehouse to the interim warehouses and from the intermediate warehouses to the central warehouse. The problem addressed according to these features is the heterogeneous simultaneous ball-deliver-vehicle routing problem. The problem was solved by modeling with mathematical programming [11]. Ahad Furug, in his study, addressed the problem of periodic vehicle routing with capacity constraints in order to minimize the transportation distance and cost of vehicles in the collection of medical wastes [12]. In the study of Şahin et al., the distribution activities of a company operating as a regional distributor were considered as a green vehicle routing problem with a heterogeneous fleet, and it was tried to obtain environmentally friendly solutions that provide lower emission values with the annealing simulation method [13].

2. Material and Method

In this study, a data set consisting of 1 warehouse, 10 node (customer) coordinates and customer demands was randomly generated in Matlab. A mathematical model was created for this data set. The model is a deterministic capacity ARP problem with a heterogeneous fleet. The minimum route distance and fuel cost of the created problem are approximately solved by simulated annealing, which is a meta-heuristic algorithm.

3. Suggested Mathematical Model for Vehicle Routing Problem

The vehicle routing problem (VRP) is defined on a graph $G = (V, A)$ where $V = \{v_1, v_2 \dots, v_n\}$is a set of vertices (nodes) and $A \subseteq \{(v_i, v_j): i \neq j, v_i, v_j \in V\}$ is the set of arcs [14]. Capacity VRP is the problem of finding the routes that a company's vehicles with limited capacity from one or more warehouses should follow in order to serve their customers with known demands [15]. In a capacity VRP, customer demands should not be more than the capacity of the vehicles on the route. For the problem in this study, the mathematical model of the capacity and heterogeneous fleet of VRP was created as follows [16] and [17].

Parameters:

K = Number of vehicles,

$f_k =$ k. fuel cost per km of the vehicle,

$\mathrm{Q}_k =$ Capacity of k vehicle,

$\mathrm{N} =$ Number of customers or stops,

$q_i = i \ (i > 0)$ customer demand quantity,

$C_{ij} = i$ distance between client i and client j,

Decision variables

$$x_{ijk} = \begin{cases} 1, & \text{if the vehicle is going from customer i to customer j} \\ 0, & \text{otherwise)} \end{cases}$$

where $i \neq j$, $i, j \in \{0, \dots, N\}$ and 0 warehouses

Objective function:

$$\text{Min} \sum_{i=0}^{N} \sum_{j=0, i \neq j}^{N} \sum_{k=1}^{K} f_k c_{ij} x_{ijk} \tag{1}$$

Constraints:

$$\sum_{k=1}^{K} \sum_{j=1}^{N} x_{ijk} = K, \qquad i = 0 \ for \tag{2}$$

$$\sum_{k=1}^{K} \sum_{i=0, i \neq j}^{N} x_{ijk} = 1, \qquad \forall j, \ j \in \{1, \dots, N\} \tag{3}$$

$$\sum_{k=1}^{K} \sum_{j=0, i \neq j}^{N} x_{ijk} = 1, \qquad \forall i, \ i \in \{1, \dots, N\} \tag{4}$$

$$\sum_{i=0}^{N} \sum_{j=0, j \in S}^{N} x_{ijk} \leq |S| - 1 \tag{5}$$

$$\sum_{i=1}^{N} q_i \sum_{j=0, i \neq j}^{N} x_{ijk} \leq Q_k \tag{6}$$

$$\sum_{i=1}^{N} x_{ijk} \leq 1, \qquad k \in \{1, \dots, K\} \tag{7}$$

$$x_{ijk} \in \{0,1\} \qquad \forall_{ij} \in V, \forall_k \in K \tag{8}$$

In the model (1), the objective function expresses that the total distance traveled and the total fuel cost should be minimized. Constraint equation (2) indicates that K vehicles will exit the operation. Constraint equations (3) and (4) indicate that a customer must be visited by only one vehicle and only one of the ways out of the customer must be used. Constraint (5) is used to eliminate rounds that do not start at the warehouse and are not completed in the warehouse, this is added for every possible subset S of customers that do not include the warehouse. (6) states that in case the vehicle capacities are different, [18] indicates that the vehicle capacities should not exceed Q_k. Constraint (7) states that a vehicle will leave the operation only once, so it will be used once on the route. The constraint (8) is related to the x_{ijk}variable being an integer [19].

The simulation annealing algorithm, which is a meta-heuristic solution that we will use to solve the problem described above, is described in the following section.,

4. Simulated Annealing (SA) meta-heuristic algorithm

SA, developed by Metropolis et al, is a probability-based optimization algorithm, which is generally used for discrete optimization problems, inspired by the slow cooling of solids after heating until crystallization [5].

The SA algorithm is used to select a better solution than the previous one in each round by scanning the solution area, based on the principle of heating the solids and then cooling them slowly until they crystallize. According to this simulation, the temperature value is used to determine the probability of accepting solutions worse than the best solution found.

An Annealing simulation algorithm generally consists of an initial solution, a neighbor solution generation method, and an annealing program. The function of the annealing process is to start from a solution at a sufficiently high temperature and gradually decrease the temperature to cycle between good and bad solutions and finally arrive at the best solution.

The algorithm is started with a sufficiently high temperature value and at each step a certain number of solutions are obtained before the temperature is reduced. New solutions are either accepted or rejected according to established criteria. Each decrease in temperature affects the probability of leaving the obtained solution and switching to a new solution. The algorithm is terminated when the temperature reaches the lowest value or when the SA algorithm runs for the desired number of repetitions [20].

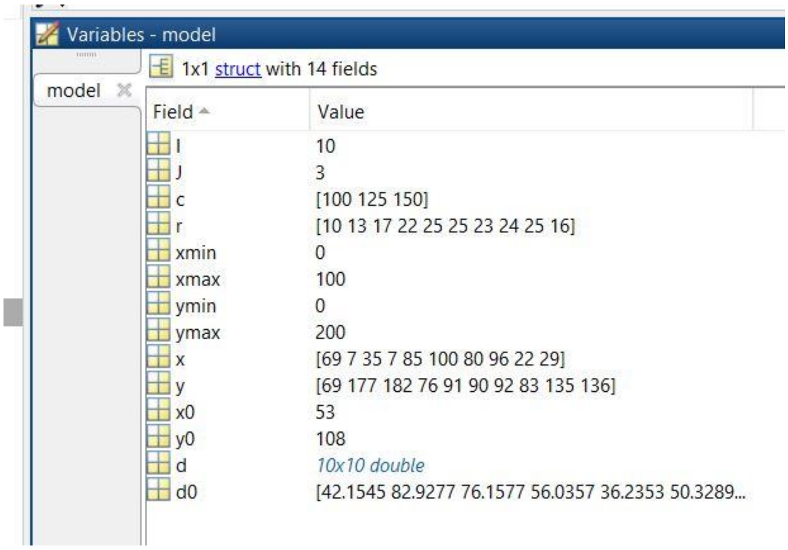

Figure1. Screenshot of the model created in Matlab

In Figure 1, I is the number of customers, j is the number of vehicles, c is the vehicle capacities, r is the demands for 10 customers, x and y is the customer coordinates, x0 and y0 is the warehouse coordinate, d is the distance matrix between customers, d0 is the distance matrix between the warehouse and customers.

When the algorithm was run, optimal route drawing was obtained for 3 vehicles in figure 2.

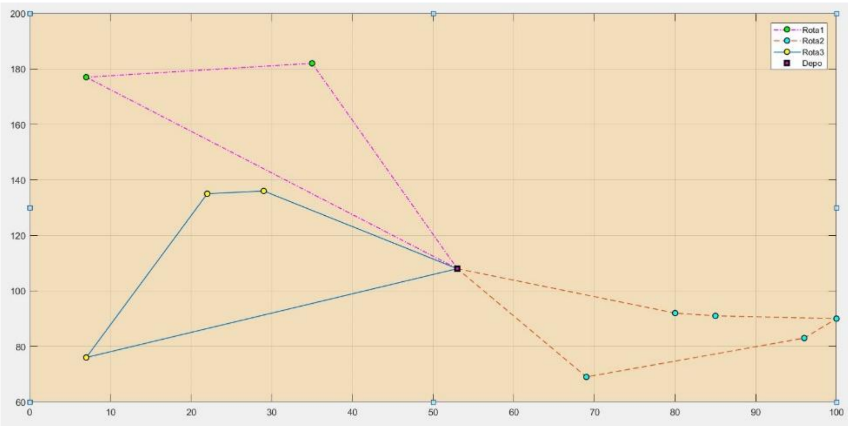

Figure 2. Routings obtained from Matlab

Figure 2: It shows the circulation shape of the 3 routes obtained from Matlab in the coordinate axis.

Optimal solutions of routing and fuel costs obtained from Matlab for the generated problem are given in Table 1.

Table1. Routing results and fuel cost with SA

Routes	Vehicle capacity	Route distance (km)	Route fuel cost TL	Route request	Total distance (km)	Total fuel cost TL
R1 : 0-2-3-0	100	187,5283	562,5849	30	480,5378	1441,6134
R2 : 0-7-5-6-8-1-0	125	132,1476	396,4428	107		
R3 : 0-4-9-10-0	150	160,8619	482,5857	63		

The depot is shown with 0 in Table1, and each vehicle route starts from the depot and ends at the depot. R1 is the route that the first vehicle should follow, R2 is the route that the second vehicle should follow, R3 is the route that the third vehicle should follow. Assuming that the fuel liter price is 26.40 TL, the fuel cost per km of the vehicles is calculated as 3 TL/km.

5. Conclusion

In this study, a model for the heterogeneous fleet vehicle routing problem of 3 different capacities, consisting of 1 warehouse, 10 customers, 100,125,150, was created. A near-optimal result was obtained with the Backgammon simulation algorithm in Matlab. The demands of each customer were met without exceeding the vehicle capacities from the shortest routes obtained from the solution of the problem. Total customer demands were met with 3 different routes at optimum distance. Total fuel costs were found to be the least at the minimum distance.

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Certain Sublasses of Multivalent Functions Defined by Deniz-Özkan Differential Operator

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Abstract

In the present paper a new extended Differential operator $\mathcal{D}_{\lambda,p}^{m}$ $\left(\lambda \geq 0;\, p \in \mathbb{N}, m \in \mathbb{N} \cup \mathbb{N}_0\right)$ of multivalent functions is introduced. Making use of the Differential operator $\mathcal{D}_{\lambda,p}^{m}$ two new subclasses $\mathcal{S}_{\lambda}^{m}(A,B;\sigma,p)$ and $\mathcal{ST}_{\lambda}^{m}(A,B;\sigma,p)$ of multivalent analytic functions are introduced and investigated in the open unit disk. Some interesting relations and characteristics such as inclusion relationships, neighborhoods, partial sums, some applications of fractional calculus belonging to each of these subclasses $\mathcal{S}_{\lambda}^{m}(A,B;\sigma,p)$ and $\mathcal{ST}_{\lambda}^{m}(A,B;\sigma,p)$ Relevant connections of the definitions and results presented in this paper with those obtained in several earlier works on the subject are also pointed out.

Keywords: Multivalent analytics, neighborhoods, partial sums, differential operator.

1. Introduction and definitions

Let $\mathcal{A}(k,p)$ denote the class of functions normalized by

$$f(z) = z^p + \sum_{n=k+p}^{\infty} a_n z^n \qquad \left(p,k \in \mathbb{N} := \{1,2,3,...\}\right) \tag{1}$$

which are analytic and $p-valent$ in the open unit disk $\mathcal{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$.

Let $f(z)$ and $g(z)$ be analytic in $\mathcal{U}$. Then we say that the function f is subordinate to g if there exists a Schwarz function $w(z)$, analytic in $\mathcal{U}$ with $w(0) = 0,\ |w(z)| < 1$ such that $f(z) = g(w(z)) \quad (z \in \mathcal{U})$. We denote this subordination $f \prec g$ or $f(z) \prec g(z) \ (z \in \mathcal{U})$.

In particular, if the function g is univalent in $\mathcal{U}$, the above subordination is equivalent to $f(0) = g(0)$, $f(\mathcal{U}) \subset g(\mathcal{U})$.

For $f \in \mathcal{A}(k,p)$ given by (1) and $g(z)$ given by

$$g(z) = z^p + \sum_{n=k+p}^{\infty} b_n z^n \quad \left(p, k \in \mathbb{N} := \{1, 2, 3, \ldots\}\right) \tag{2}$$

their convolution (or Hadamard product), denoted by $(f * g)$, is defined as

$$(f * g)(z) := z^p + \sum_{n=k+p}^{\infty} a_n b_n z^n =: (g * f)(z) \quad \left(z \in \mathcal{U}\right). \tag{3}$$

Note that $f * g \in \mathcal{A}(k, p)$. In particular, we set

$$\mathcal{A}(p,1) := \mathcal{A}_p, \quad \mathcal{A}(1,k) := \mathcal{A}(k), \quad \mathcal{A}(1,1) := \mathcal{A}_1 = \mathcal{A}.$$

Definition 1. (Deniz- Çekin differential operator) [5] Let $f \in \mathcal{A}(k, p)$. For the parameters $\lambda \geq 0$, $z \in U$ and $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ the differential operatör $\mathcal{D}_{\lambda}^{m}$ on $\mathcal{A}(k, p)$ by the following

$$\mathcal{D}_{\lambda,p}^{m} : \mathcal{A}(k, p) \to \mathcal{A}(k, p),$$

$$\mathcal{D}_{\lambda,p}^{0} f(z) = f(z) \tag{4}$$

$$\mathcal{D}_{\lambda,p}^{1} f(z) = \mathcal{D}_{\lambda,p} f(z) = \frac{1}{p^2}\left\{\lambda z^3 f'''(z) + (2\lambda+1) z^2 f''(z) + \left(1-\lambda p(p-1)\right) z f'(z)\right\}$$

$$\vdots$$

$$\mathcal{D}_{\lambda,p}^{m} f(z) = \mathcal{D}_{\lambda,p}\left(\mathcal{D}_{\lambda,p}^{m-1} f(z)\right).$$

If f is given by (1) then from the definition of the differential operator $\mathcal{D}_{\lambda,p}^{m}$ we can easily see that

$$\mathcal{D}_{\lambda,p}^{m} f(z) = z^p + \sum_{n=k+p}^{\infty} \Phi^n(\lambda, m, p) a_n z^n \tag{5}$$

where

$$\Phi^n(\lambda, m, p) = \left[\frac{n\left(\lambda(n-p)(n+p-1)+n\right)}{p^2}\right]^m. \tag{6}$$

Remark 1. It should be remarked that the operator $\mathcal{D}_{\lambda,p}^{m}$ is a generalization of many other *linear differential operators* considered earlier. In particular, for $f \in \mathcal{A}(k, p)$ we have the following:

(i) $\mathcal{D}^{m}_{\lambda,1} = \mathcal{D}^{m}_{\lambda}$, $(\lambda \geq 0)$ the Deniz-Özkan differential operator [6].

(ii) $\mathcal{D}^{m}_{0,1} = \mathcal{D}^{2m}_{1}$ $(\delta \in \mathbb{N}_0)$ the Salagean differential operator [7].

(iii) $\mathcal{D}^{m}_{0,p} = \mathcal{D}^{2m}_{p}$, $(p \in \mathbb{N})$ Shenan, Salim and Mousa oprator [19].

Now, by making use of the operator $\mathcal{D}^{m}_{\lambda,p}$, we define a new subclass of functions belonging to the class $\mathcal{A}(k,p)$.

Definition 2. Let $\lambda \geq 0$ and $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $p \in \mathbb{N}$ and for the parameters σ, A and B such that

$$-1 \leq A < B \leq 1,\ 0 < B \leq 1 \text{ and } 0 \leq \sigma < p, \tag{7}$$

we say that a function $f(z) \in \mathcal{A}(k,p)$ is in the class $\mathcal{S}^{m}_{\lambda}(A,B;\sigma,p)$ if it satisfies the following subordination condition:

$$\frac{1}{p-\sigma}\left(\frac{[\mathcal{D}^{m}_{\lambda,p} f(z)]'}{z^{p-1}} - \sigma\right) \prec \frac{1+Az}{1+Bz} \qquad (z \in \mathcal{U}). \tag{8}$$

If the following inequality holds true,

$$\left| \frac{\dfrac{[\mathcal{D}^{m}_{\lambda,p} f(z)]'}{z^{p-1}} - p}{B\dfrac{[\mathcal{D}^{m}_{\lambda,p} f(z)]'}{z^{p-1}} - [pB + (A-B)(p-\sigma)]} \right| < 1 \qquad (z \in \mathcal{U}) \tag{9}$$

the inequality (9) is equivalent the subordination condition (8).

We note that by specializing the parameters λ, m, σ, A, B and p, the subclass $\mathcal{S}^{m}_{\lambda}(A,B;\sigma,p)$ reduces to several well-known subclasses of analytic functions. Furthermore, we say that a function $f(z) \in \mathcal{S}^{m}_{\lambda}(A,B;\sigma,p)$ is in the subclass $\mathcal{ST}^{m}_{\lambda}(A,B;\sigma,p)$ if $f(z)$ is of the following form:

$$f(z) = z^p - \sum_{n=k+p}^{\infty} |a_n| z^n \qquad \left(p, k \in \mathbb{N} := \{1,2,3,\ldots\}\right). \tag{10}$$

In our present paper, we shall make use of the familiar *integral operator* $\mathcal{I}_{\vartheta,p}$ defined by (see, for details, [2, 11, 13]; see also [25])

$$(\mathcal{I}_{\vartheta,p})(z) := \frac{\vartheta + p}{z^p} \int_0^z t^{\vartheta-1} f(t)\,dt \qquad f \in \mathcal{A}(k,p);\ \vartheta + p > 0;\ p \in \mathbb{N}) \tag{11}$$

as well as the fractional calculus operator $\mathcal{D}_z^{\nu}$ for which it is well known that (see, for details, [16,23] and [21]; see also Section 7)

$$\mathcal{D}_z^{\nu}\{z^{\rho}\} = \frac{\Gamma(\rho+1)}{\Gamma(\rho+1-\nu)} z^{\rho-\nu} \quad (\rho > -1; \nu \in \mathbb{R}) \tag{12}$$

in terms of Gamma function.

The main object of the present paper is to investigate the various important properties and characteristics of two subclasses of $\mathcal{A}(k,p)$ of normalized analytic functions in $\mathcal{U}$ with negative and positive coefficients, which are introduced here by making use of the *differential operator* defined by (4). Inclusion relationships for the class $\mathcal{S}_{\lambda}^{m}(A,B;\sigma,p)$ are investigated by applying the techniques of convolution. Furthermore, several properties involving generalized neighborhoods and partial sums for functions belonging to these subclasses are investigated. Finally, some applications of fractional calculus operators are considered. Relevant connections of the definitions and results presented here with those obtained in several earlier works are also pointed out.

2. Basic properties of the function class $\mathcal{S}_{\lambda}^{m}(A,B;\sigma,p)$

We first determine a necessary and sufficient condition for a function $f(z) \in \mathcal{A}(k,p)$ of the form (10) to be in the class $\mathcal{ST}_{\lambda}^{m}(A,B;\sigma,p)$.

Theorem 1. Let the function $f(z) \in \mathcal{A}(k,p)$ be defined by (10). Then the function $f(z)$ is in the class $\mathcal{ST}_{\lambda}^{m}(A,B;\sigma,p)$. if and only if

$$\sum_{n=k+p}^{\infty} \left[(n-p)(1+B)-(B-A)(p-\sigma)\right]\Phi^{n}(\lambda,m,p)|a_n| \leq (B-A)(p-\sigma) \tag{13}$$

where $\Phi^{n}(\lambda,m,p)$ is given by (6).

Proof. If the condition (13) hold true, we find from (10) and (13) that

$$\left|z[\mathcal{D}_{\lambda,p}^{m} f(z)]' - p\mathcal{D}_{\lambda,p}^{m} f(z)\right| - \left|Bz[\mathcal{D}_{\lambda,p}^{m} f(z)]' - [pB+(A-B)(p-\sigma)\mathcal{D}_{\lambda,p}^{m} f(z)]\right|$$

$$= \left|-\sum_{n=k+p}^{\infty}(n-p)\Phi^{n}(\lambda,m,p)|a_n|z^n\right|$$
$$-\left|-\left[(B-A)(p-\sigma)z^p + \sum_{n=k+p}^{\infty} B(n-p)-(B-A)(p-\sigma)\right]\Phi^{n}(\lambda,m,p)|a_n|z^n\right|$$

$$\leq \sum_{n=k+p}^{\infty}\left[(n-p)(1+B)-(B-A)(p-\sigma)\right]\Phi^{n}(\lambda,m,p)|a_n|-(B-A)(p-\sigma)\leq 0$$

$$\left(z\in\partial\mathcal{U}=\{z:z\in\mathbb{C}\text{ and }|z|=1\}\right).$$

Hence, by the *Maximum Modulus Theorem*, we have

$$f(z)\in\mathcal{ST}_{\lambda}^{m}(A,B;\sigma,p).$$

Conversely, assume that the function $f(z)$ defined by (10) is in the class $\mathcal{ST}_{\lambda}^{m}(A,B;\sigma,p)$. Then we have

$$\left|\frac{\frac{z[\mathcal{D}_{\lambda,p}^{m}f(z)]'}{\mathcal{D}_{\lambda,p}^{m}f(z)}-p}{B\frac{z[\mathcal{D}_{\lambda,p}^{m}f(z)]'}{\mathcal{D}_{\lambda,p}^{m}f(z)}-[pB+(A-B)(p-\sigma)]}\right|$$

$$=\left|\frac{\sum_{n=k+p}^{\infty}(n-p)\Phi^{n}(\lambda,m,p)|a_n|z^{n}}{\sum_{n=k+p}^{\infty}\left[-B(n-p)+(B-A)(p-\sigma)\right]\Phi^{n}(\lambda,m,p)|a_n|z^{n}+(B-A)(p-\sigma)z^{p}}\right|<1\quad(z\in\mathcal{U}).\tag{14}$$

Now, since $|\Re(z)|\leq|z|$ for all z, we have

$$\Re\left(\frac{\sum_{n=k+p}^{\infty}(n-p)\Phi^{n}(\lambda,m,p)|a_n|z^{n-p}}{\sum_{n=k+p}^{\infty}\left[-B(n-p)+(B-A)(p-\sigma)\Phi^{n}(\lambda,m,p)|a_n|z^{n-p}+(B-A)(p-\sigma)\right]}\right)<1.\tag{15}$$

We choose values of z on the real axis so that the following expression:

$$\frac{z\left[\mathcal{D}_{\lambda,p}^{m}f(z)\right]'}{\mathcal{D}_{\lambda,p}^{m}f(z)}$$

is real. Then, upon clearing the denominator in (15) and letting $z\to 1^{-}$ though real values, we get the following inequality

$$\sum_{n=k+p}^{\infty}\left[(n-p)(1+B)-(B-A)(p-\sigma)\right]\Phi^{n}(\lambda,m,p)|a_n|\leq(B-A)(p-\sigma)$$

This completes the proof of Theorem 1.

Corollary 1. Let the function $f(z) \in \mathcal{A}(k,p)$ be defined by (1). If the function $f(z) \in \mathcal{S}_{\lambda}^{m}(A,B;\sigma,p),$ then

$$\sum_{n=k+p}^{\infty}\left[(n-p)(1+B)-(B-A)(p-\sigma)\right]\Phi^{n}(\lambda,m,p)\left|a_n\right| \le (B-A)(p-\sigma) \tag{16}$$

where $\Phi^{n}(\lambda,m,p)$ is given by (6).

Corollary 2. Let the function $f(z) \in \mathcal{A}(k,p)$ be defined by (10). If the function $f(z) \in \mathcal{ST}_{\lambda}^{m}(A,B;\sigma,p),$ then

$$\left|a_n\right| \le \frac{(B-A)(p-\sigma)}{\sum\limits_{n=k+p}^{\infty}\left[(n-p)(1+B)-(B-A)(p-\sigma)\right]\Phi^{n}(\lambda,m,p)} \quad \left(n,p \in \mathbb{N}\right) \tag{17}$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z^{p} - \frac{(B-A)(p-\sigma)}{\left[(n-p)(1+B)-(B-A)(p-\sigma)\right]\Phi^{n}(\lambda,m,p)} z^{n} \quad \left(n,p \in \mathbb{N}\right) \tag{18}$$

We next prove the following growth and distortion properties for the class $\mathcal{ST}_{\lambda}^{m}(A,B;\sigma,p).$

Theorem 2. If a function $f(z)$ be defined by (1) is in the class $\mathcal{ST}_{\lambda}^{m}(A,B;\sigma,p).$ then

$$\left(\frac{p!}{(p-q)!} - \frac{(B-A)(p-\sigma)(k+p)!}{(k+p-q)!\left[k(1+B)-(B-A)(p-\sigma)\right]\Phi^{k+p}(\lambda,m,p)}\left|z\right|^{k}\right)\left|z\right|^{p-q} \le \left|f^{(q)}(z)\right|$$

$$\le \left(\frac{p!}{(p-q)!} + \frac{(B-A)(p-\sigma)(k+p)!}{(k+p-q)!\left[(k)(1+B)-(B-A)(p-\sigma)\right]\Phi^{k+p}(\lambda,m,p))}\left|z\right|^{k}\right)\left|z\right|^{p-q}$$

(19)

for $q \in \mathbb{N}_0, p > q$ and all $z \in \mathcal{U}.$ The result is sharp for the function $f(z)$ given by

$$f(z) = z^{p} - \frac{(B-A)(p-\sigma)}{\left[k(1+B)-(B-A)(p-\sigma)\right]\Phi^{k+p}(\lambda,m,p))} z^{k+p} \quad \left(p \in \mathbb{N}\right). \tag{20}$$

Proof. In view of Theorem 1, we have

$$\frac{(n-p)(1+B)-(B-A)(p-\sigma)\Phi^{k+p}(\lambda,m,p)}{(B-A)(p-\sigma)}\left|a_n\right|\leq 1$$

which readily yields we obtain

$$\frac{(k+p-q)!\left[k\left(1+B\right)-\left(B-A\right)(p-\sigma)\right]\Phi^{k+p}(\lambda,m,p)}{(B-A)(p-\sigma)(k+p)!}\sum_{n=k+p}^{\infty}\frac{n!}{(n-q)!}\left|a_n\right|\leq 1, \tag{21}$$

or

$$\sum_{n=k+p}^{\infty}\frac{n!}{(n-q)!}\left|a_n\right|\leq\frac{(B-A)(p-\sigma)(k+p)!)}{(k+p-q)!\left[k\left(1+B\right)-\left(B-A\right)(p-\sigma)\right]\Phi^{k+p}(\lambda,m,p)}.$$

From last inequality,

$$\left|f^{(q)}(z)\right|=\left|\frac{p!}{(p-q)!}z^{p-q}-\sum_{n=k+p}^{\infty}\frac{n!}{(n-q)!}\left|a_n\right|z^{n-q}\right| \quad \left(q\in\mathbb{N}_0;\ p>q\right) \tag{22}$$

$$\leq\frac{p!}{(p-q)!}+\frac{(B-A)(p-\sigma)(k+p)!)}{(k+p-q)!\left[k\left(1+B\right)-\left(B-A\right)(p-\sigma)\right]\Phi^{k+p}(\lambda,m,p)}.$$

Similarly,

$$\left|f^{(q)}(z)\right|=\left|\frac{p!}{(p-q)!}z^{p-q}-\sum_{n=k+p}^{\infty}\frac{n!}{(n-q)!}\left|a_n\right|z^{n-q}\right| \quad \left(q\in\mathbb{N}_0;\ p>q\right) \tag{23}$$

$$\geq\frac{p!}{(p-q)!}-\frac{(B-A)(p-\sigma)(k+p)!)}{(k+p-q)!\left[k\left(1+B\right)-\left(B-A\right)(p-\sigma)\right]\Phi^{k+p}(\lambda,m,p)}.$$

This complete the proof of Theorem 2.

For $q=0$ in Theorem 2, we get obtain the following distortion result for $\mathcal{ST}_{\lambda}^{m}(A,B;\sigma,p)$.

Corollary 3. Let $f(z)\in\mathcal{ST}_{\lambda}^{m}(A,B;\sigma,p)$. Then we have

$$1-\frac{(B-A)(p-\sigma)}{\left[k\left(1+B\right)-\left(B-A\right)(p-\sigma)\right]\Phi^{k+p}(\lambda,m,p)}\left|z\right|^{k+p}\leq\left|f(z)\right|$$

$$\leq 1+\frac{(B-A)(p-\sigma)}{\left[k\left(1+B\right)-\left(B-A\right)(p-\sigma)\right]\Phi^{k+p}(\lambda,m,p)}\left|z\right|^{k+p}.$$

For $q=1$ in Theorem 2, we get obtain the following growth result for $\mathcal{ST}_{\lambda}^{m}(A,B;\sigma,p)$.

Corollary 4. Let $f(z)\in\mathcal{ST}_{\lambda}^{m}(A,B;\sigma,p)$. Then we have

$$p-\frac{(B-A)(p-\sigma)(k+p)!}{\left[k(1+B)-(B-A)(p-\sigma)\right]\Phi^{k+p}(\lambda,m,p)}|z|^{k+p}\le|f'(z)|$$
$$\le p+\frac{(B-A)(p-\sigma)(k+p)}{\left[k(1+B)-(B-A)(p-\sigma)\right]\Phi^{k+p}(\lambda,m,p)}|z|^{k+p}.$$

4. Inclusion relations involving neighborhoods

Following the earlier investigations (based upon the familiar concept of neighborhoods of analytic functions) by Goodman [10], and Ruscheweyh [17], and others including Srivastava *et al.* [23, 24], Orhan [14, 15], Deniz *et al.* [9], Aouf *et al*. [1] (see also [3]).

Firstly, we define the $(k,\eta)-$neighborhood of function k of the form (1) by means of Definition 3 below.

Definition 3. For $\eta>0$ and a non-negative sequence $\mathcal{S}=\{s_n\}_{n=1}^{\infty}$, where

$$s_n:=\frac{\left[(n-p)(1+B)-(B-A)(p-\sigma)\right]\Phi^{k+p}(\lambda,m,p)}{(B-A)(p-\sigma)}\qquad(n\in\mathbb{N}).\tag{24}$$

The $(k,\eta)-$neighborhood of a function $f(z)\in\mathcal{A}(k,p)$ of the form (1) is defined as follows:

$$\mathcal{N}_{k,p}^{\eta}(f):=\left\{g:g(z)=z^p+\sum_{n=k+p}^{\infty}a_nz^n\in\mathcal{A}(k,p)\text{ and }\sum_{n=k+p}^{\infty}s_n|b_n-a_n|\le\eta\ (\eta>0)\right\}.\tag{25}$$

For $s_n=n$, Definition 3 would correspond to the $\mathcal{N}_{\eta}-$neighborhood considered by Ruscheweyh [17].

Our first result based upon the familiar concept of neighborhood defined by (24).

Theorem 2. Let $f(z)\in\mathcal{S}_{\lambda}^{m}(A,B;\sigma,p)$ be given by (1). If f satisfies the inclusion condition:

$$\left(f(z)+\varepsilon z^p\right)(1+\varepsilon)^{-1}\in\mathcal{S}_{\lambda}^{m}(A,B;\sigma,p)\qquad\left(\varepsilon\in\mathbb{C};|\varepsilon|<\eta;\eta>0\right),\tag{26}$$

then

$$\mathcal{N}_{k,p}^{\eta}(f) \subset \mathcal{S}_{\lambda}^{m}(A,B;\sigma,p). \tag{27}$$

Proof. It is not difficult to see that a function f belongs to $\mathcal{S}_{\lambda}^{m}(A,B;\sigma,p)$ if and only if

$$\frac{[\mathcal{D}_{\lambda,p}^{m} f(z) f(z)]' - pz^{p-1}}{B[\mathcal{D}_{\lambda,p}^{m} f(z) f(z)]' - z^{p-1}[pB + (A-B)(p-\sigma)]} \neq \tau \qquad \left(z \in \mathcal{U};\, \tau \in \mathbb{C}, |\tau| = 1\right), \tag{28}$$

which is equivalent to,

$$(f * h)(z) \big/ z^{p} \neq 0 \quad (z \in \mathcal{U}), \tag{29}$$

where for convenience,

$$h(z) := z^{p} + \sum_{n=k+p}^{\infty} c_{n} z^{n} = z^{p} + \sum_{n=k+p}^{\infty} \frac{\left[(n-1) - (nB - \left(B-A\right)(p-\sigma)\right]\Phi^{n}(\lambda,m,p)}{\tau(B-A)(p-\sigma)} z^{n}. \tag{30}$$

We easily find from (30) that

$$\begin{aligned} |c_{n}| &\leq \left| \frac{\left[(n-1) - \left(nB - \left(B-A\right)(p-\sigma)\right)\tau\right]\Phi^{n}(\lambda,m,p)}{\tau(B-A)(p-\sigma)} \right| \\ &\leq \frac{\left[(n-1) - \left(nB - (B-A)(p-\sigma)\right)\right]\Phi^{n}(\lambda,m,p)}{(B-A)(p-\sigma)} \qquad (n \in \mathbb{N}). \end{aligned} \tag{31}$$

Furthermore, under the hypotheses of theorem, (26) and (29) yields the following inequalities:

$$\frac{\left((f(z) + \varepsilon z^{p})(1+\varepsilon)^{-1}\right) * h(z)}{z^{p}} \neq 0 \quad (z \in \mathcal{U})$$

or

$$\frac{f(z) * h(z)}{z^{p}} \neq \varepsilon \quad (z \in \mathcal{U}),$$

which is equivalent to the following:

$$\frac{f(z) * h(z)}{z^{p}} \geq \eta \qquad (z \in \mathcal{U}; \eta > 0). \tag{32}$$

Now, if we let

$$g(z) := z^{p} + \sum_{n=k+p}^{\infty} b_{n} z^{n} \in \mathcal{N}_{k,p}^{\eta}(f),$$

then we have

$$\left|\frac{\left(f(z)-g(z)\right)*h(z)}{z^p}\right| = \left|\sum_{k=n+p}^{\infty}(a_k-b_k)c_k z^{k-p}\right|$$
$$\leq \sum_{k=n+p}^{\infty}\frac{\left[\left(n-1\right)-\left(nB-(B-A)(p-\sigma)\right)\tau\right]\Phi^n(\lambda,m,p)}{\tau(B-A)(p-\sigma)}|a_n-b_n||z|^{n-p}$$
$$<\eta$$
$$(z\in\mathcal{U};\eta>0).$$

Thus, for any complex number τ such that $|\tau|=1,$ we have $(g*h)(z)/z^p \neq 0 \quad (z\in\mathcal{U}),$

which implies that $g\in\mathcal{S}_{\lambda}^{m}(A,B;\sigma,p).$ The proof is complete. □

We now define the $(k,\eta)-$neighborhood of a function $f(z)\in\mathcal{A}(k,p)$ of the form (11) as follows

Definition 4. For $\eta>0,$ the $(k,\eta)-$neighborhood of a function $f(z)\in\mathcal{A}(k,p)$ of the form (10) is given by

$$\widetilde{\mathcal{N}}_{k,p}^{\eta}(f):=\left\{g:g(z)=z^p-\sum_{n=k+p}^{\infty}|a_n|z^n\in\mathcal{A}(k,p)\text{ and}\right.$$
$$\left.\sum_{n=k+p}^{\infty}\frac{\left[\left(n-p\right)\left(1+B\right)-(B-A)(p-\sigma)\right]\Phi^n(\lambda,m,p)}{(B-A)(p-\sigma)}\left||b_n|-|a_n|\right|\leq\eta\ \ (\eta>0)\right\}. \tag{33}$$

Next, we prove

Theorem 3. If the function $f(z)$ defined by (11) is in the class $\mathcal{ST}_{\lambda}^{m}(A,B;\sigma,p).$ then

$$\widetilde{\mathcal{N}}_{k,p}^{\eta}(f)\subset\mathcal{ST}_{\lambda}^{m}(A,B;\sigma,p). \tag{34}$$

where

$$\eta:=\frac{k[\lambda\mu(k+p)+\lambda-\mu]}{k[\lambda\mu(k+p)+\lambda-\mu]+p+l}.$$

Proof. For a function $f(z)\in\mathcal{ST}_{\lambda}^{m}(A,B;\sigma,p)$ of the form (10) Theorem 1 immediately yields

$$\sum_{n=k+p}^{\infty} \frac{\left[\left(n-p\right)\left(1+B\right)-(B-A)(p-\sigma)\Phi^{n}(\lambda,m,p)\right]}{(B-A)(p-\sigma)}\left|a_n\right| \\ \leq \frac{p+l}{k[\lambda\mu(k+p)+\lambda-\mu]+p+l}. \tag{35}$$

Similarly, by taking

$$g(z) := z^p - \sum_{n=k+p}^{\infty} \left|b_n\right| z^n \in \widetilde{\mathcal{N}}^{\eta}_{k,p}(f) \qquad \left(\eta = \frac{k[\lambda\mu(k+p)+\lambda-\mu]}{k[\lambda\mu(k+p)+\lambda-\mu]+p+l}\right),$$

we find from the Definition 4 that

$$\sum_{n=k+p}^{\infty} \frac{\left[\left(n-p\right)\left(1+B\right)-(B-A)(p-\sigma)\Phi^{n}(\lambda,m,p)\right]}{(B-A)(p-\sigma)}\left\|b_n\right|-\left|a_n\right\| \leq \eta \ \ (\eta > 0). \tag{36}$$

With the help of (35) and (36), we have

$$\sum_{n=k+p}^{\infty} \frac{\left[\left(n-p\right)\left(1+B\right)-(B-A)(p-\sigma)\Phi^{n}(\lambda,m,p)\right]}{(B-A)(p-\sigma)}\left|b_n\right| \\ \leq \sum_{n=k+p}^{\infty} \frac{\left[\left(n-p\right)\left(1+B\right)-(B-A)(p-\sigma)\Phi^{n}(\lambda,m,p)\right])}{(B-A)(p-\sigma)}\left|a_n\right| \\ + \sum_{n=k+p}^{\infty} \frac{\left[\left(n-p\right)\left(1+B\right)-(B-A)(p-\sigma)\Phi^{n}(\lambda,m,p)\right]}{(B-A)(p-\sigma)}\left\|b_n\right|-\left|a_n\right\| \\ \leq \frac{p+l}{k[\lambda\mu(k+p)+\lambda-\mu]+p+l}+\eta = 1.$$

Hence, in view of the Theorem 1 again, we see that $g(z) \in \mathcal{S}_{\lambda}^{m}(A,B;\sigma,p)$.

To show the sharpness of the assertion of Theorem 1, we consider the functions $f(z)$ and $g(z)$ given by

$$f(z) = z^p - \left[\frac{(B-A)(p-\sigma)}{\left[k\left(1+B\right)-(B-A)(p-\sigma)\Phi^{k+p}(\lambda,m,p)\right]}\right] z^{k+p} \in f(z) \in \mathcal{ST}_{\lambda}^{m+1}(A,B;\sigma,p) \tag{37}$$

and

$$g(z)=z^p-\left[\begin{array}{l}\dfrac{(B-A)(p-\sigma)}{\left[k\left(1+B\right)-(B-A)(p-\sigma)\Phi^{k+p}(\lambda,m,p)\right]}\\ +\dfrac{(B-A)(p-\sigma)}{\left[k\left(1+B\right)-(B-A)(p-\sigma)\Phi^{k+p}(\lambda,m,p)\right]}\eta^*\end{array}\right]z^{k+p} \tag{38}$$

where $\eta^*>\eta$.

Clearly, the function $g(z)$ belong to $\widetilde{\mathcal{N}}_{k,p}^{\eta^*}(f)$. On the other hand, we find from Theorem 1 that $g(z)\notin \mathcal{ST}_{\lambda}^{m}(A,B;\sigma,p)$. This evidently completes the proof of Theorem 3.

5. Partial sums of the function class $\mathcal{ST}_{\lambda}^{m}(A,B;\sigma,p)$.

Following the earlier works by Silverman [20], Liu [12] and Deniz *et al*. [9], in this section we investigate the ratio of real parts of functions involving (10) and its sequence of partial sums defined by

$$\kappa_m(z)=\begin{cases} z^p, & m=1,2,...,k+p-1; \\ z^p-\sum\limits_{n=k+p}^{m}\left|a_n\right|z^n, & m=k+p,k+p+1,.... \end{cases} \qquad (n\geq k+p;\, k,p\in\mathbb{N}) \tag{39}$$

and determine sharp lower bounds for $\Re\left\{f(z)/\kappa_m(z)\right\}, \Re\left\{\kappa_m(z)/f(z)\right\}$.

Theorem 4. Let $f\in\mathcal{A}(k,p)$ and $\kappa_m(z)$ be given by (10) and (39), respectively. Suppose also that

$$\sum_{n=k+p}^{\infty}\lambda_n\left|a_n\right|\leq 1$$

$$\left(\text{where } \lambda_n=\frac{\left[\left(n-p\right)\left(1+B\right)-(B-A)(p-\sigma)\Phi^n(\lambda,m,p)\right]}{(B-A)(p-\sigma)}\right). \tag{40}$$

Then for $m\geq n+p$, we have

$$\Re\left(\frac{f(z)}{\kappa_m(z)}\right)>1-\frac{1}{\lambda_{m+1}} \tag{41}$$

and

$$\Re\left(\frac{\kappa_m(z)}{f(z)}\right)>\frac{\lambda_{m+1}}{1+\lambda_{m+1}}. \tag{42}$$

The result are sharp for every m, with the extremal functions given by

$$f(z)=z^p-\frac{1}{\lambda_{m+1}}z^{m+1}. \tag{43}$$

Proof. Under the hypothesis of the theorem, we can see from (40) that

$$\lambda_{n+1}>\lambda_n>1 \qquad (n\geq k+p).$$

Therefore, we have

$$\sum_{n=k+p}^{m}|a_n|+\lambda_{m+1}\sum_{n=m+1}^{\infty}|a_n|\leq\sum_{n=k+p}^{\infty}\lambda_n|a_n|\leq 1, \tag{44}$$

by using hypothesis (40) again.

Upon setting

$$\begin{aligned}\omega(z)&=\lambda_{m+1}\left[\frac{f(z)}{\kappa_m(z)}-\left(1-\frac{1}{\lambda_{m+1}}\right)\right]\\ &=1-\frac{\lambda_{m+1}\sum_{n=m+1}^{\infty}|a_n|z^{n-p}}{1-\sum_{n=k+p}^{m}|a_n|z^{n-p}}.\end{aligned} \tag{45}$$

By applying (44) and (45), we find that

$$\begin{aligned}\left|\frac{\omega(z)-1}{\omega(z)+1}\right|&=\left|\frac{-\lambda_{m+1}\sum_{n=m+1}^{\infty}|a_n|z^{n-p}}{2-2\sum_{n=k+p}^{m}|a_n|z^{n-p}-\lambda_{m+1}\sum_{n=m+1}^{\infty}|a_n|z^{n-p}}\right|\\ &\leq\frac{\lambda_{m+1}\sum_{n=m+1}^{\infty}|a_n|z^{n-p}}{2-2\sum_{n=k+p}^{m}|a_n|z^{n-p}-\lambda_{m+1}\sum_{n=m+1}^{\infty}|a_n|z^{n-p}}\leq 1 \quad (z\in\mathcal{U};\, n\geq k+p),\end{aligned} \tag{46}$$

which shows that $\Re(\omega(z))>0\,(z\in\mathcal{U})$. From (45), we immediately obtain the inequality (41).

To see that the function f given by (43) gives the sharp result, we observe for $z\to 1^-$ that

$$\frac{f(z)}{\kappa_m(z)} = 1 - \frac{1}{\lambda_{m+1}} z^{m-p+1} \to 1 - \frac{1}{\lambda_{m+1}},$$

which shows that the bound in (41) is the best possible.

Similarly, if we put

$$\begin{aligned}\phi(z) &= (1+\lambda_{m+1})\left[\frac{\kappa_m(z)}{f(z)} - \frac{\lambda_{m+1}}{1+\lambda_{m+1}}\right] \\ &= 1 + \frac{(1+\lambda_{m+1})\sum_{n=m+1}^{\infty}|a_n|z^{n-p}}{1-\sum_{n=k+p}^{m}|a_n|z^{n-p}},\end{aligned} \tag{47}$$

and make use of (44), we can deduce that

$$\begin{aligned}\left|\frac{\phi(z)-1}{\phi(z)+1}\right| &= \left|\frac{(1+\lambda_{m+1})\sum_{n=m+1}^{\infty}|a_n|z^{n-p}}{2-2\sum_{n=k+p}^{m}|a_n|z^{n-p}+(\lambda_{m+1}-1)\sum_{n=m+1}^{\infty}|a_n|z^{n-p}}\right| \\ &\leq \frac{(1+\lambda_{m+1})\sum_{n=m+1}^{\infty}|a_n|z^{n-p}}{2-2\sum_{n=k+p}^{m}|a_n|z^{n-p}-(\lambda_{m+1}-1)\sum_{n=m+1}^{\infty}|a_n|z^{n-p}} \leq 1 \quad (z\in\mathcal{U};\, n\geq k+p),\end{aligned} \tag{48}$$

which leads us immediately to assertion (42) of the theorem.

The bound in (42) is sharp with the extremal function given by (43). The proof of theorem is thus completed.

□

6. Applications of fractional calculus operators

Various operators of fractional calculus (that is, fractional integral and fractional derivatives) have been studied in the literature rather extensively (*cf., e.g.*, [16, 23, 21]; see also [9, 22] the various references cited therein). For our present investigation, we recall the following definitions.

Definition 5. Let $f(z)$ be analytic in a simply connected region of the z*-plane* containing the origin. The fractional integral of f of order ν is defined by

$$D_z^{-\nu} f(z) = \frac{1}{\Gamma(\nu)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\nu}} d\zeta \qquad (\nu > 0), \tag{49}$$

where the multiplicity of $(z-\zeta)^{\nu-1}$ is removed by requiring that $\log(z-\zeta)$ is real for $z-\zeta > 0$.

Definition 6. Let $f(z)$ be analytic in a simply connected region of the z*-plane* containing the origin. The fractional derivative of f of order ν is defined by

$$D_z^{\nu} f(z) = \frac{1}{\Gamma(1-\nu)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\nu}} d\zeta \qquad (0 \le \nu < 1), \tag{50}$$

where the multiplicity of $(z-\zeta)^{-\nu}$ is removed by requiring that $\log(z-\zeta)$ is real for $z-\zeta > 0$.

Definition 7. Under the hypotheses of Definition 6, the fractional derivative of order $n+\nu$ is defined, for a function $f(z)$, by

$$D_z^{n+\nu} f(z) = \frac{d^n}{dz^n}\{D_z^{\nu} f(z)\} \qquad (0 \le \nu < 1; n \in \mathbb{N}_0). \tag{51}$$

In this section, we shall investigate the growth and distortion properties of functions in the class $\mathcal{ST}_{\lambda}^{m}(A,B;\sigma,p)$ which involving the operators $\mathcal{I}_{\vartheta,p}$ and D_z^{ν}. In order to derive our results, we need the following lemma given by Chen *et al.* [4].

Lemma 1 (*see* [16]). Let the function $f(z)$ defined by (10). Then

$$D_z^{\nu}\{(\mathcal{I}_{\vartheta,p} f)(z)\} = \frac{\Gamma(p+1)}{\Gamma(p+1-\nu)} z^{p-\nu} - \sum_{n=k+p}^{\infty} \frac{(\vartheta+p)\Gamma(n+1)}{(\vartheta+n)\Gamma(n+1-\nu)} a_n z^{n-\nu} \tag{52}$$

$$(\nu \in \mathbb{R}; \vartheta > -p; \ p, n \in \mathbb{N})$$

and

$$\mathcal{I}_{\vartheta,p}\{(D_z^{\nu} f)(z)\} = \frac{(\vartheta+p)\Gamma(p+1)}{(\vartheta+p-\nu)\Gamma(p+1-\nu)} z^{p-\nu} - \sum_{n=k+p}^{\infty} \frac{(\vartheta+p)\Gamma(n+1)}{(\vartheta+n-\nu)\Gamma(n+1-\nu)} a_n z^{n-\nu} \tag{53}$$

$$(\nu \in \mathbb{R}; \vartheta > -p; \ p, n \in \mathbb{N})$$

provided that no zeros appear in the denominators in (52) and (53).

Theorem 5. Let the functions $f(z)$ defined by (10) be in the class $\mathcal{ST}_{\lambda}^{m}(A,B;\sigma,p)$. Then

$$\left|D_z^{-\nu}\{(\mathcal{I}_{\vartheta,p}f)(z)\}\right| \geq \left\{\frac{\Gamma(p+1)}{\Gamma(p+1+\nu)} - \frac{(\vartheta+p)\Gamma(n+p+1)(B-A)(p-\sigma)}{(\vartheta+n+p)\Gamma(n+p+1+\nu)(n+p)(1+B)\Phi^{k+p}(\lambda,m)}|z|^n\right\}|z|^{p+\nu} \tag{54}$$

$$(z\in\mathcal{U};\ \nu>0;\ \vartheta>-p;\ p,n\in\mathbb{N})$$

and

$$\left|D_z^{-\nu}\{(\mathcal{I}_{\vartheta,p}f)(z)\}\right| \leq \left\{\frac{\Gamma(p+1)}{\Gamma(p+1+\nu)} + \frac{(\vartheta+p)\Gamma(n+p+1)(B-A)(p-\sigma)}{(\vartheta+n+p)\Gamma(n+p+1+\nu)(n+p)(1+B)\Phi^{k+p}(\delta,m)}|z|^n\right\}|z|^{p+\nu}. \tag{55}$$

$$(z\in\mathcal{U};\ \nu>0;\ \vartheta>-p;\ p,n\in\mathbb{N})$$

Each of the assertions (54) and (55) is sharp.

Proof. In view of Theorem 1, we have

$$\frac{\left[k\left(1+B\right)-(B-A)(p-\sigma)\Phi^n(\lambda,m,p)\right]}{(B-A)(p-\sigma)}\sum_{n=k+p}^{\infty}|a_n| \leq \sum_{n=k+p}^{\infty}\frac{\left[\left(n-p\right)\left(1+B\right)-(B-A)(p-\sigma)\Phi^n(\lambda,m,p)\right]}{(B-A)(p-\sigma)}|a_n| \leq 1, \tag{56}$$

which readily yields

$$\sum_{n=k+p}^{\infty}|a_n| \leq \frac{(B-A)(p-\sigma)}{\left[k\left(1+B\right)-(B-A)(p-\sigma)\Phi^n(\lambda,m,p)\right]}. \tag{57}$$

Consider the function $\mathcal{F}(z)$ defined in $\mathcal{U}$ by

$$\mathcal{F}(z):=\frac{\Gamma(p+1-\nu)}{\Gamma(p+1)}z^{-\nu}D_z^{\nu}\{(\mathcal{I}_{\vartheta,p}f)(z)\}$$

$$=z^p-\sum_{n=k+p}^{\infty}\frac{(\vartheta+p)\Gamma(n+1)\Gamma(p+1+\nu)}{(\vartheta+n)\Gamma(n+1+\nu)\Gamma(p+1)}|a_n|z^n$$

$$= z^p - \sum_{n=k+p}^{\infty} \Theta(n)\left|a_n\right| z^n \qquad (z \in \mathcal{U}).$$

where

$$\Theta(n) := \frac{(\vartheta + p)\Gamma(n+1)\Gamma(p+1+\nu)}{(\vartheta + n)\Gamma(n+1+\nu)\Gamma(p+1)} \qquad (n \geq p+k;\ p,k \in \mathbb{N};\ \nu > 0). \tag{58}$$

Since $\Theta(n)$ is a *decreasing* function of n when $\nu > 0$, we get

$$0 < \Theta(n) \leq \Theta(k+p) = \frac{(\vartheta + p)\Gamma(k+p+1)\Gamma(p+1+\nu)}{(\vartheta + k + p)\Gamma(k+p+1+\nu)\Gamma(p+1)} \tag{59}$$

$$(\nu > 0;\ \vartheta > -p;\ p, n \in \mathbb{N})$$

Thus, by using (57) and (59), for all $z \in \mathcal{U}$, we deduce that

$$\left|\mathcal{F}(z)\right| \geq |z|^p - \Theta(k+p)|z|^{k+p} \sum_{n=k+p}^{\infty} \left|a_n\right|$$

$$\geq |z|^p - \frac{(\vartheta + p)\Gamma(k+p+1)\Gamma(p+1+\nu)(B-A)(p-\sigma)}{(\vartheta + k + p)\Gamma(k+p+1+\nu)\Gamma(p+1)\left[k\left(1+B\right) - (B-A)(p-\sigma)\Phi^{k+p}(\lambda, m, p)\right]}|z|^{k+p}$$

and

$$\left|\mathcal{F}(z)\right| \geq |z|^p + \Theta(k+p)|z|^{k+p} \sum_{n=k+p}^{\infty} \left|a_n\right|$$

$$\leq |z|^p + \frac{(\vartheta + p)\Gamma(k+p+1)\Gamma(p+1+\nu)(B-A)(p-\sigma)}{(\vartheta + k + p)\Gamma(k+p+1+\nu)\Gamma(p+1)\left[k\left(1+B\right) - (B-A)(p-\sigma)\Phi^{k+p}(\lambda, m, p)\right]}|z|^{k+p}$$

which yield the inequalities (54) and (55) of Theorem 5. Equalities in (54) and (55) are attained for the function $f(z)$ given by

$$D_z^{-\nu}\{(\mathcal{I}_{\vartheta,p} f)(z)\} = \left\{\frac{\Gamma(p+1)}{\Gamma(p+1+\nu)} - \frac{(\vartheta + p)\Gamma(k+p+1)(B-A)(p-\sigma)}{(\vartheta + k + p)\Gamma(k+p+1+\nu)\left[k\left(1+B\right) - (B-A)(p-\sigma)\Phi^{k+p}(\lambda, m, p)\right]} z^k\right\} z^{p+\nu}$$

or, equivalently, by

$$(\mathcal{I}_{\vartheta,p} f)(z) = z^p - \frac{(\vartheta+p)(B-A)(p-\sigma)}{(\vartheta+k+p)\left[k\left(1+B\right)-(B-A)(p-\sigma)\Phi^{k+p}(\lambda,m,p)\right]} z^{k+p}.$$

Thus we complete the proof of Theorem 5.

Theorem 6. Let the functions $f(z)$ defined by (10) be in the class $\mathcal{ST}_{\lambda}^{m}(A,B;\sigma,p).$ Then

$$\left|D_z^{\nu}\{(\mathcal{I}_{\vartheta,p} f)(z)\}\right| \geq \left\{\frac{\Gamma(p+1)}{\Gamma(p+1-\nu)} z^{p-\nu} - \frac{(\vartheta+p)\Gamma(k+p)(B-A)(p-\sigma)}{(\vartheta+k+p)\Gamma(k+p+1-\nu)\left[k\left(1+B\right)-(B-A)(p-\sigma)\Phi^{k+p}(\lambda,m,p)\right]}|z|^k\right\}|z|^{p-\nu} \tag{60}$$

$$(z \in \mathcal{U};\, \nu > 0;\, \vartheta > -p;\; p, n \in \mathbb{N})$$

and

$$\left|D_z^{\nu}\{(\mathcal{I}_{\vartheta,p} f)(z)\}\right| \leq \left\{\frac{\Gamma(p+1)}{\Gamma(p+1-\nu)} z^{p-\nu} + \frac{(\vartheta+p)\Gamma(k+p)(B-A)(p-\sigma)}{(\vartheta+k+p)\Gamma(k+p+1-\nu)\left[k\left(1+B\right)-(B-A)(p-\sigma)\Phi^{n}(\lambda,m,p)\right]}|z|^k\right\}|z|^{p-\nu}. \tag{61}$$

$$(z \in \mathcal{U};\, \nu > 0;\, \vartheta > -p;\; p, n \in \mathbb{N})$$

Each of the assertions (60) and (61) is sharp.

Proof. It follows from Theorem 1 that

$$\sum_{n=k+p}^{\infty} n|a_n| \leq \frac{(B-A)(p-\sigma)}{\left[k\left(1+B\right)-(B-A)(p-\sigma)\Phi^{k+p}(\lambda,m,p)\right]}. \tag{62}$$

Consider the function $\mathcal{Q}(z)$ defined in $\mathcal{U}$ by

$$\mathcal{Q}(z) := \frac{\Gamma(p+1-\nu)}{\Gamma(p+1)} z^{\nu} D_z^{\nu}\{(\mathcal{I}_{\vartheta,p} f)(z)\}$$

$$= z^p - \sum_{n=k+p}^{\infty} \frac{(\vartheta+p)\Gamma(n)\Gamma(p+1-\nu)}{(\vartheta+n)\Gamma(n+1-\nu)\Gamma(p+1)} n|a_n| z^n$$

$$= z^p - \sum_{n=k+p}^{\infty} \wp(n) n |a_n| z^n \qquad (z \in \mathcal{U}).$$

where, for convenience,

$$\wp(n) := \frac{(\vartheta + p)\Gamma(n)\Gamma(p+1-\nu)}{(\vartheta + n)\Gamma(n+1-\nu)\Gamma(p+1)} \qquad (n \geq p+n;\ p,k \in \mathbb{N};\ 0 \leq \nu < 1). \tag{63}$$

Since $\wp(n)$ is a *decreasing* function of ⫽ when $0 \leq \nu < 1,$ we find that

$$0 < \wp(n) \leq \wp(k+p) = \frac{(\vartheta + p)\Gamma(k+p)\Gamma(p+1-\nu)}{(\vartheta + k + p)\Gamma(k+p+1-\nu)\Gamma(p+1)} \tag{64}$$

$(0 \leq \nu < 1; \vartheta > -p;\ p,n \in \mathbb{N})$

Hence, with the aid of (62) and (64), for all $z \in \mathcal{U},$ we have

$$|\mathcal{Q}(z)| \geq |z|^p - \wp(k+p)|z|^{k+p} \sum_{n=k+p}^{\infty} n|a_n|$$

$$\geq |z|^p - \frac{(\vartheta + p)\Gamma(k+p)\Gamma(p+1-\nu)(B-A)(p-\sigma)}{(\vartheta + k + p)\Gamma(k+p+1-\nu)\Gamma(p+1)\left[k\left(1+B\right)-(B-A)(p-\sigma)\Phi^{k+p}(\lambda,m,p)\right]}|z|^{k+p}$$

and

$$|\mathcal{Q}(z)| \geq |z|^p + \wp(k+p)|z|^{k+p} \sum_{n=k+p}^{\infty} n|a_n|$$

$$\geq |z|^p + \frac{(\vartheta + p)\Gamma(k+p)\Gamma(p+1-\nu)(B-A)(p-\sigma)}{(\vartheta + n + p)\Gamma(k+p+1-\nu)\Gamma(p+1)\left[k\left(1+B\right)-(B-A)(p-\sigma)\Phi^{k+p}(\lambda,m,p)\right])}|z|^{k+p}$$

which yield the inequalities (63) and (64) of Theorem 6. Equalities in (63) and (64) are attained for the function $f(z)$ given by

$$D_z^{\nu}\{(\mathcal{I}_{\vartheta,p} f)(z)\} = \left\{\frac{\Gamma(p+1)}{\Gamma(p+1-\nu)} - \frac{(\vartheta + p)\Gamma(k+p+1)(B-A)(p-\sigma)}{(\vartheta + k + p)\Gamma(k+p+1-\nu)(1+B)\Phi^{k+p}(\lambda,m)} z^k\right\} z^{p-\nu}$$

or, equivalently, by

$$(\mathcal{I}_{\vartheta,p} f)(z) = z^p - \frac{(\vartheta + p)(B - A)(p - \sigma)}{(\vartheta + k + p)(k + p)(1 + B)\Phi^{k+p}(\lambda, m)} z^{k+p}.$$

Consequently, we complete the proof of Theorem 6. □

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Coefficient Estimates For A Certain Subclass of Bi-Univalent Functions Defined By using The Generalized Jung-Kim-Srivastava Integral Operator

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Abstract

In this paper, we investigate a new subclass $\mathcal{B}_{\Sigma}^{\beta}(\delta,\lambda;\varphi)$ of bi-univalent functions in the open unit disk U defined by the generalized Jung-Kim-Srivastava integral operator. We obtain initial coefficients bounds for functions belonging to this class.

Keywords: Analytic function, Univalent function, Bi-univalent function, Coefficient inequality.

1. Introduction

Let A denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. Further, by S we shall denote the class of all functions in A which are univalent in U. It is well known that every function $f \in S$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \qquad (z \in U)$$

and

$$f(f^{-1}(w)) = w \qquad \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4}\right)$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \ldots$$

A function $f \in A$ is said to be in Σ, the class of bi-univalent functions in U, if both $f(z)$ and $f^{-1}(z)$ are univalent in U. Lewin [13] showed that $|a_2| < 1.51$ for every function $f \in \Sigma$ given by (1). Posteriorly, Brannan and Clunie [1] improved Lewin's result and conjectured that $|a_2| \leq \sqrt{2}$ for every function $f \in \sum$ given by (1). Later, Netanyahu [15] showed that $\max_{f \in \Sigma} |a_2| = \frac{4}{3}$ The coefficient estimate problem for each of the following Taylor-Maclaurin coefficients:

$$|a_n| \quad (n \in N = \{1, 2, ...\}; n \geq 4)$$

is still an open problem (see, for details, [19]). Since then, many researchers (see [2,5,6-9,18,20,21]) investigated several interesting subclasses of the class $\sum$ and found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. Also, many researchers (see [3,4,12,16,17]) investigated the upper bounds of combination of initial coefficients. In fact, its worth to mention that by making use of the Faber polynomial coefficient expansions Jahangiri and Hamidi [10] have obtained estimates for the general coefficients $|a_n|$ for bi-univalent functions subject to certain gap series.

Let P denote the class of function of p analytic in U such that $p(0) = 1$ and $\mathrm{Re}\{p(z)\} > 0$, where

$$p(z) = 1 + p_1 z + p_2 z^2 + ... \quad (z \in U).$$

If f and g are analytic in U, we say that f is subordinate to g, written symbolically as

$$f \prec g \qquad \text{or} \qquad f(z) \prec g(z) \quad (z \in U),$$

if there exists a Schwarz function $w(z)$, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ in U such that $f(z) = g(w(z)), z \in U$.

In particular, if the function $g(z)$ is univalent in U, then we have that:

$$f(z) \prec g(z) \quad (z \in U) \quad \text{if and only if} \quad f(0) = g(0) \text{ and } f(U) \subseteq g(U).$$

Let φ be an analytic function with positive real part in the unit disk U such that

$$\varphi(0) = 1, \varphi'(0) > 0$$

and $\varphi(U)$ is symmetric with respect to the real axis and has a series expansion of the form (see [14]):

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + ... \quad (B_1 > 0).$$

Let $u(z)$ and $v(z)$ be two analytic functions in the unit disk U with $u(0) = v(0) = 0$ $|u(z)| < 1, |v(z)| < 1$, and suppose that

$$u(z) = 1 + b_1 z + b_2 z^2 + b_3 z^3 + ... \text{ and } v(w) = 1 + c_1 w + c_2 w^2 + c_3 w^3 + ... \cdot \tag{2}$$

For above functions, well-known inequalities are

$$|b_1| \leq 1, |b_2| \leq 1-|b_1|^2, |c_1| < 1 \text{ and } |c_2| \leq 1-|c_1|^2. \tag{3}$$

Further we have

$$\varphi(u(z)) = 1 + B_1 b_1 z + \left(B_1 b_2 + B_2 b_1^2\right) z^2 + \ldots \left(|z| < 1\right) \tag{4}$$

and

$$\varphi(v(w)) = 1 + B_1 c_1 w + \left(B_1 c_2 + B_2 c_1^2\right) w^2 + \ldots \left(|w| < 1\right) \tag{5}$$

In this study, we consider the generalized Jung-Kim-Srivastava integral operator Q_δ^β [11] defined by

$$\begin{aligned} Q_\delta^\beta f(z) &= \frac{\Gamma(\beta+\delta+1)}{z\Gamma(\beta)\Gamma(\delta+1)} \int_0^z t^{\delta-1}(1-\frac{t}{z})^{\beta-1} f(t)dt,\ \beta \geq 0,\ \delta > -1 \\ &= z + \sum_{n=2}^{\infty} \frac{\Gamma(\beta+\delta+1)\Gamma(\delta+n)}{\Gamma(\beta+\delta+n)\Gamma(\delta+1)} a_n z^n \end{aligned} \tag{6}$$

and for $\beta = 0$, we have $Q_\delta^0 f(z) = f(z)$.

The main object of this paper is to introduce the following new subclass of bi-univalent functions involving Jung-Kim-Srivastava integral operator Q_δ^β [11] and to obtain initial bounds for the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ of the functions belonging to this class.

2. Preliminaries and Definitions

The function class $B_\Sigma^\beta(\delta,\lambda;\varphi)$ defined as follows:

Definition 1. A function $f(z) \in \Sigma$ is said to be in the class $B_\Sigma^\beta(\delta,\lambda;\varphi)$ if and only if

$$(1-\lambda)\frac{Q_\delta^\beta f(z)}{z} + \lambda\left(Q_\delta^\beta f(z)\right)' \prec \varphi(z)$$

and

$$(1-\lambda)\frac{Q_\delta^\beta g(w)}{w} + \lambda\left(Q_\delta^\beta g(w)\right)' \prec \varphi(w)$$

where $0 \leq \lambda \leq 1,\ z,\ w \in U$ and $g(w) = f^{-1}(w)$.

Theorem 1. If $f(z)$ given by (1) is in the class $B_{\Sigma}^{\beta}(\delta,\lambda;\varphi)$, then

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{\chi + B_1\left((1+\lambda)\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+2)}{\Gamma(\beta+\delta+2)\Gamma(\delta+1)}\right)^2}} \tag{7}$$

and

$$|a_3| \leq \begin{cases} \dfrac{B_1\Gamma(\beta+\delta+3)\Gamma(\delta+1)}{(1+2\lambda)\Gamma(\beta+\delta+1)\Gamma(\delta+3)} & if \quad B_1 < \Upsilon \\ \dfrac{\chi B_1 + B_1^3(1+2\lambda)\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+3)}{\Gamma(\beta+\delta+3)\Gamma(\delta+1)}}{(1+2\lambda)\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+3)}{\Gamma(\beta+\delta+3)\Gamma(\delta+1)}\left(\chi + B_1\left((1+\lambda)\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+2)}{\Gamma(\beta+\delta+2)\Gamma(\delta+1)}\right)^2\right)} & if \quad B_1 \geq \Upsilon \end{cases} \tag{8}$$

where

$$\chi = \left|(1+2\lambda)\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+3)}{\Gamma(\beta+\delta+3)\Gamma(\delta+1)}B_1^2 - B_2\left((1+\lambda)\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+2)}{\Gamma(\beta+\delta+2)\Gamma(\delta+1)}\right)^2\right|$$

and

$$\Upsilon = \left(\frac{(1+\lambda)\Gamma(\delta+2)}{\Gamma(\beta+\delta+2)}\right)^2 \frac{\Gamma(\beta+\delta+1)\Gamma(\beta+\delta+3)}{(1+2\lambda)\Gamma(\delta+1)\Gamma(\delta+3)}.$$

Proof: Let $f(z) \in B_{\Sigma}^{\beta}(\delta,\lambda;\varphi)$. Then, there are analytic functions u and v with $u(0)=v(0)=0$, $|u(z)|<1, |v(w)|<1$ given by (2) and satisfying the following conditions:

$$(1-\lambda)\frac{Q_{\delta}^{\beta} f(z)}{z} + \lambda\left(Q_{\delta}^{\beta} f(z)\right)' = \varphi(u(z)) \tag{9}$$

and

$$(1-\lambda)\frac{Q_{\delta}^{\beta} g(w)}{w} + \lambda\left(Q_{\delta}^{\beta} g(w)\right)' = \varphi(v(w)), \tag{10}$$

where $g(w) = f^{-1}(w)$. Since

$$\begin{aligned}&(1-\lambda)\frac{Q_\delta^\beta f(z)}{z}+\lambda\left(Q_\delta^\beta f(z)\right)'\\ &=1+(1+\lambda)\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+2)}{\Gamma(\beta+\delta+2)\Gamma(\delta+1)}a_2z+(1+2\lambda)\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+3)}{\Gamma(\beta+\delta+3)\Gamma(\delta+1)}a_3z^2+\ldots\end{aligned} \tag{11}$$

and

$$\begin{aligned}&(1-\lambda)\frac{Q_\delta^\beta g(w)}{w}+\lambda\left(Q_\delta^\beta g(w)\right)'\\ &=1-(1+\lambda)\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+2)}{\Gamma(\beta+\delta+2)\Gamma(\delta+1)}a_2w+(1+2\lambda)\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+3)}{\Gamma(\beta+\delta+3)\Gamma(\delta+1)}\left(2a_2^2-a_3\right)w^2+\ldots,\end{aligned} \tag{12}$$

it follows from (4), (5), (11) and (12) that

$$(1+\lambda)\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+2)}{\Gamma(\beta+\delta+2)\Gamma(\delta+1)}a_2=B_1b_1, \tag{13}$$

$$(1+2\lambda)\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+3)}{\Gamma(\beta+\delta+3)\Gamma(\delta+1)}a_3=B_1b_2+B_2b_1^2, \tag{14}$$

$$-(1+\lambda)\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+2)}{\Gamma(\beta+\delta+2)\Gamma(\delta+1)}a_2=B_1c_1, \tag{15}$$

and

$$(1+2\lambda)\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+3)}{\Gamma(\beta+\delta+3)\Gamma(\delta+1)}\left(2a_2^2-a_3\right)=B_1c_2+B_2c_1^2. \tag{16}$$

From (13) and (15), we get

$$c_1=-b_1 \tag{17}$$

$$2\left[(1+\lambda)\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+2)}{\Gamma(\beta+\delta+2)\Gamma(\delta+1)}\right]^2a_2^2=B_1^2\left(b_1^2+c_1^2\right). \tag{18}$$

By adding (14) to (16), we have

$$2(1+2\lambda)\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+3)}{\Gamma(\beta+\delta+3)\Gamma(\delta+1)}a_2^2=B_1\left(b_2+c_2\right)+B_2\left(b_1^2+c_1^2\right). \tag{19}$$

Therefore, from equalities (18) and (19) we find that

$$\left[2(1+2\lambda)\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+3)}{\Gamma(\beta+\delta+3)\Gamma(\delta+1)}B_1^2-2B_2\left((1+\lambda)\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+2)}{\Gamma(\beta+\delta+2)\Gamma(\delta+1)}\right)^2\right]a_2^2=B_1^3(b_2+c_2) \tag{20}$$

Then, in view of (13), (17) and (3), we obtain

$$\left|2(1+2\lambda)\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+3)}{\Gamma(\beta+\delta+3)\Gamma(\delta+1)}B_1^2-2B_2\left((1+\lambda)\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+2)}{\Gamma(\beta+\delta+2)\Gamma(\delta+1)}\right)^2\right||a_2|^2$$
$$\le B_1^3(|b_2|+|c_2|)\le 2B_1^3\left(1-|b_1|^2\right)=2B_1^3-2B_1\left((1+\lambda)\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+2)}{\Gamma(\beta+\delta+2)\Gamma(\delta+1)}\right)^2|a_2|^2.$$

Thus, we get

$$|a_2|\le\frac{B_1\sqrt{B_1}}{\sqrt{\chi+B_1\left((1+\lambda)\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+2)}{\Gamma(\beta+\delta+2)\Gamma(\delta+1)}\right)^2}},$$

where

$$\chi=\left|(1+2\lambda)\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+3)}{\Gamma(\beta+\delta+3)\Gamma(\delta+1)}B_1^2-B_2\left((1+\lambda)\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+2)}{\Gamma(\beta+\delta+2)\Gamma(\delta+1)}\right)^2\right|.$$

Next, in order to find the bound on $|a_3|$, subtracting (16) from (14) and using (17), we get

$$2(1+2\lambda)\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+3)}{\Gamma(\beta+\delta+3)\Gamma(\delta+1)}a_3=2(1+2\lambda)\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+3)}{\Gamma(\beta+\delta+3)\Gamma(\delta+1)}a_2^2+B_1(b_2-c_2). \tag{21}$$

Then in view of (3) and (17), we have

$$2(1+2\lambda)\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+3)}{\Gamma(\beta+\delta+3)\Gamma(\delta+1)}|a_3|\le 2(1+2\lambda)\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+3)}{\Gamma(\beta+\delta+3)\Gamma(\delta+1)}|a_2|^2+B_1(|b_2|+|c_2|)$$
$$\le 2(1+2\lambda)\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+3)}{\Gamma(\beta+\delta+3)\Gamma(\delta+1)}|a_2|^2+2B_1\left(1-|b_1|^2\right).$$

From (13), we immediately have

$$B_1(1+2\lambda)\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+3)}{\Gamma(\beta+\delta+3)\Gamma(\delta+1)}|a_3|$$
$$\leq\left[B_1(1+2\lambda)\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+3)}{\Gamma(\beta+\delta+3)\Gamma(\delta+1)}-\left((1+\lambda)\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+2)}{\Gamma(\beta+\delta+2)\Gamma(\delta+1)}\right)^2\right]|a_2|^2+B_1^2.$$

Now the assertion (8) follows from (7). This evidently completes the proof of Theorem 1.

By taking $\lambda=1$ in Theorem 1, we have

Corollary 1. If $f(z)$ given by (1) is in the class, $B_\Sigma^\beta(\delta,\lambda;\varphi)$ then

$$|a_2|\leq\frac{B_1\sqrt{B_1}}{\sqrt{\chi+4B_1\left(\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+2)}{\Gamma(\beta+\delta+2)\Gamma(\delta+1)}\right)^2}}$$

and

$$|a_3|\leq\begin{cases}\dfrac{B_1\Gamma(\beta+\delta+3)\Gamma(\delta+1)}{3\Gamma(\beta+\delta+1)\Gamma(\delta+3)} & if \quad B_1<\Upsilon\\[2ex] \dfrac{\chi B_1+3B_1^3\dfrac{\Gamma(\beta+\delta+1)\Gamma(\delta+3)}{\Gamma(\beta+\delta+3)\Gamma(\delta+1)}}{3\dfrac{\Gamma(\beta+\delta+1)\Gamma(\delta+3)}{\Gamma(\beta+\delta+3)\Gamma(\delta+1)}\left(\chi+4B_1\left(\dfrac{\Gamma(\beta+\delta+1)\Gamma(\delta+2)}{\Gamma(\beta+\delta+2)\Gamma(\delta+1)}\right)^2\right)} & if \quad B_1\geq\Upsilon\end{cases}$$

where

$$\chi=\left|3\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+3)}{\Gamma(\beta+\delta+3)\Gamma(\delta+1)}B_1^2-4B_2\left(\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+2)}{\Gamma(\beta+\delta+2)\Gamma(\delta+1)}\right)^2\right|$$

and

$$\Upsilon=\left(\frac{2\Gamma(\delta+2)}{\Gamma(\beta+\delta+2)}\right)^2\frac{\Gamma(\beta+\delta+1)\Gamma(\beta+\delta+3)}{3\Gamma(\delta+1)\Gamma(\delta+3)}.$$

Putting $\beta=0$ in Theorem 1, we have

Corollary 2. If $f(z)$ given by (1) is in the class $B_{\Sigma}^{0}(\delta,\lambda;\varphi)=B_{\Sigma}^{\beta}(\lambda;\varphi)$, then

$$|a_2|\leq\frac{B_1\sqrt{B_1}}{\sqrt{\left|(1+2\lambda)B_1^2-B_2(1+\lambda)^2\right|+B_1(1+\lambda)^2}}$$

and

$$|a_3|\leq\begin{cases}\dfrac{B_1}{(1+2\lambda)} & if \quad B_1<\dfrac{(1+\lambda)^2}{1+2\lambda}\\[2ex] \dfrac{\left|(1+2\lambda)B_1^2-B_2(1+\lambda)^2\right|B_1+B_1^3(1+2\lambda)}{(1+2\lambda)\left(\left|(1+2\lambda)B_1^2-B_2(1+\lambda)^2\right|+B_1(1+\lambda)^2\right)} & if \quad B_1\geq\dfrac{(1+\lambda)^2}{1+2\lambda}\end{cases}.$$

Putting $\beta=0$ in Corollary 1, we have

Corollary 3. If $f(z)$ given by (1) is in the class $B_{\Sigma}^{0}(\delta,1;\varphi)=H_{\Sigma}(\varphi)$, then

$$|a_2|\leq\frac{B_1\sqrt{B_1}}{\sqrt{\chi+4B_1}}$$

and

$$|a_3|\leq\begin{cases}\dfrac{B_1}{3} & if \quad B_1<\dfrac{4}{3}\\[2ex] \dfrac{\left|3B_1^2-4B_2{}^2\right|B_1+3B_1^3}{3\left(\left|3B_1^2-4B_2{}^2\right|+4B_1\right)} & if \quad B_1\geq\dfrac{4}{3}\end{cases}.$$

3. References

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Coefficient Inequalities For A Subclass of Bi-univalent Functions Involving Laguerre Polynomials

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Abstract

In this article, we establish the bounds for the initial Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ for a new family of analytic and bi-univalent functions in the open unit disk which involve Laguerre polynomials. Furthermore, we investigate the special cases and consequences for the new family functions.

Keywords: Analytic and bi-univalent functions, subordination, coefficient inequality, Laguerre polynomials.

1. Introduction and Preliminaries

Let A represents the class of functions whose members are of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad (z \in U), \tag{1}$$

which are analytic in $U = \{z \in \mathbb{C} : |z| < 1\}$, and let S be the subclass of A whose members are univalent in U. The Koebe one quarter theorem [3] ensures that the image of U under every univalent function $f \in$ A contains a disk of radius $\frac{1}{4}$. Thus every univalent function f has an inverse f^{-1} satisfying

$$f^{-1}(f(z)) = z, \ (z \in U) \ \text{ and } \ f(f^{-1}(\omega)) = \omega, \ (|\omega| < r_0(f), \ r_0(f) \geq \tfrac{1}{4}).$$

A function $f \in$ A is said to be bi-univalent in U if both f and f^{-1} are univalent in U, and let Σ denote the class of bi–univalent functions defined in the unit disk U. Since $f \in \Sigma$ has the Maclaurin series given by (1), a computation shows that its inverse $g = f^{-1}$ has the expansion

$$g(\omega) = f^{-1}(\omega) = \omega - a_2\omega^2 + (2{a_2}^2 - a_3)\omega^3 + \cdots. \tag{2}$$

We notice that the class Σ is not empty. For instance, the functions

$$f_1(z) = \frac{z}{z-1}, \quad f_2(z) = \frac{1}{2}\log\frac{1+z}{1-z}, \quad f_3(z) = -\log(1-z)$$

with their corresponding inverses

$$f_1^{-1}(\omega) = \frac{\omega}{1+\omega'}, \; f_2^{-1}(\omega) = \frac{e^{2\omega}-1}{e^{2\omega}+1} \;, \; f_3^{-1}(\omega) = \frac{e^{\omega}-1}{e^{\omega}}$$

are elements of Σ. However, the Koebe function is not a member of Σ. Lately, Srivastava et al. $[16]$ have essentially revived the study of analytic and bi-univalent functions; this was followed by such works as those of $[1-15]$. Several authors have introduced and examined subclasses of bi-univalent functions and obtained bounds for the initial coefficients (see $[16]$, bi-close-to-convex functions $[6,9]$, and bi-prestarlike functions by Jahangiri and Hamidi $[7]$.

Let f and g be analytic functions in U. We define that the function f is subordinate to g in U and denoted by

$$f(z) \prec g(z) \quad (z \in U),$$

if there exists a Schwarz function w, which is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$ such that

$$f(z) = g(w(z)) \quad (z \in U).$$

If g is a univalent function in Δ, then

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U).$$

The generalized Laguerre polynomial $L_n^{\gamma}(\beta)$ is the polynomial solution $\phi(\beta)$ of the differential equation (see $[10]$)

$$\beta\phi'' + (1+\gamma-\beta)\phi' + n\phi = 0,$$

where $\gamma > -1$ and n is non-negative integers.

The generating function of generalized Laguerre polynomial $L_n^{\gamma}(\beta)$ is defined by

$$H_{\gamma}(\beta, z) = \sum_{n=0}^{\infty} L_n^{\gamma}(\beta) z^n = \frac{e^{-\frac{\beta z}{1-z}}}{(1-z)^{\gamma+1}}, \tag{3}$$

where $\beta \in \mathbb{R}$ and $z \in U$. Generalized Laguerre polynomials can also be defined by the following recurrence relations:

$$L_{n+1}^{\gamma}(\beta) = \frac{2n+1+\gamma-\beta}{n+1} L_n^{\gamma}(\beta) - \frac{n+\gamma}{n+1} L_{n-1}^{\gamma}(\beta) \quad (n \geq 1), \tag{4}$$

with the initial conditions

$$L_0^{\gamma}(\beta) = 1, \quad L_1^{\gamma}(\beta) = 1 + \gamma - \beta \text{ and } L_1^{\gamma}(\beta) = \frac{\beta^2}{2} - (\gamma+2)\beta + \frac{(\gamma+1)(\gamma+1)}{2}. \tag{5}$$

Clearly, when $\gamma = 0$ the generalized Laguerre polynomials leads to the simply Laguerre polynomial, i.e., $L_n^0(\beta) = L_n(\beta)$.

The analytic function $h(z)$ with positive real part in U such that $h(0) = 1, h'(0) > 0$ and $h(U)$ is symmetric with respect to real axis, which is of the type:

$$h(z) = 1 + e_1 z + e_2 z^2 + \cdots \tag{6}$$

where

$$e_1 = 1 + \gamma - \beta, \; e_2 = \frac{\beta^2}{2} - (\gamma+2)\beta + \frac{(\gamma+1)(\gamma+1)}{2}. \tag{7}$$

First, we define a new subclass of bi-univalent functions in the open unit disk, associated with Laguerre polynomials as below.

Definition 1. For $\tau \in \mathbb{C}\backslash\{0\}$, $0 \leq \vartheta \leq 1$ and h is analytic in U, $h(0) = 1$, a function $f \in \Sigma$ the form (1) is said to be in the class $M_{\Sigma}(\tau, \vartheta, e_1, e_2)$ if the following subordinations hold:

$$1 + \frac{1}{\tau}(f'(z) + \vartheta z f''(z) - 1) \prec h(z) \tag{8}$$

and

$$1 + \frac{1}{\tau}(g'(\omega) + \vartheta \omega f''(\omega) - 1) \prec h(\omega) \tag{9}$$

where $z, \omega \in U$, e_1, e_2 are given by (7), and $g = f^{-1}$ is given by (2).

2. Initial Taylor Coefficients Estimates for the Functions of $M_{\Sigma}(\tau, \vartheta, e_1, e_2)$

To obtain our first results, we need the following lemma:

Lemma 1 ([13], p.172). Assume that $w(z) = \sum_{n=1}^{\infty} w_n z^n$, $z \in U$, is an analytic function in U such that $|w(z)| < 1$ for all $z \in U$. Then,

$$|w_1| \le 1, \ \ |w_n| \le 1 - |w_1|^2, \quad n = 2,3, \ldots.$$

In the following result, we obtain upper bounds for the modules of the first two coefficients for the functions to belong to a class $M_\Sigma(\tau, \vartheta, e_1, e_2)$.

Theorem 1. Assume that $\tau \in \mathbb{C}\backslash\{0\}$, $0 \le \vartheta \le 1$. If $f \in \Sigma$ of the form (1) is in the family $M_\Sigma(\tau, \vartheta, e_1, e_2)$**,** with $h(z) = 1 + e_1 z + e_2 z^2 + \cdots$, then

$$|a_2| \le \frac{|\tau| e_1 \sqrt{e_1}}{\sqrt{|3\tau(1+2\vartheta)e_1{}^2 - 4(1+\vartheta)^2 e_2|}}, \tag{10}$$

and

$$|a_3| \le \frac{e_1|\tau|}{3|1+2\vartheta|} + \frac{|\tau|^2 e_1{}^2}{4|1+\vartheta|^2}, \tag{11}$$

where e_1, e_2 are given by (7).

Proof. Suppose that $f \in M_\Sigma(\tau, \vartheta, e_1, e_2)$. From the definition in formulas (8) and (9), we have

$$1 + \frac{1}{\tau}(f'(z) + \vartheta z f''(z) - 1) = h(\varphi(z)) \tag{12}$$

and

$$1 + \frac{1}{\tau}(g'(\omega) + \vartheta \omega g''(\omega) - 1) = h(x(\omega)), \tag{13}$$

where there exsist two holomorphic functions $\varphi, \chi\colon U \to U$ given by

$$\varphi(z) = r_1 z + r_2 z^2 + \cdots, \tag{14}$$

$$x(\omega) = s_1 \omega + s_2 \omega^2 + \cdots, \tag{15}$$

with $\varphi(0) = 0 = x(0)$, and $|\varphi(z)| < 1$, $|x(\omega)| < 1$, for all $z, \omega \in U$. From Lemma 1 it follows that

$$|r_j| \le 1 \ \text{ and } \ |s_j| \le 1, \text{ for all } \ j \in \mathbb{N}. \tag{16}$$

Replacing (14) and (15) in (12) and (13), respectively, we have

$$1 + \frac{1}{\tau}(f'(z) + \vartheta z f''(z) - 1) = 1 + e_1 \varphi(z) + e_2 \varphi^2(z) + \cdots, \tag{17}$$

and

$$1+\frac{1}{\tau}(g'(\omega)+\vartheta\omega g''(\omega)-1)=\ 1+e_1x(\omega)+e_2x^2(\omega)+\cdots. \tag{18}$$

In view of (1) and (2), from (17) and (18), we obtain

$$1+\frac{1}{\tau}(2a_2(1+\vartheta)z+3a_3(1+2\vartheta)z^2)=1+e_1r_1z+[e_1r_2+e_2{r_1}^2]z^2$$

and

$$1+\frac{1}{\tau}(-2a_2(1+\vartheta)\omega+3(2{a_2}^2-a_3)(1+2\vartheta)\omega^2)$$

$$=1+e_1s_1\omega+[e_1s_2+e_2{s_1}^2]\omega^2$$

which yields the following relations :

$$2a_2(1+\vartheta)=\tau e_1r_1\ , \tag{19}$$

$$3a_3(1+2\vartheta)=\tau e_1r_2+\tau e_2{r_1}^2, \tag{20}$$

and

$$-2a_2(1+\vartheta)=\tau e_1s_1\ , \tag{21}$$

$$3(2{a_2}^2-a_3)(1+2\vartheta)=\ \tau e_1s_2+\tau e_2{s_1}^2. \tag{22}$$

From (19) and (21), it follows that

$$r_1=-s_1, \tag{23}$$

and

$$8{a_2}^2(1+\vartheta)^2=\tau^2{e_1}^2({r_1}^2+{s_1}^2)$$

$${a_2}^2=\frac{\tau^2{e_1}^2({r_1}^2+{s_1}^2)}{8(1+\vartheta)^2}. \tag{24}$$

Adding (20) and (22), using (24), we obtain

$${a_2}^2=\frac{\tau^2{e_1}^3(r_2+s_2)}{6\tau(1+2\vartheta){e_1}^2-8(1+\vartheta)^2e_2}\ . \tag{25}$$

Applying (16) for the coefficients r_2 and s_2 and using (7), we obtain the inequality (10).

By subtracting (22) from (20), using (23) and (24), we get

$$\begin{aligned} a_3 &= \frac{\tau e_1(r_2-s_2)+\tau e_2({r_1}^2-{s_1}^2)}{6(1+2\vartheta)} + {a_2}^2 \\ &= \frac{\tau e_1(r_2-s_2)+\tau e_2({r_1}^2-{s_1}^2)}{6(1+2\vartheta)} + \frac{\tau^2 {e_1}^2({r_1}^2+{s_1}^2)}{8(1+\vartheta)^2}. \end{aligned} \tag{26}$$

Using (7) and once again applying (16) for the coefficients r_1, r_2 and s_1 , s_2 , we deduce the required inequality (11).

For $\tau = 1$ in Theorem 1, we obtain the following corollary.

Corollary 1. Assume that $0 \le \vartheta \le 1$. If $f \in \Sigma$ of the form (1) is in the family $M_\Sigma(1, \vartheta, e_1, e_2)$**,** with $h(z) = 1 + e_1 z + e_2 z^2 + \cdots$, then

$$|a_2| \le \frac{e_1\sqrt{e_1}}{\sqrt{|3(1+2\vartheta){e_1}^2 - 4(1+\vartheta)^2 e_2|}},$$

and

$$|a_3| \le \frac{e_1}{3|1+2\vartheta|} + \frac{{e_1}^2}{4|1+\vartheta|^2},$$

where e_1, e_2 are given by (7).

For $\tau = \vartheta = 1$ in Theorem 1, we get the following corollary.

Corollary 2. If $f \in \Sigma$ of the form (1) is in the family $M_\Sigma(1,1, e_1, e_2)$**,** with $h(z) = 1 + e_1 z + e_2 z^2 + \cdots$, then

$$|a_2| \le \frac{e_1\sqrt{e_1}}{\sqrt{|9{e_1}^2 - 16 e_2|}},$$

and

$$|a_3| \le \frac{e_1}{9} + \frac{{e_1}^2}{16},$$

where e_1, e_2 are given by (7).

For $\tau = 1 \; and \; \vartheta = 0$ in Theorem 1, we have the following corollary.

Corollary 3. If $f \in \Sigma$ of the form (1) is in the family $M_\Sigma(1,0, e_1, e_2)$**,** with $h(z) = 1 + e_1 z + e_2 z^2 + \cdots$, then

$$|a_2| \le \frac{e_1\sqrt{e_1}}{\sqrt{|3{e_1}^2 - 4 e_2|}},$$

and

$$|a_3| \leq \frac{e_1}{3} + \frac{{e_1}^2}{4},$$

where e_1, e_2 are given by (7).

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Commutativity of Fourth-Order Discrete-Time Linear Time-Varying Systems

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Abstract

In this proceeding, two fourth-order discrete-time linear time-varying systems (systems A and B) are considered. These systems are described by fourth-order linear difference equations. Formulations of the cascaded-connected systems AB and BA are presented. Necessary conditions of commutativity of fourth-order discrete-time linear time-varying systems are given.

Keywords: Commutativity, Discrete-time, Fourth-order, Linear system

1. Introduction

Cascade-connected system is a common method for the realization of many engineering designs and this is important for the synthesis of especially electronic and electrical systems. The order of connection of subsystems may be arbitrary or depend on the specific design methods and traditional techniques. However, when system performance parameters such as sensitivity, linearity, stability, noise quality, and robustness are important, drastic differences may occur. Therefore, the proper order should be chosen to obtain the best performance whilest the main function of the overall system remains the same (commutativity). For this reason, commutativity is important with regard to technical applications.

As shown in Fig. 1, by changing the connection order of two cascade-connected time-varying linear systems A and B, we say that A and B are commutative systems and (A, B) constitutes a commutative pair if input-output relations of the assembled systems AB and BA are identical.

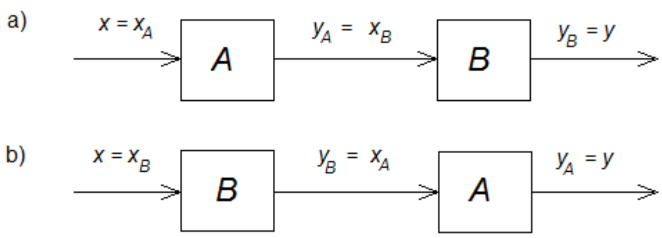

Figure 1. Cascade connections of differential systems

The cooncept of commutativity was studied for the first time by Marshal [1] in 1977. He developed commutativity conditions of first-order continuous-time linear time-varying systems. After that, the results for first-order systems were extended by Koksal for second-order [2], third-order [3] and fourth-order [4] continuous-time lienar time-varying systems in 1982, 1984 and 1985, respectively. After about two and a half decade, commutativity of Euler differential systems was investigated, and explicit commutativity conditions of fifth-order continuous-time linear time-varying ystems were studied in [5]. The last literature about the commutativity of continuous time-varying linear systems, sixth-order systems were studied in [6] in 2021.

Even though there are many papers on the commutativity of continuous-time (analog) linear time-varying systems, there are only a few studies on the commutativity of discrete-time (digital) linear time-varying systems. The trend in the new technology is moving to the digital world from the analog world. There are many advantages of digital systems over analog systems. Some of these advantages are reproduciblity of the results and accuracy, easy of design, flexibility and functionality, programmability, speed, economy, etc. From this point of view, the investigation of commutativity conditions of discrete-time linear time-varying systems is important.

The first literature on the commutativity of digital systems appeared in 2015 [7] where explicit commutativity conditions of second-order discrete-time linear time-varying systems were presented. After that, in 2019, commutativity of first-order discrete-time linear time-varying system was studied in [8] and theoritical reults were supported by illustrative examples. Finally, in [9], third-order discrete-time linear time-varying systems are considered. Some commutativity conditions are given.

2. Commutativity Conditions

Let systems A and B be described by the following linear fourth-order difference equations:

$$A: a_4(k)y_A(k+4) + a_3(k)y_A(k+3) + a_2(k)y_A(k+2) + a_1(k)y_A(k+1) + a_0(k)y_A(k) = x_A(k), \quad (1)$$
$$B: b_4(k)y_B(k+4) + b_3(k)y_B(k+3) + b_2(k)y_B(k+2) + b_1(k)y_B(k+1) + b_0(k)y_B(k) = x_B(k). \quad (2)$$

In Eq. (2), writing $k+1$, $k+2$, $k+3$, $k+4$ instead of k,

$$\begin{gathered}
b_4(k+1)y_B(k+5) + b_3(k+1)y_B(k+4) + b_2(k+1)y_B(k+3) + b_1(k+1)y_B(k+2) \\
+b_0(k+1)y_B(k+1) = x_B(k+1) = y_A(k+1), \\
b_4(k+2)y_B(k+6) + b_3(k+2)y_B(k+5) + b_2(k+2)y_B(k+4) + b_1(k+2)y_B(k+3) \\
+b_0(k+2)y_B(k+2) = x_B(k+2) = y_A(k+2), \\
b_4(k+3)y_B(k+7) + b_3(k+3)y_B(k+6) + b_2(k+3)y_B(k+5) + b_1(k+3)y_B(k+4) \\
+b_0(k+3)y_B(k+3) = x_B(k+3) = y_A(k+3), \\
b_4(k+4)y_B(k+8) + b_3(k+4)y_B(k+7) + b_2(k+4)y_B(k+6) + b_1(k+4)y_B(k+5) \\
+b_0(k+4)y_B(k+4) = x_B(k+4) = y_A(k+4).
\end{gathered}$$

Substituting these values of $y_A(i)$ for $i = k+1, k+2, k+3, k+4$ in Eq. (1)

$$a_4(k)[b_4(k+4)y_B(k+8)+b_3(k+4)y_B(k+7)+b_2(k+4)y_B(k+6)+b_1(k+4)y_B(k+5)$$
$$+b_0(k+4)y_B(k+4)]+a_3(k)[b_4(k+3)y_B(k+7)+b_3(k+3)y_B(k+6)+b_2(k+3)y_B(k+5)$$
$$+b_1(k+3)y_B(k+4)+b_0(k+3)y_B(k+3)]+a_2(k)[b_4(k+2)y_B(k+6)+b_3(k+2)y_B(k+5)$$
$$+b_2(k+2)y_B(k+4)+b_1(k+2)y_B(k+3)+b_0(k+2)y_B(k+2)]+a_1(k)[b_4(k+1)y_B(k+5)$$
$$+b_3(k+1)y_B(k+4)+b_2(k+1)y_B(k+3)+b_1(k+1)y_B(k+2)+b_0(k+1)y_B(k+1)]$$
$$+a_0(k)[b_4(k)y_B(k+4)+b_3(k)y_B(k+3)+b_2(k)y_B(k+2)+b_1(k)y_B(k+1)+b_0(k)y_B(k)]$$
$$=x_A(k).$$

Using $x_A(k)=x(k), x_B(k)=y_A(k), y_B(k)=y(k)$ and rearranging the terms, we obtain the formula of system AB as follows:

$$AB: a_4(k)\ b_4(k+4)y(k+8)$$
$$+[a_4(k)\ b_3(k+4)+a_3(k)b_4(k+3)]y(k+7)$$
$$+[a_4(k)b_2(k+4)+a_3(k)b_3(k+3)+a_2(k)b_4(k+2)]y(k+6)$$
$$+[a_4(k)b_1(k+4)+a_3(k)b_2(k+3)+a_2(k)b_3(k+2)+a_1(k)b_4(k+1)]y(k+5)$$
$$+[a_4(k)b_0(k+4)+a_3(k)b_1(k+3)++a_2(k)b_2(k+2)a_1(k)b_3(k+1)+a_0(k)b_4(k)]y(k+4)$$
$$+[a_3(k)b_0(k+3)+a_2(k)b_1(k+2)+a_1(k)b_2(k+1)+a_0(k)b_3(k)]y(k+3)$$
$$+[a_2(k)b_0(k+2)+a_1(k)b_1(k+1)+a_0(k)b_2(k)]y(k+2)$$
$$+[a_1(k)b_0(k+1)+a_0(k)b_1(k)]y(k+1)$$
$$+a_0(k)b_0(k)y(k)=x(k). \tag{3}$$

To obtain the difference equation describing the connection AB, we may either follow the above procedure for the connection BA, rather than this we may better the inter-change the coefficients a_i and b_i; thus, we obtain

$$BA: b_4(k)a_4(k+4)y(k+8)$$
$$+[b_4(k)a_3(k+4)+b_3(k)a_4(k+3)]y(k+7)$$
$$+[b_4(k)a_2(k+4)+b_3(k)a_3(k+3)+b_2(k)a_4(k+2)]y(k+6)$$
$$+[b_4(k)a_1(k+4)+b_3(k)a_2(k+3)+b_2(k)a_3(k+2)+b_1(k)a_4(k+1)]y(k+5)$$
$$+[b_4(k)a_0(k+4)+b_3(k)a_1(k+3)+b_2(k)a_2(k+2)+b_1(k)a_3(k+1)+b_0(k)a_4(k)]y(k+4)$$
$$+[b_3(k)a_0(k+3)+b_2(k)a_1(k+2)+b_1(k)a_2(k+1)+b_0(k)a_3(k)]y(k+3)$$
$$+[b_2(k)a_0(k+2)+b_1(k)a_1(k+1)+b_0(k)a_2(k)]y(k+2)$$
$$+[b_1(k)a_0(k+1)+b_0(k)a_1(k)]y(k+1)$$
$$+b_0(k)a_0(k)y(k)=x(k). \tag{4}$$

Comparing the coefficients of $y(i)$ for connections AB and BA, with $i=k+8, k+7, \dots k+1$ (the case of $i=k$ is ommitted since it gives an identity), we obtain 8 equations between $2x5=10$ coefficients of the subsytems A and B:

$$a_4(k)\ b_4(k+4)=b_4(k)a_4(k+4), \tag{5.1}$$

$$a_4(k)b_3(k+4) + a_3(k)b_4(k+3) = b_4(k)a_3(k+4) + b_3(k)a_4(k+3), \tag{5.2}$$

$$\begin{aligned} &a_4(k)b_2(k+4) + a_3(k)b_3(k+3) + a_2(k)b_4(k+2) \\ &= b_4(k)a_2(k+4) + b_3(k)a_3(k+3) + b_2(k)a_4(k+2), \end{aligned} \tag{5.3}$$

$$\begin{aligned} &a_4(k)b_1(k+4) + a_3(k)b_2(k+3) + a_2(k)b_3(k+2) + a_1(k)b_4(k+1) \\ &= b_4(k)a_1(k+4) + b_3(k)a_2(k+3) + b_2(k)a_3(k+2) + b_1(k)a_4(k+1), \end{aligned} \tag{5.4}$$

$$\begin{aligned} &a_4(k)b_0(k+4) + a_3(k)b_1(k+3) + a_2(k)b_2(k+2) + a_1(k)b_3(k+1) + a_0(k)b_4(k) \\ &= b_4(k)a_0(k+4) + b_3(k)a_1(k+3) + b_2(k)a_2(k+2) + b_1(k)a_3(k+1) + b_0(k)a_4(k), \end{aligned} \tag{5.5}$$

$$\begin{aligned} &a_3(k)b_0(k+3) + a_2(k)b_1(k+2) + a_1(k)b_2(k+1) + a_0(k)b_3(k) \\ &= b_3(k)a_0(k+3) + b_2(k)a_1(k+2) + b_1(k)a_2(k+1) + b_0(k)a_3(k), \end{aligned} \tag{5.6}$$

$$\begin{aligned} &a_2(k)b_0(k+2) + a_1(k)b_1(k+1) + a_0(k)b_2(k) \\ &= b_2(k)a_0(k+2) + b_1(k)a_1(k+1) + b_0(k)a_2(k), \end{aligned} \tag{5.7}$$

$$a_1(k)b_0(k+1) + a_0(k)b_1(k) = b_1(k)a_0(k+1) + b_0(k)a_1(k), \tag{5.8}$$

3. Solution of Commutativity Conditions

The problem of finding all the fourth order commutative conjugates of a given fourth order discrete time system A, we consider the solutions of the difference equations in Eq. (5). It is assumed that the coefficients of A is known. The coefficients of B are found in the following order:

i) Assuming $b_4(0)$, $b_4(1)$, $b_4(2)$, $b_4(3)$ are arbitrary nonzero constants, solve the fourth-degree difference equation (5.1) for $b_4(k)$ for $k \geq 4$.

ii) Knowing $b_4(k)$ and assuming $b_3(0)$, $b_3(1)$, $b_3(2)$, $b_3(3)$ are arbitrary constants, solve the fourth-degree difference equation (5.2) for $b_3(k)$ for $k \geq 4$.

iii) Knowing $b_4(k)$, $b_3(k)$, and assuming $b_2(0)$, $b_2(1)$, $b_2(2)$, $b_2(3)$ are arbitrary constants, solve the fourth-degree difference equation (5.3) for $b_2(k)$ for $k \geq 4$.

iv) Knowing $b_4(k)$, $b_3(k)$, $b_2(k)$, and assuming $b_1(0)$, $b_1(1)$, $b_1(2)$, $b_1(3)$ are arbitrary constants, solve the fourth-degree difference equation (5.4) for $b_1(k)$ for $k \geq 4$.

v) Knowing $b_4(k)$, $b_3(k)$, $b_2(k)$, $b_1(k)$, and assuming $b_0(0)$, $b_0(1)$, $b_0(2)$, $b_0(3)$ are arbitrary constants, solve the fourth-degree difference equation (5.5) for $b_0(k)$ for $k \geq 4$.

The method of z-transform may be used to find the solutions. Since the complexity of difference equations is rather high and it is further increasing with substitutions of previous solutions each time, a proper computer toolbox, in MATLAB or Mathematica for example, is absolutely needed to obtain correct results.

4. Existence of the Commutative Conjugate

It is mentioned in the previous section that $5x4 = 20$ arbitrary constants are needed for finding the coefficients of the commutative conjugates B of the subsystem A. On the other hand, there are three remaining equations in (5) which have not been used, namely Eqs. (5.6), (5.7), (5.8). For the existence of the commutative conjugate B, these equations must also be satisfied for all $k \geq 0$. To fix the 20 arbitrary selected initial values at least 20 equations are needed. These equations can be chosen in the subset of $\{\text{Eqs.}\,(5.6), (5.7), (5.8)\ with\ k \in [0, 19]\}$. If a solution is found, remember that Eqs. (5.6), (5.7), (5.8) must be satisfied for all $k \geq 0$. With the found 20 initial values of coefficients of B, if the remaining equations which have not been used in their computation are not satisfied or result with contradictions, this is sufficient to decide that the given fourth order discrete time system A has not a fourth order commutative conjugate other than itself.

5. Conclusions

The formulation of commutativity conditions is presented for 4-th order linear discrete time systems. It is shown that unlike the case of continuous time systems, the problem of finding the commutative conjugates of a given 4-th order discrete time system is very cumbersome. General formulas for explicit commutativity conditions cannot be obtained as for the continuous time varying systems [3], [5], [6]. And it is hardly possible to find them for discrete time case by hand computation. Although the commutativity problem for first order discrete time systems is fully solved [8], no such explicit results have been presented so far for the higher order case. Therefore, Computer tools are necessary such as MATLAB or Mathematica. As it is shown in this presentation for the fourth-order discrete time systems, the conditions of commutativity are shown to be very stringent and very few systems have commutative conjugates. The difficulty will surely increase if the case of nonzero initial conditions is treated. This case and illustrative example which are relevant to the subject is reserved for future work and it will be presented in another issue.

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Convulation Properties of Certain Sublasses of Multivalent Functions

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Abstract

In this study, we consider certain subclasses of multivalent functions defined by Deniz- Özkan differential operator. We obtain convulation (or modified Hadamard products) of functions belonging to these subclasses

Keywords: Modified Hadamard product, Differential operator.

1. Introduction and definitions

Let $\mathcal{A}(k,p)$ denote the class of functions normalized by

$$f(z) = z^p + \sum_{n=k+p}^{\infty} a_n z^n \qquad \left(p,k \in \mathbb{N} := \{1,2,3,...\}\right) \tag{1}$$

which are analytic and $p-valent$ in the open unit disk $\mathcal{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$.

Let $f(z)$ and $g(z)$ be analytic in $\mathcal{U}$. Then we say that the function f is subordinate to g if there exists a Schwarz function $w(z)$, analytic in $\mathcal{U}$ with $w(0)=0, |w(z)|<1$ such that $f(z) = g(w(z)) \quad (z \in \mathcal{U})$. We denote this subordination $f \prec g$ or $f(z) \prec g(z) \ (z \in \mathcal{U})$.

In particular, if the function g is univalent in $\mathcal{U}$, the above subordination is equivalent to $f(0) = g(0)$, $f(\mathcal{U}) \subset g(\mathcal{U})$.

For $f \in \mathcal{A}(k,p)$ given by (1) and $g(z)$ given by

$$g(z) = z^p + \sum_{n=k+p}^{\infty} b_n z^n \quad \left(p,k \in \mathbb{N} := \{1,2,3,...\}\right) \tag{2}$$

their convolution (or Hadamard product), denoted by $(f * g)$, is defined as

$$(f*g)(z) := z^p + \sum_{n=k+p}^{\infty} a_n b_n z^n =: (g*f)(z) \quad (z \in \mathcal{U}). \tag{3}$$

Note that $f*g \in \mathcal{A}(k,p)$. In particular, we set

$$\mathcal{A}(p,1) := \mathcal{A}_p, \quad \mathcal{A}(1,k) := \mathcal{A}(k), \quad \mathcal{A}(1,1) := \mathcal{A}_1 = \mathcal{A}.$$

Definition 1. (Deniz- Çekin differential operator) [5] Let $f \in \mathcal{A}(k,p)$. For the parameters $\lambda \geq 0$, $z \in U$ and $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ the differential operatör $\mathcal{D}_\lambda^m$ on $\mathcal{A}(k,p)$ by the following

$$\mathcal{D}_{\lambda,p}^m : \mathcal{A}(k,p) \to \mathcal{A}(k,p),$$

$$\mathcal{D}_{\lambda,p}^0 f(z) = f(z) \tag{4}$$

$$\mathcal{D}_{\lambda,p}^1 f(z) = \mathcal{D}_{\lambda,p} f(z) = \frac{1}{p^2}\left\{\lambda z^3 f'''(z) + (2\lambda+1)z^2 f''(z) + (1-\lambda p(p-1))zf'(z)\right\}$$

$$\vdots$$

$$\mathcal{D}_{\lambda,p}^m f(z) = \mathcal{D}_{\lambda,p}\left(\mathcal{D}_{\lambda,p}^{m-1} f(z)\right).$$

If f is given by (1) then from the definition of the differential operator $\mathcal{D}_{\lambda,p}^m$ we can easily see that

$$\mathcal{D}_{\lambda,p}^m f(z) = z^p + \sum_{n=k+p}^{\infty} \Phi^n(\lambda,m,p) a_n z^n \tag{5}$$

where

$$\Phi^n(\lambda,m,p) = \left[\frac{n\left(\lambda(n-p)(n+p-1)+n\right)}{p^2}\right]^m . \tag{6}$$

Remark 1. It should be remarked that the operator $\mathcal{D}_{\lambda,p}^m$ is a generalization of many other *linear differential operators* considered earlier. In particular, for $f \in \mathcal{A}(k,p)$ we have the following:

(i) $\mathcal{D}_{\lambda,1}^m = \mathcal{D}_\lambda^m$, $(\lambda \geq 0)$ the Deniz-Özkan differential operator [2].

(ii) $\mathcal{D}_{0,1}^m = \mathcal{D}_1^{2m}$ $(\delta \in \mathbb{N}_0)$ the Salagean differential operator [4].

(iii) $\mathcal{D}_{0,p}^m = \mathcal{D}_p^{2m}$, $(p \in \mathbb{N})$ Shenan, Salim and Mousa oprator [5].

Now, by making use of the operator $\mathcal{D}_{\lambda,p}^{m}$, we define a new subclass of functions belonging to the class $\mathcal{A}(k,p)$.

Definition 2. Let $\lambda \geq 0$ and $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $p \in \mathbb{N}$ and for the parameters σ, A and B such that

$$-1 \leq A < B \leq 1,\ 0 < B \leq 1 \text{ and } 0 \leq \sigma < p, \tag{7}$$

we say that a function $f(z) \in \mathcal{A}(k,p)$ is in the class $\mathcal{S}_{\lambda}^{m}(A,B;\sigma,p)$ if it satisfies the following subordination condition:

$$\frac{1}{p-\sigma}\left(\frac{[\mathcal{D}_{\lambda,p}^{m} f(z)]'}{z^{p-1}} - \sigma\right) \prec \frac{1+Az}{1+Bz} \qquad (z \in \mathcal{U}). \tag{8}$$

If the following inequality holds true,

$$\left| \frac{\dfrac{[\mathcal{D}_{\lambda,p}^{m} f(z)]'}{z^{p-1}} - p}{B\dfrac{[\mathcal{D}_{\lambda,p}^{m} f(z)]'}{z^{p-1}} - [pB + (A-B)(p-\sigma)]} \right| < 1 \qquad (z \in \mathcal{U}) \tag{9}$$

the inequality (9) is equivalent the subordination condition (8).

We note that by specializing the parameters λ, m, σ, A, B and p, the subclass $\mathcal{S}_{\lambda}^{m}(A,B;\sigma,p)$ reduces to several well-known subclasses of analytic functions. Furthermore, we say that a function $f(z) \in \mathcal{S}_{\lambda}^{m}(A,B;\sigma,p)$ is in the subclass $\mathcal{ST}_{\lambda}^{m}(A,B;\sigma,p)$ if $f(z)$ is of the following form:

$$f(z) = z^p - \sum_{n=k+p}^{\infty} |a_n| z^n \qquad \left(p,k \in \mathbb{N} := \{1,2,3,\ldots\}\right). \tag{10}$$

In our present paper, we shall make use of the familiar *integral operator* $\mathcal{I}_{\vartheta,p}$ defined by (see, for details, [2, 11, 13]; see also [25])

$$(\mathcal{I}_{\vartheta,p})(z) := \frac{\vartheta + p}{z^p}\int_0^z t^{\vartheta-1} f(t)dt \qquad f \in \mathcal{A}(k,p);\ \vartheta + p > 0;\ p \in \mathbb{N}) \tag{11}$$

as well as the fractional calculus operator $\mathcal{D}_z^{\nu}$ for which it is well known that (see, for details, [16,23] and [21]; see also Section 7)

$$\mathcal{D}_z^{\nu}\{z^p\} = \frac{\Gamma(\rho+1)}{\Gamma(\rho+1-\nu)} z^{\rho-\nu} \quad (\rho > -1; \nu \in \mathbb{R}) \tag{12}$$

in terms of Gamma function.

The main object of the present paper is to investigate the various important properties and characteristics of two subclasses of $\mathcal{A}(k,p)$ of normalized analytic functions in $\mathcal{U}$ with negative and positive coefficients, which are introduced here by making use of the *differential operator* defined by (4). Inclusion relationships for the class $\mathcal{S}_{\lambda}^{m}(A,B;\sigma,p)$ are investigated by applying the techniques of convolution. Furthermore, several properties involving generalized neighborhoods and partial sums for functions belonging to these subclasses are investigated. Finally, some applications of fractional calculus operators are considered. Relevant connections of the definitions and results presented here with those obtained in several earlier works are also pointed out.

2. Basic properties of the function class $\mathcal{S}_{\lambda}^{m}(A,B;\sigma,p)$

We first determine a necessary and sufficient condition for a function $f(z) \in \mathcal{A}(k,p)$ of the form (10) to be in the class $\mathcal{ST}_{\lambda}^{m}(A,B;\sigma,p)$.

Theorem 1. Let the function $f(z) \in \mathcal{A}(k,p)$ be defined by (10). Then the function $f(z)$ is in the class $\mathcal{ST}_{\lambda}^{m}(A,B;\sigma,p)$. if and only if

$$\sum_{n=k+p}^{\infty} \left[(n-p)(1+B) - (B-A)(p-\sigma)\right] \Phi^n(\lambda,m,p)|a_n| \le (B-A)(p-\sigma) \tag{13}$$

where $\Phi^n(\lambda,m,p)$ is given by (6).

Proof. If the condition (13) hold true, we find from (10) and (13) that

$$\left|z[\mathcal{D}_{\lambda,p}^{m} f(z)]' - p\mathcal{D}_{\lambda,p}^{m} f(z)\right| - \left|Bz[\mathcal{D}_{\lambda,p}^{m} f(z)]' - [pB + (A-B)(p-\sigma)\mathcal{D}_{\lambda,p}^{m} f(z)]\right|$$

$$= \left|-\sum_{n=k+p}^{\infty} (n-p)\Phi^n(\lambda,m,p)|a_n|z^n\right|$$

$$-\left|-\left[(B-A)(p-\sigma)z^p + \sum_{n=k+p}^{\infty} B(n-p) - (B-A)(p-\sigma)\right]\Phi^n(\lambda,m,p)|a_n|z^n\right|$$

$$\le \sum_{n=k+p}^{\infty} \left[(n-p)(1+B) - (B-A)(p-\sigma)\right] \Phi^n(\lambda,m,p)|a_n| - (B-A)(p-\sigma) \le 0$$

$$\left(z \in \partial\mathcal{U} = \{z : z \in \mathbb{C} \text{ and } |z| = 1\}\right).$$

Hence, by the *Maximum Modulus Theorem*, we have

$$f(z) \in \mathcal{ST}_{\lambda}^{m}(A, B; \sigma, p).$$

Conversely, assume that the function $f(z)$ defined by (10) is in the class $\mathcal{ST}_{\lambda}^{m}(A, B; \sigma, p)$. Then we have

$$\left| \frac{\dfrac{z[\mathcal{D}_{\lambda,p}^{m} f(z)]'}{\mathcal{D}_{\lambda,p}^{m} f(z)} - p}{B\dfrac{z[\mathcal{D}_{\lambda,p}^{m} f(z)]'}{\mathcal{D}_{\lambda,p}^{m} f(z)} - [pB + (A-B)(p-\sigma)]} \right|$$

$$= \left| \frac{\sum_{n=k+p}^{\infty} (n-p)\Phi^{n}(\lambda, m, p)|a_n| z^n}{\sum_{n=k+p}^{\infty} \left[-B(n-p) + (B-A)(p-\sigma)\right]\Phi^{n}(\lambda, m, p)|a_n| z^n + (B-A)(p-\sigma)z^p} \right| < 1 \qquad (z \in \mathcal{U}). \qquad (14)$$

Now, since $|\Re(z)| \le |z|$ for all z, we have

$$\Re\left(\frac{\sum_{n=k+p}^{\infty} (n-p)\Phi^{n}(\lambda, m, p)|a_n| z^{n-p}}{\sum_{n=k+p}^{\infty} \left[-B(n-p) + (B-A)(p-\sigma)\Phi^{n}(\lambda, m, p)|a_n| z^{n-p} + (B-A)(p-\sigma)\right]} \right) < 1. \qquad (15)$$

We choose values of z on the real axis so that the following expression:

$$\frac{z[\mathcal{D}_{\lambda,p}^{m} f(z)]'}{\mathcal{D}_{\lambda,p}^{m} f(z)}$$

is real. Then, upon clearing the denominator in (15) and letting $z \to 1^{-}$ though real values, we get the following inequality

$$\sum_{n=k+p}^{\infty} \left[(n-p)(1+B) - (B-A)(p-\sigma)\right]\Phi^{n}(\lambda, m, p)|a_n| \le (B-A)(p-\sigma)$$

This completes the proof of Theorem 1.

2. Properties associated with Quasi-convolution

In this part, we establish certain results concerning the Quasi-convolution of function is in the class $\mathcal{ST}_{\lambda}^{m}(A,B;\sigma,p)$.

For the functions $f_j(z)\in\mathcal{A}(k,p)$ given by

$$f_j(z)=z^p-\sum_{n=k+p}^{\infty}\left|a_{n,j}\right|z^n \quad (j=\overline{1,t},\ p\in\mathbb{N}), \tag{16}$$

we denote by $(f_1\bullet f_2)(z)$ the Quasi-convolution of functions $f_1(z)$ and $f_2(z)$, that is,

$$(f_1\bullet f_2)(z)=z^p-\sum_{n=k+p}^{\infty}\left|a_{n,1}\right|\left|a_{n,2}\right|z^n. \tag{17}$$

Theorem 2. If $f_j(z)\in\mathcal{ST}_{\lambda}^{m}(A,B;\sigma,p)$ $(j=\overline{1,t})$, then

$$(f_1\bullet f_2\bullet\ldots\bullet f_t)(z)\in\mathcal{ST}_{\lambda}^{m}(A,B;\Upsilon,p), \tag{18}$$

where

$$\Upsilon:=p-\frac{\prod_{j=1}^{m}(B-A)(p-\sigma_j)}{(B-A)\left[k(1+B)-\left(B-A\right)\right]\Phi^{k+p}(\lambda,m,p)]^{t-1}} \tag{19}$$

The result is sharp for the functions $f_j(z)$ given by

$$f_j(z)=z^p-\frac{(B-A)(p-\sigma_j)}{\left[k(1+B)-(B-A)\right]\Phi^{k+p}(\lambda,m,p)}z^{p+k} \quad (j=\overline{1,t}). \tag{20}$$

Proof. For $t=1,$ we see that $\Upsilon=\sigma_1.$ For $t=2,$ Theorem 1, gives

$$\sum_{n=k+p}^{\infty}\frac{\left[\left(n-p\right)\left(1+B\right)-\left(B-A\right)\left(P-\sigma\right)\right]\Phi^n(\lambda,m,p)}{(B-A)(p-\sigma_j)}\left|a_{n,j}\right|\le 1 \quad (j=1,2). \tag{21}$$

Therefore, by the Cauchy-Schwarz inequality, we obtain

$$\sum_{n=k+p}^{\infty}\frac{\left[\left(n-p\right)\left(1+B\right)-\left(B-A\right)\left(P-\sigma\right)\right]\Phi^n(\lambda,m,p)}{\sqrt{\prod_{j=1}^{2}(B-A)(p-\sigma_j)}}\sqrt{\left|a_{n,1}\right|\left|a_{n,2}\right|}\le 1. \tag{22}$$

To prove the case when $t=2,$ we have to find the largest Υ such that

$$\sum_{n=k+p}^{\infty}\frac{\left[(n-p)(1+B)-(B-A)(P-\sigma)\right]\Phi^{n}(\lambda,m,p)}{(B-A)(p-\Upsilon)}\left|a_{n,1}\right|\left|a_{n,2}\right|\leq 1, \tag{23}$$

or such that

$$\frac{\left|a_{n,1}\right|\left|a_{n,2}\right|}{(B-A)(p-\Upsilon)}\leq\frac{\sqrt{\left|a_{n,1}\right|\left|a_{n,2}\right|}}{\sqrt{\prod_{j=1}^{2}(B-A)(p-\sigma_j)}}, \tag{24}$$

this, equivalently, that

$$\sqrt{\left|a_{n,1}\right|\left|a_{n,2}\right|}\leq\frac{(B-A)(p-\Upsilon)}{\sqrt{\prod_{j=1}^{2}(B-A)(p-\sigma_j)}}. \tag{25}$$

Further, by using (22), we need to find the largest Υ such that

$$\frac{\sqrt{\prod_{j=1}^{2}(B-A)(p-\sigma_j)}}{\left[(n-p)(1+B)-(B-A)(P-\sigma)\right]\Phi^{n}(\lambda,m,p)}\leq\frac{(B-A)(p-\Upsilon)}{\sqrt{\prod_{j=1}^{2}(B-A)(p-\sigma_j)}}$$

or, equivalently, that

$$\frac{1}{(B-A)(p-\Upsilon)}\leq\frac{\left[(n-p)(1+B)-(B-A)(P-\sigma)\right]\Phi^{n}(\lambda,m,p)}{\prod_{j=1}^{2}(B-A)(p-\sigma_j)}. \tag{26}$$

It follows from (24) that

$$\Upsilon\leq p-\frac{\prod_{j=1}^{2}(B-A)(p-\sigma_j)}{(B-A)\left[(n-p)(1+B)-(B-A)(P-\sigma)\right]\Phi^{n}(\lambda,m,p)}. \tag{27}$$

Now, defining the function $\psi(n)$ by

$$\psi(n)=p-\frac{\prod_{j=1}^{2}(B-A)(p-\sigma_j)}{(B-A)\left[(n-p)(1+B)-(B-A)(P-\sigma)\right]\Phi^n(\lambda,m,p)}, \tag{28}$$

we see that $\psi'(n)\geq 0$ for $n\geq p+k$. This implies that

$$\Upsilon\leq\psi(k+p)=p-\frac{\prod_{j=1}^{2}(B-A)(p-\sigma_j)}{(B-A)\left[k(1+B)-(B-A)(P-\sigma)\right]\Phi^{k+p}(\lambda,m,p)}.$$

Therefore, the result is true for $t=2$.

Suppose that the result is true for any positive integer t. Then we have $(f_1\bullet f_2\bullet\ldots\bullet f_t\bullet f_{t+1})(z)\in\mathcal{ST}_\lambda^m(A,B;\gamma,p)$, when

$$\gamma=p-\frac{(B-A)(p-\Upsilon)(B-A)(p-\sigma_{t+1})}{(B-A)\left[k(1+B)-(B-A)(P-\sigma)\right]\Phi^{k+p}(\lambda,m,p)}$$

where, Υ is given by (19). After a simple calculation, we have

$$\gamma\leq p-\frac{\prod_{j=1}^{t+1}(B-A)(p-\sigma_j)}{(B-A)\left[k(1+B)-(B-A)(P-\sigma)\right]\Phi^{k+p}(\lambda,m,p)}.$$

Thus, the result is true for $t+1$. Therefore, by using the mathematical induction, we conclude that the result is true for any positive integer t.

Finally, taking the functions $f_j(z)$ defined by (20), we have

$$\begin{aligned}(f_1\bullet f_2\bullet\ldots\bullet f_t)(z)&=z^p-\left\{\prod_{j=1}^{m}\frac{(B-A)(p-\sigma_j)}{\left[k(1+B)-(B-A)(P-\sigma)\right]\Phi^{k+p}(\lambda,m,p)}\right\}z^{p+k}\\&=z^p-\mathrm{A}_{k+p}z^{k+p},\end{aligned}$$

which shows that

$$\sum_{n=k+p}^{\infty}\frac{\left[(n+p)(1+B)-(B-A)(P-\sigma)\right]\Phi^{n}(\lambda,m,p)}{(B-A)(p-\Upsilon)}\mathrm{A}_n$$
$$=\frac{\left[k(1+B)-(B-A)(P-\sigma)\right]\Phi^{k+p}(\lambda,m,p)}{(B-A)(p-\Upsilon)}\mathrm{A}_{k+p}$$
$$=\frac{\left[k(1+B)-(B-A)(P-\sigma)\right]\Phi^{k+p}(\lambda,m,p)}{(B-A)(p-\Upsilon)}\left\{\prod_{j=1}^{2}\frac{(B-A)(p-\sigma_j)}{\left[k(1+B)-(B-A)(P-\sigma)\right]\Phi^{k+p}(\lambda,m,p)}\right\}=1.$$

Consequently, the result is sharp. □

Putting $\sigma_j=\sigma \quad (j=\overline{1,t}),$ in Theorem 2, we have;

Corollary 2. If $f_j(z)\in\mathcal{ST}_{\lambda}^{m}(A,B;\sigma,p) \quad (j=\overline{1,t}),$ then

$$(f_1\bullet f_2\bullet\ldots\bullet f_t)(z)\in\mathcal{ST}_{\lambda}^{m}(A,B;\Upsilon,p),$$

where

$$\Upsilon:=p-\frac{[(B-A)(p-\sigma)]^{t}}{(B-A)\left[\left[k(1+B)-(B-A)(P-\sigma)\right]\Phi^{k+p}(\lambda,m,p)\right]^{t-1}}$$

The result is sharp for the functions $f_j(z)$ given by

$$f_j(z)=\frac{(B-A)(p-\sigma)}{\left[k(1+B)-(B-A)(P-\sigma)\right]\Phi^{k+p}(\lambda,m,p)}z^{k+p} \qquad (j=\overline{1,t}).$$

Theorem 3. Let the functions $f_j(z)\ (j=\overline{1,t})$ given by (6.1) be in the class $\mathcal{ST}_{\lambda}^{m}(A,B;\sigma_j,p)$. Then the function

$$h(z)=z^{p}-\sum_{n=k+p}^{\infty}\left(\sum_{j=1}^{t}\left|a_{n,j}\right|^{2}\right)z^{n} \tag{29}$$

belongs to the class $\mathcal{ST}_{\lambda}^{m}(A,B;\chi,p),$ where

$$\chi:=p-\frac{t(B-A)(p-\sigma^{*})^{2}}{\left[k(1+B)-(B-A)(P-\sigma)\right]\Phi^{k+p}(\lambda,m,p)} \quad (\sigma^{*}:=\min\{\sigma_1,\sigma_2,\ldots,\sigma_t\}). \tag{30}$$

The result is sharp for the functions $f_j(z)\ \ (j=\overline{1,t})$ given by (20).

Proof. By virtue of Theorem 1 we have

$$\sum_{n=k+p}^{\infty}\left\{\frac{\left[(n-p)(1+B)-(B-A)(p-\sigma)\right]\Phi^n(\lambda,m,p)}{(B-A)(p-\sigma_j)}\right\}^2\left|a_{n,j}\right|^2$$
$$\leq\left\{\sum_{n=k+p}^{\infty}\frac{\left[(n-p)(1+B)-(B-A)(p-\sigma)\right]\Phi^n(\lambda,m,p)}{(B-A)(p-\sigma_j)}\left|a_{n,j}\right|\right\}^2\leq 1. \tag{31}$$

Then it follows that for $(j=\overline{1,t})$,

$$\frac{1}{t}\sum_{n=k+p}^{\infty}\left\{\frac{\left[(n-p)(1+B)-(B-A)(p-\sigma)\right]\Phi^n(\lambda,m,p)}{(B-A)(p-\sigma_j)}\right\}^2\left(\sum_{j=1}^{t}\left|a_{n,j}\right|^2\right)\leq 1. \tag{32}$$

Therefore, we need to find the largest χ such that

$$\frac{1}{t}\sum_{n=k+p}^{\infty}\left\{\frac{\left[(n-p)(1+B)-(B-A)(p-\sigma)\right]\Phi^n(\lambda,m,p)}{(B-A)(p-\chi)}\right\}\left(\sum_{j=1}^{t}\left|a_{n,j}\right|^2\right)\leq 1. \tag{33}$$

This implies that

$$\chi\leq p-\frac{t(B-A)(p-\sigma^*)^2}{\left[(n-p)(1+B)-(B-A)(p-\sigma)\right]\Phi^n(\lambda,m,p)}\quad(\sigma^*:=\min\{\sigma_1,\sigma_2,...,\sigma_t\},\ n\geq k+p). \tag{34}$$

Now, defining the function $\Im(n)$ by

$$\Im(n):=p-\frac{t(B-A)(p-\sigma^*)^2}{\left[(n-p)(1+B)-(B-A)(p-\sigma)\right]\Phi^n(\lambda,m,p)}, \tag{35}$$

we see that $\Im(n)$ is an increasing function of $n,\ \ n\geq k+p.$ Setting $n=k+p$ in (31) we have

$$\chi\leq\Im(k+p):=p-\frac{t(B-A)(p-\sigma^*)^2}{\left[k(1+B)-(B-A)(P-\sigma)\right]\Phi^{k+p}(\lambda,m,p)}$$

which completes the proof of Theorem 3.

Setting $\sigma_j=\sigma\ \ (j=\overline{1,t}),$ in Theorem 3, we arrive at the following result.

Corollary 3. Let the functions $f_j(z)\ (j=\overline{1,t})$ given by (16) be in the class $\mathcal{ST}_{\lambda}^{m}(A,B;\sigma,p)$. Then the function

$$h(z)=z^p-\sum_{n=k+p}^{\infty}\left(\sum_{j=1}^{t}\left|a_{n,j}\right|^2\right)z^n$$

belongs to the class $\mathcal{ST}_{\lambda}^{m}(A,B;\chi,p)$, where

$$\chi:=p-\frac{t(B-A)(p-\sigma)^2}{\left[k(1+B)-(B-A)(P-\sigma)\right]\Phi^{k+p}(\lambda,m,p)}.$$

The result is sharp for the functions $f_j(z)\ \ (j=\overline{1,t})$ given by (20).

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Data Envelopment Analysis in the presence of q-rung fuzzy inputs and outputs

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Abstract

This work is the first to investigate the Data Envelopment Model (DEA) in the presence of q-Rung fuzzy inputs and outputs. Fundamental CCR and BCC models are presented in the context of q-Rung triangular fuzzy numbers (qRTFNs), which take into account the truth and falsity membership degrees of each input and output and provide a novel solution approach for it. This method divides a q-Rung fuzzy DEA (qR-FDEA) model into two crisp DEA models, first evaluating data value efficiency and the second evaluating membership grade efficiency. The efficiency score of the DMU is a combination of the efficiency score of the data value and the efficiency score of the membership grade. Furthermore, an example shows the applicability and validity of this unique technique, and DMUs are ranked based on their combined overall efficiency score.

Keywords: Data Envelopment Analysis; q-Rung Fuzzy Set (qRFS); Efficiency Analysis; q-Rung Triangular Fuzzy Number (qRTFN); Ranking.

1. Introduction

Decision-makers (DMs) cannot conduct their own cognitive evaluation in a completely precise environment in multi-attribute decision-making (MADM) situations due to the ambiguity and complexity of the problems. Many researchers have used fuzzy sets in MADM situations to tackle this challenge and have gotten a lot of research findings as a consequence. Many distinct kinds of higher fuzzy sets were proposed after Zadeh's [1] 1965 introduction of fuzzy sets. Atanassov [2] 1986 modified the fuzzy set into an intuitionistic fuzzy set (IFS), including membership and nonmembership. Yager [3] in 2013 proposed the pythogorean fuzzy set (PFS) as a human decision-making model for voting, which includes membership and non membership grade and satisfy the condition sum of squar of membership and non membership grade is at most one. Yager [4] proposed the q-rung fuzzy set (q-RFS) as a variant of the classical fuzzy set and IFS in which the qth order power summation of the membership and nonmembership functions is at most one. The domain of uncertainty increases as the rung q increases. As a result, q-RFS allows decision-makers to express opinions more effectively. The q-RFS is successfully used in many MCDM problems such as AHP [5], TOPSIS [5], ELECTRE [6], PROMETHEE [7], etc.

Data envelopment analysis (DEA) is a nonparametric system analysis method for evaluating the relative efficiency of decision-making units (DMUs) with multiple inputs and outputs. For performance evaluation of a certain production activity, DMUs aim to produce higher outputs with as few inputs as possible. Obviously, the optimal alternative. The original DEA model was put forward by Charnes et al. [8], defined as the CCR model. The CCR model assumes the constant return to scale (CRS), which means that when inputs increase by times, outputs also increase by the same proportion. Later, Banker et al. [9] put forward the BCC model with variable returns to scale (VRS) to apply to the production situation where outputs and inputs do not need to change in the same proportion. The proposal of two models opened up a new field for efficiency evaluation systems. In the process of continuous improvement, many DEA models have been proposed to adapt to different evaluation scenarios, such as the additive model, superefficiency model, cross-efficiency model, SBM model, and so on. The DEA is a powerful and efficient MCDM approach that has been widely implemented in various fields such as a variety of industries, including banking institutions [10], the insurance business [11], education [12], supply chain manegment [13], crises management [14], sustainability [15], energy [16] and healthcare services [17].

Sengupta [18], in 1992, used fuzzy sets in DEA for the first time. The DEA techniques employing fuzzy theory may be grouped into four basic groups, according to Hatami-Marbini et al. [19]: parametric approaches, possibility approaches, ranking approaches, defuzzification approaches, and many additional approaches have been brought to fuzzy DEA advancement. Emrouznejad et al. [20] in 2014 categorized the fuzzy DEA approach into six types: "the tolerance technique, the α $-$level-based approach, the fuzzy ranking approach, the possible approach, the fuzzy arithmetic, and the fuzzy random/type-2 fuzzy set," and reviewed the literature during the last 30 years. Zhou and Xu [21] summarizes the fuzzy data envelopment analysis research and its successful implementations. Several ways to deal with inaccurate, ambiguous, partial, and/or missing data in DEA have been proposed. Stochastic approaches and interval DEA models are widely utilized to detect inaccurate input and output data. Additionally, a number of research publications have been published in DEA that use intuitionistic fuzzy sets [22]. Gandotra et al. [23] proposed the data envelopment analysis in the context of the intuitionistic fuzzy weighted entropy approach. Sahil et al. [24] proposed the Parabolic Intuitionistic Fuzzy based Data Envelopment Analysis based on a parametric approach. Puri and Yadav [25] presented the optimistic and pessimistic efficiencies with intuitionistic fuzzy input/output data in DEA. Arya and Yadav [26] proposed the intuitionistic fuzzy data envelopment analysis (IFDEA) and dual IFDEA (DIFDEA) models based on α- and β-cuts, and the index ranking approach is used to rank the DMUs. Javaherian et al. [27] proposed the fuzzy network two-stage DEA model based on the expected value of the Intuitionistic fuzzy inputs and outputs. Shakouri et al. [28] proposed the intuitionistic fuzzy network DEA model based on a parametric approach. Santos Arteaga et al. [29] proposed a novel method for solving Intuitionistic Fuzzy Data Envelopment Analysis. Edalatpanah [30] proposed a ranking approach for solving the intuitionistic fuzzy DEA model.

After reviewing the literature, we were unable to find any articles that did not examine the data envelopment analysis model using a q-Rung fuzzy set. As a result, we investigated and developed a model for data envelopment analysis based on q-rung fuzzy inputs and outputs, which is called the q-Rung fuzzy data envelopment analysis (qR-FDEA) model. The suggested qR-FDEA model evaluates the effectiveness of a group of homogenous DMUs when q-Rung triangular fuzzy inputs and outputs are present. The qR-

FDEA model is solved using an innovative solution approach. This qR-FDEA model enables the decision-maker to describe their level of uncertainty in a wider domain.

This paper is organised as follows: Section 2 explained some preliminary findings and essential concepts. In Section 3, we constructed q-Rung fuzzy data envelopment analysis on the basis of the aggrigation operator. Section 4 suggested solution methodology of the proposed q-Rung fuzzy data envelopment analysis.. Section 5 presents a case study to demonstrate the applicability and validity. Finally, several possible future study directions were discussed in the conclusion section..

2. Preliminaries

In this section, we studied the preliminary results and operation of the q-Rung fuzzy set. The aggregation operator for the q-Rung fuzzy set is defined here.

Definition 1 [1] Let U be a universe. A fuzzy set F in U is given by

$$F = \{(x, \mu_F(x)) : x \in U\}$$

where $\mu_F : U \to [0,1]$ is the membership function. The nonmembership function of any $x \in F$ is defined as $\overline{\mu_F}(x) = 1 - \mu_F(x)$.

Definition 2 [2] Let U be a universe. An intuitionistic fuzzy set I in U is given by:

$$I = \{(x, \mu_I(x), \nu_I(x)) : x \in U\}$$

where the function $\mu_I, \nu_I : U \to [0,1]$ defines the membership and nonmembership degree of $x \in I$, respectively, and satisfies $0 \leq \mu_I(x) + \nu_I(x) \leq 1, \forall x \in U$. The hesitancy degree $\pi_I(x) = 1 - \mu_I(x) - \nu_I(x)$.

Definition 3 [3] Let U be a universe. A Pythagorean fuzzy set P in U is defined as

$$P = \{\langle x, \mu_P(x), \nu_P(x) \rangle : x \in U\}$$

where μ_P and ν_P are the membership and nonmembership degrees, respectively. $\mu_P, \nu_P : U \to [0,1]$, and satisfy the condition $0 \leq \mu_P^2(x) + \nu_P^2(x) \leq 1$ for all $x \in U$. Pythagorean fuzzy index $\pi_P(x) = \sqrt{1 - \mu_P^2(x) - \nu_P^2(x)}$.

Definition 4 [4] A q-Rung fuzzy set Q in a finite universe of discourse U is defined as follows by

$$Q = \left\{\left\langle x, \mu_Q(x), \nu_Q(x) \right\rangle : x \in U\right\} \tag{1}$$

where the function $\mu_Q : U \to [0,1]$ denotes the degree of membership and $\nu_Q : U \to [0,1]$ denotes the degree of nonmembership of the element $x \in U$ to the set Q, respectively, with the condition that $0 \leq \mu_Q^q(x) + \nu_Q^q(x) \leq 1, (q \geq 1)$ for every $x \in U$. The degree of indeterminacy is given as $\pi_P(x) = \sqrt[q]{1 - \mu_P(x)^q - \nu_P(x)^q}$.

Definition 5 A q-Rung Triangular Fuzzy Numbers (qRTFNs) is denoted by $\hat{X} = \langle x^L, x^{\mathrm{M}}, x^U; \phi_x, \psi_x \rangle$, where the three membership functions for the truth, falsity, and indeterminacy of x can be defined as follows:

$$\tau(x) = \begin{cases} \frac{x-x^L}{x^{\mathrm{M}}-x^L}\phi_x, \text{if } \ x \in [x^L, x^{\mathrm{M}}] \\ \phi_x, \quad \text{if } \ x = x^{\mathrm{M}} \\ \frac{x^U-x}{x^U-x^{\mathrm{M}}}\phi_x, \text{if } \ x \in [x^{\mathrm{M}}, x^U] \\ 0, \text{ otherwise} \end{cases}$$

$$\nu(x) = \begin{cases} \frac{x^{\mathrm{M}}-x+(x-x^L)\psi_x}{x^M-x^L}, \ \text{if } \ x \in [x^L, x^{\mathrm{M}}] \\ \psi_x, \quad \text{if } \ x = x^{\mathrm{M}} \\ \frac{x^U-x+(x-x^M)\psi_x}{x^U-x^{\mathrm{M}}}, \text{if } \ x \in [x^{\mathrm{M}}, x^U] \\ 1, \text{ otherwise} \end{cases}$$

where $0 \leq \tau(x)^q + \nu(x)^q \leq 1$.

Definition 6 Suppose $\hat{X}_1 = \langle x_1^L, x_1^M, x_1^U; \phi_{x_1}, \psi_{x_1} \rangle$, and $\hat{X}_2 = \langle x_2^L, x_2^M, x_2^U; \phi_{x_2}, \psi_{x_2} \rangle$, be two qRTFNs. The arithmetic relationships are defined as follows:

1. $\hat{X}_1 \oplus \hat{X}_2 = \left\langle x_1^L + x_2^L, x_1^M + x_2^M, x_1^U + x_2^U; \sqrt[q]{\phi_{x_1}^q + \phi_{x_2}^q - \phi_{x_1}^q \phi_{x_2}^q}, \psi_{x_1}\psi_{x_2} \right\rangle$.
2. $\hat{X}_1 - \hat{X}_2 = \left\langle x_1^L - x_2^U, x_1^M - x_2^M, x_1^U - x_2^L; \sqrt[q]{\phi_{x_1}^q + \phi_{x_2}^q - \phi_{x_1}^q \phi_{x_2}^q}, \psi_{x_1}\psi_{x_2} \right\rangle$.
3. $\hat{X}_1 \otimes \hat{X}_2 = \left\langle x_1^L x_2^L, x_1^M x_2^{\mathrm{M}}, x_1^U x_2^U; \phi_{x_1}\phi_{x_2}, \ \sqrt[q]{\psi_{x_1}^q + \psi_{x_2}^q - \psi_{x_1}^q \psi_{x_2}^q} \right\rangle$.
4. $\lambda\hat{X}_1 = \begin{cases} \left(\left\langle \lambda x_1^L, \lambda x_1^M, \lambda x_1^U; \sqrt[q]{1-(1-\phi_{x_1}^q)^\lambda}, \psi_{x_1}^\lambda \right\rangle, \ \lambda > 0. \right. \\ \left(\left\langle \lambda x_1^U, \lambda x_1^M, \lambda x_1^L; \sqrt[q]{1-(1-\phi_{x_1}^q)^{-\lambda}}, \psi_{x_1}^{-\lambda} \right\rangle, \ \lambda < 0. \right. \end{cases}$
5. $\hat{X}_1^\lambda = \langle (x_1^L)^\lambda, (x_1^M)^\lambda, (x_1^U)^\lambda; \phi_{x_1}^\lambda, \sqrt[q]{1-(1-\psi_{x_1}^q)^\lambda} \rangle$.

Lemma 1 Suppose $\hat{X}_i = \langle x_i^L, x_i^{\mathrm{M}}, x_i^U; \phi_{x_i}, \psi_{x_i} \rangle$, for $i = 1,2,\cdots,n$ are n qRTFNs. Then the aggregation operator is defined as

$$\sum_{i=1}^n \lambda_i \hat{X}_i = \langle \sum_{i=1}^n \lambda_i x_i^L, \sum_{i=1}^n \lambda_i x_i^M, \sum_{i=1}^n \lambda_i x_i^U;$$

$$\sqrt[q]{1-\prod_{i=1}^{n}\left(1-\phi_{x_i}^{q}\right)^{\lambda_i}}, \prod_{i=1}^{n}\psi_{x_i}^{\lambda_i}\rangle,\ \forall\lambda_i \geq 0. \quad (2)$$

and

$$\prod_{i=1}^{n}\hat{X}_i^{\lambda_i} = \langle\prod_{i=1}^{n}(x_i^L)^{\lambda_i}, \prod_{i=1}^{n}(x_i^M)^{\lambda_i}, \prod_{i=1}^{n}(x_i^U)^{\lambda_i};$$

$$\prod_{i=1}^{n}\phi_{x_i}^{\lambda_i}, \sqrt[q]{1-\prod_{i=1}^{n}\left(1-\psi_{x_i}^{q}\right)^{\lambda_i}}\rangle,\ \forall\lambda_i \geq 0. \quad (3)$$

where $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ is the weight vector.

Proof. The process of induction can be used to prove this.

3. qRung Fuzzy Data Envelopment Analysis (qR-FDEA)

Data envelopment analysis (DEA) is one of the most effective nonparametric mathematical approaches for calculating the overall performance of homogenous decision-making units (DMUs) with multiple inputs and outputs. The original DEA approaches were designed to deal with information based on crisp data, but they lack the capacity to deal with indeterminacy, impreciseness, ambiguity, inconsistency, and incomplete information. Various well-known DEA techniques, such as CCR and BCC models, may be found in the traditional DEA literature.

Let us assume there are n DMUs, each with m inputs and s outputs specified by the vectors $x_i = (x_{1i}, x_{2i}, \cdots, x_{mi})^T \in R^m$ and $y_i = (y_{1i}, y_{2i}, \cdots, y_{si})^T \in R^s$ respectively. The input and output matrices are specified as $x = (x_1, x_2, \cdots, x_m) \in R^{m\times n}$, and $y = (y_1, y_2, \cdots, y_s) \in R^{s\times n}$with $x > 0$ and $y > 0$. The CCR model is proposed by Charnes et al. [8], and the production possibility set of the CCR model is as follows:

$$P_{CCR} = \left\{(x,y): \sum_{j=1}^{n}\lambda_j x_j \leq x,\ \sum_{j=1}^{n}\lambda_j y_j \geq y,\ \lambda_j \geq 0\right\} \quad (4)$$

The CCR model for DMU_o can be defined as following linear programming.

$$\left.\begin{array}{ll} \min & \theta_o \\ \text{s.t.} & \sum_{j=1}^{n}\lambda_j x_{ij} \leq \theta_o x_{io},\ \ i = 1,2,3,\cdots,m \\ & \sum_{j=1}^{n}\lambda_j y_{rj} \geq y_{ro},\ \ r = 1,2,3,\cdots,s \\ & \lambda_j \geq 0,\ j = 1,2,3,\cdots,n. \end{array}\right\} \quad (5)$$

The BCC model is proposed by Banker et al. [9], which is the extension of the CCR model, and the production possibility set of the BCC model is as follows:

$$P_{BCC} = \left\{(x,y): \sum_{j=1}^{n}\lambda_j x_j \leq x,\ \sum_{j=1}^{n}\lambda_j y_j \geq y,\ \sum_{j=1}^{n}\lambda_j,\ \lambda_j \geq 0\right\} \quad (6)$$

The CCR model for DMU_o can be defined as following linear programming.

$$\left.\begin{array}{ll} \min & \theta_o \\ \text{s.t.} & \sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta_o x_{io}, \;\; i = 1,2,3,\cdots,m \\ & \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{ro}, \;\; r = 1,2,3,\cdots,s \\ & \sum_{j=1}^{n} \lambda_j = 1, \text{ and } \quad \lambda_j \geq 0, \; j = 1,2,3,\cdots,n. \end{array}\right\} \quad (7)$$

This Model is calculated the optimum solution θ_o^* and $\lambda_j^*, j = 1,2,\cdots,n$ for each $DMU_o, o = 1,2,\cdots,n$. The DMU_o is efficient if $\theta_o^* = 1$, else it is inefficient.

Suppose the inputs and outputs are qRung fuzzy numbers that are $x_{ij} = \left\langle x_{ij}^L, x_{ij}^M, x_{ij}^U; \phi_{x_{ij}}, \psi_{x_{ij}} \right\rangle$ and $y_{rj} = \left\langle y_{rj}^L, y_{rj}^M, y_{rj}^U; \phi_{y_{rj}}, \psi_{y_{rj}} \right\rangle, \forall i = 1,2,\cdots,n, \forall j = 1,2,\cdots,m, \forall r = 1,2,\cdots,s.$

The CCR model becomes

$$\left.\begin{array}{ll} \min & \theta_o \\ \text{s.t.} & \sum_{j=1}^{n} \lambda_j \langle x_{ij}^L, x_{ij}^M, x_{ij}^U; \phi_{x_{ij}}, \varphi_{x_{ij}}, \psi_{x_{ij}} \rangle \\ & \qquad \leq \theta_o \langle x_{io}^L, x_{io}^M, x_{io}^U; \phi_{x_{io}}, \varphi_{x_{io}}, \psi_{x_{io}} \rangle, \;\; i = 1,2,3,\cdots,m \\ & \sum_{j=1}^{n} \lambda_j \langle y_{rj}^L, y_{rj}^M, y_{rj}^U; \phi_{y_{rj}}, \varphi_{y_{rj}}, \psi_{y_{rj}} \rangle \\ & \qquad \geq \langle y_{ro}^L, y_{ro}^M, y_{ro}^U; \phi_{y_{ro}}, \varphi_{y_{ro}}, \psi_{y_{ro}} \rangle, \;\; r = 1,2,3,\cdots,s \\ \text{and} & \quad \lambda_j \geq 0, \; j = 1,2,3,\cdots,n. \end{array}\right\} \quad (8)$$

which is called the qRung fuzzy CCR (qR-FCCR) model.

The qRung fuzzy BCC (qR-FBCC) model is defined as

$$\left.\begin{array}{ll} \min & \theta_o \\ \text{s.t.} & \sum_{j=1}^{n} \lambda_j \langle x_{ij}^L, x_{ij}^M, x_{ij}^U; \phi_{x_{ij}}, \varphi_{x_{ij}}, \psi_{x_{ij}} \rangle \\ & \qquad \leq \theta_o \langle x_{io}^L, x_{io}^M, x_{io}^U; \phi_{x_{io}}, \varphi_{x_{io}}, \psi_{x_{io}} \rangle, \;\; i = 1,2,3,\cdots,m \\ & \sum_{j=1}^{n} \lambda_j \langle y_{rj}^L, y_{rj}^M, y_{rj}^U; \phi_{y_{rj}}, \varphi_{y_{rj}}, \psi_{y_{rj}} \rangle \\ & \qquad \geq \langle y_{ro}^L, y_{ro}^M, y_{ro}^U; \phi_{y_{ro}}, \varphi_{y_{ro}}, \psi_{y_{ro}} \rangle, \;\; r = 1,2,3,\cdots,r \\ & \sum_{j=1}^{n} \quad \lambda_j = 1, \text{ and } \quad \lambda_j \geq 0, \; j = 1,2,3,\cdots,n. \end{array}\right\} \quad (9)$$

Theorem *The DEA model given in equation (5) and the qR-FDEA Model in equation (8) are equivalents.*

Proof. When the aggregation operator is applied, it's easy to see that every qR-FCCR Model's optimum feasible solution is also an optimum feasible solution for the CCR model and vice versa.

The qR-FCCR Model in equation (8) is a non-linear programming problem that is difficult to solve. We separated equation (8) into two crisp models by applying definition (7) and theorem (1).

$$\left.\begin{array}{ll}\min & \eta_o \\ \text{s.t.} & \sum_{j=1}^{n}\lambda_j x_{ij}^{L} \le \eta_o x_{io}^{L};\ \sum_{j=1}^{n}\lambda_j x_{ij}^{M} \le \eta_o x_{io}^{M}; \\ & \sum_{j=1}^{n}\lambda_j x_{ij}^{U} \le \eta_o x_{io}^{U},\ \ i = 1,2,3,\cdots,m \\ & \sum_{j=1}^{n}\lambda_j y_{rj}^{L} \ge y_{ro}^{L};\ \sum_{j=1}^{n}\lambda_j y_{rj}^{M} \ge y_{ro}^{M}; \\ & \sum_{j=1}^{n}\lambda_j y_{rj}^{U} \ge y_{ro}^{U},\ \ r = 1,2,3,\cdots,s \\ & \text{and} \quad \lambda_j \ge 0,\ j = 1,2,3,\cdots,n. \end{array}\right\} \tag{10}$$

and

$$\left.\begin{array}{ll}\min & \rho_o \\ \text{s.t} & \sqrt[q]{1-\prod_{j=1}^{n}(1-\phi_{x_{ij}}^{\mathrm{q}})^{\lambda_j}} \le \sqrt[q]{1-(1-\phi_{x_{io}}^{\mathrm{q}})^{\rho_o}} \\ & \prod_{j=1}^{n}\psi_{x_{ij}}^{\lambda_j} \le \psi_{x_{io}}^{\rho_o},\ \ i = 1,2,3,\cdots,m \\ & \sqrt[q]{1-\prod_{j=1}^{n}(1-\phi_{y_{rj}}^{\mathrm{q}})^{\lambda_j}} \ge \phi_{y_{ro}} \\ & \prod_{j=1}^{n}\psi_{y_{rj}}^{\lambda_j} \ge \psi_{y_{ro}},\ \ r = 1,2,3,\cdots,s \\ & \text{and} \quad \lambda_j \ge 0,\ j = 1,2,3,\cdots,n. \end{array}\right\} \tag{11}$$

Here, equation (10) represents an LP problem, while equation (11) represents a non-linear programming problem.

The qR-FBCC Model is the extension of the qR-FCCR Model with an additional convexity condition $\sum_{j=1}^{n}\lambda_j = 1$, which becomes

$$\left.\begin{array}{ll}\min & \eta_o \\ \text{s.t.} & \sum_{j=1}^{n}\lambda_j x_{ij}^{L} \le \eta_o x_{io}^{L};\ \sum_{j=1}^{n}\lambda_j x_{ij}^{M} \le \eta_o x_{io}^{M}; \\ & \sum_{j=1}^{n}\lambda_j x_{ij}^{U} \le \eta_o x_{io}^{U},\ \ i = 1,2,3,\cdots,m \\ & \sum_{j=1}^{n}\lambda_j y_{rj}^{L} \ge y_{ro}^{L};\ \sum_{j=1}^{n}\lambda_j y_{rj}^{M} \ge y_{ro}^{M}; \\ & \sum_{j=1}^{n}\lambda_j y_{rj}^{U} \ge y_{ro}^{U},\ \ r = 1,2,3,\cdots,s \\ & \sum_{j=1}^{n}\lambda_j = 1,\ \text{and} \quad \lambda_j \ge 0,\ j = 1,2,3,\cdots,n. \end{array}\right\} \tag{12}$$

and

$$\left.\begin{array}{ll}\min & \rho_o \\ \text{s.t} & \sqrt[q]{1-\prod_{j=1}^{n}\left(1-\phi_{x_{ij}}^{\mathrm{q}}\right)^{\lambda_j}} \leq \sqrt[q]{1-\left(1-\phi_{x_{io}}^{\mathrm{q}}\right)^{\rho_o}} \\ & \prod_{j=1}^{n}\psi_{x_{ij}}^{\lambda_j} \leq \psi_{x_{io}}^{\rho_o},\ \ i=1,2,3,\cdots,m \\ & \sqrt[q]{1-\prod_{j=1}^{n}\left(1-\phi_{y_{rj}}^{\mathrm{q}}\right)^{\lambda_j}} \geq \phi_{y_{ro}} \\ & \prod_{j=1}^{n}\psi_{y_{rj}}^{\lambda_j} \geq \psi_{y_{ro}},\ \ r=1,2,3,\cdots,s \\ & \sum_{j=1}^{n}\lambda_j=1,\ \text{and}\quad \lambda_j\geq 0,\ j=1,2,3,\cdots,n. \end{array}\right\} \tag{13}$$

Definition 7 *The efficiency score* (θ_o) *of the* $DMU_o, o=1,2,\cdots,n$ *is the average of* η_o *and* ρ_o *, i.e.* $\theta_o=\frac{\eta_o+\rho_o}{2}$.

Definition 8 *A DMU is said to be efficient iff the efficiency score is 1 (i.e., The DMU is efficient in both data value and membership degree). Otherwise, it is inefficient*

4. Method for Solving qR-FDEA model

Let us consider the qRTFNs as the inputs and outputs of the DMUs. The efficiency score of the DMU may be calculated using the following steps outlined below.

1. Transform the DEA model into the qR-FDEA model as shown in equation (8).
2. Using definition (2) and theorem (1), split the qR-FDEA Model into two crisp models, one of which is an LP problem in equation (10) and the other a non-linear programming problem in equation (11).
3. The logarithm function is used to convert the non-linear programming problem into linear programming, as shown in equation (14).

$$\left.\begin{array}{ll}\min & \rho_o \\ \text{s.t} & \sum_{j=1}^{n}\lambda_j\log\left(1-\phi_{x_{ij}}^{2}\right) \geq \rho_o\log\left(1-\phi_{x_{io}}^{2}\right), \\ & \sum_{j=1}^{n}\lambda_j\log\left(\psi_{x_{ij}}\right) \leq \rho_o\log\left(\psi_{x_{io}}\right),\ \ i=1,2,3,\cdots,m. \\ & \sum_{j=1}^{n}\lambda_j\log\left(1-\phi_{y_{rj}}^{2}\right) \leq \log\left(1-\phi_{y_{ro}}^{2}\right), \\ & \sum_{j=1}^{n}\lambda_j\log\left(\psi_{y_{rj}}\right) \geq \log\left(\psi_{y_{ro}}\right),\ \ r=1,2,3,\cdots,s. \\ & \text{and}\quad \lambda_j\geq 0,\ j=1,2,3,\cdots,n. \end{array}\right\} \tag{14}$$

4. Solve these two crisp linear programming problems, and the efficiency score for $DMU_o o=1,2,\cdots,n$ is the product of the optimal solutions of the equation (10) and (14). The DMUs are ranked according to their efficiency score.

   By adding convexity condition $\sum_{j=1}^{n}\lambda_j=1$ in equations (10) and (14) and solve them to calculate the efficiency score in the qR-FBCC Model. The data value efficiency and membership degree efficiency of the DMUs are calculated. The DMUs are ranked according to their efficiency score.

The solution method for the qR-FDEA Model is depicted in the given flow chart.

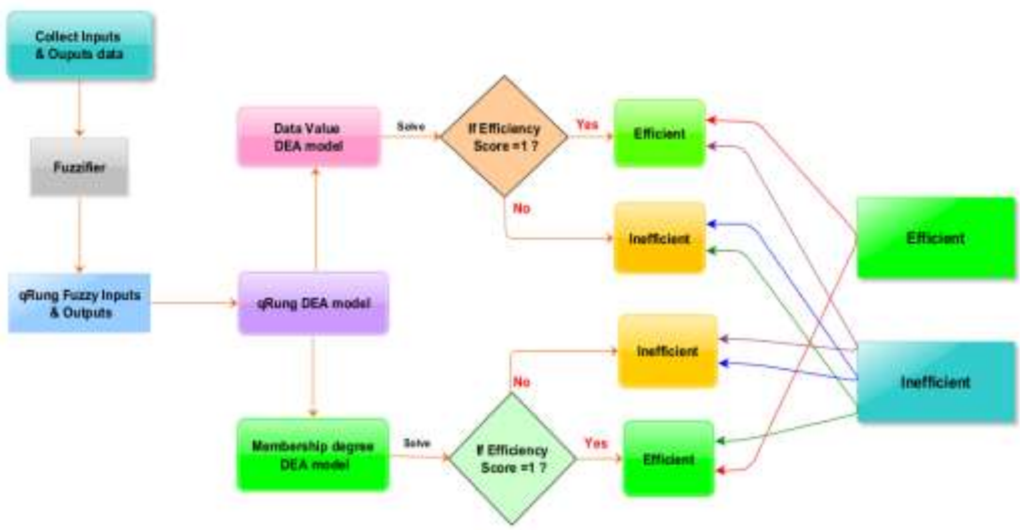

Figure 1: The method of solution for the qR-FDEA model

5. Numerical Example

To demonstrate the applicability and practicality of the novel approach proposed in this research, we added the truth membership and false membership degrees of each DMU to the example offered by Guo and Tanaka [32]. There are five DMUs, each of which has two inputs and two outputs. These inputs and outputs are represented by triangular intuitionistic numbers, and the information for this case is given in Table 1. In order to solve the given problem, we must take q = 1, which turns into a triangular intuitionistic fuzzy number because the given data satisfies the condition for an intuitionistic set.

DMU	Input 1	Input 2	Output 1	Output 2
DMU 1	$\langle 3.5,4.0,4.5;\ 0.75,0.1 \rangle$	$\langle 1.9,2.1,2.3;\ 0.8,0.05 \rangle$	$\langle 2.4,2.6,2.8;\ 0.7,0.15 \rangle$	$\langle 3.8,4.1,4.4;\ 0.8,0.1 \rangle$
DMU 2	$\langle 2.9,2.9,2.9;\ 0.8,0.2 \rangle$	$\langle 1.4,1.5,1.6;\ 0.9,0.1 \rangle$	$\langle 2.2,2.2,2.2;\ 0.6,0.2 \rangle$	$\langle 3.3,3.5,3.7;\ 0.55,0.15 \rangle$
DMU 3	$\langle 4.4,4.9,5.4;\ 0.9,0.01 \rangle$	$\langle 2.2,2.6,3.0;\ 0.98,0.01 \rangle$	$\langle 2.7,3.2,3.7;\ 0.45,0.25 \rangle$	$\langle 4.3,5.1,5.9;\ 0.7,0.05 \rangle$
DMU 4	$\langle 3.4,4.1,4.8;\ 0.6,0.4 \rangle$	$\langle 2.2,2.3,2.4;\ 0.8,0.05 \rangle$	$\langle 2.5,2.9,3.3;\ 0.5,0.35 \rangle$	$\langle 5.5,5.7,5.9;\ 0.55,0.3 \rangle$
DMU 5	$\langle 5.9,6.5,7.1;\ 0.7,0.2 \rangle$	$\langle 3.6,4.1,4.6;\ 0.5,0.2 \rangle$	$\langle 4.4,5.1,5.8;\ 0.6,0.2 \rangle$	$\langle 6.5,7.4,8.3;\ 0.85,0.15 \rangle$

Table 1: q-Rung fuzzy inputs and outputs data

The following procedures can be used to determine the efficiency score of DMU 1.

Step 1.

The given DEA model was constructed using the qRTFNs data.

$$\min\ \theta_1$$

$$\text{s.t. } \lambda_1\langle 3.5,4.0,4.5;0.75,0.1\rangle \oplus \lambda_2\langle 2.9,2.9,2.9;0.8,0.2\rangle \oplus \lambda_3\langle 4.4,4.9,5.4;0.9,0.01\rangle \oplus \lambda_4\langle 3.4,4.1,4.8;0.6,0.4\rangle \oplus \lambda_5\langle 5.9,6.5,7.1;0.7,0.2\rangle \le \theta_1\langle 3.5,4.0,4.5;0.75,0.1\rangle,$$

$$\lambda_1\langle 1.9,2.1,2.3;0.8,0.05\rangle \oplus \lambda_2\langle 1.4,1.5,1.6;0.9,0.1\rangle \oplus \lambda_3\langle 2.2,2.6,3.0;0.98,0.01\rangle \oplus \lambda_4\langle 2.2,2.3,2.4;0.8,0.05\rangle \oplus \lambda_5\langle 3.6,4.1,4.6;0.5,0.2\rangle \le \theta_1\langle 1.9,2.1,2.3;0.8,0.05\rangle,$$

$$\lambda_1\langle 2.4,2.6,2.8;0.7,0.15\rangle \oplus \lambda_2\langle 2.2,2.2,2.2;0.6,0.2\rangle \oplus \lambda_3\langle 2.7,3.2,3.7;0.45,0.25\rangle \oplus \lambda_4\langle 2.5,2.9,3.3;0.5,0.35\rangle \oplus \lambda_5\langle 4.4,5.1,5.8;0.6,0.2\rangle \ge \langle 2.4,2.6,2.8;0.7,0.15\rangle,$$

$$\lambda_1\langle 3.8,4.1,4.4;0.8,0.1\rangle \oplus \lambda_2\langle 3.3,3.5,3.7;0.55,0.15\rangle \oplus \lambda_3\langle 4.3,5.1,5.9;0.7,0.05\rangle \oplus \lambda_4\langle 5.5,5.7,5.9;0.55,0.3\rangle \oplus \lambda_5\langle 6.5,7.4,8.3;0.85,0.15\rangle \ge \langle 3.8,4.1,4.4;0.8,0.1\rangle,$$

and $\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5 \ge 0$

which is the qR-FCCR Model for DMU D1.

Step 2. Which can be converted into two crips DEA model as shown below.

$$\min\ \eta_1$$

$$\text{s.t. } 3.5\lambda_1 + 2.9\lambda_2 + 4.4\lambda_3 + 3.4\lambda_4 + 5.9\lambda_5 \le 3.5\eta_1$$

$$4\lambda_1 + 2.9\lambda_2 + 4.9\lambda_3 + 4.1\lambda_4 + 6.5\lambda_5 \le 4\eta_1$$

$$4.5\lambda_1 + 2.9\lambda_2 + 5.4\lambda_3 + 4.8\lambda_4 + 7.1\lambda_5 \le 4.5\eta_1$$

$$1.9\lambda_1 + 1.4\lambda_2 + 2.2\lambda_3 + 2.2\lambda_4 + 3.6\lambda_5 \le 1.9\eta_1$$

$$2.1\lambda_1 + 1.5\lambda_2 + 2.6\lambda_3 + 2.3\lambda_4 + 4.1\lambda_5 \le 2.1\eta_1$$

$$2.3\lambda_1 + 1.6\lambda_2 + 3\lambda_3 + 2.4\lambda_4 + 4.6\lambda_5 \le 2.3\eta_1$$

$$2.4\lambda_1 + 2.2\lambda_2 + 2.7\lambda_3 + 2.5\lambda_4 + 4.4\lambda_5 \ge 2.4$$

$$2.6\lambda_1 + 2.2\lambda_2 + 3.2\lambda_3 + 2.9\lambda_4 + 5.1\lambda_5 \ge 2.6$$

$$2.8\lambda_1 + 2.2\lambda_2 + 3.7\lambda_3 + 3.3\lambda_4 + 5.8\lambda_5 \ge 2.8$$

$$3.8\lambda_1 + 3.3\lambda_2 + 4.3\lambda_3 + 5.5\lambda_4 + 6.5\lambda_5 \ge 3.8$$

$$4.1\lambda_1 + 3.5\lambda_2 + 5.1\lambda_3 + 5.7\lambda_4 + 7.4\lambda_5 \geq 4.1$$

$$4.4\lambda_1 + 3.7\lambda_2 + 5.9\lambda_3 + 5.9\lambda_4 + 8.3\lambda_5 \geq 4.4$$

and $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \geq 0.$

and

$$\min \quad \rho_1$$

$$\text{s.t.}\ 1 - (1-0.75)^{\lambda_1}(1-0.8)^{\lambda_2}(1-0.9)^{\lambda_3}(1-0.6)^{\lambda_4}(1-0.7)^{\lambda_5} \leq 1-(1-0.75)^{\rho_1}$$

$$(0.1)^{\lambda_1}(0.2)^{\lambda_2}(0.01)^{\lambda_3}(0.4)^{\lambda_4}(0.32 \leq (0.1)^{\rho_1}$$

$$1 - (1-0.8)^{\lambda_1}(1-0.9)^{\lambda_2}(1-0.98)^{\lambda_3}(1-0.8)^{\lambda_4}(1-0.5)^{\lambda_5} \leq 1-(1-0.8)^{\rho_1}$$

$$(0.05)^{\lambda_1}(0.1)^{\lambda_2}(0.01)^{\lambda_3}(0.05)^{\lambda_4}(0.2)^{\lambda_5} \leq (0.05)^{\rho_1}$$

$$1 - (1-0.7)^{\lambda_1}(1-0.6)^{\lambda_2}(1-0.45)^{\lambda_3}(1-0.5)^{\lambda_4}(1-0.6)^{\lambda_5} \geq 0.7$$

$$(0.15)^{\lambda_1}(0.2)^{\lambda_2}(0.25)^{\lambda_3}(0.35)^{\lambda_4}(0.2)^{\lambda_5} \geq 0.15$$

$$1 - (1-0.8)^{\lambda_1}(1-0.55)^{\lambda_2}(1-0.7)^{\lambda_3}(1-0.55)^{\lambda_4}(1-0.85)^{\lambda_5} \geq 0.8$$

$$(0.1)^{\lambda_1}(0.15)^{\lambda_2}(0.05)^{\lambda_3}(0.3)^{\lambda_4}(0.15)^{\lambda_5} \geq 0.1$$

and $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \geq 0,$

which is a non-linear programming problem.

Step 3.
Convert non-linear programming problem into LP problem by using logarithm function.
$\min\ \eta_1$

s.t.

$$\log(1-0.75)\lambda_1 + \log(1-0.8)\lambda_2 + \log(1-0.9)\lambda_3 + \log(1-0.6)\lambda_4 + \log(1-0.7)\lambda_5 \geq \log(1-0.75)\eta_1$$

$$\log(0.1)\lambda_1 + \log(0.2)\lambda_2 + \log(0.01)\lambda_3 + \log(0.4)\lambda_4 + \log(0.2)\lambda_5 \leq \log(0.1)\eta_1$$

$$\log(1-0.8)\lambda_1 + \log(1-0.9)\lambda_2 + \log(1-0.98)\lambda_3 + \log(1-0.8)\lambda_4 + \log(1-0.5)\lambda_5 \geq \log(1-0.8)\eta_1$$

$$\log(0.05)\lambda_1 + \log(0.1)\lambda_2 + \log(0.01)\lambda_3 + \log(0.05)\lambda_4 + \log(0.2)\lambda_5 \leq \log(0.05)\eta_1$$

$$\log(1-0.7)\lambda_1 + \log(1-0.6)\lambda_2 + \log(1-0.45)\lambda_3 + \log(1-0.5)\lambda_4 + \log(1-0.6)\lambda_5 \leq \log(1-0.7)$$

$$\log(0.15)\lambda_1 + \log(0.2)\lambda_2 + \log(0.25)\lambda_3 + \log(0.35)\lambda_4 + \log(0.2)\lambda_5 \geq \log(0.15)$$

$$\log(1-0.8)\lambda_1 + \log(1-0.55)\lambda_2 + \log(1-0.7)\lambda_3 + \log(1-0.55)\lambda_4 + \log(1-0.85)\lambda_5 \leq \log(1-0.8)$$

$$\log(0.1)\lambda_1 + \log(0.15)\lambda_2 + \log(0.05)\lambda_3 + \log(0.3)\lambda_4 + \log(0.15)\lambda_5 \geq \log(0.1)$$

and $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \geq 0.$

Step 4.

The efficiency of each DMU was computed using MATLAB, and the above crisp LP problem evaluated the data value efficiency (η) and membership grade efficiency (ρ), as shown in Table (3), and compared the efficiency score in Figure (2).

DMU	qR-FCCR Model				qR-FBCC Model			
	η	ρ	$Mean$	$Type$	η	ρ	$Mean$	$Type$
DMU 1	0.9297	1	0.96485	Inefficient	0.9432	1	0.9716	Inefficient
DMU 2	1	0.5363	0.76815	Inefficient	1	0.5934	0.7967	Inefficient
DMU 3	1	0.3226	0.6613	Inefficient	1	1	1	Efficient
DMU 4	1	1	1	Efficient	1	1	1	Efficient
DMU 5	1	1	1	Efficient	1	1	1	Efficient

Table 2: Efficiency Score of the DMUs

The DMUs are ranked based on their mean efficiency Score. In the qR-FCCR Model, DMU 4 and DMU 5 are efficient, and other DMUs are inefficient. The DMUs are raked as follows

$$DMU\ 4 = DMU\ 5 > DMU\ 1 > DMU\ 2 > DMU\ 3.$$

In the qR-FBCC Model, DMU 3, DMU 4, and DMU 5 are efficient, and other DMUs are inefficient. The DMUs are ranked as follows:

$$DMU\ 4 = DMU\ 5 = DMU\ 3 > DMU\ 1 > DMU\ 2.$$

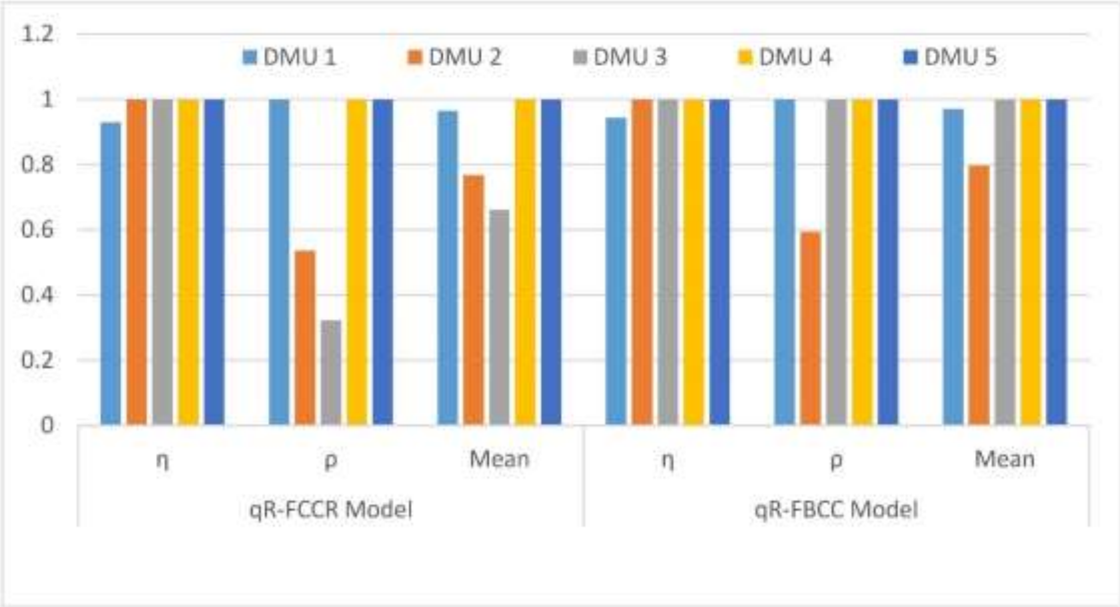

Figure 2: Efficiency Score in qR-FCCR and qR-FBCC models

6. Conclusion

A wide range of decision-making problems, including operation research, management science, industrial engineering, etc., use the q-Rung fuzzy set (qRFS), a comparatively recent academic concept quickly gaining popularity. In real-world applications, accurate input-output data are not always accessible, and certain subjective, linguistic, or ambiguous inputs and outputs may also have an q-Rung fuzzy essence in addition to simple fuzziness. Consequently, The q-Rung fuzzy DEA (qR-FDEA) model based on the aggregation operator is developed and gives a unique approach to solving them. This research focuses on extending the classic DEA models into FDEA with q-Rung fuzzy inputs and outputs. The recommended qR-FCCR and qR-FBCC models were used to calculate the efficiency score of the DMUs, which were then ranked according to their mean efficiency scores. We concluded by providing an example to demonstrate the method's viability and usefulness. Based on the results, the Model may be considered as appropriate and beneficial.

It is encouraged that more research is done on calculating the efficiency score of the DMU in various DEA models, including SBM, Additive, and Super-efficiency models. Future studies should concentrate on the practical use of our method. This method will help decision-makers assess the performance of several industries, including banking institutions, the insurance industry, financial services, education, supply chain management, crisis management, sustainability, energy, and healthcare services.

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Decomposition Formulas of a Third-order Discrete-time Linear Time-varying Systems into its First- and Second-order Commutative Pairs

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Abstract

In this study, a third-order discrete-time linear time-varying system is considered. Decomposition formulas of this system into its first-order and second-order commutative pairs are presented.

Keywords: Decomposition, discrete-time, linear system, difference equations.

1. Introduction

The realization of many engineering systems is done by cascaded-connected systems. This is very important in the design of systems [1]. Although the connection order of these subsystems depends on the specific design and engineering experience used, changing the connection order of the subsystems without changing the main function of the total system (commutativity) can have positive results when system performances are taken into account. For this reason, commutativity is very important in terms of practical applications especially in electrical and electronic engineering.

As shown in Fig. 1, by changing the connection order of two cascade-connected time-varying linear systems A and B, we say that A and B are commutative systems and (A, B) constitutes a commutative pair if input-output relations of the assembled systems AB and BA are identical.

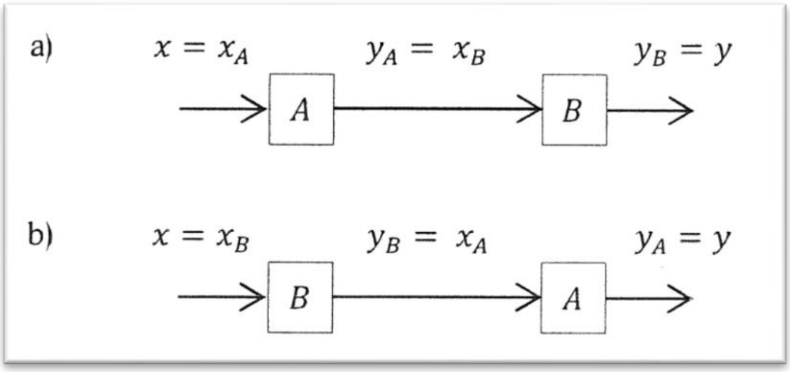

Figure 1. Subsystems A and B are connected in cascade way

The first publication on commutativity was reviewed in [2] in 1977 for the first-order continuous-time linear time-varying systems and the results here were generalized to second-order [3], third-order [4] and fourth-order [5] systems in 1982, 1984 and 1985, respectively. After a long period of time, fifth-order systems were investigated in [6] in 2011. The last study on the commutaitvity of continuous-time linear time-varying systems was appeared in [7] in 2021 for sixth-order systems.

Since all the studies on commutativity conditions [2]-[7] are related to the commutativity of analog systems and today's technology is based on digital systems rather than analog systems, there is a serious gap that needs to be studied on the commutativity of digital systems. Commutativity conditions of second-order [8] and first-order [9] discrete-time linear time-varying systems were studied in 2015 and 2019 respectively.

On the other hand, the higher the degree of the system, the more difficult it becomes to examine the system. Hence, the concept of decomposition is used as a very important tool for solving and analysing many problems and improving system properties to break down a high-order linear time-varying system into lower-order commutative pairs. Then, dealing with low-order parts becomes naturally simpler.

The decompostiion of the second-order continuous-time linear time-varying systems into two first-order commutative subsystems were proved in [10] in 2016 by Koksal. And for a third-order system, similar work to decompose it into first-order and second-order subsystems were presented in [11] in 2019. Finally, the decomposition of a fourth-order continuous-time linear time-varying differential system by cascaded two second-order commutative pairs and by cascaded first- and third-order commutative pairs was studied in [12].

2. Decomposition

Let the subsystems A and B in Figure 1 be defined by the following linear discrete-time time-varying difference equations:

$$A: a_1(k)y_A(k+1) + a_0(k)y_A(k) = x_A(k), k \geq 0 \tag{1}$$

$$B: b_2(k)y_B(k+2) + b_1(k)y_B(k+1) + b_0(k)y_B(k) = x_B(k), k \geq 0 \tag{2}$$

Note that $a_1(k)$ and $b_2(k)$ must not be zero for all $k \geq 0$ for the solvability of A and B.
From the connection AB shown in Figure 1, it follows that

$$x(k) = x_A(k),\ y_A(k) = x_B(k),\ y_B(k) = y(k), k \geq 0 \tag{3}$$

With this and Eqs. (1) and (2), the difference equation between $x(k)$ and y(k) can be obtained as follows:

$$b_2(k+1)y_B(k+3) + b_1(k+1)y_B(k+2) + b_0(k+1)y_B(k+1) = x_B(k+1), k \geq 1.$$

Inserting $y_B(k) = y(k)$, $x_B(k) = y_A(k)$, and then eliminating $y_A(k+1)$ and rearranging the terms, we obtain the difference formula of the cascaded-connected system AB as follows:

$$\begin{aligned} AB\colon & a_1(k)b_2(k+1\,)y(k+3) + [a_1(k)b_1(k+1\,) + a_0(k)b_2(k)]y(k+2) \\ & +[a_1(k)b_0(k+1) + a_0(k)b_1(k)]y(k+1) + a_0(k)b_0(k)y(k) = x(k). \end{aligned} \tag{4}$$

In a similar manner, we obtain the formula of the system BA as follows:

$$\begin{aligned} BA\colon & b_2(k)a_1(k+2)y(k+3) + [b_2(k\,)a_0(k+2) + b_1(k)a_1(k+1)]y(k+2) \\ & +[b_1(k\,)a_0(k+1) + b_0(k)a_1(k)]y(k+1) + b_0(k)a_0(k)y(k) = x(k). \end{aligned} \tag{5}$$

On the other hand, the connection AB and BA of sub-systems A and B are requested to be equivalent to the same system (say C) due to the commutativity. Let C be defined

$$C\colon c_3(k)y(k+3) + c_2(k)y(k+2) + c_1(k)y(k+1) + c_0(k)y(k) = x(k), k \geq 0. \tag{6}$$

If Eqs. (4) and (5) defining AB and BA are compared with Eq. (6) defining C, we have

$$c_3(k) = a_1(k)b_2(k+1\,) = b_2(k)a_1(k+2), \tag{7}$$
$$c_2(k) = a_1(k)b_1(k+1\,) + a_0(k)b_2(k) = b_2(k)a_0(k+2) + b_1(k)a_1(k+1), \tag{8}$$
$$c_1(k) = a_1(k)b_0(k+1) + a_0(k)b_1(k) = b_1(k)a_0(k+1) + b_0(k)a_1(k), \tag{9}$$
$$c_0(k) = a_0(k)b_0(k) = b_0(k)a_0(k). \tag{10}$$

3. Decomposition Formulas

From Eq. (7), we have

$$b_2(k+1) = \frac{b_2(k)a_1(k+2)}{a_1(k)}.$$

Generalizing the above equation, we get

$$b_2(k) = \frac{b_2(0)a_1(k)}{a_1(0)a_1(1)}a_1(k+1) \tag{11}$$

Again, by using Eq. (7) and replacing $b_2(k)$ in Eq. (7) with its equivalence in Eq. (11)

$$a_1(k+2) = \frac{c_3(k)a_1(0)a_1(1)}{b_2(0)a_1(k)a_1(k+1)};$$

which yields the following general expression for $a_1(k)$:

$$a_1(k) = \begin{cases} \dfrac{c_3(0)}{b_2(0)}, & k = 2 \\ a_1(0)\displaystyle\prod_{i=1}^{\frac{k}{3}} \dfrac{c_3(3i-2)}{c_3(3i-3)}, & k = 3,6,9,\cdots \\ a_1(1)\displaystyle\prod_{i=1}^{\frac{k-1}{3}} \dfrac{c_3(3i-1)}{c_3(3i-2)}, & k = 4,7,10,\cdots \\ \dfrac{c_3(0)}{b_2(0)}\displaystyle\prod_{i=1}^{\frac{k-2}{3}} \dfrac{c_3(3i)}{c_3(3i-1)}, & k = 5,8,11,\cdots \end{cases}, \tag{12}$$

Solving Eq. (8) for $b_1(k+1)$, replacing $b_2(k)$ with Eq. (11) and then generalizing the resulting equation, we find the formula of $b_1(k)$ as follows:

$$b_1(k) = \frac{b_2(0)a_1(k)}{a_1(0)a_1(1)}[a_0(k+1) - a_0(1) + a_0(k) - a_0(0)] + \frac{b_1(0)}{a_1(0)}a_1(k), \tag{13}$$

Similarly, if we solve $b_0(k+1)$ in Eq. (9), replace $b_1(k)$ with Eq. (13), generalize the resulting equation, we obtain

$$b_0(k) = \frac{b_2(0)}{a_1(0)a_1(1)}[a_0(k) - a_0(0)][a_0(k) - a_0(1)] + \frac{b_1(0)}{a_1(0)}[a_0(k) - a_0(0)] + b_0(0). \tag{14}$$

On the other hand, solving Eq. (8) for $a_0(k+2)$, substituting $b_2(k)$ and $b_1(k)$ in Eqs. (11) and (13) respectively, and generalizing the resulting equation, we obtain

$$a_0(k)=\frac{a_1(0)a_1(1)}{b_2(0)}\begin{cases}\frac{1}{a_1(0)}\left[\frac{c_2(0)}{a_1(1)}-b_1(0)\right], k=2\\ \sum_{i=0}^{\frac{k-3}{3}}\frac{1}{a_1(3i+1)}\left[\frac{c_2(3i+1)}{a_1(3i+2)}-\frac{c_2(3i)}{a_1(3i)}\right]+\frac{a_0(0)}{a_1(1)}, k=3,6,9\cdots\\ \sum_{i=1}^{\frac{k-1}{3}}\frac{1}{a_1(3i-1)}\left[\frac{c_2(3i-1)}{a_1(3i)}-\frac{c_2(3i-2)}{a_1(3i-2)a_1}\right]+\frac{a_0(1)}{a_1(1)}, k=4,7,10\cdots\\ \sum_{i=0}^{\frac{k-2}{3}}\frac{1}{a_1(3i)}\left[\frac{c_2(3i)}{a_1(3i+1)}-\frac{c_2(3i-1)}{a_1(3i-1)}\right]-\frac{b_1(0)}{b_2(0)}, k=5,8,11\cdots\end{cases} \quad (15)$$

The decomposition sequence will be as follows:

i) Compute $a_1(k)$ using Eq. (12).

ii) Then compute $a_0(k)$ using Eq. (15), $b_2(k)$ using Eq. (11), $b_1(k)$ using Eq. (13).

iii) Compute $b_0(k)$ using Eq. (14).

In the computations, $a_1(1), a_1(0), a_0(1), a_0(0), b_2(0), b_1(0), b_0(0)$ are undefined and they seem to be chosen arbitrary constants (nonzero for a_1 and b_2). However, this is not true since these constants should also satisfy Eq. (9) for $c_1(k)$ and Eq. (10) for $c_0(k)$ for all $k \geq 0$. If no consistent solutions present, then we say that the given third-order discrete-time system cannot be decomposed into its commutative pairs.

4. Conclusion

In this study, the method of synthesizing a third-order discrete-time system as a commutative cascaded pair of first- and second-order subsystems. It is also shown that decomposibility conditions are very stringent and hence not every system is decomposable. The case of non-zero initial values are not considered in this contribution. Otherwise, the decomposition process will introduce extra constraints which may be the subject of further research.

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Demonstration of Several Properties on Special Number Theoretic Functions

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Abstract

The theory of numbers called arithmetic is important in mathematical sciences. The importance and implementation of computer science in recent years (as a result of rapid development their fields such as process systems, communication their systems are financial predictions and quantum computers) are increasing related with number theory rapidly. Also, it is easily seen that arithmetic functions have practices (since they related with the coefficients of the power series) in engineering, IT, architecture, economics, physics as well as number theory, combinatorics, game theory, probability theory, analysis in mathematics.

Main aim in this work is to consider Euler totient arithmetic function and Divisor arithmetic function (the number of positive divisors of natural numbers) with their relations to each others. Firstly, some of the fundamental notations with theoretical results on these arithmetic functions are given. Then, several results are demonstrated by comparing values of these functions for the same natural number. Obtained results are also supported by numerical instances.

Keywords: Arithmetic Functions, Properties of Arithmetic Functions, Primes, Natural Numbers, Euler's Totient Arithmetic Function, Divisor Arithmetic Function, Multiplication..

1. Introduction

Number theory and algebra are one of the most active parts of mathematics. Published workings/documents (there is a rich literature consisting of books, monographs, journals, separate articles, reviews) and the continuity of them are indicators of how popular the studies in this field are. One of the topics in analytical number theory is arithmetic functions. Number theory and especially the theory of arithmetic functions, is full of rediscovery.Also, in the theory of numbers, the function is the basis in all areas of mathematics. It is treated as an arithmetic function or theoretical number function if it is defined on a set of positive integers and valued of the subset in the set of complex numbers.

The generalizations and results obtained in the theory of arithmetic functions are not only enable new open problems to be raised, but also aim to continue active studies in this field. Many mathematicians contribute to the theory of arithmetic functions by using different methods and terminological notations. Recently, the results obtained on this subject are used not only in the field of mathematics (Group theory, Lattice theory, Partially ordered sets, analysis, geometry, applied mathematics, etc.) but also in computer sciences, natural sciences, coding theory, cryptography and engineering...so on…

There are still many unsolved problems and assumptions about arithmetic functions theory such as concepts, definitions, unresolved problems, questions, theorems, formulas, assumptions, examples, mathematical criteria, etc. So, workings on this subject is still in progress.

The main purpose of the studies on the theory of arithmetic functions is to find the algebraic properties of these functions and their connections with each other, to obtain the existing theories with different methods and to facilitate the difficulties encountered. In addition, revealing the connections of these functions with functional analysis or geometry elements and obtaining new results with analytical approaches contribute to the literature of the subject.

In the literature, the study of the arithmetic functions are defined with algebraic properties/structures and some of these special functions can be seen with their specific names such as Euler Totient function, Tau function, Prime divisor or Prime Counting Functions, Mobius function, Mangoldt function, Liouville function etc…They are also categorized as multiplicative /full multiplicative or additive /full additive with single variable or two variables.

Another issue on arithmetic functions, based on the concept of unit division unit multiplication with a type of multiplication can be given. Also two variables, which are generalized of single variable arithmetic functions arithmetic functions are defined. From two variable arithmetic functions Nagell and Ramanujan are two in some special cases, as are their total functions these variable functions can be converted into single variable basic functions. In addition, the single variable and two variable arithmetic is treated with the unit region the functions can be functionally similar.

In this paper, (as it is seen in the abstract) , we consider relations between $div\ (\Phi(n))$ and $div\ (n)$ for searching n natural numbers. Also, some general results as theoretical and practical with numerical examples for this equation are given/ demonstrated. Obtained results will give different perspectives for readers in the literature of arithmetic functions theory.

2. Preliminaries

Euler phi function and Divisor function are *multiplicative* arithmetic functions. They have an importance roles in the applications of number theory, prime numbers, large calculations and cryptographic systems. To use in the next section (main results) , some basic and useful notions are mentioned for these arithmetic functions as follows.

Definition. If there is a function which is defined on the set of positive integers and also takes values in the set of real numbers or complex numbers, then it is called as an ***arithmetic function***. For example; a function defined as $g : \mathbb{Z}^+ \to \mathbb{R}$ with g (n)= *cos n* is an arithmetic function.

Definition (Euler phi (totient) function): $\Phi(n)$ is defined by $\sum_{m=1,gcd(m,n)=1}^{n} 1$ where the number of positive integers satisfy $m \leq n$ that are relatively prime to n.

Note: The multiplicative Euler phi function Φ holds many properties with Carmichael's conjecture. It can be seen some of them as follows:

- $\sum_{d|s} \Phi(d) = s$, for all natural numbers s.
- $\Phi(1) = 1$
- Φ (s) = $\sum_{d|s}$ d|s $\mu(d)(s/d)$ where μ is möbius function.
- Φ (s) = $\prod_{q^m||s}(q^m - q^{m-1})$ for all s ∈ N and q prime.

Definition (Divisor function): $div(s)$ is defined by $\sum_{d|s} 1$ for the number of positive divisors of *s* natural numbers.

Note: The multiplicative divisor function div satisfies many properties like Euler phi function. For practically, following result can be reminded;

Assume that q be a prime number and s $=q_1^{k_1} q_2^{k_2} \dots q_t^{k_t}$ be a positive integer. Then, we get

$$div\ (q^k) = k+1 \text{ and } div\ (s) = \prod_{i=1}^{t}(k_i + 1).$$

3. Main Results

Before starting our results, it will be better to calculate some values of divisor function, Euler phi function and composition of them. So, following table is obtained.

n	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**22**
$div(n)$	1	2	2	3	2	4	2	4	3	4	2	6	2	4	4	4
$\Phi(n)$	1	1	2	2	4	2	6	4	6	4	10	4	12	6	8	10
$div(\Phi(n))$	1	1	2	2	3	2	4	3	4	3	4	3	6	4	4	4

Table 1. Some values of divisor function, Euler phi function and composition of them.

From table, it is seen that $div(n) = div(\Phi(n))$, $div(n) < div(\Phi(n))$ or $div(n) > div(\Phi(n))$ for natural *n* numbers. Considering these three aspects, some new results can be given (some of them will be given without proof) as follows:

Theorem 1. Suppose that $(r_1, r_2, \dots, r_t)$ and $(s_1, s_2, \dots, s_t)$ be two different t-tuples include primes. Then divisor function satisfies $div\ (r_1 r_2 \dots r_t . s_1 s_2 \dots s_t) \geq 3^t$.

Proof. By using mathematical induction on t, inequality is easily demonstrated. If we start with t=1, then we get

$$div(r_1 s_1) = \begin{cases} 4, & if\ s_1 \neq r_1, \\ 3, & if\ s_1 = r_{1.} \end{cases}$$

Also, if there exists primes such that $s_i = r_j$, then

$$div(r_1 r_2 \dots s_1 s_2 \dots s_t) = div(s_1^2 s_2^2 \dots s_k^2) = 3^k$$

for every $i \in \{1, \dots, t\}$ and $j \in \{1, \dots, t\}$. In the contrary case, if $s_i \neq r_i$ for $i = 1,2, \dots, t-1$ and $div(p_1 p_2 \dots p_{k-1} q_1 q_2 \dots q_{k-1}) \geq 3^{k-1}$ then,

$$div\ (r_1 r_2 \dots r_t s_1 s_2 \dots s_t) = div\ (r_1 r_2 \dots r_{t-1} s_1 s_2 \dots s_{t-1})\ div\ (r_t s_t) \geq 3^{t-1}.3 = 3^t.$$

since $gcd\ (r_t, s_1 s_2 \dots s_{t-1}) = 1$ and $gcd\ (s_t, s_1 s_2 \dots s_{t-1}) = 1$.

This completes the proof.

Theorem 2 If $div\ (s) = div(\Phi(s))$ for s odd ıntegers, then s is divisible by 3 for $s \geq 3$.

Proof. Let us $3 \nmid s$ and s is odd integer. We can consider several cases for the proof, some of them can be given as follows:

Case 1. Assume that $s = q^\alpha$ with $q \geq 5, \alpha \geq 1$ such that $div\ (s) = div(\Phi(s))$. We obtain

$$\alpha + 1 = div\left((q-1)q^{\alpha-1}\right) = div\left((q-1)\right)div\ (q^{\alpha-1}) = div\ ((q-1))\alpha \ \geq 3\alpha$$

due to $gcd\ (q-1, q) = 1$.

This means that $\alpha + 1 > 2\alpha$. It is a contradiction. It means if $3 \nmid s$ then $div\ (s) \neq div(\Phi(s))$.

So, theorem is satisfied in the case of $s = q^\alpha$ with above conditions.

Case 2. Supposing that $s = q^\alpha r^\beta$ with $5 \leq q < r$, $\alpha, \beta \geq 1$ such that $div\,(s) = div(\Phi(s))$.

If we use $gcd\big((q-1)(r-1)q, s\big) = 1$, then we have

$$(\alpha+1)(\beta+1) = div\left((q-1)(r-1)q^{\alpha-1}r^{\beta-1}\right) = div\left((q-1)(r-1)q^{\alpha-1}\right)div\,(r^{\beta-1}\right)$$

$$= div\left((q-1)(r-1)q^{\alpha-1}\right)\beta$$

If $gcd(r-1,q) = 1$, then $(\alpha+1)(\beta+1) \geq 8\alpha\beta$ is satisfied. Otherwise, $(\alpha+1)(\beta+1) \geq 4(\alpha+1)\beta$.

If calculations and cases are considered/continued, then the proof of the theorem is completed.

Conjecture 3: There are infinitely many squarefree s such that $div\,(s) = div(\Phi(s))$.

Theorem 4. Assume that s be a positive integer having at most two distinct prime factors. Then, **t**here are 1,3,15, $2^\alpha r$ with r $\geq$7 is a safe prime and $\alpha \geq 1$ numbers which are some of the solutions of $div\,(s) = div(\Phi(s))$.

Proof. Suppose that s be a positive integer having at most two distinct prime factors for $div\,(s) = div(\Phi(s))$. It can be distinguished the following cases:

It is trivial that $div\,(s) = div(\Phi(s))$ is satisfied for $s = 1$.

Case 1. Suppose that $s = q^\alpha$ be a prime power. Then,

$\alpha + 1 = div\big((q-1)q^{\alpha-1}\big) = \alpha.\,div\,(q-1)$,

that is only true and satisfied for q =3 and $\alpha = 1$. So, it is obtained that $s = 3$.

Case2. Assume that $s = q.r$ where q and r are odd primes. Then, we have

$$4 = div\,((q-1)(r-1)),$$

which is only true and satisfied for q = 3 and r= 5. Hence, $s = 15$.

Case 3. $s = 2^{\alpha}r$ where r is odd prime and $\alpha \geq 1$.Then

$2(\alpha + 1) = div\ (2^{\alpha-1}(r - 1))$.

…

The rest is left to the reader.

Note. Even there are some different special types of numbers which they are solutions of the $div\ (s) = div(\Phi(s))$, it is not given all of them in this work.

Note: As we see from the table of $div(n)$ and $div(\Phi(n))$'s values, 5,7, 9, 11, 13, … are solutions of the $div(n) < div(\Phi(n))$. It seems that primes can be solutions of the inequality $div(n) < div(\Phi(n))$.

Then, we can give following theorem:

Theorem 5. Let q be a prime greater than 3. Then, every q prime satisfies the inequality $div(q) < div(\Phi(q))$.

Proof. It is trivial and proof is easily obtained using properties and definitions from prelimineries section.

Remark: It can be obtained new and similar results on divisibility or solutions for inequality $div(n) < div(\Phi(n))$.

Note: In a similar way, from the table of $div(n)$, $div(\Phi(n))$ values, 2,4,5,8,10,12 are solutions of the $div(n) > div(\Phi(n))$.

So, following theorem can be given:

Theorem 6. Let α be an integer greater than 0. Then, every 2^{α} holds the inequality $div(q) > div(\Phi(q))$.

Proof. Trivially, proof is got by using properties and definitions given in the preliminaries section.

Remark: We can easily verify many properties by considering these three statements [$div(n) = div(\Phi(n))$, $div(n) < div(\Phi(n))$ or $div(n) > div(\Phi(n))$ for natural *n* numbers.] Thus, this work just demonstrate several results on them to entrance this work with details. One may consider this kind of problems and get new results.

4. Conclusion

As a result (in this work) we obtain some common properties of two special arithmetic functions according to their connection with each other. We also give some numerical examples to support this type of results, which are included newly in the literature. This study will guide our next work and will add a different perspective to the readers.

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Double Hausdorff Deferred Statistical Equivalence of Order μ

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Abstract

In this study, first of all, the concepts of asymptotical Hausdorff deferred statistical equivalence of order μ and asymptotical Hausdorff deferred Cesàro equivalence of order μ $(0 < \mu \leq 1)$ for double sequences of sets are introduced, some properties of these concepts are given and the relations between them are examined. Then, the relation between the concepts of asymptotical Hausdorff deferred statistical equivalence of order μ and asymptotical Wijsman deferred statistical equivalence of order μ for double sequences of sets is showed.

Keywords: Wijsman convergence, Hausdorff convergence, double sequences of sets, asymptotical equivalence, order μ, deferred Cesàro mean, deferred statistical convergence.

1. Introduction

The concept of deferred Cesàro mean for real (or complex) sequences was presented by Agnew [1]. Recently, this concept has been attractive for researchers. In 2016, the concept of deferred statistical convergence was introduced by Küçükaslan and Yılmaztürk [2]. Then, the concepts of deferred Cesàro summability and deferred statistical convergence for double sequences was studied by Dağadur and Sezgek in [3, 4]. Also, for sequences, basic asymptotical deferred equivalence definitions were given by Koşar et al. [5].

The order of statistical convergence of a number sequence was given by Gadjiev and Orhan [6]. Then, the concepts of statistical convergence of order α and strongly p-Cesàro summability of order α were studied by Çolak [7] and Çolak-Bektaş [8]. Also, the concept of Wijsman $\mathcal{I}$-statistical convergence of order α was introduced by Savaş [9] and Şengül-Et [10]. The concepts of Wijsman statistical convergence of order α, Hausdorff statistical convergence of order α and Wijsman strongly p-Cesàro summability of order α for double sequences of sets were presented by Ulusu and Gülle [11].

The concept of deferred Cesàro mean was extended to sequences of sets by Altınok et al. [12] and they introduced the concepts of Wijsman deferred Cesàro summability and Wijsman deferred statistical convergence. Then, the concepts of Wijsman asymptotical deferred equivalence and Wijsman asymptotical deferred statistical equivalence for sequences of sets were presented by Altınok et al. [13]. Recently, Ulusu and Gülle [14] studied on the concepts of Wijsman deferred Cesàro summability and Wijsman deferred statistical convergence for double sequences of sets.

Also, the concepts of Wijsman asymptotical strongly deferred Cesàro equivalence and Wijsman asymptotical deferred statistical equivalence for double sequences of sets studied by Ulusu [15] and Gülle [16], respectively.

The study is aimed to study on some basic asymptotical Hausdorff deferred equivalence types of order μ for double set sequences.

More information on the notions of convergence and asymptotical equivalence in this study can be found in [17-38].

2. Definitions and Notations

First of all, let's start by recalling some basic definitions and notations to make our study easier to understand (See, [3, 14, 29, 30, 33, 37]).

The deferred Cesàro mean $D_{\phi,\psi}$ of a double sequence $\boldsymbol{a} = \{a_{uv}\}$ is defined by

$$(D_{\phi,\psi}a)_{kj} = \frac{1}{\phi_k\psi_j}\sum_{u=p_k+1}^{r_k}\sum_{v=q_j+1}^{s_j} a_{uv} \coloneqq \frac{1}{\phi_k\psi_j}\sum_{\substack{u=p_k+1\\ v=q_j+1}}^{r_k,s_j} a_{uv}\,,$$

where $[p_k]$, $[r_k]$, $[q_j]$ and $[s_j]$ are sequences of non-negative integers satisfying following conditions:

$$p_k < r_k,\ \lim_{k\to\infty} r_k = \infty; \quad q_j < s_j,\ \lim_{j\to\infty} s_j = \infty \tag{2.1}$$

$$r_k - p_k = \phi_k; \quad s_j - q_j = \psi_j. \tag{2.2}$$

Throughout the paper, unless otherwise specified, $[p_k]$, $[r_k]$, $[q_j]$ and $[s_j]$ are considered as sequences of non-negative integers satisfying (2.1) and (2.2).

For a metric space $(\mathcal{X},\rho)$, $d(x,E)$ represents the distance from x to E where

$$d(x,E) = \inf_{y\in E}\rho(x,y) \colon= d_x(E)$$

for any $x \in \mathcal{X}$ and any non-empty $E \subseteq \mathcal{X}$.

For a non-empty set $\mathcal{X}$, let a function $h\colon \mathbb{N} \to P(\mathcal{X})$ is defined by $h(v) = E_v \in P(\mathcal{X})$ for each $v \in \mathbb{N}$. Then, the sequence $\{E_v\} = \{E_1, E_2, \ldots\}$, which is the range elements of h, is called sequences of sets.

Throughout the study, unless otherwise stated, $(\mathcal{X},\rho)$ is considered as a metric space and E, E_{uv} $(u, v \in \mathbb{N})$ are considered as any non-empty closed subsets of $\mathcal{X}$.

A double sequence of sets $\{E_{uv}\}$ is said to be Hausdorff convergent to a set E provided that

$$\lim_{u,v\to\infty} \sup_{x\in\mathcal{X}} |d_x(E_{uv}) - d_x(E)| = 0.$$

It is denoted by $E_{uv} \xrightarrow{H_2} E$.

A double sequence of sets $\{E_{uv}\}$ is said to be Hausdorff Cesàro summable of order μ $(0 < \mu \leq 1)$ to a set E provided that

$$\lim_{m,n\to\infty} \frac{1}{(mn)^\mu} \sum_{\substack{u=1\\v=1}}^{m,n} \sup_{x\in\mathcal{X}} |d_x(E_{uv}) - d_x(E)| = 0.$$

It is denoted by $E_{uv} \xrightarrow{H_2(C)^\mu} E$. For $\mu = 1$, we obtain the concept of Hausdorff Cesàro summability for double sequences of sets.

A double sequence of sets $\{E_{uv}\}$ is said to be Hausdorff statistical convergence of order μ $(0 < \mu \leq 1)$ to a set E provided that for every $\varepsilon > 0$,

$$\lim_{m,n\to\infty} \frac{1}{(mn)^\mu} \left| \left\{ (u,v) : u \leq m, v \leq n : \sup_{x\in\mathcal{X}} |d_x(E_{uv}) - d_x(E)| \geq \varepsilon \right\} \right| = 0.$$

It is denoted by $E_{uv} \xrightarrow{H_2 S^\mu} E$. For $\mu = 1$, we obtain the concept of Hausdorff statistical convergence for double sequences of sets.

For any non-empty closed subsets $E_{uv}, F_{uv} \in \mathcal{X}$ such that $d_x(E_{uv}) > 0$ and $d_x(F_{uv}) > 0$ for each $x \in \mathcal{X}$, double sequences of sets $\{E_{uv}\}$ and $\{F_{uv}\}$ are said to be Wijsman asymptotical equivalent to multiple L if for each $x \in \mathcal{X}$

$$\lim_{u,v\to\infty} \frac{d_x(E_{uv})}{d_x(F_{uv})} := \lim_{u,v\to\infty} d_x\left(\frac{E_{uv}}{F_{uv}}\right) = L\,.$$

3. Main Results

In this section, the concepts of asymptotical Hausdorff deferred statistical equivalence of order μ and asymptotical Hausdorff deferred Cesàro equivalence of order μ $(0 < \mu \leq 1)$ for double sequences of sets are introduced, some properties of these concepts are given and the relations between them are examined. Then, the relation between the concepts of asymptotical Hausdorff deferred statistical equivalence of order μ and asymptotical Wijsman deferred statistical equivalence of order μ for double sequences of sets is showed.

From now on, we will consider that $d_x(E_{uv}) > 0$ and $d_x(F_{uv}) > 0$ in the following definitions, for each $x \in \mathcal{X}$ and any non-empty closed subsets $E_{uv}, F_{uv} \in \mathcal{X}$.

Definition 3.1 Double sequences of sets $\{E_{uv}\}$ and $\{F_{uv}\}$ are said to be asymptotical Hausdorff deferred statistical equivalent to multiple L order μ $(0 < \mu \leq 1)$ if for every $\varepsilon > 0$

$$\lim_{k,j\to\infty} \frac{1}{(\phi_k \psi_j)^\mu} \left| \left\{ (u,v) : u \in (p_k, r_k], v \in \left(q_j, s_j\right], \sup_{x\in\mathcal{X}} \left| d_x\left(\frac{E_{uv}}{F_{uv}}\right) - L \right| \geq \varepsilon \right\} \right| = 0$$

In this case, the notation $E_{uv} \overset{H_2^L DS^\mu}{\sim} F_{uv}$ is used. For $\mu = 1$, we obtain the concept of asymptotical Hausdorff deferred statistical equivalence to multiple L $(H_2^L DS)$ for double sequences of sets which has never been mentioned before.

Remark 3.1 The concept of asymptotical Hausdorff deferred statistical equivalence order μ for double sequences of sets is coincides with;

- the notion of asymptotical Hausdorff statistical equivalence of order μ for double sequences of sets which has never been studied before, for $p_k = 0, r_k = k$ and $q_j = 0, s_j = j$.
- the notion of asymptotical Hausdorff statistical equivalence for double sequences of sets which has never been studied before, for $\mu = 1$, and $p_k = 0, r_k = k$ and $q_j = 0, s_j = j$.

Theorem 3.1 If $0 < \mu \leq \nu \leq 1$, then

$$E_{uv} \overset{H_2^L DS^\mu}{\sim} F_{uv} \Rightarrow E_{uv} \overset{H_2^L DS^\nu}{\sim} F_{uv}\,.$$

Proof. Let $0 < \mu \leq \nu \leq 1$ and suppose that $E_{uv} \overset{H_2^L DS^\mu}{\sim} F_{uv}$. For every $\varepsilon > 0$, we can write the following inequality

$$\frac{1}{(\phi_k \psi_j)^\mu} \left| \left\{ (u,v) : u \in (p_k, r_k], v \in (q_j, s_j], \sup_{x\in\mathcal{X}} \left| d_x\left(\frac{E_{uv}}{F_{uv}}\right) - L \right| \geq \varepsilon \right\} \right|$$
$$\geq \frac{1}{(\phi_k \psi_j)^\nu} \left| \left\{ (u,v) : u \in (p_k, r_k], v \in (q_j, s_j], \sup_{x\in\mathcal{X}} \left| d_x\left(\frac{E_{uv}}{F_{uv}}\right) - L \right| \geq \varepsilon \right\} \right|.$$

Hence, by our assumption, we get $E_{uv} \overset{H_2^L DS^\nu}{\sim} F_{uv}$.

If $\nu = 1$ is taken in Theorem 3.1, then the following corollary is obtained.

Corollary 3.1 If double sequences of sets $\{E_{uv}\}$ and $\{F_{uv}\}$ are $H_2^L DS^\mu$-equivalent $(0 < \mu \leq 1)$, then these sequences are $H_2^L DS$-equivalent.

Theorem 3.2 Let $\{E_{uv}\}$, $\{F_{uv}\}$ and $\{G_{uv}\}$ are double sequences of sets such that

$$E_{uv} \subset F_{uv} \subset G_{uv} \quad (\text{for every } u, v \in \mathbb{N}).$$

If $E_{uv} \overset{H_2^L DS^\mu}{\sim} H_{uv}$ and $G_{uv} \overset{H_2^L DS^\mu}{\sim} H_{uv}$, then $F_{uv} \overset{H_2^L DS^\mu}{\sim} H_{uv}$ where $0 < \mu \leq 1$.

Proof. Let $E_{uv} \subset F_{uv} \subset G_{uv}$, $E_{uv} \overset{H_2^L DS^\mu}{\sim} H_{uv}$ and $G_{uv} \overset{H_2^L DS^\mu}{\sim} H_{uv}$. From the inclusion, it is obvious that

$$E_{uv} \subset F_{uv} \subset G_{uv} \Rightarrow d_x(G_{uv}) \leq d_x(F_{uv}) \leq d_x(E_{uv}) \quad (\text{for each } x \in \mathcal{X})$$

$$\Rightarrow \left| d_x\left(\frac{G_{uv}}{H_{uv}}\right) - L \right| \leq \left| d_x\left(\frac{F_{uv}}{H_{uv}}\right) - L \right| \leq \left| d_x\left(\frac{E_{uv}}{H_{uv}}\right) - L \right|.$$

Then, for every $\varepsilon > 0$ we have

$$\left\{(u,v): u \in (p_k, r_k], v \in (q_j, s_j], \left| d_x\left(\frac{F_{uv}}{H_{uv}}\right) - L \right| \geq \varepsilon \right\}$$

$$= \left\{(u,v): u \in (p_k, r_k], v \in (q_j, s_j], d_x\left(\frac{F_{uv}}{H_{uv}}\right) \geq L + \varepsilon \right\}$$

$$\cup \left\{(u,v): u \in (p_k, r_k], v \in (q_j, s_j], d_x(\frac{F_{uv}}{H_{uv}}) \leq L - \varepsilon \right\}$$

$$\subset \left\{(u,v): u \in (p_k, r_k], v \in (q_j, s_j], d_x\left(\frac{F_{uv}}{H_{uv}}\right) \geq L + \varepsilon \right\}$$

$$\cup \left\{(u,v): u \in (p_k, r_k], v \in (q_j, s_j], d_x\left(\frac{G_{uv}}{H_{uv}}\right) \leq L - \varepsilon \right\}$$

for each $y \in \mathcal{Y}$ and so

$$\frac{1}{(\phi_k \psi_j)^\mu} |\{(u,v): u \in (p_k, r_k], v \in (q_j, s_j], \sup_{x \in \mathcal{X}} \left| d_x\left(\frac{F_{uv}}{H_{uv}}\right) - L \right| \geq \varepsilon\}|$$

$$\leq \frac{1}{(\phi_k \psi_j)^\mu} |\{(u,v): u \in (p_k, r_k], v \in (q_j, s_j], \sup_{x \in \mathcal{X}} \left| d_x\left(\frac{E_{uv}}{H_{uv}}\right) - L \right| \geq \varepsilon\}|$$

$$+ \frac{1}{(\phi_k \psi_j)^\mu} |\{(u,v): u \in (p_k, r_k], v \in (q_j, s_j], \sup_{x \in \mathcal{X}} \left| d_x\left(\frac{G_{uv}}{H_{uv}}\right) - L \right| \geq \varepsilon\}|.$$

Hence, by our assumptions, we get $F_{uv} \overset{H_2^L DS^\mu}{\sim} H_{uv}$.

Definition 3.2 Double sequences of sets $\{E_{uv}\}$ and $\{F_{uv}\}$ are said to be asymptotical Hausdorff deferred Cesàro equivalent to multiple L order μ $(0 < \mu \leq 1)$ if

$$\lim_{u,v\to\infty} \frac{1}{(\phi_k \psi_j)^\mu} \sum_{\substack{u=p_k+1 \\ v=q_j+1}}^{r_k, s_j} \sup_{x\in X} \left| d_x\left(\frac{E_{uv}}{F_{uv}}\right) - L \right| = 0.$$

In this case, the notation $E_{uv} \overset{H_2^L D^\mu}{\sim} F_{uv}$ is used. For $\mu = 1$, we obtain the concept of asymptotical Hausdorff deferred Cesàro equivalence to multiple L $(H_2^L D)$ for double sequences of sets which has never been mentioned before.

Remark 3.2 The concept of asymptotical Hausdorff deferred Cesàro equivalence order μ for double sequences of sets is coincides with;

- the notion of asymptotical Hausdorff Cesàro equivalence of order μ for double sequences of sets which has never been studied before, for $p_k = 0, r_k = k$ and $q_j = 0, s_j = j$.

the notion of asymptotical Hausdorff Cesàro equivalence for double sequences of sets which has never been studied before, for $\mu = 1$, and $p_k = 0, r_k = k$ and $q_j = 0, s_j = j$.

Theorem 3.1 If $0 < \mu \leq \nu \leq 1$, then

$$E_{uv} \overset{H_2^L D^\mu}{\sim} F_{uv} \Rightarrow E_{uv} \overset{H_2^L D^\nu}{\sim} F_{uv}.$$

Proof. Let $0 < \mu \leq \nu \leq 1$ and suppose that $E_{uv} \overset{H_2^L D^\mu}{\sim} F_{uv}$. Here, we can write the following inequality

$$\frac{1}{(\phi_k \psi_j)^\mu} \sum_{\substack{u=p_k+1 \\ v=q_j+1}}^{r_k, s_j} \sup_{x\in X} \left| d_x\left(\frac{E_{uv}}{F_{uv}}\right) - L \right| \geq \frac{1}{(\phi_k \psi_j)^\nu} \sum_{\substack{u=p_k+1 \\ v=q_j+1}}^{r_k, s_j} \sup_{x\in X} \left| d_x\left(\frac{E_{uv}}{F_{uv}}\right) - L \right|.$$

Hence, by our assumption, we get $E_{uv} \overset{H_2^L D^\nu}{\sim} F_{uv}$.

If $\nu = 1$ is taken in Theorem 3.3, then we obtain the following corollary.

Corollary 3.2 If double sequences of sets $\{E_{uv}\}$ and $\{F_{uv}\}$ are $H_2^L D^\mu$-equivalent $(0 < \mu \leq 1)$, then these sequences are $H_2^L D$-equivalent.

Theorem 3.4 Let $0 < \mu \leq 1$. If double sequences of sets $\{E_{uv}\}$ and $\{F_{uv}\}$ are $H_2^L D^\mu$-equivalent, then these sequences are $H_2^L DS^\mu$-equivalent.

Proof. Let $0 < \mu \leq 1$ and suppose that $E_{uv} \overset{H_2^L D^\mu}{\sim} F_{uv}$. For every $\varepsilon > 0$, we can write the following inequality

$$\sum_{\substack{u=p_k+1 \\ v=q_j+1}}^{r_k,s_j} \sup_{x\in\mathcal{X}} \left|d_x\left(\frac{E_{uv}}{F_{uv}}\right) - L\right| \geq \sum_{\substack{u=p_k+1 \\ v=q_j+1 \\ |d_x(\frac{E_{uv}}{F_{uv}})-L|\geq\varepsilon}}^{r_k,s_j} \sup_{x\in\mathcal{X}} \left|d_x\left(\frac{E_{uv}}{F_{uv}}\right) - L\right|$$

$$\geq \varepsilon \left|\left\{(u,v): u \in (p_k, r_k], v \in (q_j, s_j], \sup_{x\in\mathcal{X}} \left|d_x\left(\frac{E_{uv}}{F_{uv}}\right) - L\right| \geq \varepsilon\right\}\right|$$

and so

$$\frac{1}{\varepsilon}\frac{1}{(\phi_k\psi_j)^\mu} \sum_{\substack{u=p_k+1 \\ v=q_j+1}}^{r_k,s_j} \sup_{x\in\mathcal{X}} \left|d_x\left(\frac{E_{uv}}{F_{uv}}\right) - L\right|$$

$$\geq \frac{1}{(\phi_k\psi_j)^\mu} \left|\left\{(u,v): u \in (p_k, r_k], v \in (q_j, s_j], \sup_{x\in\mathcal{X}} \left|d_x\left(\frac{E_{uv}}{F_{uv}}\right) - L\right| \geq \varepsilon\right\}\right|.$$

Hence, by our assumption, we get $E_{uv} \overset{H_2^L DS^\mu}{\sim} F_{uv}$.

Corollary 3.3 If $E_{uv} \overset{H_2^L}{\sim} F_{uv}$, then $E_{uv} \overset{H_2^L DS^\mu}{\sim} F_{uv}$.

The converse of Theorem 3.4 is true only in the cases $\mu = 1$ and $\{E_{uv}\}, \{F_{uv}\} \in L_\infty^2$ (the class of all bounded double sequences of sets).

The sequence $\{E_{uv}\}$ is called bounded if $\sup_{u,v}\{d_x(E_{uv})\} < \infty$ for each $x \in \mathcal{X}$.

Theorem 3.5 Let $\mu = 1$ and $\{E_{uv}\}, \{F_{uv}\} \in L_\infty^2$. If double sequences of sets $\{E_{uv}\}$ and $\{F_{uv}\}$ are $H_2^L DS$-equivalent, then these sequences are $H_2^L D$-equivalent.

Proof. Let $\mu = 1$, $\{E_{uv}\}, \{F_{uv}\} \in L_\infty^2$ and $E_{uv} \overset{H_2^L DS}{\sim} F_{uv}$. Since $\{E_{uv}\}, \{F_{uv}\} \in L_\infty^2$, there is an $\mathcal{M} > 0$ such that

$$\left|d_x\left(\frac{E_{uv}}{F_{uv}}\right) - L\right| \leq \mathcal{M}$$

for all $u, v \in \mathbb{N}$ and each $x \in \mathcal{X}$. Thus, for every $\varepsilon > 0$ we can write the following inequality

$$\frac{1}{\phi_k\psi_j}\sum_{\substack{u=p_k+1\\v=q_j+1}}^{r_k,s_j}\sup_{x\in\mathcal{X}}\left|d_x\left(\frac{E_{uv}}{F_{uv}}\right)-L\right|$$

$$=\frac{1}{\phi_k\psi_j}\sum_{\substack{u=p_k+1\\v=q_j+1\\|d_x(\frac{E_{uv}}{F_{uv}})-L|\geq\varepsilon}}^{r_k,s_j}\sup_{x\in\mathcal{X}}\left|d_x\left(\frac{E_{uv}}{F_{uv}}\right)-L\right|$$

$$+\frac{1}{\phi_k\psi_j}\sum_{\substack{u=p_k+1\\v=q_j+1\\|d_x(\frac{E_{uv}}{F_{uv}})-L|<\varepsilon}}^{r_k,s_j}\sup_{x\in\mathcal{X}}\left|d_x\left(\frac{E_{uv}}{F_{uv}}\right)-L\right|$$

$$\leq\frac{\mathcal{M}}{\phi_k\psi_j}\left|\{(u,v):u\in(p_k,r_k],v\in(q_j,s_j],\sup_{x\in\mathcal{X}}\left|d_x\left(\frac{E_{uv}}{F_{uv}}\right)-L\right|\geq\varepsilon\}\right|+\varepsilon.$$

Hence, by our assumptions, we get $E_{uv}\overset{H_2^LD}{\sim}F_{uv}$.

Finally, we showed the relation between the concepts of asymptotical Hausdorff deferred statistical equivalence of order μ and Wijsman asymptotical deferred statistical equivalence of order μ for double sequences of sets. Before this, let's recall the concept of Wijsman asymptotical deferred statistical equivalence for $\mu=1$ in [16].

Definition 3.3 [16] Double sequences of sets $\{E_{uv}\}$ and $\{F_{uv}\}$ are said to be Wijsman asymptotical deferred statistically equivalent to multiple L if for every ε >0 and each $x\in\mathcal{X}$

$$\lim_{i,j\to\infty}\frac{1}{\phi_k\psi_j}\left|\left\{(u,v):u\in(p_k,r_k],v\in\left(q_j,s_j\right],\left|d_x\left(\frac{E_{uv}}{F_{uv}}\right)-L\right|\geq\varepsilon\right\}\right|=0.$$

It is denoted by $E_{uv}\overset{W_2^LDS}{\sim}F_{uv}$.

Theorem 3.6 Let $0<\mu\leq 1$. If double sequences of sets $\{E_{uv}\}$ and $\{F_{uv}\}$ are $H_2^LDS^\mu$-equivalent, then these sequences are $W_2^LDS^\mu$-equivalent.

Proof. Let $0<\mu\leq 1$ and suppose that $E_{uv}\overset{H_2^LDS^\mu}{\sim}F_{uv}$. For every $\varepsilon>0$, we can write the following inequality

$$\frac{1}{(\phi_k\psi_j)^\mu}|\{(u,v): u \in (p_k,r_k], v \in (q_j,s_j], \sup_{x\in\mathcal{X}} \left|d_x\left(\frac{E_{uv}}{F_{uv}}\right) - L\right| \geq \varepsilon\}|$$
$$\geq \frac{1}{(\phi_k\psi_j)^\mu}|\{(u,v): u \in (p_k,r_k], v \in (q_j,s_j], \left|d_x\left(\frac{E_{uv}}{F_{uv}}\right) - L\right| \geq \varepsilon\}|.$$

for each $x \in \mathcal{X}$. Hence, by our assumption, we get $E_{uv} \overset{W_2^L DS^\mu}{\sim} F_{uv}$.

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Double Wijsman Deferred Cesàro Summability and Statistical Convergence of Order α

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Abstract

The aim of this study is to introduce the notions of deferred Cesàro summability of order α and deferred statistical convergence of order α $(0 < \alpha \leq 1)$ in the Wijsman sense for double set sequences, to give some properties of these notions and to examine the relationship between them.

Keywords: Deferred Cesàro mean, deferred statistical convergence, order α, double sequences of sets, Wijsman convergence.

1. Introduction

In [1], Agnew first introduced the notion of deferred Cesàro mean for real (or complex) sequences. Long after this, Küçükaslan and Yılmaztürk [2] presented the notion of deferred statistical convergence and showed the relationship of this notion with the strongly deferred Cesàro summability. Also, for double sequences, Dağadur and Sezgek [3] introduced and studied on the notions of deferred Cesàro summability and deferred statistical convergence. Furthermore, using order α, Et et al. [4] studied on the notions of deferred strongly Cesàro summability and deferred statistical convergence of order α in metric spaces.

For sequences of sets, Altınok et al. [5] studied on the notions of deferred statistical convergence and strongly deferred Cesàro summability in the Wijsman sense. Also, for double sequences of sets, Ulusu and Gülle [6] introduced and studied on similar notions. Furthermore, using order α, Yılmazer et al. [7] studied on the notions of Wijsman deferred statistical convergence and Wijsman strongly deferred Cesàro summability of order α for sequences of sets.

The purpose of this work is to introduce the notions of deferred Cesàro summability of order α and deferred statistical convergence of order α in the Wijsman sense for double set sequences, and to study on these notions.

More information on the notions in this study can be found in [8-19].

2. Basic Notions

Let's start by recalling some fundamental definitions and notations firstly (See, [6, 20-24]).

For a metric space $(\mathcal{X}, d)$, $\rho(x, B)$ indicates the distance from x to B where

$$\rho(x, B) = \inf_{b \in B} d(x, b) := \rho_x(B)$$

for any $x \in \mathcal{X}$ and any non-empty $B \subseteq \mathcal{X}$.

For a non-empty set $\mathcal{X}$, let a function $h: \mathbb{N} \to P(\mathcal{X})$ is defined by $h(v) = B_v \in P(\mathcal{X})$ for each $v \in \mathbb{N}$. Then, the sequence $\{B_v\} = \{B_1, B_2, \dots\}$, which is the codomain elements of h, is called sequences of sets.

Throughout the study, $(\mathcal{X}, \rho)$ will be considered as a metric space and B, B_{uv} $(u, v \in \mathbb{N})$ will be considered as any non-empty closed subsets of $\mathcal{X}$.

The double sequence $\{B_{uv}\}$ is said to be Wijsman convergent to the set B if

$$\lim_{u,v \to \infty} \rho_x(B_{uv}) = \rho_x(B),$$

for each $x \in \mathcal{X}$ and it is denoted by $B_{uv} \stackrel{W_2}{\longrightarrow} B$.

As an example to this notion, the following sequence of squares in $\mathbb{R}^2$ can be given.

Let $\mathcal{X} = \mathbb{R}^2$ and a double sequence $\{B_{uv}\}$ be defined as following:

$$B_{uv} := \left\{(x, y) \in \mathbb{R}^2 : |x| + |y| = \frac{1}{uv}\right\}.$$

Since

$$\lim_{u,v \to \infty} \rho_x(B_{uv}) = \rho_x(\{0,0\})$$

for each $x \in \mathcal{X}$, the double sequence $\{B_{uv}\}$ is Wijsman convergent to the set $B = \{(0,0)\}$.

The double sequence $\{B_{uv}\}$ is said to be Wijsman Cesàro summable of order α $(0 < \alpha \leq 1)$ to the set B if

$$\lim_{m,n \to \infty} \frac{1}{(mn)^\alpha} \sum_{u=1}^{m} \sum_{v=1}^{n} \rho_x(B_{uv}) = \rho_x(B),$$

for each $x \in \mathcal{X}$ and it is denoted by $B_{uv} \stackrel{W_2(C)^\alpha}{\longrightarrow} B$.

The double sequence $\{B_{uv}\}$ is said to be Wijsman strong Cesàro summable of order α $(0 < \alpha \leq 1)$ to the set B if

$$\lim_{m,n \to \infty} \frac{1}{(mn)^\alpha} \sum_{u=1}^{m} \sum_{v=1}^{n} |\rho_x(B_{uv}) - \rho_x(B)| = 0,$$

for each $x \in \mathcal{X}$ and it is denoted by $B_{uv} \stackrel{W_2[C]^\alpha}{\longrightarrow} B$.

The double sequence $\{B_{uv}\}$ is said to be Wijsman statistically convergent of order α $(0 < \alpha \leq 1)$ to the set B if for every $\varepsilon > 0$

$$\lim_{m,n\to\infty} \frac{1}{(mn)^{\alpha}} |\{(u,v): u \leq m, v \leq n: |\rho_x(B_{uv}) - \rho_x(B)| \geq \varepsilon\}| = 0,$$

for each $x \in \mathcal{X}$ and and it is denoted by $B_{uv} \overset{W_2(S)^{\alpha}}{\longrightarrow} B$.

The double sequence $\{B_{uv}\}$ is said to be bounded if $\sup_{u,v}\{\rho_x(B_{uv})\} < \infty$ for each $x \in \mathcal{X}$. Also, L_{∞}^{2} denotes the class of all bounded double set sequences.

The deferred Cesàro mean $D_{\phi,\psi}$ of a double sequence $\mathcal{B} = \{B_{uv}\}$ is defined by

$$(D_{\phi,\psi}\mathcal{B})_{uv} = \frac{1}{\phi_k \psi_j} \sum_{u=p_k+1}^{r_k} \sum_{v=q_j+1}^{s_j} \rho_x(B_{uv}),$$

where $[p_k]$, $[r_k]$, $[q_j]$ and $[s_j]$ are sequences of non-negative integers satisfying following conditions:

$$p_k < r_k,\ \lim_{k\to\infty} r_k = \infty; \quad q_j < s_j,\ \lim_{j\to\infty} s_j = \infty \tag{2.1}$$

$$r_k - p_k = \phi_k; \quad s_j - q_j = \psi_j. \tag{2.2}$$

Throughout the paper, unless otherwise specified, $[p_k]$, $[r_k]$, $[q_j]$ and $[s_j]$ are considered as sequences of non-negative integers satisfying (2.1) and (2.2).

3. New Concepts

In this section, we have introduced the notions of deferred Cesàro summability of order α and deferred statistical convergence of order α $(0 < \alpha \leq 1)$ in the Wijsman sense for double set sequences.

Definition 3.1 The double sequence $\{B_{uv}\}$ is said to be Wijsman deferred Cesàro summable of order α to the set B $(0 < \alpha \leq 1)$ provided that

$$\lim_{k,j\to\infty} \frac{1}{(\phi_k \psi_j)^{\alpha}} \sum_{u=p_k+1}^{r_k} \sum_{v=q_j+1}^{s_j} \rho_x(B_{uv}) = \rho_x(B),$$

for each $x \in \mathcal{X}$. In this case, the notation $B_{uv} \overset{W_2(D)^{\alpha}}{\longrightarrow} B$ is used.

Definition 3.2 The double sequence $\{B_{uv}\}$ is said to be Wijsman strong deferred Cesàro summable of order α to the set B $(0 < \alpha \leq 1)$ provided that

$$\lim_{k,j\to\infty} \frac{1}{(\phi_k \psi_j)^\alpha} \sum_{u=p_k+1}^{r_k} \sum_{v=q_j+1}^{s_j} |\rho_x(B_{uv}) - \rho_x(B)| = 0,$$

for each $x \in \mathcal{X}$. In this case, the notation $B_{uv} \overset{W_2[D]^\alpha}{\longrightarrow} B$ is used.

Remark 3.1 The notion of Wijsman strongly deferred Cesàro summability of order α for double set sequences is reduced to;

- the notion of Wijsman strongly deferred Cesàro summability in [6], for $\alpha = 1$.
- the notion of Wijsman strongly Cesàro summability of order α in [23], for $p_k = 0, r_k = k$ and $q_j = 0, s_j = j$.
- the notion of Wijsman strongly Cesàro summability in [22], for $\alpha = 1$, and $p_k = 0, r_k = k$ and $q_j = 0, s_j = j$.

Definition 3.3 The double sequence $\{B_{uv}\}$ is said to be Wijsman deferred statistically convergent of order α to the set B $(0 < \alpha \leq 1)$ provided that for every $\varepsilon > 0$

$$\lim_{k,j\to\infty} \frac{1}{(\phi_k \psi_j)^\alpha} \left|\left\{(u,v): u \in (p_k, r_k], v \in (q_j, s_j], |\rho_x(B_{uv}) - \rho_x(B)| \geq \varepsilon\right\}\right| = 0,$$

for each $x \in \mathcal{X}$. In this case, the notation $B_{uv} \overset{W_2(DS)^\alpha}{\longrightarrow} B$ is used.

Remark 3.2 The notion of Wijsman deferred statistical convergence of order α for double set sequences is reduced to;

- the notion of Wijsman deferred statistical convergence in [6], for $\alpha = 1$.
- the notion of Wijsman statistical convergence of order α in [23], for $p_k = 0, r_k = k$ and $q_j = 0, s_j = j$.
- the notion of Wijsman statistical convergence in [25], for $\alpha = 1$, and $p_k = 0, r_k = k$ and $q_j = 0, s_j = j$.

4. Main Results

In this section, we have given some properties of the notions of Wijsman deferred Cesàro summability of order α and Wijsman deferred statistical convergence of order α $(0 < \alpha \leq 1)$ for double set sequences, and have examined the relationship between these notions.

Theorem 4.1 If $0 < \alpha \leq \beta \leq 1$, then

$$B_{uv} \overset{W_2[D]^\alpha}{\longrightarrow} B \Rightarrow B_{uv} \overset{W_2[D]^\beta}{\longrightarrow} B.$$

Proof. Let $0 < \alpha < \beta \leq 1$ and assume that $B_{uv} \overset{W_2[D]^\alpha}{\longrightarrow} B$. For each $x \in \mathcal{X}$, we can write

$$\frac{1}{(\phi_k \psi_j)^\beta} \sum_{u=p_k+1}^{r_k} \sum_{v=q_j+1}^{s_j} |\rho_x(B_{uv}) - \rho_x(B)| \leq \frac{1}{(\phi_k \psi_j)^\alpha} \sum_{u=p_k+1}^{r_k} \sum_{v=q_j+1}^{s_j} |\rho_x(B_{uv}) - \rho_x(B)|.$$

Therefore, by our assumption, we get $B_{uv} \overset{W_2[D]^\beta}{\longrightarrow} B$.

If $\beta = 1$ is taken in Theorem 4.1, then the following corollary is obtained.

Corollary 4.1 If a double sequence $\{B_{uv}\}$ is $W_2[D]^\alpha$-summable to a set B $(0 < \alpha \leq 1)$, then the sequence is $W_2[D]$-summable to same set.

Theorem 4.2 If $0 < \alpha < \beta \leq 1$, then

$$B_{uv} \overset{W_2(DS)^\alpha}{\longrightarrow} B \Rightarrow B_{uv} \overset{W_2(DS)^\beta}{\longrightarrow} B.$$

Proof. Let $0 < \alpha < \beta \leq 1$ and assume that $B_{uv} \overset{W_2(DS)^\alpha}{\longrightarrow} B$. For every $\varepsilon > 0$ and each $x \in \mathcal{X}$, we can write

$$\frac{1}{(\phi_k \psi_j)^\beta} \left| \left\{ (u,v) : u \in (p_k, r_k], v \in (q_j, s_j], |\rho_x(B_{uv}) - \rho_x(B)| \geq \varepsilon \right\} \right|$$

$$\leq \frac{1}{(\phi_k \psi_j)^\alpha} \left| \left\{ (u,v) : u \in (p_k, r_k], v \in (q_j, s_j], |\rho_x(B_{uv}) - \rho_x(B)| \geq \varepsilon \right\} \right|.$$

Therefore, by our assumption, we get $B_{uv} \overset{W_2(DS)^\beta}{\longrightarrow} B$.

If $\beta = 1$ is taken in Theorem 4.2, then the following corollary is obtained.

Corollary 4.2 If a double sequence $\{B_{uv}\}$ is $W_2(DS)^\alpha$-convergent to a set B $(0 < \alpha \leq 1)$, then the sequence is $W_2(DS)$-convergent to same set.

Theorem 4.3 Let $0 < \alpha \leq 1$. If a double sequence $\{B_{uv}\}$ is $W_2[D]^\alpha$-summable to a set B, then the sequence is $W_2(DS)^\alpha$-convergent to the same set.

Proof. Let $0 < \alpha \leq 1$ and assume that $B_{uv} \overset{W_2[D]^\alpha}{\longrightarrow} B$. For every $\varepsilon > 0$ and each $x \in \mathcal{X}$, we can write

$$\sum_{u=p_k+1}^{r_k} \sum_{v=q_j+1}^{s_j} |\rho_x(B_{uv}) - \rho_x(B)| \geq \sum_{\substack{u=p_k+1 \\ |\rho_x(B_{kj}) - \rho_x(B)| \geq \varepsilon}}^{r_k} \sum_{v=q_j+1}^{s_j} |\rho_x(B_{uv}) - \rho_x(B)|$$

$$\geq \varepsilon \left|\left\{(u,v): u \in (p_k, r_k], v \in \left(q_j, s_j\right], |\rho_x(B_{uv}) - \rho_x(B)| \geq \varepsilon\right\}\right|$$

and so

$$\frac{1}{\varepsilon} \frac{1}{(\phi_k \psi_j)^\alpha} \sum_{u=p_k+1}^{r_k} \sum_{v=q_j+1}^{s_j} |\rho_x(B_{uv}) - \rho_x(B)|$$

$$\geq \frac{1}{(\phi_k \psi_j)^\alpha} \left|\left\{(u,v): u \in (p_k, r_k], v \in (q_j, s_j], |\rho_x(B_{uv}) - \rho_x(B)| \geq \varepsilon\right\}\right|.$$

Therefore, by our assumption, we get $B_{uv} \overset{W_2(DS)^\alpha}{\longrightarrow} B$.

Remark 4.1 The converse of Theorem 4.3 is true only in the case $\alpha = 1$ and $\{B_{uv}\} \in L_\infty^2$, which has already been shown in [6].

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Epitrochoidal Surfaces

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Abstract

We consider the epitrochoidal surfaces in 3-dimensional Euclidean space $\mathbb{E}^3$. We give fundamental notations of a Euclidean space. Defining a helicoidal surface, we reveal the epitrochoidal surface, and find its Gauss map, Gaussian curvature, and the mean curvature. Then, we indicate some relations between the curvatures of that kind surfaces.

Keywords: 3-space, helicoidal surface, epitrochoidal surface, Gauss map, Gaussian curvature, mean curvature.

1. Introduction

The surface theory has been studied for years. It can be seen the books about the topic in literature, such as [1-7].

In this research, we reveal the epitrochoid helical surface in three dimensional Euclidean space $\mathbb{E}^3$.

We indicate the notions of 3-space in Section 1. In Section 2, we give helicoidal surface. Then, we reveal epitrochoid helical surface, compute its Gauss and mean curvatures in Section 3. We give some relations for the curvatures of the surface. We serve a conclusion in the end.

In this work, with its transpose we equivalent a vector $(\mathrm{p}, \mathrm{q}, \mathrm{r})$. Next, in $\mathbb{E}^3$, we describe the fundamental forms *I*, *II*, shape operator matrix $\boldsymbol{S}$, Gauss curvature *K*, mean curvature *H* of the surface $\boldsymbol{\sigma} = \boldsymbol{\sigma}(u, v)$.

Let $\boldsymbol{\sigma}$ be an immersion of surface M^2 in $\mathbb{E}^3$. The vector product of $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ and $\vec{\beta} = (\beta_1, \beta_2, \beta_3)$ of $\mathbb{E}^3$ is defined by

$$\vec{\alpha} \times \vec{\beta} = \det \begin{pmatrix} e_1 & e_2 & e_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{pmatrix}.$$

We consider the following matrices

$$I = \left(g_{ij}\right)_{2\times 2}, \text{ and } II = \left(h_{ij}\right)_{2\times 2},$$

where

$$g_{11} = \boldsymbol{\sigma}_u \cdot \boldsymbol{\sigma}_u,$$

$$g_{12} = \boldsymbol{\sigma}_u \cdot \boldsymbol{\sigma}_v = g_{21},$$

$$g_{22} = \boldsymbol{\sigma}_v \cdot \boldsymbol{\sigma}_v,$$

$$h_{11} = \boldsymbol{\sigma}_{uu} \cdot \mathcal{u},$$

$$h_{12} = \boldsymbol{\sigma}_{uv} \cdot \mathcal{u} = h_{21},$$

$$h_{22} = \boldsymbol{\sigma}_{vv} \cdot \mathcal{u},$$

" $\cdot$ " is a Euclidean inner product, the unit normal (i.e., the Gauss map) of the surface is given by

$$\mathcal{u} = \frac{\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v}{\|\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v\|}.$$

We calculate $I^{-1}.II$, then it supplies the following matrix of the shape operator

$$\boldsymbol{\mathcal{S}} = \frac{1}{g_{11}g_{22} - {g_{12}}^2}\begin{pmatrix} g_{22}h_{11} - g_{12}h_{12} & g_{22}h_{12} - g_{12}h_{22} \\ g_{11}h_{12} - g_{12}h_{11} & g_{11}h_{22} - g_{12}h_{12} \end{pmatrix}.$$

Finally, we obtain the following formula of Gaussian curvature

$$\begin{aligned} K &= det(\boldsymbol{\mathcal{S}}) \\ &= \frac{h_{11}h_{22} - {h_{12}}^2}{g_{11}g_{22} - {g_{12}}^2} \end{aligned}$$

and the mean curvature formula

$$\begin{aligned} H &= \frac{1}{2} tr(\boldsymbol{\mathcal{S}}) \\ &= \frac{g_{11}h_{22} + g_{22}h_{11} - 2g_{12}h_{12}}{2(g_{11}g_{22} - {g_{12}}^2)}, \end{aligned}$$

respectively. The surface $\boldsymbol{\sigma}(u, v)$ is flat when $K(u) = 0$, and it is minimal when $H(u) = 0$.

2. Helical Surface

In this section, we present the surface of rotation and the helical surface in $\mathbb{E}^3$.

Consider open interval I, let $\gamma : \mathrm{I} \subset \mathbb{R} \longrightarrow \Pi$ be a curve, and ℓ be a line in Π. We define the surface of rotation as a surface rotating the generating curve γ about the axis ℓ.

While the generating curve rotates about ℓ, it replaces parallel lines orthogonal to ℓ, then the accelerate of replacement is in proportion to the accelerate of rotation. Therefore, the above surface is named the *helical surface* having axis ℓ, pitch $\wp \in \mathbb{R}^+$.

The orthogonal matrix is given by

$$\mathcal{O}(v) = \begin{pmatrix} cosv & -sinv & 0 \\ sinv & cosv & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Here, $v \in \mathbb{R}$. $\mathcal{O}$ holds the following

$$\mathcal{O}.\ell = \ell, \quad \mathcal{O}^t.\mathcal{O} = \mathcal{O}.\mathcal{O}^t = \mathfrak{I}_3, \ det\mathcal{O} = 1,$$

where $\mathfrak{I}_3$ is the identity matrix.

When the rotation axis be ℓ, there is a transformation transformed ℓ to the axis x_3. The generating curve is given by

$$\gamma(u) = (\mathcal{f}(u), 0, \mathcal{g}(u)),$$

where $\mathcal{f}(u), \mathcal{g}(u) \in C^k(\mathrm{I}, \mathbb{R})$. Hence, the helical surface spanned by the $(0,0,1)$ having pitch $\wp$, is defined by

$$\mathcal{H}(u, v) = \mathcal{O}(v).\gamma(u) + \wp\, v\, \ell^t,$$

where $u \in \mathrm{I}$, $v \in [0, 2\pi)$. So, we have the following helicoidal surface

$$\mathcal{H}(u, v) = \begin{pmatrix} \mathcal{f}(u)cosv \\ \mathcal{f}(u)sinv \\ \mathcal{g}(u) + \wp v \end{pmatrix}.$$

When $\wp = 0$, the helical surface is transform to the surface of rotation.

3. Helical Surface Having Epitrochoid Curve

In $\mathbb{E}^2$, epitrochoid curve is defined by

$$\gamma[a,b,h](u) = \left((a+b)\cos u - h\cos\left(\frac{a+b}{b}u\right), (a+b)\sin u - h\sin\left(\frac{a+b}{b}u\right)\right),$$

where $a, b, h \in \mathbb{R}$.

In $\mathbb{E}^3$, the epitrochoid helical surface (see Figure 1) spanned by the $(0,0,1)$, has pitch $\wp \in \mathbb{R}^+$, (see Figure 2 for $\wp = 0$) is defined by

$$\mathfrak{E}(u,v) = \begin{pmatrix} \left((a+b)\cos u - h\cos\left(\frac{a+b}{b}u\right)\right)\cos v \\ \left((a+b)\cos u - h\cos\left(\frac{a+b}{b}u\right)\right)\sin v \\ \left((a+b)\sin u - h\sin\left(\frac{a+b}{b}u\right)\right) + \wp v \end{pmatrix},$$

where the generating curve is presented by

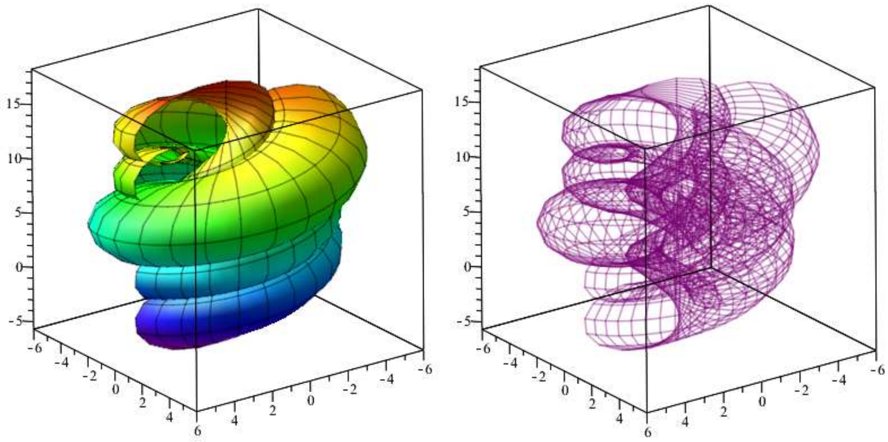

Figure. 1. Epitrochoid helical surface
Left: Outside view, **Right:** Inside view

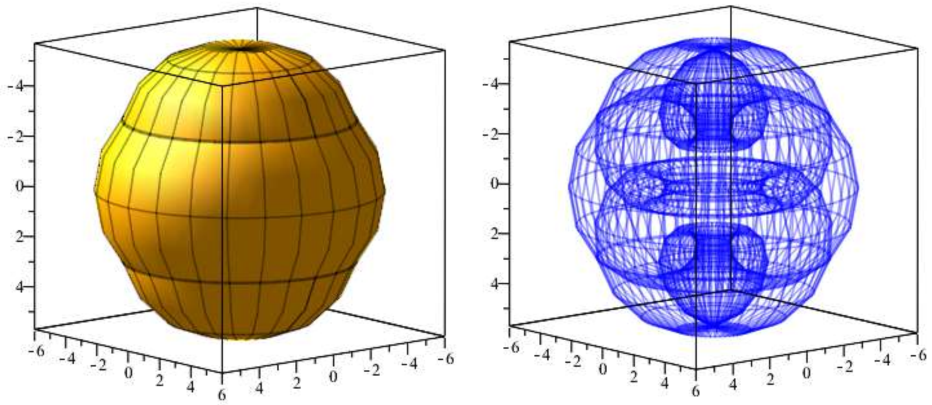

Figure. 1. Epitrochoid rotational surface
Left: Outside view, **Right:** Inside view

$$\gamma(u) = \left((a+b)\cos u - h\cos\left(\frac{a+b}{b}u\right), 0, (a+b)\sin u - h\sin\left(\frac{a+b}{b}u\right)\right),$$

$u \in \mathrm{I}$, $v \in [0, 2\pi)$.

We take $[a, b, h] = [3,1,2]$. By using the first differentials of the epitrochoid helical surface $\mathfrak{E}(u, v)$ depends on u and v, we reveal the following first quantities

$$g_{11} = 80 - 64\cos(3u),$$

$$g_{12} = (4\cos u - 8\cos(4u))\wp,$$

$$g_{22} = 8\cos(2u) - 8\cos(5u) - 8\cos(3u) + 2cos(8u) + 10 + \wp^2.$$

Then, we get

$$\det(g_{ij}) = -64\cos(11u) + (-32\wp^2 + 416)\cos(8u) + 256\cos(6u)$$
$$+(32\wp^2 - 960)\cos(5u) + (-32\wp^2 - 1280)\cos(3u)$$
$$+(-8\wp^2 + 896)\cos(2u) - 256\cos u + 40c^2 + 1056.$$

Hence, the Gauss map of the surface is given by

$$\mathcal{u}(u,v)=\frac{1}{(\tau(\mathrm{u}))^{1/2}}\begin{pmatrix}\mathcal{u}_1(u,v)\\ \mathcal{u}_2(u,v)\\ \mathcal{u}_3(u,v)\end{pmatrix},$$

where

$$\mathcal{u}_1=\wp(\cos(u+v)-\cos(u-v)-2\cos(4u+v)+2\cos(4u-v))$$
$$-2\cos(2u+v)-2\cos(2u-v)+5\cos(3u+v)+5\cos(3u-v)$$
$$+5\cos(5u+v)+5\cos(5u-v)-2\cos(8u+v)-2\cos(8u-v)-8\cos(v),$$
$$\mathcal{u}_2=\wp\big(\sin(u+v)+\sin(u-v)-2\sin(4u+v)-2\sin(4u-v)\big)$$
$$-2\sin(2u+v)+2\sin(2u-v)+5\sin(3u+v)-5\sin(3u-v)$$
$$+5\sin(5u+v)-5\sin(5u-v)-2\sin(8u+v)+2\sin(8u-v)-8\sin(v),$$
$$\mathcal{u}_3=-4\sin(8u)+10\sin(5u)+6\sin(3u)-4\sin(2u).$$

and

$$\tau(\mathrm{u})=2\wp^2(-4\cos(8u)+4\cos(5u)-4\cos(3u)-\cos(2u)+5)$$
$$-16\cos(11u)+104\cos(8u)+64\cos(6u)-240\cos(5u)$$
$$-320\cos(3u)+224\cos(2u)-64\cos(u)+264.$$

In the end, the mean curvature of the epitrochoid helical surface $\mathfrak{E}(u,v)$ is given by

$$H(u)=\frac{\hbar(u)}{2(W)^{3/2}},$$

where

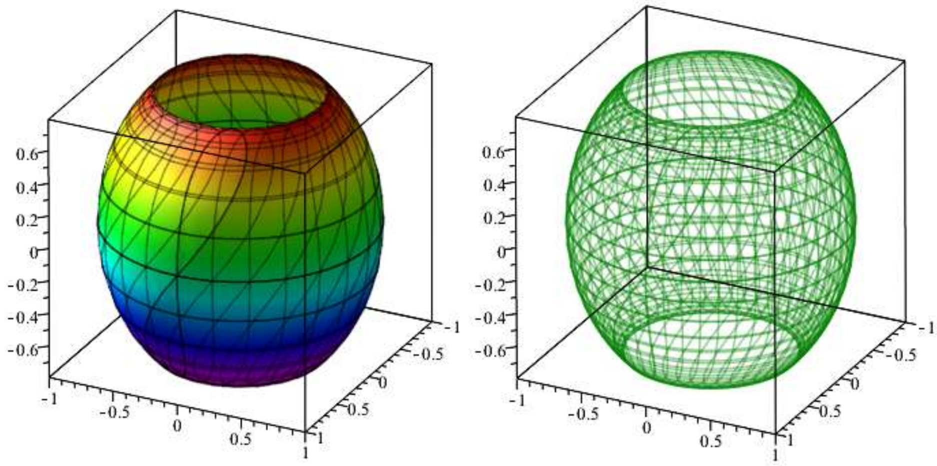

Figure. 1. Gauss map of the epitrochoid helical surface
Left: Outside view, **Right:** Inside view

$$\begin{aligned}\mathscr{h}(u) = {} & 212992\cos^{15}(u) - 798720\cos^{13}(u) + (16384\mathscr{p}^2 - 210944)\cos^{12}(u)\\ & +1198080\cos^{11}(u) + (-49152\mathscr{p}^2 + 599040)\cos^{10}(u)\\ & +(-3072\mathscr{p}^2 - 835072)\cos^{9}(u) + (55296\mathscr{p}^2 - 627456)\cos^{8}(u)\\ & +(7488\mathscr{p}^2 + 222720)\cos^{7}(u) + (-28480\mathscr{p}^2 + 280960)\cos^{6}(u)\\ & +(-6192\mathscr{p}^2 + 8256)\cos^{5}(u) + (6120\mathscr{p}^2 - 47712)\cos^{4}(u)\\ & +(1940\mathscr{p}^2 - 6272)\cos^{3}(u) + (-180\mathscr{p}^2 + 3840)\cos^{2}(u)\\ & +(-120\mathscr{p}^2 + 456)\cos(u) - 25\mathscr{p}^2 - 148,\end{aligned}$$

and the Gaussian curvature of the Eptrochoid helical surface $\mathfrak{E}(u, v)$ is given by

$$K(u) = \frac{\mathscr{k}(u)}{[W(u)]^2},$$

where

$$\begin{aligned}\mathscr{k}(u) = {} & -655360\cos^{19}(u) + 3112960\cos^{17}(u) + (-65536\mathscr{p}^2 + 811008)\cos^{16}(u)\\ & -6225920\cos^{15}(u) + (262144\mathscr{p}^2 - 3110912)\cos^{14}(\mathrm{u})\end{aligned}$$

$$+(16384\wp^2 + 6419456)\cos^{13}(u) + (-425984\wp^2 + 4805632)\cos^{12}(u)$$

$$+(-57344\wp^2 - 3301888)\cos^{11}(u) + (358912\wp^2 - 3728384)\cos^{10}(u)$$

$$+(77824\wp^2 + 539264)\cos^9(u) + (-163328\wp^2 + 1484032)\cos^8(u)$$

$$+(-51136\wp^2 + 165696)\cos^7(u) + (35968\wp^2 - 287648)\cos^6(u)$$

$$+(16224\wp^2 - 63536)\cos^5(u) + (-1793\wp^2 + 30320)\cos^4(u)$$

$$+(-1920\wp^2 + 7456)\cos^3(u) + (-382\wp^2 - 1936)\cos^2(u)$$

$$+(-32\wp^2 - 322)\cos(u) - \wp^2 + 68,$$

and

$$\mathrm{W}(u) = \wp^2 \left\{ \begin{array}{c} 256\, cos^8(u) - 512 cos^6(u) - 32\, cos^5(u) + 320\, cos^4(u) \\ +48\, cos^3(u) - 63\, cos^2(u) - 16\, cos(u) - 1 \end{array} \right\}$$

$$+4096\cos^{11}(u) - 11264\cos^9(u) - 3328\cos^8(u)$$

$$+11264\cos^7(u) + 6144\cos^6(\mathrm{u}) - 3968\cos^5(u)$$

$$-3392\cos^4(u) + 432\cos^2(u) + 32 cos(u) - 20.$$

4. Conclusion

By using above findings, we get the following.

Corollary 1. Let $\mathfrak{E} : \mathrm{M}^2 \longrightarrow \mathbb{E}^3$ be an immersion defined by $\mathfrak{E}(u, v)$. M^2 is minimal iff

$$\wp = \mp\frac{2\left(-\left\{\begin{matrix}16384\cos^{12}(u)-49152\cos^{10}(u)\\-3072\cos^{9}(u)+55296\cos^{8}(u)\\+7488\cos^{7}(u)-28480\cos^{6}(u)\\-6192\cos^{5}(u)+6120\cos^{4}(u)\\+1940\cos^{3}(u)-180\cos^{2}(u)\\-120\cos(u)-25\end{matrix}\right\}.\left\{\begin{matrix}832\cos^{7}(u)\\-1456\cos^{5}(u)\\-408\cos^{4}(u)\\+728\cos^{3}(u)\\+380\cos^{2}(u)\\-34\cos(u)-37\end{matrix}\right\}\right)^{1/2}.\left\{\begin{matrix}8\cos^{4}(u)\\-8\cos^{2}(u)\\-2\cos(u)+1\end{matrix}\right\}}{\left\{\begin{matrix}16384\cos^{12}(u)-49152\cos^{10}(u)-3072\cos^{9}(u)+55296\cos^{8}(u)\\+7488\cos^{7}(u)-28480\cos^{6}(u)-6192\cos^{5}(u)+6120\cos^{4}(u)\\+1940\cos^{3}(u)-180\cos^{2}(u)-120\cos(u)-25\end{matrix}\right\}}.$$

Proof. Computing $H=0$, we obtain $\Bbbk(u)=0$. Then, we reveal $\wp$.

Corollary 2. Let $\mathfrak{E}:\ \mathrm{M}^2 \longrightarrow \mathbb{E}^3$ be an immersion given by $\mathfrak{E}(u,v)$. M^2 is flat iff

$$\wp = \mp\frac{\{2\cos(4u)-4\cos(u)\}.\left\{\begin{matrix}-10\cos(11u)+59\cos(8u)+25\cos(6u)-105\cos(5u)\\-125\cos(3u)+59\cos(2u)-10\cos(u)+93\end{matrix}\right\}^{1/2}}{4\cos(8u)-4\cos(5u)+4\cos(3u)+\cos(2u)-5}.$$

Proof. Solving $K(u)=0$, we have $\Bbbk(u)=0$. Hence, we obtain $\wp$.

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Fekete-Szegö Inequalities For A Subclass of Bi-univalent Functions Defined by Laguerre Polynomials

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Abstract

In this paper, we obtain the bounds for Fekete-Szegö inequalities for a new subclass of analytic and bi-univalent functions in the open unit disk defined by Laguerre polynomials. Furthermore, we investigate the special cases and consequences for a new subclass.

Keywords: Analytic and bi-univalent functions, subordination, Fekete-Szegö problem, Laguerre polynomials.

1. Introduction and Preliminaries

Let A represents the class of functions whose members are of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad (z \in \Delta), \tag{1}$$

which are analytic in $\Delta = \{z \in \mathbb{C} : |z| < 1\}$, and let S be the subclass of A whose members are univalent in Δ. The Koebe one quarter theorem [3] ensures that the image of Δ under every univalent function $f \in A$ contains a disk of radius $\frac{1}{4}$. Thus every univalent function f has an inverse f^{-1} satisfying

$$f^{-1}(f(z)) = z, (z \in \Delta) \text{ and } f(f^{-1}(\omega)) = \omega, \ (|\omega| < r_0(f), \ r_0(f) \geq \tfrac{1}{4}).$$

A function $f \in A$ is said to be bi-univalent in Δ if both f and f^{-1} are univalent in Δ, and let Σ denote the class of bi – univalent functions defined in the unit disk Δ. Since $f \in \Sigma$ has the Maclaurin series given by (1), a computation shows that its inverse $g = f^{-1}$ has the expansion

$$g(\omega) = f^{-1}(\omega) = \omega - a_2\omega^2 + (2{a_2}^2 - a_3)\omega^3 + \cdots. \tag{2}$$

We notice that the class Σ is not empty. For instance, the functions

$$f_1(z) = \frac{z}{z-1}, \quad f_2(z) = \frac{1}{2}\log\frac{1+z}{1-z}, \quad f_3(z) = -\log(1-z)$$

with their corresponding inverses

$$f_1^{-1}(\omega) = \frac{\omega}{1+\omega'}, \quad f_2^{-1}(\omega) = \frac{e^{2\omega}-1}{e^{2\omega}+1} \quad , \quad f_3^{-1}(\omega) = \frac{e^{\omega}-1}{e^{\omega}}$$

are elements of Σ. However, the Koebe function is not a member of Σ. Lately, Srivastava et al. $[17]$ have essentially revived the study of analytic and bi-univalent functions; this was followed by such works as those of $[1-16]$. Several authors have introduced and examined subclasses of bi-univalent functions and obtained bounds for the initial coefficients (see $[17]$, bi-close-to-convex functions $[6,11]$, and bi-prestarlike functions by Jahangiri and Hamidi $[7]$.

Let f and g be analytic functions in Δ. We define that the function f is subordinate to g in Δ and denoted by

$$f(z) \prec g(z) \quad (z \in \Delta),$$

if there exists a Schwarz function w, which is analytic in Δ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \Delta)$ such that

$$f(z) = g(w(z)) \quad (z \in \Delta).$$

If g is a univalent function in Δ, then

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(\Delta) \subset g(\Delta).$$

The classical Fekete-Szegö inequality [4], presented by means of Loewner's method, for the coefficients of $f \in S$ is that

$$|a_3 - \mu a_2^2| \leq 1 + 2exp\left(\frac{-2\mu}{1-\mu}\right) \; for \;\; 0 \leq \mu < 1.$$

As $\mu \to 1^-$, we have the elementary inequality $|a_3 - {a_2}^2| \leq 1$. Moreover, the coefficient functional

$$F_\mu(f) = a_3 - \mu {a_2}^2$$

on the normalized analytic functions, f in the open unit disk Δ plays an important role in geometric function theory. The problem of maximizing the absolute value of the functional $F_\mu(f)$ is called the Fekete-Szegö problem.

In geometric function theory, there have been numerous interesting and fruitful us-ages of a wide variety of special functions, q-calculus, and special polynomials; for example, the Fibonacci polynomials, the Faber polynomials, the Lucas polynomials, the Pell polynomials, the Pell–Lucas polynomials, and the

Chebyshev polynomials of the second kind. The Horadam polynomials are potentially important in various disciplines in the mathematical, physical, statistical, and engineering sciences.

The generalized Laguerre polynomial $L_n^{\gamma}(\beta)$ is the polynomial solution $\phi(\beta)$ of the differential equation (see [12])

$$\beta\phi'' + (1+\gamma-\beta)\phi' + n\phi = 0,$$

where $\gamma > -1$ and n is non-negative integers.

The generating function of generalized Laguerre polynomial $L_n^{\gamma}(\beta)$ is defined by

$$H_{\gamma}(\beta, z) = \sum_{n=0}^{\infty} L_n^{\gamma}(\beta) z^n = \frac{e^{-\frac{\beta z}{1-z}}}{(1-z)^{\gamma+1}}, \tag{3}$$

where $\beta \in \mathbb{R}$ and $z \in \Delta$. Generalized Laguerre polynomials can also be defined by the following recurrence relations:

$$L_{n+1}^{\gamma}(\beta) = \frac{2n+1+\gamma-\beta}{n+1} L_n^{\gamma}(\beta) - \frac{n+\gamma}{n+1} L_{n-1}^{\gamma}(\beta) \quad (n \geq 1), \tag{4}$$

with the initial conditions

$$L_0^{\gamma}(\beta) = 1, \quad L_1^{\gamma}(\beta) = 1+\gamma-\beta \quad \text{and} \quad L_1^{\gamma}(\beta) = \frac{\beta^2}{2} - (\gamma+2)\beta + \frac{(\gamma+1)(\gamma+1)}{2}. \tag{5}$$

Clearly, when $\gamma = 0$ the generalized Laguerre polynomials leads to the simply Laguerre polynomial, i.e., $L_n^0(\beta) = L_n(\beta)$.

The analytic function $h(z)$ with positive real part in Δ such that $h(0) = 1, h'(0) > 0$ and $h(\Delta)$ is symmetric with respect to real axis, which is of the type:

$$h(z) = 1 + e_1 z + e_2 z^2 + \cdots \tag{6}$$

where

$$e_1 = 1+\gamma-\beta, \quad e_2 = \frac{\beta^2}{2} - (\gamma+2)\beta + \frac{(\gamma+1)(\gamma+1)}{2}. \tag{7}$$

First, we define a new subclass of bi-univalent functions in the open unit disk, associated with Laguerre polynomials as below.

Definition 1. For $\tau \in \mathbb{C}\backslash\{0\}$, $0 \leq \vartheta \leq 1$ and h is analytic in Δ, $h(0) = 1$, a function $f \in \Sigma$ the form (1) is said to be in the class $M_{\Sigma}(\tau, \vartheta, e_1, e_2)$ if the following subordinations hold:

$$1 + \frac{1}{\tau}(f'(z) + \vartheta z f''(z) - 1) \prec h(z) \tag{8}$$

and

$$1 + \frac{1}{\tau}(g'(\omega) + \vartheta \omega f''(\omega) - 1) \prec \; h(\omega) \tag{9}$$

where $z, \omega \in \Delta$, e_1, e_2 are given by (7), and $g = f^{-1}$ is given by (2).

To obtain our first results, we need the following lemma:

Lemma 1 ([15], p.172). Assume that $w(z) = \sum_{n=1}^{\infty} w_n z^n$, $z \in \Delta$, is an analytic function in Δ such that $|w(z)| < 1$ for all $z \in \Delta$. Then,

$$|w_1| \leq 1, \quad |w_n| \leq 1 - |w_1|^2, \quad n = 2,3, \ldots.$$

In the following result, we obtain upper bounds for the modules of the first two coefficients for the functions to belong to a class $M_\Sigma(\tau, \vartheta, e_1, e_2)$.

2. Fekete-Szegö Inequality for the Function Class $M_\Sigma(\tau, \vartheta, e_1, e_2)$

Due to the result of Zaprawa [18], in this section, we obtain the Fekete-Szegö inequality for the function classes $M_\Sigma(\tau, \vartheta;\ \phi_l^\lambda)$.

Theorem 1. Assume that $\tau \in \mathbb{C}\backslash\{0\}$, $0 \leq \vartheta \leq 1$. Let f given by (1) be in the class $M_\Sigma(\tau, \vartheta, e_1, e_2)$ with $h(z) = 1 + e_1 z + e_2 z^2 + \cdots$ and $\mu \in \mathbb{R}$. Then, we have

$$|a_3 - \mu {a_2}^2| \leq \begin{cases} \frac{|\tau| e_1}{3|1+2\vartheta|}, & if \quad |l(\mu)| \leq \frac{1}{6|1+2\vartheta|}, \\ 2|\tau||l(\mu)| e_1, & if \quad |l(\mu)| \geq \frac{1}{6|1+2\vartheta|}, \end{cases} \tag{10}$$

where e_1, e_2 are given by (7) and

$$l(\mu) = \frac{(1-\mu)\tau^2 {e_1}^3}{6\tau(1+2\vartheta){e_1}^2 - 8(1+\vartheta)^2 e_2}. \tag{11}$$

Proof. Suppose that $f \in M_\Sigma(\tau, \vartheta, e_1, e_2)$. From the definition in formulas (8) and (9), we have

$$1 + \frac{1}{\tau}(f'(z) + \vartheta z f''(z) - 1) = h(\varphi(z)) \tag{12}$$

and

$$1 + \frac{1}{\tau}(g'(\omega) + \vartheta \omega g''(\omega) - 1) = h(x(\omega)), \tag{13}$$

where there exsist two holomorphic functions $\varphi, \chi: \Delta \to \Delta$ given by

$$\varphi(z) = r_1 z + r_2 z^2 + \cdots, \tag{14}$$

$$x(\omega) = s_1 \omega + s_2 \omega^2 + \cdots, \tag{15}$$

with $\varphi(0) = 0 = x(0)$, and $|\varphi(z)| < 1$, $|x(\omega)| < 1$, for all $z, \omega \in \Delta$. From Lemma 1, it follows that

$$|r_j| \leq 1 \text{ and } |s_j| \leq 1, \text{ for all } j \in \mathbb{N}. \tag{16}$$

Replacing (14) and (15) in (12) and (13), respectively, we have

$$1 + \frac{1}{\tau}(f'(z) + \vartheta z f''(z) - 1) = 1 + e_1 \varphi(z) + e_2 \varphi^2(z) + \cdots, \tag{17}$$

and

$$1 + \frac{1}{\tau}(g'(\omega) + \vartheta \omega g''(\omega) - 1) = 1 + e_1 x(\omega) + e_2 x^2(\omega) + \cdots, \tag{18}$$

In view of (1) and (2), from (17) and (18), we obtain

$$1 + \frac{1}{\tau}(2a_2(1+\vartheta)z + 3a_3(1+2\vartheta)z^2) = 1 + e_1 r_1 z + [e_1 r_2 + e_2 {r_1}^2] z^2$$

and

$$1 + \frac{1}{\tau}(-2a_2(1+\vartheta)\omega + 3(2{a_2}^2 - a_3)(1+2\vartheta)\omega^2)$$

$$= 1 + e_1 s_1 \omega + [e_1 s_2 + e_2 {s_1}^2]\omega^2$$

which yields the following relations :

$$2a_2(1+\vartheta) = \tau e_1 r_1, \tag{19}$$

$$3a_3(1+2\vartheta) = \tau e_1 r_2 + \tau e_2 {r_1}^2, \tag{20}$$

and

$$-2a_2(1+\vartheta) = \tau e_1 s_1, \tag{21}$$

$$3(2{a_2}^2 - a_3)(1+2\vartheta) = \tau e_1 s_2 + \tau e_2 {s_1}^2 \tag{22}$$

From (19) and (21), it follows that

$$r_1 = -s_1, \tag{23}$$

and

$$8{a_2}^2(1+\vartheta)^2 = \tau^2{e_1}^2({r_1}^2+{s_1}^2)$$

$${a_2}^2 = \frac{\tau^2{e_1}^2({r_1}^2+{s_1}^2)}{8(1+\vartheta)^2} \tag{24}$$

Adding (20) and (22), using (24), we obtain

$${a_2}^2 = \frac{\tau^2{e_1}^3(r_2+s_2)}{6\tau(1+2\vartheta){e_1}^2-8(1+\vartheta)^2e_2}\,. \tag{25}$$

By subtracting (22) from (20), using (23) and (24), we get

$$\begin{aligned} a_3 &= \frac{\tau e_1(r_2-s_2)+\tau e_2({r_1}^2-{s_1}^2)}{6(1+2\vartheta)} + {a_2}^2 \\ &= \frac{\tau e_1(r_2-s_2)+\tau e_2({r_1}^2-{s_1}^2)}{6(1+2\vartheta)} + \frac{\tau^2{e_1}^2({r_1}^2+{s_1}^2)}{8(1+\vartheta)^2} \end{aligned} \tag{26}$$

From (25) and (26), we have

$$a_3-\mu{a_2}^2 = \frac{\tau e_1(r_2-s_2)}{6(1+2\vartheta)} + (1-\mu){a_2}^2$$

$$= \frac{\tau e_1(r_2-s_2)}{6(1+2\vartheta)} + \frac{(1-\mu)\tau^2{e_1}^3(r_2+s_2)}{6\tau(1+2\vartheta){e_1}^2-8(1+\vartheta)^2e_2}$$

$$= \tau e_1\left[\frac{r_2}{6(1+2\vartheta)} - \frac{s_2}{6(1+2\vartheta)} + \frac{(1-\mu)\tau^2{e_1}^3r_2}{6\tau(1+2\vartheta){e_1}^2-8(1+\vartheta)^2e_2} + \frac{(1-\mu)\tau^2{e_1}^3s_2}{6\tau(1+2\vartheta){e_1}^2-8(1+\vartheta)^2e_2}\right]$$

$$= \tau e_1\left[\left(l(\mu)+\frac{1}{6(1+2\vartheta)}\right)r_2 + \left(l(\mu)-\frac{1}{6(1+2\vartheta)}\right)s_2\right],$$

where

$$l(\mu) = \frac{(1-\mu)\tau^2{e_1}^3}{6\tau(1+2\vartheta){e_1}^2-8(1+\vartheta)^2e_2}.$$

Now, by using (7)

$$a_3-\mu{a_2}^2 = \tau(1+\gamma-\beta)\left[\left(l(\mu)+\frac{1}{6(1+2\vartheta)}\right)r_2 + \left(l(\mu)-\frac{1}{6(1+2\vartheta)}\right)s_2\right],$$

where

$$l(\mu) = \frac{(1-\mu)\tau^2(1+\gamma-\beta)^3}{6\tau(1+2\vartheta)(1+\gamma-\beta)^2-8(1+\vartheta)^2(\frac{\beta^2}{2}-(\gamma+2)\beta+\frac{(\gamma+1)(\gamma+1)}{2})}.$$

Therefore, in view of (7) and (16), we conclude that the required inequality holds.

For $\tau = 1$ in Theorem 1, we obtain the following corollary.

Corollary 1. Assume that $0 \leq \vartheta \leq 1$. Let f given by (1) be in the class $M_\Sigma(1, \vartheta, e_1, e_2)$ with $h(z) = 1 + e_1 z + e_2 z^2 + \cdots$ and $\mu \in \mathbb{R}$. Then, we have

$$|a_3 - \mu a_2{}^2| \leq \begin{cases} \dfrac{e_1}{3|1+2\vartheta|}, & if \qquad |l(\mu)| \leq \dfrac{1}{6|1+2\vartheta|}, \\ 2|l(\mu)|e_1, & if \qquad |l(\mu)| \geq \dfrac{1}{6|1+2\vartheta|}, \end{cases}$$

where e_1, e_2 are given by (7) and

$$l(\mu) = \frac{(1-\mu)e_1{}^3}{6(1+2\vartheta)e_1{}^2 - 8(1+\vartheta)^2 e_2}.$$

For $\tau = \vartheta = 1$ in Theorem 1, we get the following corollary.

Corollary 2. Let f given by (1) be in the class $M_\Sigma(1,1, e_1, e_2)$ with $h(z) = 1 + e_1 z + e_2 z^2 + \cdots$ and $\mu \in \mathbb{R}$. Then, we have

$$|a_3 - \mu a_2{}^2| \leq \begin{cases} \dfrac{e_1}{9}, & if \qquad |l(\mu)| \leq \dfrac{1}{18}, \\ 2|l(\mu)|e_1, & if \qquad |l(\mu)| \geq \dfrac{1}{18}, \end{cases}$$

where e_1, e_2 are given by (7) and

$$l(\mu) = \frac{(1-\mu)e_1{}^3}{18e_1{}^2 - 32e_2}.$$

For $\tau = 1 \; and \; \vartheta = 0$ in Theorem 1, we have the following corollary.

Corollary 3. Let f given by (1) be in the class $M_\Sigma(1,0, e_1, e_2)$ with $h(z) = 1 + e_1 z + e_2 z^2 + \cdots$ and $\mu \in \mathbb{R}$. Then, we have

$$|a_3 - \mu a_2{}^2| \leq \begin{cases} \dfrac{e_1}{3}, & if \qquad |l(\mu)| \leq \dfrac{1}{6}, \\ 2|l(\mu)|e_1, & if \qquad |l(\mu)| \geq \dfrac{1}{6}, \end{cases}$$

where e_1, e_2 are given by (7) and

$$l(\mu)=\frac{(1-\mu){e_1}^3}{6{e_1}^2-8e_2}.$$

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Fekete-Szegö Problem For Some Subclasses of Bi-Univalent Functions Defined By The generalized Integral Operator

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Abstract

In this study, we solve Fekete-Szegö problem for a new subclass $\mathcal{B}_{\Sigma}^{\beta}(\delta,\lambda;\varphi)$ of bi-univalent functions in the open unit disk U defined by generalized Jung-Kim-Srivastava integral operator.

Keywords: Analytic Function, Univalent function, Bi-Univalent function, Integral operator, Fekete-Szegö problem.

1. Introduction and Preliminaries

Let A denote the class of functions of the form:

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n, \tag{1}$$

which are analytic in the open unit disk $U=\{z\in\mathbb{C}:|z|<1\}$. Further, by S we shall denote the class of all functions in A which are univalent in U. It is well known that every function $f\in S$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z))=z \qquad (z\in U)$$

and

$$f(f^{-1}(w))=w \qquad \left(|w|<r_0(f);r_0(f)\geq\frac{1}{4}\right)$$

where

$$f^{-1}(w)=w-a_2w^2+\left(2a_2^2-a_3\right)w^3-\left(5a_2^3-5a_2a_3+a_4\right)w^4+\dots$$

A function $f\in A$ is said to be in $\sum$, the class of bi-univalent functions in U, if both $f(z)$ and $f^{-1}(z)$ are univalent in U. Lewin [10] showed that $|a_2|<1.51$ for every function $f\in\sum$ given by (1). Posteriorly, Brannan and Clunie [1] improved Lewin's result and conjectured that $|a_2| \le \sqrt{2}$ for every function $f \in \Sigma$ given by (1). Later, Netanyahu [11] showed that $\max_{f\in\Sigma} |a_2| = \frac{4}{3}$ The coefficient estimate problem for each of the following Taylor-Maclaurin coefficients:

$$|a_n| \quad \left(n \in N = \{1,2,...\}; n \ge 4\right)$$

is still an open problem (see, for details, [14]). Since then, many researchers (see [2,4,6,8,15]) investigated several interesting subclasses of the class $\sum$ and found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. One of the most important problem on coefficients of univalent functions as known Fekete-Szegö problem. Very recently, some results have obtained by [3,5,8,9,13] for this problem.

Let P denote the class of function of p analytic in U such that $p(0)=1$ and $\mathrm{Re}\{p(z)\}>0$, where

$$p(z) = 1 + p_1 z + p_2 z^2 + ... \quad (z \in U).$$

If f and g are analytic in U, we say that f is subordinate to g, written symbolically as

$$f \prec g \qquad \text{or} \qquad f(z) \prec g(z) \quad (z \in U),$$

if there exists a Schwarz function $w(z)$, which (by definition) is analytic in U with $w(0)=0$ and $|w(z)|<1$ in U such that $f(z) = g(w(z)),\ z \in U$.

In particular, if the function $g(z)$ is univalent in U, then we have that:

$$f(z) \prec g(z) \quad (z \in U) \quad \text{if and only if} \quad f(0) = g(0) \ \text{ and } \ f(U) \subseteq g(U).$$

Let φ be an analytic function with positive real part in the unit disk U such that

$$\varphi(0) = 1, \varphi'(0) > 0$$

and $\varphi(U)$ is symmetric with respect to the real axis and has a series expansion of the form (see [11]):

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + ... \quad (B_1 > 0).$$

Let $u(z)$ and $v(z)$ be two analytic functions in the unit disk U with $u(0) = v(0) = 0$ $|u(z)| < 1, |v(z)| < 1$, and suppose that

$$u(z) = 1 + b_1 z + b_2 z^2 + b_3 z^3 + ... \text{ and } \mathrm{v}(w) = 1 + c_1 w + c_2 w^2 + c_3 w^3 + ... \cdot \tag{2}$$

For above functions, well-known inequalities are

$$|b_1| \le 1, |b_2| \le 1 - |b_1|^2, |c_1| < 1 \text{ and } |c_2| \le 1 - |c_1|^2. \tag{3}$$

Further we have

$$\varphi(u(z)) = 1 + B_1 b_1 z + (B_1 b_2 + B_2 b_1^2) z^2 + \dots (|z| < 1) \tag{4}$$

and

$$\varphi(v(w)) = 1 + B_1 c_1 w + (B_1 c_2 + B_2 c_1^2) w^2 + \dots (|w| < 1) \tag{5}$$

In this study, we consider the generalized Jung-Kim-Srivastava integral operator Q_δ^β [7] defined by

$$\begin{aligned} Q_\delta^\beta f(z) &= \frac{\Gamma(\beta+\delta+1)}{z\Gamma(\beta)\Gamma(\delta+1)} \int_0^z t^{\delta-1} (1 - \frac{t}{z})^{\beta-1} f(t) dt, \ \beta \ge 0, \ \delta > -1 \\ &= z + \sum_{n=2}^{\infty} \frac{\Gamma(\beta+\delta+1)\Gamma(\delta+n)}{\Gamma(\beta+\delta+n)\Gamma(\delta+1)} a_n z^n \end{aligned} \tag{6}$$

and for $\beta = 0$, we have $Q_\delta^0 f(z) = f(z)$.

The main object of this paper is to introduce the following new subclass of bi-univalent functions involving Jung-Kim-Srivastava integral operator Q_δ^β [7] and discuss Fekete-Szegö functional problem for functions in this new class (see [5]).

2. Fekete-Szegö problem for the functions class $B_\Sigma^\beta(\delta, \lambda; \varphi)$

Definition 1. A function $f(z) \in \Sigma$ is said to be in the class $B_\Sigma^\beta(\delta, \lambda; \varphi)$ if and only if

$$(1-\lambda) \frac{Q_\delta^\beta f(z)}{z} + \lambda \left(Q_\delta^\beta f(z)\right)' \prec \varphi(z)$$

and

$$(1-\lambda) \frac{Q_\delta^\beta g(w)}{w} + \lambda \left(Q_\delta^\beta g(w)\right)' \prec \varphi(w)$$

where $0 \le \lambda \le 1, \ z, \ w \in U$ and $g(w) = f^{-1}(w)$.

Now, we are ready to find the sharp bounds of Fekete–Szegö functional $a_3 - \mu a_2^2$ defined for $f \in B_\Sigma^\beta(\delta, \lambda; \varphi)$ given by (1).

Theorem 1. Let $f(z)$ given by (1) be in the class $B_{\Sigma}^{\beta}(\delta,\lambda;\varphi)$. Then

$$\left|a_3-\mu a_2^2\right|\le\begin{cases}\dfrac{\Gamma(\beta+\delta+3)\Gamma(\delta+1)B_1}{(1+2\lambda)\Gamma(\beta+\delta+1)\Gamma(\delta+3)} & \text{for} \quad 0\le\left|h(\mu)\right|<\dfrac{\Gamma(\beta+\delta+3)\Gamma(\delta+1)}{2(1+2\lambda)\Gamma(\beta+\delta+1)\Gamma(\delta+3)}\\ 2B_1\left|h(\mu)\right| & \text{for} \quad \left|h(\mu)\right|\ge\dfrac{\Gamma(\beta+\delta+3)\Gamma(\delta+1)}{2(1+2\lambda)\Gamma(\beta+\delta+1)\Gamma(\delta+3)}\end{cases} \tag{7}$$

where

$$h(\mu)=\frac{B_1^2(1-\mu)}{2B_1^2\dfrac{(1+2\lambda)\Gamma(\beta+\delta+1)\Gamma(\delta+3)}{\Gamma(\beta+\delta+3)\Gamma(\delta+1)}-2B_2\left[\dfrac{(1+\lambda)\Gamma(\beta+\delta+1)\Gamma(\delta+2)}{\Gamma(\beta+\delta+2)\Gamma(\delta+1)}\right]^2}.$$

Proof. Let $f(z)\in B_{\Sigma}^{\beta}(\delta,\lambda;\varphi)$. By the definition of subordination, there are analytic functions u and v with $u(0)=v(0)=0$, $\left|u(z)\right|<1$, $\left|v(w)\right|<1$, given by (2) and satisfying the following conditions:

$$(1-\lambda)\frac{Q_{\delta}^{\beta}f(z)}{z}+\lambda\left(Q_{\delta}^{\beta}f(z)\right)'=\varphi\left(u(z)\right)$$

and

$$(1-\lambda)\frac{Q_{\delta}^{\beta}g(w)}{w}+\lambda\left(Q_{\delta}^{\beta}g(w)\right)'=\varphi\left(v(w)\right),$$

where $g(w)=f^{-1}(w)$. Since

$$\begin{aligned}&(1-\lambda)\frac{Q_{\delta}^{\beta}f(z)}{z}+\lambda\left(Q_{\delta}^{\beta}f(z)\right)'\\&=1+(1+\lambda)\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+2)}{\Gamma(\beta+\delta+2)\Gamma(\delta+1)}a_2z+(1+2\lambda)\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+3)}{\Gamma(\beta+\delta+3)\Gamma(\delta+1)}a_3z^2+\dots\end{aligned} \tag{8}$$

and

$$\begin{aligned}&(1-\lambda)\frac{Q_{\delta}^{\beta}g(w)}{w}+\lambda\left(Q_{\delta}^{\beta}g(w)\right)'\\&=1-(1+\lambda)\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+2)}{\Gamma(\beta+\delta+2)\Gamma(\delta+1)}a_2w+(1+2\lambda)\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+3)}{\Gamma(\beta+\delta+3)\Gamma(\delta+1)}\left(2a_2^2-a_3\right)w^2+\dots,\end{aligned} \tag{9}$$

it follows from (4), (5), (8) and (9) that

$$\left(1+\lambda\right)\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+2)}{\Gamma(\beta+\delta+2)\Gamma(\delta+1)}a_2 = B_1 b_1, \tag{10}$$

$$\left(1+2\lambda\right)\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+3)}{\Gamma(\beta+\delta+3)\Gamma(\delta+1)}a_3 = B_1 b_2 + B_2 b_1^2, \tag{11}$$

$$-\left(1+\lambda\right)\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+2)}{\Gamma(\beta+\delta+2)\Gamma(\delta+1)}a_2 = B_1 c_1, \tag{12}$$

and

$$\left(1+2\lambda\right)\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+3)}{\Gamma(\beta+\delta+3)\Gamma(\delta+1)}\left(2a_2^2 - a_3\right) = B_1 c_2 + B_2 c_1^2. \tag{13}$$

From (10) and (12), we get

$$c_1 = -b_1 \tag{14}$$

$$2\left[\left(1+\lambda\right)\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+2)}{\Gamma(\beta+\delta+2)\Gamma(\delta+1)}\right]^2 a_2^2 = B_1^2\left(b_1^2 + c_1^2\right). \tag{15}$$

By adding (11) to (13), we have

$$2\left(1+2\lambda\right)\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+3)}{\Gamma(\beta+\delta+3)\Gamma(\delta+1)}a_2^2 = B_1\left(b_2 + c_2\right) + B_2\left(b_1^2 + c_1^2\right). \tag{16}$$

Therefore, from equalities (15) and (16) we find that

$$\left[2\left(1+2\lambda\right)\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+3)}{\Gamma(\beta+\delta+3)\Gamma(\delta+1)}B_1^2 - 2B_2\left(\left(1+\lambda\right)\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+2)}{\Gamma(\beta+\delta+2)\Gamma(\delta+1)}\right)^2\right]a_2^2 = B_1^3\left(b_2 + c_2\right) \tag{17}$$

We conclude that, from (17)

$$a_2^2 = \frac{B_1^3\left(b_2 + c_2\right)}{2\left(1+2\lambda\right)\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+3)}{\Gamma(\beta+\delta+3)\Gamma(\delta+1)}B_1^2 - 2B_2\left(\left(1+\lambda\right)\frac{\Gamma(\beta+\delta+1)\Gamma(\delta+2)}{\Gamma(\beta+\delta+2)\Gamma(\delta+1)}\right)^2} \tag{18}$$

and subtracting (13) from (11) and using (14)

$$a_3 = a_2^2 + \frac{\Gamma(\beta+\delta+3)\Gamma(\delta+1)B_1\left(b_2-c_2\right)}{2\left(1+2\lambda\right)\Gamma(\beta+\delta+1)\Gamma(\delta+3)}. \tag{19}$$

From the Eqs. (18) and (19), it follows that

$$a_3-\mu a_2^2 = B_1\left[\left(h\left(\mu\right)+\frac{\Gamma(\beta+\delta+3)\Gamma(\delta+1)}{2\left(1+2\lambda\right)\Gamma(\beta+\delta+1)\Gamma(\delta+3)}\right)b_2+\left(h\left(\mu\right)-\frac{\Gamma(\beta+\delta+3)\Gamma(\delta+1)}{2\left(1+2\lambda\right)\Gamma(\beta+\delta+1)\Gamma(\delta+3)}\right)c_2\right],$$

where

$$h\left(\mu\right)=\frac{B_1^2\left(1-\mu\right)}{2B_1^2\dfrac{\left(1+2\lambda\right)\Gamma(\beta+\delta+1)\Gamma(\delta+3)}{\Gamma(\beta+\delta+3)\Gamma(\delta+1)}-2B_2\left[\dfrac{\left(1+\lambda\right)\Gamma(\beta+\delta+1)\Gamma(\delta+2)}{\Gamma(\beta+\delta+2)\Gamma(\delta+1)}\right]^2}.$$

Since all B_i are real and $B_1>0$, which implies the assertion (7). This completes the proof of Theorem 1.

By taking $\lambda=1$ in Theorem 1, we have

Corollary 1. Let $f\left(z\right)$ given by (1) be in the class $B_\Sigma^\beta\left(\delta,1;\varphi\right).$ Then

$$\left|a_3-\mu a_2^2\right|\le\begin{cases}\dfrac{\Gamma(\beta+\delta+3)\Gamma(\delta+1)B_1}{3\Gamma(\beta+\delta+1)\Gamma(\delta+3)} & for \quad 0\le\left|h\left(\mu\right)\right|<\dfrac{\Gamma(\beta+\delta+3)\Gamma(\delta+1)}{6\Gamma(\beta+\delta+1)\Gamma(\delta+3)}\\ 2B_1\left|h\left(\mu\right)\right| & for \quad \left|h\left(\mu\right)\right|\ge\dfrac{\Gamma(\beta+\delta+3)\Gamma(\delta+1)}{6\Gamma(\beta+\delta+1)\Gamma(\delta+3)}\end{cases}$$

where

$$h\left(\mu\right)=\frac{B_1^2\left(1-\mu\right)}{6B_1^2\dfrac{\Gamma(\beta+\delta+1)\Gamma(\delta+3)}{\Gamma(\beta+\delta+3)\Gamma(\delta+1)}-8B_2\left[\dfrac{\Gamma(\beta+\delta+1)\Gamma(\delta+2)}{\Gamma(\beta+\delta+2)\Gamma(\delta+1)}\right]^2}.$$

Putting $\beta=0$ in Theorem 1., we have

Corollary 2. Let $f\left(z\right)$ given by (1) be in the class $B_\Sigma^0\left(\delta,\lambda;\varphi\right)=B_\Sigma\left(\lambda;\varphi\right).$ Then

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} \dfrac{B_1}{(1+2\lambda)} & \text{for} \quad 0 \le \left|h(\mu)\right| < \dfrac{1}{2(1+2\lambda)} \\ 2B_1\left|h(\mu)\right| & \text{for} \quad \left|h(\mu)\right| \ge \dfrac{1}{2(1+2\lambda)} \end{cases}$$

where

$$h(\mu)=\frac{B_1^2(1-\mu)}{2B_1^2(1+2\lambda)-2B_2(1+\lambda)^2}.$$

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Fekete-Szegö Problem For Some Subclasses of Bi-Univalent Functions

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Abstract

In this study, we solve Fekete-Szegö problem for a new subclass $\mathcal{B}_{\Sigma}^{\delta}(c,\beta;\varphi)$ of bi-univalent functions in the open unit disk U defined by an integral operator.

Keywords: Analytic Function, Univalent function, Bi-Univalent function, Integral operator, Fekete-Szegö problem.

1. Introduction and Preliminaries

Let A denote the class of functions of the form:

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n, \qquad (1)$$

which are analytic in the open unit disk $U=\{z\in\mathbb{C}:|z|<1\}$. Further, by S we shall denote the class of all functions in A which are univalent in U. It is well known that every function $f\in S$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z))=z \qquad (z\in U)$$

and

$$f(f^{-1}(w))=w \qquad \left(|w|<r_0(f);r_0(f)\geq\frac{1}{4}\right)$$

where

$$f^{-1}(w)=w-a_2w^2+(2a_2^2-a_3)w^3-(5a_2^3-5a_2a_3+a_4)w^4+\ldots$$

A function $f\in A$ is said to be in $\sum$, the class of bi-univalent functions in U, if both $f(z)$ and $f^{-1}(z)$ are univalent in U. Lewin [10] showed that $|a_2|<1.51$ for every function $f\in\sum$ given by (1). Posteriorly, Brannan and Clunie [2] improved Lewin's result and conjectured that $|a_2|\leq\sqrt{2}$ for every function $f\in\sum$

given by (1). Later, Netanyahu [11] showed that $\max_{f\in\Sigma}|a_2|=\frac{4}{3}$ The coefficient estimate problem for each of the following Taylor-Maclaurin coefficients:

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If f and g are analytic in U, we say that f is subordinate to g, written symbolically as

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if there exists a Schwarz function $w(z)$, which (by definition) is analytic in U with $w(0)=0$ and $|w(z)|<1$ in U such that $f(z)=g(w(z)),\ z\in U.$

In particular, if the function $g(z)$ is univalent in U, then we have that:

$$f(z)\prec g(z) \quad (z\in U) \quad \text{if and only if} \quad f(0)=g(0) \text{ and } f(U)\subseteq g(U).$$

Let φ be an analytic function with positive real part in the unit disk U such that

$$\varphi(0)=1, \varphi'(0)>0$$

and $\varphi(U)$ is symmetric with respect to the real axis and has a series expansion of the form (see [11]):

$$\varphi(z)=1+B_1z+B_2z^2+B_3z^3+... \quad (B_1>0).$$

Let $u(z)$ and $v(z)$ be two analytic functions in the unit disk U with $u(0)=v(0)=0$ $|u(z)|<1, |v(z)|<1$, and suppose that

$$u(z)=1+b_1z+b_2z^2+b_3z^3+... \text{ and } \mathrm{v}(w)=1+c_1w+c_2w^2+c_3w^3+...\cdot \tag{2}$$

For above functions, well-known inequalities are

$$|b_1| \le 1, |b_2| \le 1-|b_1|^2, |c_1| < 1 \text{ and } |c_2| \le 1-|c_1|^2. \tag{3}$$

Further we have

$$\varphi(u(z)) = 1 + B_1 b_1 z + \left(B_1 b_2 + B_2 b_1^2\right) z^2 + \ldots (|z| < 1) \tag{4}$$

and

$$\varphi(v(w)) = 1 + B_1 c_1 w + \left(B_1 c_2 + B_2 c_1^2\right) w^2 + \ldots (|w| < 1) \tag{5}$$

For $f(z) \in A,$ Al-Shaqsi [1] defined the following integral operator:

$$\begin{aligned} L_c^\delta &= (1+c)^\delta \Phi_\delta(c;z) * f(z) \\ &= -\frac{(1+c)^\delta}{\Gamma(\delta)} \int_0^1 t^{c-1} \log\left(\frac{1}{t}\right)^{\delta-1} f(zt)\,dt \end{aligned} \tag{6}$$

$(c > 0, \quad \delta > 1, z \in U)$

where Γ standarts for the usual gamma function, $\Phi_\delta(c;z)$ is the well known generalization of the Riemann- zeta and polylogarithm functions, or the δth polylogarithm function, given by

$$\Phi_\delta(c;z) = \sum_{k=1}^{\infty} \frac{z^k}{(k+c)^\delta}$$

where any term without $k+c=0$ is excluded. Also, $\Phi_{-1}(0;z) = \dfrac{z}{(1-z)^2}$ is Koebe function.

We also state that the operator $L_c^\delta f(z)$ given by (6) can be expressed by the series expansions as follows:

$$L_c^\delta f(z) = z + \sum_{k=2}^{\infty} \left(\frac{1+c}{k+c}\right)^\delta a_k z^k.$$

The main object of this paper is to introduce the following new subclass of bi-univalent functions involving the integral operator L_c^δ [1] and discuss Fekete-Szegö functional problem for functions in this new class (see [5]).

2. Fekete-Szegö problem for the functions class $\mathcal{B}_{\Sigma}^{\delta}(c,\beta;\varphi)$

Definition 1. A function $f(z)\in\Sigma$ is said to be in the class $\mathcal{B}_{\Sigma}^{\delta}(c,\beta;\varphi)$ if and only if

$$(1-\beta)\frac{L_c^{\delta} f(z)}{z}+\beta\left(L_c^{\delta} f(z)\right)' \prec \varphi(z)$$

and

$$(1-\beta)\frac{L_c^{\delta} g(w)}{w}+\beta\left(L_c^{\delta} g(w)\right)' \prec \varphi(w)$$

where $0\le\beta\le 1,\ z,\ w\in U$ and $g(w)=f^{-1}(w)$.

Now, we are ready to find the sharp bounds of Fekete–Szegö functional $a_3-\mu a_2^2$ defined for $f\in\mathcal{B}_{\Sigma}^{\delta}(c,\beta;\varphi)$ given by (1).

Theorem 1. Let $f(z)$ given by (1) be in the class $\mathcal{B}_{\Sigma}^{\delta}(c,\beta;\varphi)$. Then

$$\left|a_3-\mu a_2^2\right|\le\begin{cases}\dfrac{B_1}{(1+2\beta)\left(\dfrac{1+c}{3+c}\right)^{\delta}} & \textit{for}\quad 0\le\left|h(\mu)\right|<\dfrac{1}{2(1+2\beta)\left(\dfrac{1+c}{3+c}\right)^{\delta}}\\[2ex] 2B_1\left|h(\mu)\right| & \textit{for}\quad \left|h(\mu)\right|\ge\dfrac{1}{2(1+2\beta)\left(\dfrac{1+c}{3+c}\right)^{\delta}}\end{cases}\tag{7}$$

where

$$h(\mu)=\frac{B_1^2(1-\mu)}{2B_1^2(1+2\beta)\left(\dfrac{1+c}{3+c}\right)^{\delta}-2B_2(1+\beta)^2\left(\dfrac{1+c}{2+c}\right)^{2\delta}}.$$

Proof. Let $f(z)\in\mathcal{B}_{\Sigma}^{\delta}(c,\beta;\varphi)$. By the definition of subordination, there are analytic functions u and v with $u(0)=v(0)=0,\ \left|u(z)\right|<1,\ \left|v(w)\right|<1$, given by (2) and satisfying the following conditions:

$$(1-\beta)\frac{L_c^{\delta} f(z)}{z}+\beta\left(L_c^{\delta} f(z)\right)'=\varphi(u(z))$$

and

$$(1-\beta)\frac{L_c^{\delta}g(w)}{w}+\beta\left(L_c^{\delta}g(w)\right)'=\varphi\left(v(w)\right),$$

where $g(w)=f^{-1}(w)$. Since

$$\begin{aligned}&(1-\beta)\frac{L_c^{\delta}f(z)}{z}+\beta\left(L_c^{\delta}f(z)\right)'\\&=1+(1+\beta)\left(\frac{1+c}{2+c}\right)^{\delta}a_2z+(1+2\beta)\left(\frac{1+c}{3+c}\right)^{\delta}a_3z^2+\ldots\end{aligned}\tag{8}$$

and

$$\begin{aligned}&(1-\lambda)\frac{L_c^{\delta}g(w)}{w}+\lambda\left(L_c^{\delta}g(w)\right)'\\&=1-(1+\beta)\left(\frac{1+c}{2+c}\right)^{\delta}a_2w+(1+2\beta)\left(\frac{1+c}{3+c}\right)^{\delta}\left(2a_2^2-a_3\right)w^2+\ldots,\end{aligned}\tag{9}$$

it follows from (4), (5), (8) and (9) that

$$(1+\beta)\left(\frac{1+c}{2+c}\right)^{\delta}a_2=B_1b_1,\tag{10}$$

$$(1+2\beta)\left(\frac{1+c}{3+c}\right)^{\delta}a_3=B_1b_2+B_2b_1^2,\tag{11}$$

$$-(1+\beta)\left(\frac{1+c}{2+c}\right)^{\delta}a_2=B_1c_1,\tag{12}$$

and

$$(1+2\beta)\left(\frac{1+c}{3+c}\right)^{\delta}\left(2a_2^2-a_3\right)=B_1c_2+B_2c_1^2.\tag{13}$$

From (10) and (12), we get

$$c_1=-b_1\tag{14}$$

$$2\left[\left(\frac{1+c}{2+c}\right)^{\delta}(1+\beta)\right]^2 a_2^2 = B_1^2\left(b_1^2+c_1^2\right). \tag{15}$$

By adding (11) to (13), we have

$$2\left(\frac{1+c}{3+c}\right)^{\delta}(1+2\beta)a_2^2 = B_1\left(b_2+c_2\right)+B_2\left(b_1^2+c_1^2\right). \tag{16}$$

Therefore, from equalities (15) and (16) we find that

$$\left[2\left(\frac{1+c}{3+c}\right)^{\delta}(1+2\beta)B_1^2-2B_2\left(\left(\frac{1+c}{2+c}\right)^{\delta}(1+\beta)\right)^2\right]a_2^2 = B_1^3\left(b_2+c_2\right) \tag{17}$$

We conclude that, from (17)

$$a_2^2 = \frac{B_1^3\left(b_2+c_2\right)}{2(1+2\beta)\left(\frac{1+c}{3+c}\right)^{\delta}B_1^2-2B_2(1+\beta)^2\left(\frac{1+c}{2+c}\right)^{2\delta}} \tag{18}$$

and subtracting (13) from (11) and using (14)

$$a_3 = a_2^2+\frac{B_1\left(b_2-c_2\right)}{2(1+2\beta)\left(\frac{1+c}{3+c}\right)^{\delta}}. \tag{19}$$

From the Eqs. (18) and (19), it follows that

$$a_3-\mu a_2^2 = B_1\left[\left(h(\mu)+\frac{1}{2(1+2\beta)\left(\frac{1+c}{3+c}\right)^{\delta}}\right)b_2+\left(h(\mu)-\frac{1}{2(1+2\beta)\left(\frac{1+c}{3+c}\right)^{\delta}}\right)c_2\right],$$

where

$$h(\mu)=\frac{B_1^2(1-\mu)}{2B_1^2(1+2\beta)\left(\frac{1+c}{3+c}\right)^{\delta}-2B_2(1+\beta)^2\left(\frac{1+c}{2+c}\right)^{2\delta}}.$$

Since all B_i are real and $B_1>0$, which implies the assertion (7). This completes the proof of Theorem 1.

By taking $\beta=1$ in Theorem 1, we have

Corollary 1. Let $f(z)$ given by (1) be in the class $\mathcal{B}_{\Sigma}^{\delta}(c,1;\varphi)$. Then

$$\left|a_3-\mu a_2^2\right|\le\begin{cases}\dfrac{B_1}{3\left(\frac{1+c}{3+c}\right)^{\delta}} & for\quad 0\le\left|h(\mu)\right|<\dfrac{1}{6\left(\frac{1+c}{3+c}\right)^{\delta}}\\ 2B_1\left|h(\mu)\right| & for\quad \left|h(\mu)\right|\ge\dfrac{1}{6\left(\frac{1+c}{3+c}\right)^{\delta}}\end{cases}$$

where

$$h(\mu)=\frac{B_1^2(1-\mu)}{6B_1^2\left(\frac{1+c}{3+c}\right)^{\delta}-8B_2\left(\frac{1+c}{2+c}\right)^{2\delta}}.$$

Putting $\delta=0$ in Theorem 1., we have

Corollary 2. Let $f(z)$ given by (1) be in the class $\mathcal{B}_{\Sigma}^{0}(c,\beta;\varphi)=\mathcal{B}_{\Sigma}(c;\varphi)$. Then

$$\left|a_3-\mu a_2^2\right|\le\begin{cases}\dfrac{B_1}{(1+2\beta)} & for\quad 0\le\left|h(\mu)\right|<\dfrac{1}{2(1+2\beta)}\\ 2B_1\left|h(\mu)\right| & for\quad \left|h(\mu)\right|\ge\dfrac{1}{2(1+2\beta)}\end{cases}$$

where

$$h(\mu)=\frac{B_1^2(1-\mu)}{2B_1^2(1+2\beta)-2B_2(1+\beta)^2}.$$

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Focal Curves of the Principal-Direction Curves

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Abstract

In this study, we firstly characterize focal curves of principal-direction curves in 3D ordinary space. Then, we obtain the relation of curvatures of principal-direction curve in terms of focal curvatures. Finally, we give some conditions with constant curvatures in the ordinary space.

Keywords: Serret-Frenet frame, focal curve, principal-direction curve.

1. Preliminaries

Associated curves provide meaningful expressions in the study of the characterization of curves and surfaces, their behavior, and their motion in space-time. Fundamentally, it allows a second curve geometrically related to a curve to be defined with the help of the 1st curve. One of the most obvious examples of associated curves is integral curves, which is a superscript of adjoint curves. Integral curves are important in terms of the possibilities they provide for the solution of some differential equations that we encounter in geometric problems. The principal-direction curves that we will consider in our study are the curves determined by the integral of the normal vector field of a curve [1-13].

By way of design and style, this is model to kind of a moving frame with regards to a particle. In the quick stages of regular differential geometry, the Frenet-Serret frame was applied to create a curve in location. After that, Serret-Frenet frame is established by way of subsequent equations for a presented framework [4],

$$\begin{bmatrix} \nabla_{\boldsymbol{T}}\boldsymbol{T} \\ \nabla_{\boldsymbol{T}}\boldsymbol{N} \\ \nabla_{\boldsymbol{T}}\boldsymbol{B} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{T} \\ \boldsymbol{N} \\ \boldsymbol{B} \end{bmatrix},$$

where $\kappa = \|\boldsymbol{T}\|$ and τ are the curvature and torsion of γ, respectively.

Definition 1. Let γ be a regular curve arc-length parametrized. Then, the principal-direction curve of γ is given as $\gamma_d(s) = \int \boldsymbol{N}(s)ds$ [6].

Teorem 2. Let γ be a regular curve arc-length parametrized, $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}\}$ be Serret-Frenet frame of γ and γ_d be principal-direction curve of γ. Denote by $\{\boldsymbol{T}_d, \boldsymbol{T}_d, \boldsymbol{B}_d\}$ Serret-Frenet frame of γ_d. Then, relation with $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}\}$ of Serret-Frenet frame elements of γ_d can given by [6]

$$\begin{aligned} \boldsymbol{T}_d &= \boldsymbol{N}, \\ \boldsymbol{N}_d &= \frac{-\kappa\boldsymbol{T}+\tau\boldsymbol{B}}{\sqrt{\kappa^2+\tau^2}}, \\ \boldsymbol{B}_d &= \frac{\tau\boldsymbol{T}+\kappa\boldsymbol{B}}{\sqrt{\kappa^2+\tau^2}}, \end{aligned} \tag{1}$$

and

$$\kappa_d = \sqrt{\kappa^2+\tau^2}, \qquad \tau_d = \frac{\tau\kappa'-\kappa\tau'}{\sqrt{\kappa^2+\tau^2}}. \tag{2}$$

Proof. From Serret-Frenet frame formulas and Definition 1, the proof is plainly obtained.

2. Focal Curves of Principal-Direction Curves

The focal curve of γ is given by

$$F_\gamma = \gamma + \varphi_1\boldsymbol{N} + \varphi_2\boldsymbol{B}, \tag{3}$$

where the coefficients φ_1, φ_2 are smooth functions of the parameter of the curve γ, called the first and second focal curvatures of γ, respectively.

Teorem 3. Let $\gamma_d\colon I \to E^3$ be principal-direction curve of γ and F_{γ_d} be its focal curve on E^3. Then,

$$\begin{aligned} F_{\gamma_d} &= \gamma_d + \frac{-\kappa\sqrt{\kappa^2+\tau^2}+\tau(\kappa\kappa'+\tau\tau')(\tau\kappa'-\kappa\tau')}{(\kappa^2+\tau^2)^{3/2}}\boldsymbol{T} \\ &\quad + \frac{\tau\sqrt{\kappa^2+\tau^2}+\kappa(\kappa\kappa'+\tau\tau')(\tau\kappa'-\kappa\tau')}{(\kappa^2+\tau^2)^{3/2}}\boldsymbol{B}, \end{aligned} \tag{4}$$

where $\kappa,\ \tau$ are curvatures of γ.

Proof. Assume that γ_d is a unit speed curve and F_{γ_d} its focal curve in E^3. So, by differentiating of the formula $F_{\gamma_d} = \gamma_d + \varphi_{d1}\boldsymbol{N}_d + \varphi_{d2}\boldsymbol{B}_d$, we get

$$F'_{\gamma_d} = (1-\kappa_d\varphi_{d1})\boldsymbol{T}_d + (\varphi'_{d1} - \tau_d\varphi_{d2})\boldsymbol{N}_d + (\varphi'_{d2} + \tau_d\varphi_{d1})\boldsymbol{B}_d$$

From above equation, the first 2 components vanish, we get

$$\begin{aligned} 1-\kappa_d\varphi_{d1} &= 0, \\ \varphi'_{d1} - \tau_d\varphi_{d2} &= 0. \end{aligned}$$

Using the above equations, we obtain

$$\varphi_{d1} = \frac{1}{\kappa_d},$$

$$\varphi_{d2} = \frac{\varphi'_{d1}}{\tau_d} = \frac{-\kappa'_d}{\kappa_d^2\tau_d}.$$

By using (2) in this equations, we find

$$\varphi_{d1} = \frac{1}{\sqrt{\kappa^2+\tau^2}},$$

$$\varphi_{d2} = \frac{(\kappa\kappa'+\tau\tau')(\tau\kappa'-\kappa\tau')}{\kappa^2+\tau^2}.$$

By means of obtained equations and using (1), we express (4). This completes the proof of the theorem.

As an immediate consequence of the above theorem, we have:

Corollary 4. Let $\gamma_d : I \to E^3$ be a unit speed curve and F_{γ_d} its focal curve on E^3. Then, the focal curvatures of γ_d are

$$\varphi_{d1} = \frac{1}{\sqrt{\kappa^2+\tau^2}},$$

$$\varphi_{d2} = \frac{(\kappa\kappa' + \tau\tau')(\tau\kappa' - \kappa\tau')}{\kappa^2+\tau^2}.$$

Proof. From above theorem, we have above system, which completes the proof.

In the light of Theorem 3, we express the following corollary without proof:

Corollary 5. Let $\gamma : I \to E^3$ be unit speed curve, γ_d be the principal-direction curve of γ and F_{γ_d} its focal curve on E^3. Then, if curvature κ and torsion τ of γ are constant, then the focal curve ofγ_d is given by

$$F_{\gamma_d} = \gamma_d - \frac{\kappa}{\kappa^2+\tau^2}\boldsymbol{T} + \frac{\tau}{\kappa^2+\tau^2}\boldsymbol{B}.$$

Example. Let a unit speed curve γ be given as

$$\gamma(t) = \left(\frac{1}{\sqrt{2}}\cos t, \frac{1}{\sqrt{2}}\sin t, \frac{1}{\sqrt{2}}t\right).$$

Then, Serret-Frenet frame and curvatures of γ are obtained as

$$\boldsymbol{T} = \gamma' = \left(-\frac{1}{\sqrt{2}}\sin t, \frac{1}{\sqrt{2}}\cos t, \frac{1}{\sqrt{2}}\right),$$

$$\boldsymbol{N} = \frac{\gamma''}{\|\gamma''\|} = (-\cos t, -\sin t, 0),$$

$$\boldsymbol{B} = \boldsymbol{T} \times \boldsymbol{N} = \left(\frac{1}{\sqrt{2}}\sin t, \frac{-1}{\sqrt{2}}\cos t, \frac{1}{\sqrt{2}}\right),$$

and

$$\kappa = \langle \boldsymbol{T}', \boldsymbol{N} \rangle = \frac{1}{\sqrt{2}},$$

$$\tau = \langle \boldsymbol{N}', \boldsymbol{B} \rangle = \frac{1}{\sqrt{2}}.$$

Let γ_d be principal-direction curve of γ. Then, we obtain

$$\begin{aligned}
\gamma_d &= \int \boldsymbol{N}(t)dt = (-\sin t, \cos t, c), \\
\boldsymbol{T}_d &= (-\cos t, -\sin t, 0) = \boldsymbol{N}, \\
\boldsymbol{N}_d &= (\sin t, -\cos t, 0) = \frac{-\kappa\boldsymbol{T}+\tau\boldsymbol{B}}{\sqrt{\kappa^2+\tau^2}}, \\
\boldsymbol{B}_d &= (0,0,1) = \frac{\tau\boldsymbol{T}+\kappa\boldsymbol{B}}{\sqrt{\kappa^2+\tau^2}},
\end{aligned}$$

and

$$\kappa_d =< \boldsymbol{T}'_d, \boldsymbol{N}_d >= 1 = \sqrt{\kappa^2+\tau^2},$$
$$\tau_d =< \boldsymbol{N}'_d, \boldsymbol{B}_d >= 0 = \frac{\tau\kappa' - \kappa\tau'}{\sqrt{\kappa^2+\tau^2}}.$$

Hence, equations of (1) and (2) are provided. Also, the focal curve of γ_d is given by

$$F_{\gamma_d} = \gamma_d + \varphi_{d1}\boldsymbol{N}_d + \varphi_{d2}\boldsymbol{B}_d.$$

Here, $\varphi_{d1} = \frac{1}{\kappa_d} = 1$ and $\varphi_{d2} = \frac{-\kappa'_d}{\kappa_d^2\tau_d} = 0$. Hence the focal curve F_{γ_d} is obtained as

$$F_{\gamma_d} = (0,0,c).$$

On the other hand, since curvature and torsion are constant, Corollary 5 is provided:

$$\begin{aligned} F_{\gamma_d} &= \gamma_d - \frac{\kappa}{\kappa^2+\tau^2}\boldsymbol{T} + \frac{\tau}{\kappa^2+\tau^2}\boldsymbol{B} \\ &= (-\sin t, \cos t, c) - \frac{1}{2}(-\sin t, \cos t, 1) \\ &\quad + \frac{1}{2}(\sin t, -\cos t, 1) \\ &= (-\sin t, \cos t, c) + (\sin t, -\cos t, 0) \\ &= (0,0,c). \end{aligned}$$

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Generalized spherical fuzzy topological spaces with their applications to the multi-criteria decision-making problems

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Abstract

The aim of this study is to introduce the generalized spherical fuzzy topological spaces by defining some basic concepts such as generalized spherical fuzzy subspace, generalized spherical fuzzy interior, generalized spherical fuzzy closure and generalized spherical fuzzy boundary. Then, we obtain some properties of these concepts and explain them with examples. We also establish an algorithm to solve multi-criteria decision-making problems based on generalized spherical fuzzy topological spaces. Finally, we compare the proposed method with the generalized spherical fuzzy TOPSIS method under a numerical example to demonstrate the validity and reliability of this new method.

Keywords: Generalized spherical fuzzy sets, generalized spherical fuzzy topological spaces, multi-attribute decision-making, TOPSIS.

1. Introduction

We have needed to more powerful and useful set theories than the classical set theory to handle the real-life problems that contain uncertain and ambiguous data such as decision-making problems, data mining, pattern recognition, image filtering and etc. For this reason, Zadeh [20] introduced the fuzzy set theory which can be used in a wide range of domains in which information is incomplete or imprecise. After Zadeh [20] introduced the fuzzy set theory to handle the uncertainty in the real-life problems, different set theories which are extensions of fuzzy set theory have been presented by many authors: intuitionistic fuzzy set theory (Atanassov [5]), Pythagorean fuzzy set theory (Yager [19]), picture fuzzy set theory (Cuong [6]), spherical fuzzy set theory (Ashraf et al. [1], Kutlu Gündoğdu and Kahraman [15]), spherical fuzzy soft set theory (Perveen et al. [18]) and etc. Some important decision-making methods established on spherical fuzzy (soft) sets was given in [2, 3, 4, 8, 9, 14-18]. Recently, Hague et al. [12] initiated the generalized spherical fuzzy (GSF) set theory as a generalization of the spherical fuzzy set to use when this theory cannot enough to handle the data in the problems consisting of uncertain information. Some recent studies on generalized spherical fuzzy set theories and decision-making approaches can be found in [7, 10, 11, 13].

In this study, we introduce the generalized spherical fuzzy topological spaces by defining some basic concepts such as generalized spherical fuzzy subspace, generalized spherical fuzzy interior, generalized spherical fuzzy closure and generalized spherical fuzzy boundary. Then, we obtain some properties of these concepts and explain them with examples. We also establish an algorithm to solve multi-criteria decision-making problems based on generalized spherical fuzzy topological spaces. Finally, we compare the proposed method with the generalized spherical fuzzy TOPSIS method under a numerical example to demonstrate the validity and reliability of this new method.

2. Preliminaries

In this section, we recall some fundamental definitions which will be used in the main sections. Throughout this paper U will refer the set of the discourse.

Definition 2.1. [1, 12] Let $\mu: U \to [0,1]$, $\iota: U \to [0,1]$ and $\nu: U \to [0,1]$ be three mappings. A set $G = \{< x, \mu(x), \iota(x)\ \nu(x) > | x \in U\}$ is called a

(i) spherical fuzzy set (SFS) if the condition $0 \leq \mu^2(x) + \iota^2(x) + \nu^2(x) \leq 1$ hold for all $x \in U$.

(ii) generalized spherical fuzzy set (GSFS) if the condition $0 \leq \mu^2(x) + \iota^2(x) + \nu^2(x) \leq 3$ hold for all $x \in U$.

The values $\mu(x), \iota(x), \nu(x) \in [0,1]$ denote the positive membership degree, neutral membership degree and negative membership degree of x to G, respectively.

The triplet $G =< \mu, \iota, \nu >$ where $\mu, \iota, \nu \in [0,1]$ and $\mu^2 + \iota^2 + \nu^2 \leq 3$ (or $\mu + \iota + \nu \leq 1$ and $\mu^2 + \iota^2 + \nu^2 \leq 1$, resp.), is called a generalized spherical fuzzy number (GSFN) (or spherical fuzzy number (SFN)).

Definition 2.2. A GSFS on U of the form $\{< x, 1,0,0 > : x \in U\}$ is called an absolute GSFS and $\{ < x, 0,1,1 > : x \in U\}$ a null GSFS. We will denote the absolute GSFS and null GSFS by 1_U and 0_U, respectively.

Definition 2.3. [12] Let $G_1 = \{< x, \mu_1(x), \iota_1(x), \nu_1(x) > | x \in U\}$ and $G_2 = \{< x, \mu_2(x), \iota_2(x), \nu_2(x) > | x \in U\}$ be two GSFSs on U. Then the set operations between GSFSs are defined as follows:

(i) G_1 is called a subset of G_2 and denoted by $G_1 \sqsubseteq G_2$ if $\mu_1(x) \leq \mu_2(x)$, $\iota_1(x) \geq \iota_2(x)$ and $\nu_1(x) \geq \nu_2(x)$ for all $x \in U$.

(ii) G_1 is called equal to G_2 and denoted by $G_1 = G_2$ if $G_1 \sqsubseteq G_2$ and $G_2 \sqsubseteq G_1$.

(iii) The union of G_1 and G_2 is denoted by $G_1 \sqcup G_2$ and defined by $G_1 \sqcup G_2 = \{(x, \mu_1(x) \vee \mu_2(x), \iota_1(x) \wedge \iota_2(x), \nu_1(x) \wedge \nu_2(x) | x \in X\}$.

(iv) The intersection of G_1 and G_2 is denoted by $G_1 \sqcup G_2$ and defined by $G_1 \sqcup G_2 = \{(x, \mu_1(x) \wedge \mu_2(x), \iota_1(x) \vee \iota_2(x), \nu_1(x) \vee \nu_2(x) | x \in X\}$.

Definition 2.4. Let $G = \{< x, \mu(x), \iota(x)\ \nu(x) > | x \in U\}$ be a GSFS on U. Then the complement of G is denoted by G^c and defined by $G^c = \{< x, \nu(x), \iota(x), \mu(x) > | x \in U\}$.

Definition 2.5. [12] Let $G =< \mu, \iota, \nu >$, $G_1 =< \mu_1, \iota_1, \nu_1 >$, $G_2 =< \mu_2, \iota_2, \nu_2 >$ be three GSFNs and $a \geq 0$. Then the algebraic operations between GSFNs are defined as follows:

(i) $G_1 \oplus G_2 =< \sqrt{\mu_1^2 + \mu_2^2 - \mu_1^2\mu_2^2}, \iota_1\iota_2, \nu_1\nu_2 >$,

(ii) $G_1 \odot G_2 =< \mu_1\mu_2, \iota_1\iota_2, \sqrt{\nu_1^2 + \nu_2^2 - \nu_1^2\nu_2^2} >$,

(iii) $a \times G =< \sqrt{1 - (1 - \mu^2)^a}, \iota^a, \nu^a >$,

(iv) $G^a =< \mu^a, \iota^a, \sqrt{1 - (1 - \nu^2)^a} >$.

Lemma 2.1. Let $G_1 =< \mu_1, \iota_1, \nu_1 >$, $G_2 =< \mu_2, \iota_2, \nu_2 >$ be two GSFNs and $a, a_1, a_2 \geq 0$. Then the following properties hold:

(i) $G_1 + G_2 = G_2 + G_1$,

(ii) $a \times (G_1 + G_2) = a \times G_1 + a \times G_2$,

(iii) $(a_1 + a_2) \times G_1 = a_1 \times G_1 + \ a_2 \times G_2$,

(vi) $\left(G_1^{a_1}\right)^{a_2} = G_1^{a_1 a_2}$.

Definition 2.6. [12] Let $\mathcal{G}$ be a collection of all GSFNs and $(G_1, G_2, \dots, G_n) \in \ \mathcal{G}^n$ where $G_i =< \mu_i, \iota_i, \nu_i >$ for all $i = 1,2, \dots, n$ and $w = (w_1, w_2, \dots, w_n)^T$ the weight vector corresponding to $(G_i)_{i=1}^n$ such that $w_i \geq 0$ for all i and $\sum_{i=1}^n w_i = 1$. A mapping $GSWA_w: \mathcal{G}^\mathrm{n} \to \mathcal{G}$ is said to be a GSF-weighted averaging (GSWA) operator and defined by

$$GSWA_w(G_1, G_2, \dots, G_n) = w_1 \times G_1 \oplus w_2 \times G_2 \oplus \dots w_n \times G_n = \oplus_{i=1}^n \ w_i \times G_i.$$

Definition 2.7. [12] Let $\mathcal{G}$ be the family of all GSFNs and $G \in \mathcal{G}$ where $G =< \mu, \iota, \nu >$.

(i) A score function $SF: \mathcal{G} \to [-1,1]$ is defined as $SF(G) = \frac{3\mu^2 - 2\iota^2 - \nu^2}{3}$.

(ii) An accuracy function $AF: \mathcal{G} \to [0,1]$ is defined as $AF(G) = \frac{1 + 3\mu^2 - \nu^2}{4}$.

Definition 2.8. [12] Let $G_1 =< \mu_1, \iota_1, \nu_1 >$ and $G_2 =< \mu_2, \iota_2, \nu_2 >$ be any two GSFNs. Then the ranking method (comparison technique) as follows:

(i) $SF(G_1) < SF(G_2) \Rightarrow G_1 < G_2$,

(ii) $SF(G_1) > SF(G_2) \Rightarrow G_1 > G_2$,

(iii) If $SF(G_1) = SF(G_2)$, then; (a) $AF(G_1) < AF(G_2) \ \Rightarrow G_1 < G_2$,

(b) $AF(G_1) > AF(G_2) \ \Rightarrow G_1 > G_2$,

(c) $AF(G_1) = AF(G_2) \ \Rightarrow G_1 = G_2$.

3. Generalized spherical fuzzy topological spaces

In this section, we give the definitions of generalized spherical fuzzy topology, generalized spherical fuzzy closure and generalized fuzzy interior to use in the methodology for solving MCGDM problems.

Definition 3.1. Let $\mathcal{G}$ be the family of GSFSs on U. If the collection $\tau \subseteq \mathcal{G}$ satisfies the following conditions, then τ is called a generalized spherical fuzzy topology (GSFT) on U.

(T1) $0_u, 1_U \in \tau$,

(T2) If $G_1, G_2 \in \tau$, then $G_1 \sqcap \ G_2 \in \tau$,

(T3) If $G_i \in \tau$ for all $i \in I$, then $\sqcup_{i \in I} G_i \in \tau$.

Then the pair (U, τ) is called a generalized spherical fuzzy topological space (GSFTS). A member of τ is said to be generalized spherical fuzzy open (GSF-open) and if $G^c \in \tau$ then G is called generalized spherical fuzzy closed (GSF-closed) set.

Example 3.1. Let U be a non-empty set. Then the topology $\tau = \{0_u, 1_U\}$ is called a trivial GSFT on U and denoted by τ_t. Also, the topology $\tau = \mathcal{G}$ is called a discrete topology on U and denoted by τ_ρ.

Proposition 3.1. If τ_1 and τ_2 are two GSFTs on U, then $\tau_1 \cap \tau_2$ is also GSFT on U. But $\tau_1 \cup \tau_2$ may not be a GSFT on U.

Definition 3.2. Let (U, τ) be a GSFTS and $Y \subset X$. Then the subspace topology τ_Y on Y is defined by $\tau_Y = \{ G \sqcap 1_y : G \in \tau\}$.

Definition 3.3. Let (U, τ) be a GSFTS and $G \in \mathcal{G}$. The generalized spherical fuzzy closure of G is denoted by $\overline{G}$ and defined by $\overline{G} = \sqcap \ \{H: H \ is \ closed \ and \ G \sqsubseteq H\}$. It is clear from definition that $\overline{G}$ is the smallest generalized spherical fuzzy closed set which contains G.

Definition 3.4. Let (U, τ) be a GSFTS and $G \in \mathcal{G}$. The generalized spherical fuzzy interior of G is denoted by G^o and defined by $G^o = \sqcup \{H: H \ is \ open \ and \ H \sqsubseteq G\}$. It is clear from definition that G^o is the largest generalized spherical fuzzy open set which is contained in G.

Theorem 3.1. Let (U, τ) be a GSFTS and $G, G_1, G_2 \in \ \mathcal{G}$. Then the followings are satisfied.

(i) $\overline{0_U} = 0_U, \overline{1_U} = 1_U, 0_U^o = 0_U$,
(ii) $G \sqsubseteq \overline{G}, G^o \sqsubseteq G$
(iii) $\overline{\overline{G}} = \overline{G}, (G^o)^o = G^o$,
(iv) If $G_1 \sqsubseteq \ G_2$, then $\overline{G}_1 \sqsubseteq \overline{G}_2$ and $G_1^o \sqsubseteq \ G_2^o$,
(v) If G is a GSF-open set, then $G^o = G$,
(vi) If G is a GSF-closed set, then $\overline{G} = G$,
(vii) $\overline{G_1 \sqcup \ G_2} = \overline{G_1} \sqcup \overline{G}_2$,
(viii) $(G_1 \sqcap \ G_2)^o = \ G_1^o \sqcap \ G_2^o$,
(ix) $(G^o)^c = \overline{G^c}$,
(x) $\left(\overline{G}\right)^c = \ (G^c)^o$.

4. Application of the GSFTS to the MCGDM problems

In this section, we construct a novel method to solve the MCDM problems based on GSFTS where we consider the interior of the GSFTSs. Then we give an example to illustrate the method step by step. Finally, we compare the proposed method with the TOPSIS method in GSF environment by analyzing the result of the given numerical example.

4.1. Methodology

Let $U = \{x_1, x_2, \ldots, x_m\}$ be the set of m alternatives, $c = \{c_1, c_2, \ldots, c_n\}$ be the set of n criteria and $D = \{D_1, D_2, \ldots, D_r\}$ be the set of r experts (DMs). The proposed method is consisting of the following steps:

Step I: Input the GSFS-data by considering the alternatives and criteria to construct a GSFTS for suitable numbers (k) of the attributes. Denote the GSFS-data by G_k.

Step II: Construct the GSFT τ_G on U as $\tau_G = \{0_U, 1_U, G_1, G_2, \dots, G_k\}$.

Step III: Each DM $(D_i, i = 1,2,\dots,r)$ establishes the decision matrix individually. Denote these matrices by M_{D_i} as follow:

***Table 1**. Decision Matrix $D^{(r)}$.*

Alternatives	Attributes			
	E_1	E_2	$\dots$	E_n
A_1	D_{11}^r	D_{12}^r	$\dots$	D_{1n}^r
A_2	D_{21}^r	D_{22}^r	$\dots$	D_{2n}^r
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
A_m	D_{m1}^r	D_{m2}^r	$\dots$	D_{mn}^r

And also, if there exist cost type criteria then normalize the each decision matrices by taking complement of related element.

Step IV: If the weights of DMs are given as $w = (w_1, w_2, \dots, w_l)$, calculate the weighted decision matrix

$(\delta_{D_i}, i = 1,2,\dots,r)$ as $\delta_{D_i} = w_i \times M_{D_i}$, for all $i = 1,2,\dots,r$. If the weights of DMs are equal, then this step is skipped.

Step V: Calculate the aggregated weighted decision matrices according to the weights of criteria and denote each aggregated weighted decision matrices as GSFS (say Δ_{D_i}). If weights of criteria are given by $\epsilon = (\epsilon_1, \dots, \epsilon_n)$, then the aggregated weighted decision matrices are obtained by using GSWA (GSWG) operator.

Step VI: Find the interior of each $\Delta_{D_i}(\Delta_{D_i}^o)$ according to the GSFT τ_G constructed in the Step 2.

Step VII: Find $\sqcap \Delta_{D_i}^o = R$ (Resultant GSFS).

Step VIII: Calculate the score values of each raw of the weighted resultant GSFS R^* by using score function.

Step IX: Rank the alternatives according to the score values calculated in the last step.

4.2. A numerical example

In this subsection, we solve a MCGDM problem given in [7] to explain the proposed method step by step.

There is a three-shareholder company in which the rates of share are effective at the decisions to be made by shareholders and the sharing of the earnings. Let the shareholders be denoted by D_1, D_2 and D_3. The shareholder D_1 has 35% share rate, the shareholder D_2 has 45% share rate and the shareholder D_3 has 20% share rates. This company is planning to make an investment in an area where the alternatives are A_1:Development of small business, A_2: Information Technology, A_3: Tourism, A_4: Transportation. They are taking into consideration the degree of risk, volume of income and investment recovery period when making an investment in these areas. Let the degree of risk, volume of income and investment recovery period be denoted by c_1, c_2 and c_3, respectively. A prioritization relationship among the criteria c_i $(i = 1,2,3)$ which satisfies $c_2 > c_1 > c_3$ was determined according to the shareholder's preferences. So, assume that $w = (0{,}3, 0{,}45, 0{,}25)$ is the weight vector of the attribute $\{c_1, c_2, c_3\}$. In this problem, whereas the attributes c_1 and c_3 are non-benefit (cost) types, c_2 is benefit type. In order to choose the optimum investment, the shareholder's D_1, D_2 and D_3 with the decision-makers weight vector $\delta = (0{,}35, 0{,}45, 0{,}2)$ evaluate the four investment options based on these criteria considering the aggregation operators.

Let us start the decision-making process:

Step I: By considering the alternatives and criteria, initial data is inputted to construct a GSFTS as follows:

$G_1 = \{< x_1, 0.2, 0.84, 0.84 >, < x_2, 0.44, 0.81, 0.89 >, < x_3, 0.27, 0.74, 0.83 >, < x_4, 0.35, 0.88, 0.83 >\}$,

$G_2 = \{< x_1, 0.5, 0.84, 0.84 >, < x_2, 0.6, 0.55, 0.58 >, < x_3, 0.27, 0.85, 0.88 >, < x_4, 0.61, 0.62, 0.7 >\}$,

$G_3 = \{< x_1, 0.19, 0.93, 0.94 >, < x_2, 0.16, 0.94, 0.92 >, < x_3, 0.26, 0.95, 0.94 >, < x_4, 0.28, 0.93, 0.94 >\}$.

Step II: The GSFT $\tau_G = \{0_U, 1_U, G_1, G_2, G_3\}$ on U is established.

Step III: Each DMs (D_i for all $i = 1,2,3$) establishes the decision matrix individually. The decision matrices are shown in the following table:

Table 2. *Decision Matrices.*

D_1	E_1	E_2	E_3
A_1	<0.6,0.8,0,2>	<0.4, 0.3, 0.7>	<0.2, 0.7, 0.4>
A_2	<0.55,0.2,0.8>	<0.8,0.75,0.65>	<0.9, 0.8, 0.2>
A_3	<0.7, 0.4, 0.4>	<0.55,0.2,0.45>	<0.5, 0.7, 0.8>
A_4	<0.35,0.6,0.5>	<0,7, 0.8,0.55>	<0.8, 0.6, 0.5>
D_2	E_1	E_2	E_3
A_1	<0.85, 0.7,0.8>	<0.4, 0.75, 0.8>	<0.6, 0.8, 0.5>
A_2	<0.3, 0.4, 0.4>	<0.8, 0.2, 0.45>	<0.5, 0.6, 0.8>
A_3	<0.9, 0, 8, 0,2>	<0.4, 0.8, 0.7>	<0.8, 0.7, 0.4>
A_4	<0.75,0.3, 0.5>	<0.8, 0.5, 0.45>	<0.5, 0.6, 0.8>

D_3	E_1	E_2	E_3
A_1	<0.75, 0.4, 0.5>	<0.8, 0.8, 0.45>	<0.8, 0.6, 0.8>
A_2	<0.9, 0, 6, 0, 4>	<0.4, 0.6, 0.9>	<0.2, 0.7, 0.4>
A_3	<0.55, 0.5, 0.8>	<0.8,0.75,0.85>	<0.6, 0.8, 0.2>
A_4	<0.75, 0.4, 0.8>	<0.4, 0.8, 0.45>	<0.8, 0.6, 0.6>

Step IV: We calculate the weighted decision matrices by considering the decision-makers weight vector $\delta = (0{,}35, 0{,}45, 0{,}2)$. So, we obtain the following matrices:

$$\delta_{D_1} = \begin{pmatrix} < 0.1191, 0.9249, 0.8363 > & < 0.2433, 0.6561, 0.8826 > & < 0.2433, 0.8826, 0.5693 > \\ < 0.5483, 0.5693, 0.8112 > & < 0.5483, 0.9042, 0.8600 > & < 0.1191, 0.9249, 0.9638 > \\ < 0.2433, 0.7256, 0.8826 > & < 0.3442, 0.5693, 0.7562 > & < 0.5483, 0.8826, 0.7846 > \\ < 0.3095, 0.8363, 0.6925 > & < 0.4582, 0.9249, 0.8112 > & < 0.3095, 0.8363, 0.9249 > \end{pmatrix},$$

$$\delta_{D_2} = \begin{pmatrix} < 0.6071, 0.8517, 0.9295 > & < 0.2747, 0.8786, 0.9045 > & < 0.3485, 0.9045, 0.7946 > \\ < 0.2747, 0.6621, 0.5817 > & < 0.6071, 0.4847, 0.6981 > & < 0.6071, 0.7946, 0.7320 > \\ < 0.1349, 0.9045, 0.9537 > & < 0.2747, 0.9045, 0.8517 > & < 0.2747, 0.8517, 0.9045 > \\ < 0.3485, 0.5817, 0.8786 > & < 0.6071, 0.7320, 0.6981 > & < 0.6071, 0.7946, 0.7320 > \end{pmatrix},$$

$$\delta_{D_3} = \begin{pmatrix} < 0.2365, 0.8326, 0.9441 > & < 0.4299, 0.9564, 0.8524 > & < 0.4299, 0.9029, 0.9564 > \\ < 0.1851, 0.9029, 0.9791 > & < 0.1851, 0.9029, 0.9791 > & < 0.1851, 0.9311, 0.7248 > \\ < 0.4299, 0.8706, 0.8873 > & < 0.4299, 0.9441, 0.9680 > & < 0.0902, 0.9564, 0.9029 > \\ < 0.4299, 0.8326, 0.9441 > & < 0.1851, 0.9564, 0.8524 > & < 0.2922, 0.9029, 0.9564 > \end{pmatrix}$$

Step V: We calculate the aggregated weighted decision matrices according to the weights of criteria $w = (0.3, 0.45, 0.25)$ and denote each aggregated weighted decision matrices as GSFS $\left(\Delta_{D_i}\right)$ as follows:

	Δ_{D_1}	Δ_{D_2}	Δ_{D_3}
A_1	$< 0.2002, 0.8292, 0.7710 >$	$< 0.5094, 0.8389, 0.8340 >$	$< 0.3317, 0.9208, 0.9366 >$
A_2	$< 0.4433, 0.8004, 0.8877 >$	$< 0.6015, 0.5407, 0.5793 >$	$< 0{,}1607, 0.9334, 0.9130 >$
A_3	$< 0.3883, 0.7389, 0.8236 >$	$< 0.2754, 0.8499, 0.8704 >$	$< 0.3146, 0.9416, 0.9384 >$
A_4	$< 0.3505, 0.8775, 0.8217 >$	$< 0.6134, 0.6141, 0.6975 >$	$< 0.2808, 0.9208, 0.9366 >$

Step VI: We find the interior of each $\Delta_{D_i}\left(\Delta^o_{D_i}\right)$ for all $i = 1,2,3$ according to the GSFT τ_G constructed in the Step 2 as follows: $\Delta^o_{D_1} = G_1, \Delta^o_{D_2} = G_2 \; and \; \Delta^o_{D_3} = G_3$.

Step VII: We find $\Delta^o_{D_1} \sqcap \Delta^o_{D_2} \sqcap \Delta^o_{D_3} = G_3$.

Step VIII: Then, we calculate the score values of each raw of the weighted resultant GSFS G_3 by using the function $SF(G) = \frac{|3\mu^2 - 2\iota^2 - \nu^2|}{3}$.

Step IX: Finally, we rank the alternatives according to the score values $SF(A_1) = 0{,}5467$, $SF(A_2) = 0{,}5511$, $SF(A_3) = 0{,}5278$ and $SF(A_4) = 0{,}5044$. As a conclude, A_2 is the best solution of this problem.

In [11], Güner and Aygün established the TOPSIS method in the GSF environment. If we solve the same problem according to the TOPSIS method, we obtain the ranking result as $A_2 > A_4 > A_1 > A_3$ which shows the best solution are the same in these two approaches.

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Geometric Properties of Generalized Integral Operator Involving The Rabotnov Function

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Abstract

A useful family of integral operators and special functions plays a crucial role on the study of mathematical and applied sciences. The purpose of the present paper is to give sufficient conditions for an integral operator, which involves the normalized form of the Rabotnov function to be univalent in the open unit disk. Furthermore, we determine the order of the convexity of this operator. In order to prove main results, we use differential inequalities for the Rabotnov functions together with some known properties in connection with the integral operator which we have considered in this paper. Moreover, some graphical illustrations are provided in support of the results proved in this paper.

Keywords: Univalent functions, Integral operator, Rabotnov function, Convexity.

1. Introduction

Let $\mathcal{A}$ denote the family of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $\mathbb{D}$ and satisfy the usual normalization condition:

$$f(0) = f'(0) - 1 = 0.$$

We denote by $\mathcal{S}$ the subclass of the normalized analytic function class $\mathcal{A}$ consisting of functions which are also univalent in $\mathbb{D}$. A function $f \in \mathcal{S}$ is said to be convex of order $\delta\left(0 \le \delta < 1\right)$ if it satisfies

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \delta \quad (z \in \mathbb{D}).$$

Let $n \in \mathbb{N} := \{1, 2, 3, \ldots\}$ and

$$\mathcal{A}^n = \left\{\left(f_1, f_2, \cdots, f_n\right) : f_j \in \mathcal{A}\left(j = 1, 2, \ldots, n\right)\right\}.$$

For the functions $f_j \in \mathcal{A}\left(j=1,2,\ldots,n\right)$, the parameters $\alpha_j, \beta_j \in \mathbb{C}\left(j=1,2,\ldots,n\right)$ and $\gamma \in \mathbb{C}$, we define the following integral operator:

$$\mathcal{L}_{\alpha_1,\alpha_2,\ldots,\alpha_n;\beta_1,\beta_2,\ldots,\beta_n;n;\gamma} : \mathcal{A}^n \longrightarrow \mathcal{A}$$

by

$$\mathcal{L}_{\alpha_1,\alpha_2,\ldots,\alpha_n;\beta_1,\beta_2,\ldots,\beta_n;n;\gamma}\left[f_1, f_2,\ldots, f_n\right](z) := \left[\gamma \int_0^z t^{\gamma-1} \prod_{j=1}^{n} \left(f_j'(t)\right)^{\alpha_j} \left(e^{f_j(t)}\right)^{\beta_j} dt\right]^{1/\gamma}. \tag{1}$$

Moreover, Deniz et al. [4] introduced certain integral operators by using an obvious parametric variation of the generalized Bessel functions of the first kind and of order ν and studied the univalence criteria of the corresponding integral operators. On the other hand, Deniz [5] and Raza et al. [15] discussed the convexity, starlikeness and uniform convexity of integral operators involving these equivalent forms of the classical Bessel function $J_\nu(z)$. Recently, Al-Khrasani et al. [1] investigated some sufficient conditions for univalence of some linear fractional derivative operators involving the normalized forms of the same obvious parametric variation of the classical Bessel function $J_\nu(z)$ of the first kind and of order ν. Additionally, the theory of integrals and derivatives of an arbitrary real or complex order (see, for details, [18]) has been applied not only in geometric function theory of complex analysis, but has also emerged as a potentially useful direction in the mathematical modeling and analysis of real-world problems in applied sciences (see, for example,[17]).

Motivated by the works mentioned above, in this paper, we will investigate some mapping and geometric properties for the integral operator defined by (1), associated with the Rabotnov function which is defined as follows:

Denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ which consists of univalent functions in $\mathbb{D}$.

Consider the function $R_{\mu,c}\left(z\right)$ defined by

$$R_{\mu,c}\left(z\right) = z^{\mu} \sum_{n=0}^{\infty} \frac{c^n}{\Gamma\left(\left(1+\mu\right)\left(n+1\right)\right)} z^{n(1+\mu)} \tag{2}$$

where Γ stands for the Euler gamma function and $\mu > -1$, $c \in \mathbb{C}$ and $z \in \mathbb{D}$. This function was introduced by Rabotnov in 1948 [14] and is therefore known as the Rabotnov function.

The function defined by (2) does not belong to the class $\mathcal{A}$. Therefore, we consider the following normalization of the Rabotnov function $R_{\mu,c}\left(z\right)$: for $z \in \mathbb{D}$,

$$\mathbb{R}_{\mu,c}\left(z\right)=\Gamma\left(1+\mu\right)z^{1/(1+\mu)}R_{\mu,c}\left(z^{1/(1+\mu)}\right)=\sum_{n=0}^{\infty}\frac{c^{n}\Gamma\left(1+\mu\right)}{\Gamma\left(\left(1+\mu\right)\left(n+1\right)\right)}z^{n+1} \tag{3}$$

where $\mu>-1$ and $c\in\mathbb{C}$.

Note that some special cases of $\mathbb{R}_{\mu,c}\left(z\right)$ are:

$$\begin{cases}\mathbb{R}_{0,-\frac{1}{3}}\left(z\right)=ze^{-\frac{z}{3}}\\ \mathbb{R}_{1,\frac{1}{2}}\left(z\right)=\sqrt{2z}\sinh\sqrt{\frac{z}{2}}\\ \mathbb{R}_{1,-\frac{1}{4}}\left(z\right)=2\sqrt{z}\sin\frac{\sqrt{z}}{2}\\ \mathbb{R}_{1,1}\left(z\right)=\sqrt{z}\sinh\sqrt{z}\\ \mathbb{R}_{1,2}\left(z\right)=\frac{\sqrt{2z}\sinh\sqrt{2z}}{2}\end{cases}.$$

Let $j=1,2,\ldots,n$ and let $\mu_j>-1$ and $c\in\mathbb{C}$. Consider the functions $\mathbb{R}_{\mu_j,c}\left(j=1,2,\ldots,n\right)$ defined by

$$\mathbb{R}_{\mu_j,c}\left(z\right)=\Gamma\left(1+\mu_j\right)z^{1/\left(1+\mu_j\right)}R_{\mu_j,c}\left(z^{1/\left(1+\mu_j\right)}\right).$$

Using the functions $\mathbb{R}_{\mu_j,c}$ and integral operator defined by (1), we define the function

$\mathcal{L}^{\mu_1,\mu_2,\ldots,\mu_n;c}_{\alpha_1,\alpha_2,\ldots,\alpha_n;\beta_1,\beta_2,\ldots,\beta_n;n;\gamma}:\mathbb{D}\longrightarrow\mathbb{C}$ as follows:

$$\begin{aligned}\mathcal{L}^{\mu_1,\mu_2,\ldots,\mu_n;c}_{\alpha_1,\alpha_2,\ldots,\alpha_n;\beta_1,\beta_2,\ldots,\beta_n;n;\gamma}(z)&:=\mathcal{L}_{\alpha_1,\alpha_2,\ldots,\alpha_n;\beta_1,\beta_2,\ldots,\beta_n;n;\gamma}\left[\mathbb{R}_{\mu_1,c},\mathbb{R}_{\mu_2,c},\ldots,\mathbb{R}_{\mu_n,c}\right](z)\\ &=\left[\gamma\int_0^z t^{\gamma-1}\prod_{j=1}^{n}\left(\mathbb{R}'_{\mu_j,c}(t)\right)^{\alpha_j}\left(e^{\mathbb{R}_{\mu_j,c}(t)}\right)^{\beta_j}dt\right]^{1/\gamma}.\end{aligned} \tag{4}$$

In this paper, we drive some sufficient conditions for the following operator:

$$\mathcal{L}^{\mu_1,\mu_2,\ldots,\mu_n;c}_{\alpha_1,\alpha_2,\ldots,\alpha_n;\beta_1,\beta_2,\ldots,\beta_n;n;\gamma}$$

defined by (4), respectively, to be univalent in $\mathbb{D}$. We also determine the order of convexity of the functions defined by using the above-mentioned integral operator (4).

Recently, many mathematicians have set the univalence criteria of several those integral operators which preserve the class $\mathcal{S}$. By using a variety of different analytic techniques, operators and special functions, several authors have studied univalence criterion, a few of them are as mentioned below.

In 2010 Baricz and Frasin [3] studied some integral operators involving Bessel functions. These integral operators were defined by using the normalized Bessel functions of the first kind. Frasin [8] and Arif and Raza [2] studied the convexity and strongly convexity of the integral operators defined in [3]. There is an extensive literature in geometric function theory that deals with the geometric properties of the integral operators defined by different kinds of special functions like Bessel functions [2-5,8], Struve function [6,9,10], Lommel function [7,11] and Mittag-Leffler function [16]. Motivated by the work of these above authors, we contribute to this univalence theory by studying the univalence and convexity of integral operator involving the Rabotnov function.

2. A set of lemmas

The following lemmas will be required in our present investigations.

Lemma 1. (See Pescar [13]) Let ζ and η be complex number such that

$$\Re(\zeta) > 0 \text{ and } |\eta| \leqq 1 \ (\eta \neq -1).$$

If the function $h \in \mathcal{A}$ satisfies the following inequality:

$$\left| \eta |z|^{2\zeta} + \left(1-|z|^{2\zeta}\right) \frac{zh''(z)}{\zeta h'(z)} \right| \leqq 1$$

for all $z \in \mathbb{D}$, then the function $F_\zeta \in \mathcal{A}$ defined by

$$F_\zeta(z) = \left(\zeta \int_0^z t^{\zeta-1} h'(t)dt\right)^{1/\zeta} \tag{5}$$

is in the normalized univalent function class $\mathcal{S}$.

Lemma 2. (See Pascu [12]) Let $\lambda \in \mathbb{C}$ such that $\Re(\lambda) > 0$. If $f \in \mathcal{A}$ satisfies the following inequality:

$$\left(\frac{1-|z|^{2\Re(\lambda)}}{\Re(\lambda)}\right)\left|\frac{zh''(z)}{h'(z)}\right| \leqq 1$$

for all $z \in \mathbb{D}$. Then, for all $\zeta \in \mathbb{C}$ such that

$$\Re(\zeta) \geqq \Re(\lambda),$$

the function F_ζ defined by (5) is in the normalized univalent function class $\mathcal{S}$.

Lemma 3. Let $\mu > -1$ and $c \in \mathbb{C}$. Then, for all $z \in \mathbb{D}$, the function $\mathbb{R}_{\mu,c}$ defined by (3) satisfies the following inequalities:

$$\left|z\mathbb{R}'_{\mu,c}(z)\right| \leqq \left(\frac{1+\mu+|c|}{1+\mu}\right)e^{\frac{|c|}{1+\mu}} \tag{6}$$

and

$$\left|\frac{z\mathbb{R}''_{\mu,c}(z)}{\mathbb{R}'_{\mu,c}(z)}\right| \leqq \frac{\left(\frac{|c|\left(2+2\mu+|c|\right)}{(1+\mu)^2}\right)e^{\frac{|c|}{1+\mu}}}{2-\left(\frac{1+\mu+|c|}{1+\mu}\right)e^{\frac{|c|}{1+\mu}}}, \quad \left(|c| < (1+\mu)\ln\left(\frac{2(1+\mu)}{1+\mu+|c|}\right)\right). \tag{7}$$

Proof. By using the well-known triangle inequality and the following inequality

$$\frac{\Gamma(1+\mu)}{\Gamma\left((1+\mu)(n+1)\right)} \leq \frac{1}{(1+\mu)^n (n)!}, \ \mathrm{n} \in \mathbb{N},$$

we have

$$\begin{aligned} \left|z\mathbb{R}'_{\mu,c}(z)\right| &= \left|z + \sum_{n=1}^{\infty} \frac{(n+1)c^n\Gamma(1+\mu)}{\Gamma\left((1+\mu)(n+1)\right)} z^{n+1}\right| \\ &\leq 1 + \sum_{n=1}^{\infty} \frac{(n+1)|c|^n\Gamma(1+\mu)}{\Gamma\left((1+\mu)(n+1)\right)} \\ &\leq 1 + \sum_{n=1}^{\infty} \frac{(n+1)|c|^n}{n!(1+\mu)^n} = \left(\frac{1+\mu+|c|}{1+\mu}\right)e^{\frac{|c|}{1+\mu}}. \end{aligned} \tag{8}$$

Finally, using the same inequalities to prove (7), we arrive at the following

$$\begin{aligned}\left|z\mathbb{R}''_{\mu,c}(z)\right| &= \left|\sum_{n=1}^{\infty}\frac{(n+1)nc^n\Gamma(1+\mu)}{\Gamma((1+\mu)(n+1))}z^n\right| \\ &\le \sum_{n=1}^{\infty}\frac{(n+1)n|c|^n\,\Gamma(1+\mu)}{\Gamma((1+\mu)(n+1))} \\ &\le \sum_{n=1}^{\infty}\frac{(n+1)n|c|^n}{n!(1+\mu)^n} = \left(\frac{|c|(2+2\mu+|c|)}{(1+\mu)^2}\right)e^{\frac{|c|}{1+\mu}}.\end{aligned} \tag{9}$$

Moreover, if we use reverse triangle inequality, then we have

$$\begin{aligned}\left|\mathbb{R}'_{\mu,c}(z)\right| &= \left|1+\sum_{n=1}^{\infty}\frac{(n+1)c^n\Gamma(1+\mu)}{\Gamma((1+\mu)(n+1))}z^n\right| \\ &\ge 1-\sum_{n=1}^{\infty}\frac{(n+1)|c|^n\,\Gamma(1+\mu)}{\Gamma((1+\mu)(n+1))} \\ &\ge 1-\sum_{n=1}^{\infty}\frac{(n+1)|c|^n}{n!(1+\mu)^n} = 2-\left(\frac{1+\mu+|c|}{1+\mu}\right)e^{\frac{|c|}{1+\mu}},\left(|c|<(1+\mu)\ln\left(\frac{2(1+\mu)}{1+\mu+|c|}\right)\right).\end{aligned} \tag{10}$$

Next, by combining the inequalities (9) with (10), we immediately deduce that

$$\left|\frac{z\mathbb{R}''_{\mu,c}(z)}{\mathbb{R}'_{\mu,c}(z)}\right| \leqq \frac{\left(\frac{|c|(2+2\mu+|c|)}{(1+\mu)^2}\right)e^{\frac{|c|}{1+\mu}}}{2-\left(\frac{1+\mu+|c|}{1+\mu}\right)e^{\frac{|c|}{1+\mu}}}, \quad \left(|c|<(1+\mu)\ln\left(\frac{2(1+\mu)}{1+\mu+|c|}\right)\right).$$

This completes the proof.

3. Univalence and convexity conditions for the integral operator in (4)

In this section, we investigate univalence and convexity conditions for the integral operator defined by (4).

Theorem 1. Let $j=1,2,\dots,n,$ $\mu_j>-1,$ $c\in\mathbb{C}$ and $|c|<(1+\mu_j)\ln\left(\frac{2(1+\mu_j)}{1+\mu_j+|c|}\right)$. Also, let γ,η,α_j and β_j be in $\mathbb{C}$ such that

$$\Re(\gamma) > 0, \ |\eta| \leq 1 \ (\eta \neq -1), \quad \alpha_j, \beta_j \neq 0.$$

Suppose that these numbers satisfy the following inequality:

$$|\eta| + \frac{1}{|\gamma|}\left(\frac{\left(\frac{|c|\left(2+2\mu+|c|\right)}{(1+\mu)^2}\right)e^{\frac{|c|}{1+\mu}}}{2-\left(\frac{1+\mu+|c|}{1+\mu}\right)e^{\frac{|c|}{1+\mu}}} \sum_{j=1}^{n}|\alpha_j| + \left(\frac{1+\mu+|c|}{1+\mu}\right)e^{\frac{|c|}{1+\mu}} \sum_{j=1}^{n}|\beta_j| \right) \leqq 1, \tag{11}$$

where $\mu = \min\{\mu_1, \mu_2, \ldots, \mu_n\}$. Then the function:

$$\mathcal{L}^{\mu_1,\mu_2,\ldots,\mu_n;c}_{\alpha_1,\alpha_2,\ldots,\alpha_n;\beta_1,\beta_2,\ldots,\beta_n;n;\gamma}$$

defined by (4) is in the normalized univalent function class $\mathcal{S}$.

Proof. Let us define the function ϕ by

$$\phi(z) := \mathcal{L}^{\mu_1,\mu_2,\ldots,\mu_n;c}_{\alpha_1,\alpha_2,\ldots,\alpha_n;\beta_1,\beta_2,\ldots,\beta_n;n;1}(z) = \int_0^z \prod_{j=1}^{n}\left(\mathbb{R}'_{\mu_j,c}(t)\right)^{\alpha_j}\left(e^{\mathbb{R}_{\mu_j,c}(t)}\right)^{\beta_j} dt, \tag{12}$$

so that

$$\phi'(z) = \prod_{j=1}^{n}\left(\mathbb{R}'_{\mu_j,c}(z)\right)^{\alpha_j}\left(e^{\mathbb{R}_{\mu_j,c}(z)}\right)^{\beta_j}. \tag{13}$$

We observe that

$$\mathbb{R}_{\mu_j,c}(0) = \mathbb{R}'_{\mu_j,c}(0) - 1 = 0 \quad \left(\forall\ j = 1, 2, \ldots, n\right),$$

since $\mathbb{R}_{\mu_j,c} \in \mathcal{A}$ for all $j = 1, 2, \ldots, n.$ Therefore, clearly, $\phi(z) = \phi'(z) - 1 = 0.$

Now, upon Differentiating both sides of (13) logarithmically, we obtain

$$\frac{z\phi''(z)}{\phi'(z)} = \sum_{j=1}^{n}\alpha_j \frac{z\mathbb{R}''_{\mu_j,c}(z)}{\mathbb{R}'_{\mu_j,c}(z)} + \sum_{j=1}^{n}\beta_j z\mathbb{R}'_{\mu_j,c}(z). \tag{14}$$

Furthermore, by (6) and (7), we have

$$\left|\frac{z\phi''(z)}{\phi'(z)}\right| \leqq \sum_{j=1}^{n}\left|\alpha_j\right|\left|\frac{z\mathbb{R}''_{\mu_j,c}(z)}{\mathbb{R}'_{\mu_j,c}(z)}\right| + \sum_{j=1}^{n}\left|\beta_j\right|\left|z\mathbb{R}'_{\mu_j,c}(z)\right|.$$

$$\leqq \sum_{j=1}^{n}\left(\left|\alpha_j\right|\frac{\left(\frac{|c|\left(2+2\mu_j+|c|\right)}{\left(1+\mu_j\right)^2}\right)e^{\frac{|c|}{1+\mu_j}}}{2-\left(\frac{1+\mu_j+|c|}{1+\mu_j}\right)e^{\frac{|c|}{1+\mu_j}}} + \left|\beta_j\right|\left(\frac{1+\mu_j+|c|}{1+\mu_j}\right)e^{\frac{|c|}{1+\mu_j}}\right)$$

$$\leqq \sum_{j=1}^{n}\left(\left|\alpha_j\right|\frac{\left(\frac{|c|\left(2+2\mu+|c|\right)}{\left(1+\mu\right)^2}\right)e^{\frac{|c|}{1+\mu}}}{2-\left(\frac{1+\mu+|c|}{1+\mu}\right)e^{\frac{|c|}{1+\mu}}} + \left|\beta_j\right|\left(\frac{1+\mu+|c|}{1+\mu}\right)e^{\frac{|c|}{1+\mu}}\right) \tag{15}$$

where $z \in \mathbb{D}; |c| < \left(1+\mu_j\right)\ln\left(\frac{2\left(1+\mu_j\right)}{1+\mu_j+|c|}\right), \left(j=1,2,\ldots,n\right)$. We have also used the fact that the functions $\Theta_1, \Theta_2 : \left(-1,\infty\right) \longrightarrow \mathbb{R},$ defined by

$$\Theta_1(x) = \frac{\left(\frac{|c|\left(2+2x+|c|\right)}{\left(1+x\right)^2}\right)e^{\frac{|c|}{1+x}}}{2-\left(\frac{1+x+|c|}{1+x}\right)e^{\frac{|c|}{1+x}}} \quad \text{and} \quad \Theta_2(x) = \left(\frac{1+x+|c|}{1+x}\right)e^{\frac{|c|}{1+x}},$$

are decreasing and, consequently, we have

$$\frac{\left(\frac{|c|\left(2+2\mu+|c|\right)}{\left(1+\mu\right)^2}\right)e^{\frac{|c|}{1+\mu}}}{2-\left(\frac{1+\mu+|c|}{1+\mu}\right)e^{\frac{|c|}{1+\mu}}} < \frac{\left(\frac{|c|\left(2+2\mu_j+|c|\right)}{\left(1+\mu_j\right)^2}\right)e^{\frac{|c|}{1+\mu_j}}}{2-\left(\frac{1+\mu_j+|c|}{1+\mu_j}\right)e^{\frac{|c|}{1+\mu_j}}}$$

and

$$\left(\frac{1+\mu+|c|}{1+\mu}\right)e^{\frac{|c|}{1+\mu}}<\left(\frac{1+\mu_j+|c|}{1+\mu_j}\right)e^{\frac{|c|}{1+\mu_j}}.$$

Therefore, we have

$$\begin{aligned}&\left|\eta|z|^{2\gamma}+\left(1-|z|^{2\gamma}\right)\frac{z\phi''(z)}{\gamma\phi'(z)}\right|\\&\leqq|\eta|+\left|\frac{z\phi''(z)}{\phi'(z)}\right|\\&\leqq|\eta|+\frac{1}{|\gamma|}\left(\frac{\left(\frac{|c|\left(2+2\mu+|c|\right)}{(1+\mu)^2}\right)e^{\frac{|c|}{1+\mu}}}{2-\left(\frac{1+\mu+|c|}{1+\mu}\right)e^{\frac{|c|}{1+\mu}}}\sum_{j=1}^{n}\left|\alpha_j\right|+\left(\frac{1+\mu+|c|}{1+\mu}\right)e^{\frac{|c|}{1+\mu}}\sum_{j=1}^{n}\left|\beta_j\right|\right)\leqq 1.\end{aligned}\tag{16}$$

By Lemma 1., the inequalities in (16) imply that the function $\phi\in\mathcal{S}$.

Theorem 2. Let $j=1,2,\ldots,n,\ \mu_j>-1,\ c\in\mathbb{C}$ and $|c|<\left(1+\mu_j\right)\ln\left(\frac{2\left(1+\mu_j\right)}{1+\mu_j+|c|}\right)$. Also, let γ,α_j and β_j be in $\mathbb{C}$ such that $\Re(\gamma)>0$ and $\alpha_j,\beta_j\neq 0$. Suppose that these numbers satisfy the following inequality:

$$\Re(\gamma)\geqq\left\{\frac{\left(\frac{|c|\left(2+2\mu+|c|\right)}{(1+\mu)^2}\right)e^{\frac{|c|}{1+\mu}}}{2-\left(\frac{1+\mu+|c|}{1+\mu}\right)e^{\frac{|c|}{1+\mu}}}\sum_{j=1}^{n}\left|\alpha_j\right|+\left(\frac{1+\mu+|c|}{1+\mu}\right)e^{\frac{|c|}{1+\mu}}\sum_{j=1}^{n}\left|\beta_j\right|\right\}\tag{17}$$

where $\mu=\min\{\mu_1,\mu_2,\ldots,\mu_n\}$. Then the function:

$$\mathcal{L}^{\mu_1,\mu_2,\ldots,\mu_n;c}_{\alpha_1,\alpha_2,\ldots,\alpha_n;\beta_1,\beta_2,\ldots,\beta_n;n;\gamma}$$

defined by (4) is in the normalized univalent function class $\mathcal{S}$.

Proof. Let us define the function ϕ as in (12). By using similar methods in (15), we get

$$
\begin{aligned}
&\frac{1-|z|^{2\Re(\gamma)}}{\Re(\gamma)}\left|\frac{z\phi''(z)}{\phi'(z)}\right| \\
&\leqq \frac{1-|z|^{2\Re(\gamma)}}{\Re(\gamma)}\left\{\frac{\left(\frac{|c|\left(2+2\mu+|c|\right)}{(1+\mu)^2}\right)e^{\frac{|c|}{1+\mu}}}{2-\left(\frac{1+\mu+|c|}{1+\mu}\right)e^{\frac{|c|}{1+\mu}}}\sum_{j=1}^{n}\left|\alpha_j\right|+\left(\frac{1+\mu+|c|}{1+\mu}\right)e^{\frac{|c|}{1+\mu}}\sum_{j=1}^{n}\left|\beta_j\right|\right\} \\
&\leqq \frac{1}{\Re(\gamma)}\left\{\frac{\left(\frac{|c|\left(2+2\mu+|c|\right)}{(1+\mu)^2}\right)e^{\frac{|c|}{1+\mu}}}{2-\left(\frac{1+\mu+|c|}{1+\mu}\right)e^{\frac{|c|}{1+\mu}}}\sum_{j=1}^{n}\left|\alpha_j\right|+\left(\frac{1+\mu+|c|}{1+\mu}\right)e^{\frac{|c|}{1+\mu}}\sum_{j=1}^{n}\left|\beta_j\right|\right\}\leqq 1.
\end{aligned}
\tag{18}
$$

By Lemma 2., the inequalities in (18) imply that the function $\phi \in \mathcal{S}$.

Theorem 3. Let $j=1,2,\ldots,n,\ \mu_j>-1,\ c\in\mathbb{C}$ and $|c|<\left(1+\mu_j\right)\ln\left(\frac{2\left(1+\mu_j\right)}{1+\mu_j+|c|}\right)$. Also, let γ,α_j and β_j be in $\mathbb{C}$ such that

$$\Re(\gamma)>0 \text{ and } \alpha_j,\beta_j\neq 0.$$

Suppose that these numbers satisfy the following inequality:

$$0<\sum_{j=1}^{n}\left|\alpha_j\right|\frac{\left(\frac{|c|\left(2+2\mu+|c|\right)}{(1+\mu)^2}\right)e^{\frac{|c|}{1+\mu}}}{2-\left(\frac{1+\mu+|c|}{1+\mu}\right)e^{\frac{|c|}{1+\mu}}}+\sum_{j=1}^{n}\left|\beta_j\right|\left(\frac{1+\mu+|c|}{1+\mu}\right)e^{\frac{|c|}{1+\mu}}\leqq 1, \tag{19}$$

where $\mu=\min\left\{\mu_1,\mu_2,\ldots,\mu_n\right\}$. Then the function:

$$\mathcal{L}^{\mu_1,\mu_2,\ldots,\mu_n;c}_{\alpha_1,\alpha_2,\ldots,\alpha_n;\beta_1,\beta_2,\ldots,\beta_n;n;1}$$

defined by (4) with $\gamma=1$, is convex of order δ given by

$$\delta = 1 - \sum_{j=1}^{n} \left|\alpha_j\right| \frac{\left(\frac{|c|\left(2+2\mu+|c|\right)}{\left(1+\mu\right)^2} \right) e^{\frac{|c|}{1+\mu}}}{2 - \left(\frac{1+\mu+|c|}{1+\mu} \right) e^{\frac{|c|}{1+\mu}}} + \sum_{j=1}^{n} \left|\beta_j\right| \left(\frac{1+\mu+|c|}{1+\mu} \right) e^{\frac{|c|}{1+\mu}}. \tag{20}$$

Proof. Let us define the function ϕ as in (12). By using similar methods in (15), we obtain

$$\begin{aligned} \left| \frac{z\phi''(z)}{\phi'(z)} \right| &\leqq \sum_{j=1}^{n} \left|\alpha_j\right| \frac{\left(\frac{|c|\left(2+2\mu+|c|\right)}{\left(1+\mu\right)^2} \right) e^{\frac{|c|}{1+\mu}}}{2 - \left(\frac{1+\mu+|c|}{1+\mu} \right) e^{\frac{|c|}{1+\mu}}} + \sum_{j=1}^{n} \left|\beta_j\right| \left(\frac{1+\mu+|c|}{1+\mu} \right) e^{\frac{|c|}{1+\mu}} \\ &= 1-\delta. \end{aligned} \tag{21}$$

Therefore, the function ϕ is convex of order δ.

From Theorem 1. with $n=1,$ $\mu_1 = 0$ and $c = -1/3,$ we can obtain the following result.

Corollary 1. Let γ, η, α and β be in $\mathbb{C}$ such that $\Re(\gamma) > 0,$ $|\eta| \leq 1$ $(\eta \neq -1)$. If these numbers satisfy the following inequality:

$$|\eta| + \frac{1}{|\gamma|} \left(\frac{7e^{1/3}}{18-12e^{1/3}} |\alpha| + \frac{4e^{1/3}}{3} |\beta| \right) \leqq 1$$

then the function:

$$\left[\frac{\gamma}{3^{\alpha}} \int_0^z t^{\gamma-1} (3-t)^{\alpha} \left(e^{-t/3}\right)^{\alpha} \left(e^{te^{-t/3}}\right)^{\beta} dt \right]^{1/\gamma}$$

is in the normalized univalent function class $\mathcal{S}$.

Example 1. From Corollary 1.: we can easily get the following consequence (see Figure 1 below):

$$f_1(z) = \left[\frac{1}{3} \int_0^z (3-t)\left(e^{-t/3}\right)\left(e^{te^{-t/3}}\right) dt \right] \in \mathcal{S}.$$

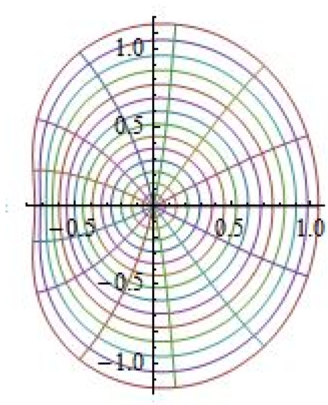

Figure 1: The image of f_1 on $\mathbb{D}$.

From Theorem 3. with $n=1, \mu_1 = 0$ and $c=-1/3,$ we can obtain the following result.

Corollary 2. Let α and β be complex numbers such that

$$0 < \frac{7e^{1/3}}{18-12e^{1/3}}|\alpha| + \frac{4e^{1/3}}{3}|\beta| \leqq 1.$$

Then the function:

$$3^{-\alpha} \int_0^z (3-t)^{\alpha} \left(e^{-t/3}\right)^{\alpha} \left(e^{te^{-t/3}}\right)^{\beta} dt$$

is convex of order δ given by

$$\delta = 1 - \frac{7e^{1/3}}{18-12e^{1/3}}|\alpha| - \frac{4e^{1/3}}{3}|\beta|.$$

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Hardy Space of Miller-Ross Function

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Abstract

In this paper, we obtain conditions for the normalized Miller-Ross function to belong to the Hardy space $\mathcal{H}^{\infty}$.

Keywords: Analytic function, starlike and convex functions, Miller-Ross function, Hardy space.

1. Introduction

Denote by $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$ the open unit disk and $\mathcal{H}$ be set of all analytic functions on $\mathbb{D}$. Let $\mathcal{A}$ be a class of functions f in $\mathbb{D}$ which satisfy the usual normalization conditions $f(0)=f'(0)-1=0.$ Traditionally, the subclass of $\mathcal{A}$ consisting of univalent functions is denoted by $\mathcal{S}$. The classes of starlike and convex functions in $\mathbb{D}$ are two important $\kappa\left(\kappa\in[0,1)\right)$ in $\mathbb{D}$ are defined by $\mathcal{S}^*(\kappa):=\left\{f: f\in\mathcal{S} \text{ and } \Re\left(zf'(z)/f(z)\right)>\kappa\right\}$ and $\mathcal{C}(\kappa):=\left\{f: f\in\mathcal{S} \text{ and } 1+\Re\left(zf''(z)/f'(z)\right)>\kappa\right\}$, respectively. The familiar classes $\mathcal{S}^*:=\mathcal{S}^*(0)$ and $\mathcal{C}:=\mathcal{C}(0)$ are known, respectively, as the classes of starlike and convex functions in $\mathbb{D}$. In [1], for $\gamma<1,$ the author introduced the classes:

$$\mathcal{P}(\gamma):=\left\{p\in\mathcal{H}:\exists\eta\in\mathbb{R} \text{ such that } p(0)=1,\ \Re\left[e^{i\eta}p(z)\right]>\gamma,\ z\in\mathbb{D}\right\}$$

and $\mathcal{R}(\gamma):=\{g\in\mathcal{A}: g'\in\mathcal{P}(\gamma)\}.$

When $\eta=0,$ the classes $\mathcal{P}(\gamma)$ and $\mathcal{R}(\gamma)$ will be denoted by $\mathcal{P}_0(\gamma)$ and $\mathcal{R}_0(\gamma)$, respectively. Also, for $\gamma=0$ we denote $\mathcal{P}_0(\gamma)$ and $\mathcal{R}_0(\gamma)$ simply $\mathcal{P}$ and $\mathcal{R}$, respectively. Moreover, the Hadamard product (or convolution) of two power series belongs to the class $\mathcal{A}$ given by $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ and $g(z)=z+\sum_{n=2}^{\infty}b_nz^n$ defined as

$$(f*g)(z):=z+\sum_{n\geq 2}a_nb_nz^n=:(g*f)(z),\quad (z\in\mathbb{D}).$$

Let $\mathcal{H}^p\left(0<p\le\infty\right)$ denote the Hardy space of all analytic functions $f\left(z\right)$ in $\mathbb{D}$ and define the integral means $M_p\left(r,f\right)$ by

$$M_p\left(r,f\right)=\begin{cases}\left(\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\left|f\left(re^{i\theta}\right)\right|^p d\theta\right)^{\frac{1}{p}} & \left(0<p<\infty\right)\\ \sup\limits_{0\le\theta\le2\pi}\left|f\left(re^{i\theta}\right)\right| & \left(p=\infty\right)\end{cases}.$$

An analytic function $f\left(z\right)$ in $\mathbb{D}$, is said to belong to the Hardy space $\mathcal{H}^p\left(0<p\le\infty\right)$, if the set $\left\{M_p\left(r,f\right):r\in\left[0,1\right)\right\}$ is bounded. It is important to remind here that $\mathcal{H}^p$ is Banach space with the norm defined by (see [2, p. 23])

$$\left\|f\right\|_p=\lim_{r\to1^-}M_p\left(r,f\right)$$

for $1\le p\le\infty$. On the other hand, we known that $\mathcal{H}^\infty$ is the class of bounded analytic functions in $\mathbb{D}$, while $\mathcal{H}^2$ is the class of power series $\sum a_nz^n$ such that $\sum\left|a_n\right|^2<\infty$. In addition, it is known from [2] that $\mathcal{H}^q$ is a subset of $\mathcal{H}^p$ for $0<p\le q\le\infty$. Also, two well-known results about the Hardy space $\mathcal{H}^p$ are the following (see [2]):

$$\Re\left\{f'(z)\right\}>0\Rightarrow\begin{cases}f'\in\mathcal{H}^q & \left(q<1\right)\\ f^{\frac{q}{1-q}}\in\mathcal{H}^q & \left(q\in\left(0,1\right)\right)\end{cases}. \tag{1}$$

2. Preliminaries

In 1993, Miller and Ross [7] function is defined as

$$E_{\nu,c}=z^\nu e^{cz}\gamma^*\left(\nu,cz\right),$$

where γ^* is the incomplete gamma function.

$E_{\nu,c}\left(z\right)$ is a solution of the ordinary differential equation

$$Dy-cy=\frac{z^{\nu-1}}{\Gamma\left(\nu\right)},\quad \nu>0.$$

Since $e^{cz}\gamma^*\left(\nu,cz\right)$ is an entire function, if $\nu>-1$ then $E_{\nu,c}\left(z\right)$ is a function of class $\mathcal{C}$, which is the class of functions that have both a fractional integral and a fractional derivative of any order, (see Miller and Ross [7, p.88]). It is clear that we can write

$$E_{\nu,c}\left(z\right)=z^\nu\sum_{n=0}^{\infty}\frac{\left(cz\right)^n}{\Gamma\left(\nu+n+1\right)}, \tag{2}$$

where $\nu > -1$ and $c, z \in \mathbb{C}$.

It is clear that the Miller-Ross function $E_{\nu,c}(z)$ does not belong to the family $\mathcal{A}$. Thus, it is natural to consider the following normalization of the Miller-Ross function:

$$\mathbb{E}_{\nu,c}(z) = z^{1-\nu}\Gamma(\nu+1)E_{\nu,c}(z) = z + \sum_{n=2}^{\infty}\frac{c^{n-1}\Gamma(\nu+1)}{\Gamma(\nu+n)}z^n, \quad (\nu > -1,\ c \in \mathbb{C},\ z \in \mathbb{D}). \tag{3}$$

In this recent years, the authors in [1,3,5,6,8-10,13,14] studied the Hardy space of some special functions as normalized; Hypergeometric, Bessel, Struve, Lommel, Wright, Mittag-Leffler and Rabotnov. Motivated by above studies, our main aim is to determine some conditions on the parameters such that the Miller-Ross function $\mathbb{E}_{\nu,c}(z)$ is convex of order κ, Also, we find some conditions for the Hadamard products $\mathbb{E}_{\nu,c}(z) * f(z)$ to belong to $\mathcal{H}^\infty \cap \mathcal{R}$, where f is an analytic function in $\mathcal{R}$. Moreover, we investigate the Hardy space of the mentioned the normalized Miller-Ross function $\mathbb{E}_{\nu,c}(z)$.

In order to prove the main results we need the following preliminary results.

Lemma 1. (Silverman [11]) Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$. If

$$\sum_{n=2}^{\infty} n(n-\kappa)|a_n| \le 1-\kappa,$$

then the function $f(z)$ is in the class $\mathcal{C}(\kappa)$.

Lemma 2. (Eenigenburg and Keogh, [4]) Let $\kappa \in [0,1)$. If the function $f \in \mathcal{C}(\kappa)$ is not of the form

$$\begin{cases} f(z) = k + lz\left(1 - ze^{i\theta}\right)^{2\kappa-1} & \left(\kappa \ne \dfrac{1}{2}\right) \\ f(z) = k + l\log\left(1 - ze^{i\theta}\right) & \left(\kappa = \dfrac{1}{2}\right) \end{cases} \tag{4}$$

for some $k, l \in \mathbb{C}$ and $\theta \in \mathbb{R}$, then the following statements hold:

a: There exist $\delta = \delta(f) > 0$ such that $f' \in \mathcal{H}^{\delta + \frac{1}{2(1-\kappa)}}$.

b: If $\kappa \in \left[0, \dfrac{1}{2}\right)$, then there exist $\tau = \tau(f) > 0$ such that $f' \in \mathcal{H}^{\tau + \frac{1}{1-2\kappa}}$.

c: If $\kappa \ge \dfrac{1}{2}$, then $f \in \mathcal{H}^\infty$.

Lemma 3. (Stankiewich and Stankiewich, [12]) $\mathcal{P}_0(\lambda) * \mathcal{P}_0(\mu) \subset \mathcal{P}_0(\gamma)$, where $\gamma = 1 - 2(1-\lambda)(1-\mu)$. The value of γ is the best possible.

2. Main Results

In this section, we present our main results related to the some geometric properties and Hardy class of the normalized Miller-Ross function $\mathbb{E}_{\nu,c}(z)$.

Theorem 1. Let $\kappa \in [0,1)$, $\nu > -1$, $c \in \mathbb{C}$ and $|c| < \nu + 1$. The following inequality is true:

$$1 - 2\left(\frac{\nu - |c| + 1}{\nu + 1}\right)^2 + \frac{2|c|}{\nu - |c| + 1} \leq \kappa \tag{5}$$

holds, then the normalized Miller-Ross function $\mathbb{E}_{\nu,c}(z)$ is convex of order κ in $\mathbb{D}$.

Proof. By virtue of the Silverman's result which is given in Lemma 1, in order to prove the convex of order κ of the function, $\mathbb{E}_{\nu,c}(z)$ it is enough to show that the following inequality

$$\sum_{n=2}^{\infty} n(n-\kappa)\left|\frac{c^{n-1}\Gamma(\nu+1)}{\Gamma(\nu+n)}\right| \leq 1-\kappa \tag{6}$$

is satisfied under our assumptions. According to the hypothesis of theorem, by using the inequality

$$\Gamma(\nu+1)(\nu+1)^{n-1} \leq \Gamma(\nu+n)$$

and thus

$$\frac{\Gamma(\nu+1)}{\Gamma(\nu+n)} \leq \frac{1}{(\nu+1)^{n-1}}, \quad n \in \mathbb{N}, \tag{7}$$

we have

$$\begin{aligned}
\sum_{n=2}^{\infty} n(n-\kappa)\left|\frac{c^{n-1}\Gamma(\nu+1)}{\Gamma(\nu+n)}\right| &= \sum_{n=2}^{\infty} n(n-\kappa)\frac{|c|^{n-1}\Gamma(\nu+1)}{\Gamma(\nu+n)} \\
&\leq \sum_{n=2}^{\infty} n(n-\kappa)\frac{|c|^{n-1}}{(\nu+1)^{n-1}} \\
&= \sum_{n=2}^{\infty} n^2\left(\frac{|c|}{\nu+1}\right)^{n-1} - \sum_{n=2}^{\infty} \kappa n\left(\frac{|c|}{\nu+1}\right)^{n-1} \\
&= \left(\frac{\nu+1}{\nu-|c|+1}\right)^2 + \frac{2|c|(\nu+1)^2}{(\nu-|c|+1)^3} - \kappa\left(\frac{\nu+1}{\nu-|c|+1}\right)^2 - \kappa - 1, \quad (|c| < \nu+1).
\end{aligned}$$

The inequality (5) implies that the last sum is bounded above by $1-\kappa$. Therefore the inequality (6) is satisfied, that is, $\mathbb{E}_{\alpha,\beta}(z)$ is convex of order κ in $\mathbb{D}$.

Theorem 2. Let $\kappa \in [0,1)$, $\nu > -1$, $c \in \mathbb{C}$ and $|c| < \nu + 1$. If inequality

$$\kappa < \frac{\nu - 2|c| + 1}{\nu - |c| + 1} \tag{8}$$

holds, then $\dfrac{\mathbb{E}_{\nu,c}(z)}{z} \in \mathcal{P}_0(\kappa)$.

Proof. In order to prove $\dfrac{\mathbb{E}_{\nu,c}(z)}{z} \in \mathcal{P}_0(\kappa)$. it is enough to show that $|p(z) - 1| < 1$, where $p(z) = \dfrac{1}{1-\kappa}\left(\dfrac{\mathbb{E}_{\nu,c}(z)}{z} - \kappa\right)$. Now, using the inequalities (7), we have

$$\begin{aligned} |p(z)-1| &= \left| \frac{1}{1-\kappa}\left(1 + \sum_{n=2}^{\infty} \frac{c^{n-1}\Gamma(\nu+1)}{\Gamma(\nu+n)} z^{n-1} - \kappa\right) - 1 \right| \\ &\leq \frac{1}{1-\kappa} \sum_{n=2}^{\infty} \frac{|c|^{n-1}\Gamma(\nu+1)}{\Gamma(\nu+n)} \\ &\leq \frac{1}{1-\kappa} \sum_{n=2}^{\infty} \left(\frac{|c|}{\nu+1}\right)^{n-1} = \frac{|c|}{(1-\kappa)(\nu - |c| + 1)}, \quad (|c| < \nu + 1). \end{aligned}$$

Consequently, from (8) $\dfrac{\mathbb{E}_{\nu,c}(z)}{z}$ is in the class $\mathcal{P}_0(\kappa)$, and the proof is completed.

Theorem 3. Let $\kappa \in [0,1)$, $\nu > -1$, $c \in \mathbb{C}$. If the inequality (5) is satisfied, then

$$\mathbb{E}_{\nu,c}(z) \in \begin{cases} \mathcal{H}^{\frac{1}{1-2\kappa}}, & \kappa \in \left[0, \frac{1}{2}\right) \\ \mathcal{H}^{\infty}, & \kappa \in \left[\frac{1}{2}, 1\right) \end{cases}.$$

Proof. It is known that Gauss hypergeometric function is defined by

$${}_2F_1(a,b,c;z) = \sum_{n \geq 0} \frac{(a)_n (b)_n}{(c)} \frac{z^n}{n!} \quad (z \in \mathbb{C}). \tag{9}$$

Now, using the equality (9) it is possible to show that the function $\mathbb{E}_{\nu,c}(z)$ can not be written in the forms which are given by (4) for corresponding values of κ. More precisely, we can write that the following equalities:

$$k + lz\left(1 - ze^{i\theta}\right)^{2\kappa - 1} = k + l\sum_{n \geq 0} \frac{(1-2\kappa)_n}{n!} e^{in\theta} z^{n+1} \tag{10}$$

and

$$k+l\log\left(1-ze^{i\theta}\right)=k-l\sum_{n\geq 0}\frac{1}{n+1}e^{in\theta}z^{n+1} \tag{11}$$

hold true for $k,l\in\mathbb{C}$ and $\theta\in\mathbb{R}$. If we consider the series representation of the function $\mathbb{E}_{\nu,c}(z)$ which is given by (3), then we see that the function $\mathbb{E}_{\nu,c}(z)$ is not of the forms (10) for $\kappa\neq\frac{1}{2}$ and (11) for $\kappa=\frac{1}{2}$, respectively. On the other hand, Theorem 1, states that the function $\mathbb{E}_{\nu,c}(z)$ is convex of order under hypothesis. Therefore, the proof is completed by applying Lemma 2.

Theorem 4. Let $\nu>-1,\ c\in\mathbb{C},\ \lambda\in[0,1),\ \mu<1,\ |c|<\nu+1$ and $\gamma=1-2(1-\lambda)(1-\mu)$. Suppose that the function $f(z)\in\mathcal{P}_0(\mu)$. If the inequality

$$\lambda<\frac{\nu-2|c|+1}{\nu-|c|+1} \tag{13}$$

holds, then $\mathbb{E}_{\nu,c}(z)*f(z)\in\mathcal{R}_0(\gamma)$.

Proof. If $f(z)\in\mathcal{R}_0(\mu)$, then this implies that $f'(z)\in\mathcal{P}_0(\mu)$. We know from the Theorem 2 that the function $\frac{\mathbb{E}_{\nu,c}(z)}{z}\in\mathcal{P}_0(\lambda)$. Since $u'(z)=\frac{\mathbb{E}_{\nu,c}(z)}{z}*f'(z)$, taking into account the Lemma 3 we may write that $u'(z)\in\mathcal{P}_0(\gamma)$. This implies that $u(z)\in\mathcal{R}_0(\gamma)$.

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Initial Bounds For A Certain Subclass of Bi-Univalent Functions Defined By An Integral Operator

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Abstract

In this study, we obtained initial coefficients bounds for a new subclass of bi-univalent functions defined by an integral operator in the open unit disk U .

Keywords: Analytic function, Univalent function, Bi-univalent function, Coefficient inequality.

1. Introduction

Let A denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. Further, by S we shall denote the class of all functions in A which are univalent in U . It is well known that every function $f \in S$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \qquad (z \in U)$$

and

$$f(f^{-1}(w)) = w \qquad \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4}\right)$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \ldots$$

A function $f \in A$ is said to be in $\sum$, the class of bi-univalent functions in U , if both $f(z)$ and $f^{-1}(z)$ are univalent in U. Lewin [14] showed that $|a_2| < 1.51$ for every function $f \in \sum$ given by (1). Posteriorly, Brannan and Clunie [2] improved Lewin’s result and conjectured that $|a_2| \le \sqrt{2}$ for every function $f \in \Sigma$ given by (1). Later, Netanyahu [16] showed that $\max_{f\in\Sigma} |a_2| = \frac{4}{3}$ The coefficient estimate problem for each of the following Taylor-Maclaurin coefficients:

$$|a_n| \quad \left(n \in N = \{1,2,...\}; n \ge 4\right)$$

is still an open problem (see, for details, [21]). Since then, many researchers (see [3,5,8-11,22,23]) investigated several interesting subclasses of the class $\sum$ and found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. Also, many researchers (see [4,13,17,18]) investigated the upper bounds of combination of initial coefficients. In fact, its worth to mention that by making use of the Faber polynomial coefficient expansions Jahangiri and Hamidi [12] have obtained estimates for the general coefficients $|a_n|$ for bi-univalent functions subject to certain gap series.

Let P denote the class of function of p analytic in U such that $p(0)=1$ and $\operatorname{Re}\{p(z)\}>0$, where

$$p(z)=1+p_1z+p_2z^2+... \quad (z \in U).$$

If f and g are analytic in U, we say that f is subordinate to g, written symbolically as

$$f \prec g \qquad \text{or} \qquad f(z) \prec g(z) \quad (z \in U),$$

if there exists a Schwarz function $w(z)$, which (by definition) is analytic in U with $w(0)=0$ and $|w(z)|<1$ in U such that $f(z)=g(w(z)),\ z \in U$.

In particular, if the function $g(z)$ is univalent in U, then we have that:

$$f(z) \prec g(z) \quad (z \in U) \quad \text{if and only if} \quad f(0)=g(0) \ \text{ and } \ f(U) \subseteq g(U).$$

Let φ be an analytic function with positive real part in the unit disk U such that

$$\varphi(0)=1, \varphi'(0)>0$$

and $\varphi(U)$ is symmetric with respect to the real axis and has a series expansion of the form (see [15]):

$$\varphi(z)=1+B_1z+B_2z^2+B_3z^3+... \quad (B_1>0).$$

Let $u(z)$ and $v(z)$ be two analytic functions in the unit disk U with $u(0)=v(0)=0$ $|u(z)|<1, |v(z)|<1$, and suppose that

$$u(z)=1+b_1z+b_2z^2+b_3z^3+\ldots \text{ and } \mathrm{v}(w)=1+c_1w+c_2w^2+c_3w^3+\ldots \cdot \tag{2}$$

For above functions, well-known inequalities are

$$|b_1|\le 1, |b_2|\le 1-|b_1|^2, |c_1|<1 \text{ and } |c_2|\le 1-|c_1|^2. \tag{3}$$

Further we have

$$\varphi(u(z))=1+B_1b_1z+\left(B_1b_2+B_2b_1^2\right)z^2+\ldots(|z|<1) \tag{4}$$

and

$$\varphi(v(w))=1+B_1c_1w+\left(B_1c_2+B_2c_1^2\right)w^2+\ldots(|w|<1) \tag{5}$$

For $f(z)\in A$, Al-Shaqsi [1] defined the following integral operator:

$$\begin{aligned} L_c^\delta &= (1+c)^\delta \Phi_\delta(c;z) * f(z) \\ &= -\frac{(1+c)^\delta}{\Gamma(\delta)}\int_0^1 t^{c-1}\log\left(\frac{1}{t}\right)^{\delta-1} f(zt)dt, \quad (c>0, \quad \delta>1, z\in U) \end{aligned} \tag{6}$$

where Γ standarts for the usual gamma function, $\Phi_\delta(c;z)$ is the well known generalization of the Riemann- zeta and polylogarithm functions, or the δth polylogarithm function, given by

$$\Phi_\delta(c;z)=\sum_{k=1}^{\infty}\frac{z^k}{(k+c)^\delta}$$

where any term without $k+c=0$ is excluded. Also, $\Phi_{-1}(0;z)=\dfrac{z}{(1-z)^2}$ is Koebe function.

We also state that the operator $L_c^\delta f(z)$ given by (6) can be expressed by the series expansions as follows:

$$L_c^\delta f(z)=z+\sum_{k=2}^{\infty}\left(\frac{1+c}{k+c}\right)^\delta a_k z^k.$$

The main object of this paper is to introduce the following new subclass of bi-univalent functions involving integral operator L_c^δ and to obtain initial bounds for the Taylor- Maclaurin coefficients $|a_2|$ and $|a_3|$ of the functions belonging to this class.

2. Preliminaries and Definitions

The function class $B_{\Sigma}^{\delta}(c,\beta;\varphi)$ defined as follows:

Definition 1. A function $f(z)\in\Sigma$ is said to be in the class $B_{\Sigma}^{\delta}(c,\beta;\varphi)$ if and only if

$$(1-\beta)\frac{L_c^{\delta}f(z)}{z}+\beta\left(L_c^{\delta}f(z)\right)'\prec\varphi(z)$$

and

$$(1-\beta)\frac{L_c^{\delta}g(w)}{w}+\beta\left(L_c^{\delta}g(w)\right)'\prec\varphi(w)$$

where $0\le\beta\le 1,\ z,w\in U$ and $g(w)=f^{-1}(w)$.

Theorem 1. If $f(z)$ given by (1) is in the class $B_{\Sigma}^{\delta}(c,\beta;\varphi)$, then

$$|a_2|\le\frac{B_1\sqrt{B_1}}{\sqrt{\chi+B_1(1+\beta)^2\left(\frac{1+c}{2+c}\right)^{2\delta}}} \tag{7}$$

and

$$|a_3|\le\begin{cases}\dfrac{B_1}{(1+2\beta)\left(\frac{1+c}{3+c}\right)^{\delta}} & \text{if}\quad B_1<\dfrac{(1+\beta)^2\left(\frac{1+c}{2+c}\right)^{2\delta}}{(1+2\beta)\left(\frac{1+c}{3+c}\right)^{\delta}}\\[2ex] \dfrac{\chi B_1+(1+2\beta)\left(\frac{1+c}{3+c}\right)^{\delta}B_1^3}{(1+2\beta)\left(\frac{1+c}{3+c}\right)^{\delta}\left(\chi+(1+\beta)^2\left(\frac{1+c}{2+c}\right)^{2\delta}B_1\right)} & \text{if}\quad B_1\ge\dfrac{(1+\beta)^2\left(\frac{1+c}{2+c}\right)^{2\delta}}{(1+2\beta)\left(\frac{1+c}{3+c}\right)^{\delta}}\end{cases} \tag{8}$$

where

$$\chi=\left|B_1^2(1+2\beta)\left[9(2\lambda+1)\right]^m-B_2(1+\beta)^2\left[4(\lambda+1)\right]^{2m}\right|.$$

Proof: Let $f(z)\in B_{\Sigma}^{\delta}(c,\beta;\varphi)$. Then, there are analytic functions u and v with $u(0)=v(0)=0,\ |u(z)|<1,\ |v(w)|<1$, given by (2) and satisfying the following conditions:

$$(1-\beta)\frac{L_c^\delta f(z)}{z}+\beta\left(L_c^\delta f(z)\right)'=\varphi(u(z)) \tag{9}$$

and

$$(1-\beta)\frac{L_c^\delta g(w)}{w}+\beta\left(L_c^\delta g(w)\right)'=\varphi(v(w)), \tag{10}$$

where $g(w)=f^{-1}(w)$. Since

$$\begin{aligned}&(1-\beta)\frac{L_c^\delta f(z)}{z}+\beta\left(L_c^\delta f(z)\right)'\\&=1+(1+\beta)\left(\frac{1+c}{2+c}\right)^\delta a_2z+(1+2\beta)\left(\frac{1+c}{3+c}\right)^\delta a_3z^2+\ldots\end{aligned} \tag{11}$$

and

$$\begin{aligned}&(1-\beta)\frac{L_c^\delta g(w)}{w}+\beta\left(L_c^\delta g(w)\right)'\\&=1-(1+\beta)\left(\frac{1+c}{2+c}\right)^\delta a_2w+(1+2\beta)\left(\frac{1+c}{3+c}\right)^\delta\left(2a_2^2-a_3\right)w^2+\ldots,\end{aligned} \tag{12}$$

it follows from (4), (5), (11) and (12) that

$$(1+\beta)\left(\frac{1+c}{2+c}\right)^\delta a_2=B_1b_1, \tag{13}$$

$$(1+2\beta)\left(\frac{1+c}{3+c}\right)^\delta a_3=B_1b_2+B_2b_1^2, \tag{14}$$

$$-(1+\beta)\left(\frac{1+c}{2+c}\right)^\delta a_2=B_1c_1, \tag{15}$$

and

$$(1+2\beta)\left(\frac{1+c}{3+c}\right)^\delta\left(2a_2^2-a_3\right)=B_1c_2+B_2c_1^2. \tag{16}$$

From (13) and (15), we get

$$c_1=-b_1 \tag{17}$$

$$2\left[\left(\frac{1+c}{2+c}\right)^{\delta}(1+\beta)\right]^2 a_2^2 = B_1^2\left(b_1^2+c_1^2\right). \tag{18}$$

By adding (13) to (16), we have

$$2\left(\frac{1+c}{3+c}\right)^{\delta}(1+2\beta)\,a_2^2 = B_1\left(b_2+c_2\right)+B_2\left(b_1^2+c_1^2\right). \tag{19}$$

Therefore, from equalities (18) and (19) we find that

$$\left[2\left(\frac{1+c}{3+c}\right)^{\delta}(1+2\beta)\,B_1^2-2B_2\left(\left(\frac{1+c}{2+c}\right)^{\delta}(1+\beta)\right)^2\right]a_2^2 = B_1^3\left(b_2+c_2\right). \tag{20}$$

Then, in view of (13), (17) and (3), we obtain

$$\left|2\left(\frac{1+c}{3+c}\right)^{\delta}(1+2\beta)\,B_1^2-2B_2\left(\left(\frac{1+c}{2+c}\right)^{\delta}(1+\beta)\right)^2\right|\left|a_2\right|^2$$

$$\le B_1^3\left(\left|b_2\right|+\left|c_2\right|\right)\le 2B_1^3\left(1-\left|b_1\right|^2\right)=2B_1^3-2B_1\left(\left(\frac{1+c}{2+c}\right)^{\delta}(1+\beta)\right)^2\left|a_2\right|^2.$$

Thus, we get

$$\left|a_2\right|\le\frac{B_1\sqrt{B_1}}{\sqrt{\chi+B_1(1+\beta)^2\left(\frac{1+c}{2+c}\right)^{2\delta}}},$$

where

$$\chi=\left|B_1^2(1+2\beta)\left(\frac{1+c}{3+c}\right)^{\delta}-B_2(1+\beta)^2\left(\frac{1+c}{2+c}\right)^{2\delta}\right|.$$

Next, in order to find the bound on $\left|a_3\right|$, subtracting (16) from (14) and using (17), we get

$$2(1+2\beta)\left(\frac{1+c}{3+c}\right)^{\delta}a_3 = 2(1+2\beta)\left(\frac{1+c}{3+c}\right)^{\delta}a_2^2+B_1\left(b_2-c_2\right). \tag{21}$$

Then in view of (3) and (7), we have

$$2(1+2\beta)\left(\frac{1+c}{3+c}\right)^{\delta}|a_3| \le 2(1+2\beta)\left(\frac{1+c}{3+c}\right)^{\delta}|a_2|^2 + B_1\left(|b_2|+|c_2|\right)$$

$$\le 2(1+2\beta)\left(\frac{1+c}{3+c}\right)^{\delta}|a_2|^2 + 2B_1\left(1-|b_1|^2\right).$$

From (13), we immediately have

$$B_1(1+2\beta)\left(\frac{1+c}{3+c}\right)^{\delta}|a_3|$$

$$\le \left|B_1(1+2\beta)\left(\frac{1+c}{3+c}\right)^{\delta} - (1+\beta)^2\left(\frac{1+c}{2+c}\right)^{\delta}\right||a_2|^2 + B_1^2.$$

Now the assertion (8) follows from (7). This evidently completes the proof of Theorem 1.

By taking $\beta = 1$ in Theorem 1, we have

Corollary 1. If $f(z)$ given by (1) is in the class $B_{\Sigma}^{\delta}(c,1;\varphi)$, then

$$|a_2| \le \frac{B_1\sqrt{B_1}}{\sqrt{\tau + 4B_1\left(\frac{1+c}{2+c}\right)^{2\delta}}} \tag{22}$$

and

$$|a_3| \le \begin{cases} \dfrac{B_1}{3\left(\frac{1+c}{3+c}\right)^{\delta}} & \text{if} \quad B_1 < \dfrac{4\left(\frac{1+c}{2+c}\right)^{2\delta}}{3\left(\frac{1+c}{3+c}\right)^{\delta}} \\[3ex] \dfrac{\tau B_1 + 3\left(\frac{1+c}{3+c}\right)^{\delta} B_1^3}{3\left(\frac{1+c}{3+c}\right)^{\delta}\left(\tau + 4\left(\frac{1+c}{2+c}\right)^{2\delta} B_1\right)} & \text{if} \quad B_1 \ge \dfrac{4\left(\frac{1+c}{2+c}\right)^{2\delta}}{3\left(\frac{1+c}{3+c}\right)^{\delta}} \end{cases} \tag{23}$$

where

$$\tau = \left|3B_1^2\left(\frac{1+c}{3+c}\right)^{\delta} - 4B_2\left(\frac{1+c}{2+c}\right)^{2\delta}\right|.$$

Putting $\delta = 0$ in Theorem 1, we have

Corollary 2. [19] If $f(z)$ given by (1) is in the class $B_{\Sigma}^{0}(c,\beta;\varphi)=B_{\Sigma}(\beta;\varphi)$, then

$$|a_2|\leq\frac{B_1\sqrt{B_1}}{\sqrt{\left|B_1^2(1+2\beta)-B_2(1+\beta)^2\right|+B_1(1+\beta)^2}} \tag{24}$$

and

$$|a_3|\leq\begin{cases}\dfrac{B_1}{(1+2\beta)} & \text{if}\quad B_1<\dfrac{(1+\beta)^2}{1+2\beta}\\[2ex] \dfrac{\left|B_1^2(1+2\beta)-B_2(1+\beta)^2\right|B_1+(1+2\beta)B_1^3}{(1+2\beta)\left(\left|B_1^2(1+2\beta)-B_2(1+\beta)^2\right|+B_1(1+\beta)^2\right)} & \text{if}\quad B_1\geq\dfrac{(1+\beta)^2}{1+2\beta}\end{cases} \tag{25}$$

Putting $\delta=0$ in Corollary 1, we have

Corollary 3. [19] If $f(z)$ given by (1) is in the class $B_{\Sigma}^{0}(c,1;\varphi)=H_{\Sigma}(\varphi)$, then

$$|a_2|\leq\frac{B_1\sqrt{B_1}}{\sqrt{\left|3B_1^2-4B_2\right|+4B_1}} \tag{26}$$

and

$$|a_3|\leq\begin{cases}\dfrac{B_1}{3} & \text{if}\quad B_1<\dfrac{4}{3}\\[2ex] \dfrac{\left|3B_1^2-4B_2\right|B_1+3B_1^3}{3\left(\left|3B_1^2-4B_2\right|+4B_1\right)} & \text{if}\quad B_1\geq\dfrac{4}{3}\end{cases} \tag{27}$$

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Multi-Objective Optimal Sizing of SC's Considering Compressive and Tensile Deformation Using A Metaheuristic Algorithm

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Abstract

Optimization is significant to many applications such as engineering design, computer science, artificial intelligence, business planning, and industries. Product designs must maximize energy efficiency, performance, sustainability, and cost and waste minimization. Furthermore, optimization is crucial for engineering applications. Steel cushion (SC) is a major device to improve the seismic performance of structures since it is an easy-to-produce and plug-and-play damper with stable hysteretic loops and big displacement capacity, but SC is less effective in the transverse direction. As a result, optimal sizing utilizing intelligent optimization techniques improves the damper's efficiency. Also, we have to optimize multiple objectives. Finding solutions to a multi-objective optimization problem, even using a simple method is frequently difficult. Other potential methods, particularly metaheuristic methods such as genetic algorithms (GAs), particle swarm optimization (PSO), simulated annealing, firefly algorithm and cuckoo search work well for multi-objective optimization problems. The study's aim is to use a metaheuristic method, simulated annealing, to optimal size the SC that is subjected to transversal loads. By optimal sizing considering the dissipated cumulative energy as an objective, the efficiency of SC that is subjected to transversal loads in energy dissipation is increased.

Keywords: Multi-objective optimization, transversal loading, metaheuristic method.

1. Introduction

Sacrificial dampers are used in today's earthquake-resistant structure design to diffuse a significant portion of seismic energy. The seismic performance of structures improves because energy is dispersed by dampers rather than core structural components. Using extra damping, it is also possible to lessen the seismic demands of structures [1]. In the literature, many energy dissipative devices and isolation systems, such as lead extrusion dampers, metallic dampers, and spring-type isolators have been created to suit this purpose [2-5]. For different loadings and connections, the effectiveness of steel cushion (SC), which is a hysteretic damper, for seismic energy dissipation has been proved. Furthermore, it was discovered that the damping ratio in the transversal direction is roughly 18 percent, whereas the longitudinal damping ratio is 50 percent. As a result, optimization, which is a natural tool for developing methodologies to accomplish safe and cost-

effective design as the fundamental goal of multi-objective structural engineering, may improve the efficiency of SC [6-8].
Because optimization is significant in many disciplines, such as engineering design, computer science, artificial intelligence, and industries, it is a part of many applied fields. The majority of studies have concentrated on classic optimization strategies; nevertheless, during the last few decades, optimization techniques have developed, and some new techniques have begun to show their usefulness and have become an integral element of new mainstream methodologies. The terminology employed in the optimization literature, as well as the classification of optimization issues, can be confusing.
Many factors, as well as the actual perspective of mathematical formulations, might influence whether an optimization issue is regarded as easy or difficult. In fact, three elements make a problem more difficult: the objective function's nonlinearity, the problem's high dimensionality, and the search domain's complicated structure. Other factors, such as the number of an objective, are also important. In many cases, a single objective's evaluation is insufficient, hence multi-objective functions can be utilized. In fact, we frequently have to optimize multiple objectives at the same time. For example, we might aim to improve a product's performance while also attempting to reduce costs. We're working with multiobjective optimization problems in this situation. To solve multiobjective optimization, many new concepts are necessary. Moreover, these multiobjectives may conflict, necessitating some trade-offs. As a consequence, rather than a single solution, a series of Pareto-optimal solutions must be found. This frequently necessitates numerous iterations of the solution methods. As a result, selecting efficient methods for multi-objective optimization problems becomes critical. Because iterative algorithms for handling optimization problems are common, multiple objective evaluations are required, generally hundreds or thousands of evaluations.
For optimal size of energy dissipative SCs loaded in the longitudinal direction, Güllü et al. used different optimization techniques and multiple objective functions (maximizing dissipated energy, cumulative dissipated energy, and damping ratio) [9]. The damping ratio of the damper and the amount of energy dissipated by the damper were both enhanced as a result of proper sizing. Güllü and Körpeoğlu studied simultaneous multi-objective optimal sizing of energy dissipative steel cushions for transversal loading. Finite element analysis was used to assess the optimization's effectiveness (FEA). It was demonstrated that the optimally sized SC dissipates energy better [10].
Optimization is the most essential instrument for increasing the capability and effectiveness of sacrificial damping devices, according to studies published in the literature. As a result, the best device sizing or configuration throughout the structure is of practical significance. The study's aim is to a metaheuristic algorithm to size the SC that is exposed to transversal loading. Since the objective functions may be expressed in closed-form equations, the optimization process is primarily a mathematical problem solution.

2. Metaheuristic Algorithms

Different algorithms may have varying efficiencies and requirements. Newton's gradient-based technique is particularly efficient for solving smooth objective functions, but if the objective is very multimodal, it can get trapped in local modes. The trust-region approach, the interior-point method, and others are examples of nonlinear optimization algorithms, but they are mainly local search methods. Quadratic programming (QP) use convexity properties to their advantage. In practice, the simplex method for solving LP can be effective, but it requires that all problem functions be linear.

Because traditional approaches are primarily local search algorithms, heuristic and metaheuristic algorithms are becoming increasingly popular. Meta denotes "beyond" or "higher level," thus meta heuristics outperform simple heuristics. It's worth noting that there are no universally accepted definitions of heuristics and metaheuristics in the literature, and some authors use the terms interchangeably. However, in recent years, any stochastic algorithms involving randomization and local search have been referred to as metaheuristics. Almost all metaheuristic algorithms claim to be capable of global optimization, while in fact, global optimality may be difficult to accomplish for most tasks [11].

The majority of metaheuristic algorithms are nature-inspired, as they were created based on a natural abstraction. Nature has evolved over millions of years and has perfected solutions to nearly every difficulty encountered. As a result, we can learn from nature about problem-solving success and design nature-inspired heuristic and/or metaheuristic algorithms. They are referred regarded as biologically inspired or simply bio-inspired [11].

Randomization and selection of the best solutions are two essential components of any metaheuristic algorithm. The best solution is chosen, ensuring that the solutions will converge to the optimality, while randomization prevents solutions from becoming stuck at local optima while also increasing the diversity of the solutions. In most cases, a solid combination of these two components will result in global optimality.

Many different types of metaheuristic algorithms exist. One method is to divide them into two categories: trajectory-based and population-based. Because they use a collection of strings, genetic algorithms are population-based, as are the firefly algorithm (FA) and particle swarm optimization (PSO), which use many agents or particles. Agent-based algorithms are also known as PSO and FA.

Simulated annealing (SA), on the other hand, employs a single agent or solution that advances piecewise through the search space or design space. A better action or solution is always accepted, although a less-than-ideal move can be accepted with a high degree of certainty. The steps or moves follow a course in the search space that has a nonzero chance of reaching the global optimum [11].

$$
\text{Algorithms}\begin{cases}
\text{Deterministic}\begin{cases}
\text{Linear programming (Simplex method, ...)}\\
\text{Integer programming (Branch and bound, ...)}\\
\text{Convex optimization (QP, ...)}\\
\text{Gradient} - \text{based (Newton's method, ...)}\\
\cdots\\
\text{Nonlinear programming (Newton's method, ...)}
\end{cases}\\
\text{Stochastic}\begin{cases}
\text{Heuristic (evaluation strategy, ...)}\\
\text{Metaheuristic}\begin{cases}
\text{Population} - \text{based (PSO, FA, ...)}\\
\text{Trajectory} - \text{based (SA, ...)}
\end{cases}
\end{cases}\\
\text{Stochastic learning}\begin{cases}
\text{Supervised learning(Regression, ANN, ...)}\\
\text{Unsupervised learning (Clustering, ANN, ...)}
\end{cases}
\end{cases}
$$

Figure 1. Classification of optimization algorithms [11].

3. Energy Dissipative SCs

SC has been used in the literature for both retrofitting and improving the seismic performance of frame-type structures. SCs, which are made by bending steel sheets, disperse seismic energy by utilizing the ductile properties of mild steel and relative movement between fixed surfaces [12-13]. Its geometry (Figure 2(a)) consists of two half-circles with straight portions connecting them. The relative movement of the bolts through each other on SC was used to name the loading types (Figures. 2(b) and 2(c)). A transversal load is one in which the bolts stay in the same vertical portion as they were when the load was applied. It's possible that the transverse load is compressive or tensile. The loading is called longitudinal if there is shearing action between the fixed surfaces. The out-of-plane direction is the third direction. SC's hysteretic behavior varies according on the loading type. In real-world applications, SC was subjected to several deformation mechanisms for longitudinal, transverse (compressive and tensile) as well as bending moment loadings. SC may be subjected to longitudinal or transverse deformations depending on the connection detail. When SC is used under shear walls/panels, the wall/rocking panel's behavior causes transversal deformations in SC (Figure 3).

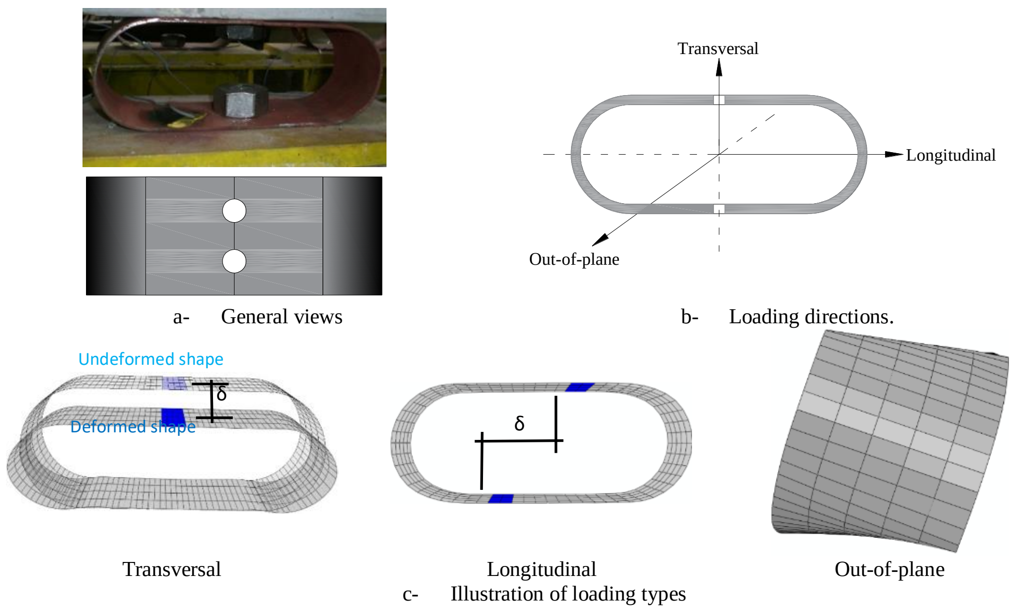


Figure 2. Steel cushions [10].

By optimizing size with the dissipated cumulative energy as an objective, the efficiency of SC that is subjected to transversal loads in energy dissipation might be increased. The optimization method is carried out by solving the closed form equations for SC under transversal loads, which are already available in the literature [6-8]. Equations (1)–(8) provide the relevant equations for determining the hysteretic behavior of SC under the influence of transversal loads.

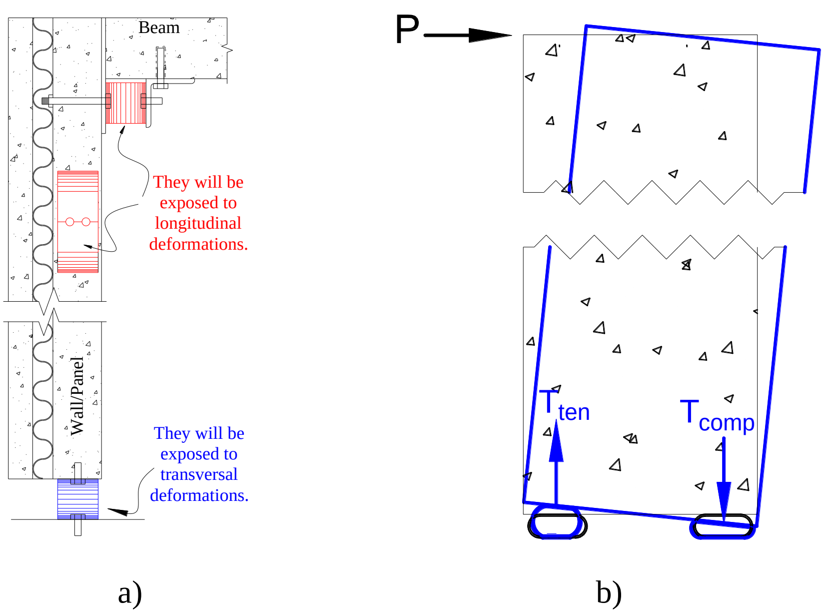


Figure 3. a) Possible connection details of SCs and b) transversal loading mechanism [10].

Moment (M)	$M(\varphi)=\frac{N}{2}\left(\frac{2r^2-a^2}{\pi r+2a}-r\sin\varphi\right)$	(1)
Location of extremum moment (φ_{ext})	$\varphi_{ext}=\frac{\pi}{2}$	(2)
Vertical disp. under compression ($\delta_{v,c}$)	$\delta_{v,c}=\frac{Nr}{2EI}\left(\frac{\pi a^3+8a^2r+2\pi ar+\{\pi^2-8\}r^3}{2\{2a+\pi r\}}\right)$	(3)
Compressive yielding force ($N_{y,c}$)	$N_{y,c}=f_{yd}\frac{bt^2}{4}$	(4)
Compressive ultimate force ($N_{u,c}$)	$N_{u,c}=f_{ud}\frac{bt^2}{4}$	(5)
Vertical disp. under tension ($\delta_{v,t}$)	$\delta_{v,t}=\frac{Nr}{EI}\left(\frac{\pi a^3+8a^2r+2\pi ar+\{\pi^2-8\}r^3}{2\{2a+\pi r\}}\right)$	(6)
Tensile yielding force ($N_{y,t}$)	$N_{y,t}=f_{yd}\frac{bt^2}{4}\times\frac{D_n}{2a}$	(7)
Compressive ultimate force ($N_{u,t}$)	$N_{u,t}=f_{ud}\frac{bt^2}{4}\times\frac{D_n}{2a}$	(8)

In the equations, b represents the width, a represents the half-length of the straight part, r represents the radius of the half-circle, D_n represents the diameter of the nuts, t represents the thickness of the steel sheet, φ represents the angle between the vertical axis and any point on the half-circle, N represents the transversal force, and f_{yd} and f_{ud} represent the yielding and ultimate strengths of the base material.

4. Optimization of SCs for Transversal Loading

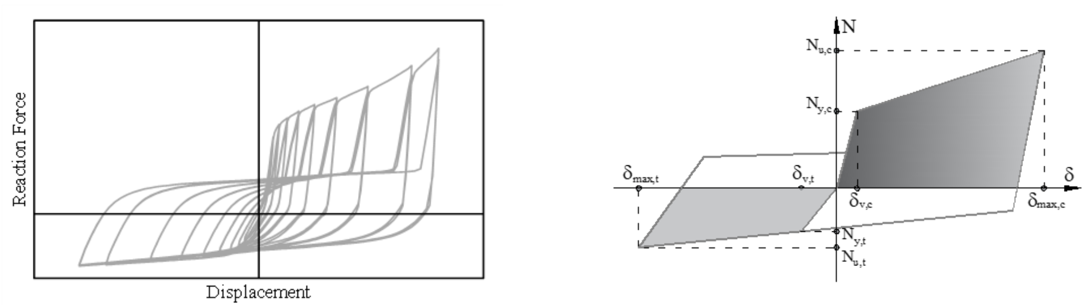
This paper proposes a multi-objective optimization technique for selecting the optimal SC sizes. The half-length of the straight component (a), the width (b), the thickness (t), and the radius of the half-circle are the operating variables (r). Two objective functions are studied, both of which attempt to maximize the cumulative plastic energy dissipated. Finding the design variables that maximize equation (9) and meet the equality constraints given by equations (1)–(8) results in optimal sizing:

$$F(x)=\left[f_1(x),\, f_2(x)\right]^T \tag{9}$$

Figures 4(a) and 4(b) illustrate a general hysteresis produced from a cyclic experiment, as well as a schematic diagram showing the considered parameters. Equations (10) and (11) are the objective functions of the optimization problem that aim to maximize the dissipated cumulative energy (hatched areas in Figure 4b.).

$$f_1(x) = \frac{N_{y,c} + N_{u,c}}{2}\left(\delta_{\max,c} - \frac{N_{u,c}}{N_{y,c}}\delta_{v,c}\right) \tag{10}$$

$$f_2(x) = \frac{N_{y,t} + N_{u,t}}{2}\left(\delta_{\max,t} - \frac{N_{u,t}}{N_{y,t}}\delta_{v,t}\right) \tag{11}$$



a- Experimental hysteresis b- Schematic representation

Figure 4. General hysteretic behavior of SCs under transversal loading [10].

5. Application of the Optimization Technique

Simulated Annealing Algorithm

For global optimization problems, simulated annealing (SA) is a random search approach. It is similar to the annealing process in materials processing, which occurs when a metal cools and freezes into a crystalline state with the least amount of energy and a bigger crystal size to decrease flaws in metallic structures [11]. The annealing process necessitates precise temperature and cooling rate control.

Kirk Patrick et al. were the first to apply SA to optimization problems in 1983 [14]. There have been numerous investigations since then. With the exception of gradient-based and other deterministic search methods, which are susceptible to becoming stuck in local minima, SA's key advantage is its ability to resist becoming trapped in local minima. In reality, it has been proven that if enough randomization is utilized in combination with very slow cooling, SA will converge to its global optimality. In terms of metaphor, the iterations in SA are analogous to scattering bouncing balls throughout a landscape. The balls will settle down to some local minima as they bounce and lose energy. If the balls are allowed to bounce for long enough and lose energy slowly enough, some of them will eventually fall to the lowest global locations, resulting in the global minimum.

In this part, the simulated annealing algorithm is introduced for solving optimization problems [11]. The simulated annealing algorithm works on the principle of random search, which accepts modifications that improve the objective function while still keeping some changes that aren't perfect. In a minimization task, for example, any better moves or adjustments that reduce the cost (or value) of the objective function f will be accepted with a probability p; but, some changes that raise f will also be accepted with a probability p. This probability p, also known as the transition probability, is calculated as follows:

$$p = \exp\left(-\frac{\delta E}{k_B T}\right), \tag{12}$$

where T is the temperature used to control the annealing process and k_B is the Boltzmann constant. The change in energy level is denoted by δE. The Boltzmann distribution in physics is used to calculate this transition probability. The most straightforward way to connect δE to the change in the objective function δf is to use $\delta E = \gamma \delta f$, where γ is a real constant. We can use $k_B = 1$ and $\gamma = 1$ for convenience without sacrificing generality.

As a result, the probability p is simply

$$p(\delta f, T) = e^{-\frac{\delta f}{T}}. \tag{13}$$

We commonly use a random number r (taken from a uniform distribution in $[0,1]$) as a threshold to determine whether or not to accept a change. As a result, if $p > r$, it is acceptable.

The selection of the appropriate temperature is critical in this case. If T is too high ($T \to \infty$), then $p \to 1$, suggests that practically all changes will be accepted for a given change δf. If T is too low ($T \to 0$), any $\delta f > 0$ (worse solution) will be rarely accepted as $p \to 0$, limiting the solution's variety, but any improvement in δf will almost always be accepted. In fact, because only better solutions are allowed and the system is effectively climbing or descending a hill, the exceptional case $T \to 0$ corresponds to the gradient-based technique. As a result, if T is too high, the system is in a high-energy state on the topological landscape, making it difficult to attain the minima. If T is too low, the system may become trapped in a local minimum (rather than a global minimum), with insufficient energy to jump out of the local minimum and explore other possible global minima. As a consequence, a correct starting temperature must be computed.

Another critical challenge is how to control the cooling process so that the system gradually cools down from a higher temperature to eventually freeze to a global lowest condition. There are numerous methods for controlling the rate of cooling or temperature decline.

Linear and geometric cooling are two regularly utilized cooling schemes.

- We have a linear cooling process

$$T = T_0 - \beta t. \tag{14}$$

The initial temperature is T_0, and the pseudo time for iterations is t. The cooling rate, β, should be chosen so that $T \to 0$ when $t \to t_f$ (maximum number of iterations) is reached, which usually results in $\beta = T_0/t_f$.

- The geometric cooling process reduces the temperature by a factor of $0 < \alpha < 1$ so that T is replaced by αT.

$$T(t) = T_0\alpha^t, \ \left(t = 1,2,\dots,t_f\right). \tag{15}$$

The benefit of the second way is that $T \to 0$ when $\to \infty$, eliminating the requirement to provide the maximum number of iterations t_f. As a result, we'll follow this geometric cooling strategy. The cooling process should be slow enough that the system can easily stabilize.

Furthermore, several objective function evaluations are required for a given temperature. There is a possibility that the system may not stabilize and, as a result, will not converge to its global optimality if there are too few evaluations. It is time-consuming to perform too many evaluations, and the system will usually converge too slowly since the number of iterations required to attain stability may be exponentially proportional to the problem size.

As a result, there is a ratio between the number of assessments and the quality of the solutions. We can either perform a large number of evaluations at a few temperature levels or a small number of evaluations at a large number of temperature levels. The number of iterations can be set in one of two ways: fixed or variable. The first employs a set number of iterations at each temperature, but the second is meant to increase the number of repeats at lower temperatures in order to completely investigate the local minima. The simulated annealing algorithm's main procedure can be summarized using the pseudocode presented in Algorithm 1 [11].

We can utilize any knowledge about the objective function to obtain an appropriate starting temperature T_0. We can estimate an initial temperature T_0 for a given probability p_0 if we know the objective function's maximum change $max(\delta f)$.

$$T_0 \approx -\frac{\max(\delta f)}{lnp_0}. \tag{16}$$

We can apply a heuristic technique if we don't know the greatest feasible change in the objective function. We can begin evaluations at a very high temperature (such that practically all changes are allowed), then rapidly lower the temperature until around 50 or 60 percent of the worst moves are accepted, and then use this temperature as the new initial temperature T_0 for proper and reasonably slow cooling.

In theory, the final temperature should be zero, such that no worse move can be tolerated. In practice, depending on the desired quality of the answers and time restrictions, we just set a very small value.

Algorithm 1. Simulated Annealing Algorithm.

Objective function $F(x)$, $x = (x_1, x_2, \dots,)^T$
Initialize initial temperature T_0 and initial guess x_0
Set final temperature T_f and max number of iterations N
Define cooling schedule $T \rightarrow \alpha T$, $(0 < \alpha < 1)$
while ($T > T_f$ or $t < t_f$)
Move randomly to new location x_{t+1}
Calculate $\delta f = F(x_{t+1}) - F(x_t)$
Accept the new solution if better
if not improved
Generate a random number r
Accept if $p = exp[-\delta F / T] > r$
end if
Update the best x* and F*
end while

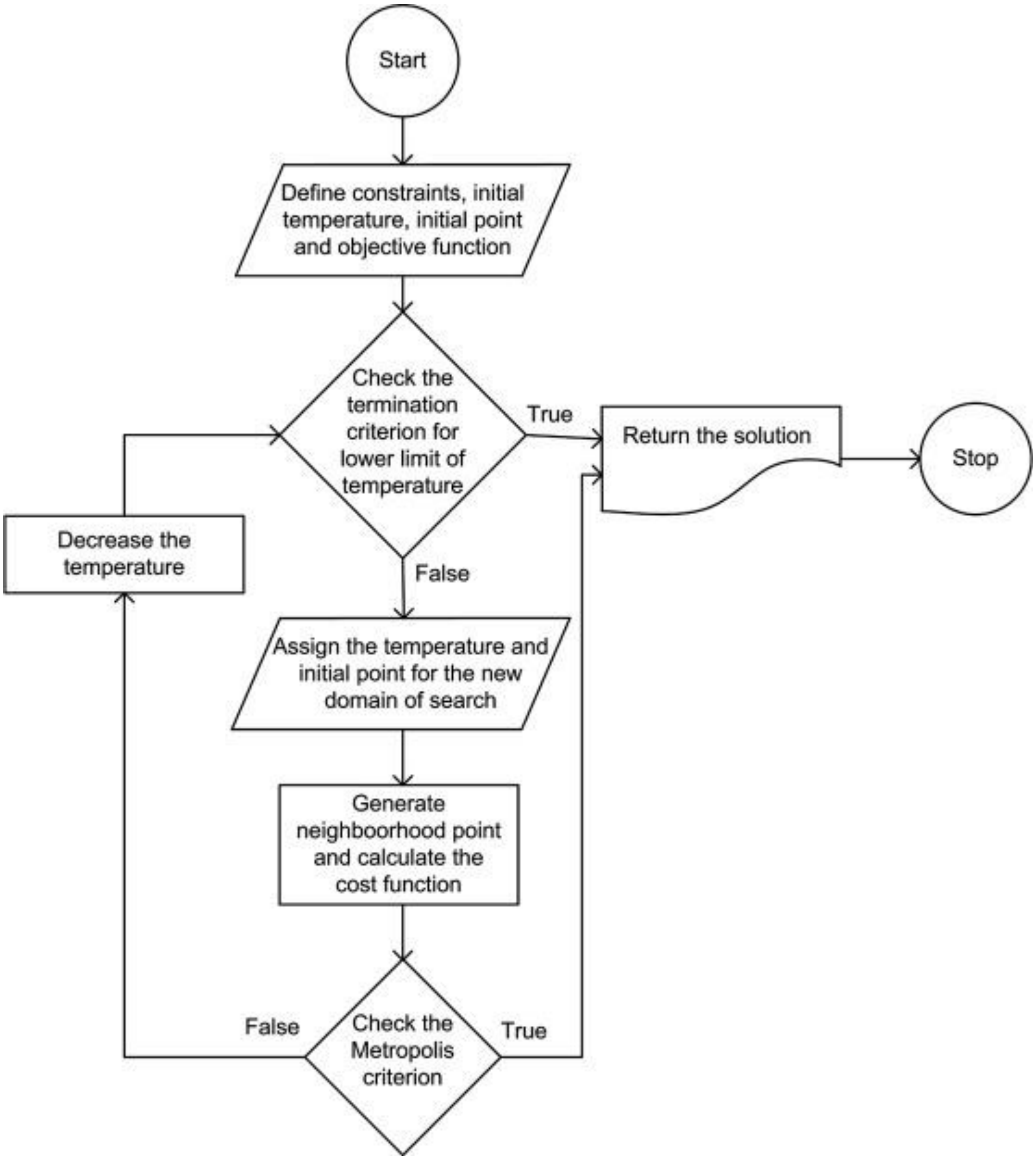


Figure 5. Flowchart of Simulated Annealing Algorithm [15].

MATLAB program can be used to demonstrate the use of SA to optimize the functions in equations (10)-(11). We chose T_0 = 1.0 as the initial temperature, $t_f = 10^{-10}$ as the final temperature, and a geometric cooling plan with α= 0.8 as the cooling rate.

Table 1. Optimization results for design variables of SCs.

Objective function	a/r	b/t	a/b
$f_1(x)$	3.0003	20.0	1.0000
$f_2(x)$	3.0002	20.0	1.4039

Table 2. Smoothed geometric dimensions of optimized SCs.

Objective function	a(mm)	b(mm)	t(mm)	r(mm)
$f_1(x)$	20.0	20.0	10.0	6.6
$f_2(x)$	28.0	20.0	10.0	9.3

Table 3. Resultant object function values.

	$f_1(x)$	$f_2(x)$
Simulated annealing algorithm	1.3459e14	1.2112e12

6. Conclusions

In the literature, energy dissipative SC was created to improve the seismic performance of cladding systems, existing buildings, and other structures. The efficiency of transversal loading is improved in this work by optimizing the size of multiple object functions related to cumulative dissipated energy. Considering unitless ratios of $a/r, b/t$, and a/b, the optimal sizing is reached. An optimal energy dissipative SC can be designed easily for applications in practice using the achieved optimal ratios, the closed-form equations, and structural constraints. The optimal SC ratios are 3, 20, 1.40 and 3, 20, 1 for tensile and compressive load, respectively, for half of the straight part length to the radius of the circular portion (a/r), width to thickness (b/t), and half of the straight part length to width (a/b).

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Neighborhoods and Partial Sums for Certain Subclasses of Meromorphic Functions

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Abstract

In this study, using the fractional derivative operator, we define a new subclass of meromorphic functions. Some properties neighborhoods and partial sums of functions in this subclass are given.

Keywords: Meromorphic, Neighborhood, Fractional derivative operator, Partial sum.

1. Introduction

Let Σ denote the class of functions of the form

$$f(z) = z^{-1} + \sum_{n=0}^{\infty} a_n z^n \tag{1}$$

which are analytic in the punctured disc $\mathbb{D} = \{z \in \mathbb{C} : 0 < |z| < 1\}$.

The meromorphic analouge of the fractional derivative of order α, $0 \le \alpha \le 1$, is defined in [3] for a function $f(z)$ by

$$D_z^\alpha f(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \left\{ z^{\alpha-1} \int_0^z (z-\xi)^{-\alpha} {}_2F_1\left(1-\alpha, 1, 1-\alpha; 1-\frac{\xi}{2}\right) \xi^2 f(\xi)\, d\xi \right\},$$

where $f(z)$ is analytic function in a simply connected domain of the $z-$plane containing the origin and the multiplicity of $(z-\xi)^{-\alpha}$ is removed by requiring $\log(z-\xi)$ to be real when $(z-\xi) > 0$. Using $D_z^\alpha f(z)$, Noor, Ahmad and Khan [9] defined an operator $\Omega_z^\alpha f(z) : \Sigma \to \Sigma$, as follows:

$$\begin{aligned} \Omega_z^\alpha f(z) &= \frac{\Gamma(2-\alpha)}{\Gamma(2)} z D_z^\alpha f(z) \\ &= z^{-1} + \sum_{n=0}^{\infty} \frac{(2)_{n+1}}{(2-\alpha)_{n+1}} a_n z^n \\ &= \phi(2, 2-\alpha; z) * f(z), \ \alpha \ne 2, 3, 4, \ldots \end{aligned}$$

where

$$\phi(2,2-\alpha;z)=z^{-1}+\sum_{n=0}^{\infty}\frac{(2)_{n+1}}{(2-\alpha)_{n+1}}z^{n}.$$

We now define the following classes of functions.

Let $-1\le B<A\le 1$. A function $f(z)=z^{-1}+\sum_{n=0}^{\infty}a_n z^n\in\Sigma$ is said to be in the class $T_m(\alpha,A,B)$ if it satisfies the condition

$$\left|\frac{z\left(\Omega_z^{\alpha}f(z)\right)'+\Omega_z^{\alpha}f(z)}{Bz\left(\Omega_z^{\alpha}f(z)\right)'+A\Omega_z^{\alpha}f(z)}\right|<1 \tag{2}$$

for all $z\in\mathbb{D}$.

Furthermore, a function $f(z)=z^{-1}+\sum_{n=1}^{\infty}|a_n|z^n\in\Sigma$ is said to be in the class $T_m^{*}(\alpha,A,B)$ if it satisfies the condition (2).

It should be remarked in passing that the definition (2) is motivated essentially by the recent work of Morga [8] and Srivastava and co-authors [11].

In recent years, many important properties and characteristics of various interesting subclasses of the class Σ of meromorphically functions were inverstigated extensively by (among others) Aouf et al. [2], Dziok et al. [4], El-Ashwah and Aouf [5], He et al. [7] and also [12].

The main object of this paper is to present neighborhoods and partial sums of functions in the classes $T_m(\alpha,A,B)$ and $T_m^{*}(\alpha,A,B)$ which we introduced here.

2. Neighborhoods and partial sums

Following the earlier works (based upon the familiar concept of neighborhoods of analytic functions) by Goodman [6] and Ruscheweyh [10] and (more recently)by Altıntas and Owa [1] and Srivastava and Owa [12] ,we begin by introducing here the δ-neighborhood of a function $f\in\Sigma$ of the form (1) by means of the definition

$$\mathrm{N}_{\delta}(f)=\left\{g(z)=z^{-1}+\sum_{n=0}^{\infty}b_n z^n\in\Sigma:\sum_{n=0}^{\infty}\frac{[(1-A)+n(1-B)]}{(A-B)}\frac{(2)_{n+1}}{(2-\alpha)_{n+1}}|b_n-a_n|\le\delta,-1\le B<A\le 1;\delta\ge 0\right\}$$

where $0\le\alpha\le 1$.

Making use of this definition, we now prove that:

Theorem 1. Let $\delta>0$ and $-1<A\le 0$. If $f(z)=z^{-1}+\sum_{n=0}^{\infty}a_n z^n\in\Sigma$ satisfies the condition

$$\frac{f(z)+\varepsilon z^{-1}}{1+\varepsilon} \in T_m(\alpha, A, B) \tag{3}$$

for any complex number ε such that $|\varepsilon| < \delta$, then $\mathrm{N}_\delta(f) \subset T_m(\alpha, A, B)$.

Proof. It is obvious from (2) that $g(z) \in T_m(\alpha, A, B)$ if and only if for any complex number σ with $|\sigma| = 1$

$$\frac{z\left(\Omega_z^\alpha g(z)\right)' + \Omega_z^\alpha g(z)}{Bz\left(\Omega_z^\alpha g(z)\right)' + A\Omega_z^\alpha g(z)} \neq \sigma \qquad (z \in \mathbb{D}),$$

which is equivalent to

$$\frac{g(z) * h(z)}{z^{-1}} \neq 0 \qquad (z \in \mathbb{D}) \tag{4}$$

where

$$\begin{aligned} h(z) &= z^{-1} + \sum_{n=0}^{\infty} c_n z^n \\ &= z^{-1} + \sum_{n=0}^{\infty} \frac{[(1+n) - \sigma(A+nB)]}{\sigma(B-A)} \frac{(2)_{n+1}}{(2-\alpha)_{n+1}} z^n. \end{aligned} \tag{5}$$

From (5), we have

$$\begin{aligned} |c_n| &= \left| \frac{[(1+n) - \sigma(A+nB)}{\sigma(B-A)} \frac{(2)_{n+1}}{(2-\alpha)_{n+1}} \right| \\ &\leq \frac{(1-A) + n(1-B)}{(A-B)} \frac{(2)_{n+1}}{(2-\alpha)_{n+1}}. \end{aligned}$$

If $f(z) = z^{-1} + \sum_{n=0}^{\infty} a_n z^n \in \Sigma$ satisfies the condition (3), then (4) yields

$$\left| \frac{f(z) * h(z)}{z^{-1}} \right| \geq \delta \qquad (z \in \mathbb{D}). \tag{6}$$

Now let $p(z) = z^{-1} + \sum_{n=0}^{\infty} b_n z^n \in \mathrm{N}_\delta(f)$, then

$$\begin{aligned} \left| \frac{\left(p(z) - f(z)\right) * h(z)}{z^{-1}} \right| &= \left| \sum_{n=0}^{\infty} (b_n - a_n) c_n z^{n+1} \right| \\ &\leq |z| \sum_{n=0}^{\infty} \frac{[(1-A) + n(1-B)]}{(A-B)} \frac{(2)_{n+1}}{(2-\alpha)_{n+1}} |b_n - a_n| < \delta. \end{aligned}$$

Thus for any complex number σ such that $|\sigma| = 1$, we have

$$\frac{p(z)*h(z)}{z^{-1}} \neq 0 \qquad (z \in \mathbb{D}),$$

which implies that $p(z) \in T_m(\alpha, A, B)$.

Theorem 2. Let $-1 < A \leq 0$, $f(z) = z^{-1} + \sum_{n=0}^{\infty} a_n z^n \in \Sigma$, $s_1(z) = z^{-1}$ and $s_k(z) = z^{-1} + \sum_{n=0}^{k-2} a_n z^n \quad (k \geq 2)$. Suppose that

$$\sum_{n=0}^{\infty} c_n |a_n| \leq 1 \tag{7}$$

where

$$c_n = \frac{(1-A)+n(1-B)}{(A-B)} \frac{(2)_{n+1}}{(2-\alpha)_{n+1}}.$$

Then we have

i. $f(z) \in T_m(\alpha, A, B)$

ii. $$\operatorname{Re}\left\{\frac{f(z)}{s_k(z)}\right\} > 1 - \frac{1}{c_{k-1}} \tag{8}$$

and

$$\operatorname{Re}\left\{\frac{s_k(z)}{f(z)}\right\} > \frac{c_{k-1}}{1+c_{k-1}}. \tag{9}$$

The estimates are sharp.

Proof. i. It is obvious that $z^{-1} \in T_m(\alpha, A, B)$. Thus from Theorem 1. and the condition (7), we have $\mathrm{N}_1(z^{-1}) \subset T_m(\alpha, A, B)$. This gives $f(z) \in T_m(\alpha, A, B)$.

ii. It is easy to see that $c_{n+1} > c_n > 1$. Thus

$$\sum_{n=0}^{k-2} |a_n| + c_{k-1} \sum_{n=k-1}^{\infty} |a_n| \leq \sum_{n=0}^{\infty} c_n |a_n| \leq 1. \tag{10}$$

Let

$$h_1(z) = c_{k-1}\left\{\frac{f(z)}{s_k(z)} - \left(1 - \frac{1}{c_{k-1}}\right)\right\}$$

$$= 1 + \frac{c_{k-1} \sum_{n=k-1}^{\infty} a_n z^{n+1}}{1 + \sum_{n=0}^{k-2} a_n z^{n+1}}.$$

It follows from (10) that

$$\left|\frac{h_1(z)-1}{h_1(z)+1}\right| \le \frac{c_{k-1}\sum\limits_{n=k-1}^{\infty}|a_n|}{2-2\sum\limits_{n=0}^{k-2}|a_n|-c_{k-1}\sum\limits_{n=k-1}^{\infty}|a_n|} \le 1 \qquad (z\in\mathbb{D}).$$

From this we obtain the inequality (8).
If we take

$$f(z)=z^{-1}-\frac{z^{k-1}}{c_{k-1}}, \tag{11}$$

then

$$\frac{f(z)}{s_k(z)}=1-\frac{z^k}{c_{k-1}}\to 1-\frac{1}{c_{k-1}} \quad \text{as} \quad z\to 1^-.$$

This shows that the bound in (8) is best possible for each *k*.
Similarly, if we take

$$h_2(z)=(1+c_{k-1})\left\{\frac{s_k(z)}{f(z)}-\frac{c_{k-1}}{1+c_{k-1}}\right\}$$

$$=1-\frac{(1+c_{k-1})\sum\limits_{n=k-1}^{\infty}a_n z^{n+1}}{1+\sum\limits_{n=0}^{\infty}a_n z^{n+1}}$$

then we deduce that

$$\left|\frac{h_2(z)-1}{h_2(z)+1}\right| \le \frac{(1+c_{k-1})\sum\limits_{n=k-1}^{\infty}|a_n|}{2-2\sum\limits_{n=0}^{k-2}|a_n|+(1-c_{k-1})\sum\limits_{n=k-1}^{\infty}|a_n|} \le 1 \qquad (z\in\mathbb{D}),$$

which yields (9). The estimate (9) is sharp with the extramal function $f(z)$ given by (11).

Theorem 3. Let $f(z)=z^{-1}+\sum\limits_{n=1}^{\infty}|a_n|z^n$ be analytic in $\mathbb{D}=\{z:0<|z|<1\}$. Then $f(z)\in T_m^*(\alpha,A,B)$ if and only if

$$\sum_{n=1}^{\infty}\left[(1-A)+n(1-B)\right]\frac{(2)_{n+1}}{(2-\alpha)_{n+1}}|a_n| \le (A-B) \tag{12}$$

The result is sharp for the function $f(z)$ given by

$$f(z)=z^{-1}+\frac{(2-\alpha)_{n+1}(A-B)}{(2)_{n+1}\left[(1-A)+n(1-B)\right]}z^n \qquad (n\ge 1)$$

Proof. Let $f(z)=z^{-1}+\sum_{n=1}^{\infty}|a_n|z^n \in T_m^*(\alpha,A,B)$. Then

$$\left|\frac{z(\Omega_z^{\alpha}f(z))'+\Omega_z^{\alpha}f(z)}{Bz(\Omega_z^{\alpha}f(z))'+A\Omega_z^{\alpha}f(z)}\right|=\left|\frac{\sum_{n=1}^{\infty}(1+n)\frac{(2)_{n+1}}{(2-\alpha)_{n+1}}|a_n|z^{n+1}}{(A-B)+\sum_{n=1}^{\infty}(A+Bn)\frac{(2)_{n+1}}{(2-\alpha)_{n+1}}|a_n|z^{n+1}}\right|. \tag{13}$$

Since $|\operatorname{Re} z|\le|z|$ for any z, choosing z to be real letting $z\to 1^-$ throuh real values (12) yields

$$\sum_{n=1}^{\infty}(1+n)\frac{(2)_{n+1}}{(2-\alpha)_{n+1}}|a_n|\le(A-B)+\sum_{n=1}^{\infty}(A+Bn)\frac{(2)_{n+1}}{(2-\alpha)_{n+1}}|a_n|,$$

which gives (13).
On the other hand, we have that

$$\left|\frac{z(\Omega_z^{\alpha}f(z))'+\Omega_z^{\alpha}f(z)}{Bz(\Omega_z^{\alpha}f(z))'+A\Omega_z^{\alpha}f(z)}\right|\le\frac{\sum_{n=1}^{\infty}(1+n)\frac{(2)_{n+1}}{(2-\alpha)_{n+1}}|a_n|}{(A-B)+\sum_{n=1}^{\infty}(A+Bn)\frac{(2)_{n+1}}{(2-\alpha)_{n+1}}|a_n|}<1.$$

This shows that $f(z)\in T_m^*(\alpha,A,B)$.

For $\delta\ge 0, -1\le B<A\le 1$ and $f(z)=z^{-1}+\sum_{n=0}^{\infty}|a_n|z^n\in\Sigma$, we define neighborhood of $f(z)$ by

$$\mathrm{N}_{\delta}^*(f)=\left\{g(z)=z^{-1}+\sum_{n=0}^{\infty}|b_n|z^n\in\Sigma:\sum_{n=0}^{\infty}\frac{[(1-A)+n(1-B)]}{(A-B)}\frac{(2)_{n+1}}{(2-\alpha)_{n+1}}\big||b_n|-|a_n|\big|\le\delta\right\}.$$

Theorem 4. Let $A+B\le 0$. If $f(z)=z^{-1}+\sum_{n=1}^{\infty}|a_n|z^n\in T_{m+1}^*(\alpha,A,B)$, then $N_{\delta}^*(f)\subset T_m^*(\alpha,A,B)$, where $\delta=\frac{2}{3}$. The result is sharp.

Proof. Using the same method as in Theorem 1., we would have

$$h(z)=z^{-1}+\sum_{n=1}^{\infty}c_nz^n=z^{-1}+\sum_{n=1}^{\infty}\frac{(1+n)-\sigma(A+nB)}{\sigma(B-A)}\frac{(2)_{n+1}}{(2-\alpha)_{n+1}}z^n.$$

Under the hypothesis $A+B\le 0$, we obtain that

$$\left|\frac{f(z)*h(z)}{z^{-1}}\right|=\left|1+\sum_{n=1}^{\infty}c_n|a_n|z^{n+1}\right|$$
$$\ge 1-\frac{1}{3}\sum_{n=1}^{\infty}\frac{[(1-A)+n(1-B)]}{(A-B)}\frac{(2)_{n+1}}{(2-\alpha)_{n+1}}|a_n|.$$

From Theorem 3., we get

$$\left|\frac{f(z)*h(z)}{z^{-1}}\right| \geq \frac{2}{3} = \delta.$$

The remaining part of the proof is similar to that of Theorem 1.
To show the sharpness,we consider the function

$$f(z) = z^{-1} + \frac{A-B}{\left(2-(A+B)\right)3^{n+1}} z \in T^*_{m+1}(\alpha, A, B)$$

and

$$g(z) = z^{-1} + \left[\frac{A-B}{(2-(A+B))3^{n+1}} + \frac{(A-B)\delta'}{(2-(A+B))3^{n}}\right] z$$

where $\delta' > \frac{2}{3}$. Then the function $g(z)$ belong to $\mathrm{N}^*_{\delta'}(f)$.

On the other hand, we find from Theorem 3. that $g(z)$ is not in $T^*_n(\alpha, A, B)$. Now the proof is complete.

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Normal Surfaces of Principal-Donor Curves

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Abstract

In this study, we work on the surfaces determined in relation to associated curves. We study normal surfaces defined with the help of donor curves, a special type of associated curve. For this, we first remember the basic equations of the 3-dimensional Euclidean space and the donor curve issue. Then, by obtaining the first and second fundamental forms, principal curvatures and Gaussian and mean curvatures of the normal surface of a donor curve, we give the characterizations of this surface and some results.

Keywords: Normal surface, Principal-donor curve, Serret-Frenet frame.

1. Introduction

Surfaces and curves in differantial geometry is a valuable topic that paves the way for studies in these fields by providing geometric expressions to many applied sciences such as physics, engineering and geophysics that serve technology. The subject of surfaces associated with curves, which we will discuss in this study, is one of the special examples of this. These surfaces, which are formed as a result of the movement of a line or curve depending on another curve, provide important conveniences in terms of giving geometric expressions to the subject [1-12].

Tangent, normal and binormal surfaces, which are formed as a result of the movement of a curve in the direction of the tangent, normal and binormal vector field due to the change of the time parameter, can be given as examples of surfaces associated with curves. Associated curves have an important place in determining the behavior and characterization of surfaces. The surfaces established with the help of donor curves, which is a type of associated curve, form the framework of our study [13-15].

2. Preliminaries

In this part, we remember the basic definitions and formulas related to the frame elements and the concept of principal-donor curves that we have studied in 3D Euclidean space. Next, we give some basic reminders that have an important place in determining the behavior and characterization of a surface.

The Serret-Frenet(SF) formulas in 3D Euclidean are given as

$$\begin{bmatrix} \nabla_s \boldsymbol{T} \\ \nabla_s \boldsymbol{N} \\ \nabla_s \boldsymbol{B} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{T} \\ \boldsymbol{N} \\ \boldsymbol{B} \end{bmatrix},$$

where κ, τ are curvature and torsion of λ, respectively. Let s be the length of the arc [1]. Then, the SF frame formulas are given as

$$\boldsymbol{T} = \lambda'(s), \quad \boldsymbol{N} = \frac{\lambda''(s)}{\|\lambda''(s)\|}, \quad \boldsymbol{B} = \boldsymbol{T} \times \boldsymbol{N}.$$

Definition 1. Let $\{\boldsymbol{T}_\lambda, \boldsymbol{N}_\lambda, \boldsymbol{B}_\lambda\}$ be the SF frame of λ curve with s parameter. Then, the principal-donor curve of λ according to the SF frame is defined as [2]

$$\delta(s) = -\int_{s_0}^{s} \boldsymbol{N}_\lambda(s) \cos(\alpha) ds + \int_{s_0}^{s} \boldsymbol{B}_\lambda(s) \sin(\alpha) ds, \qquad \alpha = \tau_\lambda(s) ds.$$

Theorem 2. Let $\{\boldsymbol{T}_\lambda, \boldsymbol{N}_\lambda, \boldsymbol{B}_\lambda\}$ be the SF frame of λ curve with s parameter, δ be principal-donor curve of λ according to the SF frame and κ_λ and τ_λ be curvature and torsion of λ. Denote by $\{\boldsymbol{T}_\delta, \boldsymbol{N}_\delta, \boldsymbol{B}_\delta\}$ the SF frame elements for δ and denote by τ_δ and κ_δ be torsion and curvature of δ. Then, relationship between δ and λ can be given by following equations [2]:

$$\begin{aligned} \mathbf{T}_\delta &= -\cos(\alpha)\mathbf{N}_\lambda + \sin(\alpha)\mathbf{B}_\lambda, \\ \mathbf{N}_\delta &= \frac{\kappa_\lambda}{\kappa_\delta}\cos(\alpha)\mathbf{T}_\lambda, \\ \mathbf{B}_\delta &= \frac{\kappa_\lambda}{2\kappa_\delta}\sin(2\alpha)\mathbf{N}_\lambda + \frac{\kappa_\lambda}{\kappa_\delta}\cos^2(\alpha)\mathbf{B}_\lambda, \end{aligned}$$

and

$$\kappa_\delta = \kappa_\lambda |\cos(\alpha)|, \quad \tau_\delta = \kappa_\lambda \sin(\alpha).$$

The normal(unit) vector field for any surface $\varphi(s,t)$ is defined by the equation

$$n = \frac{\varphi_s \wedge \varphi_t}{\|\varphi_s \wedge \varphi_t\|},$$

where $\varphi_t = \partial\varphi/\partial t$, $\varphi_s = \partial\varphi/\partial s$ and, t is parameter representing time. Then, first and second fundamental forms of φ are given by following equations:

$$\begin{aligned} \boldsymbol{I} &= E ds^2 + 2F ds dt + G dt^2, \\ \boldsymbol{II} &= e ds^2 + 2f ds dt + g dt^2, \end{aligned}$$

where

$$\begin{aligned} E &= \langle \varphi_s, \varphi_s \rangle, \ F = \langle \varphi_s, \varphi_t \rangle, \ G = \langle \varphi_t, \varphi_t \rangle, \\ e &= \langle \varphi_{ss}, n \rangle, \ f = \langle \varphi_{st}, n \rangle, \ g = \langle \varphi_{tt}, n \rangle. \end{aligned} \tag{1}$$

Also, Gaussian and mean curvatures K, and H are given as

$$H = \frac{Eg - 2Ff + Ge}{2(EG - F^2)}, \ K = \frac{eg - f^2}{EG - F^2} \tag{2}$$

and principal curvatures are defined by [3-7]

$$k_1 = \sqrt{H^2 - K} + H,\ k_2 = H - \sqrt{H^2 - K}. \tag{3}$$

Theorem 3. A surface is minimal surface if and only if it has vanished mean curvature of this surface [1].
Theorem 4. A surface is a developable (flat) surface if and only if it has vanished Gaussian curvature of this surface [1].
Definition 5. The normal surface describing with normal vector field of a regular curve λ is defined as $\varphi_{(st)} = \lambda + t\boldsymbol{N}$ [5].

3. Normal Surfaces of Donor Curves with The SF Frame in $\boldsymbol{E}^3$

In this section, we give certain characterizations and results for the normal surface of a principal-donor curve with the help of reminders from the previous section.

Theorem 6. Let δ be principal-donor curve of λ curve with arc length parameter. Then, first and second fundamental forms of normal surface of δ are given by following equations:

$$\begin{aligned}
\mathbf{I}_\delta &= ((1 - t\kappa_\lambda|\cos(\alpha)|)^2 + \kappa_\lambda^2 \sin^2(\alpha))ds^2 + dt^2, \\
\mathbf{II}_\delta &= \frac{t\kappa_\lambda^2 \sin(\alpha)|\cos(\alpha)| + (\kappa_\lambda' \sin(\alpha) + \kappa_\lambda \cos(\alpha))(1 + t\kappa_\lambda|\cos(\alpha)|)}{\sqrt{(1 - t\kappa_\lambda|\cos(\alpha)|)^2 + \kappa_\lambda^2 \sin^2(\alpha)}} ds^2 \\
&\quad + \frac{2\kappa_\lambda^2 \sin(\alpha)|\cos(\alpha)|}{\sqrt{(1 - t\kappa_\lambda|\cos(\alpha)|)^2 + \kappa_\lambda^2 \sin^2(\alpha)}} dsdt.
\end{aligned}$$

Proof. From the definition of normal surface, the normal surface of δ is written as

$$\varphi^\delta(s,t) = \delta + t\boldsymbol{N}_\delta.$$

Therefore, the following equalities are obtained:

$$\begin{aligned}
\varphi_s^\delta &= \tau_\delta \boldsymbol{B}_\delta + (1 - t\kappa_\delta)\boldsymbol{T}_\delta, \\
\varphi_{ss}^\delta &= -t\kappa_\delta \boldsymbol{T}_\delta + (\kappa_\delta - t\kappa_\delta^2 - \tau_\delta^2)\boldsymbol{N}_\delta + \tau_\delta' \boldsymbol{B}_\delta, \\
\varphi_t^\delta &= \boldsymbol{N}_\delta, \quad \varphi_{tt}^\delta = 0, \quad \varphi_{st}^\delta = -\kappa_\delta \boldsymbol{T}_\delta,
\end{aligned}$$

and, from the equalities, unit standart normal vector field of φ^δ surface is found as

$$n_\delta = \frac{\varphi_s^\delta \times \varphi_t^\delta}{\|\varphi_s^\delta \times \varphi_t^\delta\|} = \frac{-\tau_\delta \boldsymbol{T}_\delta + (1 - t\kappa_\delta)\boldsymbol{B}_\delta}{\sqrt{\tau_\delta^2 + (1 - t\kappa_\delta)^2}}.$$

These equalities are obtained similarly for the normal surface of λ curve. Then, with the help of Theorem 2 and of the equations we gave at the beginning of this section, we obtain

$$\begin{aligned}
&E_\delta = \tau_\delta^2 + (1 - t\kappa_\delta)^2,\ F_\delta = 0,\ G_\delta = 1, \\
&e_\delta = \frac{t\kappa_\delta \tau_\delta + \tau_\delta' + t\kappa_\delta \tau_\delta'}{\sqrt{\tau_\delta^2 + (1 - t\kappa_\delta)^2}},\ f_\delta = \frac{\kappa_\delta \tau_\delta}{\sqrt{\tau_\delta^2 + (1 - t\kappa_\delta)^2}},\ g_\delta = 0.
\end{aligned} \tag{4}$$

Hence, from Theorem 2, the first and second fundamental forms of normal surfaces of δ are obtained as

$$\begin{aligned}\mathbf{I}_\delta &= ((1-t\kappa_\lambda|\cos(\alpha)|)^2+\kappa_\lambda^2\sin^2(\alpha))ds^2+dt^2,\\ \mathbf{II}_\delta &= \frac{t\kappa_\lambda^2\sin(\alpha)|\cos(\alpha)|+(\kappa_\lambda'\sin(\alpha)+\kappa_\lambda\tau_\lambda\cos(\alpha))(1+t\kappa_\lambda|\cos(\alpha)|)}{\sqrt{(1-t\kappa_\lambda|\cos(\alpha)|)^2+\kappa_\lambda^2\sin^2(\alpha)}}ds^2\\ &+\frac{2\kappa_\lambda^2\sin(\alpha)|\cos(\alpha)|}{\sqrt{(1-t\kappa_\lambda|\cos(\alpha)|)^2+\kappa_\lambda^2\sin^2(\alpha)}}dsdt.\end{aligned}$$

Corollary 7. Let δ be principal-donor curve of λ curve with arc length parameter. Then, mean (H_δ) and Gaussian curvatures (K_δ) of the normal surfaces of δ are given by following equations:

$$\begin{aligned}K_\delta &= \frac{\kappa_\lambda^4\sin^2(2\alpha)}{4((1-t\kappa_\lambda|\cos(\alpha)|)^2+\kappa_\lambda^2\sin^2(\alpha))^2},\\ H_\delta &= \frac{t\kappa_\lambda^2\sin(\alpha)|\cos(\alpha)|+(\kappa_\lambda'\sin(\alpha)+\kappa_\lambda\tau_\lambda\cos(\alpha))(1+t\kappa_\lambda|\cos(\alpha)|)}{2\sqrt{((1-t\kappa_\lambda|\cos(\alpha)|)^2+\kappa_\lambda^2\sin^2(\alpha))^3}}.\end{aligned} \tag{5}$$

Proof. Using equations (4), we obtain

$$\begin{aligned}K_\delta &= \frac{e_\delta g_\delta-f_\delta^2}{E_\delta G_\delta-F_\delta^2}=\frac{\kappa_\delta^2\tau_\delta^2}{(\tau_\delta^2+(1-t\kappa_\delta)^2)^2},\\ H_\delta &= \frac{E_\delta g_\delta-2F_\delta f_\delta+G_\delta e_\delta}{2(E_\delta G_\delta-F_\delta^2)}=\frac{t\kappa_\delta\tau_\delta+\tau_\delta'+t\kappa_\delta\tau_\delta'}{2\sqrt{(\tau_\delta^2+(1-t\kappa_\delta)^2)^3}}.\end{aligned}$$

Hence, by using equations of $\kappa_\delta=\kappa_\lambda|cos(\alpha)|$, $\tau_\delta=\kappa_\lambda\, sin(\alpha)$, the proof is completed.

Theorem 8. Let δ be principal-donor curve of λ curve with arc length parameter. If the curve λ is planar, then the normal surface of δ is minimal and flat.

Proof. Since the curve λ is planar, $\tau_\lambda=0$. Then, it's obtained

$$\tau_\delta=\kappa_\lambda\, sin(0)=0 \quad \text{and} \quad \kappa_\delta=\kappa_\lambda|cos(0)|=\kappa_\lambda.$$

From Theorem 3 and Theorem 4, $H_\delta=0$ and $K_\delta=0$. Hence, the normal surface of δ is minimal and flat.

We can easily obtain the following results with the aid of equations (5):

Corollary 9. Let δ be principal-donor curve of λ curve with arc length parameter. If the torsion of the curve λ is constant. Then, principal curvatures of normal surface of δ are given by

$$\begin{aligned}k_{\delta_1} &= \frac{\kappa_\lambda^4\sin^2(2\alpha)\sqrt{t^2-4((1-t\kappa_\lambda|\cos(\alpha)|)^2+\kappa_\lambda^2\sin^2(\alpha))}}{4\sqrt{((1-t\kappa_\lambda|\cos(\alpha)|)^2+\kappa_\lambda^2\sin^2(\alpha))^3}}\\ &+\frac{t\kappa_\lambda^4\sin^2(2\alpha)}{2\sqrt{((1-t\kappa_\lambda|\cos(\alpha)|)^2+\kappa_\lambda^2\sin^2(\alpha))^3}},\end{aligned}$$

$$k_{\delta_1} = \frac{t\kappa_\lambda^4 \sin^2(2\alpha)}{2\sqrt{((1-t\kappa_\lambda|\cos(\alpha)|)^2+\kappa_\lambda^2\sin^2(\alpha))^3}} - \frac{\kappa_\lambda^4\sin^2(2\alpha)\sqrt{t^2-4((1-t\kappa_\lambda|\cos(\alpha)|)^2+\kappa_\lambda^2\sin^2(\alpha))}}{4\sqrt{((1-t\kappa_\lambda|\cos(\alpha)|)^2+\kappa_\lambda^2\sin^2(\alpha))^3}}.$$

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On Double Wijsman Deferred Invariant Equivalences

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Abstract

The aim of this paper is introduce the concepts of asymptotical deferred invariant equivalence, strongly deferred invariant equivalence and deferred invariant statistical equivalence in the Wijsman sense for double set sequences. Additionally, some properties and based relations among these concepts have been established.

Keywords: Deferred Cesàro mean, deferred statistical convergence, asymptotical equivalence, invariant convergence, Wijsman convergence, double sequences of sets.

1. Preliminaries

Long after the concept of deferred Cesàro mean for real or complex-valued sequences was introduced by Agnew [1], Küçükaslan and Yılmaztürk [2] studied on the concept of deferred statistical convergence. Then, using the invariant mean, Nuray [3] gave the definitions of strongly deferred invariant and deferred invariant statistical convergence. In addition, for non-negative sequences, Koşar et al. [4] presented new concepts named asymptotical deferred and asymptotical deferred statistical equivalence. Also, Dağadur and Sezgek [5, 6] extended the concepts of deferred Cesàro mean and deferred statistical convergence to the double sequences. Furthermore, on the concepts of strongly double deferred invariant and double deferred invariant statistical convergence were studied by Savaş [7].

For sequences of sets, Altınok et al. [8] firstly introduced the concepts of Wijsman strongly deferred Cesàro summability and Wijsman deferred statistical convergence. Using the asymptotical equivalence, Altınok et al. [9] also gave the definitions of asymptotical deferred and asymptotical deferred statistical equivalence in the Wijsman sense for sequences of sets. In addition, for sequences of sets, some asymptotical deferred invariant equivalence types in the Wijsman sense were presented by Gülle and Ulusu [10]. Furthermore, Ulusu and Gülle [11] studied on the concepts of Wijsman deferred Cesàro summability and Wijsman deferred statistical convergence for double sequences of sets.

More information on these concepts can be found in [12-35].

The deferred Cesàro mean $D_{\phi,\psi}$ of a double real sequence $\mathrm{x} = (x_{uv})$ is defined by

$$(D_{\phi,\psi}\mathrm{x})_{ij} = \frac{1}{\phi(i)\psi(j)} \sum_{u=p_i+1}^{q_i} \sum_{v=r_j+1}^{s_j} x_{uv} := \sum_{\substack{u=p_i+1 \\ v=r_j+1}}^{q_i,s_j} x_{uv},$$

where (p_i), (q_i), (r_j) and (s_j) are sequences of non-negative integers satisfying following conditions:

$$p_i < q_i,\ \lim_{i\to\infty} q_i = \infty; \quad r_j < s_j,\ \lim_{j\to\infty} s_j = \infty \tag{1}$$

$$q_i - p_i = \phi(i); \quad s_j - r_j = \psi(j). \tag{2}$$

Note here that the method $D_{\phi,\psi}$ is openly regular for any selection of the above sequences of integers.

Throughout the paper, unless otherwise specified, (p_i), (q_i), (r_j) and (s_j) are considered as sequences of non-negative integers satisfying (1) and (2).

For a metric space $(\mathcal{X}, d)$, distance from x to B is denoted by $\mu_x(B)$ where

$$\mu_x(B) := \mu(x, B) = \inf_{y\in B} d(x, y)$$

for any $x \in \mathcal{X}$ and any non-empty $B \subseteq \mathcal{X}$.

For a non-empty set $\mathcal{X}$, let a function $f: \mathbb{N} \to 2^{\mathcal{X}}$ is defined by $f(u) = B_u \in 2^{\mathcal{X}}$ for each $u \in \mathbb{N}$ where $2^{\mathcal{X}}$denotes the power set of $\mathcal{X}$. Then, the sequence $\{B_u\} = \{B_1, B_2, \dots\}$ is called sequence of sets.

Throughout the paper, we will take that $(\mathcal{X}, d)$ as a metric space and B, B_{uv}, C_{uv} as any non-empty closed subsets of $\mathcal{X}$.

The double sequence $\{B_{uv}\}$ is called bounded if $\sup_{u,v} \mu_x(B_{uv}) < \infty$ for each $x \in \mathcal{X}$. Also, L_∞^2 denotes the class of all bounded double sequences of sets.

The double sequence $\{B_{uv}\}$ is said to be Wijsman convergent to the set B if for each $x \in \mathcal{X}$

$$\lim_{u,v\to\infty} \mu_x(B_{uv}) = \mu_x(B)$$

and it is denoted by $B_{uv} \xrightarrow{W_2} B$.

Let σ be a mapping of the set of positive integers into itself. A continuous linear functional ϕ on ℓ_∞, the space of real bounded sequences, is called an invariant mean or a σ-mean if and only if

i. $\phi(x_u) \geq 0$, when the sequence (x_u) has $x_u \geq 0$ for all u,
ii. $\phi(e) = 1$, where $e = (1,1,1, \dots)$, and
iii. $\phi\big(x_{\sigma(u)}\big) = \phi(x_u)$ for all $(x_u) \in \ell_\infty$.

The mappings σ are assumed to be one to one and $\sigma^u(m) \neq m$ for all positive integers u and m, where $\sigma^u(m)$ denotes the u th iterate of the mapping σ at m. Thus, ϕ extends the limit functional on c, the space of convergent sequences, in the sense that $\phi(x_u) = \lim x_u$ for all $(x_u) \in c$.

The double sequence $\{B_{uv}\}$ is said to be Wijsman invariant convergent to the set B if for each $x \in \mathcal{X}$

$$\lim_{i,j\to\infty} \frac{1}{ij} \sum_{u,v=1,1}^{i,j} \mu_x(B_{\sigma^u(m)\sigma^v(n)}) = \mu_x(B)$$

uniformly in m, n.

The double sequence $\{B_{uv}\}$ is said to be Wijsman strongly invariant convergent to the set B if for each $x \in \mathcal{X}$

$$\lim_{i,j\to\infty} \frac{1}{ij} \sum_{u,v=1,1}^{i,j} \left|\mu_x(B_{\sigma^u(m)\sigma^v(n)}) - \mu_x(B)\right| = 0$$

uniformly in m, n.

The double sequence $\{B_{uv}\}$ is said to be Wijsman invariant statistically convergent to the set B if for every $\varepsilon > 0$ and each $x \in \mathcal{X}$

$$\lim_{i,j\to\infty} \frac{1}{ij} \left|\left\{(u,v): u \leq i, v \leq j, |\mu_x(B_{\sigma^u(m)\sigma^v(n)}) - \mu_x(B)| \geq \varepsilon\right\}\right| = 0$$

uniformly in m, n.

For any non-empty closed subsets $B_{uv}, C_{uv} \in \mathcal{X}$ such that $\mu_x(B_{uv}) > 0$ and $\mu_x(C_{uv}) > 0$ for each $x \in \mathcal{X}$, the double sequences $\{B_{uv}\}$ and $\{C_{uv}\}$ are said to be Wijsman asymptotically equivalent if

$$\lim_{u,v\to\infty} \mu_x\left(\frac{B_{uv}}{C_{uv}}\right) := \lim_{u,v\to\infty} \frac{\mu_x(B_{uv})}{\mu_x(C_{uv})} = 1$$

for each $x \in \mathcal{X}$. It is denoted by $B_{uv} \overset{W_2}{\sim} C_{uv}$.

A double sequence $\theta_2 = \{(u_i, v_j)\}$ is called double lacunary sequence if there exists increasing integers sequences (u_i) and (v_j) such that

$$u_0 = 0,\ h_i = u_i - u_{i-1} \to \infty \quad \text{and} \quad v_0 = 0,\ \bar{h}_j = v_j - v_{j-1} \to \infty \quad \text{as} \quad i,j \to \infty.$$

2. New Concepts

In this section, we introduced the concepts of asymptotical deferred invariant equivalence, strongly deferred invariant equivalence and deferred invariant statistical equivalence in the Wijsman sense for double set sequences. Additionally, we established some properties and based relations among these concepts.

Throughout the section, we regarded that $\mu_x(B_{uv}) > 0$ and $\mu_x(C_{uv}) > 0$ for each $x \in \mathcal{X}$.

Definition 2.1 The double set sequences $\{B_{uv}\}$ and $\{C_{uv}\}$ are said to be Wijsman asymptotical deferred invariant statistically equivalent of multiple λ if for every $\varepsilon > 0$ and each $x \in \mathcal{X}$

$$\lim_{i,j\to\infty} \frac{1}{\phi(i)\psi(j)} \left| \left\{ (u,v): p_i < u \leq q_i, r_j < v \leq s_j, \left| \mu_x\left(\frac{B_{\sigma^u(m)\sigma^v(n)}}{C_{\sigma^u(m)\sigma^v(n)}} \right) - \lambda \right| \geq \varepsilon \right\} \right| = 0$$

uniformly in m, n. The notation $B_{uv} \overset{W_2^\lambda DS_\sigma}{\sim} C_{uv}$ is used for this case and these sequences are called Wijsman asymptotical deferred invariant statistically equivalent if $\lambda = 1$.

The set of all double set sequences that Wijsman asymptotical deferred invariant statistically equivalent is denoted by $\{W_2^\lambda DS_\sigma\}$.

Remark 2.1

- For $p_i = 0, q_i = i$ and $r_j = 0, s_j = j$, the concept of Wijsman asymptotical deferred invariant statistical equivalence coincides with the concept of Wijsman asymptotical invariant statistical equivalence for double set sequences in [34].
- For $p_i = u_{i-1}, q_i = u_i$ and $r_j = v_{j-1}, s_j = v_j$ where $\{(u_i, v_j)\}$ is a double lacunary sequence, the concept of Wijsman asymptotical deferred invariant statistical equivalence coincides with the concept of Wijsman asymptotical lacunary invariant statistical equivalence for double set sequences in [34].

Definition 2.2 The double set sequences $\{B_{uv}\}$ and $\{C_{uv}\}$ are said to be Wijsman asymptotically deferred invariant equivalent of multiple λ if for each $x \in \mathcal{X}$

$$\lim_{k,j\to\infty} \frac{1}{\phi(i)\psi(j)} \sum_{\substack{u=p_i+1 \\ v=r_j+1}}^{q_i,s_j} \mu_x\left(\frac{B_{\sigma^u(m)\sigma^v(n)}}{C_{\sigma^u(m)\sigma^v(n)}} \right) = \lambda$$

uniformly in m, n. The notation $B_{uv} \overset{W_2^\lambda D_\sigma}{\sim} C_{uv}$ is used for this case and these sequences are called Wijsman asymptotically deferred invariant equivalent if $\lambda = 1$.

Definition 2.3 The double set sequences $\{B_{uv}\}$ and $\{C_{uv}\}$ are said to be Wijsman asymptotical strongly deferred invariant equivalent of multiple λ if for each $x \in \mathcal{X}$

$$\lim_{k,j\to\infty} \frac{1}{\phi(i)\psi(j)} \sum_{\substack{u=p_i+1 \\ v=r_j+1}}^{q_i,s_j} \left| \mu_x\left(\frac{B_{\sigma^u(m)\sigma^v(n)}}{C_{\sigma^u(m)\sigma^v(n)}}\right) - \lambda \right| = 0$$

uniformly in m,n. The notation $B_{uv} \overset{W_2^\lambda[D_\sigma]}{\sim} C_{uv}$ is used for this case and these sequences are called Wijsman asymptotical strongly deferred invariant equivalent if $\lambda = 1$.

The set of all double set sequences that Wijsman asymptotical strongly deferred invariant equivalent is denoted by $\{W_2^\lambda[D_\sigma]\}$.

Remark 2.2

- For $p_i = 0, q_i = i$ and $r_j = 0, s_j = j$, the concept of Wijsman asymptotical strongly deferred invariant equivalence coincides with the concept of Wijsman asymptotical strongly invariant equivalence for double set sequences in [34].
- For $p_i = u_{i-1}, q_i = u_i$ and $r_j = v_{j-1}, s_j = v_j$ where $\{(u_i, v_j)\}$ is a double lacunary sequence, the concept of Wijsman asymptotical strongly deferred invariant equivalence coincides with the concept of Wijsman asymptotical strongly lacunary invariant equivalence for double set sequences in [34].

3. Main Theorems

In this section, firstly, we gave two theorems associated with the concept of $W_2^\lambda DS_\sigma$-equivalence.

Theorem 3.1 Let $A_{uv} \subseteq B_{uv}$ for all $u, v \in \mathbb{N}$. If $A_{uv} \overset{W_2^\lambda DS_\sigma}{\sim} C_{uv}$, then $B_{uv} \overset{W_2^\lambda DS_\sigma}{\sim} C_{uv}$.

Proof. Suppose that $A_{uv} \overset{W_2^\lambda DS_\sigma}{\sim} C_{uv}$. Since $A_{uv} \subseteq B_{uv}$ for all $u, v \in \mathbb{N}$, we have

$$\begin{aligned} A_{uv} \subseteq B_{uv} &\Rightarrow A_{\sigma^u(m)\sigma^v(n)} \subseteq B_{\sigma^u(m)\sigma^v(n)} \quad \text{(for all } m,n) \\ &\Rightarrow \mu_x(B_{\sigma^u(m)\sigma^v(n)}) \leq \mu_x(A_{\sigma^u(m)\sigma^v(n)}) \quad \text{(for each } x \in \mathcal{X}) \\ &\Rightarrow \left| \mu_x\left(\frac{B_{\sigma^u(m)\sigma^v(n)}}{C_{\sigma^u(m)\sigma^v(n)}}\right) - \lambda \right| \leq \left| \mu_x\left(\frac{A_{\sigma^u(m)\sigma^v(n)}}{C_{\sigma^u(m)\sigma^v(n)}}\right) - \lambda \right|. \end{aligned}$$

So, for every $\varepsilon > 0$

$$\left\{(u,v): p_i < u \leq q_i, r_j < u \leq s_j, \left|\mu_x\left(\frac{B_{\sigma^u(m)\sigma^v(n)}}{C_{\sigma^u(m)\sigma^v(n)}}\right) - \lambda\right| \geq \varepsilon\right\}$$
$$\subseteq \left\{(u,v): p_i < u \leq q_i, r_j < u \leq s_j, \left|\mu_x\left(\frac{A_{\sigma^u(m)\sigma^v(n)}}{C_{\sigma^u(m)\sigma^v(n)}}\right) - \lambda\right| \geq \varepsilon\right\}.$$

Then, the following inequality is obtained:

$$\frac{1}{\phi(i)\psi(j)}\left|\left\{(u,v): p_i < u \leq q_i, r_j < u \leq s_j, \left|\mu_x\left(\frac{B_{\sigma^u(m)\sigma^v(n)}}{C_{\sigma^u(m)\sigma^v(n)}}\right) - \lambda\right| \geq \varepsilon\right\}\right|$$
$$\leq \frac{1}{\phi(i)\psi(j)}\left|\left\{(u,v): p_i < u \leq q_i, r_j < u \leq s_j, \left|\mu_x\left(\frac{A_{\sigma^u(m)\sigma^v(n)}}{C_{\sigma^u(m)\sigma^v(n)}}\right) - \lambda\right| \geq \varepsilon\right\}\right|.$$

Thus, by the assumption, $B_{uv} \overset{W_2^\lambda DS_\sigma}{\sim} C_{uv}$.

Theorem 3.2 Let $B_{uv} \subseteq C_{uv}$ for all $u, v \in \mathbb{N}$. If $A_{uv} \overset{W_2^\lambda DS_\sigma}{\sim} C_{uv}$, then $A_{uv} \overset{W_2^\lambda DS_\sigma}{\sim} B_{uv}$.

Proof. Suppose that $A_{uv} \overset{W_2^\lambda DS_\sigma}{\sim} C_{uv}$. Since $B_{uv} \subseteq C_{uv}$ for all $u, v \in \mathbb{N}$, we have

$$B_{uv} \subseteq C_{uv} \Rightarrow B_{\sigma^u(m)\sigma^v(n)} \subseteq C_{\sigma^u(m)\sigma^v(n)} \quad \text{(for all } m, n)$$
$$\Rightarrow \mu_x(C_{\sigma^u(m)\sigma^v(n)}) \leq \mu_x(B_{\sigma^u(m)\sigma^v(n)}) \quad \text{(for each } x \in \mathcal{X})$$
$$\Rightarrow \left|\mu_x\left(\frac{A_{\sigma^u(m)\sigma^v(n)}}{B_{\sigma^u(m)\sigma^v(n)}}\right) - \lambda\right| \leq \left|\mu_x\left(\frac{A_{\sigma^u(m)\sigma^v(n)}}{C_{\sigma^u(m)\sigma^v(n)}}\right) - \lambda\right|.$$

So, for every $\varepsilon > 0$

$$\left\{(u,v): p_i < u \leq q_i, r_j < u \leq s_j, \left|\mu_x\left(\frac{A_{\sigma^u(m)\sigma^v(n)}}{B_{\sigma^u(m)\sigma^v(n)}}\right) - \lambda\right| \geq \varepsilon\right\}$$
$$\subseteq \left\{(u,v): p_i < u \leq q_i, r_j < u \leq s_j, \left|\mu_x\left(\frac{A_{\sigma^u(m)\sigma^v(n)}}{C_{\sigma^u(m)\sigma^v(n)}}\right) - \lambda\right| \geq \varepsilon\right\}.$$

Then, the following inequality is obtained:

$$\frac{1}{\phi(i)\psi(j)}\left|\left\{(u,v): p_i < u \leq q_i, r_j < u \leq s_j, \left|\mu_x\left(\frac{A_{\sigma^u(m)\sigma^v(n)}}{B_{\sigma^u(m)\sigma^v(n)}}\right) - \lambda\right| \geq \varepsilon\right\}\right|$$
$$\leq \frac{1}{\phi(i)\psi(j)}\left|\left\{(u,v): p_i < u \leq q_i, r_j < u \leq s_j, \left|\mu_x\left(\frac{A_{\sigma^u(m)\sigma^v(n)}}{C_{\sigma^u(m)\sigma^v(n)}}\right) - \lambda\right| \geq \varepsilon\right\}\right|.$$

Thus, by the assumption, $A_{uv} \overset{W_2^\lambda DS_\sigma}{\sim} B_{uv}$.

Finally, we compared the concepts of $W_2^\lambda DS_\sigma$-equivalence and $W_2^\lambda[D_\sigma]$-equivalence.

Theorem 3.3 If double sequences $\{B_{uv}\}$ and $\{C_{uv}\}$ are Wijsman asymptotical strongly deferred invariant equivalent of multiple λ, then these sequences are Wijsman asymptotical deferred invariant statistically equivalent of multiple λ.

Proof. Suppose that $B_{uv} \overset{W_2^\lambda[D_\sigma]}{\sim} C_{uv}$. For every $\varepsilon > 0$ and each $x \in \mathcal{X}$, we have the following equality:

$$\sum_{\substack{u=p_i+1\\v=r_j+1}}^{q_i,s_j} \left|\mu_x\left(\frac{B_{\sigma^u(m)\sigma^v(n)}}{C_{\sigma^u(m)\sigma^v(n)}}\right)-\lambda\right| = \sum_{\substack{u=p_i+1\\v=r_j+1\\ \left|\mu_x\left(\frac{B_{\sigma^u(m)\sigma^v(n)}}{C_{\sigma^u(m)\sigma^v(n)}}\right)-\lambda\right|\geq\varepsilon}}^{q_i,s_j} \left|\mu_x\left(\frac{B_{\sigma^u(m)\sigma^v(n)}}{C_{\sigma^u(m)\sigma^v(n)}}\right)-\lambda\right| + \sum_{\substack{u=p_i+1\\v=r_j+1\\ \left|\mu_x\left(\frac{B_{\sigma^u(m)\sigma^v(n)}}{C_{\sigma^u(m)\sigma^v(n)}}\right)-\lambda\right|<\varepsilon}}^{q_i,s_j} \left|\mu_x\left(\frac{B_{\sigma^u(m)\sigma^v(n)}}{C_{\sigma^u(m)\sigma^v(n)}}\right)-\lambda\right| \tag{3}$$

From the equality (3), we have

$$\sum_{\substack{u=p_i+1\\v=r_j+1}}^{q_i,s_j} \left|\mu_x\left(\frac{B_{\sigma^u(m)\sigma^v(n)}}{C_{\sigma^u(m)\sigma^v(n)}}\right)-\lambda\right| \geq \sum_{\substack{u=p_i+1\\v=r_j+1\\ \left|\mu_x\left(\frac{B_{\sigma^u(m)\sigma^v(n)}}{C_{\sigma^u(m)\sigma^v(n)}}\right)-\lambda\right|\geq\varepsilon}}^{q_i,s_j} \left|\mu_x\left(\frac{B_{\sigma^u(m)\sigma^v(n)}}{C_{\sigma^u(m)\sigma^v(n)}}\right)-\lambda\right|$$
$$\geq \varepsilon \left|\left\{(u,v): p_i < u \leq q_i, r_j < u \leq s_j, \left|\mu_x\left(\frac{B_{\sigma^u(m)\sigma^v(n)}}{C_{\sigma^u(m)\sigma^v(n)}}\right)-\lambda\right| \geq \varepsilon\right\}\right|$$

and

$$\frac{1}{\varepsilon}\frac{1}{\phi(i)\psi(j)}\sum_{\substack{u=p_i+1\\v=r_j+1}}^{q_i,s_j} \left|\mu_x\left(\frac{B_{\sigma^u(m)\sigma^v(n)}}{C_{\sigma^u(m)\sigma^v(n)}}\right)-\lambda\right|$$
$$\geq \frac{1}{\phi(i)\psi(j)}\left|\left\{(u,v): p_i < u \leq q_i, r_j < u \leq s_j, \left|\mu_x\left(\frac{B_{\sigma^u(m)\sigma^v(n)}}{C_{\sigma^u(m)\sigma^v(n)}}\right)-\lambda\right| \geq \varepsilon\right\}\right|$$

Thus, by the assumption, $B_{uv} \overset{W_2^\lambda DS_\sigma}{\sim} C_{uv}$.

Theorem 3.4 Let $\{B_{uv}\},\{C_{uv}\} \in L_\infty^2$. If double sequences $\{B_{uv}\}$ and $\{C_{uv}\}$ are Wijsman asymptotical deferred invariant statistically equivalent of multiple λ, then these sequences are Wijsman asymptotical strongly deferred invariant equivalent of multiple λ.

Proof. Since $\{B_{uv}\},\{C_{uv}\} \in L_\infty^2$, there is a positive real number $\mathcal{K}$ such that for all $u,v \in \mathbb{N}$ and each $x \in \mathcal{X}$

$$\left|\mu_x\left(\frac{B_{\sigma^u(m)\sigma^v(n)}}{C_{\sigma^u(m)\sigma^v(n)}}\right)-\lambda\right| \leq \mathcal{K}$$

uniformly in m,n. Suppose that $B_{uv} \overset{W_2^\lambda DS_\sigma}{\sim} C_{uv}$. From the equality (3), for every $\varepsilon > 0$ and each $x \in \mathcal{X}$ we have

$$\frac{1}{\varepsilon}\frac{1}{\phi(i)\psi(j)}\sum_{\substack{u=p_i+1\\v=r_j+1}}^{q_i,s_j}\left|\mu_x\left(\frac{B_{\sigma^u(m)\sigma^v(n)}}{C_{\sigma^u(m)\sigma^v(n)}}\right)-\lambda\right|$$

$$\leq \frac{\mathcal{K}}{\phi(i)\psi(j)}\left|\left\{(u,v): p_i < u \leq q_i, r_j < u \leq s_j, \left|\mu_x\left(\frac{B_{\sigma^u(m)\sigma^v(n)}}{C_{\sigma^u(m)\sigma^v(n)}}\right)-\lambda\right| \geq \varepsilon\right\}\right| + \varepsilon$$

Thus, by the assumptions, we get $B_{uv} \overset{W_2^\lambda[D_\sigma]}{\sim} C_{uv}$.

Corollary 3.1 $\{W_2^\lambda DS_\sigma\} \cap L_\infty^2 = \{W_2^\lambda[D_\sigma]\} \cap L_\infty^2$.

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On gs-Essential Submodules

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Abstract

In this work, some new properties of gs-essential submodules are studied. Every ring has an unity and every module is an unitary left module, in this work. It is proved that the finite intersection of gs-essential submodules is gs-essential.

Keywords: Essential Submodules, Small Submodules, g-Small Submodules.

2020 Mathematics Subject Classification: 16D10, 16D80.

1. INTRODUCTION

Throughout this paper all rings will be associative with identity and all modules will be unital left modules.

Let R be a ring and M be an R-module. We will denote a submodule N of M by $N \leq M$. Let M be an R-module and $N \leq M$. If $L=M$ for every submodule L of M such that $M=N+L$, then N is called a *small* submodule of M and denoted by $N \ll M$. Let M be an R-module and $N \leq M$. If there exists a submodule K of M such that $M=N+K$ and $N \cap K=0$, then N is called a *direct summand* of M and it is denoted by $M=N \oplus K$. A submodule N of an R-module M is called an *essential* submodule of M, denoted by $N \trianglelefteq M$, if $K=0$ for every $K \leq M$ with $K \cap N=0$. Let M be an R-module and K be a submodule of M. K is called a *generalized small* (briefly, *g-small*) submodule of M if for every $T \trianglelefteq M$ with $M=K+T$ implies that $T=M$, this is written by $K \ll_g M$ (in [6], it is called an *e-small* submodule of M and denoted by $K \ll_e M$). Let M be an R-module. M is called an *hollow* module if every proper submodule of M is small in M. M is called a *generalized hollow* (briefly, *g-hollow*) module, if every proper submodule of M is g-small in M. M is called a *local* module if M has the largest submodule, i. e. a proper submodule which contains all other proper submodules. The intersection of all maximal submodules of an R-module M is called the *radical* of M and denoted by $RadM$. If M have no maximal submodules, then we denote $RadM=M$. The intersection of all essential maximal submodules of an R-module M is called the *generalized radical* (briefly, *g-radical*) of M and denoted by $Rad_g M$ (in [6], it is denoted by $Rad_e M$). If M have no maximal essential submodules, then we denote $Rad_g M=M$.Let M be an R-module and N be a submodule of M. If $L=0$ for every $L \ll M$

with $N\cap L=0$, then N is called a *small essential* (briefly, *s-essential*) submodule of M and denoted by $N\trianglelefteq_s M$.

More informations about small and essential submodules are in [1] and [5]. More details about g-small submodules are in [2] and [3]. More informations about s-essential submodules are in [6].

Lemma 1.1. Let M be an R-module.

(1) If $K\leq L\leq M$, then $K\trianglelefteq M$ if and only if $K\trianglelefteq L\trianglelefteq M$.

(2) Let N be an R-module and $f: M\to N$ be an R-module homomorphism. If $K\trianglelefteq N$, then $f^{-1}(K)\trianglelefteq M$.

(3) For $N\leq K\leq M$, if $K/N\trianglelefteq M/N$, then $K\trianglelefteq M$.

(4) If $K_1\trianglelefteq L_1\leq M$ and $K_2\trianglelefteq L_2\leq M$, then $K_1\cap K_2\trianglelefteq L_1\cap L_2$.

(5) If $K_1\trianglelefteq M$ and $K_2\trianglelefteq M$, then $K_1\cap K_2\trianglelefteq M$.

Proof. See [5, 17.3].

Lemma 1.2. Let M be an R-module. The following assertions are hold.

(1) Every small submodule in M is g-small in M.

(2) If $K\leq L\leq M$ and $L\ll_g M$, then $K\ll_g M$ and $L/K\ll_g M/K$.

(3) Let N be an R-module and $f: M\to N$ be an R-module homomorphism. If $K\ll_g M$, then $f(K)\ll_g N$.

(4) If $K\ll_g M$, then $(K+L)/L\ll_g M/L$ for every $L\leq M$.

(5) If $L\leq M$ and $K\ll_g L$, then $K\ll_g M$.

(6) If $K_1,K_2,...,K_n\ll_g M$, then $K_1+K_2+...+K_n\ll_g M$.

(7) Let $K_1,K_2,...,K_n,L_1,L_2,...,L_n\leq M$. If $K_i\ll_g L_i$ for every $i=1,2,...,n$, then $K_1+K_2+...+K_n\ll_g L_1+L_2+...+L_n$.

Proof. See [2] and [3].

2. gs-ESSENTIAL SUBMODULES

Definition 2.1. Let M be an R-module and N be a submodule of M. If $L=0$ for every $L\ll_g M$ with $N\cap L=0$, then N is called a *g-small essential* (briefly, *gs-essential*) submodule of M and denoted by $N\trianglelefteq_{gs} M$. (See also [4])

Proposition 2.2. Every essential submodule is gs-essential.

Proof. Let M be an R-module and $K \trianglelefteq M$. Let $K \cap L=0$ with $L \ll_g M$. Since $K \trianglelefteq M$ and $L \leq M$, $L=0$. Hence K is a gs-essential submodule of M, as desired.

Proposition 2.3. Every gs-essential submodule is s-essential.

Proof. Let M be an R-module and $K \trianglelefteq_{gs} M$. Let $K \cap L=0$ with $L \ll M$. Since $L \ll M$, $L \ll_g M$. Since $K \trianglelefteq_{gs} M$ and $L \ll_g M$, $L=0$. Hence K is a s-essential submodule of M, as desired.

Proposition 2.4. Let M be an R-module and $K \leq L \leq M$. If $K \trianglelefteq M$, then $K \trianglelefteq_{gs} L \trianglelefteq_{gs} M$.

Proof. By Lemma 1.1(1), $K \trianglelefteq L \trianglelefteq M$. Since $K \trianglelefteq L$, by Proposition 2.2, $K \trianglelefteq_{gs} L$. Since $L \trianglelefteq M$, by Proposition 2.2, $L \trianglelefteq_{gs} M$. Hence $K \trianglelefteq_{gs} L \trianglelefteq_{gs} M$, as desired.

Proposition 2.5. Let M be an R-module and $K \leq L \leq M$. If $K \trianglelefteq L \trianglelefteq M$, then $K \trianglelefteq_{gs} M$.

Proof. Since $K \trianglelefteq L \trianglelefteq M$, by Lemma 1.1(1), $K \trianglelefteq M$. Then by Proposition 2.2, $K \trianglelefteq_{gs} M$, as desired.

Proposition 2.6. Let $f: M \rightarrow N$ be an R-module homomorphism. If $K \trianglelefteq N$, then $f^{-1}(K) \trianglelefteq_{gs} M$.

Proof. Since $K \trianglelefteq N$, by Lemma 1.1(2), $f^{-1}(K) \trianglelefteq M$. Then by Proposition 2.2, $f^{-1}(K) \trianglelefteq_{gs} M$, as desired.

Proposition 2.7. Let M be an R-module and $K \leq L \leq M$. If $L/K \trianglelefteq M/K$, then $L \trianglelefteq_{gs} M$.

Proof. Since $L/K \trianglelefteq M/K$, by Lemma 1.1, $L \trianglelefteq M$. Then by Proposition 2.2, $L \trianglelefteq_{gs} M$, as required.

Proposition 2.8. Let M be an R-module. If $K_1 \trianglelefteq L_1 \leq M$ and $K_2 \trianglelefteq L_2 \leq M$, then $K_1 \cap K_2 \trianglelefteq_{gs} L_1 \cap L_2$.

Proof. Since $K_1 \trianglelefteq L_1 \leq M$ and $K_2 \trianglelefteq L_2 \leq M$, by Lemma 1.1(4), $K_1 \cap K_2 \trianglelefteq L_1 \cap L_2$. Then by Proposition 2.2, $K_1 \cap K_2 \trianglelefteq_{gs} L_1 \cap L_2$, as desired.

Proposition 2.9. Let M be an R-module. If $K_1 \trianglelefteq M$ and $K_2 \trianglelefteq M$, then $K_1 \cap K_2 \trianglelefteq_{gs} M$.

Proof. Since $K_1 \trianglelefteq M$ and $K_2 \trianglelefteq M$, by Lemma 1.1(4), $K_1 \cap K_2 \trianglelefteq M$. Then by Proposition 2.2, $K_1 \cap K_2 \trianglelefteq_{gs} M$, as desired.

Proposition 2.10. Let M be an R-module. If $K_1 \trianglelefteq_{gs} M$ and $K_2 \trianglelefteq_{gs} M$, then $K_1 \cap K_2 \trianglelefteq_{gs} M$.

Proof. Let $K_1 \cap K_2 \cap L=0$ with $L \ll_g M$. Since $L \ll_g M$, by Lemma 1.2, $K_2 \cap L \ll_g M$ and since $K_1 \cap K_2 \cap L=0$ and $K_1 \trianglelefteq M$, $K_2 \cap L=0$. Then by $K_2 \trianglelefteq M$ and $L \ll_g M$, $L=0$ holds. Hence $K_1 \cap K_2 \trianglelefteq_{gs} M$, as required.

Corollary 2.11. Let M be an R-module and $K_i \trianglelefteq_{gs} M$ for $i=1,2,...,n$. Then $K_1 \cap K_2 \cap ... \cap K_n \trianglelefteq_{gs} M$.

Proof. Clear from Proposition 2.10.

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On Hausdorff Deferred Statistical Convergence of Order η of Double Set Sequences

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Abstract

In this study, we firstly introduced the concepts of Hausdorff deferred Cesàro summability of order η and Hausdorff deferred statistical convergence of order η $(0 < \eta \leq 1)$ for double set sequences, gave some properties of these concepts and examined the relations between them. Finally, we showed the relation between the concepts of Hausdorff deferred statistical convergence of order η and Wijsman deferred statistical convergence of order η for double sequences of sets.

Keywords: Double sequences of sets, Hausdorff convergence, Deferred Cesàro mean, Order η, Deferred statistical convergence, Wijsman convergence.

1. Introduction

For real (or complex) sequences, the concept of deferred Cesàro mean was first introduced by Agnew [1]. Long after this, Küçükaslan and Yılmaztürk [2] presented the concept of deferred statistical convergence and showed the relationship of this concept with the strongly deferred Cesàro summability. Also, for double sequences, on the concepts of deferred Cesàro summability and deferred statistical convergence were introduced and studied by Dağadur and Sezgek [3]. Furthermore, using order α, Et et al. [4] studied on the concepts of deferred strongly Cesàro summability and deferred statistical convergence of order α in metric spaces.

The concepts of Hausdorff convergence and Wijsman convergence which are considered in this study are two of the important convergence concepts for sequences of sets [5-7]. Nuray and Rhoades [8] extended these concepts to statistical convergence and gave some fundamental theorems. In [9, 10], Nuray et al. also introduced and studied on the concepts of Wijsman convergence, Wijsman Cesàro summability, Wijsman statistical convergence and Hausdorff statistical convergence for double sequences of sets. Also, for double sequences of sets, the concepts of Hausdorff convergence was presented by Sever et al. [11]. Furthermore, using order α, on similar concepts for double sequences of sets were studied by Ulusu and Gülle [12].

For sequences of sets, on the concepts of strongly deferred Cesàro summability and deferred statistical convergence in the Wijsman sense were studied by Altınok et al. [13]. Also, for double sequences of sets, Ulusu and Gülle [14] introduced and studied on similar concepts. Furthermore, using order α, on the concepts of Wijsman strongly deferred Cesàro summability and Wijsman deferred statistical convergence of order α for sequences of sets were studied by Yılmazer et al. [15].

In this work, for double sequences of sets, we introduced the concept of Hausdorff deferred statistical convergence of order α and studied on this concept.

More information on the concepts in this study can be found in [16-30].

2. Definitions and Notations

First of all, let's start by recalling some basic definitions and notations to make our study easier to understand (See, [3, 5, 10-12]).

The deferred Cesàro mean $D_{\psi,\phi}$ of a double sequence $\mathrm{x} = (x_{ij})$ is defined by

$$(D_{\psi,\phi}\mathrm{x})_{uv} = \frac{1}{\psi_u \phi_v} \sum_{i=p_u+1}^{q_u} \sum_{j=r_v+1}^{s_v} x_{ij},$$

where $[p_u]$, $[q_u]$, $[r_v]$ and $[s_v]$ are sequences of non-negative integers satisfying following conditions:

$$p_u < q_u, \ \lim_{u\to\infty} q_u = \infty; \qquad r_v < s_v, \ \lim_{v\to\infty} s_v = \infty \tag{2.1}$$

$$q_u - p_u = \psi_u; \qquad s_v - r_v = \phi_v. \tag{2.2}$$

Throughout the paper, unless otherwise specified, $[p_u]$, $[q_u]$, $[r_v]$ and $[s_v]$ are considered as sequences of non-negative integers satisfying (2.1) and (2.2).

For a metric space $(\mathcal{Y}, \rho)$, $d(y, C)$ represents the distance from y to C where

$$d(y, C) = \inf_{c\in C} \rho(y, c) := d_y(C)$$

for any $y \in \mathcal{Y}$ and any non-empty $C \subseteq \mathcal{Y}$.

For a non-empty set $\mathcal{Y}$, let a function $g: \mathbb{N} \to 2^{\mathcal{Y}}$ is defined by $g(i) = C_i \in 2^{\mathcal{Y}}$ for each $i \in \mathbb{N}$. Then, the sequence $\{C_i\} = \{C_1, C_2, ...\}$, which is the codomain elements of g, is called sequences of sets.

Throughout the study, unless otherwise stated, $(\mathcal{Y}, \rho)$ is considered as a metric space and C, C_{ij} $(i, j \in \mathbb{N})$ are considered as any non-empty closed subsets of $\mathcal{Y}$.

A double sequence of sets $\{C_{ij}\}$ is said to be Hausdorff convergent to a set C provided that

$$\lim_{i,j\to\infty} \sup_{y\in\mathcal{Y}} \left| d_y(C_{ij}) - d_y(C) \right| = 0.$$

It is denoted by $C_{ij} \xrightarrow{H_2} C$.

A double sequence of sets $\{C_{ij}\}$ is said to be Hausdorff Cesàro summable of order η $(0 < \eta \leq 1)$ to a set C provided that

$$\lim_{m,n\to\infty} \frac{1}{(mn)^\eta} \sum_{i=1}^{m} \sum_{j=1}^{n} \sup_{y\in\mathcal{Y}} \left| d_y(C_{ij}) - d_y(C) \right| = 0.$$

It is denoted by $C_{ij} \xrightarrow{H_2(C)^\eta} C$. For $\eta = 1$, we obtain the concept of Hausdorff Cesàro summability for double sequences of sets.

A double sequence of sets $\{C_{ij}\}$ is said to be Hausdorff statistical convergence of order η $(0 < \eta \leq 1)$ to a set C provided that for every $\varepsilon > 0$,

$$\lim_{m,n\to\infty} \frac{1}{(mn)^\eta} \left| \left\{ (i,j) \colon i \leq m, j \leq n \colon \sup_{y\in\mathcal{Y}} \left| d_y(C_{ij}) - d_y(C) \right| \geq \varepsilon \right\} \right| = 0.$$

It is denoted by $C_{ij} \xrightarrow{H_2 S^\eta} C$. For $\eta = 1$, we obtain the concept of Hausdorff statistical convergence for double sequences of sets.

3. Main Results

In this section, we firstly introduce the concepts of Hausdorff deferred Cesàro summability of order η and Hausdorff deferred statistical convergence of order η $(0 < \eta \leq 1)$ for double set sequences, give some properties of these concepts and examine relations between them. Finally, we show the relation between the concepts of Hausdorff deferred statistical convergence of order η and Wijsman deferred statistical convergence of order η for double sequences of sets.

Definition 3.1 A double sequence of sets $\{C_{ij}\}$ is said to be Hausdorff deferred Cesàro summable of order η $(0 < \eta \leq 1)$ to a set C if

$$\lim_{u,v\to\infty} \frac{1}{(\psi_u \phi_v)^\eta} \sum_{i=p_u+1}^{q_u} \sum_{j=r_v+1}^{s_v} \sup_{y\in\mathcal{Y}} \left| d_y(C_{ij}) - d_y(C) \right| = 0.$$

In this case, the notation $C_{ij} \xrightarrow{H_2 D^\eta} C$ is used. For $\eta = 1$, we obtain the concept of Hausdorff deferred Cesàro summability $(H_2 D)$ for double sequences of sets which has never been mentioned before.

Remark 3.1 The concept of Hausdorff deferred Cesàro summability of order η for double set sequences is coincides with;

- the notion of Hausdorff Cesàro summability of order η for double set sequences in [12], for $p_u = 0, q_u = u$ and $r_v = 0, s_v = v$.

- the notion of Hausdorff Cesàro summabililty for double set sequences in [10], for $\eta = 1$, and $p_u = 0, q_u = u$ and $r_v = 0, s_v = v$.

Theorem 3.1 If $0 < \eta \leq \mu \leq 1$, then

$$C_{ij} \overset{H_2D^\eta}{\longrightarrow} C \Rightarrow C_{ij} \overset{H_2D^\mu}{\longrightarrow} C.$$

Proof. Let $0 < \eta \leq \mu \leq 1$ and suppose that $C_{ij} \overset{H_2D^\eta}{\longrightarrow} C$. Here, we can write following inequality

$$\frac{1}{(\psi_u \phi_v)^\eta} \sum_{i=p_u+1}^{q_u} \sum_{j=r_v+1}^{s_v} \sup_{y \in \mathcal{Y}} \left| d_y(C_{ij}) - d_y(C) \right| \geq \frac{1}{(\psi_u \phi_v)^\mu} \sum_{i=p_u+1}^{q_u} \sum_{j=r_v+1}^{s_v} \sup_{y \in \mathcal{Y}} \left| d_y(C_{ij}) - d_y(C) \right|.$$

Hence, by our assumption, we get $C_{ij} \overset{H_2D^\mu}{\longrightarrow} C$.

If $\mu = 1$ is taken in Theorem 3.1, then we obtain the following corollary.

Corollary 3.1 If a double sequence of sets $\{C_{ij}\}$ is H_2D^η-summable to a set C $(0 < \eta \leq 1)$, then the sequence is H_2D-summable to same set.

Definition 3.2 A double sequence of sets $\{C_{ij}\}$ is said to be Hausdorff deferred statistically convergent of order η to the set C $(0 < \eta \leq 1)$ if for every $\varepsilon > 0$,

$$\lim_{u,v \to \infty} \frac{1}{(\psi_u \phi_v)^\eta} \left| \left\{ (i,j) \colon i \in (p_u, q_u], j \in (r_v, s_v], \sup_{y \in \mathcal{Y}} \left| d_y(C_{ij}) - d_y(C) \right| \geq \varepsilon \right\} \right| = 0.$$

In this case, the notation $C_{ij} \overset{H_2DS^\eta}{\longrightarrow} C$ is used. For $\eta = 1$, we obtain the concept of Hausdorff deferred statistical convergence (H_2DS) for double sequences of sets which has never been mentioned before.

Remark 3.2 The concept of Hausdorff deferred statistical convergence of order η for double set sequences is coincides with;

- the notion of Hausdorff statistical convergence of order η for double set sequences in [12], for $p_u = 0, q_u = u$ and $r_v = 0, s_v = v$.
- the notion of Hausdorff statistical convergence for double set sequences in [10], for $\eta = 1$, and $p_u = 0, q_u = u$ and $r_v = 0, s_v = v$.

Theorem 3.2 If $0 < \eta \leq \mu \leq 1$, then

$$C_{ij} \overset{H_2DS^\eta}{\longrightarrow} C \Rightarrow C_{ij} \overset{H_2DS^\mu}{\longrightarrow} C.$$

Proof. Let $0 < \eta \leq \mu \leq 1$ and suppose that $C_{ij} \overset{H_2DS^\eta}{\longrightarrow} C$. For every $\varepsilon > 0$, we can write following inequality

$$\frac{1}{(\psi_u \phi_v)^\eta}\left|\left\{(i,j): i \in (p_u, q_u], j \in (r_v, s_v], \sup_{y \in \mathcal{Y}} \left|d_y(C_{ij}) - d_y(C)\right| \geq \varepsilon\right\}\right|$$

$$\geq \frac{1}{(\psi_u \phi_v)^\mu}\left|\left\{(i,j): i \in (p_u, q_u], j \in (r_v, s_v], \sup_{y \in \mathcal{Y}} \left|d_y(C_{ij}) - d_y(C)\right| \geq \varepsilon\right\}\right|.$$

Hence, by our assumption, we get $C_{ij} \overset{H_2DS^\mu}{\longrightarrow} C$.

If $\mu = 1$ is taken in Theorem 3.2, then the following corollary is obtained.

Corollary 3.2 If a double sequence of sets $\{C_{ij}\}$ is H_2DS^η-convergent to a set C $(0 < \eta \leq 1)$, then the sequence is H_2DS-convergent to same set.

Theorem 3.3 Let $\{A_{ij}\}, \{B_{ij}\}$ and $\{C_{ij}\}$ are double sequences of sets such that

$$A_{ij} \subset B_{ij} \subset C_{ij} \quad (\text{for every } i, j \in \mathbb{N}).$$

In this case; if $A_{ij} \overset{H_2DS^\eta}{\longrightarrow} B$ and $C_{ij} \overset{H_2DS^\eta}{\longrightarrow} B$, then $B_{ij} \overset{H_2DS^\eta}{\longrightarrow} B$ where $0 < \eta \leq 1$.

Proof. Let $A_{ij} \subset B_{ij} \subset C_{ij}$, $A_{ij} \overset{H_2DS^\eta}{\longrightarrow} B$ and $C_{ij} \overset{H_2DS^\eta}{\longrightarrow} B$. From the inclusion, it is obvious that

$$d_y(C_{ij}) \leq d_y(B_{ij}) \leq d_y(A_{ij})$$

for each $y \in \mathcal{Y}$. Then, for every $\varepsilon > 0$ we have

$$\{(i,j): i \in (p_u, q_u], j \in (r_v, s_v], |d_y(B_{ij}) - d_y(B)| \geq \varepsilon\}$$

$$= \{(i,j): i \in (p_u, q_u], j \in (r_v, s_v], d_y(B_{ij}) \geq d_y(B) + \varepsilon\}$$

$$\cup \{(i,j): i \in (p_u, q_u], j \in (r_v, s_v], d_y(B_{ij}) \leq d_y(B) - \varepsilon\}$$

$$\subset \{(i,j): i \in (p_u, q_u], j \in (r_v, s_v], d_y(A_{ij}) \geq d_y(B) + \varepsilon\}$$

$$\cup \{(i,j): i \in (p_u, q_u], j \in (r_v, s_v], d_y(C_{ij}) \leq d_y(B) - \varepsilon\}$$

for each $y \in \mathcal{Y}$ and so

$$\frac{1}{(\psi_u\phi_v)^\mu}\left|\left\{(i,j): i\in(p_u,q_u], j\in(r_v,s_v], \sup_{y\in\mathcal{Y}}\left|d_y(B_{ij})-d_y(B)\right|\geq\varepsilon\right\}\right|$$
$$\leq\frac{1}{(\psi_u\phi_v)^\mu}\left|\left\{(i,j): i\in(p_u,q_u], j\in(r_v,s_v], \sup_{y\in\mathcal{Y}}\left|d_y(A_{ij})-d_y(B)\right|\geq\varepsilon\right\}\right|$$
$$+\frac{1}{(\psi_u\phi_v)^\mu}\left|\left\{(i,j): i\in(p_u,q_u], j\in(r_v,s_v], \sup_{y\in\mathcal{Y}}\left|d_y(C_{ij})-d_y(B)\right|\geq\varepsilon\right\}\right|.$$

Hence, by our assumption, we get $B_{ij}\overset{H_2DS^\mu}{\longrightarrow}B$.

Theorem 3.4 Let $0<\eta\leq1$. If a double sequence of sets $\{C_{ij}\}$ is H_2D^η-summable to a set C, then the sequence is H_2DS^η-convergent to the same set.

Proof. Let $0<\eta\leq1$ and suppose that $C_{ij}\overset{H_2D^\eta}{\longrightarrow}C$. For every $\varepsilon>0$, we can write following inequality

$$\sum_{i=p_u+1}^{q_u}\sum_{j=r_v+1}^{s_v}\sup_{y\in\mathcal{Y}}\left|d_y(C_{ij})-d_y(C)\right|\geq\sum_{\substack{i=p_u+1\\ |d_y(C_{ij})-d_y(C)|\geq\varepsilon}}^{q_u}\sum_{j=r_v+1}^{s_v}\sup_{y\in\mathcal{Y}}\left|d_y(C_{ij})-d_y(C)\right|$$
$$\geq\varepsilon\left|\left\{(i,j): i\in(p_u,q_u], j\in(r_v,s_v], \sup_{y\in\mathcal{Y}}\left|d_y(C_{ij})-d_y(C)\right|\geq\varepsilon\right\}\right|$$

and so

$$\frac{1}{\varepsilon}\frac{1}{(\psi_u\phi_v)^\eta}\sum_{i=p_u+1}^{q_u}\sum_{j=r_v+1}^{s_v}\sup_{y\in\mathcal{Y}}\left|d_y(C_{ij})-d_y(C)\right|$$
$$\geq\frac{1}{(\psi_u\phi_v)^\eta}\left|\left\{(i,j): i\in(p_u,q_u], j\in(r_v,s_v], \sup_{y\in\mathcal{Y}}\left|d_y(C_{ij})-d_y(C)\right|\geq\varepsilon\right\}\right|.$$

Hence, by our assumption, we get $C_{ij}\overset{H_2DS^\eta}{\longrightarrow}C$.

Corollary 3.3 If $C_{ij}\overset{H_2}{\longrightarrow}C$, then $C_{ij}\overset{H_2DS^\eta}{\longrightarrow}C$.

The converse of Theorem 3.4 is true only in the case $\eta=1$ and $\{C_{ij}\}\in L_\infty^2$ (the class of all bounded double sequences of sets).

The sequence $\{C_{ij}\}$ is said to be bounded if $\sup_{i,j}\{d_y(C_{ij})\} < \infty$ for each $y \in \mathcal{Y}$.

Theorem 3.5 If a double sequence of sets $\{C_{ij}\} \in L_\infty^2$ is H_2DS-convergent to a set C, then the sequence is H_2D-summable to the same set.

Proof. Let the double sequence of sets $\{C_{ij}\}$ is bounded and $C_{ij} \xrightarrow{H_2DS} C$. Since $\{C_{ij}\} \in L_\infty^2$, there is an $\mathcal{M} > 0$ such that

$$\left|d_y(C_{ij}) - d_y(C)\right| \le \mathcal{M}$$

for all $i, j \in \mathbb{N}$ and each $y \in \mathcal{Y}$. Thus, for every $\varepsilon > 0$ we can write the following inequality

$$\frac{1}{\psi_u \phi_v} \sum_{i=p_u+1}^{q_u} \sum_{j=r_v+1}^{s_v} \sup_{y\in\mathcal{Y}} \left|d_y(C_{ij}) - d_y(C)\right|$$

$$= \sum_{\substack{i=p_u+1 \\ |d_y(C_{ij})-d_y(C)|\ge\varepsilon}}^{q_u} \sum_{j=r_v+1}^{s_v} \sup_{y\in\mathcal{Y}} \left|d_y(C_{ij}) - d_y(C)\right| + \sum_{\substack{i=p_u+1 \\ |d_y(C_{ij})-d_y(C)|<\varepsilon}}^{q_u} \sum_{j=r_v+1}^{s_v} \sup_{y\in\mathcal{Y}} \left|d_y(C_{ij}) - d_y(C)\right|$$

$$\le \frac{\mathcal{M}}{\psi_u \phi_v} \left|\left\{(i,j): i \in (p_u, q_u], j \in (r_v, s_v], \sup_{y\in\mathcal{Y}} \left|d_y\left(C_{ij}\right) - d_y(C)\right| \ge \varepsilon\right\}\right| + \varepsilon.$$

Hence, by our assumption, we get $C_{ij} \xrightarrow{H_2D} C$.

Corollary 3.4 Let $\{C_{ij}\} \in L_\infty^2$, then

$$C_{ij} \xrightarrow{H_2DS} C \Leftrightarrow C_{ij} \xrightarrow{H_2D} C.$$

Finally, we show the relation between the concepts of Hausdorff deferred statistical convergence of order η and Wijsman deferred statistical convergence of order η for double sequences of sets. Before show the relation, let's recall the concept of Wijsman deferred statistical convergence of order η for double sequences of sets from [14].

Definition 3.3 *[14]* A double sequence of sets $\{C_{ij}\}$ is said to be Wijsman deferred statistically convergent of order η to a set C $(0 < \eta \le 1)$ if for every $\varepsilon > 0$

$$\lim_{u,v\to\infty} \frac{1}{(\psi_u \phi_v)^\eta} \left|\left\{(i,j): i \in (p_u, q_u], j \in (r_v, s_v], \left|d_y(C_{ij}) - d_y(C)\right| \ge \varepsilon\right\}\right| = 0,$$

for each $y \in \mathcal{Y}$ and it is denoted by $C_{ij} \overset{W_2DS^\eta}{\longrightarrow} C$.

Theorem 3.6 Let $0 < \eta \leq 1$. If a double sequence of sets $\{C_{ij}\}$ is H_2DS^η-convergent to a set C, then the sequence is W_2DS^η-convergent to the same set.

Proof. Let $0 < \eta \leq 1$ and suppose that $C_{ij} \overset{H_2DS^\eta}{\longrightarrow} C$. For every $\varepsilon > 0$, we can write following inequality

$$\frac{1}{(\psi_u \phi_v)^\eta} \left| \left\{ (i,j): i \in (p_u, q_u], j \in (r_v, s_v], \sup_{y \in \mathcal{Y}} \left| d_y(C_{ij}) - d_y(C) \right| \geq \varepsilon \right\} \right|$$

$$\geq \frac{1}{(\psi_u \phi_v)^\eta} \left| \left\{ (i,j): i \in (p_u, q_u], j \in (r_v, s_v], \left| d_y(C_{ij}) - d_y(C) \right| \geq \varepsilon \right\} \right|.$$

for each $y \in \mathcal{Y}$. Hence, by our assumption, we get $C_{ij} \overset{W_2DS^\eta}{\longrightarrow} C$.

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On Lacunary Statistical Convergence for triple sequences on L − Fuzzy Normed Space

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Abstract

The idea of lacunary statistical convergence for triple sequences, which is a development of statistical convergence, is examined and expanded in this study on L − fuzzy normed spaces, which is a generalization of normed spaces.

Keywords: Triple sequences, lacunary sequence, lacunary statistically convergence

1. Introduction

To date, studies have been conducted on statistical convergence in many different spaces [1],[3],[5],[6],[12],[13],[15],[19],[26],[28],[30]. These studies have a very important place in the field of analysis and continue to be of great interest to mathematicians.

The concept of lacunary statistical convergence, which is a generalization of statistical convergence, was first introduced by Fridy, John Albert, and Cihan Orhan in 1993[7],[8] and very important studies have been conducted on this concept again [6],[16],[17],[24],[27],[29].

The fuzzy concept was first introduced to the mathematical community by Zadeh [32], and then the intuitionistic fuzzy set concept was introduced [2] along with the L-fuzzy set concept [9]. In subsequent years, studies have been conducted on these notions.

2. Preliminaries

Preliminaries on L − fuzzy normed spaces are presented in this section.

Definition 2.1. [25] Assume that $K : [0, 1] \times [0, 1] \to [0, 1]$ is a function that satisfies the following

1. $K(a, b) = K(b, a)$

2. $K(K(a, b), c) = K(a, K(b, c))$

3. $K(a, 1) = K(1, a) = x$

4. $a \leq b$, $c \leq d$ then $K(a, c) \leq K(b, d)$

is known as a t− norm.

Example 2.2. [25] K1, K2 and K3 are the functions that given with,

K1(a, b) = min{a, b},

K2(a, b) = ab,

K3(a, b) = max{a + b − 1, 0}

are the samples, which are well known of t− norms.

Definition 2.3. [25] Let L = (L, ⪯) be a complete lattice and let a set A be called the universe. An L−fuzzy set, on A is defined with a function

$$X : A \to L.$$

On a set A, the family of all L−sets is denoted by L A.

Two L− sets on A intersect and union

$$(C \cap D)(x) = C(x) \cap D(x)\ ,\ (C \cup D)(x) = C(x) \cup D(x)$$

for all x ∈ A

Definition 2.4. [25] Let L = (L, ⪯) be a complete lattice. Therefore, t− norm is a function

$$K : L \times L \to L$$

that satisfies the following for all a, b, c, d ∈ L:

1. K (a, b) = K (b, a)

2. K (K (a, b), c) = K (a, K (b, c))

3. K (a, 1L) = K (1L, a) = a

4. a ⪯ b and c ⪯ d, then K (a, c) ⪯ K (b, d).

Definition 2.6. [25] The function N : L → L is defined as a negator on L = (L, ⪯) if,

N1) N $(0_L) = 1_L$

N2) N (1_L) = 0_L

N3) a ⪯ b implies N (b) ⪯ N (a) for all a, b ∈ L.

If in addition,

N4) N (N (a)) = a for all a ∈ L.

Therefore N is known as an involutive.

Definition 2.7. [25] Let L = (L, ⪯) be a complete lattice and V be a real vector space. K be a continuous t−norm on L and μ be an L−set on V × (0, ∞) satisfying the following

(a) μ(a, t) ≻ 0_L for all a ∈ V, t > 0

(b) μ(a, t) = 1_L for all t > 0 if and only if a = θ

(c) μ(αa, t) = μ(a, t |α|) for all a ∈ V, t > 0 and α ∈ R − {0}

(d) K (μ(a, t), μ(b, s)) ⪯ ν(a + b, t + s), for all a, b ∈ V and t, s > 0

(e) limt→∞ μ(a, t) = 1L and limt→0 μ(a, t) = 0L for all a ∈ V − {θ}

(f) The functions fa : (0, ∞) → L which is f(t) = μ(a, t) are continuous.

The triple (V, μ, K) is referred to as an L − fuzzy normed space or L − normed space in this context. 3

Lacunary Statistical Convergence for triple sequences on L −Fuzzy Normed Space

Recently, many studies show us that the notion of lacunary statistical convergence has been introduced and investigated in many fields [7], [8] both for double [17], [1] and also triple sequences [24], [6]. In this section we define and study lacunary statistical convergence for triple sequences on L − fuzzy normed space.

Definition 3.1. By a lacunary sequence we mean an increasing integer sequence θ = (k_r) such that k_0 = 0 and $h_r = k_r - k_{r-1} \to \infty$ as r → ∞. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by q_r. For any subset of natural numbers, the number

$$\delta_\theta(N) = \lim_r \frac{1}{h_r} |k \in I_r : k \in N|$$

is called the θ density of the set N, provided the limit exists. A sequence a = (a_k) is said to be lacunary statistically convergent to a number ℓ provided that for each ϵ > 0, δ_θ {k ∈ N : $|a_k - \ell| \geq \epsilon\}| = 0$. In this case the number ℓ is called lacunary statistical limit of the sequence a = (a_k).

Definition 3.2. Let (V, μ, K) be a L −fuzzy normed space. Then a sequence a = (a_k) is lacunary statistically convergent to l ∈ V with respect to μ fuzzy norm, provided that, for each ϵ ∈ L − {0L} and t> 0,

$$\delta_\theta\{k \in N : \mu(a_k - l, t) \nsucc N(\epsilon)\} = 0.$$

The triple sequence θ = $\{(k_r, l_s, t_q)\}$ is called triple lacunary if there exist three increasing integer sequence such that

$$k_0 = 0,\ h_r = k_r - k_{r-1} \to \infty \text{ as } r \to \infty,$$

$$l_0 = 0,\ m_s = l_s - l_{s-1} \to \infty, \text{ as } s \to \infty,$$

$$t_0 = 0,\ ,\ n_q = t_q - t_{q-1} \to \infty, \text{ as } q \to \infty.$$

The intervals are determined by θ, $I_r = \{(k): k_{r-1} < k < k_r\}$ Ir = {(k) : kr−1 < k ≤ kr},, $I_s = \{(l): l_{s-1} < l < l_s\}$, $I_q = \{(t): t_{q-1} < t < t_q\}$, , $I_{r,s,q} = \{(k,l,t): k_{r-1} < k < k_r, l_{s-1} < l < l_s, t_{q-1} < t < t_q\}$, $q_r = \frac{k_r}{k_{r-1}}$, $u_s = \frac{l_s}{l_{s-1}}$ and $y_q = \frac{t_q}{t_{q-1}}$. Note that the triple θ− density will be denoted by δ_{θ_3}.

Definition 3.3. Let (V, μ, K) be a L −fuzzy normed space. Then a triple sequence $a = (a_{mnk})$ is lacunary statistically convergent to l ∈ V with respect to ν fuzzy norm, provided that, for each ϵ ∈ L − $\{0_L\}$ and t > 0,

$$\delta_{\theta_3}\{(m, n, k) \in N \times N \times N : \mu(a_{mnk}) - l, t) \nsucc N(\epsilon)\} = 0.$$

In this case, we write $S_{\theta_{3_L}}$ − lim a = l.

Proposition 3.4. Let (V, ν, K) be a L −fuzzy normed space. Then, the following statements are equivalent, for every ϵ ∈ L − {0L} and t > 0:

(a) $S_{\theta_{3_L}}$ − lim a = ℓ.

(b) δ_{θ_3} {(m, n, k) ∈ N × N × N : $\mu(a_{mnk} - \ell, t) \nsucc N(\epsilon)\} = 0$.

(c) δ_{θ_3} {(m, n, k) ∈ N × N × N : $\mu(a_{mnk} - \ell, t) \succ N(\epsilon)\} = 1$.

(d) $S_{\theta_{3_L}}$ − lim $\mu(a_{mnk} - \ell, t) = 1_L$.

Theorem 3.5. Let (V, μ, K) be a L − fuzzy normed space and a = (a_{mnk}) be a triple sequence. If lim a = ℓ, then $S_{\theta_{3_L}}$ − lim a = ℓ.

Proof. Let lim a = ℓ. Then for every ϵ ∈ L − $\{0_L\}$ and t > 0, there is a number t_0 ∈ N such that

$$\mu(a_{mnk} - \ell, t) \succ N(\epsilon)$$

for all m, n, k ≥ t_0. Therefore, {(m, n, k) ∈ N × N × N : $\mu(a_{mnk} - \ell, t) \nsucc N(\epsilon)$} has at most finitely many terms. We can see right away that any finite subset of the natural numbers has triple θ− density zero. Hence,

$$S_{\theta_{3_L}}\{(m, n, k) \in N \times N \times N : \mu(a_{mnk} - l, t) \nsucc N(\epsilon)\} = 0.$$

Theorem 3.7. Let (V, μ, K) be a L −fuzzy normed space. If a triple sequence $a = (a_{mnk})$ is lacunary statistically convergent with respect to the L − fuzzy norm μ, then $S_{\theta_{3_L}}$ − limit is unique.

Proof. Suppose that $S_{\theta_{3_L}}$ − lim a = ℓ_1 and $S_{\theta_{3_L}}$ − lim a = ℓ_2 where $\ell_1 \neq \ell_2$. For any given ϵ ∈ L − $\{0_L\}$

and t > 0, we can choose a r ∈ L − $\{0_L\}$ such that K (N (r), N (r)) ≻ N (ϵ). Define the following sets

$$K\_1 = \{(m, n, k) \in N \times N \times N : \mu(a_{mnk} - \ell_1, t)) \nsucc N(r)\}$$

and

$$K\_2 = \{(m, n, k) \in N \times N \times N : \mu(a_{mnk} - \ell_2, t)) \nsucc N(r)\}$$

for any t > 0. Since for elements of the set K(ϵ, t) = K_1(ϵ, t) ∪ K_2(ϵ, t) we have

$$\mu(\ell_1 - \ell_2, t) \geq K(\mu(a_{mnk} - \ell_1, t\ 2), \mu(a_{mnk} - \ell_2, t\ 2)) \succ K(N(r), N(r)) \succ N(\epsilon).$$

It can be concluded that ℓ1 = ℓ2.

6. Conclusion

In this study, the properties of Lacunary statistical convergence for triple sequences, which is a generalization of statistical convergence, are defined on L − fuzzy normed spaces, which are a generalization of normed, and their properties are examined.

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On Quasi Einstein Spacetimes Admitting M-Projective Curvature Tensor

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Abstract

The object of the present paper is to study quasi-Einstein spacetimes admitting M-projective curvature tensor. In the first section, we give the definition and some properties of M-projective curvature tensor. In the second section, the definition of quasi-Einstein manifold admitting M-projective curvature tensor is given. Some geometric properties of these spacetimes have been studied under special conditions. In the third section, perfect fluid spacetimes satisfying the Einstein's field equations without cosmological constant are examined.

Keywords: Quasi-Einstein manifold, divergence-free, M-projective curvature tensor.

1. Introduction

This paper is dedicated to certain investigations in general relativity by the coordinate free method of differential geometry. In this method, the spacetime of general relativity is regarded as a connected four dimensional semi-Riemannian manifold (M^4, g) with the Lorentzian metric g of signature $(-,+,+,+)$. The geometry of the Lorentzian manifold ([1]) begins with the study of the causal character of the vectors of the manifold. This makes the Lorentzian manifolds a convenient choice in the study of general relativity. The Einstein's equations [2] (p.337), imply that the energy-momentum tensor is of vanishing divergence. We get this easily if the energy-momentum tensor is covariantly constant [3]. In the paper [3], M. C. Chaki and Sarbari Ray showed that a general relativistic spacetime with the covariant-constant energy-momentum tensor is Ricci symmetric, that is, $\Delta S = 0$, where S is the Ricci tensor of the spacetime. Many authors studied about spacetimes and their properties such as spacetimes with semisymmetric energy momentum tensor by De and Velimirovi [4], M-Projectively flat spacetimes by Özen Zengin [5], pseudo Z symmetric spacetimes by Mantica and Suh [6], Mixed generalized quasi-Einstein manifold and some properties on it by Debnath, De and Bhattacharyya [7], On Ricci-symmetric mixed generalized quasi-Einstein spacetime by Chattopadhyay, Bhunia and Bhattacharyya [8], Concircular Curvature Tensor and Fluid Spacetimes by Z. Ahsan and S. A. Siddiqui in [9] and many more.

An Einstein manifold is a Riemannian or pseudo-Riemannian manifold whose Ricci tensor S of type (0, 2) is non-zero and proportional to the metric tensor. Einstein manifolds form a natural sub-class of various classes of Riemannian or semi-Riemannian manifolds by a curvature condition imposed on their Ricci tensor [1]. Also in Riemannian geometry as well as in general relativity theory, the Einstein manifold plays a very important role. In [10], Karcher stated that a conformally flat perfect fluid spacetime has the geometric structure of quasi-constant curvature. A manifold of quasi-constant curvature is a natural sub-class of quasi Einstein manifold [11]. Studying on quasi Einstein manifolds helps us to have a deeper understanding of the global characteristics of the universe including its topology [14]. Hence, Chaki and Maity [11] generalized the concept of Einstein manifolds and introduced the notion of a quasi-Einstein manifold. According to them, a Riemannian or semi-Riemannian manifold is said to be a quasi-Einstein manifold if its Ricci tensor S of type (0, 2) is non-zero and satisfies the condition

$$S(X,Y) = \alpha g(X,Y) + \beta A(X)A(Y) \tag{1}$$

where α, β are two non-zero real-valued scalar functions and A is a nowhere vanishing 1-form defined by $g(X,\rho) = A(X)$, $g(\rho,\rho) = 1$ for all vector fields X; ρ being a unit vector field, called the generator of the manifold. An n-dimensional manifold of this kind is denoted by $(QE)_n$. The scalars α, β are known as the associated scalars. The notion of quasi-Einstein manifolds arose during the study of exact solutions of the Einstein's field equations as well as during the considerations of quasi-umbilical hypersurfaces of semi-Euclidean spaces. For instance, the Robertson-Walker spacetime is a quasi-Einstein manifold. Also, quasi-Einstein manifolds can be taken as a model of perfect fluid spacetimes in general relativity. The importance of quasi-Einstein spacetimes lies in the fact that 4-dimensional semi-Riemannian manifolds are related to study of general relativistic fluid spacetimes, where the unit vector field ρ is taken as a timelike velocity vector field, that is, $g(\rho,\rho) = -1$.

As a generalization of quasi Einstein manifolds, in [12], De and Ghosh introduced and studied the notion of generalized quasi Einstein manifolds. A Riemannian manifold is said to be a generalized quasi Einstein manifold if its Ricci tensor S of type (0, 2) is not identically zero and satisfies the following:

$$S(X,Y) = \alpha g(X,Y) + \beta A(X)A(Y) + \gamma B(X)B(Y), \tag{2}$$

where α, β, γ are scalars of which $\beta \neq 0$, $\gamma \neq 0$ and A, B are nowhere vanishing 1-forms such that $g(X,\rho) = A(X)$, $g(X,\mu) = B(X)$ for all vector fields X. The unit vectors ρ and μ corresponding to the 1-forms A and B are orthogonal to each other. Also ρ and μ are known as the generators of the manifold. Such an n-dimensional manifold is denoted by $G(QE)_n$.

In Cosmology, spacetime models are studied in order to represent the different phases in the evolution of the Universe which can be divided into three phases:

- **Initial Phase.** The initial phase is just after the big bang when the effects of both viscosity and heat flux were quite pronounced.

- **Intermediate Phase.** The effect of viscosity was no longer significant but the heat flux was still not negligible.
- **Final Phase.** This phase extends to the present state of the universe. In this phase, both the effects of viscosity and the heat flux have become negligible and the matter content of the universe may be assumed to be a perfect fluid.

The significance of the study of $G(QE)_n$ and $(QE)_n$ lies in the fact that $G(QE)_n$ spacetime manifold represents the second phase while $(QE)_n$ the spacetime manifold corresponds to the third phase in the evolution of the universe [13]. One way to understand the geometric properties of such manifolds is to study the tensors for these manifolds.

In 1971, Pokhariyal and Mishra [15] introduced a new curvature tensor of type (1,3) denoted by $\widetilde{M}$ in an n-dimensional Riemannian manifold (M^n, g), $n > 2$ and defined by

$$\widetilde{M}(Y,Z)U = \tilde{R}(Y,Z)U - \frac{1}{2(n-1)}[S(Z,U)Y - S(Y,U)Z + g(Z,U)QY - g(Y,U)QZ] \tag{3}$$

where $\tilde{R}$ and S denote the Riemannian curvature tensor of type (1, 3) and the Ricci operator defined by $g(QX,Y) = S(X,Y)$, respectively. Such a tensor $\widetilde{M}$ is known as the M-projective curvature tensor. The M-projective curvature tensor have been studied by J.P. Singh [16], S.K. Chaubey and R.H. Ojha [17], S.K. Chaubey [18], and many others.

From (3) we can define the M-projective curvature tensor of type (0,4) as follows:

$$\begin{aligned} M(Y,Z,U,V) = R(Y,Z,U,V) - \frac{1}{2(n-1)}[S(Z,U)g(Y,V) - S(Y,U)g(Z,V) + S(Y,V)g(Z,U) - \\ -S(Z,V)g(Y,U)], \end{aligned} \tag{4}$$

where R denotes the Riemannian curvature tensor of type (0,4) defined by

$$R(Y,Z,U,V) = g(\tilde{R}(Y,Z)U,V).$$

Thus, from (4), we have

$$M(Y,Z,U,V) = g(\widetilde{M}(Y,Z)U,V),$$

where $\tilde{R}$ is the Riemannian curvature tensor of type (1,3) and S denotes the Ricci tensor of type (0,2), respectively.

Let $\{e_i, \quad i = 1,2,\dots,n\}$ be an orthonormal basis of the tangent space at each point of the manifold. From (3) we can easily verify that the tensor M satisfies the following property

$$\widetilde{M}(Y,Z)U = -\widetilde{M}(Z,Y)U,$$

$$\widetilde{M}(Y,Z)U + \widetilde{M}(Z,U)Y + \widetilde{M}(U,Y)Z = 0. \tag{5}$$

From (4) and (5) it follows that

$$\begin{aligned}
&(i) \quad M(Y,Z,U,V) = -M(Z,Y,U,V),\\
&(ii) \quad M(Y,Z,U,V) = -M(Y,Z,V,U),\\
&(iii) \quad M(Y,Z,U,V) = M(U,V,Y,Z),\\
&(iv) \quad M(Y,Z,U,V) + M(Z,U,Y,V) + M(U,Y,Z,V) = 0.
\end{aligned} \tag{6}$$

Also from the equation (4) we have

$$\sum_{i=1}^{n} M(Y,Z,e_i,e_i) = 0 = \sum_{i=1}^{n} M(e_i,e_i,U,V) \tag{7}$$

and

$$\begin{aligned}
\sum_{i=1}^{n} M(e_i,Z,U,e_i) &= \sum_{i=1}^{n} M(Z,e_i,e_i,U)\\
&= \frac{n}{2(n-1)}\left[S(Z,U) - \frac{r}{n}g(Z,U)\right],
\end{aligned} \tag{8}$$

where r is the scalar curvature.

2. M-projective Curvature Tensor on Quasi Einstein Spacetime

In this section, we consider a spacetime $n = 4$ admitting the M-projective curvature tensor. Then, from the equation (4), we have

$$M_{hijk} = R_{hijk} + \frac{1}{6}(S_{ik}g_{hj} - S_{ij}g_{hk} + S_{hj}g_{ik} - S_{hk}g_{ij}). \tag{9}$$

Theorem 2.1 *If a spacetime is a Lorentzian infinitesimally isotropic relative to a unit timelike vector field A^l then this spacetime reduces to M-projectively flat spacetime.*

Proof. If our spacetime is a Lorentzian infinitesimally isotropic relative to a unit timelike vector field A^l then we have [10]

$$R_{hijk} = \gamma(g_{ik}g_{hj} - g_{ij}g_{hk}). \tag{10}$$

Multiplying (10) by g^{hk}, we get

$$S_{ij} = -3\gamma g_{ij}. \tag{11}$$

Putting (10) and (11) in (4), we find $M_{hijk} = 0$. Thus, the proof is completed.

Let us assume that our spacetime is a quasi Einstein. Here, we denote this spacetime which is quasi-Einstein as (V_4, g). From (1) and (9), we get

$$M_{hijk} = R_{hijk} + \frac{\alpha}{3}(g_{ik}g_{hj} - g_{ij}g_{hk}) + \frac{\beta}{6}(g_{hj}A_iA_k - g_{hk}A_iA_j + g_{ik}A_hA_j - g_{ij}A_hA_k) \tag{12}$$

In this case, the covariant derivative of (12) can be obtained as follows

$$\begin{aligned} M_{hijk,l} = R_{hijk,l} &+ \frac{\alpha_l}{3}(g_{ik}g_{hj} - g_{ij}g_{hk}) + \frac{\beta_l}{6}(g_{hj}A_iA_k - g_{hk}A_iA_j + g_{ik}A_hA_j - g_{ij}A_hA_k) \\ &+ \frac{\beta}{6}(g_{hj}(A_{i,l}A_k + A_iA_{k,l}) - g_{hk}(A_{i,l}A_j + A_iA_{j,l}) \\ &+ g_{ik}(A_{h,l}A_j + A_hA_{j,l}) - g_{ij}(A_{h,l}A_k + A_hA_{k,l})) \end{aligned} \tag{13}$$

where α and β are the associated scalars for the quasi Einstein manifold.

Theorem 2.2 *If the M-projective curvature tensor of* (V_4, g) *is covariantly constant then the associated scalar function* β *must be constant.*

Proof. Assuming that the M-projective curvature tensor is covariantly constant then from (13), we get

$$\begin{aligned} R_{hijk,l} &= \frac{\alpha_l}{3}(g_{ij}g_{hk} - g_{ik}g_{hj}) + \frac{\beta_l}{6}(g_{hk}A_iA_j - g_{hj}A_iA_k + g_{ij}A_hA_k - g_{ik}A_hA_j) \\ &+ \frac{\beta}{6}(g_{hk}(A_{i,l}A_j + A_iA_{j,l}) - g_{hj}(A_{i,l}A_k + A_iA_{k,l}) + g_{ij}(A_{h,l}A_k + A_hA_{k,l}) \\ &- g_{ik}(A_{h,l}A_j + A_hA_{j,l})) \end{aligned} \tag{14}$$

Since $A^i A_i = -1$ then $A_{i,l} A^i = 0$. Multiplying (14) by g^{hk}, we obtain

$$S_{ij,l} = \alpha_l g_{ij} + \frac{\beta_l}{3} A_i A_j - \frac{\beta_l}{6} g_{ij} + \frac{\beta}{3}(A_{i,l} A_j + A_i A_{j,l}). \tag{15}$$

On the other hand, by taking the covariant derivative of (1), we also have

$$S_{ij,l} = \alpha_l g_{ij} + \beta_l A_i A_j + \beta(A_{i,l} A_j + A_i A_{j,l}). \tag{16}$$

Comparing the equations (15) and (16), it can be found

$$\beta_l(4A_i A_j + g_{ij}) + 4\beta(A_{i,l} A_j + A_i A_{j,l}) = 0. \tag{17}$$

Multiplying (2.9) by $A^i A^j$, we can see that $\beta_l = 0$. Thus, β must be constant. This completes the proof.

Theorem 2.3 *If the M-projective curvature tensor of* (V_4, g) *is covariantly constant then the covariant derivative of* A_j *vanishes.*

Proof. Since (V_4, g) is a quasi Einstein then, we obtain from (1) and Theorem2.2,

$$r = 4\alpha - \beta. \tag{18}$$

Comparing (4), (8) and (18), we can get

$$M_{ij} = \frac{\beta}{6}(g_{ij} + 4A_i A_j). \quad (\beta \neq 0) \tag{19}$$

By taking the covariant derivative of (19), since $M_{ij,l} = 0$, we obtain

$$A_{i,l} A_j + A_i A_{j,l} = 0. \tag{20}$$

Finally, multiplying (20) by A^i, it can be seen that

$$A_{j,l} = 0. \tag{21}$$

Thus, the proof is completed.

Theorem 2.4 *Let the M-projective curvature tensor of* (V_4, g) *be divergence-free. The vector field* A^l *is divergence-free if and only if* β_l *is orthogonal to* A^l.

Proof. Multiplying the equation (13) by g^{hl} and assuming that the M-projective curvature tensor of (V_4, g) is divergence-free, we get

$$R^l_{ijk,l} = \frac{\alpha_k}{3} g_{ij} - \frac{\alpha_j}{3} g_{ik} + \frac{1}{6}(\beta_k A_i A_j - \beta_j A_i A_k + g_{ij} A^l \beta_l A_k - g_{ik} A^l \beta_l A_j)$$
$$+ \frac{\beta}{6}(A_{i,k} A_j + A_i A_{j,k} - A_{i,j} A_k - A_i A_{k,j} + g_{ij} A^l_{,l} A_k + g_{ij} A^l A_{k,l} - g_{ik} A^l_{,l} A_j - g_{ik} A^l A_{j,l}). \tag{22}$$

Now, multiplying (22) by g^{ij}, we find

$$S^l_{k,l} = \alpha_k + \frac{\beta_l}{3} A^l A_k - \frac{\beta_k}{6} + \frac{\beta}{3}(A^l_{,l} A_k - A^l A_{k,l}). \tag{23}$$

On the other hand, by taking the covariant derivative of (1) and after that multiplying the last equation by g^{il}, we obtain

$$S^l_{k,l} = \alpha_k + \beta_l A^l A_k + \beta(A^l_{,l} A_k - A^l A_{k,l}). \tag{24}$$

Comparing the equations (23) and (24), one can find

$$4\beta_l A^l A_k + 4\beta(A^l_{,l} A_k - A^l A_{k,l}) + \beta_k = 0. \tag{25}$$

Multiplying (25) by A^k, we obtain

$$A^l_{,l} = -\frac{3}{4\beta}(\beta_l A^l). \qquad (\beta \neq 0) \tag{26}$$

If we assume that A^l is divergence-free then we get from (26), $\beta_l A^l = 0$. The converse is also true. Thus, the proof is completed.

Theorem 2.5 *Let the M-projective curvature tensor of* (V_4, g) *be divergence-free. If the non-zero associated scalar* β *of* (V_4, g) *is constant then the vector field* A^l *is divergence-free.*

Proof. Let the M-projective curvature tensor of (V_4, g) be divergence-free. Then from Theorem2.4, the equation (25) reduces to

$$4\beta(A^l_{,l}A_k - A^l A_{k,l}) = 0. \quad (\beta \neq 0) \tag{27}$$

If we multiply (27) by A^k, we get $A^l_{,l} = 0$. This completes the proof.

Theorem 2.6 *Let the M-projective curvature tensor of* (V_4, g) *be divergence-free. If the associated vector field* A^l *of* (V_4, g) *is also divergence-free then* α_l *is orthogonal to* A^l.

Proof. Let (V_4, g) be of divergence-free M-projective curvature tensor. Multiplying (1) by g^{ij}, we obtain

$$r = 4\alpha - \beta. \tag{28}$$

By taking the covariant derivative of (28), one can get

$$r_{,j} = 4\alpha_j - \beta_j. \tag{29}$$

From the Ricci identity $S^l_{k,l} = \frac{1}{2}r_{,k}$ and the equation (24), we find

$$r_{,j} = 2(\alpha_j + \beta_l A^l A_j + \beta(A^l_{,l}A_j - A^l A_{j,l})). \tag{30}$$

Comparing the equations (29) and (30) gives

$$2\alpha_j = \beta_j + 2\beta_l A^l A_j + 2\beta(A^l_{,l}A_j - A^l A_{j,l}). \tag{31}$$

And multiplying (31) by A^j,

$$(2\alpha_j + \beta_j)A^j = -2\beta A^l_{,l}. \tag{32}$$

If we assume that the divergence of A^l is zero then

$$(2\alpha_j + \beta_j)A^j = 0. \tag{33}$$

From Theorem2.4 and the equation (33), $\alpha_j A^j = 0$. This completes the proof.

3. Perfect Fluid Quasi Einstein Spacetime with the M-Projective Curvature Tensor

In this section, we consider a perfect fluid (V_4, g) with the M-projective curvature tensor. To find a model of universe, Einstein obtained the field equations of general relativity. The universe on a large scale shows isotropy and homogeneity and the matter contents of the universe(stars, galaxies, nebulas, etc.) can be assumed to be that of a perfect fluid. Assume that the Einstein's field equations without the cosmological constant is given by

$$S(X,Y) - \frac{r}{2} g(X,Y) = kT(X,Y), \tag{34}$$

where S and r denote the Ricci tensor and the scalar curvature, respectively and T is the energy momentum tensor.

The energy momentum tensor T of a perfect fluid is given by [2]

$$T(X,Y) = (\sigma + p)A(X)A(Y) + pg(X,Y), \tag{35}$$

where σ is the energy density, p is the isotropic pressure, $g(X,\rho) = A(X)$ and ρ is a unit timelike vector field.

Now, if we compare the equations (34) and (35) then we find

$$S_{ij} = k(\sigma + p)A_i A_j + (kp + \frac{r}{2})g_{ij}. \tag{36}$$

From (36), we get

$$r = k(\sigma - 3p). \tag{37}$$

Thus, putting (37) in (36), it can be found that

$$S_{ij} = k(\sigma + p)A_i A_j + \frac{k}{2}(\sigma - p)g_{ij}. \tag{38}$$

For a perfect fluid (V_4, g) admitting the M-projective curvature tensor, we get from (9)

$$M_{ij} = \frac{2}{3} S_{ij} - \frac{r}{6} g_{ij}. \tag{39}$$

Comparing the equations (18) and (37), we get

$$4\alpha - \beta = k(\sigma - 3p). \tag{40}$$

Also, considering the equations (1) and (36), we obtain

$$-\alpha + \beta = \frac{k}{2}(\sigma + 3p). \tag{41}$$

In this case, from (40) and (41), we can see that

$$\sigma = \frac{2\alpha+\beta}{2\mathrm{k}} \tag{42}$$

and

$$p = \frac{\beta-2\alpha}{2\mathrm{k}} \tag{43}$$

If we multiply the equation (19) by A^i, we obtain

$$M_{ij}A^i = -\frac{\beta}{2}A_j. \tag{44}$$

Hence, we have the following theorem:

Theorem 3.1 *In a perfect fluid* (V_4, g) *with the M-projective curvature tensor,* $-\frac{\beta}{2}$ *is an eigenvalue of this tensor corresponding to the eigenvector* A_j.

Theorem 3.2 *Let* (V_4, g) *be a perfect fluid spacetime with the M-projective curvature tensor. If* (V_4, g) *describes*

1. a fluid of strings then $\frac{\alpha}{2}$,
2. a fluid of cosmic walls then $-\frac{\alpha}{5}$,
3. a radiation fluid then -2α,
4. a dust then $-\alpha$

is an eigenvalue of the M-projective curvature tensor corresponding to the eigenvector A_j.

Proof. Let (V_4, g) be a perfect fluid spacetime with the M-projective curvature tensor. Then, from Theorem3.1, we have (44). If (V_4, g) describes

1. a fluid of strings, i.e., $\sigma = -\frac{1}{3}p$ then from (42) and (43), we get $\beta = -\alpha$.
   Then $A^i M_{ij} = \frac{\alpha}{2} A_j$,
2. a fluid of cosmic walls, i.e., $p = -\frac{2}{3}\sigma$ then from (42) and (43), we get $\beta = \frac{2\alpha}{5}$.
   Then $A^i M_{ij} = -\frac{\alpha}{5} A_j$,
3. a radiation fluid, i.e., $\sigma = 3p$, from (42) and (43), we get $\beta = 4\alpha$.
   Then $A^i M_{ij} = -2\alpha A_j$,
4. a dust, i.e., $p = 0$, then we get $\beta = 2\alpha$. Thus, $A^i M_{ij} = -\alpha A_j$,

Thus, the proof is completed.

Theorem 3.3 *In a perfect fluid* (V_4, g) *with the divergence-free M-projective curvature tensor, the following*

$$A^l_{,l} = -\frac{3(\beta_l - 2\alpha_l)}{2\beta} A^l$$

holds.

Proof. Differentiating covariantly of the equation (8), we have

$$M_{ij,l} = \frac{2}{3}\left(S_{ij,l} - \frac{r_{,l}}{4} g_{ij}\right). \tag{45}$$

Multiplying the equation (45) by g^{il} and using the Ricci identity, we find

$$M^l_{j,l} = \frac{1}{6} r_{,l}. \tag{46}$$

Now, taking the covariant derivative of the equation (37), we have

$$r_{,l} = k(\sigma_l - 3p_l). \tag{47}$$

If we put the equation (47) in the equation (46), we get

$$M^l_{j,l} = \frac{k}{6}(\sigma_l - 3p_l). \tag{48}$$

Now, assume that the M-projective curvature tensor is divergence-free. From the equation (48), we find

$$\sigma_l = 3p_l, \tag{49}$$

and

$$r_{,l} = 0. \tag{50}$$

Taking the covariant derivative of the equation (38), we get

$$S_{ij,l} = k(\sigma_l + p_l)A_iA_j + k(\sigma + p)(A_{i,l}A_j + A_iA_{j,l}) + \frac{k}{2}(\sigma_l - p_l)g_{ij}. \tag{51}$$

Multiplying the equation (51) by A^i, it can be found

$$S_{ij,l}A^i = -3kp_lA_j - k(\sigma + p)A_{j,l}. \tag{52}$$

Again, multiplying the equation (52) by g^{jl} and using the equations (42), (43) and (50) we obtain

$$A^l_{,l} = -\frac{3(\beta_l - 2\alpha_l)}{2\beta}A^l \tag{53}$$

Thus, the proof is completed.

7. References

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On Some Subclasses of Meromorphic Functions Involving The Fractional Derivative Operator

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Abstract

Let Σ denote the class of functions of the form $f(z)=\frac{1}{z}+\sum_{k=0}^{\infty}a_k z^k$ which are analytic in the punctured disc $\mathbb{D}=\{z:0<|z|<1\}$. We introduce and study some new families of meromorphic functions defined by the fractional derivative operator. A number of useful characteristics of functions in these families are obtained.

Keywords: Meromorphic, Coefficients estimates, Fractional Derivative Operator, Radii problems.

1. Introduction

Let Σ denote the class of functions of the form

$$f(z)=z^{-1}+\sum_{n=0}^{\infty}a_n z^n \tag{1}$$

which are analytic in the punctured disc $\mathbb{D}=\{z\in\mathbb{C}:0<|z|<1\}$.

The meromorphic analouge of the fractional derivative of order $\alpha,\ 0\leq\alpha\leq 1,$ is defined in [2] for a function $f(z)$ by

$$D_z^{\alpha}f(z)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dz}\left\{z^{\alpha-1}\int_0^z(z-\xi)^{-\alpha}\ {}_2F_1\left(1-\alpha,1,1-\alpha;1-\frac{\xi}{2}\right)\xi^2 f(\xi)\,d\xi\right\},$$

where $f(z)$ is analytic function in a simply connected domain of the $z-$plane containing the origin and the multiplicity of $(z-\xi)^{-\alpha}$ is removed by requiring $\log(z-\xi)$ to be real when $(z-\xi)>0.$ Γ is Gamma function and ${}_2F_1(a,b,c,;z)$ is Gauss-Hypergeometric function. Using $D_z^{\alpha}f(z),$ Noor, Ahmad and Khan [11] defined an operator $\Omega_z^{\alpha}f(z):\Sigma\to\Sigma,$ as follows:

$$\begin{aligned}\Omega_z^{\alpha} f(z) &= \frac{\Gamma(2-\alpha)}{\Gamma(2)} zD_z^{\alpha} f(z) \\ &= z^{-1} + \sum_{n=0}^{\infty} \frac{(2)_{n+1}}{(2-\alpha)_{n+1}} a_n z^n \\ &= \phi(2, 2-\alpha; z) * f(z), \ \alpha \neq 2, 3, 4, \ldots\end{aligned}$$

where

$$\phi(2, 2-\alpha; z) = z^{-1} + \sum_{n=0}^{\infty} \frac{(2)_{n+1}}{(2-\alpha)_{n+1}} z^n$$

and $(\beta)_n$ is Pochhammer symbol that is defined by $(\beta)_n = \dfrac{\Gamma(\beta+n)}{\Gamma(\beta)} = \beta(\beta+1)\cdots(\beta+n-1), \ \beta \in \mathbb{C}, \ n \in \mathbb{N}.$

We now define the following classes of functions.

Let $-1 \leq B < A \leq 1.$ A function $f(z) = z^{-1} + \sum_{n=0}^{\infty} a_n z^n \in \Sigma$ is said to be in the class $T_m(\alpha, A, B)$ if it satisfies the condition

$$\left| \frac{z\left(\Omega_z^{\alpha} f(z)\right)' + \Omega_z^{\alpha} f(z)}{Bz\left(\Omega_z^{\alpha} f(z)\right)' + A\Omega_z^{\alpha} f(z)} \right| < 1 \tag{2}$$

for all $z \in \mathbb{D}$.

Furthermore, a function $f(z) = z^{-1} + \sum_{n=1}^{\infty} |a_n| z^n \in \Sigma$ is said to be in the class $T_m^*(\alpha, A, B)$ if it satisfies the condition (2).

It should be remarked in passing that the definition (2) is motivated essentially by the recent work of Morga [10] and Srivastava and co-authors [12].

In recent years, many important properties and characteristics of various interesting subclasses of the class Σ of meromorphically functions were inverstigated extensively by (among others) Aouf et al [1], Chen et al. [3], Cho and Owa [4], Dziok et al. [5], El-Ashwah and Aouf [6], He et al. [7], Liu and Srivastava [9], Joshi and Srivastava [8] and also [13].

The main object of this paper is to present coefficients estimates, growth and distortion theorems and radii problems of functions in the classes $T_m(\alpha, A, B)$ and $T_m^*(\alpha, A, B)$ which we introduced here.

1. Properties of the class $T_m^*(\alpha, A, B)$

Theorem 1. Let $f(z) = z^{-1} + \sum_{n=1}^{\infty} |a_n| z^n$ be analytic in $\mathbb{D} = \{z : 0 < |z| < 1\}$. Then $f(z) \in T_m^*(\alpha, A, B)$ if and only if

$$\sum_{n=1}^{\infty} \left[(1-A) + n(1-B)\right] \frac{(2)_{n+1}}{(2-\alpha)_{n+1}} |a_n| \le A - B \tag{3}$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z^{-1} + \frac{(2-\alpha)_{n+1}(A-B)}{(2)_n\left[(1-A) + n(1-B)\right]} z^n \qquad (n \ge 1). \tag{4}$$

Proof. Let $f(z) = z^{-1} + \sum_{k=n}^{\infty} |a_n| z^n \in T_m^*(\alpha, A, B)$. Then

$$\left| \frac{z(\Omega_z^\alpha f(z))' + \Omega_z^\alpha f(z)}{Bz(\Omega_z^\alpha f(z))' + A\Omega_z^\alpha f(z)} \right| = \left| \frac{\sum_{n=1}^{\infty} (1+n) \frac{(2)_{n+1}}{(2-\alpha)_{n+1}} |a_n| z^{n+1}}{(A-B) + \sum_{n=1}^{\infty} (A+Bn) \frac{(2)_{n+1}}{(2-\alpha)_{n+1}} |a_n| z^{n+1}} \right|. \tag{5}$$

Since $|\operatorname{Re} z| \le |z|$ for any z, choosing z to be real letting $z \to 1^-$ throuh real values (5) yields

$$\sum_{n=1}^{\infty} (1+n) \frac{(2)_{n+1}}{(2-\alpha)_{n+1}} |a_n| \le (A-B) + \sum_{n=1}^{\infty} (A+Bn) \frac{(2)_{n+1}}{(2-\alpha)_{n+1}} |a_n|,$$

which gives (3).

On the other hand,we have that

$$\left|\frac{z(\Omega_z^\alpha f(z))' + \Omega_z^\alpha f(z)}{Bz(\Omega_z^\alpha f(z))' + A\Omega_z^\alpha f(z)}\right| \le \frac{\sum_{n=1}^{\infty}(1+n)\frac{(2)_{n+1}}{(2-\alpha)_{n+1}}|a_n|}{(A-B)+\sum_{n=1}^{\infty}(A+Bn)\frac{(2)_{n+1}}{(2-\alpha)_{n+1}}|a_n|} < 1.$$

This shows that $f(z) \in T_m^*(\alpha, A, B)$.

Next, we prove the following growth and distortion property for the class $T_m^*(\alpha, A, B)$.

Theorem 2. If $f(z) \in T_m^*(\alpha, A, B),$ then for $0 < |z| = r < 1$

$$\frac{1}{r} - \frac{A-B(2-\alpha)_2}{(2-(A+B))(2)_2} r \le |f(z)| \le \frac{1}{r} + \frac{A-B(2-\alpha)_2}{(2-(A+B))(2)_2} r \tag{6}$$

and

$$\frac{1}{r^2} - \frac{A-B(2-\alpha)_2}{(1-B)(2)_2} \le |f'(z)| \le \frac{1}{r^2} + \frac{A-B(2-\alpha)_2}{(1-B)(2)_2} \tag{7}$$

Proof. Let $f(z) \in T_m^*(\alpha, A, B)$. Then, we find from Theorem 1. that

$$(2-(A+B))\frac{(2)_2}{(2-\alpha)_2}\sum_{n=1}^{\infty}|a_n| \le \sum_{n=1}^{\infty}\left[(1-A)+n(1-B)\right]\frac{(2)_{n+1}}{(2-\alpha)_{n+1}}|a_n| \le A-B$$

which yields

$$\sum_{n=1}^{\infty}|a_n| \le \frac{A-B(2-\alpha)_2}{(2-(A+B))(2)_2}. \tag{8}$$

Also, by applying the triangle inequality, we have

$$|f(z)| = \left|z^{-1} + \sum_{n=0}^{\infty} a_n z^n\right| \le \frac{1}{|z|} + \sum_{n=0}^{\infty}|a_n||z|^n.$$

Since $|z| = r < 1,$ we can see that $r^n \leq r.$ Thus, we have

$$\left| f(z) \right| \leq \frac{1}{r} + r \sum_{n=0}^{\infty} |a_n|$$

and

$$\left| f(z) \right| \geq \frac{1}{r} - r \sum_{n=0}^{\infty} |a_n|.$$

From the inequality (8), we obtain the result of (6).

On the other hand, we get

$$(1-A)\frac{(2)_2}{(2-\alpha)_2}\sum_{n=1}^{\infty}|a_n| + (1-B)\frac{(2)_2}{(2-\alpha)_2}\sum_{n=1}^{\infty}n|a_n| \leq \sum_{n=1}^{\infty}\left[(1-A)+n(1-B)\right]\frac{(2)_{n+1}}{(2-\alpha)_{n+1}}|a_n| \leq A-B$$

and, so from $(1-A)\dfrac{(2)_2}{(2-\alpha)_2} \geq 0$

$$\begin{aligned}(1-B)\frac{(2)_2}{(2-\alpha)_2}\sum_{n=1}^{\infty}n|a_n| &\leq A-B-(1-A)\frac{(2)_2}{(2-\alpha)_2}\sum_{n=1}^{\infty}|a_n|. \\ &\leq A-B\end{aligned}$$

Thus, we have

$$\sum_{n=1}^{\infty} n|a_n| \leq \frac{A-B(2-\alpha)_2}{(1-B)(2)_2}. \tag{9}$$

By applying the triangle inequality, we obtain

$$\left| f'(z) \right| \leq \frac{1}{r^2} + \sum_{n=0}^{\infty} n|a_n|$$

and

$$\left| f'(z) \right| \leq \frac{1}{r^2} - \sum_{n=0}^{\infty} n|a_n|.$$

From the inequality (9), we obtain the result of (7).

Finally, we determine the radius of meromorphically starlikeness and convexity for functions in the class $T_m^*(\alpha, A, B)$.

Theorem 3. Let $f(z) \in T_m^*(\alpha, A, B)$. Then

(i) $f(z)$ is meromorphically starlike of order δ in $|z| < r_1$, that is

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} < -\delta \qquad \left(|z| < r_1\right) \tag{10}$$

where $0 \le \delta < 1$ and

$$r_1 = \inf_{n\ge 1}\left\{\frac{(1-\delta)[(1-A)+n(1-B)]\left(2\right)_{n+1}}{(A-B)(n+\delta)\left(2-\alpha\right)_{n+1}}\right\}^{\frac{1}{n+1}}$$

(ii) $f(z)$ is meromorphically convex of order δ in $|z| < r_2$, that is

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} < -\delta \qquad \left(|z| < r_2\right) \tag{11}$$

where $0 \le \delta < 1$ and

$$r_2 = \inf_{n\ge 1}\left\{\frac{(1-\delta)[(1-A)+n(1-B)]\left(2\right)_{n+1}}{n(A-B)(n+\delta)\left(2-\alpha\right)_{n+1}}\right\}^{\frac{1}{n+1}}.$$

Each of these results is sharp for the function $f(z)$ given by (4).

Proof. (i) From Theorem 1. we have

$$\sum_{n=1}^{\infty}\frac{n+\delta}{(1-\delta)}|a_n||z|^{n+1}<\sum_{k=1}^{\infty}\frac{((1-A)+n(1-B)](2)_{n+1}}{(A-B)(2-\alpha)_{n+1}}|a_n|\leq 1 \quad (|z|<r_1).$$

Therefore for $|z|<r_1$ we have

$$\left|\frac{zf'(z)/f(z)+1}{zf'(z)/f(z)-(1-2\delta)}\right|\leq\frac{\sum_{n=1}^{\infty}(n+1)|a_n||z|^{n+1}}{2(1-\delta)-\sum_{n=1}^{\infty}[n-(1-2\delta)]|a_n||z|^{n+1}}<1$$

which shows that (10) is true

(ii) It follows from Theorem 1. that

$$\sum_{n=1}^{\infty}\frac{n(n+\delta)}{(1-\delta)}|a_n||z|^{n+1}<\sum_{n=1}^{\infty}\frac{((1-A)+n(1-B)](2)_{n+1}}{(A-B)(2-\alpha)_{n+1}}|a_n|\leq 1 \quad (|z|<r_2)$$

Thus for $|z|<r_2$, we obtain

$$\left|\frac{1+zf''(z)/f'(z)+1}{1+zf''(z)/f'(z)-(1-2\delta)}\right|\leq\frac{\sum_{n=1}^{\infty}n(n+1)|a_n||z|^{n+1}}{2(1-\delta)-\sum_{n=1}^{\infty}n[n-(1-2\delta)]|a_n||z|^{n+1}}<1$$

which shows that (11) is true.

Sharpness can be verified easily.

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On The Univalence Criteria For Analytic Functions Defined by Deniz-Özkan Operator

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Abstract

In this study, we obtained some sufficient conditions for univalence of analytic functions defined by Deniz-Özkan differential operatör.

1 Introduction

Let A denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open disk $E = \{z \in \mathbb{C} : |z < 1|\}$.

Let S denote the subclass of A, which consists of functions of the form (1) that are univalent and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$ in E.

In [7], Deniz and Özkan defined the differential operator D_λ^m (say: Deniz-Özkan differential operator) as follows:

For the parametres $\lambda \geq 0$ and $m \in N_0 = N \cup \{0\}$ the differential operatör D_λ^m on A defined by

$$D_\lambda^0 f(z) = f(z)$$

$$D_\lambda^1 f(z) = \lambda z^3 f'''(z) + (2\lambda + 1) z^2 f''(z) + z f'(z) \tag{2}$$

$$D_\lambda^m f(z) = D(D_\lambda^{m-1} f(z)$$

for $z \in U$.

For a function f in A, from the definition of the differential operatör D_λ^m, we can easily see that

$$D_\lambda^m f(z) = z + \sum_{n=2}^{\infty} B_n(\lambda, m) a_n z^n, \tag{3}$$

where $B_n(\lambda, m) = n^{2m}(\lambda(n-1)+1)^m$.

Also, $D_\lambda^m f(z) \in A$. For the special cases of $\lambda = 0,1$ we obtain Salagean differential operator (see [13]).

In this paper we derive sufficient conditions of univalence for the generalized operator $D_\lambda^m f(z)$. Also, a number of known univalent conditions would follow upon specializing the parameters involved. In order to prove our results we need the following Lemmas.

Lemma 1.1 [5] Let $f \in A$.If for all $z \in E$

$$\left(1-|z|^2\right)\left|\frac{zf''(z)}{f'(z)}\right| \leq 1, \tag{4}$$

then the function f is univalent in E.

Lemma 1.2 [10] Let $f \in A$. If for all $z \in E$

$$\left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| \leq 1, \tag{5}$$

then the function f is univalent in E.

Lemma 1.3 [14] Let μ be a real number $\mu > \frac{1}{2}$ and $f \in A$.If for all $z \in E$

$$\left|\left(1-|z|^{2\mu}\right)\frac{zf''(z)}{f'(z)} + 1 - \mu\right| \leq \mu,, \tag{6}$$

then the function f is univalent in E.

Lemma 1.4 [8] If $f \in S$ (the class of univalent functions) and

$$\frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n, \tag{7}$$

then $\sum_{n=1}^{\infty}(n-1)|b_n|^2 \leq 1$.

Lemma 1.5 [11] Let $v \in \mathbb{C}$, $\operatorname{Re}\{v\} \geq 0$ and $f \in A$. If for all $z \in E$

$$\frac{\left(1-|z|^{\operatorname{Re}(v)}\right)}{\operatorname{Re}(v)}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1, \tag{8}$$

then function

$$F_v(z) = \left(v\int_0^z u^{v-1} f'(u)\,du\right)^{\frac{1}{v}}$$

is univalent in E.

2. Main Results

In this section, we establish the sufficient conditions to obtain a univalence for analytic functions involving the differential operator (2).

Theorem 2. 1 Let $f \in A$. If for all $z \in E$

$$\sum_{n=2}^{\infty} B_n(\lambda, m)\left[n(2n-1)\right]|a_n| \leq 1. \tag{9}$$

Then $D_\lambda^m f(z)$ is univalent in E.

Proof. Let $f \in A$. If for all $z \in E$, we have

$$\left(1-|z|^2\right)\left|\frac{z\left(D_\lambda^m f(z)\right)''}{\left(D_\lambda^m f(z)\right)'}\right| \leq \frac{2\sum_{n=2}^{\infty} n(n-1)B_n(\lambda, m)|a_n|}{1-\sum_{n=2}^{\infty} nB_n(\lambda, m)|a_n|}$$

the last inequality is less than 1 if the assertion (9) is hold. Thus in view of Lemma 1.1, $D_\lambda^m f(z)$ is univalent in E.

Theorem 2. 2 Let $f \in A$. If for all $z \in E$

$$\sum_{n=2}^{\infty} B_n(m, \lambda)\left[n(2n-1)\right]|a_n| \leq \frac{1}{\sqrt{7}}. \tag{10}$$

Then $D_\lambda^m f(z)$ is univalent in E.

Let $f \in A$. It suffices to show that

$$\left|\frac{z^2\left(D_\lambda^m f(z)\right)'}{2\left(D_\lambda^m f(z)\right)^2}\right| \leq 1.$$

Now

$$\left|\frac{z^2\left(D_\lambda^m f(z)\right)'}{2\left(D_\lambda^m f(z)\right)^2}\right| \leq \frac{1+\sum_{n=2}^{\infty} nB_n(\lambda,m)|a_n|}{2\left(1-2\sum_{n=2}^{\infty}\left[B_n(\lambda,m)\right]\right)^m |a_n| - \left(\sum_{n=2}^{\infty} B_n(\lambda,m)|a_n|^2\right)}.$$

The last inequality is less than 1 if the assertion (10) is hold. Thus in view of Lemma 1. 2, $D_\lambda^m f(z)$ is univalent in E.

Theorem 2. 3 Let $f \in A$. If for all $z \in E$

$$\sum_{n=2}^{\infty} n\left[2(n-1)+(2\mu-1)\right]B_n(\lambda,m)|a_n| \leq 2\mu-1, \quad \mu > \frac{1}{2}.$$

(11)

then $D_\lambda^m f(z)$ is univalent in E.

Proof. Let $f \in A$. Then for all $z \in E$, we have

$$\left(1-|z|^{2\mu}\right)\left|\frac{z\left(D_\lambda^m f(z)\right)''}{\left(D_\lambda^m f(z)\right)'}+1-\mu\right| \leq \left(1+|z|^2\right)\left|\frac{z\left(D_\lambda^m f(z)\right)''}{\left(D_\lambda^m f(z)\right)'}\right|+|1-\mu|$$

$$\leq \frac{2\sum_{n=2}^{\infty} B_n(\lambda,m)\left[n(n-1)\right]|a_n|}{1-\sum_{n=2}^{\infty} nB_n(\lambda,m)|a_n|}$$

the last inequality is less than μ if the assertion (11) is hold. Thus in view of Lemma 1. 3, $D_\lambda^m f(z)$ is univalent in E.

As applications of Theorems 2. 1, 2. 2 and 2.3, we have the following Theorem.

Theorem 2. 4 Let $f \in A$. If for all $z \in E$ one of the inequality (9-11) holds then

$$\sum_{n=1}^{\infty}(n-1)|b_n|^2 \le 1, \tag{12}$$

where $\dfrac{z}{D_\lambda^m} = 1 + \sum_{n=1}^{\infty} b_n z^n$.

Proof. Let $f \in A$. Then in view of Theorems 2. 1, 2. 2 or 2. 3, $D_\lambda^m f(z)$ is Hence by Lemma 1. 4, we optain the result.

Theorem 2. 5 Let $f \in A$. If for all $z \in E$

$$\sum_{n=2}^{\infty} n\left[2(n-1)+\operatorname{Re}(v)\right] B_m(\lambda,m)|a_n| \le \operatorname{Re}(v), \ \operatorname{Re}(v) > 0. \tag{13}$$

Then

$$G_v(z) = \left(v\int_0^z u^{v-1}\left[D_\lambda^m f(z)\right]' du\right)^{\frac{1}{v}}$$

is univalent in E.

Let $f \in A$. Then for all $z \in E$,

$$\left(\frac{1-|z|^{2\operatorname{Re}(v)}}{\operatorname{Re}(v)}\right)\left|\frac{z\left(D_\lambda^m f(z)\right)''}{\left(D_\lambda^m f(z)\right)'}\right| \le \left(\frac{1+|z|^{2\operatorname{Re}(v)}}{\operatorname{Re}(v)}\right)\left|\frac{z\left(D_\lambda^m f(z)\right)''}{\left(D_\lambda^m f(z)\right)'}\right|$$

$$\le \frac{2}{\operatorname{Re}(v)} \frac{\sum_{n=2}^{\infty} n(n-1) B_n(\lambda,m)|a_n|}{1-\sum_{n=2}^{\infty} n B_n(\lambda,m)|a_n|}$$

the last inequality is less than 1 if the assertion (13) is hold. Thus in view of Lemma 1. 5, $G_v(v)$ is univalent in E.

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One of the Special Type of D(2) Diophantine Pairs

(Extendibility of Them and Their Properties)

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Abstract

Diophantus (who was widely influenced by Mesopotamian mathematics) from Alexandria is defined as the father of algebra. Algebraic equations have been in the Diophantus's book Arithmetica (included 130 equivalents and their solutions) which is first book on the theory of numbers. Diophantus Equations, also known as Diophantine equations, was named on Diophantus after A.D. third century. After him, this famous theory has been worked to extend and generalize more by a lot of mathematicians.

The aim of this work is to get some new results on the special type of D(2) Diophantine pairs ({2, a} where a is a positive integer). Many significant and useful results are obtained while working on the problem but some of them are shown in this work. Firstly, it is demonstrated that such type of pairs can not be extended to D(2) Diophantine quadruple even they are regular D(2) Diophantine 3-tuples. These results are demonstrated by using solvability of the pell equations. Also, numerical results are given to support our obtained implications (as we explained above) in this work too.

Keywords: D(n) Diophantine m-Tuples, Pell Equations, Natural Numbers, Regularity of D(n) Diophantine Triples, Iteration, Primes

1. Introduction

The subject "Diophantine m-tuples" (also called as D(1)-sets or simply Diophantine m-tuples) was found out by Diophantus of Alexandria roughly five centuries after Euclid's era. He didn't just consider integer numbers of simply Diophantine m-tuples but also he worked on rational numbers for quadruples. This problem starts with Diophantine m-tuples with property D(1) and obtained many results on Diophantine D(1)-triples, Diophantine D(1)-quadruples, Diophantine D(1)—quintuples, Diophantine D(1)-sextuple, etc… even still some open problems in this topic with different perspectives, techniques and approximations such as elliptic curves and their ranks, quadratic number fields, linear form and logarithms, etc.

There are some useful and basic results on D(1)-sets as follows:

- Diophantine D(1) pair $\{u, v\}$ is extendable to Diophantine D(1) triple if $u + v + 2t$ is in the D(1)-set for $t^2 = u.v + 1$.
- Diophantine D(1) triple $\{u, v, w\}$ is extendable to Diophantine D(1) quadruple as $\{u, v, w,\ d_+\}$ and $\{u, v, w,\ d_-\}$ where equations $t^2 = u.v + 1$, $r^2 = u.w + 1$, $s^2 = w.v + 1$ $(r, s, t \in Z^+)$ are satisfied for $d_\mp = u + v + w + 2uvw \mp 2rst$ and $d_- \neq 0$.
- Numbers of Diophantine D(1) quintuples are most finitely many.
- There aren't any Diophantine D(1) sextuples.
- All Diophantine D(1) quadruples are regular (conjecture)

  … so on

It is seen that above mentioned results were/are demonstrated for integers. It is also known that the subject "Diophantine m-tuples" is considered on commutative rings such as the set of rational numbers, the set of Gaussian integers and more…

This theory still has unproved traditional problems such as either finite or infinite solutions, preparing an easy algorıthm to calculate solutions in practice, so on for centuries. Since the descriptions are not simple, these problems have been indicated to Diophantine sets. Main problem is returned to classify and characterize D(n) Diophantine m-tuples. Immensely significant and useful results have been proved by using a lot of different methods such as hypergeometric method, linear forms in logarithms, elliptic curves, binary quadratic forms etc. in recent years.

The purpose of this work is to mention some useful and basic notations firstly. Then, It is to give some numerical calculations and theories (to generalize results in our next study) for certain types of pairs of the Diophantine D(2) sets. It is also used number theoreticals' techniques in the work. One may consider other types of D(2) or D(*n*) for natural number *n*.

2. Preliminaries

Definition. A Diophantine *m*-tuples with the property *D*(*r*) for an integer *r* is an *m*-tuples of distinct positive integers $\{c_1, \dots, c_m\}$ such that $c_i c_j + r$ is always a square of an integer for every $i \neq j$.

Specifically, it is called Diophantine *D*(*r*) – triple for *m=3*, Diophantine *D*(*r*) – quadruple for *m=4*.

Definition. If Diophantine $D(r)$- triple $\{a, b, c\}$ satisfies the following condition

$$(c - b - a)^2 = 4(a.b + r)$$

then, Diophantine $D(r)$- triple is named as regular.

Definition. (Quadratic Residue) Assume that q be an odd prime, $b \not\equiv 0 \pmod{q}$. We call that b is a quadratic residue $\text{modulo } q$ if a nonzero number b is a square $\text{modulo } q$. Otherwise, it is defined as quadratic non-residue.

Definition. (Legendre Symbol) Legendre symbol is defined as following equation for prime q:

$$\left(\frac{a}{q}\right) = \begin{cases} 1 & \text{if } a \text{ is a quadratic residue mod } q \\ -1 & \text{if } a \text{ is a quadratic non} - \text{residue mod } q \end{cases}$$

Theorem. (Quadratic Reciprocity Theorem and Quadratic Reciprocity Law) If x, y are distinct odd primes, then we have following result.

$$\left(\frac{x}{y}\right)\left(\frac{y}{x}\right) = (-1)^{\frac{(x-1)}{2}\cdot\frac{(y-1)}{2}}$$

Also, let *a*,*b* be odd numbers such that they are coprimes. Followings are satisfied.

$$\left(\frac{2}{a}\right) = (-1)^{\frac{a^2-1}{8}} \quad \text{and} \quad \left(\frac{a}{b}\right)\left(\frac{b}{a}\right) = (-1)^{\frac{(a-1)}{2}\cdot\frac{(b-1)}{2}}$$

3. Main Results

Used Technique:

In this work, the techniques and notations such as quadratic congruence's solutions, quadratic reciprocity theorem, Legendre symbol, etc are used from number theory. For solving Diophantine equations $ax^2+bxy+cy^2=n$ (where n is non-zero, b is zero, a is positive integer and discriminant is greater than zero and nonsquare), "Determination of Stolt fundamental solutions method" (found out by Bengt Stolt) is used. Readers can be found more information about this method in the paper "On fundamental solutions of binary quadratic form equations" (ref [21]).

3.Main Results

In this work we consider a type of Diophantine D(2) pairs which are started with number two (2) and continues with positive integers. Even we consider this type of diophantine D(2) sets, there are many (finitely number) different kinds of Diophantine D(2) pairs as follows too:

$$\{1,2\}\ ,\ \{1,7\}\ ,\{1,14\}\ldots\ldots\ldots,\{2,7\}\ ,\ \{2,17\}\ ,\{2,31\}\ \ldots\ldots$$

$$\{7,14\}\ ,\{7,17\}\quad,\{7,41\}\ldots\ \ldots\ \ldots,\{14,\ 23\},\ \{14,41\},\{14,73\}\ldots\ldots$$

$$\ldots\ \ldots\ \ldots\ \ \ldots\ \ \ldots$$

If we investigate Diophantine D(2) sets (pairs, triples, quadruples, quintuples or more), it is easily seen that some positive integers have not been in such sets. There are a lot of integers not belong to Diophantine D(2) sets. In the following theorem, we demonstrate some of them with theoritical proofs.

Note: Following theorem can be extended for other numbers as well

Theorem 1. Following statements are true and satisfied.

- **i.** D(2) Diophantine sets don't contain numbers that include the number 3 or any multiple of 3.
- **ii.** D(2) Diophantine sets do not have numbers containing the number 4 or any multiple of 4.
- **iii.** D(2) Diophantine sets have not got numbers including the number 5 or any multiple of 5.
- **iv.** ….

Proof.

i. Assuming that $3n\ (n \in \mathbb{Z}^+)$ and $k\ (k \in \mathbb{Z}^+)$ be in the Diophantine D(2) sets. Then following equation has to be satisfied for some integers $\mathcal{A}$.

$$3nk + 2 = \mathcal{A}^2$$

If we deduce $3nk + 2 = \mathcal{A}^2$ in (mod 3), then we get

$$\mathcal{A}^2 \equiv 2(mod\ 3)$$

To see solvability of quadratic congruence $\mathcal{A}^2 \equiv 2(mod\ 3)$, we have to consider residue classes of (mod 3) (i.e. $\{0,1,2\}$) and evaluate whether or not these numbers satisfy the $\mathcal{A}^2 \equiv 2(mod\ 3)$. If we put $\{0,1,2\}$ into the $\mathcal{A}^2 \equiv 2(mod\ 3)$ respectively, then we have a contradiction.

Hence , there isn't any Diophantine $D(2)$ set contain numbers that include the numer 3 or any multiple of 3.

ii. Supposing that t $(t \in \mathbb{Z}^+)$ and $5u$ $(u \in \mathbb{Z}^+)$ be integers in the Diophantine $D(2)$ set. So, we obtain

$$5tu + 2 = \mathcal{B}^2$$

for at least an integer $\mathcal{B}$. Similarly, we have

$$\mathcal{B}^2 \equiv 2(mod\ 5)$$

Using Legendre symbol and its properties, we get

$$\left(\frac{2}{5}\right) = (-1)^{5^2-1/8} = -1$$

It implies that there isn't any $\mathcal{B}$ integer holds quadratic congruence $\mathcal{B}^2 \equiv 2(mod\ 5)$. Therefore, there is no Diophantine $D(2)$ sets include numbers 5 or multiple of 5.

iii. Proof is left for readers.

Theorem 2. The set $\{2, 7\}$ is a Diophantine D(2) pair and extendable to a Diophantine D(2) triple.

Proof. First of all show whether or not $\{2, 7\}$ is Diophantine D(2) pair. If definition of the Diophantine D(2) set is applied to $\{2, 7\}$, then we have $2.7 + 2 = (\pm 4)^2$. It implies that $\{2, 7\}$ is a Diophantine D(2) pair.

Assuming that d be the smallest positive integer that makes $\{2, 7, d\}$ Diophantine D(2) triple. Then, following equations are satisfied for some X, Y integers.

$$2d + 2 = X^2 \qquad \text{and} \qquad 7d + 2 = Y^2$$

It is seen that $d = 17$ is the smallest positive integer so that it makes X and Y values integers in the given equations. So, Diophantine $\{2, 7\}$ pair can be extended to Diophantine D(2) triple as $\{2, 7, 17\}$.

Theorem 3. The set $\{2, 7, 17\}$ is regular Diophantine D(2) triple but can not be extended to Diophantine D(2) quadruple.

Proof. By using regularity condition determined for Diophantine D(n) triples, it is easily seen that $\{2, 7, 17\}$ is a Diophantine D(2) regular triple.

Supposing that $\{2, 7, 17\}$ can be extended for any positive integer w so that $\{2, 7, 17, w\}$ is a Diophantine $D(2)$ quadruple. Then, there exist X, Y, Z integers such that;

$$2w + 2 = X^2$$

$$7w + 2 = Y^2$$

$$17w + 2 = Z^2$$

If we simplify and edit these equations, then we obtain following Diophantine equations.

$$7X^2 - 2Y^2 = 10$$

$$17X^2 - 2Z^2 = 30$$

$$17Y^2 - 7Z^2 = 20.$$

We can give fundamental solutions of them using Stolt fundamental solution technique as we mentioned above.

For $7X^2 - 2Y^2 = 10$, we get Pell equation $x^2 - 56Y^2 = 280$ using the transformation $x = 14X,\ y = Y$. It is also obtained that there are two Stolt fundamental solutions of Pell equation as $(28, 3)$ and $(-28, 3)$. Using them, we get two (2) fundamental solutions for $7X^2 - 2Y^2 = 10$ as $(2, 3)$ and $(-2, 3)$.

In the same manner;

For $17X^2 - 2Z^2 = 30$, we get Pell equation $x^2 - 136Z^2 = 2040$ using the transformation $x = 34X,\ z = Z$. Then, we get two Stolt fundamental solutions of Pell equation as $(136, 11)$ and $(-136, 11)$. Considering them, we obtain two (2) fundamental solutions for $17X^2 - 2Z^2 = 30$ as $(4, 11)$ and $(-4, 11)$.

For $17Y^2 - 7Z^2 = 20$, we have Pell equation $y^2 - 476Z^2 = 1360$ applying the transformation $y = 34Y,\ z = Z$. Then, we get two Stolt fundamental solutions of Pell equation as $(136, 6)$ and $(-136, 6)$.So, we obtain two (2) fundamental solutions for $17Y^2 - 7Z^2 = 20$ as $(4, 6)$ and $(-4, 6)$ by using them.

If we consider the above solutions and write them interchangeably in the equations, it is seen that there is no common solution that makes X, Y and Z be integers at the same time.

Hence, there isn't any such positive integer w. Therefore, $\{2, 7, 17\}$ can not be extended to Diophantine D(2) quadruple.

Theorem 4. The set $\{2, 17\}$ is a Diophantine D(2) pair. Also, it is extendable to a Diophantine D(2) triple.

Proof. Let us consider $\{2, 17\}$. Is it Diophantine D(2) pair? . From the definition of the Diophantine D(2) set, then we obtain $2.17 + 2 = (\pm 6)^2$. It gives us that $\{2, 17\}$ is a Diophantine D(2) pair.

Supposing that e be the smallest positive integer that makes $\{2, 7, e\}$ Diophantine D(2) triple. Then, following equations are satisfied for some X, Y integers.

$$2e + 2 = A^2 \qquad \text{and} \qquad 17e + 2 = B^2.$$

We can easily see that $e = 31$ is the smallest positive integer so that it makes A and B values integers in the given equations. Hence, Diophantine $\{2, 17\}$ pair is extendable to Diophantine D(2) triple as $\{2, 17, 31\}$.

Theorem 5. The set $\{2, 17, 31\}$ is a regular Diophantine D(2) 3-tuples even it isn't extended to Diophantine D(2) 4-tuples.

Proof. From the regularity condition mentioned in preliminaries, it is obtained that $\{2, 17, 31\}$ is a Diophantine D(2) regular triple.

Supposing that $\{2, 17, 31\}$ can be extended for any positive integer ∂ so that $\{2, 17, 31, \partial\}$ is a $D(2)$ Diophantine quadruple. So, there are A, B, C integers such that;

$$2\partial + 2 = A$$

$$17\partial + 2 = B^2$$

$$31\partial + 2 = C^2$$

By dropping ∂, following Diophantine equations are obtained.

$$17A^2 - 2B^2 = 30$$

$$31A^2 - 2C^2 = 58$$

$$31B^2 - 17C^2 = 28$$

By using Stolt fundamental solution technique, we have results as follows:

For $17A^2 - 2B^2 = 30$, we get Pell equation $a^2 - 136B^2 = 2040$ using the transformation $a = 34A$, $b = B$. It is also obtained that there are two Stolt fundamental solutions of Pell equation as $(136, 11)$ and $(-136, 11)$. Using them, we have two fundamental solutions for $17A^2 - 2B^2 = 30$ as $(4, 11)$ and $(-4, 11)$.

Similarly;

For $31A^2 - 2C^2 = 58$, Pell equation $a^2 - 248C^2 = 7192$ using the transformation $a = 62A$, $c = C$. Then, we have two Stolt fundamental solutions of Pell equation as $(372, 23)$ and $(-372, 23)$. Using them, we obtain two (2) fundamental solutions for $31A^2 - 2C^2 = 58$ as $(6, 23)$ and $(-6, 23)$.

For $31B^2 - 17C^2 = 28$, we obtain two fundamental solutions as $(6, 8)$ and $(-6, 8)$.

If we consider the above solutions and write them interchangeably in the equations, it is obtained that there is no common solution that makes A, B and C be integers at the same time.

So, there is no such positive integer ∂. Therefore, $\{2,17,31\}$ is not extended to Diophantine D(2) quadruple.

Theorem 6. The set $\{2,31\}$ is a diophantine D(2) pair extendable to a Diophantine D(2) triple as $\{2,31,49\}$.

Proof. In a similar way of the proofs of Theorem 2 and Theorem 4, it is easily to got result. So, proof is left for readers.

Theorem 7. The set $\{2,31,49\}$ is a regular Diophantine D(2) triple although it is not extendable to Diophantine D(2) quadruple.

Proof. Using similarity of previous proofs, proof of the Theorem 7 can be easily obtained by readers. Hence, proof is left for readers.

5. Conclusion

In this study, it is proven that a special type of Diophantine D(2) pairs (i.e. ($\{2, a\}$ where a is a positive integer) can be extended to Diophantine D(2) regular triples but not extendable to Diophantine D(2) quadruple. This work gives us a new idea to get more general results on such pairs for our next work and it forms the basis for our next study. Therefore, all of them will be contributed more to the literature of this subject. We also kindly recommend readers to follow the continuation of this work.

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Quantile regression neural network models in the description of biological networks with outliers observations

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Abstract

The quantile regression is one of the well-known regression methods to describe data including outliers when the conditions of ordinary linear regression are not met such as linearity, homoscedasticity or normality assumptions. Then, this idea has been recently combined by the neural network model, which is a popular machine learning method applied in many fields, and later, the quantile regression neural network model has been developed to use the advantage of both popular approaches. Hereby, in this study, we implement this novel approach in the representation of complex biological network which includes outliers. In our analyses, we generate outliers under different dimensions and topologies. Later, we evaluate the performance of suggested approach in generated biological networks. Finally, we assess the results by accuracy and sensitivity measures.

Keywords: Quantile regression, Outlier detection, Artificial neural networks, Biological networks.

1. Introduction

The quantile regression is a very important modelling tool to describe data having anomalies via specific loss function. Briefly, this regression presents the relationship between a set of predictor (independent) variables and particular percentiles or "quantiles" of a dependent variable, generally taken as the median. This description can be challenging when the number of parameter is greater than the number of observations values. More recently, quantile regression neural networks (QRNN) has been developed and applied environmental sciences (Cannon, 2011; Cannon, 2018), but, this model has not been yet implemented for the construction of biological networks. Hence, in this work, we consider quantile regression neural networks (QRNN) in the description of biological networks which includes outliers in observations.

Artificial neural network is a very important mathematical expression that enables us to presents the estimated data more accurately (McCulloch and Walter Pitts, 1943). Basically, this models creates a computational model for neural networks based on algorithm, called, the threshold logic. However, more recently, nonparametric regression models under Generalized Additive Models (GAM) such as multivariate adaptive regression splines (MARS) are applied in place of the artificial neural networks as performed in the studies of Schmidt-Hieber (2020) and Bauer and Kohler (2019). Although these complex non-parametric regression models are successful in the presentation of various datasets, they have some limitations when data have outlier values. To solve this problem, Cannon (2011, 2018) has proposed quantile regression neural network. This model can detect outlier values via regression and can perform computationally faster regression procedure. In this regression, the conditional median has a crucial role to minimize the mean absolute error in the estimation and to predict the data. For this reason, asymmetric weights via positive/negative errors are used by controlling the conditional median under distinct shape of data distribution.

Biological networks are one of modelling fields when the distribution of data is very different and its structure is nonlinear. Protein-protein interaction (PPI) networks are special types of networks in biological systems. More recently, Ayyıldız et al. (2018), Kaygusuz et al. (2021) and Seçilmiş et al. (2022) have suggested loop based multivariate adaptive regression spline, neural networks and random forest, respectively, to present underlying systems. Kaygusuz and Purutçuoğlu (2022a, 2022b) have proposed bootstrap procedure in order to improve the performance of models under PPI systems under various model selection criteria. In this work, we suggest quantile regression neural networks for biological networks while data have outlier values.

Accordingly, the organization of the paper is as follows. In the second section, we define quantile regression and in the third section we give the definition of quantile regression neural networks. The fourth section includes data analysis for the proposed model. Lastly, we represent the fifth section for the conclusion and future work.

2.Quantile regression

Quantile regression models the relationship between a set of predictor (independent) variables and specific percentiles or "quantiles" of a dependent variable, most often the median. The aim of quantile regression is to estimate a given quantile such as the median of Y conditional on X. In other words, the conditional distribution function Y given X=x is found as follows:

$$F(Y|X=x) = P(Y<y|X=x) \tag{1}$$

while τ -th conditional quantile function

$$q_{\tau}(x) = \inf(\ Y \ \in \ R \ ; F(Y|X=x) > \tau). \tag{2}$$

Traditional regression models estimate the conditional mean of the dependent variable Y_{n+1} given the features $X_{n+1} = X$ by minimizing the sum of squared residuals on the *n* training points so that the mean μ under estimated model parameter can be found as below:

$$\hat{\mu}\ (x) = \mu(x,\ \hat{\theta}\ ) \text{ and } \hat{\theta}\ = argmin\ \frac{1}{n}\sum_{i=1}^{n}\ \ (\ Y_i - \mu(\ X_i\,;\theta\ ))^2\ . \tag{3}$$

Here, θ is the estimator of regression model and $\mu(x;\theta\ )$ is te estimator of the regression function when *argmin*(.) denotes the set of values where the function gives the minimum. Thus we can calculate a conditional quantile function $q_{\tau}(x)$ of Y_{n+1} given $X_{n+1} = X$ via

$\hat{q}\ (x) = f(x,\ \hat{\theta}\ )$ while $\hat{\theta}\ = argmin\ \frac{1}{n}\sum_{i=1}^{n}\ \ \rho_{\tau}(\ Y_i\ - \mu(\ \ X_i\,; \theta\ ))^2$ (4)

in which $f(x,\ \hat{\theta}\ )$ is regression function and ρ_{τ} is a loss function which is defined as below.

$$\rho_{\tau}\ (u)=\begin{cases} \tau u & if\ u > 1 \\ (1-\tau)u & if\ u < 1 \end{cases}. \quad (5)$$

In Equation 5, τ lies on $0 < \tau < 1$, So, if *x(t)* shows an independent variable, b and m_i present the intercept and the slope, respectively, we can define the quantile regression by

$$y(t)=f\,(\textstyle\sum_{i=1}^{l}\ m_i\, x_{\,i}\,(t)\ + b). \quad (6)$$

Hence, the quantile regression error function (E) can be computed by

$$E = 1/N(\textstyle\sum_{t=1}^{N}\ \ \rho\ (y\ (t)\text{-}y(t)) \quad (7)$$

for N observational datasets.

3. Quantile regression neural networks (QRNN)

Artificial neural networks have a lot of attention due to its success in applicational area such as signal progressing, machine learning and bioinformatics (Goodfellow et al., 2016; Lecun et al., 2015). On the other hand, in this work, we study classical artificial neural network for the quantile regression. The quantile regression estimation process starts with the central median case in which the median regressor estimator minimizes a sum of absolute errors, as opposed to ordinay least square (OLS) approach that minimizes the sum of squared errors. Hence, Quantile Regression Neural Network (QRNN) combines quantile regression (QR), which can model data with non-homogeneous variance, with neural network (NN) approach, which can capture nonlinear patterns in the data succesfully.

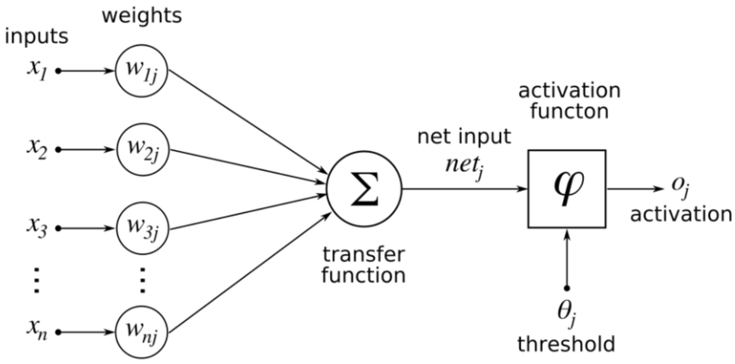

Figure 1 An example of the architecture of artificial neural network with one hidden layer.

In QRNN, if *x(t)* denotes the input vector and *y(t)* is the output vector, the output from the *j*th hidden layer, i.e., *g(t)* vector, can have a hyperbolic tangent (tanh) activation function as shown in Equation 8. Indeed, in the calculation, other functions can be also selected such as sigmoid or rectified linear activation functions.

$$g_j(t) = \tanh\left(\sum_{i=1}^{l} x_i(t)\, w_{ij}^{(h)} + b_j^{(h)}\right). \tag{8}$$

in which $w_{ij}^{(h)}$ is hidden layers weights vector and $b_j^{(h)}$ is a hidden layer bias vector. So, the estimated output layer of transfer function has the following form.

$$f_i^{(1)}(x) = f\left(\sum_{j=1}^{d} g_j(t)\, w_j^{(0)} + b^{(0)}\right), \tag{9}$$

where $w_j^{(0)}$ represents the output layer vector and $b^{(0)}$ is the output layer bias vector.

Accordingly, in modeling via QRNN, we construct a regression model per protein separately in such a way that each protein is taken as response in the QRNN and the remaining proteins are used for predictors of the selected response. Then, the regression coefficients are computed. These coefficient values are the weigths in the neural network. Finally, we repeat this process for every protein in the system sequentially so that the total number of regression models can be equal to the total number of proteins in the system. In this network construction, we interpret the regression coefficients as the strength of the interaction between proteins while each response protein can autoregulate as well. Hereby, we compute our models with an intercept term. In the assessment of the model, we consider that coefficients having values greater than 0.1 in absolute value can be taken as zero, otherwise, they can set to one. By this way, we can convert the estimated coefficients as binary value so that the generated adjacency matrix can be compared with the true adjacency matrix of the network.

4. Data analyses

4.1. Description of the data

In the application of QRNN in PPI networks, we use four different simulated datasets. We generate data under the following scenerious:

a) We generate system with 12 and 24 proteins where each protein has 100 observations. In the first 12-protein system, we simulate multivariate normally distributed data whose 10 proteins have random topology and 2 proteins have scale-free topology. In the second dataset, we have 20 proteins, which are random, and 4 proteins, which are sclae-free. By this way we aim to generate 2 systems having outliers under mixture distribution and outliers ratios are not greather than % 25 of the data.
b) We simulate mixture data with the above scenerios, but, with the following composition: 10 proteins from scale-free network and 2 proteins from random network. And in the fourth data set, we take 20

proteins from scale-free network and 4 proteins from random network. By this way, we can generate 2 more data sets whose dimensions are 12 and 24, respectively.

4.2. Results

The regression coefficients of quantile regression neural networks from different scales of biological data such as random network and scale-free network are presented in Tables 1, 2, 3 and 4 while the observations of networks are 100 and the numbers of variable 12 and 24 as described in the previous section. For the first two datasets, we estimate the intercept via the notation of β_0 and estimated coefficients of 11 independent variables via the notations of β_i $(i = 1, 2, \dots, 10)$ for totally 11 cases. Here, for 12-protein system, since we consider each protein as a responce once, we can use 11 predictors for each protein. On the other side, since we repeat this process sequentially for each protein in the system as described in previous section, we construct 11 different models in the end. The underlying distinct models are shown as **case i (i=1, 2, ..., 11)** in Tables 1 and 2. Similarly, for the second data set, we estimate an intercept (β_0) as before and coeffcients of 23 indepedent variables for totally 24-protein systems. Finally, as we construct the model for each protein in the system separately, indeed, we generate 23 distinct models, i.e., **case j (j=1, 2, ..., 23)**, in Tables 3 and 4. In all these tabulated values, τ value in QRNN is taken 0.5 which implies the median.

Coeff.	**case1**	**case2**	**case3**	**case4**	**case5**	**case6**	**case7**	**case8**	**case9**	**case10**	**case11**
$\boldsymbol{\beta_0}$	0.135	-0.392	-1.096	-1.417	-0.255	-0.037	0.148	-0.751	0.094	-0.393	-0.254
$\boldsymbol{\beta_1}$	0.424	-0.169	1.3402	-0.279	0.376	-0.562	-0.198	0.353	0.260	-0.169	0.376
$\boldsymbol{\beta_2}$	0.739	0.184	-0.995	-0.473	-0.267	0.129	0.388	-0.638	-0.057	0.183	-0.267
$\boldsymbol{\beta_3}$	0.107	-1.038	-0.128	-0.983	-0.685	-0.057	-0.155	0.175	0.021	-1.038	-0.685
$\boldsymbol{\beta_4}$	-0.093	0.333	-0.186	-0.368	-1.102	-0.181	1.938	-5.075	-0.259	0.333	-1.102
$\boldsymbol{\beta_5}$	0.141	-0.396	1.078	-0.333	-0.424	-0.946	-0.135	1.711	-1.123	-0.396	-0.423
$\boldsymbol{\beta_6}$	0.110	0.250	-0.820	-0.271	-0.317	0.224	-0.867	0.282	-0.559	0.250	-0.317
$\boldsymbol{\beta_7}$	0.249	0.268	-0.757	-0.499	-0.057	0.371	0.649	-1.551	-0.115	0.268	-0.057
$\boldsymbol{\beta_8}$	-0.041	0.326	-1.669	-0.392	-0.763	0.109	-0.326	-0.693	-0.458	0.326	-0.763
$\boldsymbol{\beta_9}$	-0.061	-0.119	-0.849	0.260	-0.117	-0.087	0.008	-0.903	-0.239	-0.119	-0.117
$\boldsymbol{\beta_{10}}$	0.258	0.145	-0.693	-0.171	-0.127	0.061	0.259	-0.320	0.228	0.145	-0.127
$\boldsymbol{\beta_{11}}$	-0.580	0.120	0.039	-0.048	-0.871	-0.549	-0.411	0.182	-0.170	0.120	-0.870

Table 1 Regression coefficients of QRNN for scale-free structural 10 proteins and random network structural 2 proteins while the sample size per protein is n=100.

Coeff.	case1	case2	case3	case4	case5	case6	case7	case8	case9	case10	case11
β_0	0.228	0.777	-1.500	-1.036	-0.063	-0.220	-0.205	0.067	0.442	0.212	-0.254
β_1	0.539	1.34	-0.525	0.136	0.249	0.341	0.030	-0.106	0.094	-0.943	0.376
β_2	0.888	-0.608	-1.046	-0.170	0.762	0.356	-0.142	0.730	-1.127	-0.555	-0.267
β_3	-0.112	-0.599	-0.641	-0.275	0.063	-1.235	-0.057	0.840	0.398	1.824	-0.685
β_4	-0.139	1.121	-0.775	0.275	-0.09	0.302	-0.499	1.120	-1.763	0.172	-1.102
β_5	0.753	-0.377	0.344	-0.91	-0.17	0.044	-0.835	0.115	-0.925	-0.460	-0.423
β_6	0.174	-0.378	0.043	-0.366	0.356	-1.336	-1.209	0.285	0.632	0.901	-0.317
β_7	-0.702	-0.930	-1.167	-0.541	-0.335	0.061	-0.120	0.420	0.067	0.705	-0.057
β_8	-0.585	-0.786	-0.661	-0.182	0.786	0.181	-0.439	0.491	-0.765	0.688	-0.763
β_9	0.095	0.750	-0.403	0.372	0.367	0.173	-0.160	-0.004	-0.929	-0.805	-0.117
β_{10}	0.254	0.437	-0.072	-0.067	-0.17	-0.284	-0.274	-0.165	0.907	0.018	-0.127
β_{11}	0.004	-0.140	-0.378	0.53	0.058	0.138	0.031	-0.198	-0.020	-0.322	-0.870

Table 2 Regression coefficients of QRNN for random network structural 10 proteins and scale-free structural 2 proteins while the sample size per protein is n=100.

weights	case1	case2	case3	case4	case5	case6	case7	case8	case9	case10	case11
β_0	1.006	-1.69	1.006	-1.405	-0.343	0.797	-0.056	0.193	-0.280	2.010	-1.871
β_1	0.006	-0.19	0.006	0.150	1.204	-0.072	-2.097	-0.356	-0.078	1.303	-0.840
β_2	1.131	1.105	1.131	0.212	0.462	-0.552	-1.139	1.470	0.239	-1.883	0.238
β_3	0.576	-0.186	0.576	-0.167	-0.652	-0.144	-0.700	-0.286	2.478	2.842	-0.215
β_4	-0.116	0.839	-0.117	-0.58	-0.499	-0.698	1.942	0.727	-1.020	0.442	1.517
β_5	-0.016	-1.068	-0.016	-0.56	1.236	-0.824	-0.019	-0.026	-0.233	-2.409	1.176
β_6	-0.102	0.142	-0.10	0.430	-0.801	0.733	0.024	0.126	0.248	-0.877	-1.048
β_7	-0.068	-0.107	-0.068	-0.715	0.402	0.044	-0.241	0.881	0.799	0.552	0.005
β_8	0.134	-0.034	0.134	-0.923	0.265	2.023	-0.757	0.205	0.582	-0.366	-1.118
β_9	0.100	0.015	0.100	0.324	-1.75	-1.859	1.694	-0.077	-1.078	-0.840	-1.006
β_{10}	0.624	0.211	0.625	0.646	0.084	-0.528	1.099	1.059	0.371	0.397	0.604
β_{11}	0.155	-0.597	0.155	-1.10	-0.062	-0.415	-0.377	0.146	0.886	-0.573	-0.543
β_{12}	-0.114	0.135	-1.114	-1.218	-1.273	-0.220	0.486	0.492	0.408	-2.123	1.214
β_{13}	0.996	-0.41	-0.996	0.556	0.710	-1.350	0.024	0.014	-1.715	1.598	-0.417
β_{14}	0.141	-0.218	0.141	-0.691	-0.85	0.838	-2.403	0.560	0.606	-0.987	0.245
β_{15}	0.888	-0.366	0.888	-0.189	0.041	-1.342	0.455	-0.409	1.257	0.455	-0.614
β_{16}	-0.215	-0.860	-0.216	0.178	0.035	-0.011	0.214	0.344	0.033	-0.269	0.864
β_{17}	0.434	-0.796	0.434	-0.438	-0.237	0.061	0.093	-0.145	0.101	1.530	0.191
β_{18}	-0.400	-0.427	-0.39	0.159	-0.369	0.611	1.163	0.550	0.076	0.987	2.554
β_{19}	0.108	0.370	1.108	0.004	-0.103	0.058	1.345	0.231	1.259	0.202	0.245
β_{20}	0.126	-0.369	0.126	0.228	0.105	-0.573	-0.136	-0.749	-2.138	1.248	-1,396
β_{21}	-0.412	0.304	-0.412	-0.17	0.181	0.317	0.132	0.953	1.000	-0.247	-0.301
β_{22}	0.157	0.179	0.156	-0.172	-0.051	0.265	0.126	-0.171	-0.044	0.184	0.215
β_{23}	-0.048	-0.205	-0.048	-0.11	-0.051	-0.043	-0.263	0.032	-0.041	-0.299	-0.016

Coeff.	case12	case13	case14	case15	case16	case17	case18	case19	case20	case21	case22
β_0	-1.785	-0.896	0.775	-1.994	-0.393	-1.768	0.687	0.437	-0.319	-1.135	-0.349
β_1	-0.521	0.880	0.926	-1.148	1.126	-0.514	0.895	2.332	-0.922	-0.509	-0.278
β_2	0.923	-0.199	-1.785	-0.347	1.365	3.725	1.138	-0.757	0.339	0.145	0.549
β_3	1.146	-0.826	-0.030	1.141	0.233	-0.029	-0.011	0.194	0.705	-0.131	0.131
β_4	-1.082	0.639	-0.692	0.219	-0.125	1.652	0.189	0.566	0.312	0.920	0.275
β_5	1.183	0.139	-0.976	0.206	0.324	-1.259	-0.311	-1.198	-0.063	0.713	-0.409
β_6	1.866	-0.685	-0.055	0.537	1.859	-0.292	0.339	-0.218	0.007	-0.635	-0.157
β_7	-1.123	-0.149	-0.863	0.502	0.263	1.154	1.270	-0.975	0.492	0.003	0.307
β_8	-0.253	-0.008	-0.760	1.024	0.922	-1.825	0.020	-0.458	0.210	-0.677	0.603
β_9	-0.591	0.184	0.959	-0.042	0.040	-1.023	-0.375	0.295	0.065	-0.610	-0.002
β_{10}	-1.664	0.035	-1.017	0.475	-0.384	1.778	0.311	1.327	1.745	0.366	0.541
β_{11}	3.339	0.627	-1.571	0.125	-0.011	-1.620	-0.341	-0.990	0.262	-0.329	0.863
β_{12}	-2.193	0.326	-0.462	1.066	0.194	0.819	0.655	0.971	1.175	0.736	-0.479
β_{13}	1,151	0.179	1.657	0.856	-0.081	1.771	-1.007	0.366	0.802	-0.253	0.728
β_{14}	-0.671	-1.022	-0.134	0.706	0.322	0.367	2.667	-0.434	-0.689	0.148	-1.300
β_{15}	-1.007	0.654	0.724	-0.033	-0.688	-0.831	2.328	-0.297	0.517	-0.373	0.672
β_{16}	-0.316	-0.182	0.558	-0.719	2.274	-0.680	0.188	-1.569	-0.003	0.524	0.283
β_{17}	0.531	-0.243	-0.463	0.244	-0.388	0.492	-1.187	-0.903	-0.223	0.116	0.041
β_{18}	-0.948	-1.021	-0.285	1.203	0.420	-0.064	0.815	0.083	-0.185	1.549	-0.330
β_{19}	-3.099	0.509	-1.058	0.734	0.005	-0.034	0.716	-0.762	-0.186	0.149	0.334
β_{20}	3.606	-0.551	1.450	-1.457	-1,510	1.589	-0.294	1.362	0.900	-0.847	0.315
β_{21}	0.905	-0.054	0.385	-0.066	0.525	0.238	-0.773	1.330	-0.018	-0.182	0.676
β_{22}	1.015	-0.143	1.024	-0.095	-0.142	3.026	0.103	1.005	-0.813	0.130	-0.893
β_{23}	-0.119	-0.069	-0.029	-0.033	0.042	-0.068	-0.109	-0.228	-0.124	-0.428	-0.021

Table 3 Regression coefficients of QRNN for scale-free structural 20 proteins and random network structural 4 proteins while the sample size per protein is n=100.

Coeff.	case1	case2	case3	case4	case5	case6	case7	case8	case9	case10	case11
β_0	0.420	-0.431	-0.759	0.492	-0.167	1.572	0.998	-2.081	0.757	-0.630	-0.630
β_1	0.612	0.514	-0.423	0.612	0.234	-0.201	0.978	0.621	-0.035	-0.852	-0.852
β_2	0.118	-1.278	-0.334	0.118	0.146	1.617	-1.067	1.364	0.221	-1.156	-1.156
β_3	-0.025	0.139	-0.572	-0.025	-0.285	0.406	-2.178	0.378	0.574	2.014	2.014
β_4	0.886	-0.475	1.012	0.885	-0.383	0.003	-2.649	-0.484	0.941	-0.668	-0.668
β_5	0.376	-1.070	-1.130	0.376	-1.837	0.451	0.725	1.108	-2.322	0.657	0.657
β_6	-1.692	0.105	-0.131	-1.692	-0.821	-0.347	-1.894	-0.839	0.894	3.898	3.898
β_7	1.261	-0.189	0.292	1.261	0.616	-3.251	1.248	2.284	1.225	-0.331	-0.332
β_8	-0.137	-0.609	0.210	-0.137	0.199	-0.151	-2.324	0.924	-0.596	1.149	1.149
β_9	0.638	0.153	0.832	0.638	0.475	0.322	0.713	0.959	-2.817	0.365	0.365
β_{10}	-2.099	-0.027	0.200	-2.099	-1.007	-1.111	0.689	-0.234	1.512	0.356	0.356
β_{11}	0.833	1.182	-0.399	0.833	0.589	0.760	1.037	0.314	-3.522	0.231	0.232
β_{12}	-0.543	-0.187	-1.541	-0.544	-1.083	0.531	0.076	0.549	-2.001	0.697	0.697

β_{13}	0.695	0.838	-0.102	0.695	0.292	-0.094	0.520	-0.593	-2.311	-0.321	-0.321
β_{14}	1.624	-1.067	-0.315	1.624	0.212	0.446	-1.111	0.182	-0.965	-0.803	-0.802
β_{15}	0.842	-1.370	0.461	0.842	0.436	-0.214	0.418	0.186	-1.438	0.104	0.104
β_{16}	-0.081	2.736	0.333	-0.082	-0.935	-0.743	0.558	0.724	0.307	1.442	1.442
β_{17}	-2.062	0.763	0.408	-2.061	-0.518	-0.369	0.497	-0.093	0.299	-0.311	-0.311
β_{18}	1.956	0.784	-1.956	1.956	-0.099	-0.755	-0.550	0.668	-0.171	-0.444	0.444
β_{19}	-0.779	0.386	-0.474	-0.779	0.277	0.048	0.287	-1.389	0.931	-0.332	-0.332
β_{20}	-0.467	0.474	0.560	-0.467	-0.330	-1.434	0.736	-0.080	-0.376	-0.254	-0.254
β_{21}	-0.404	0.127	-0.536	0.404	-0.987	0.180	-1.631	-0.001	0.713	-0.550	-0.550
β_{22}	0.681	-0.320	-0.591	0.681	-0.192	0.743	-0.453	1.848	-0.629	-1.030	-1.031
β_{23}	-0.152	-0.185	-0.249	-0.153	-0.064	-0.094	0.006	-0.302	-1.265	0.219	0.219

Coeff.	**case12**	**case13**	**case14**	**case15**	**case16**	**case17**	**case18**	**case19**	**case20**	**case21**	**case22**
β_0	-0.213	0.214	0.172	0.730	-1.883	-0.127	-0.169	-0.946	-0.453	5.613	5.613
β_1	0.618	1.096	0.042	0.094	-1.860	-0.236	-1.406	-0.574	0.464	0.513	0.514
β_2	0.391	0.279	0.186	0.248	-1.023	0.462	0.391	-0.122	1.604	-4.468	-4.468
β_3	-0.125	-0.368	0.542	0.109	-0.039	-0.393	-0.883	-0.088	0.241	-2.711	-2.712
β_4	-1.835	-0.598	0.061	0.779	0.218	-0.155	-0.830	-0.425	0.161	-2.703	-2.703
β_5	-0.216	-0.001	-0.617	-0.033	-0.314	-0.343	-0.141	0.310	-0.006	-1.721	-1.722
β_6	-0.294	-0.648	0.242	1.235	-0.692	0.528	0.053	0.809	-0.061	-2.290	-2.290
β_7	-0.043	-0.354	0.249	-0.381	-0.359	0.721	0.103	-0.519	-0.225	1.897	1.897
β_8	0.801	1.422	0.266	-0.608	2.082	0.472	0.238	0.839	0.609	-2.967	-2.967
β_9	-0.413	0.030	-0.492	0.329	0.696	-0.538	-0.271	0.684	-0.513	4.518	4.518
β_{10}	-0.155	-0.312	1.544	0.045	0.913	-0.051	0.373	0.730	-0.823	1.865	1.865
β_{11}	0.293	-0.286	-0.558	0.321	-0.402	-0.318	0.281	-1.312	-0.734	-0.421	-0.421
β_{12}	0.877	0.076	-0.183	-1.471	1.069	-0.961	-0.319	-0.417	0.018	-3.033	-3.033
β_{13}	0.147	0.851	-0.211	0.475	1.741	1.950	-0.045	-0.021	0.705	-0.494	0.493
β_{14}	0.111	0.458	1.060	-2.219	3.411	-0.741	-0.304	-0.283	-0.897	-3.394	-3.394
β_{15}	-0.955	-0.053	1.026	1.679	2.324	0.827	-0.361	0.798	0.241	-2.123	-2.123
β_{16}	0.369	0.348	-0.150	-0.243	-2.183	-0.673	0.251	-0.014	0.146	-0.147	-0.147
β_{17}	-1.327	0.379	-0.580	-0.963	0.100	0.040	-1.474	-2.097	-0.005	2.861	2.860
β_{18}	-0.367	-0.034	0.121	0.783	0.174	0.982	0.206	0.524	-0.664	0.939	0.939
β_{19}	-0.239	0.133	0.407	0.919	0.082	0.847	0.015	-0.256	-0.342	-5.137	-5.137
β_{20}	0.187	0.458	0.104	-0.541	-0,502	-0.083	-0.556	0.048	-0.003	0.363	0.363
β_{21}	-0.137	0.038	-0.266	-0.582	0.283	-0.657	-0.332	0.585	-0.010	0.071	0.071
β_{22}	0.496	-0.446	-0.158	-0.035	0.054	-0.448	0.085	0.078	0.171	-6.406	-6.406
β_{23}	-0.088	0.207	0.229	-0.096	0.091	0.176	-0.143	-0.080	-0.243	-0.232	-0.232

Table 4 Regression coefficients of QRNN for random network structural 20 proteins and scale free structural 4 proteins while the sample size per protein is n=100.

Once the estimated networks are constructed, we convert the strength of the interaction into binary forms in such a way that the values less that the absolute value of 0.1 are set to zero and the values greater than this threshold are equated to one, as stated beforehand. In this calculation, the threshold value is chosen by considering the average of estimated regression coefficients from both systems. The accuracy and sensitivity measures computed from the comparison of estimated systems via the actual system are shown in Table 5. In the table, the first column indicates the 12-protein system and the second column presents the 24-protein system. Thereby, for example, 0.934 and 0.912 imply the accuracy and senstivity value, respecitvely, computed under 12 –protein system where 10 proteins have scale-free and 2 proteins have random network structure, and similarly, 0.829 and 0.789 refer to the accuracy and sensitivity value, in order, found under 24-protein system where 20 proteins have scale-free and 4 proteins have random network structure. Accordingly, the tabulated results indicate that both accuracy and sensitivity values are high in modelling via QRNN. Therefore, we believe that the proposal QRNN model can be a promising alternative model in the estimation of the network structures when the system has outliers or it is generated as a mixture of different network topologies.

	Measures	**p=10**	**p=20**
Scale-free	**Accuracy**	0.934	0.829
	Sensitivity	0.912	0.789
Random	**Accuracy**	0.917	0.953
	Sensitivity	0.891	0.912

Table 5 Accuracy and sensitivity measures of simulated 4 data sets which are modelled by quantile regression neural networks.

5. Conclusion

In this study, we have proposed quantile regression neural networks when networks system have outlier values. We have performed QRNN for each protein in the system as a separate regression and evaluates the regression coefficients as the strenghth of the interactions between proteins The results have shown acceptable success via QRNN although all the simulated data have outlier values. This outcome indicates that QRNN can be used to infer complex PPI structures which have anomalies under different topologies of networks. Whereas, in order to get a general conclusion, we think to extend the study via a comprehensive simulation scenarios and real data analyses which will be done as future work.

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Riesz Valued Density

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Abstract

Riesz space and statistical convergence are the natural and efficient tools in the theory of functional analysis. The statistical convergence is handled with the natural density of subsets on the natural numbers. Natural density of sets value over the closed interval [0,1]. In this work, we aim to introduce a concept of vector valued density on Riesz spaces, and so, we define a statistical convergence by utilizing the new density. In addition to the fact that results obtained in the settings of Riesz spaces will shed light on the case of convergences on Riesz spaces and Banach lattices.

Keywords: Riesz space, Riesz valued density, weak order unit.

1. Introduction

Riesz space and statistical convergence are the natural and efficient tools in the theory of functional analysis. Riesz space that was introduced by F. Riesz in [18] is an ordered vector space having many applications in measure theory, operator theory, and applications in economics (cf. [1,2,24]). On the other hand, the statistical convergence is a generalization of the ordinary convergence of a real sequence, and the idea of statistical convergence was firstly introduced by Zygmund [25]. After then, Fast [10] and Steinhaus [21] independently improved that idea. Several applications and generalizations of the statistical convergence of sequences have been investigated by several authors (cf. [10,12,16,21,23]). In general, the statistical convergence is handled with the natural density of subsets on the natural numbers $\mathbb{N}$. Natural density of sets value over the closed interval [0,1]. In this work, we aim to introduce a concept of vector valued density on Riesz spaces. In the settings of Riesz spaces will shed light on the case of convergences on Riesz spaces and Banach lattices (cf. [4,5,17,23]). The study related to this papers are done by Schmidt in [14,19], where vector measures are introduced in Riesz spaces, and done by Tan in [22], where some properties of Riesz space valued measures are obtained.

The generalized asymptotic density was investigated by Buke [7], and Freedman and Sember [11] introduce a general concept of density. We remind that the natural (or, asymptotic) density of a subset K of natural numbers is defined by

$$\delta(K) := \lim_{n\to\infty} \frac{1}{n} |\{k \leq n : k \in A\}|$$

where the vertical bar of sets will stand for the cardinality of the given sets. We refer the reader to an exposition on the natural density of sets to [7,10,11,12,16]. In the same way, a sequence $x = (x_k)$ is called statistical convergent to L provided that

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n: |x_k - L| \geq \varepsilon\}| = 0$$

for each $\varepsilon > 0$. Then it is written by $S - \lim x_k = L$.

Recall that a real vector space E with an order relation "s" is called an ordered vector space if, for each $x, y \in E$ with $x \leq y, x + z \leq y + z$ and $\alpha x \leq \alpha y$ hold for all $z \in E$ and $\alpha \in \mathbb{R}_+$. An ordered vector space E is called a Riesz space or a vector lattice if, for any two vectors $x, y \in E$, the infimum and the supremum

$$x \wedge y = \inf\{x, y\} \text{ and } x \vee y = \sup\{x, y\}$$

exist in E, respectively. The order convergence is the basic tool of Riesz spaces.

Definition 1.1. A sequence (x_n) of a Riesz space is said to be order convergent to a vector x (in symbols $x_n \xrightarrow{o} x$) whenever there exists another sequence (y_n) with $y_n \downarrow 0$ and $|x_n - x| \leq y_n$ for all indices n.

Recall that a Riesz space E is called Archimedean whenever $\frac{1}{n} x \downarrow 0$ holds in E for each $x \in E_+$. Every Riesz space is not Archimedean. To see this, we give the following example.

Example 1.2. Consider the Riesz space $E := \mathbb{R}^2$ with lexicographical ordering: $(x_1, x_2) \leq (y_1, y_2)$ if and only if $x_1 < y_1$ or $x_1 = y_1$ and $x_2 \leq y_2$. Then E is not Archimedean because, for the positive element $(1,1) \in E$, we have $\frac{1}{n}(1,1) \downarrow$, but $\frac{1}{n}(1,1) \neq 0$

A subset I of a Riesz space E is said to be a solid set if, for each $x \in E$ and $y \in I$ with $|x| \leq |y|$, it follows that $x \in I$. A solid vector subspace is called an order ideal. A positive element e in a Riesz space E is called order unit (or, strong order unit) if the principal ideal $I_e := \{x \in E: \exists\lambda > 0 \text{ with } |x| \leq \lambda e\}$ generated by e is the whole space E, i.e., if, for every $x \in E$, there exists some positive scalar $\lambda > 0$, depending upon x, such that $|x| \leq \lambda e$(cf. [15, Def.21.4])). We refer the reader for an exposition on the order unit to [1,2,3,15,17].

Example 1.3. Consider a compact Hausdorff space K. Then any strictly positive function $T \in C(K)$ is a strong order unit. Indeed, take $\lambda := \min_{x\in K} T(x)$ for $T \in C(K)$. Then, we have $\lambda > 0$ and $T \leq \lambda \cdot \mathbb{1}_K$. i.e., T is a strong unit of $C(K)$.

Not all Riesz spaces have order unit elements. To see this, we proceed by contraposition as follows example.

Example 1.4. Take $p \in [1, \infty)$. Then the Riesz spaces c_0 and ℓ_p do not have order units.

Assume that E is a Riesz space and $\mathcal{F}$ is an algebra of subsets of any nonempty set X. Then a set function $\mu: \mathcal{F} \to E$ is called a Riesz space valued measure or additive measure whenever μ is additive for each disjoint elements in $\mathcal{F}$, i.e., $\mu(A \cup B) = \mu(A) + \mu(B)$ for all sets $A \cap B = \emptyset$. The space of additive measures is an ordered vector space with the order, $\mu_1 \leq \mu_2$ if and only if $\mu_1(A) \leq \mu_2(A)$ for every A in $\mathcal{F}$. Moreover, if the supremum $\sup_{B \in \mathcal{F}} |\mu(B)|$ exists in E then the Riesz space valued measure μ is called order bounded. Generally, the space $a(\mathcal{F}, E)$ of Riesz space valued measures fails to be a Riesz space. But, the space oba $(\mathcal{F}, E)$ a Dedekind complete Riesz space whenever E is a Dedekind complete Riesz space. We refer the reader for more information on Riesz space valued measure to [6,14,19,22]. Unless otherwise stated, we consider all algebras as infinite.

2. Riesz valued density

In this section, we introduce a new concept of density on Riesz spaces with respect to order unit elements.

Definition 2.1. Let E be a Riesz space with an order unit e and $\mathcal{F}$ be an subfield of $\mathcal{P}(\mathbb{N})$ which contains all the finite subsets of $\mathbb{N}$. Then a Riesz space valued measure $\mu: \mathcal{F} \to [0, e]$ is called Riesz valued density if it holds the following properties:

a) $\mu(k) = 0$ for all $k \in \mathbb{N}$;
b) *(b)* $\mu(A) + \mu(B) - \mu(A \cap B) \leq e$ *for all* $A, B \subseteq \mathbb{N}$
c) *(c)* $\mu(A) = e - \mu(\mathbb{N} \setminus A)$.

Remark 2.2. Let $\mu: \mathcal{F} \to [0, e]$ be a Riesz valued density.

i. It follows from Definition 2.1(a) that $(A) = \mu(B)$ whenever the symmetric difference $A \vartriangle B$ is finite (or equivalently, $A \sim B$).
ii. (ii) $\mu(A) \leq \mu(B)$ whenever $A \subseteq B$ because μ is finitely additive. Specially, $\mu(A) = 0$ whenever $A \subseteq B$ and $\mu(B) = 0$ holds for $A, B \in \mathcal{F}$.
iii. (iii) By property (ii), $\mu(A_n) \downarrow 0$ in $\mathcal{F}$ implies that $\mu(A_n) \downarrow 0$ in $[0, e]$.
iv. (iv) A measure $\gamma: \mathcal{F} \to [0, e]$, defined by $\gamma(A) = \sup_{B \in \mathcal{F}} \mu(A \cap B)$ for each $A \in \mathcal{F}$, is the smallest Riesz valued density majorizing μ.
v. It follows from [19, Cor.2.2] that every Riesz valued density has a Jordan decomposition such that $\mu = \mu^+ + \mu^-$.

Riesz valued density is an extension of natural density. To see this, consider the Riesz space E as real numbers $\mathbb{R}$ with the order unit 1 . However, the converse does not need to hold in general. To see this, we consider [6] for the next example.

Example 2.3. Take the algebra $\mathcal{F} := \mathcal{P}(\mathbb{N})$ with a sequence (x_n) that is a pairwise disjoint nonempty sets in $\mathcal{F}$. Now, choose points of the sequence and for each n denote the function from $\mathcal{F}$ to $\mathbb{R}$ by

$$q_n(A) := \begin{cases} 1, & x_n \in A \\ 0, & x_n \notin A \end{cases}$$

for all $A \in \mathcal{F}$. Hence, q_n is a real valued measure for each $n \in \mathbb{N}$. Let us define another function $u: \mathcal{F} \to \mathbb{R}$ such that

$$u(A) := \sum_{n=1}^{\infty} \frac{\pi^n}{4^n} q_n(A)$$

for every $A \in \mathcal{F}$. Thus, u is finitely additive. Moreover, the sequence $\left(\frac{\pi^n}{4^n}\right)$ is linearly independent over the rational field, and it can be imbedded the Hamel base $\mathcal{B}$ for the real numbers. Next, take the Riesz space $E := \mathbb{R}$ with an order unit e, and define the Riesz space valued function $v: \mathcal{B} \to E$ such that

$$v(x) := \begin{cases} \frac{1}{n} e, & x \in \left(\frac{\pi^n}{4^n}\right) \\ 0, & x \in \mathcal{F} \setminus \left(\frac{\pi^n}{4^n}\right) \end{cases}$$

for all $x \in \mathcal{B}$. It follows from the dominatedness of $\mathcal{B}$ in real numbers that v has a positive linear extension $\hat{v}$ to all real numbers (cf. [2, Thm.1.10]). Let us define a positive measure $\mu: \mathcal{F} \to [0, e]$ by $\mu(A) := \hat{v}(u(A))$. Then, μ is a Riesz valued density. But, it is not natural density.

Recall that a Riesz space E has the PR-property whenever there exists an increasing sequence h_n of positive elements in E such that, for each $x \in E$, there are some scalar $\lambda > 0$ and n_x such that $|x| \leq \lambda h_{n_x}$ (cf. [20,22]).

Lemma 2.4. Every Riesz space with an order unit has the PR-property.

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Rough Statistical Convergence of Order α for Complex Uncertain Double Sequences

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Abstract

Complex uncertain variables are measurable functions from an uncertainty space to the set of complex numbers and are utilized to model complex uncertain quantities. The main aim of this study is to investigate rough statistical convergence of order α $(0 < \alpha \leq 1)$ for complex uncertain double sequences and work various convergence concepts such as, rough statistical convergence of order α, rough statistical convergence of order α almost surely, rough statistical convergence of order α in measure, rough statistical convergence of order α in mean, rough statistical convergence of order α in distribution of complex uncertain double sequences. We also obtain some relationships among them.

Keywords: Statistical convergence, rough statistical convergence, complex uncertain sequence, uncertainty theory.

1. Introduction

Zygmund [1] used the term "almost convergence" to describe the concept of statistical convergence. It was formally presented by Fast [2]. The thought of statistical convergence of double sequences was proposed by Mursaleen and Edely [3].

The theory of uncertainty play a vital role not only in pure mathematics. The majority of human decisions are made in the face of uncertainty. A specific sort of mathematical measure can be used to represent the performance of an uncertainty. Fuzziness is another paradigm for uncertainty pioneered by Zadeh [4] in 1965 using membership functions. Fuzzy set theory and probability theory are undeniably valuable tools for dealing with uncertainty. However, in real life, natural language expressions such as "middle age", "about 30 kilometers", "about 15 degrees Celsius" and "roughly 6 kilograms" are commonly employed to represent imperfect knowledge or facts. But, multiple studies have demonstrated that such utterances are neither random nor fuzzy. These facts encourage the development of uncertainty theory as an axiomatic mathematics branch for representing human uncertainty. To model uncertainty, Liu [5] established an uncertainty theory that is a branch of mathematics based on normality, monotonicity, self-duality, countable subadditivity, and product measure axioms. Liu [5] defined the concept of uncertain variables as a function from a measurable space to $\mathbb{R}$. If real numbers are rebuild with a set of complex numbers, it is named a complex uncertain variable which was worked by Peng [6]. Recently, various researchers have also done significant studies based on complex uncertain variables, such as (see [7-13]). The conception of rough convergence was first investigated by Phu [14] in finite-dimensional normed spaces. Phu [15] expanded the results given in [14] to infinite-dimensional normed spaces. In [16], Aytar investigated rough statistical convergence. The notion of λ-statistical convergence was examined by Mursaleen [17]. Das et al. [18] expanded these ideas in 2015, including rough λ-statistical convergence in probability. Also Maity [19] proposed the notion of rough statistical convergence of order α $(0 < \alpha \leq 1)$. Rough convergence of double sequences was investigated by Malik and Maity [20-21]. Debnath and Das [22] examined some features of rough statistical convergence of complex uncertain sequences.

For the purpose of delve deeper into uncertainty theory, we defined rough statistical convergence of order α for complex uncertain double sequences and worked on some convergence conceptions such as rough statistical convergence of order α almost surely, rough statistical convergence of order α in measure, rough statistical convergence of order α in mean, rough statistical convergence of order α in distribution of complex uncertain double sequences, obtaining some inter-relationships between them.

2. Main Results

Definition 2.1: *Let r be a non-negative real number. A complex uncertain sequence $\{\xi_{uv}\}$ is called to be rough statistical (rst) convergent almost surely of order α to ξ with roughness degree r provided that for any event Λ with $M(\Lambda) = 1$ such that*

$$\lim_{n,m\to\infty} \frac{1}{(nm)^\alpha} |\{(u,v): u \leq n, v \leq m: \|\xi_{uv}(\gamma) - \xi(\gamma)\| \geq r + \varepsilon\}| = 0$$

for every $\gamma \in \Lambda$. If the above equation holds, ξ is an rough statistical limit point of $\{\xi_{uv}\}$ which is usually no more unique (for $r > 0$). So, we have to consider r-statistical limit set of $\{\xi_{uv}\}$ of order α itendified by

$$St_2{}^\alpha - \mathrm{LIM}_r^\alpha(\xi_{uv}) := \left\{ \xi : \xi_{uv} \xrightarrow{r-St_2{}^\alpha} \xi \right\}.$$

Example 2.1. Consider the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$. It becomes $\Gamma = \{\gamma_1, \gamma_2, \dots\}$ with $\mathcal{M}(\Lambda) = \sum_{\gamma_u, \gamma_v \in \Lambda} 2^{-(u+v)}$. We determine a complex uncertain variable by

$$\xi_{uv}(\gamma) = \begin{cases} i.(-1)^{u+v}, & \text{if } \gamma = \gamma_{u+v}, \\ 0, & \text{if not.} \end{cases}$$

for $u \neq k^2, v \neq l^2$ and

$$\xi_{uv}(\gamma) = \begin{cases} i.(u+v), & \text{if } \gamma = \gamma_{u+v}, \\ 0, & \text{if not.} \end{cases}$$

for $u = k^2, v = l^2$ and $\xi \equiv 0$. Then, $\xi_{uv} \xrightarrow{r-{St_2}^{\alpha}} \xi$ where

$${St_2}^{\alpha}\text{−LIM}_r^{\alpha}(\xi_{uv}) = \begin{cases} \emptyset, & \text{for } r < 1, \\ \xi.i, \quad \xi \in [1-r, r-1], & \text{for } r \geq 1. \end{cases}$$

Additionally, we get that the sequence $\{\xi_{uv}\}$ is not rough convergent *a.s.* of order α to ξ, hovewer it is rst-convergent *a.s.* of order α to ξ for any $r \geq 1$.

Theorem 2.1.

i. *If* $\xi_{uv} \xrightarrow{r-{St_2}^{\alpha}} \xi, a.s.$, *then* $\beta\xi_{uv} \xrightarrow{r-{St_2}^{\alpha}} \beta\xi, a.s.$, *where* $\beta \in \mathbb{C}$.
ii. *If* $\xi_{uv} \xrightarrow{r-{St_2}^{\alpha}} \xi, a.s.$ *and* $\omega_{uv} \xrightarrow{r-{St_2}^{\alpha}} \omega, a.s.$, *then* $a\xi_{uv} + b\omega_{uv} \xrightarrow{r-{St_2}^{\alpha}} a\xi + b\omega, a.s.$, *where* $a, b \in \mathbb{C}$.

Proof. It is obvious, so omitted.

Theorem 2.2. *The r-statistical limit set of a complex uncertain sequence of order α is convex.*

Proof. Presume that $\xi_0, \xi_1 \in {St_2}^{\alpha}\text{−LIM}_r^{\alpha}(\xi_{uv})$ for the complex uncertain double sequence $\{\xi_{uv}\}$ and let $\varepsilon > 0$ be given. Determine

$$A_0(\varepsilon) := \{(u,v) \in \mathbb{N} \times \mathbb{N} : \|\xi_{uv} - \xi_0\| \geq r + \varepsilon\},$$

and

$$A_1(\varepsilon) := \{(u,v) \in \mathbb{N} \times \mathbb{N} : \|\xi_{uv} - \xi_1\| \geq r + \varepsilon\}.$$

Since $\xi_0, \xi_1 \in {St_2}^{\alpha}\text{−}\mathrm{LIM}_r^{\alpha}(\xi_{uv})$, we get $\delta(A_0(\varepsilon)) = 0$ and $\delta(A_1(\varepsilon)) = 0$. So, we acquire

$$\|\xi_{uv} - [(1-\lambda)\xi_0 + \lambda\xi_1]\| \leq \|(1-\lambda)(\xi_{uv} - \xi_0) + \lambda(\xi_{uv} - \xi_1)\| < r + \varepsilon,$$

for all $(u,v) \in A_0^c(\varepsilon) \cap A_1^c(\varepsilon)$ and every $\lambda \in [0,1]$. Since $\delta(A_0^c \cap A_1^c) = 1$, we obtain

$$\delta\{(u,v) \in \mathbb{N} \times \mathbb{N} : \|\xi_{uv} - [(1-\lambda)\xi_0 + \lambda\xi_1]\| \geq r + \varepsilon\} = 0,$$

namely, $[(1-\lambda)\xi_0 + \lambda\xi_1] \in {St_2}^{\alpha}\text{−}\mathrm{LIM}_r^{\alpha}(\xi_{uv})$, which gives the convexity of the set ${St_2}^{\alpha}\text{−}\mathrm{LIM}_r^{\alpha}(\xi_{uv})$.

Definition 2.2: *A complex uncertain sequence $\{\xi_{uv}\}$ is named to be rough statistical convergent in measure of order α to ξ with roughness degree r provided that for $\varepsilon, \kappa > 0$ such that*

$$\lim_{n,m\to\infty} \frac{1}{(nm)^{\alpha}} |\{(u,v) : u \leq n, v \leq m : M\{\gamma : \|\xi_{uv}(\gamma) - \xi(\gamma)\| \geq \kappa\} \geq r + \varepsilon\}| = 0$$

for every $\gamma \in \Lambda$. We write $\xi_{uv} \xrightarrow{r-{St_2}_M^{\alpha}} \xi$.

Example 2.2. Contemplate the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\Gamma = \{\{\gamma_1, \gamma_2, \ldots\}$ *with*

$$\mathcal{M}(\Lambda)=\begin{cases} sup_{\gamma_{u+v}\in\Lambda}\dfrac{u+v}{2(u+v)+1}, & \text{if } sup_{\gamma_{u+v}\in\Lambda}\dfrac{u+v}{2(u+v)+1}<0.5,\\ 1-sup_{\gamma_{u+v}\in\Lambda^c}\dfrac{u+v}{2(u+v)+1}, & \text{if } sup_{\gamma_{u+v}\in\Lambda^c}\dfrac{u+v}{2(u+v)+1}<0.5\\ 0.5, & \text{if not.}\end{cases}$$

and think the uncertain variable $\{\xi_{uv}\}$ determined by

$$\xi_{uv}(\gamma)=\begin{cases} i.(u+v), & \text{if } \gamma=\gamma_{u+v},\\ 0, & \text{if not.}\end{cases}$$

for $u=k^2, v=l^2$ and $\xi_{uv}(\gamma)\equiv 0$, for $u\neq k^2, v\neq l^2$. Also take $\xi\equiv 0$. Then, we obtain

$$\lim_{n,m\to\infty}\frac{1}{(nm)^\alpha}|\{(u,v): u\leq n, v\leq m: \mathcal{M}\{\gamma: \|\xi_{uv}(\gamma)-\xi(\gamma)\|\geq\kappa\}\geq r+\varepsilon\}|=0.$$

for every $\gamma\in\Lambda$ and $r\geq 0$. This demonstrates that $\{\xi_{uv}\}$ is *rst*-convergent in measure of order α to ξ *for* $r\geq 0$. In addition, for $0\leq r<\frac{1}{2}$, the sequence $\{\xi_{uv}\}$ is not rough convergent in measure of order α to ξ, hovewer it is *rst*-convergent in measure of order α to ξ.

Theorem 2.3. *If* $\xi_{uv}\xrightarrow{r-St_{2_M}^\alpha}\xi_1$ *and* $\xi_{uv}\xrightarrow{r-St_{2_M}^\alpha}\xi_2$, *then* $\mathcal{M}\{\|\xi_1-\xi_2\|\geq r_1+r_2\}=0$.

Proof. Assume that ε,κ be any two positive real numbers and let

$$(i,j)\in\left\{(u,v)\in\ \mathbb{N}\times\mathbb{N}: \mathcal{M}\left(\|\xi_{uv}-\xi_1\|\geq r_1+\frac{\varepsilon}{2}\right)<\frac{\kappa}{2}\right\}$$
$$\cap\left\{(u,v)\in\ \mathbb{N}\times\mathbb{N}: \mathcal{M}\left(\|\xi_{uv}-\xi_2\|\geq r_2+\frac{\varepsilon}{2}\right)<\frac{\kappa}{2}\right\}$$

(because the asymptotic density of both sets is equal to one, the existence of (i,j) is assured). So,

$$\mathcal{M}\{\|\xi_1 - \xi_2\| \geq r_1 + r_2 + \varepsilon\} \leq \mathcal{M}\left(\|\xi_{uv} - \xi_1\| \geq r_1 + \frac{\varepsilon}{2}\right) + \mathcal{M}\left(\|\xi_{uv} - \xi_2\| \geq r_2 + \frac{\varepsilon}{2}\right) < \kappa.$$

This means that $\mathcal{M}\{\|\xi_1 - \xi_2\| \geq r_1 + r_2\} = 0$.

Theorem 2.4.

i. $\xi_{uv} \xrightarrow{r-St_{2\mathcal{M}}^{\alpha}} \xi \Leftrightarrow \alpha\xi_{uv} \xrightarrow{r-St_{2\mathcal{M}}^{\alpha}} \alpha\xi$, *where* $\alpha \in \mathbb{C}$.
ii. $\xi_{uv} \xrightarrow{r-St_{2\mathcal{M}}^{\alpha}} \xi$ *and* $w_{uv} \xrightarrow{r-St_{2\mathcal{M}}^{\alpha}} w \Rightarrow \xi_{uv} + w_{uv} \xrightarrow{r-St_{2\mathcal{M}}^{\alpha}} \xi + w$.
iii. $\xi_{uv} \xrightarrow{r-St_{2\mathcal{M}}^{\alpha}} \xi$ *and* $w_{uv} \xrightarrow{r-St_{2\mathcal{M}}^{\alpha}} w \Rightarrow \xi_{uv} - w_{uv} \xrightarrow{r-St_{2\mathcal{M}}^{\alpha}} \xi - w$.
iv. $\xi_{uv} \xrightarrow{r-St_{2\mathcal{M}}^{\alpha}} \xi$, *then for all* $\varepsilon > 0$ *there exists a* $(k,l) \in \mathbb{N} \times \mathbb{N}$ *so that any* $\kappa > 0$

$$\lim_{n,m\to\infty} \frac{1}{(nm)^{\alpha}} |\{(u,v): u \leq n, v \leq m: \mathcal{M}\{\|\xi_{uv} - \xi_{kl}\| \geq 2r + \varepsilon\} \geq \kappa\}| = 0.$$

Proof. Let $\varepsilon > 0$ be any positive real number. Then for

i. The proof is self-evident, thus it is removed.
ii. When $\alpha = 0$, then the claim is obvious. So, presuming $\alpha \neq 0$, then

$$\{(u,v) \in \mathbb{N} \times \mathbb{N}: \mathcal{M}(\|\alpha\xi_{uv} - \alpha\xi\| \geq |\alpha|r + \varepsilon) \geq \kappa\}$$
$$= \left\{(u,v) \in \mathbb{N} \times \mathbb{N}: \mathcal{M}\left(\|\xi_{uv} - \xi\| \geq r + \frac{\varepsilon}{|\alpha|}\right) \geq \kappa\right\}.$$

As a result, $\alpha\xi_{uv} \xrightarrow{r-St_{2\mathcal{M}}^{\alpha}} \alpha\xi$.

iii. $\mathcal{M}(\|(\xi_{uv} + w_{uv}) - (\xi + w)\| \geq r_1 + r_2 + \varepsilon) = \mathcal{M}(\|(\xi_{uv} - \xi) - (w_{uv} - w)\| \geq r_1 + r_2 + \varepsilon) \leq \mathcal{M}\left(\|\xi_{uv} - \xi\| \geq r_1 + \frac{\varepsilon}{2}\right) + \mathcal{M}\left(\|w_{uv} - w\| \geq r_2 + \frac{\varepsilon}{2}\right)$.

This implies

$$\{(u,v) \in \mathbb{N}\times\mathbb{N}: \mathcal{M}(\|(\xi_{uv}+w_{uv})-(\xi+w)\| \geq r_1+r_2+\varepsilon) \geq \kappa\}$$
$$\subseteq \left\{(u,v) \in \mathbb{N}\times\mathbb{N}: \mathcal{M}\left(\|\xi_{uv}-\xi\| \geq r_1+\frac{\varepsilon}{2}\right) \geq \frac{\kappa}{2}\right\}$$
$$\cup \left\{(u,v) \in \mathbb{N}\times\mathbb{N}: \mathcal{M}\left(\|w_{uv}-w\| \geq r_2+\frac{\varepsilon}{2}\right) \geq \frac{\kappa}{2}\right\}.$$

Hence, $\xi_{uv}+w_{uv} \xrightarrow{r-St_{2\mathcal{M}}^{\alpha}} \xi+w$.

iv. Similar to the preceding evidence and hence omitted.

v. Now, select $(k,l) \in \mathbb{N}\times\mathbb{N}$ be such that $\mathcal{M}\left(\|w_{kl}-w\| \geq r_1+r_2+\frac{\varepsilon}{2}\right) < \frac{\kappa}{2}$ (existence of (k,l) is guaranteed). Then the claim is obvious from the inequality

$$\mathcal{M}(\|w_{uv}-w_{kl}\| \geq 2r+\varepsilon) \leq \mathcal{M}\left(\|\xi_{uv}-\xi\| \geq r_1+\frac{\varepsilon}{2}\right)+\mathcal{M}\left(\|\xi_{kl}-\xi\| \geq r_1+\frac{\varepsilon}{2}\right)$$
$$\leq \frac{\kappa}{2}+\mathcal{M}\left(\|\xi_{uv}-\xi\| \geq r+\frac{\varepsilon}{2}\right).$$

Hence, we obtain

$$\lim_{n,m\to\infty}\frac{1}{(nm)^{\alpha}}|\{(u,v): u \leq n, v \leq m: M\{\|\xi_{uv}-\xi_{kl}\| \geq 2r+\varepsilon\} \geq \kappa\}| = 0.$$

Theorem 2.5. *Rough statistical convergence in measure of order α does not imply rough statistical convergence a.s of order α.*

Example 2.3. Contemplate the uncertainty space $(\Gamma,\mathcal{L},\mathcal{M})$ to be $\Gamma=\{\gamma_1,\gamma_2,\dots\}$ with

$$\mathcal{M}(\Lambda)=\begin{cases} sup_{\gamma_{u+v}\in\Lambda}\dfrac{u+v}{5(u+v)+1}, & \text{if } sup_{\gamma_{u+v}\in\Lambda}\dfrac{u+v}{5(u+v)+1}<\dfrac{1}{5},\\ 1-sup_{\gamma_{u+v}\in\Lambda^c}\dfrac{u+v}{5(u+v)+1}, & \text{if } sup_{\gamma_{u+v}\in\Lambda^c}\dfrac{u+v}{5(u+v)+1}<\dfrac{1}{5}\\ 0.5, & \text{if } not. \end{cases}$$

and think about the uncertain variable $\{\xi_{uv}\}$ identified by

$$\xi_{uv}(\gamma) = \begin{cases} i.(u+v)^3, & \text{for } u, v = 1,2,\dots \\ 0, & \text{if not.} \end{cases}$$

Also take $\xi_{uv}(\gamma) \equiv 0$.
Then, for any $\kappa > 0$, we get

$$M\{\gamma : \|\xi_{uv}(\gamma) - \xi(\gamma)\| \geq \kappa\} = M\{\gamma_{u+v}\} = \frac{u+v}{5(u+v)+1}.$$

Then, the sequence $\{\xi_{uv}\}$ is rst-convergent of order α to ξ for $r \geq \frac{1}{5}$. But it is not *rst*-convergent *a.s.* of order α to ξ.

Definition 2.3: *Assume that $\Phi, \Phi_1, \Phi_2, \dots$ be the complex uncertainty distributions of complex uncertain variables ξ, ξ_{uv}. A complex uncertain sequence $\{\xi_{uv}\}$ is defined as rough statistical convergence in distribution* of order *α to ξ with roughness degree r if for $\varepsilon > 0$,*

$$\lim_{n,m\to\infty} \frac{1}{(nm)^\alpha} |\{(u,v) \in \mathbb{N} \times \mathbb{N} : \|\Phi_{uv}(x) - \Phi(x)\| \geq r + \varepsilon\}| = 0,$$

for every $\gamma \in \Lambda$ and for all x at which $\Phi(x)$ is continuous.

Example 2.4. Contemplate the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\Gamma = \{\gamma_1, \gamma_2, \dots\}$ with

$$\mathcal{M}(\Lambda) = \begin{cases} sup_{\gamma_{u+v} \in \Lambda} \dfrac{u+v}{3(u+v)+1}, & \text{if } sup_{\gamma_{u+v} \in \Lambda} \dfrac{u+v}{3(u+v)+1} < \dfrac{1}{3}, \\ 1 - sup_{\gamma_{u+v} \in \Lambda^c} \dfrac{u+v}{3(u+v)+1}, & \text{if } sup_{\gamma_{u+v} \in \Lambda^c} \dfrac{u+v}{3(u+v)+1} < \dfrac{1}{3} \\ 0.5, & \text{if not.} \end{cases}$$

and think about the uncertain variable $\{\xi_{uv}\}$ determined by

$$\xi_{uv}(\gamma) = \begin{cases} i.(u+v)^2, & \text{for } u, v = 1,2,\dots \\ 0, & otherwise. \end{cases}$$

for $u = k^2, v = l^2$ and $\xi_{uv}(\gamma) \equiv 0$, for $u \neq k^2, v \neq l^2$. Also take $\xi \equiv 0$. Then, for $u = k^2, v = l^2$, we have the uncertainty distribution of uncertain variable $\{\xi_{uv}\}$ as

$$\Phi_{uv}(x) = \Phi_{uv}(a+ib) = \begin{cases} 0, \text{if } a < 0, b < \infty; \\ 0, \text{if } a \geq 0, b < 0; \\ 1 - \dfrac{u+v}{3(u+v)+1}, \quad \text{if } 0 \leq b < (u+v)^2, a \geq 0; \\ 1, \quad \text{if } a \geq 0, b \geq (u+v)^2. \end{cases} \text{otherwise}$$

$$\Phi_{uv}(x) = \begin{cases} 0, & \text{if } a < 0, b < \infty; \\ 0, & \text{if } a \geq 0, b < 0; \\ 1, & \text{if } a \geq 0, b \geq 0. \end{cases}$$

In addition, the complex uncertainty distribution of uncertain variable Φ is

$$\Phi(x) = \begin{cases} 0, & \text{if } a < 0, b < \infty; \\ 0, & \text{if } a \geq 0, b < 0; \\ 1, & \text{if } a \geq 0, b \geq 0. \end{cases}$$

Thus, we get

$$\lim_{n,m\to\infty} \frac{1}{(nm)^{\alpha}} |\{(u,v) \in \mathbb{N} \times \mathbb{N}: \|\Phi_{uv}(x) - \Phi(x)\| \geq r + \varepsilon\}| = 0,$$

for $r > 0$. Also, we have that the sequence $\{\xi_{uv}\}$ is not rough convergent in distribution of order α to ξ, but it is *rst*-convergent in distribution of order α to ξ for $0 \leq r < \frac{1}{3}$.

Definition 2.4: *A sequence $\{\xi_{uv}\}$ is said to have rough statistical convergence in mean* of order *α to ξ with roughness degree r if and only if*

$$\lim_{n,m\to\infty} \frac{1}{(nm)^{\alpha}} |\{(u,v) \in \mathbb{N} \times \mathbb{N}: E[\|\xi_{uv}(\gamma) - \xi(\gamma)\|] \geq r + \varepsilon\}| = 0,$$

for every $\gamma \in \Lambda$ and $\varepsilon > 0$.

Example 2.4. Contemplate the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\Gamma = \{\gamma_1, \gamma_2, \dots\}$ with

$$\mathcal{M}(\Lambda)=\begin{cases} sup_{\gamma_{u+v}\in\Lambda}\dfrac{u+v}{2(u+v)^2+1}, & \text{if } sup_{\gamma_{u+v}\in\Lambda}\dfrac{u+v}{2(u+v)^2+1}<0.5,\\ 1-sup_{\gamma_{u+v}\in\Lambda^c}\dfrac{u+v}{2(u+v)^2+1}, & \text{if } sup_{\gamma_{u+v}\in\Lambda^c}\dfrac{u+v}{2(u+v)^2+1}<0.5,\\ 0.5, & \text{if not.}\end{cases}$$

and think about the uncertain variable $\{\xi_{uv}\}$ determined by

$$\xi_{uv}(\gamma)=\begin{cases} i.(u+v), & \text{if } \gamma=\gamma_{u+v};\\ 0, & \text{if not.}\end{cases}$$

for $u=k^2, v=l^2$ and $\xi_{uv}(\gamma)\equiv 0$, for $u\neq k^2, v\neq l^2$. Also take $\xi\equiv 0$. Then, for $u=k^2, v=l^2$, we have the uncertainty distribution of uncertain variable $\{\xi_{uv}\}$ as

$$\Phi_{uv}(x)=\Phi_{uv}(a+ib)=\begin{cases} 0, \text{if } a<0, b<\infty;\\ 0, \text{if } a\geq 0, b<0;\\ 1-\dfrac{u+v}{2(u+v)^2+1}, & \text{if } 0\leq b<(u+v)^2, a\geq 0;\\ 1, & \text{if } a\geq 0, b\geq (u+v)^2.\end{cases} \text{otherwise}$$

$$\Phi_{uv}(x)=\begin{cases} 0, & \text{if } a<0, b<\infty;\\ 0, & \text{if } a\geq 0, b<0;\\ 1, & \text{if } a\geq 0, b\geq 0.\end{cases}$$

In addition, the complex uncertainty distribution of uncertain variable Φ is

$$\Phi(x)=\begin{cases} 0, & if\ a<0, b<\infty;\\ 0, & if\ a\geq 0, b<0;\\ 1, & if\ a\geq 0, b\geq 0.\end{cases}$$

Thus, we get for $u=k^2, v=l^2$

$$E[\|\xi_{uv}(\gamma)-\ \xi(\gamma)\|]=\frac{(u+v)^2}{2(u+v)^2+1},$$

So, we get

$$\lim_{n,m\to\infty}\frac{1}{(nm)^\alpha}|\{(u,v)\in\ \mathbb{N}\times\mathbb{N}: E[\|\xi_{uv}(\gamma)-\ \xi(\gamma)\|]\geq r+\varepsilon\}|=0,$$

for every $\gamma \in \Lambda$ and $\varepsilon > 0$. Also, we acquire that the sequence $\{\xi_{uv}\}$ is not rough statistical convergence in mean of order α to ξ, but it rough statistical convergence in mean of order α to ξ for $0 \leq r < \frac{1}{2}$.

Theorem 2.6. *Rough statistical convergence in mean of order α does not imply rough statistical convergence a.s of order α.*

Example 2.5. Contemplate the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\Gamma = \{\gamma_1, \gamma_2, \dots\}$ with

$$\mathcal{M}(\Lambda) = \begin{cases} sup_{\gamma_{u+v} \in \Lambda} \dfrac{u+v}{2(u+v)^2+1}, & \text{if } sup_{\gamma_{u+v} \in \Lambda} \dfrac{u+v}{2(u+v)^2+1} < 0.5, \\ 1 - sup_{\gamma_{u+v} \in \Lambda^c} \dfrac{u+v}{2(u+v)^2+1}, & \text{if } sup_{\gamma_{u+v} \in \Lambda^c} \dfrac{u+v}{2(u+v)^2+1} < 0.5, \\ 0.5, & \text{if not.} \end{cases}$$

and think about the uncertain variable $\{\xi_{uv}\}$ defined by

$$\xi_{uv}(\gamma) = \begin{cases} i.(u+v), & \text{if } \gamma = \gamma_{u+v}; \\ 0, & \text{if not.} \end{cases}$$

and $\xi_{uv}(\gamma) \equiv 0$. The uncertainty distribution of an uncertain variable $\{\xi_{uv}\}$ is thus obtained as

$$\Phi_{uv}(x) = \Phi_{uv}(a+ib) = \begin{cases} 0, \text{if } a < 0, b < \infty; \\ 0, \text{if } a \geq 0, b < 0; \\ 1 - \dfrac{u+v}{2(u+v)^2+1}, & \text{if } 0 \leq b < u+v, a \geq 0; \\ 1, & \text{if } a \geq 0, b \geq u+v. \end{cases}$$

for *u,v=1,2,…*. In addition, the complex uncertainty distribution of uncertain variable Φ is

$$\Phi(x) = \begin{cases} 0, & \text{if } a < 0, b < \infty; \\ 0, & \text{if } a \geq 0, b < 0; \\ 1, & \text{if } a \geq 0, b \geq 0. \end{cases}$$

Thus, we get

$$\lim_{n,m\to\infty} \frac{1}{(nm)^\alpha} |\{(u,v) \in \mathbb{N} \times \mathbb{N} : \mathcal{M}\{\gamma : \|\xi_{uv}(\gamma) - \xi(\gamma)\| \geq \delta\} \geq r + \varepsilon\}| = 0,$$

for $r \geq 0.5$, but

$$\lim_{n,m\to\infty} \frac{1}{(nm)^\alpha} |\{(u,v) \in \mathbb{N} \times \mathbb{N}: E\{\|\xi_{uv}(\gamma) - \xi(\gamma)\| \geq \delta\} \geq r + \varepsilon\}| \neq 0.$$

Theorem 2.7. *Rough statistical convergence in distribution of order α does not imply rough statistical convergence in mean of order α.*

Proof. It is quite simple to demonstrate from the preceding example, thus it has been removed.

Definition 2.4: *A sequence $\{\xi_{uv}\}$ is said to be rough λ^2-statistical convergence in measure* of order *α to ξ, with roughness degree r provided that for $\varepsilon, \kappa > 0$,*

$$\lim_{r,s\to\infty} \frac{1}{(\lambda_{rs})^\alpha} |\{u \in I_r, v \in J_s: \mathcal{M}(\|\xi_{uv} - \xi\| \geq r + \varepsilon) \geq \kappa\}| = 0.$$

In that case, we write $\xi_{uv} \xrightarrow{r-S{\lambda_2}^\alpha} \xi$.

Definition 2.5: *A sequence $\{\xi_{uv}\}$ is said to be rough (V, λ)-summable in measure* of order *α to ξ, with roughness degree r if*

$$\lim_{r,s\to\infty} \frac{1}{(\lambda_{rs})^\alpha} \sum_{u\in I_r, v\in J_s} \mathcal{M}(\|\xi_{uv} - \xi\| \geq r + \varepsilon) = 0.$$

In that case, we write $\xi_{uv} \xrightarrow{r-[V,\lambda^2]^\alpha} \xi$.

Theorem 2.8. *For any complex uncertain sequence $\{\xi_{uv}\}$ the following are equivalent:*

i. $\xi_{uv} \xrightarrow{r-S{\lambda_2}^\alpha} \xi$.

ii. $\xi_{uv} \xrightarrow{r-[V,\lambda^2]^\alpha} \xi$.

Proof. $(i) \Rightarrow (ii)$: First assume that $\xi_{uv} \xrightarrow{r-S{\lambda_2}^\alpha} \xi$. Then, we can write

$$\begin{aligned}\frac{1}{(\lambda_{rs})^{\alpha}}\sum_{u\in I_r, v\in J_s}\mathcal{M}(\|\xi_{uv}-\xi\|\geq r+\varepsilon)\\ =\frac{1}{(\lambda_{rs})^{\alpha}}\sum_{u\in I_r, v\in J_{s,\mathcal{M}(\|\xi_{uv}-\xi\|\geq r+\varepsilon)\geq\frac{\kappa}{2}}}\mathcal{M}(\|\xi_{uv}-\xi\|\geq r+\varepsilon)\\ +\frac{1}{(\lambda_{rs})^{\alpha}}\sum_{u\in I_r, v\in J_{s,\mathcal{M}(\|\xi_{uv}-\xi\|\geq r+\varepsilon)<\frac{\kappa}{2}}}\mathcal{M}(\|\xi_{uv}-\xi\|\geq r+\varepsilon)\leq\frac{1}{(\lambda_{rs})^{\alpha}}\Big|\{u\in I_r, v\\ \in J_s: \mathcal{M}(\|\xi_{uv}-\xi\|\geq r+\varepsilon)\geq r+\varepsilon\}\geq\frac{\kappa}{2}\Big|+\frac{\kappa}{2}.\end{aligned}$$

$(i)\Rightarrow(ii)$: Now assume that condition (ii) holds. Then,

$$\begin{aligned}\sum_{u\in I_r, v\in J_s}\mathcal{M}(\|\xi_{uv}-\xi\|\geq r+\varepsilon)\\ \geq\sum_{u\in I_r, v\in J_{s,\mathcal{M}(\|\xi_{uv}-\xi\|\geq r+\varepsilon)\geq\kappa}}\mathcal{M}(\|\xi_{uv}-\xi\|\geq r+\varepsilon)\\ \geq\kappa|\{u\in I_r, v\in J_s: \mathcal{M}(\|\xi_{uv}-\xi\|\geq r+\varepsilon)\}\geq\kappa|.\end{aligned}$$

Hence

$$\frac{1}{(\lambda_{rs})^{\alpha}}\sum_{u\in I_r, v\in J_s}\mathcal{M}(\|\xi_{uv}-\xi\|\geq r+\varepsilon)\geq\frac{1}{(\lambda_{rs})^{\alpha}}|\{u\in I_r, v\in J_s: \mathcal{M}(\|\xi_{uv}-\xi\|\geq r+\varepsilon)\}\geq\kappa|.$$

As a result $\xi_{uv}\xrightarrow{r-S\lambda_2{}^{\alpha}}\xi$. This completes the proof of the theorem.

Theorem 2.9. *If* $\xi_{uv}\xrightarrow{r-S\lambda_2{}^{\alpha}}\xi_1$ *and* $\xi_{uv}\xrightarrow{r-S\lambda_2{}^{\alpha}}\xi_2$, *then* $M\{\|\xi_1-\xi_2\|\geq r_1+r_2\}=0$.

3. Conclusion

The notion of rough statistical convergence of complex uncertain sequence has been worked by Debnath and Das [23]. In this study, for the first time, we define the concept of rough statistical convergence of order α $(0<\alpha\leq 1)$ for complex uncertain double sequences, which is the generalization of convergence concepts of complex uncertain variables. These results unify and generalize the existing results. It may attract the future researcher's in this direction.

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Rough Statistical Λ^2-Convergence of Double Sequences of Order α

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Abstract

In this study, we aim to propose rough statistical Λ^2-convergence of double sequences of order α $(0 < \alpha \leq 1)$ in normed linear spaces (NLS). We obtain some fundamental features and also established some examples to show that this new convergence type is more generalized than the rough statistical convergence. In addition, we demonstrate the consequences related to statistically Λ^2-bounded sets of order α and sets of rough statistically Λ^2-convergent sequences of order α.

Keywords: Rough statistical convergence, rough statistical limit points, normed linear space.

1. Introduction

The thought of statistical convergence of double sequences was proposed by Mursaleen and Edely [1]. The conception of statistical convergence of a double sequence is a generalization of the convergence of a double sequence in Pringsheim's sense [2]. Rough statistical convergence for single sequences was introduced by Aytar [3] utilizing the notion of natural density, which is a generalization of the rough convergence of single sequences [4]. Also Maity [5] proposed the notion of rough statistical convergence of order α $(0 < \alpha \leq 1)$. Rough convergence of double sequences was investigated by Malik and Maity [6]. The authors extended this idea in [7] and examined rough statistical convergence for double sequences in NLS. The statistical convergence was worked in random normed spaces [8]. In the study [9], the authors investigated rough statistical Λ-convergence of order α $(0 < \alpha \leq 1)$ for single sequences in NLS. In this work, we extend the concepts in [9] to double sequences and obtain significant results.

2. Main Results

Definition 2.1: *A sequence $w = \{w_{rs}\}$ in an NLS $(Y, \|.\|)$ is named to be rough Λ^2-convergent to $\sigma \in Y$ provided that for each $\xi > 0$ there exist $r > 0$ and $t_0 \in \mathbb{N}$ so that $\|\Lambda^2 w_{rs} - \sigma\| < r + \xi$, for all $r, s > t_0$.*

Definition 2.2: *A sequence $w = \{w_{rs}\}$ in an NLS $(Y, \|.\|)$ is named to be rough statistically Λ^2-convergent to $\sigma \in Y$ provided that for each $\xi > 0$ there is a $r > 0$ so that*

$$\lim_{u,v\to\infty} \frac{1}{uv} |\{r \leq u, s \leq v : \|\Lambda^2 w_{rs} - \sigma\| \geq r + \xi\}| = 0,$$

where ξ is known as $r - St_2(\Lambda^2)$-limit of a sequence $w = \{w_{rs}\}$.

Remark 2.1. *For the case $r = 0$, the notion of rough statistical Λ^2-convergence agrees with the statistical Λ^2-convergence.*

Definition 2.3: *A sequence $w = \{w_{rs}\}$ in an NLS $(Y, \|.\|)$ is named to be rough statistically Λ^2-convergent of order α $(0 < \alpha \leq 1)$ to $\sigma \in Y$ provided that for each $\xi > 0$ there is a $r > 0$ so that*

$$\lim_{u,v\to\infty} \frac{1}{(uv)^\alpha} |\{r \leq u, s \leq v : \|\Lambda^2 w_{rs} - \sigma\| \geq r + \xi\}| = 0,$$

where σ is denoted as $r - {St_2}^\alpha(\Lambda^2)$-limit of a sequence w. We denote $w_{rs} \xrightarrow{r-{St_2}^\alpha(\Lambda^2)} \sigma$.

In general, the $r - {St_2}^\alpha(\Lambda^2)$-limit of a sequence may not be unique. Hence, we contemplate $r - {St_2}^\alpha(\Lambda^2)$-limit set of a sequence $\mathrm{w} = \{\mathrm{w_{rs}}\}$ as

$$r - {St_2}^\alpha(\Lambda^2) - LT_w = \left\{\sigma : \mathrm{w_{rs}} \xrightarrow{r-{St_2}^\alpha(\Lambda^2)} \sigma\right\}.$$

The sequence $\mathrm{w} = \{\mathrm{w_{rs}}\}$ is named to be $r - {St_2}^\alpha(\Lambda^2)$-convergent so that $r - {St_2}^\alpha(\Lambda^2) - LT_w \neq \emptyset$. But the rough limit set is empty for unbounded sequence.

Example 2.1. Take $\mathrm{Y} = \mathbb{R}$. Determine the subsequent sequence

$$\Lambda^2 \mathrm{w_{rs}} = \begin{cases} (-1)^{r+s}, & r \neq m^2, s \neq n^2, \\ rs, & \text{otherwise.} \end{cases}$$

Let $\alpha = 1$, then

$$\mathrm{r} - \mathrm{St_2}^{\alpha}(\Lambda^2) - \mathrm{LT_w} = \begin{cases} \emptyset, & \mathrm{r} < 1, \\ [1-\mathrm{r}, \mathrm{r}-1], & \text{otherwise.} \end{cases}$$

and $r - \Lambda^2 - LT_w = \emptyset$ for all $r \geq 0$. So, this sequence is divergent in ordinary sense since it is unbounded. In addition, the sequence is not $St_2{}^{\alpha}(\Lambda^2)$-convergent for any r.

Definition 2.4: *A point σ is named to be rough statistically Λ^2-cluster point of order α $(0 < \alpha \leq 1)$ of a sequence $w = \{w_{rs}\}$ in an NLS $(Y, \|.\|)$ provided that for each $\xi > 0$ there is a $r > 0$ so that*

$$\lim_{u,v\to\infty} \frac{1}{(uv)^{\alpha}} |\{r \leq u, s \leq v \colon \|\Lambda^2 w_{rs} - \sigma\| \geq r + \xi\}| \neq 0.$$

Definition 2.5: *A sequence $w = \{w_{rs}\}$ is named to be statistically Λ^2-bounded provided that there is a $P_0 > 0$ so that*

$$\lim_{u,v\to\infty} \frac{1}{uv} |\{r \leq u, s \leq v \colon \|\Lambda^2 w_{rs}\| \geq P_0\}| = 0.$$

Definition 2.5: *A sequence $w = \{w_{rs}\}$ is named to be statistically Λ^2-bounded of order $\alpha (0 < \alpha \leq 1)$ provided that there is a $P_0 > 0$ so that*

$$\lim_{u,v\to\infty} \frac{1}{(uv)^{\alpha}} |\{r \leq u, s \leq v \colon \|\Lambda^2 w_{rs}\| \geq P_0\}| = 0.$$

As a result of above definitions, we acquired the subsequent significant results on rough statistical Λ^2-convergence.

Theorem 2.1. *Each rough Λ^2-convergent sequence is also rough statistically Λ^2-convergent of order $\alpha(0 < \alpha \leq 1)$, but converse can be not true.*

Proof. Presume that the sequence $w = \{w_{rs}\}$ be rough Λ^2-convergent in a NLS $(\mathrm{Y}, \|.\|)$. Then, for each $\xi > 0$ and some $r > 0$ there is a $P_0 > 0$ so that $\|\Lambda^2 w_{rs} - \sigma\| \geq r + \xi$ for all $r, s \geq P_0$.

The set $\{r \leq u, s \leq v: \|\Lambda^2 w_{rs} - \sigma\| \geq r + \xi\}$ has finitely many terms. So,

$$\lim_{u,v\to\infty} \frac{1}{(uv)^\alpha} |\{r \leq u, s \leq v: \|\Lambda^2 w_{rs} - \sigma\| \geq r + \xi\}| = 0.$$

As a result, the sequence $w = \{w_{rs}\}$ is rough statistically Λ^2-convergent of order $\alpha(0 < \alpha \leq 1)$.

The contrary part is false that can be verified by the subsequent example.

Example 2.2. Take $\mathrm{Y} = \mathbb{R}$. Contemplate $(\mathbb{R}, \|.\|)$ with usual norm. Determine a sequence

$$\Lambda^2 \mathrm{w}_{\mathrm{rs}} = \begin{cases} 1, & r = m^2, s = n^2, \\ 0, & \text{otherwise.} \end{cases}$$

For $\xi > 0$ and some $r > 0$, we acquire

$$\begin{aligned} K(\xi, r) &= \{r \leq u, s \leq v: \|\Lambda^2 w_{rs} - 0\| \geq r + \xi\} = \{r \leq u, s \leq v: \|\Lambda^2 w_{rs}\| \geq r + \xi > 0\} \\ &= \{r \leq u, s \leq v: \|\Lambda^2 w_{rs}\| = 1\} = \{r \leq u, s \leq v: r = m^2, s = n^2\}. \end{aligned}$$

Therefore,

$$\lim_{u,v\to\infty} \frac{1}{(uv)^\alpha} |K(\xi, r)| \leq \lim_{u,v\to\infty} \frac{\sqrt{uv}}{(uv)^\alpha} = 0.$$

So, $w = \{w_{rs}\}$ is rough statistically Λ^2-convergent of order α to 0, for $\alpha > \frac{1}{2}$.

Now, we examine the algebraic feature of rough statistically Λ^2-convergent sequences of order $\alpha(0 < \alpha \leq 1)$.

Theorem 2.2. *Assume that* $w = \{w_{rs}\}$ *and* $p = \{p_{rs}\}$ *be two sequences in a in a NLS* $(Y, \|.\|)$ *and* $(0 < \alpha \leq 1)$ *be given. For some* $r > 0$ *the subsequent supplies:*

1) *When* $w_{rs} \xrightarrow{r-St_2{}^{\alpha}(\Lambda^2)} \sigma_1$ *and* $s \in \mathbb{N}$ *then* $sw_{rs} \xrightarrow{r-St_2{}^{\alpha}(\Lambda^2)} s\sigma_1$;
2) *When* $w_{rs} \xrightarrow{r-St_2{}^{\alpha}(\Lambda^2)} \sigma_1$ *and* $p_{rs} \xrightarrow{r-St_2{}^{\alpha}(\Lambda^2)} \sigma_2$ *then* $(w_{rs} + p_{rs}) \xrightarrow{r-St_2{}^{\alpha}(\Lambda^2)} (\sigma_1 + \sigma_2)$.

Proof. 1) When $k = 0$, then it is obvious.

Consider $s \neq 0$. As $w_{rs} \xrightarrow{r-St_2{}^{\alpha}(\Lambda^2)} \sigma_1$, then for given $\xi > 0$ and some $r > 0$, we obtain the set

$$K(\xi, r\,) = \left\{r \leq u, s \leq v \colon \|\Lambda^2 w_{rs} - \sigma_1\| \geq \frac{r+\xi}{|s|}\right\} \quad \text{with} \lim_{u,v\to\infty} \frac{|K(\xi, r\,)|}{(uv)^{\alpha}} = 0.$$

Let $(r, s) \in K(\xi, r\,)^c$. Then

$$\|\Lambda^2 s w_{rs} - s\sigma_1\| = |s|\|\Lambda^2 w_{rs} - \sigma_1\| < |s|\left(\frac{r+\xi}{|s|}\right) < r + \xi.$$

This gives that

$$\lim_{u,v\to\infty} \frac{1}{(uv)^{\alpha}} \left|\left\{r \leq u, s \leq v \colon \|\Lambda^2 s w_{rs} - s\sigma_1\| < \frac{r+\xi}{|s|}\right\}\right| = 1,$$

i.e.

$$\lim_{u,v\to\infty}\frac{1}{(uv)^\alpha}\left|\left\{r\le u, s\le v: \|\Lambda^2 s w_{rs} - s\sigma_1\| \ge \frac{r+\xi}{|s|}\right\}\right| = 0.$$

As a result, $s w_{rs} \xrightarrow{r-St_2{}^\alpha(\Lambda^2)} s\sigma_1$.

2) As $w_{rs} \xrightarrow{r-St_2{}^\alpha(\Lambda^2)} \sigma_1$ and $p_{rs} \xrightarrow{r-St_2{}^\alpha(\Lambda^2)} \sigma_2$, then for given $\xi > 0$ and some $r > 0$, we obtain the set

$$K_1(\xi, r) = \left\{r\le u, s\le v: \|\Lambda^2 w_{rs} - \sigma_1\| \ge \frac{r+\xi}{2}\right\} \quad \text{with} \quad \lim_{u,v\to\infty}\frac{|K_1(\xi, r)|}{(uv)^\alpha} = 0,$$

$$K_2(\xi, r) = \left\{r\le u, s\le v: \|\Lambda^2 p_{rs} - \sigma_2\| \ge \frac{r+\xi}{2}\right\} \quad \text{with} \quad \lim_{u,v\to\infty}\frac{|K_2(\xi, r)|}{(uv)^\alpha} = 0.$$

Take $(r,s) \in K_1(\xi, r)^c \cap K_2(\xi, r)^c$. Then

$$\|\Lambda^2(w_{rs}+p_{rs}) - (\sigma_1+\sigma_2)\| \le \|\Lambda^2 w_{rs} - \sigma_1\| + \|\Lambda^2 p_{rs} - \sigma_2\| < \frac{r+\xi}{2} + \frac{r+\xi}{2} = r+\xi.$$

This gives that

$$\lim_{u,v\to\infty}\frac{1}{(uv)^\alpha}|\{r\le u, s\le v: \|\Lambda^2(w_{rs}+p_{rs}) - (\sigma_1+\sigma_2)\| < r+\xi\}| = 1,$$

i.e.

$$\lim_{u,v\to\infty}\frac{1}{(uv)^\alpha}|\{r\le u, s\le v: \|\Lambda^2(w_{rs}+p_{rs}) - (\sigma_1+\sigma_2)\| \ge r+\xi\}| = 0.$$

As a result, we acquire $(w_{rs}+p_{rs}) \xrightarrow{r-St_2{}^{\alpha}(\Lambda^2)} (\sigma_1+\sigma_2)$.

Theorem 2.3. *Assume that* $0<\alpha\leq\beta\leq 1$ *then* $rSt_2{}^{\alpha}\subseteq rSt_2{}^{\beta}$ *where* $rSt_2{}^{\alpha}$ *and* $rSt_2{}^{\beta}$ *denotes the sets of all rough statistically* Λ^2*-convergent of orders* α,β *respectively*.

Proof. Take $w=\{w_{rs}\}$ as a sequence in a NLS $(\mathrm{Y},\|.\|)$. When $0<\alpha\leq\beta\leq 1$ then for each $\xi>0$ and some $r>0$ with the limit point σ, we obtain

$$\lim_{u,v\to\infty}\frac{1}{(uv)^{\beta}}|\{r\leq u,s\leq v:\|\Lambda^2 w_{rs}-\sigma\|\geq r+\xi\}|$$
$$\leq \lim_{u,v\to\infty}\frac{1}{(uv)^{\alpha}}|\{r\leq u,s\leq v:\|\Lambda^2 w_{rs}-\sigma\|\geq r+\xi\}|.$$

So, we have $rSt_2{}^{\alpha}\subseteq rSt_2{}^{\beta}$.

Theorem 2.4. *A sequence* $w=\{w_{rs}\}$ *as a sequence in an NLS* $(Y,\|.\|)$ *is statistically* Λ^2*-bounded of order* α $(0<\alpha\leq 1)$ *iff* $r-St_2{}^{\alpha}-LT_w\neq\emptyset$, *for some* $r>0$.

Proof. Assume that the sequence $w=\{w_{rs}\}$ is statistically Λ^2-bounded of order α $(0<\alpha\leq 1)$, then there is a $P_0>0$ so that

$$\lim_{u,v\to\infty}\frac{1}{(uv)^{\alpha}}|\{r\leq u,s\leq v:\|\Lambda^2 w_{rs}\|\geq P_0\}|=0.$$

Consider $K=\{r\leq u,s\leq v:\|\Lambda^2 w_{rs}\|\geq P_0\}$. Determine $r_0=\sup\{\|\Lambda^2 w_{rs}\|:(r,s)\in K^c\}$. Since

$$0\in r_0-rSt_2{}^{\alpha}-LT_w\Rightarrow r-rSt_2{}^{\alpha}-LT_w\neq\emptyset.$$

But, presume that $r-rSt_2{}^{\alpha}-LT_w\neq\emptyset$ for some $r>0$. Then, for all $\xi>0$ there is a $\sigma\in Y$ so that $\sigma\in r_0-rSt_2{}^{\alpha}-LT_w$. Then

$$\lim_{u,v\to\infty}\frac{1}{(uv)^\alpha}|\{r\le u, s\le v: \|\Lambda^2 w_{rs}-\sigma\|\ge r+\xi\}|=0.$$

So, $w=\{w_{rs}\}$ is statistically Λ^2-bounded of order α.

Theorem 2.5. *When* $w'=\left\{w_{r_j s_k}\right\}$ *is a subsequence of a sequence* $w=\{w_{rs}\}$ *then*

$$r-{St_2}^\alpha-LT_w\subseteq r-{St_2}^\alpha-LT_{w'}.$$

Theorem 2.6. *Assume that the sequence* $w=\{w_{rs}\}$ *is sequence in an NLS* $(Y,\|.\|)$. *Then,* $r-{St_2}^\alpha-LT_w$ *(the rough statistical limit set of order* $(0<\alpha\le 1)$*) is convex.*

Proof. Take $\sigma_1,\sigma_2\in r-r{St_2}^\alpha-LT_w$ and $\xi>0$. For the convexity of the set $r-{St_2}^\alpha-LT_w$, we have to prove that $[(1-\beta)\sigma_1+\beta\sigma_2]\in r-{St_2}^\alpha-LT_w$ for some $\beta\in(0,1)$. We determine

$$K_1(\xi,r)=\left\{(r,s)\in\mathbb{N}\times\mathbb{N}: \|\Lambda^2 w_{rs}-\sigma_1\|\ge\frac{r+\xi}{2(1-\beta)}\right\},$$

$$K_2(\xi,r)=\left\{(r,s)\in\mathbb{N}\times\mathbb{N}: \|\Lambda^2 w_{rs}-\sigma_2\|\ge\frac{r+\xi}{2\beta}\right\}.$$

Since $\sigma_1,\sigma_2\in r-{St_2}^\alpha-LT_w$, we obtain

$$\lim_{u,v\to\infty}\frac{|K_1(\xi,r)|}{(uv)^\alpha}=\lim_{u,v\to\infty}\frac{|K_2(\xi,r)|}{(uv)^\alpha}=0.$$

Take $(r,s)\in K_1(\xi,r)^c\cap K_2(\xi,r)^c$. Then

$$\|\Lambda^2 w_{rs} - [(1-\beta)\sigma_1 + \beta\sigma_2]\| = \|(1-\beta)(\Lambda^2 w_{rs} - \sigma_1) + \beta(\Lambda^2 w_{rs} - \sigma_2)\|$$
$$\leq (1-\beta)\|\Lambda^2 w_{rs} - \sigma_1\| + \beta\|\Lambda^2 w_{rs} - \sigma_2\| < r + \xi.$$

As

$$\lim_{u,v\to\infty} \frac{|K_1(\xi, r)^c \cap K_2(\xi, r)^c|}{(uv)^\alpha} = 1,$$

we get

$$\lim_{u,v\to\infty} \frac{|\{(r,s) \in \mathbb{N}\times\mathbb{N}: \|\Lambda^2 w_{rs} - [(1-\beta)\sigma_1 + \beta\sigma_2]\| \geq r + \xi\}|}{(uv)^\alpha} = 0,$$

i.e.

$$[(1-\beta)\sigma_1 + \beta\sigma_2] \in r - {St_2}^\alpha - LT_w.$$

Hence, $r - {St_2}^\alpha - LT_w$ is a convex set.

Theorem 2.7. *A sequence $w = \{w_{rs}\}$ is sequence in an NLS $(Y, \|.\|)$ is rough statistically Λ^2-convergent of order α $(0 < \alpha \leq 1)$ to $\sigma \in Y$ provided that for some $r > 0$ iff there is a sequence $p = \{p_{rs}\}$ in Y that is rough statistically Λ^2-convergent of order α $(0 < \alpha \leq 1)$ to σ and $\|\Lambda^2 w_{rs} - \Lambda^2 p_{rs}\| \leq r$ for all $r, s \in \mathbb{N}$.*

Proof. Assume that $w_{rs} \xrightarrow{r-{St_2}^\alpha(\Lambda^2)} \sigma$. Then, for all $\xi > 0$ and some $r > 0$ we obtain

$$\lim_{u,v\to\infty} \frac{1}{(uv)^\alpha} |\{r \leq u, s \leq v: \|\Lambda^2 w_{rs} - \sigma\| \geq r + \xi\}| = 0.$$

Now, we identify the sequence as

$$\Lambda^2 p_{rs} = \begin{cases} \xi, & \text{when } \|\Lambda^2 \mathrm{w}_{\mathrm{rs}} - \xi\| \leq r, \\ \Lambda^2 \mathrm{w}_{\mathrm{rs}} + r\dfrac{\xi - \Lambda^2 \mathrm{w}_{\mathrm{rs}}}{\|\Lambda^2 \mathrm{w}_{\mathrm{rs}} - \xi\|}, & \text{otherwise.} \end{cases}$$

Then, we acquire

$$\Lambda^2 p_{rs} - \xi = \begin{cases} 0, & \text{when } \|\Lambda^2 \mathrm{w}_{\mathrm{rs}} - \xi\| \leq r, \\ \dfrac{\Lambda^2 \mathrm{w}_{\mathrm{rs}} - \xi}{\|\Lambda^2 \mathrm{w}_{\mathrm{rs}} - \xi\|}(\|\Lambda^2 \mathrm{w}_{\mathrm{rs}} - \xi\| - r), & \text{otherwise.} \end{cases}$$

so that $\|\Lambda^2 w_{rs} - \Lambda^2 p_{rs}\| \leq r$ for all $r, s \in \mathbb{N}$. Moreover,

$$\|\Lambda^2 p_{rs} - \xi\| = \begin{cases} 0, & \text{when } \|\Lambda^2 w_{rs} - \xi\| \leq r, \\ \|\Lambda^2 \mathrm{w}_{\mathrm{rs}} - \xi\| - r, & \text{otherwise.} \end{cases}$$

According to definition of $\Lambda^2 p_{rs}$ and utilizing

$$\lim_{u,v\to\infty} \frac{1}{(uv)^\alpha} |\{r \leq u, s \leq v: \|\Lambda^2 w_{rs} - \sigma\| \geq r + \xi\}| = 0,$$

that demonstrate that the sequence $p = \{p_{rs}\}$ is rough statistically Λ^2-convergent of order α $(0 < \alpha \leq 1)$ to σ.

As the sequence $p = \{p_{rs}\}$ is rough statistically Λ^2-convergent of order α $(0 < \alpha \leq 1)$ to σ then for $\xi > 0$ we acquire

$$\lim_{u,v\to\infty} \frac{1}{(uv)^\alpha} |\{r \leq u, s \leq v: \|\Lambda^2 p_{rs} - \sigma\| \geq r + \xi\}| = 0.$$

For some $r > 0$ and the sequence $w = \{w_{rs}\}$ with $\|\Lambda^2 w_{rs} - \Lambda^2 p_{rs}\| \leq r$, the subsequent inclusion supplies

$$\{r \leq u, s \leq v : \|\Lambda^2 w_{rs} - \sigma\| \geq r + \xi\} \subseteq \{r \leq u, s \leq v : \|\Lambda^2 p_{rs} - \sigma\| \geq r + \xi\}.$$

So, we have

$$\lim_{u,v\to\infty} \frac{1}{(uv)^\alpha} |\{r \leq u, s \leq v : \|\Lambda^2 w_{rs} - \sigma\| \geq r + \xi\}| = 0.$$

Theorem 2.8. *The set $r - St_2{}^\alpha - LT_w$ (rough statistical Λ^2-limit set of order α $(0 < \alpha \leq 1)$) is closed.*

Proof. i) When $r - St_2{}^\alpha - LT_w = \emptyset$, then we have not to prove anything.

ii) When $r - St_2{}^\alpha - LT_w \neq \emptyset$, then consider a sequence $p = \{p_{rs}\} \subseteq r - St_2{}^\alpha - LT_w$ so that $\Lambda^2 p_{rs} \to p_*$ for $r, s \to \infty$. It is adequate to denote that $p_* \in r - St_2{}^\alpha - LT_w$.

As $\Lambda^2 p_{rs} \to p_*$, then for given $\xi > 0$ there are $r_{\frac{\varepsilon}{2}}, s_{\frac{\varepsilon}{2}} \in \mathbb{N}$ so that

$$\|\Lambda^2 p_{rs} - p_*\| < \frac{r + \xi}{3}$$

For $r > r_{\frac{\varepsilon}{2}}, s > s_{\frac{\varepsilon}{2}}$. Now, select $r_0, s_0 \in \mathbb{N}$ so that $r_0 > r_{\frac{\varepsilon}{2}}, s_0 > s_{\frac{\varepsilon}{2}}$. Then, we get

$$\left\|\Lambda^2 p_{r_0 s_0} - p_*\right\| < \frac{r + \xi}{3}.$$

Also as $p = \{p_{rs}\} \subseteq r - St_2{}^\alpha - LT_w$, we get $p_{r_0 s_0} \in r - St_2{}^\alpha - LT_w$. Obviously,

$$\lim_{u,v\to\infty}\frac{1}{(uv)^{\alpha}}\left|\left\{r\le u, s\le v: \left\|\Lambda^2 w_{rs}-p_{r_0 s_0}\right\|\ge\frac{r+\xi}{3}\right\}\right|=0.$$

We demonstrate the inclusion

$$\left\{r\le u, s\le v: \left\|\Lambda^2 w_{rs}-p_{r_0 s_0}\right\|<\frac{r+\xi}{3}\right\}\subseteq\{r\le u, s\le v: \|\Lambda^2 w_{rs}-p_*\|<r+\xi\}.$$

Take

$$(i,j)\in\left\{r\le u, s\le v: \left\|\Lambda^2 w_{rs}-p_{r_0 s_0}\right\|<\frac{r+\xi}{3}\right\}\Rightarrow\left\|\Lambda^2 w_{ij}-p_{r_0 s_0}\right\|<\frac{r+\xi}{3}.$$

So,

$$\begin{aligned}\left\|\Lambda^2 w_{ij}-p_*\right\|&=\left\|\Lambda^2 w_{ij}-p_{r_0 s_0}+\Lambda^2 p_{rs}-p_*-\Lambda^2 p_{rs}+p_{r_0 s_0}\right\|\\&\le\left\|\Lambda^2 w_{ij}-p_{r_0 s_0}\right\|+\|\Lambda^2 p_{rs}-p_*\|+\left\|\Lambda^2 p_{ij}-p_{r_0 s_0}\right\|.\end{aligned}$$

Utilizing equation

$$\lim_{u,v\to\infty}\frac{1}{(uv)^{\alpha}}\left|\left\{r\le u, s\le v: \left\|\Lambda^2 w_{rs}-p_{r_0 s_0}\right\|\ge\frac{r+\xi}{3}\right\}\right|=0$$

and Theorem 2.7 we acquire $\left\|\Lambda^2 p_{ij}-p_{r_0 s_0}\right\|<\frac{r+\xi}{3}$.

Thus,

$$\left\|\Lambda^2 w_{ij} - p_*\right\| < \frac{r+\xi}{3} + \frac{r+\xi}{3} + \frac{r+\xi}{3} = r + \xi.$$

This gives that

$$(i,j) \in \left\{r \leq u, s \leq v \colon \|\Lambda^2 w_{rs} - p_*\| < \frac{r+\xi}{3}\right\}.$$

Hence, we obtain

$$\left\{r \leq u, s \leq v \colon \left\|\Lambda^2 w_{rs} - p_{r_0 s_0}\right\| < \frac{r+\xi}{3}\right\} \subseteq \{r \leq u, s \leq v \colon \|\Lambda^2 w_{rs} - p_*\| < r + \xi\}.$$

As a result,

$$\{r \leq u, s \leq v \colon \|\Lambda^2 w_{rs} - p_*\| \geq r + \xi\} \subseteq \left\{r \leq u, s \leq v \colon \left\|\Lambda^2 w_{rs} - p_{r_0 s_0}\right\| \geq \frac{r+\xi}{3}\right\}.$$

Now,

$$\begin{aligned} &\lim_{u,v\to\infty} \frac{1}{(uv)^\alpha} |\{r \leq u, s \leq v \colon \|\Lambda^2 w_{rs} - p_*\| \geq r + \xi\}| \\ &\qquad \leq \lim_{u,v\to\infty} \frac{1}{(uv)^\alpha} \left|\left\{r \leq u, s \leq v \colon \left\|\Lambda^2 w_{rs} - p_{r_0 s_0}\right\| \geq \frac{r+\xi}{3}\right\}\right|. \end{aligned}$$

Utilizing the equation

$$\lim_{u,v\to\infty}\frac{1}{(uv)^{\alpha}}\left|\left\{r\le u, s\le v : \left\|\Lambda^2 w_{rs}-p_{r_0 s_0}\right\|\ge\frac{r+\xi}{3}\right\}\right|=0,$$

we get

$$\lim_{u,v\to\infty}\frac{1}{(uv)^{\alpha}}|\{r\le u, s\le v : \|\Lambda^2 w_{rs}-p_*\|\ge r+\xi\}|=0.$$

Theorem 2.9. *Assume that $\Gamma_{\Lambda^2 w}$ be the set of all rough statistical Λ^2-cluster points of order α $(0<\alpha\le 1)$ for a sequence $w=\{w_{rs}\}$ in an NLS $(Y,\|.\|)$. Then, for an arbitrary $q\in\Gamma_{\Lambda^2 w}$ and $r>0$, we get $\|\sigma-q\|<r$ for all $\sigma\in r-{St_2}^{\alpha}-LT_w$.*

Proof. We prove by contradiction method. For given α $(0<\alpha\le 1)$, we put a point $q\in\Gamma_{\Lambda^2 w}$ and $\sigma\in r-{St_2}^{\alpha}-LT_w$ so that $\|\sigma-q\|>r$. By selecting $\xi=(\|\sigma-q\|-r)/3$, we have the subsequent inclusion

$$\{r\le u, s\le v : \|\Lambda^2 w_{rs}-\sigma\|\ge r+\xi\}\supseteq\{r\le u, s\le v : \|\Lambda^2 w_{rs}-q\|<\xi\}.$$

As $q\in\Gamma_{\Lambda^2 w}$, then

$$\lim_{u,v\to\infty}\frac{1}{(uv)^{\alpha}}|\{r\le u, s\le v : \|\Lambda^2 w_{rs}-q\|<\xi\}\neq 0|.$$

So we get

$$\lim_{u,v\to\infty}\frac{1}{(uv)^{\alpha}}|\{r\le u, s\le v : \|\Lambda^2 w_{rs}-\sigma\|<r+\xi\}\neq 0|$$

that is a contradiction to $\sigma\in r-{St_2}^{\alpha}-LT_w$.

Theorem 2.10. *Presume that* $w = \{w_{rs}\}$ *be a sequence in a strictly convex NLS* $(Y, \|.\|)$. *Take* $\alpha > 0$ *and* $r > 0$. *When any* $\sigma_0, \sigma_1 \in r - St_2{}^{\alpha} - LT_w$ *with* $\|\sigma_0 - \sigma_1\| = 2r$, *then* $w = \{w_{rs}\}$ *is rough statistically* Λ^2*-convergent of order* α $(0 < \alpha \leq 1)$ *to* $(\sigma_0 + \sigma_1)/2$.

Proof. Take $\gamma \in \Gamma_{\Lambda^2 w}$ and $\sigma_0, \sigma_1 \in r - St_2{}^{\alpha} - LT_w$ so that $\|\sigma_0 - \sigma_1\| = 2\mathrm{r}$. Then, we acquire

$$\|\sigma_0 - \gamma\| \leq r \quad \text{and} \quad \|\sigma_1 - \gamma\| \leq r, \tag{2.1}$$

and by triangle inequality, we acquire

$$\begin{aligned} &\|\sigma_0 - \sigma_1\| \leq \|\sigma_0 - \gamma\| + \|\sigma_1 - \gamma\| \\ &\quad \Rightarrow 2r \leq \|\sigma_0 - \gamma\| + \|\sigma_1 - \gamma\|. \end{aligned} \tag{2.2}$$

Hence, we obtain from (2.1) and (2.2) we get $\|\sigma_0 - \gamma\| = \|\sigma_1 - \gamma\| = r$. Also,

$$\frac{1}{2}(\sigma_1 - \sigma_0) = \frac{1}{2}[(\gamma - \sigma_0) + (\sigma_1 - \gamma)], \tag{2.3}$$

and utilizing $\|\sigma_0 - \sigma_1\| = 2\mathrm{r}$, we acquire $\frac{(\sigma_0+\sigma_1)}{2} = \mathrm{r}$.

From equation (2.3) and from strictly convexity of the NLS $(\mathrm{Y}, \|.\|)$, we obtain

3. Conclusion

In this study, we examine rough statistical Λ^2-convergence and rough statistical Λ^2-cluster points of double sequences of order α $(0 < \alpha \leq 1)$ in a (NLS). We obtain various significant results.

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Solution of a Lake Pollution Model by Artificial Neural Networks

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Abstract

In this study, a numerical approach for the solution of a lake pollution model is investigated. A system of ordinary differential equations is considered for modelling the phenomena. Artificial neural networks are used for obtaining the approximate solutions. As an activation function tangent hyperbolic function used. Illustrations are presented in order to show the results of approximation.

Keywords: Lake pollution model, artificial neural networks, deep learning, machine learning.

1. Introduction

Pollution of water resources has a great importance in environmental pollution. In this study, we consider a numerical solution of a compartment model the pollution in the interconnected three lakes with channels by using artificial neural network. The proposed system of differential equations modeling three artificial lakes is given by [1] as follows.

$$\begin{aligned}
\frac{dx_1(t)}{dt} &= \frac{F_{21}}{V_2}x_2(t) + \frac{F_{31}}{V_3}x_3(t) + f(t) - \frac{F_{12}}{V_1}x_1(t) - \frac{F_{13}}{V_1}x_1(t),\\
\frac{dx_2(t)}{dt} &= \frac{F_{12}}{V_1}x_1(t) + \frac{F_{32}}{V_3}x_3(t) - \frac{F_{21}}{V_2}x_2(t) - \frac{F_{23}}{V_2}x_2(t),\\
\frac{dx_3(t)}{dt} &= \frac{F_{13}}{V_1}x_1(t) + \frac{F_{23}}{V_2}x_2(t) - \frac{F_{31}}{V_3}x_3(t) - \frac{F_{32}}{V_3}x_3(t),
\end{aligned} \tag{1}$$

with the initial conditions $x_1(0) = p_1$, $x_2(0) = p_2$ and $x_3(0) = p_3$. Where $x_1(t)$, $x_2(t)$ and $x_3(t)$ denote the amount of pollutant in each lake at time *t*. $f(t)$ is the rate of pollutant which enters the Lake 1 per unit time *t*. F_{ij} denotes the amount of water flowing from lake *i* to lake *j* and V_i is the volume of lake *i*.

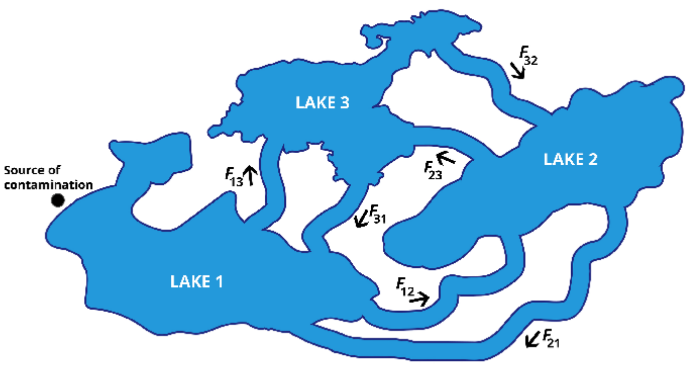


Figure 1. Illustration of the interconnected lakes 1, 2, 3, and flows $F_{12}, F_{13}, F_{21}, F_{23}, F_{31}, F_{32}$ [1].

Also note that it is assumed to ensure the constant volume in each lake as follows.

$$\begin{aligned} F_{12} &= F_{21} + F_{31} - F_{13}, \\ F_{23} &= F_{12} + F_{32} - F_{21}. \end{aligned} \tag{2}$$

The organization of the paper is the following. Initially, preliminaries about artificial neural networks are given. Then the numerical method for finding the approximate solution of the problem is proposed. Afterward, numerical results of the problem are illustrated by figures. The paper finalizes with the concluding remarks and brief discussion of results.

2. Preliminaries

Artificial Neural Networks

One of the latest products of people's efforts to research and imitate nature is artificial neural

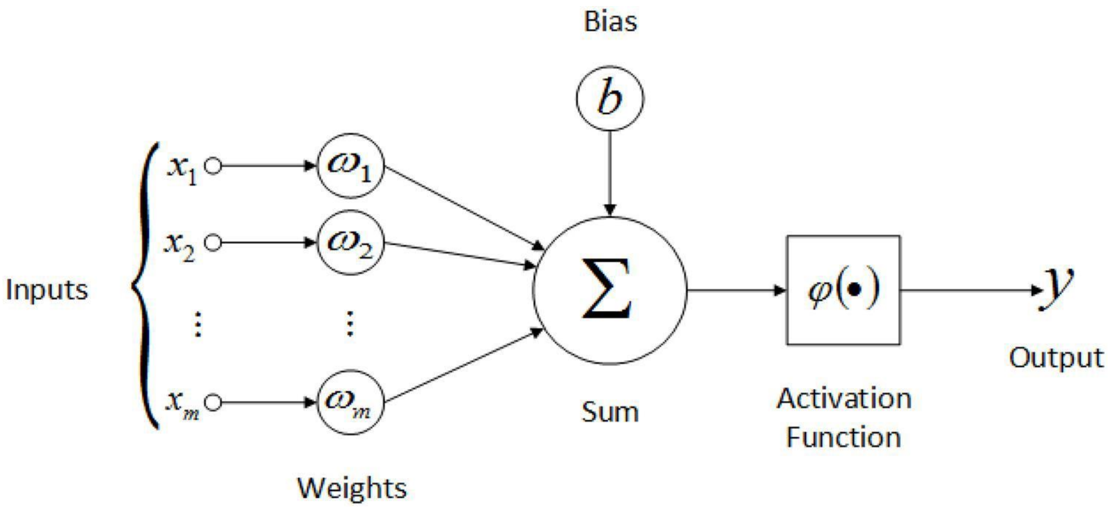


Figure 2. Model of Neuron in Neural Network, *source: (https://dzone.com/articles/the-artificial-neural-networks-handbook-part-4)*

network technology. Artificial neural networks are programs designed to simulate the way the simple biological nervous system works. This simulation contains nerve cells (neurons) and these neurons are connected to each other in various ways to form an artificial neural network. These networks have the capacity to learn, memorize and reveal the relationship between data [2].

In general, an artificial neural network is defined as a system or mathematical model that consists of a large number of non-linear artificial cells that can be arranged in a single layer or multilayer and work in parallel. The weights between cells are adjusted by various learning rules to meet the desired design objectives. With this structure, artificial neural networks are parallel processors that collect information in the learning process and store this information with the help of its weights. Recently, various artificial neural networks and learning algorithms have been developed. It is necessary to select or develop the appropriate neural network structure and learning algorithm for any application [3].

Model of neuron in neural network is given in the Figure 2, the constant b is called the bias or the threshold value of activation function. Generally, the activation function is a nonlinear function [4].

Specifying Ordinary Differential Equations by Artificial Neural Networks

The continuous dynamics of hidden units using an ordinary differential equation (ODE) specified by a neural network are parametrized as follows:

$$\frac{d\mathbf{h}(t)}{dt} = f(\mathbf{h}(t), t, \theta).$$

Starting from the input layer **h**(0), we can define the output layer **h**(T) to be the solution to this ODE initial value problem at some time T. This value can be computed by a black-box differential equation solver, which evaluates the hidden unit dynamics *f* wherever necessary to determine the solution with the desired accuracy [5].

3. Solution Procedure

Neural ordinary differential equations [5] are deep learning operations defined by the solution of an ordinary differential equation. More specifically, neural ordinary differential equation is an operation that can be used in any architecture and, given an input, defines its output as the numerical solution of the ordinary differential equation

$$x' = f(t, x, \theta)$$

for the time horizon (t_0, t_1) and the initial condition $x(t_0) = x_0$. The right-hand side $f(t, x, \theta)$ of the ordinary differential equation depends on a set of trainable parameters θ, which the model learns during the training process. Here, $f(t, x, \theta)$ is modeled with a model function containing nonlinear activations and fully connected operations. The initial condition x_0 is either the input of the entire architecture, as in the case of the investigated model (1), or is the output of a previous operation [6].

This study focuses how to train a neural network with neural ordinary differential equations to learn the pollution dynamics x of a given physical system, described by (1):

$$x' = Ax,$$

where *A* is a 3-by-3 matrix.

The neural network of (1) takes as input an initial condition and computes the ordinary differential equation solution through the learned neural ordinary differential equation model. The neural ordinary differential equation operation, given an initial condition, outputs the solution of an ordinary differential equation model. Particularly, in this study a block with a fully connected layer, a tanh layer, and another fully connected layer are specified as the ordinary differential equation model.

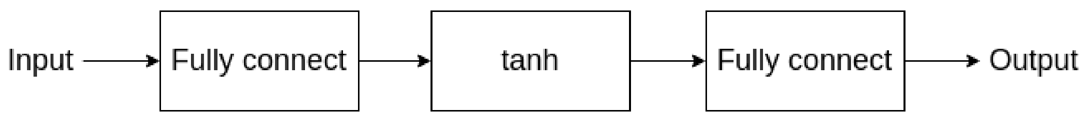


Figure 3. Structure of the neural network [6].

The model (1) is solved numerically with the explicit Runge-Kutta (4,5) pair of Dormand and Prince [7]. The backward pass uses automatic differentiation to learn the trainable parameters θ by backpropagating through each operation of the ordinary differential equation solver. The learned function $f(t, x, \theta)$ is used as the right-hand side for computing the solution of the same model for additional initial conditions [6].

4. Numerical Experiment

In this section, to show the accuracy and efficiency of the presented method, the model (1) is solved with artificial neural network. The parameters are chosen the same as in [1], as $V_1 = 2900$, $V_2 = 850$, $V_3 = 1180$, $F_{12} = 24$, $F_{13} = 22$, $F_{21} = 14$ $F_{23} = 18$, $F_{31} = 32$, $F_{32} = 8$. The pollutant function $f(t) = 100$. Besides, the initial conditions are given by $x_1(0) = 0$, $x_2(0) = 0$ and $x_3(0) = 0$. Numerical calculations and plottings were performed using MATLAB software.

Example: By adapting the parameters the model (1) turns into the following differential equation system

$$\begin{aligned}
\frac{dx_1(t)}{dt} &= \frac{14}{850}x_2(t) + \frac{32}{1180}x_3(t) + 100 - \frac{46}{2900}x_1(t),\\
\frac{dx_2(t)}{dt} &= \frac{24}{2900}x_1(t) + \frac{8}{1180}x_3(t) - \frac{32}{850}x_2(t),\\
\frac{dx_3(t)}{dt} &= \frac{22}{2900}x_1(t) + \frac{18}{850}x_2(t) - \frac{40}{1180}x_3(t).
\end{aligned} \tag{3}$$

The results obtained by artificial neural networks are illustrated in Figure 5. Regarding the graphical representation, we observe the numerical solutions of ordinary differential equation system (3) proposed by artificial neural networks, and the exact solutions are in good agreement.

Figure 4 presents the change in the value of loss function while iterations during machine learning process.

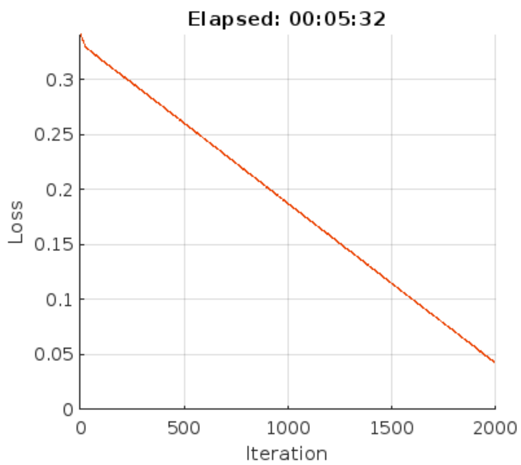
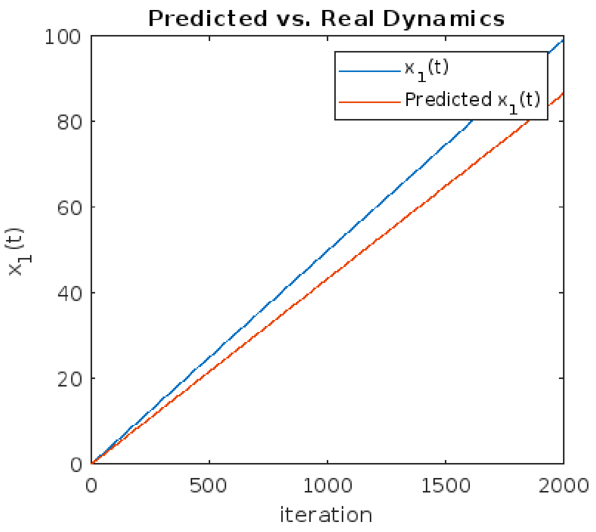
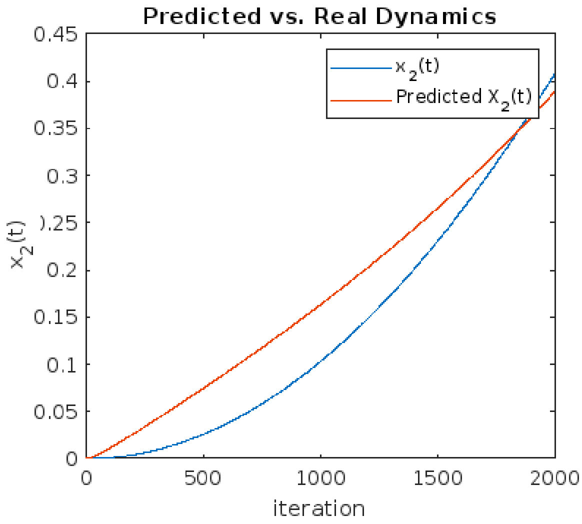


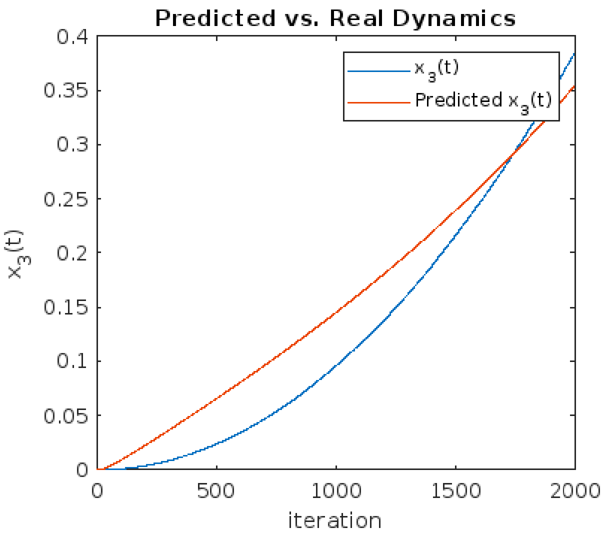
Figure 4. Change of the loss function



(a)



(b)



(c)

Figure 5. Graphical representation of predicted and exact solutions of the system (3); *(a)* the function $x_1(t)$ (pollution in Lake 1), *(b)* the function $x_2(t)$ (pollution in Lake 2), and *(c)* the function $x_3(t)$ (pollution in Lake 3).

5. Conclusion

In this study, we propose an artificial neural network in order to solve the lake pollution system given by [1]. The method has a relatively good approximation and some advantages by using machine learning algorithms. The method is applicable for further problems and their applications at interdiciplinary area.

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Solutions of fractional-order differential equation on harmonic waves and linear wave equation

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Abstract

In this thesis, we will define harmonic waves, linear wave equationon physical problem. [1]-[2] Initially, we will interrupt physical meaning of harmonic waves on time dependent and we will mantion that physical interpretation of linear wave equation with calculating. [1]-[2] After that, we will use many definitions of fractional derivative in order to need to apply on harmonic and linear waves equation.[3]-[4] Therefore, definitions of fractional-order derivative will mentioned in our thesis after interpreting harmonic waves and linear wave equation.

Keywords: Riemann-Liouville fractional derivative, Grünwald-Letnikov fractional derivative, harmonic waves, Linear wave, Oscillation, time-dependent equation.

1. Introduction

Harmonic Waves

First of all, we know that first dimensional and second dimensional wave equation but we can use only Grünwald-Letnikov and Riemann-Liouville fractional derivative on 1-dimensional harmonic waves. Therefore, we consider 1-dimensional waves solutions for using definitions of fractional derivative. Initially, a harmonic wave is defined as a harmonic is a wave with a frequency, which is a positive integer multiple of the fundamental frequency, that is the frequency of the original periodic signal, such as a sinusoidal wave [1].

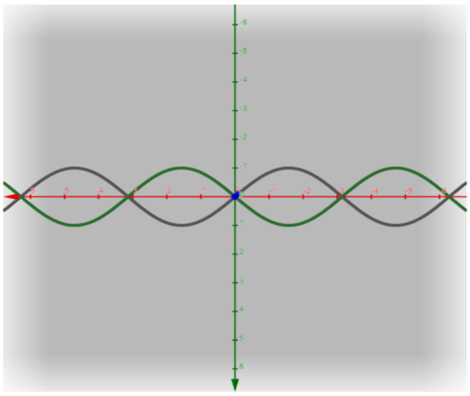

Figure 1: A harmonic wave which moves to the right direction on 1-dimensional.

Also, green wave indicate itself image at $t = 0$ and blue wave define itself image at some later time. Generally we define the displacement of the curve at $t = 0$;

$$y = csin\left(\frac{2\pi}{\lambda}x\right) \qquad (1)$$

c is named the amplitude of the wave, and λ is a wavelength of the wave. When wave moves to the right with a v velocity, wave function is defined at later t time as:

$$y = csin\left[\frac{2\pi}{\lambda}(x - vt)\right] \qquad (2)$$

Distance of the two different all waves are called as a period and it is shown as T. Period, velocity and wavelength are related with together:

$$v = \frac{\lambda}{T} \qquad (3)$$

$$\lambda = vT \qquad (4)$$

When we use (3) and (4) for substituting this into the (1), we write as:

$$y = csin\left[2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)\right] \qquad (5)$$

We are able to easily express the harmonic wave function by defining two quantities called wave number k and angular frequency ω:

$$k \equiv \frac{2\pi}{\lambda} \qquad (6)$$

$$\omega \equiv \frac{2\pi}{T} \qquad (7)$$

Eqn (6) and Eqn(7) can related with Eqn (5):

$$y = csin\left[\left(\frac{2\pi}{\lambda}x - \frac{2\pi}{T}t\right)\right] \qquad (8)$$

$$y = csin(kx - \omega t) \qquad (9)$$

Frequency is the number of waves passing a fixed place for a length of time and the formula is shown as below:

$$f = \frac{1}{T} \qquad (10)$$

f is hertz (Hz) or s^{-1}. Using Equations (6), (7) and (10), we can write the phase velocity v in the alternative forms as [1]:

$$v = \frac{\omega}{k} \qquad (11)$$

$$v = \lambda f \qquad (12)$$

Wave functions that is given by equation (9) considers that the y displacement is zero at $x = 0$ and $t = 0$. We define the wave function as below if and only if transverse displacement is not zero at $x = 0$ and $t = 0$

$$y = csin(kx - \omega t - \phi) \qquad (13)$$

ϕ is a wave constant.

2. Preliminaries

Using definitions of Fractional derivatives on Harmonic Waves solutions:

The wavelength, frequency and 1-dimensional wave was defined. Now, there are several method of solutions on harmonic wave solutions. We need to some definitions from fractional calculus. Especially, not only 1-dimensional both also 2-dimensional wave solutions are mentioned to solve different methods. We say that,

$$y = csin(kx - \omega t) \qquad (14)$$

is harmonic wave functions and this function always dependent on time variable. Especially, we use Riemann-Liouville and Grünwald-Letnikov that are in the fractional calculus for identifying transverse velocity v_y and transverse acceleration a_y.

Definition of Riemann-Liouville and Grünwald-Letnikov definition on transverse velocity:

Riemann-Liouville and Grünwald-Letnikov definition are applied when we assume that only 1-dimensional wave. In (14) when x is a constant. We use fractional derivative and fractional integral to define both transverse velocity and transverse acceleration:

$$v_y = \frac{dy}{dt}$$
$$a_y = \frac{dv_y}{dt}$$

Grünwald-Letnikov definition is [1,3,4]:

$${}_0I_t^1 y(t) = \frac{1}{\Gamma(1)} \int_0^t (t-\tau)^{1-1} y(\tau) d\tau = v_y(t) \qquad (15)$$

Take the fractional Integral of order ϕ on (15) $0 < \phi < 1$:

$${}_\alpha I_t^\phi y(t) = \frac{1}{\Gamma(\phi)} \int_\alpha^t (t-\tau)^{\phi-1}\, y(\tau) = v_y(t) \qquad (16)$$

$${}_0I_t^1 v_y(t) = \frac{1}{\Gamma(1)} \int_0^t (t-\tau)^{1-1} y(\tau) d\tau = a_y(t) \qquad (17)$$

Take the fractional Integral of order ϕ on (15) $0 < \phi < 1$:

$${}_\alpha I_t^\phi v_y(t) = \frac{1}{\Gamma(\phi)} \int_\alpha^t (t-\tau)^{\phi-1}\, v_y(t) d\tau = a_y(t) \qquad (18)$$

Riemann-Liouville fractional derivative definition is [1,3,4]:

$${}_\alpha D_t^\phi y(t) = \frac{1}{\Gamma(n-\phi)} \left(\frac{d}{dx}\right)^n \int_\alpha^t (t-\tau)^{n-\phi-1} y(\tau) d\tau = v_y(t) \qquad (19)$$

$$0 < \phi < 1:$$

$$ {}_{\alpha}D_t^{\phi} v_y(t) = \frac{1}{\Gamma(n-\phi)} \left(\frac{d}{dx}\right)^n \int_{\alpha}^{t} (t-\tau)^{n-\phi-1} v_y(\tau) d\tau = a_y(t) \qquad (19)$$

3. Solutions of fractional-order differential equation on linear wave equation:

Definition: In this part, we will mention that the concept of the wave function to represent waves travelling on a string. Wave functions of $u(x,t)$ is a solution of an equation called the linear wave equation [2]. This solution comes from different methods. Particularly, this method will be mentioned. The linear wave equation is:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 u}{\partial t^2} \qquad (20)$$

Initially, first method is Nigmatullin's fractional diffusion equation that is linear wave equation and this wave equation form is similar as Eqn (20). Let we look at detail:

$$ {}_0D_t^{\alpha} u(x,t) = \lambda^2 \frac{\partial^2 u(x,t)}{\partial x^2}, \quad (t > 0, \ -\infty < x < \infty); \quad (21)$$

$$\lim_{x \to \pm\infty} u(x,t) = 0; \quad [\ {}_0D_t^{\alpha-1} u(x,t)] = \varphi(x). \qquad (22)$$

and $t = 0$ in Eqn (22). Also, order of derivative $0 < \alpha < 1$. An equation (21) was mentioned by Nigmatullin [5] and by Westerlund [6] and studied by Mainardi [7]We will give a basic solution of the problem (21) demonstrating once again the advantage of using the Mittag-Leffler function in two parameters [3]. And we generally define Mittag-Leffler function as:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \qquad (\alpha > 0, \ \ \beta > 0) \qquad (23)$$

When we consider that the boundary conditions of Eqn. (22), take the Fourier transformation with respect to x variable then we write as:

$$ {}_0D_t^{\alpha} \acute{\ddot{u}}(\beta,t) + \lambda^2 \beta^2 \acute{\ddot{u}}(\beta,t) = 0 \qquad (24)$$

$$[\ {}_0D_t^{\alpha-1} \acute{\ddot{u}}(x,t)] = \varphi_1(\beta). \qquad (25)$$

when $t = 0$ is in Eqn. (25). Also β is the Fourier transform parameter. Now, when we take the Laplace transform of Eqn. (24) and apply the initial conditions of (25) then we satisfy:

$$\acute{U}(\beta, s) = \frac{\varphi(\beta)}{s^\alpha + \lambda^2\beta^2} \qquad (26)$$

If inverse Laplace transform of (26) using

$$\int_0^\infty e^{-pt} t^{\alpha k+\beta-1} E_{\alpha,\beta}^{(k)}(\pm\alpha t^\alpha) dt = \frac{k!\, p^{\alpha-\beta}}{(p^\alpha \mp \alpha)^{k+1}}, \qquad \left(Re(p) > |\alpha|^{1/\alpha}\right) \qquad (27)$$

therefore;

$$\acute{u}(\beta, t) = \varphi(\beta) t^{\alpha-1} E_{\alpha,\alpha}(-\lambda^2\beta^2 t^\alpha), \qquad (28)$$

After that, the inverse Fourier transform is produced from the initial value problem of (21) and (22).

$$u(x, t) = \int_{-\infty}^{\infty} G(x-\xi, t)\varphi(\xi) d\xi, \qquad (29)$$

$$G(x, t) = \frac{1}{\pi}\int_0^\infty t^{\alpha-1} E_{\alpha,\alpha}(-\lambda^2\beta^2 t^\alpha) cos\beta x d\beta, \qquad (30)$$

Take the Laplace transform of (30) then we write as;

$$L\left\{\frac{1}{\pi}\int_0^\infty t^{\alpha-1} E_{\alpha,\alpha}(-\lambda^2\beta^2 t^\alpha)\, cos\beta x d\beta\right\}$$

$$g(x, s) = \frac{1}{\pi}\int_0^\infty \frac{\cos(\beta x)\, d\beta}{\lambda^2\beta^2 + s^\alpha} = \frac{1}{2\lambda} s^{-\alpha/2} e^{-|x|\lambda^{-1} s^{\alpha/2}} \qquad (31)$$

and inverse Laplace transform of (31) is:

$$G(x, t) = \frac{1}{4\lambda\pi i}\int_{Br} e^{st} s^{-\frac{\alpha}{2}} \exp\left(-|x|\lambda^{-1} s^{\alpha/2}\right) ds \qquad (32)$$

Use the substitution of $\sigma = st$ and $z = |x|\lambda^{-1} t^{-p}$ and after we use this substitution in (32) we satisfy,

$$G(x, t) = \frac{t^{1-p}}{2\lambda}\frac{1}{2\pi i}\int_{Ha} e^{\sigma - z\sigma^p} \frac{d\sigma}{\sigma^p} = \frac{1}{2\lambda} t^{p-1} W(-z, -p, p), \qquad z = \frac{|x|}{\lambda t^p} \qquad (33)$$

Also, $w(z,\lambda,\mu)$ is defined as the Wright function and it is same as $W(-z,-p,p)$. Wright function is defined as:

$$W(z;\zeta,\eta)=\sum_{k=0}^{\infty}\frac{z^k}{k!\,\Gamma(\zeta k+\eta)} \qquad (34)$$

And the fourier cosine-transform of the function $u_1(\beta)=t^{\alpha-1}E_{\alpha,\alpha}(-\lambda^2\beta^2t^\alpha)$ is evaluated. Finally, when we use $\alpha=1$ in (33) then the fractional Green function (33) is the form [3]:

$$G(x,t)=\frac{1}{2\lambda\sqrt{\pi t}}\exp\left(-\frac{x^2}{4\lambda^2 t}\right) \qquad (35)$$

4. Using Fractional-order partial differential equations on linear wave equations

The Mainardi's fractional diffusion linear wave equation [7] is:

$$ {}_0D_t^{\alpha}u(x,t)=\lambda^2\frac{\partial^2 u(x,t)}{\partial x^2}, \qquad (|x|<\infty, t>0) \qquad (36)$$

$$u(x,0)=f(x) \qquad (|x|<\infty) \qquad (37)$$

$$\lim_{x\to\mp\infty}u(x,t)=0, \qquad (t>0) \qquad (38)$$

where $0<\alpha<1$. The Laplace transform of the formula:

$$L\{\,{}_0D_t^{\sigma_m}f(t);s\}=s^{\sigma_m}F(s)-\sum_{k=0}^{m-1}s^{\sigma_m-\sigma_{m-k}}\left[\,{}_0D_t^{\sigma_{m-k}-1}f(t)\right]\big|_{t=0} \qquad (39)$$

$${}_\alpha D_t^{\sigma_{m-k}-1}\equiv {}_\alpha D_t^{\alpha_{m-k}-1}\,{}_\alpha D_t^{\alpha_{m-k-1}}\dots {}_\alpha D_t^{\alpha_1}, \qquad (k=0,1,\dots,m-1) \quad (40)$$

Then;

$$L\{\,{}_0D_t^{\alpha}y(t);s\}=s^\alpha Y(s)-s^{\alpha-1}y(0), \qquad (41)$$

When (36) and (38) then we satisfy:

$$s^\alpha\acute{\mathrm{u}}(x,s)-s^{\alpha-1}f(x)=\lambda^2\acute{\mathrm{u}}_{xx}(x,s) \quad |x|<\infty \quad (42)$$

$$\lim_{x\to\mp\infty}\acute{\mathrm{u}}(x,s)=0, \qquad (t>0) \quad (43)$$

After that, when we apply exponential fourier transform to equation (42) and utilizing the boundary conditions (43), we obtain:

$$U(\beta,s)=\frac{s^{\alpha-1}}{s^\alpha+\lambda^2\beta^2}F(\beta), \qquad (44)$$

and $U(\beta,p)$ and $F(\beta)$ are the Fourier transforms of $\acute{\mathrm{u}}(x,s)$ and $f(x)$. When we take the inverse Laplace transform of the fraction

$$s^{\alpha-1}/(s^\alpha+\lambda^2\beta^2) \qquad (45)$$

and this Laplace transform of the fraction (45) is $E_{\alpha,1}(-\lambda^2\beta^2t^\alpha)$. And finally, the inversion of the Fourier and the Laplace transform gives the solution below:

$$u(x,t) = \int_{-\infty}^{\infty} G(x-\xi,t) f(\xi) d\xi, \qquad (46)$$

$$G(x,t) = \frac{1}{\pi}\int_{0}^{\infty} E_{\alpha,1}(-\lambda^2\beta^2 t^{\alpha}) \cos(\beta x)\, d\beta = \frac{1}{2\lambda} t^{-p} W(-z,-p,1-p), \qquad (47)$$

and $W(z,\lambda,\mu)$ is the Wright function.

5. Conclusions

In our thesis, 1-dimensional Harmonic wave and linear wave equation is defined and there are several method in fractional order ordinary and partial differential equations are used on both harmonic and linear wave equation [1]-[2]-[3]-[4]. However, only we use Riemann-Liouville and Grünwald-Letnikov definitions of fractional calculus for solution 1-dimensional Harmonic wave because, we consider only one variable of t in function. Also, Nigmatullin's and Mainardi's fractional diffusion equation are applied for solvin linear wave equation [5]-[6]-[7].

6. References

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Solutions of fractional-order differential equation on harmonic waves and linear wave equation

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Abstract

In this thesis, we will define harmonic waves, linear wave equationon physical problem. [1]-[2] Initially, we will interrupt physical meaning of harmonic waves on time dependent and we will mantion that physical interpretation of linear wave equation with calculating. [1]-[2] After that, we will use many definitions of fractional derivative in order to need to apply on harmonic and linear waves equation.[3]-[4] Therefore, definitions of fractional-order derivative will mentioned in our thesis after interpreting harmonic waves and linear wave equation.

Keywords: Riemann-Liouville fractional derivative, Grünwald-Letnikov fractional derivative, harmonic waves, Linear wave, Oscillation, time-dependent equation.

1. Introduction

Harmonic Waves

First of all, we know that first dimensional and second dimensional wave equation but we can use only Grünwald-Letnikov and Riemann-Liouville fractional derivative on 1-dimensional harmonic waves. Therefore, we consider 1-dimensional waves solutions for using definitions of fractional derivative. Initially, a harmonic wave is defined as a harmonic is a wave with a frequency, which is a positive integer multiple of the fundamental frequency, that is the frequency of the original periodic signal, such as a sinusoidal wave [1].

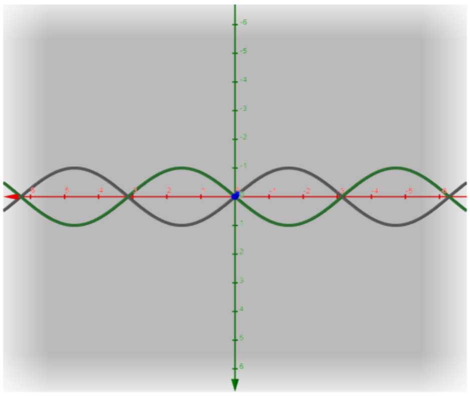

Figure 1: A harmonic wave which moves to the right direction on 1-dimensional.

Also, green wave indicate itself image at $t = 0$ and blue wave define itself image at some later time. Generally we define the displacement of the curve at $t = 0$;

$$y = csin\left(\frac{2\pi}{\lambda}x\right) \qquad (1)$$

c is named the amplitude of the wave, and λ is a wavelength of the wave. When wave moves to the right with a v velocity, wave function is defined at later t time as:

$$y = csin\left[\frac{2\pi}{\lambda}(x - vt)\right] \qquad (2)$$

Distance of the two different all waves are called as a period and it is shown as T. Period, velocity and wavelength are related with together:

$$v = \frac{\lambda}{T} \qquad (3)$$

$$\lambda = vT \qquad (4)$$

When we use (3) and (4) for substituting this into the (1), we write as:

$$y = csin\left[2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)\right] \qquad (5)$$

We are able to easily express the harmonic wave function by defining two quantities called wave number k and angular frequency ω:

$$k \equiv \frac{2\pi}{\lambda} \qquad (6)$$

$$\omega \equiv \frac{2\pi}{T} \qquad (7)$$

Eqn (6) and Eqn(7) can related with Eqn (5):

$$y = csin\left[\left(\frac{2\pi}{\lambda}x - \frac{2\pi}{T}t\right)\right] \qquad (8)$$

$$y = csin(kx - \omega t) \qquad (9)$$

Frequency is the number of waves passing a fixed place for a length of time and the formula is shown as below:

$$f = \frac{1}{T} \qquad (10)$$

f is hertz (Hz) or s^{-1}. Using Equations (6), (7) and (10), we can write the phase velocity v in the alternative forms as [1]:

$$v = \frac{\omega}{k} \qquad (11)$$

$$v = \lambda f \qquad (12)$$

Wave functions that is given by equation (9) considers that the y displacement is zero at $x = 0$ and $t = 0$. We define the wave function as below if and only if transverse displacement is not zero at $x = 0$ and $t = 0$

$$y = csin(kx - \omega t - \phi) \qquad (13)$$

ϕ is a wave constant.

2. Preliminaries

Using definitions of Fractional derivatives on Harmonic Waves solutions:

The wavelength, frequency and 1-dimensional wave was defined. Now, there are several method of solutions on harmonic wave solutions. We need to some definitions from fractional calculus. Especially, not only 1-dimensional both also 2-dimensional wave solutions are mentioned to solve different methods. We say that,

$$y = c\sin(kx - \omega t) \qquad (14)$$

is harmonic wave functions and this function always dependent on time variable. Especially, we use Riemann-Liouville and Grünwald-Letnikov that are in the fractional calculus for identifying transverse velocity v_y and transverse acceleration a_y.

Definition of Riemann-Liouville and Grünwald-Letnikov definition on transverse velocity:

Riemann-Liouville and Grünwald-Letnikov definition are applied when we assume that only 1-dimensional wave. In (14) when x is a constant. We use fractional derivative and fractional integral to define both transverse velocity and transverse acceleration:

$$v_y = \frac{dy}{dt}$$
$$a_y = \frac{dv_y}{dt}$$

Grünwald-Letnikov definition is [1,3,4]:

$$ {}_0I_t^1 y(t) = \frac{1}{\Gamma(1)} \int_0^t (t-\tau)^{1-1} y(\tau) d\tau = v_y(t) \qquad (15)$$

Take the fractional Integral of order ϕ on (15) $0 < \phi < 1$:

$$ {}_\alpha I_t^\phi y(t) = \frac{1}{\Gamma(\phi)} \int_\alpha^t (t-\tau)^{\phi-1}\, y(\tau) = v_y(t) \qquad (16)$$

$$ {}_0I_t^1 v_y(t) = \frac{1}{\Gamma(1)} \int_0^t (t-\tau)^{1-1} y(\tau) d\tau = a_y(t) \qquad (17)$$

Take the fractional Integral of order ϕ on (15) $0 < \phi < 1$:

$$ {}_\alpha I_t^\phi v_y(t) = \frac{1}{\Gamma(\phi)} \int_\alpha^t (t-\tau)^{\phi-1}\, v_y(t) d\tau = a_y(t) \qquad (18)$$

Riemann-Liouville fractional derivative definition is [1,3,4]:

$$ {}_\alpha D_t^\phi y(t) = \frac{1}{\Gamma(n-\phi)} \left(\frac{d}{dx}\right)^n \int_\alpha^t (t-\tau)^{n-\phi-1} y(\tau) d\tau = v_y(t) \qquad (19)$$

$$0 < \phi < 1:$$

$$_{\alpha}D_t^{\phi} v_y(t) = \frac{1}{\Gamma(n-\phi)}\left(\frac{d}{dx}\right)^n \int_{\alpha}^{t} (t-\tau)^{n-\phi-1} v_y(\tau) d\tau = a_y(t) \qquad (19)$$

3. Solutions of fractional-order differential equation on linear wave equation:

Definition: In this part, we will mention that the concept of the wave function to represent waves travelling on a string. Wave functions of $u(x,t)$ is a solution of an equation called the linear wave equation [2]. This solution comes from different methods. Particularly, this method will be mentioned. The linear wave equation is:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} \qquad (20)$$

Initially, first method is Nigmatullin's fractional diffusion equation that is linear wave equation and this wave equation form is similar as Eqn (20). Let we look at detail:

$$_0D_t^{\alpha} u(x,t) = \lambda^2 \frac{\partial^2 u(x,t)}{\partial x^2}, \quad (t > 0, \ -\infty < x < \infty); \quad (21)$$

$$\lim_{x\to\pm\infty} u(x,t) = 0; \quad [\ _0D_t^{\alpha-1} u(x,t)] = \varphi(x). \qquad (22)$$

and $t = 0$ in Eqn (22). Also, order of derivative $0 < \alpha < 1$. An equation (21) was mentioned by Nigmatullin [5] and by Westerlund [6] and studied by Mainardi [7]We will give a basic solution of the problem (21) demonstrating once again the advantage of using the Mittag-Leffler function in two parameters [3]. And we generally define Mittag-Leffler function as:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \qquad (\alpha > 0, \ \ \beta > 0) \qquad (23)$$

When we consider that the boundary conditions of Eqn. (22), take the Fourier transformation with respect to x variable then we write as:

$$_0D_t^{\alpha} \acute{\ddot{u}}(\beta,t) + \lambda^2\beta^2 \acute{\ddot{u}}(\beta,t) = 0 \qquad (24)$$

$$[\ _0D_t^{\alpha-1} \acute{\ddot{u}}(x,t)] = \varphi_1(\beta). \qquad (25)$$

when $t = 0$ is in Eqn. (25). Also β is the Fourier transform parameter. Now, when we take the Laplace transform of Eqn. (24) and apply the initial conditions of (25) then we satisfy:

$$\acute{\mathrm{U}}(\beta,s)=\frac{\varphi(\beta)}{s^{\alpha}+\lambda^{2}\beta^{2}} \qquad (26)$$

If inverse Laplace transform of (26) using

$$\int_{0}^{\infty} e^{-pt}t^{\alpha k+\beta-1}E_{\alpha,\beta}^{(k)}(\pm\alpha t^{\alpha})dt=\frac{k!\,p^{\alpha-\beta}}{(p^{\alpha}\mp\alpha)^{k+1}}, \qquad \left(Re(p)>|\alpha|^{1/\alpha}\right) \qquad (27)$$

therefore;

$$\acute{\mathrm{u}}(\beta,t)=\varphi(\beta)t^{\alpha-1}E_{\alpha,\alpha}(-\lambda^{2}\beta^{2}t^{\alpha}), \qquad (28)$$

After that, the inverse Fourier transform is produced from the initial value problem of (21) and (22).

$$u(x,t)=\int_{-\infty}^{\infty} G(x-\xi,t)\varphi(\xi)d\xi, \qquad (29)$$

$$G(x,t)=\frac{1}{\pi}\int_{0}^{\infty} t^{\alpha-1}E_{\alpha,\alpha}(-\lambda^{2}\beta^{2}t^{\alpha})cos\beta x d\beta, \qquad (30)$$

Take the Laplace transform of (30) then we write as;

$$L\left\{\frac{1}{\pi}\int_{0}^{\infty} t^{\alpha-1}E_{\alpha,\alpha}(-\lambda^{2}\beta^{2}t^{\alpha})\,cos\beta x d\beta\right\}$$

$$g(x,s)=\frac{1}{\pi}\int_{0}^{\infty}\frac{\cos(\beta x)\,d\beta}{\lambda^{2}\beta^{2}+s^{\alpha}}=\frac{1}{2\lambda}s^{-\alpha/2}e^{-|x|\lambda^{-1}s^{\alpha/2}} \qquad (31)$$

and inverse Laplace transform of (31) is:

$$G(x,t)=\frac{1}{4\lambda\pi i}\int_{Br} e^{st}s^{-\frac{\alpha}{2}}\exp\left(-|x|\lambda^{-1}s^{\alpha/2}\right)ds \qquad (32)$$

Use the substitution of $\sigma=st$ and $z=|x|\lambda^{-1}t^{-p}$ and after we use this substitution in (32) we satisfy,

$$G(x,t)=\frac{t^{1-p}}{2\lambda}\frac{1}{2\pi i}\int_{Ha} e^{\sigma-z\sigma^{p}}\frac{d\sigma}{\sigma^{p}}=\frac{1}{2\lambda}t^{p-1}W(-z,-p,p), \qquad z=\frac{|x|}{\lambda t^{p}} \qquad (33)$$

Also, $w(z, \lambda, \mu)$ is defined as the Wright function and it is same as $W(-z, -p, p)$. Wright function is defined as:

$$W(z; \zeta, \eta) = \sum_{k=0}^{\infty} \frac{z^k}{k!\, \Gamma(\zeta k + \eta)} \qquad (34)$$

And the fourier cosine-transform of the function $u_1(\beta) = t^{\alpha-1} E_{\alpha,\alpha}(-\lambda^2 \beta^2 t^\alpha)$ is evaluated. Finally, when we use $\alpha = 1$ in (33) then the fractional Green function (33) is the form [3]:

$$G(x, t) = \frac{1}{2\lambda\sqrt{\pi t}} \exp\left(-\frac{x^2}{4\lambda^2 t}\right) \qquad (35)$$

4. Using Fractional-order partial differential equations on linear wave equations

The Mainardi's fractional diffusion linear wave equation [7] is:

$${}_0D_t^\alpha u(x,t) = \lambda^2 \frac{\partial^2 u(x,t)}{\partial x^2}, \qquad (|x| < \infty, t > 0) \qquad (36)$$

$$u(x, 0) = f(x) \qquad (|x| < \infty) \qquad (37)$$

$$\lim_{x \to \mp\infty} u(x,t) = 0, \qquad (t > 0) \qquad (38)$$

where $0 < \alpha < 1$. The Laplace transform of the formula:

$$L\{\, {}_0D_t^{\sigma_m} f(t); s\} = s^{\sigma_m} F(s) - \sum_{k=0}^{m-1} s^{\sigma_m - \sigma_{m-k}} \left[\, {}_0D_t^{\sigma_{m-k}-1} f(t)\right]\Big|_{t=0} \qquad (39)$$

$${}_\alpha D_t^{\sigma_{m-k}-1} \equiv {}_\alpha D_t^{\alpha_{m-k}-1}\, {}_\alpha D_t^{\alpha_{m-k-1}} \dots {}_\alpha D_t^{\alpha_1}, \qquad (k = 0,1,\dots,m-1) \quad (40)$$

Then;

$$L\{\, {}_0D_t^\alpha y(t); s\} = s^\alpha Y(s) - s^{\alpha-1} y(0), \qquad (41)$$

When (36) and (38) then we satisfy:

$$s^\alpha \acute{\text{ű}}(x,s) - s^{\alpha-1} f(x) = \lambda^2 \acute{\text{ű}}_{xx}(x,s) \quad |x| < \infty \quad (42)$$

$$\lim_{x \to \mp\infty} \acute{\text{ű}}(x,s) = 0, \qquad (t > 0) \quad (43)$$

After that, when we apply exponential fourier transform to equation (42) and utilizing the boundary conditions (43), we obtain:

$$U(\beta, s) = \frac{s^{\alpha-1}}{s^\alpha + \lambda^2 \beta^2} F(\beta), \qquad (44)$$

and $U(\beta, p)$ and $F(\beta)$ are the Fourier transforms of ű(x, s) and $f(x)$. When we take the inverse Laplace transform of the fraction

$$s^{\alpha-1}/(s^\alpha + \lambda^2\beta^2) \qquad (45)$$

and this Laplace transform of the fraction (45) is $E_{\alpha,1}(-\lambda^2\beta^2 t^\alpha)$. And finally, the inversion of the Fourier and the Laplace transform gives the solution below:

$$u(x,t) = \int_{-\infty}^{\infty} G(x-\xi,t) f(\xi) d\xi, \qquad (46)$$

$$G(x,t) = \frac{1}{\pi}\int_{0}^{\infty} E_{\alpha,1}(-\lambda^2\beta^2 t^\alpha)\cos(\beta x)\, d\beta = \frac{1}{2\lambda} t^{-p} W(-z,-p,1-p), \qquad (47)$$

and $W(z,\lambda,\mu)$ is the Wright function.

5. Conclusions

In our thesis, 1-dimensional Harmonic wave and linear wave equation is defined and there are several method in fractional order ordinary and partial differential equations are used on both harmonic and linear wave equation [1]-[2]-[3]-[4]. However, only we use Riemann-Liouville and Grünwald-Letnikov definitions of fractional calculus for solution 1-dimensional Harmonic wave because, we consider only one variable of t in function. Also, Nigmatullin's and Mainardi's fractional diffusion equation are applied for solvin linear wave equation [5]-[6]-[7].

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Some Deferred Invariant Convergence Types for Double Sequences of Sets

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Abstract

In this paper, we presented the concepts of deferred invariant, strongly deferred invariant and deferred invariant statistical convergence in the Wijsman sense for double set sequences. Also, basic theorems associated with these concepts is given.

Keywords: Deferred Cesàro mean, deferred statistical convergence, invariant mean, Wijsman convergence, double sequences of sets.

1. Introduction and Backgrounds

Küçükaslan and Yılmaztürk [1] presented the concept of deferred statistical convergence long after Agnew [2] mainly introduced the concept of deferred Cesàro mean for real or complex-valued sequences. Then, Nuray [3] gave the concepts of strongly deferred invariant and deferred invariant statistical convergence using the term invariant mean. Also, the concepts of deferred Cesàro mean and deferred statistical convergence were extended to the double sequences by Dağadur and Sezgek [4, 5]. Furthermore, Savaş [6] presented the concepts of strongly double deferred invariant and double deferred invariant statistical convergence.

Recently, for sequences of sets, the concepts of Wijsman strongly deferred Cesàro summability and Wijsman deferred statistical convergence were introduced by Altınok et al. [7]. Then, Gülle [8] studied on the concepts of strongly deferred invariant and deferred invariant statistical convergence of order η in the Wijsman sense for sequences of sets. In [9], by extending to the double sequences of sets, Ulusu and Gülle also presented the concepts of Wijsman deferred Cesàro summability and Wijsman deferred statistical convergence.

More information on these concepts can be found in [10-21].

For a metric space $(\mathcal{Y}, d)$, $\rho(y, U)$ denote the distance from the point y to the set U where

$$\rho_y(U) := \rho(y, U) = \inf_{u \in U} d(y, u)$$

for any $y \in \mathcal{Y}$ and any non-empty $U \subseteq \mathcal{Y}$.

For a non-empty set $\mathcal{Y}$, let a function $g: \mathbb{N} \to 2^{\mathcal{Y}}$ (the power set of $\mathcal{Y}$) is defined by $g(i) = U_i \in 2^{\mathcal{Y}}$ for each $i \in \mathbb{N}$. Then, the sequence $\{U_i\} = \{U_1, U_2, \dots\}$ is called sequence of sets.

Throughout the study, $(\mathcal{Y}, d)$ will be considered as a metric space and U, U_{ij} $(i, j \in \mathbb{N})$ as any non-empty closed subsets of $\mathcal{Y}$.

The double sequence $\{U_{ij}\}$ is said to be Wijsman convergent to the set U if

$$\lim_{i,j\to\infty} \rho_y(U_{ij}) = \rho_y(U)$$

for each $y \in \mathcal{Y}$ and it is denoted by $U_{ij} \xrightarrow{W_2} U$.

Let σ be a mapping of the set of positive integers into itself. A continuous linear functional ϕ on ℓ_∞, the space of real bounded sequences, is called an invariant mean or a σ-mean if and only if

- $\phi(x_i) \geq 0$, when the sequence (x_i) has $x_i \geq 0$ for all i,
- $\phi(e) = 1$, where $e = (1,1,1, \dots)$, and
- $\phi\left(x_{\sigma(i)}\right) = \phi(x_i)$ for all $(x_i) \in \ell_\infty$.

The mappings σ are assumed to be one to one and $\sigma^i(k) \neq k$ for all positive integers i and k, where $\sigma^i(k)$ denotes the i th iterate of the mapping σ at k. Thus, ϕ extends the limit functional on c, the space of convergent sequences, in the sense that $\phi(x_i) = \lim x_i$ for all $(x_i) \in c$.

The double sequence $\{U_{ij}\}$ is said to be Wijsman strongly invariant convergent to the set U if for each $y \in \mathcal{Y}$

$$\lim_{m,n\to\infty} \frac{1}{mn} \sum_{i,j=1,1}^{m,n} \left|\rho_y(U_{\sigma^i(k)\sigma^j(t)}) - \rho_y(U)\right| = 0$$

uniformly in k, t.

The double sequence $\{U_{ij}\}$ is said to be Wijsman invariant statistically convergent to the set U if for every $\delta > 0$ and each $y \in \mathcal{Y}$

$$\lim_{m,n\to\infty} \frac{1}{mn} \left|\left\{(i,j): i \leq m, j \leq n, |\rho_y(U_{\sigma^i(k)\sigma^j(t)}) - \rho_y(U)| \geq \delta\right\}\right| = 0$$

uniformly in k, t.

The deferred Cesàro mean $D_{\varphi,\psi}$ of a double real sequence $\mathrm{x} = (x_{ij})$ is defined by

$$(D_{\varphi,\psi}\mathrm{x})_{mn} = \frac{1}{\varphi(m)\psi(n)} \sum_{i=p_m+1}^{r_m} \sum_{v=q_n+1}^{s_n} x_{ij} := \sum_{\substack{i=p_m+1 \\ v=q_n+1}}^{r_m,s_n} x_{ij},$$

where $\{p(m)\}$, $\{r(m)\}$, $\{q(n)\}$ and $\{s(n)\}$ are sequences of non-negative integers satisfying following conditions:

$$p(m) < r(m), \ \lim_{m\to\infty} r(m) = \infty; \quad q(n) < s(n), \ \lim_{n\to\infty} s(n) = \infty \tag{1.1}$$

and

$$r(m) - p(m) = \varphi(m); \quad s(n) - q(n) = \psi(n). \tag{1.2}$$

Note here that the method $D_{\varphi,\psi}$ is openly regular for any selection of the above sequences of integers.

Throughout the paper, unless otherwise specified, $\{p(m)\}$, $\{r(m)\}$, $\{q(n)\}$ and $\{s(n)\}$ are considered as sequences of non-negative integers satisfying (1.1) and (1.2).

A double sequence $\theta_2 = \{(i_m, j_n)\}$ is called double lacunary sequence if there exists increasing integers sequences (i_m) and (j_n) of integers such that

$$i_0 = 0, \ h_m = i_m - i_{m-1} \to \infty \ \text{ and } \ j_0 = 0, \ \bar{h}_n = j_n - j_{n-1} \to \infty \ \text{ as } \ m, n \to \infty.$$

2. Main Results

In this section, we presented the concepts of deferred invariant, strongly deferred invariant and deferred invariant statistical convergence in the Wijsman sense for double set sequences. Also, basic theorems associated with these concepts is given.

Definition 2.1 The double set sequence $\{U_{ij}\}$ is said to be

- deferred invariant convergent to the set U in the Wijsman sense if for each $y \in \mathcal{Y}$

$$\lim_{m,n\to\infty} \frac{1}{\varphi(m)\psi(n)} \sum_{\substack{i=p_m+1 \\ v=q_n+1}}^{r_m, s_n} \rho_y(U_{\sigma^i(k)\sigma^j(t)}) = \rho_y(U)$$

  uniformly in k, t and the notation $U_{ij} \overset{W_2 D_\sigma}{\longrightarrow} U$ is used.
- strongly deferred invariant convergent to the set U in the Wijsman sense if for each $y \in \mathcal{Y}$

$$\lim_{m,n\to\infty} \frac{1}{\varphi(m)\psi(n)} \sum_{\substack{i=p_m+1 \\ v=q_n+1}}^{r_m, s_n} \left|\rho_y(U_{\sigma^i(k)\sigma^j(t)}) - \rho_y(U)\right| = 0$$

  uniformly in k, t and the notation $U_{ij} \overset{W_2[D_\sigma]}{\longrightarrow} U$ is used.

The class of all double set sequences that strongly deferred invariant convergent in the Wijsman sense will be denoted by $\{W_2[D_\sigma]\}$.

Remark 2.1

- For $p(m)=0, r(m)=m$ and $q(n)=0, s(n)=n$, the concepts of deferred invariant and strongly deferred invariant convergence in the Wijsman sense coincide with the concepts of invariant and strongly invariant convergence in the Wijsman sense for double set sequences in [16].
- For $p(m)=i_{m-1}, r(m)=i_m$ and $q(n)=j_{n-1}, s(n)=j_n$ where $\{(i_m, j_n)\}$ is a double lacunary sequence, the concepts of deferred invariant and strongly deferred invariant convergence in the Wijsman sense coincide with the concepts of lacunary invariant and strongly lacunary invariant convergence in the Wijsman sense for double set sequences in [16].

Theorem 2.1 If $\{U_{ij}\}$, $\{V_{ij}\}$ and $\{Z_{ij}\}$ are double set sequences such that $U_{ij} \subset V_{ij} \subset Z_{ij}$ for all $i, j \in \mathbb{N}$, then

$$U_{ij} \overset{W_2[D_\sigma]}{\longrightarrow} V \text{ and } Z_{ij} \overset{W_2[D_\sigma]}{\longrightarrow} V \Rightarrow V_{ij} \overset{W_2[D_\sigma]}{\longrightarrow} V.$$

Proof. Assume that $U_{ij} \subset V_{ij} \subset Z_{ij}$, $U_{ij} \overset{W_2[D_\sigma]}{\longrightarrow} V$ and $Z_{ij} \overset{W_2[D_\sigma]}{\longrightarrow} V$. For all $i, j \in \mathbb{N}$,

$$U_{ij} \subset V_{ij} \subset Z_{ij} \Rightarrow U_{\sigma^i(k)\sigma^j(t)} \subset V_{\sigma^i(k)\sigma^j(t)} \subset Z_{\sigma^i(k)\sigma^j(t)} \quad \text{(for all } k, t\text{)}$$

$$\Rightarrow \rho_y(Z_{\sigma^i(k)\sigma^j(t)}) \le \rho_y(V_{\sigma^i(k)\sigma^j(t)}) \le \rho_y(U_{\sigma^i(k)\sigma^j(t)}) \quad \text{(for each } y \in \mathcal{Y}\text{)}$$

$$\Rightarrow \left|\rho_y(Z_{\sigma^i(k)\sigma^j(t)}) - \rho_y(V)\right| \le \left|\rho_y\left(V_{\sigma^i(k)\sigma^j(t)}\right) - \rho_y(V)\right| \le \left|\rho_y\left(U_{\sigma^i(k)\sigma^j(t)}\right) - \rho_y(V)\right|$$

is hold. Thus, we have

$$\frac{1}{\varphi(m)\psi(n)} \sum_{\substack{i=p(m)+1 \\ j=q(n)+1}}^{r(m),s(n)} \left|\rho_y(Z_{\sigma^i(k)\sigma^j(t)}) - \rho_y(V)\right|$$

$$\le \frac{1}{\varphi(m)\psi(n)} \sum_{\substack{i=p(m)+1 \\ j=q(n)+1}}^{r(m),s(n)} \left|\rho_y(V_{\sigma^i(k)\sigma^j(t)}) - \rho_y(V)\right|$$

$$\le \frac{1}{\varphi(m)\psi(n)} \sum_{\substack{i=p(m)+1 \\ j=q(n)+1}}^{r(m),s(n)} \left|\rho_y(U_{\sigma^i(k)\sigma^j(t)}) - \rho_y(V)\right|.$$

Hence, by our assumption, we get that $V_{ij} \overset{W_2[D_\sigma]}{\longrightarrow} V$.

Definition 2.2 The double set sequence $\{U_{ij}\}$ is said to be deferred invariant statistically convergent to the set U in the Wijsman sense if for every $\delta > 0$ and each $y \in \mathcal{Y}$

$$\lim_{m,n\to\infty} \frac{1}{\varphi(m)\psi(n)} \left|\left\{(i,j): p(m) < i \leq r(m), q(n) < j \leq s(n), |\rho_y(U_{\sigma^i(k)\sigma^j(t)}) - \rho_y(U)| \geq \delta\right\}\right| = 0$$

uniformly in k, t and the notation $U_{ij} \overset{W_2DS_\sigma}{\longrightarrow} U$ is used.

The class of all double sequences of sets that deferred invariant statistically convergent in the Wijsman sense is denoted by $\{W_2DS_\sigma\}$.

Remark 2.2

- For $p(m) = 0, r(m) = m$ and $q(n) = 0, s(n) = n$, the concept of deferred invariant statistical convergence in the Wijsman sense coincides with the concept of invariant statistical convergence in the Wijsman sense for double set sequences in [16].
- For $p(m) = i_{m-1}, r(m) = i_m$ and $q(n) = j_{n-1}, s(n) = j_n$ where $\{(i_m, j_n)\}$ is a double lacunary sequence, the concept of deferred invariant statistical convergence in the Wijsman sense coincides with the concept of lacunary invariant statistical convergence in the Wijsman sense for double set sequences in [16].

Theorem 2.2 If $\{U_{ij}\}$, $\{V_{ij}\}$ and $\{Z_{ij}\}$ are double set sequences such that $U_{ij} \subset V_{ij} \subset Z_{ij}$ for all $i, j \in \mathbb{N}$, then

$$U_{ij} \overset{W_2DS_\sigma}{\longrightarrow} V \text{ and } Z_{ij} \overset{W_2DS_\sigma}{\longrightarrow} V \Rightarrow V_{ij} \overset{W_2DS_\sigma}{\longrightarrow} V.$$

Proof. Assume that $U_{ij} \subset V_{ij} \subset Z_{ij}$, $U_{ij} \overset{W_2DS_\sigma}{\longrightarrow} V$ and $Z_{ij} \overset{W_2DS_\sigma}{\longrightarrow} V$. With the same approximation of Theorem 2.1, we have

$$\left|\rho_y(Z_{\sigma^i(k)\sigma^j(t)}) - \rho_y(V)\right| \leq \left|\rho_y(V_{\sigma^i(k)\sigma^j(t)}) - \rho_y(V)\right| \leq \left|\rho_y(U_{\sigma^i(k)\sigma^j(t)}) - \rho_y(V)\right|.$$

Then, for every $\delta > 0$ we can write

$$\begin{aligned}
&\left\{(i,j): p(m) < i \leq r(m), q(n) < j \leq s(n), |\rho_y(V_{\sigma^i(k)\sigma^j(t)}) - \rho_y(V)| \geq \delta\right\} \\
&\quad = \left\{(i,j): p(m) < i \leq r(m), q(n) < j \leq s(n), \rho_y(V_{\sigma^i(k)\sigma^j(t)}) \geq \rho_y(V) + \delta\right\} \\
&\quad\quad \cup \left\{(i,j): p(m) < i \leq r(m), q(n) < j \leq s(n), \rho_y(V_{\sigma^i(k)\sigma^j(t)}) \leq \rho_y(V) - \delta\right\} \\
&\quad \subset \left\{(i,j): p(m) < i \leq r(m), q(n) < j \leq s(n), \rho_y(U_{\sigma^i(k)\sigma^j(t)}) \geq \rho_y(V) + \delta\right\} \\
&\quad\quad \cup \left\{(i,j): p(m) < i \leq r(m), q(n) < j \leq s(n), \rho_y\left(Z_{\sigma^i(k)\sigma^j(t)}\right) \leq \rho_y(V) - \delta\right\}.
\end{aligned}$$

Hence, the following inequality is hold:

$$\frac{1}{\varphi(m)\psi(n)}\left|\left\{(i,j): p(m) < i \le r(m), q(n) < j \le s(n), |\rho_y(V_{\sigma^i(k)\sigma^j(t)}) - \rho_y(V)| \ge \delta\right\}\right|$$
$$\le \frac{1}{\varphi(m)\psi(n)}\left|\left\{(i,j): p(m) < i \le r(m), q(n) < j \le s(n), |\rho_y(U_{\sigma^i(k)\sigma^j(t)}) - \rho_y(V)| \ge \delta\right\}\right|$$
$$+\frac{1}{\varphi(m)\psi(n)}\left|\left\{(i,j): p(m) < i \le r(m), q(n) < j \le s(n), |\rho_y(Z_{\sigma^i(k)\sigma^j(t)}) - \rho_y(V)| \ge \delta\right\}\right|.$$

So, by our assumption, we get that $V_{ij} \overset{W_2DS_\sigma}{\longrightarrow} V$.

Now, we will examine relations between the concepts of $W_2[D_\sigma]$-convergence and W_2DS_σ-convergence for double set sequences.

Theorem 2.3 If a double set sequences $\{U_{ij}\}$ is $W_2[D_\sigma]$- convergent to a set U, then the sequence is W_2DS_σ -convergent to same set.

Proof. Suppose that $U_{ij} \overset{W_2[D_\sigma]}{\longrightarrow} U$. For every $\delta > 0$ and each $y \in \mathcal{Y}$, we can write the following inequality

$$\sum_{\substack{i=p(m)+1\\ j=q(n)+1}}^{r(m),s(n)} \left|\rho_y(U_{\sigma^i(k)\sigma^j(t)}) - \rho_y(U)\right|$$
$$\ge \sum_{\substack{i=p(m)+1\\ j=q(n)+1\\ \left|\rho_y\left(U_{\sigma^i(k)\sigma^j(t)}\right)-\rho_y(U)\right|\ge\delta}}^{r(m),s(n)} \left|\rho_y\left(U_{\sigma^i(k)\sigma^j(t)}\right) - \rho_y(U)\right|$$
$$\ge \delta \left|\left\{(i,j): p(m) < i \le r(m), q(n) < j \le s(n), |\rho_y(U_{\sigma^i(k)\sigma^j(t)}) - \rho_y(U)| \ge \delta\right\}\right|$$

and so

$$\frac{1}{\delta}\frac{1}{\varphi(m)\psi(n)}\sum_{\substack{i=p(m)+1\\ j=q(n)+1}}^{r(m),s(n)} \left|\rho_y(U_{\sigma^i(k)\sigma^j(t)}) - \rho_y(U)\right|$$
$$\ge \frac{1}{\varphi(m)\psi(n)}\left|\left\{(i,j): p(m) < i \le r(m), q(n) < j \le s(n), |\rho_y(U_{\sigma^i(k)\sigma^j(t)}) - \rho_y(U)| \ge \delta\right\}\right|.$$

Hence, by our assumption, we get that $U_{ij} \overset{W_2DS_\sigma}{\longrightarrow} U$.

The converse of Theorem 2.3 is provided when the double set sequence $\{U_{ij}\}$ is bounded. Otherwise it is not provided.

The double sequence $\{U_{ij}\}$ is called bounded if $\sup_{i,j} \rho_y(U_{ij}) < \infty$ for each $y \in \mathcal{Y}$. Also, L_∞^2 denotes the class of all bounded double sequences of sets.

Theorem 2.4 If a double set sequence $\{U_{ij}\} \in L_\infty^2$ is W_2D-convergent to a set U, then the sequence is $W_2[D_\sigma]$-convergent to same set.

Proof. Suppose that $\{U_{ij}\} \in L_\infty^2$ and $U_{ij} \xrightarrow{W_2DS_\sigma} U$. Since $\{U_{ij}\} \in L_\infty^2$, there is a $\mathcal{K} > 0$ such that for all $i,j \in \mathbb{N}$ and each $y \in \mathcal{Y}$

$$\left|\rho_y(U_{\sigma^i(k)\sigma^j(t)}) - \rho_y(U)\right| \leq \mathcal{K}$$

uniformly in k,t. Thus, for every $\delta > 0$ we have

$$\frac{1}{\varphi(m)\psi(n)} \sum_{\substack{i=p(m)+1 \\ j=q(n)+1}}^{r(m),s(n)} \left|\rho_y(U_{\sigma^i(k)\sigma^j(t)}) - \rho_y(U)\right|$$

$$= \frac{1}{\varphi(m)\psi(n)} \sum_{\substack{i=p(m)+1 \\ j=q(n)+1 \\ \left|\rho_y(U_{\sigma^i(k)\sigma^j(t)}) - \rho_y(U)\right| \geq \delta}}^{r(m),s(n)} \left|\rho_y(U_{\sigma^i(k)\sigma^j(t)}) - \rho_y(U)\right|$$

$$+ \frac{1}{\varphi(m)\psi(n)} \sum_{\substack{i=p(m)+1 \\ j=q(n)+1 \\ \left|\rho_y(U_{\sigma^i(k)\sigma^j(t)}) - \rho_y(U)\right| < \delta}}^{r(m),s(n)} \left|\rho_y(U_{\sigma^i(k)\sigma^j(t)}) - \rho_y(U)\right|$$

$$\leq \frac{\mathcal{K}}{\varphi(m)\psi(n)} \left|\left\{(i,j): p(m) < i \leq r(m), q(n) < j \leq s(n), \left|\rho_y(U_{\sigma^i(k)\sigma^j(t)}) - \rho_y(U)\right| \geq \delta\right\}\right| + \delta$$

for each $y \in \mathcal{Y}$. Hence, by our assumption, we get that $U_{ij} \xrightarrow{W_2[D]_\sigma} U$.

Corollary 2.1 $L_\infty^2 \cap \{W_2[D_\sigma]\} = L_\infty^2 \cap \{W_2DS_\sigma\}$.

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Some Novelty Inequalities Using Uniformly Exponentially $(\omega_1, \omega_2, h_1, h_2)$-Convex Functions Pertaining to Generalized Integral Operators And Their Applications

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Abstract

In this paper, the authors define a new generic class of functions called it uniformly exponentially $(\omega_1, \omega_2, h_1, h_2)$-convex. A useful integral identity pertaining to generalized integral operators via differentiable function is also found. Applying this as an auxiliary result, we establish some new bounds on Hermite-Hadamard type integral inequality for differentiable functions that are in absolute value at certain powers uniformly exponentially $(\omega_1, \omega_2, h_1, h_2)$-convex. Our results include several new and known results as particular cases. Finally, some applications of presented results for special means and error estimates for the mixed trapezium and midpoint formula have been analyzed.

Keywords: Hermite-Hadamard inequality, uniformly exponentially $(\omega_1, \omega_2, h_1, h_2)$-convexity, generalized integral operators, special means, error estimation.

1. Introduction

Convex functions and their generalizations have various applications in the fields of pure and applied sciences. Due to these applications, it is the most attractive area for researchers now a days. The class of convex functions is well known in the literature and is usually defined in the following way:

Definition 1. Let J be an interval in $\mathbb{R}$. A function $\psi : J \to \mathbb{R}$, is said to be convex on J, if the inequality

$$\psi(t\xi_1 + (1-t)\xi_2) \leq t\psi(\xi_1) + (1-t)\psi(\xi_2) \tag{1}$$

holds for all $\xi_1, \xi_2 \in J$ and $t \in [0,1]$. Also, we say that ψ is concave, if the inequality in (1) holds in the reverse direction. The following inequality, named Hermite-Hadamard inequality (or H-H inequality), is one of the most famous inequalities in the literature for convex functions.

Theorem 1. Let $\psi : J \to \mathbb{R}$ be a convex function and $\xi_1, \xi_2 \in J$ with $\xi_1 < \xi_2$. Then the following inequality holds:

$$\psi\left(\frac{\xi_1+\xi_2}{2}\right) \leq \frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\psi(t)dt \leq \frac{\psi(\xi_1)+\psi(\xi_2)}{2}. \tag{2}$$

This inequality (2) is also known as trapezium inequality.

The trapezium inequality has remained an area of great interest due to its wide applications in the field of mathematical analysis. Authors of recent decades have studied (2) in the premises of newly invented definitions due to motivation of convex function. Interested readers see the references [1-8].

Also, lets define a function $\phi: [0, +\infty[\longrightarrow [0, +\infty[$, which is constructed from the work of Sarikaya *et al.* [9], and fulfills the following four conditions:

$$\int_0^1 \frac{\phi(\tau)}{\tau} d\tau < +\infty,$$

$$\frac{1}{\mathcal{A}_1} \leq \frac{\phi(\tau_1)}{\phi(\tau_2)} \leq \mathcal{A}_1 \text{ for } \frac{1}{2} \leq \frac{\tau_1}{\tau_2} \leq 2, \tag{3}$$

$$\frac{\phi(\tau_2)}{\tau_2^2} \leq \mathcal{A}_2 \frac{\phi(\tau_1)}{\tau_1^2} \text{ for } \tau_1 \leq \tau_2$$

and

$$\left|\frac{\phi(\tau_2)}{\tau_2^2} - \frac{\phi(\tau_1)}{\tau_1^2}\right| \leq \mathcal{A}_3 |\tau_2 - \tau_1| \frac{\phi(\tau_2)}{\tau_2^2} \text{ for } \frac{1}{2} \leq \frac{\tau_1}{\tau_2} \leq 2,$$

where $\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{A}_3 > 0$ are independent of $\tau_1, \tau_2 > 0$.

Moreover, Sarikaya *et al.* [9] used the above function in order to define the following fractional integral operators.

Definition 2. The generalized left-side and right-side fractional integrals are given as follows:

$${}_{a_1^+}I_{\phi}\psi(x) = \int_{a_1}^{x} \frac{\phi(x-t)}{x-t} \psi(t) dt \qquad (x > a_1) \tag{4}$$

and

$${}_{a_2^-}I_{\phi}\psi(x) = \int_{x}^{a_2} \frac{\phi(t-x)}{t-x} \psi(t) dt \qquad (x < a_2), \tag{5}$$

respectively.

The most important feature of generalized integrals is that; they produce Riemann-Liouville fractional integrals, k-Riemann-Liouville fractional integrals, Katugampola fractional integrals, conformable fractional integrals, Hadamard fractional integrals, etc.

Definition 3. [10] A function $\psi : J \subseteq \mathbb{R} \to \mathbb{R}$, is called exponentially convex if

$$\psi(t\xi_1 + (1-t)\xi_2) \le t\frac{\psi(\xi_1)}{e^{\omega\xi_1}} + (1-t)\frac{\psi(\xi_2)}{e^{\omega\xi_2}} \tag{6}$$

holds true for all $\xi_1, \xi_2 \in J$, $\omega \in \mathbb{R}$, and $t \in [0,1]$.

Definition 4. [11] Let J be a convex set in $\mathbb{R}$. A function $\psi : J \to \mathbb{R}$ is said to be uniformly convex with modulus $\Phi: [0,+\infty) \longrightarrow [0,+\infty)$, if Φ is increasing, Φ vanishes only at 0, and

$$\psi(t\xi_1 + (1-t)\xi_2) \le t\,\psi(\xi_1) + (1-t)\psi(\xi_2) - t(1-t)\Phi(|\xi_2 - \xi_1|) \tag{7}$$

holds for all $\xi_1, \xi_2 \in J$, and $t \in [0,1]$.

We are now in the position to introduce a new generic class of functions called uniformly exponentially $(\omega_1, \omega_2, h_1, h_2)$-convex function.

Definition 5. Let $\psi : J \to \mathbb{R}$, $h_1, h_2: [0,1] \longrightarrow [0,+\infty)$ and $\Phi: [0,+\infty) \longrightarrow [0,+\infty)$ is an increasing function, and vanishes only at 0. If ψ satisfies the following inequality,

$$\psi(t\xi_1 + (1-t)\xi_2) \le h_1(t)\frac{\psi(\xi_1)}{e^{\omega_1\xi_1}} + h_2(t)\frac{\psi(\xi_2)}{e^{\omega_2\xi_2}} - h_1(t)h_2(t)\Phi(|\xi_2 - \xi_1|), \tag{8}$$

for all $\xi_1, \xi_2 \in J$, $\omega_1, \omega_2 \in \mathbb{R}$, and $t \in [0,1]$, then ψ is called uniformly exponentially $(\omega_1, \omega_2, h_1, h_2) -$ convex function with modulus Φ.

Remark 1. Some special cases of Definition 5 are:

i. if we take $h_1(t) = t, h_2(t) = 1-t$, $\omega_1 = \omega_2 = 0$, then we have Definiton 4.
ii. if we choose $h_1(t) = t, h_2(t) = 1-t, \omega_1 = \omega_2 = 0$, $\Phi(t) = \gamma t$, then we get the class of approximate uniformly convex function.
iii. if we take $h_1(t) = t, h_2(t) = 1-t$, $\omega_1 = \omega_2 = 0$, $\Phi(t) = \gamma t^2$, then we obtain the class of strongly uniformly convex function.
iv. if we choose $h_1(t) = h_2(t) = 1$, then we have the class of uniformly exponentially (ω_1, ω_2)–convex function with modulus Φ.
v. if we take $h_1(t) = h(t)$, $h_2(t) = h(1-t)$, then we get the class of uniformly exponentially (ω_1, ω_2, h)–convex function with modulus Φ.

vi. if we choose $h_1(t) = (1-t)^s$, $h_2(t) = t^s$, then we obtain the class of uniformly exponentially (s, ω_1, ω_2)–convex function with modulus Φ.

Motivated by the above results, we will establish in Section 2 an integral identity with three parameters pertaining to generalized integral operators. Applying this as an auxiliary result, we will derive some new bounds on Hermite-Hadamard type integral inequality for differentiable functions that are in absolute value at certain powers uniformly exponentially $(\omega_1, \omega_2, h_1, h_2)$-convex. Our results will include several new and known results as particular cases. In Section 3, some applications of presented results for special means and error estimates for the mixed trapezium and midpoint formula will be given.

2. Main Results

Throughout this study, let $J = [\xi_1, \xi_2]$, $J^\circ = (\xi_1, \xi_2)$ with $\xi_1 < \xi_2$, L(J) is the set of all integrable functions on J and $\lambda \in (0,1]$. For all $t \in [0,1]$, we define

$$\Lambda(t) := \int_0^t \frac{\phi\left(\left(\lambda\frac{\xi_1+\xi_2}{2} - \xi_1\right)\frac{u}{\lambda}\right)}{u} du < +\infty$$

and

$$\Delta(t) := \int_0^t \frac{\phi\left(\left(\xi_2 - \lambda\frac{\xi_1+\xi_2}{2}\right)\frac{u}{\lambda}\right)}{u} du < +\infty.$$

For establishing some new results regarding general fractional integrals we need to prove the following lemma.

Lemma 1. Let $\psi : J \to \mathbb{R}$ be a differentiable function on J°. If $\psi' \in L(J)$ and $\alpha, \beta \in \mathbb{R}$, then the following identity for generalized fractional integrals holds:

$$\frac{\alpha\psi(\xi_1) + \beta\psi(\xi_2)}{2} + \frac{\lambda}{2}\left[\frac{\Lambda(\lambda)}{\lambda\left(\frac{\xi_1+\xi_2}{2}\right) - \xi_1} + \frac{\Delta(\lambda)}{\xi_2 - \lambda\left(\frac{\xi_1+\xi_2}{2}\right)} - \frac{\alpha+\beta}{\lambda}\right]\psi\left(\lambda\left(\frac{\xi_1+\xi_2}{2}\right)\right)$$
$$-\frac{\lambda}{2}\left[\frac{1}{\lambda\left(\frac{\xi_1+\xi_2}{2}\right) - \xi_1} \cdot {}_{\lambda\left(\frac{\xi_1+\xi_2}{2}\right)^-}I_\phi\psi(\xi_1) + \frac{1}{\xi_2 - \lambda\left(\frac{\xi_1+\xi_2}{2}\right)} \cdot {}_{\lambda\left(\frac{\xi_1+\xi_2}{2}\right)^+}I_\phi\psi(\xi_2)\right]$$
$$= \left(\frac{\lambda\left(\frac{\xi_1+\xi_2}{2}\right)-\xi_1}{2}\right) \cdot \int_0^1 \left[\frac{\lambda\Lambda((1-t)\lambda)}{\lambda\left(\frac{\xi_1+\xi_2}{2}\right)-\xi_1} - \alpha\right]\psi'\left(\xi_1 t + \lambda(1-t)\left(\frac{\xi_1+\xi_2}{2}\right)\right)dt \tag{9}$$
$$+\left(\frac{\xi_2 - \lambda\left(\frac{\xi_1+\xi_2}{2}\right)}{2}\right) \cdot \int_0^1 \left[\beta - \frac{\lambda\Delta(\lambda t)}{\xi_2 - \lambda\left(\frac{\xi_1+\xi_2}{2}\right)}\right]\psi'\left(\lambda\left(\frac{\xi_1+\xi_2}{2}\right)t + (1-t)\xi_2\right)dt.$$

Proof. We denote

$$P_{\psi,\Lambda,\Delta}(\lambda,\alpha,\beta,\xi_1,\xi_2) := \left(\frac{\lambda\left(\frac{\xi_1+\xi_2}{2}\right)-\xi_1}{2}\right)\cdot\int_0^1\left[\frac{\lambda\Lambda\big((1-t)\lambda\big)}{\lambda\left(\frac{\xi_1+\xi_2}{2}\right)-\xi_1}-\alpha\right]\psi'\left(\xi_1 t+\lambda(1-t)\left(\frac{\xi_1+\xi_2}{2}\right)\right)dt$$
$$+\left(\frac{\xi_2-\lambda\left(\frac{\xi_1+\xi_2}{2}\right)}{2}\right)\cdot\int_0^1\left[\beta-\frac{\lambda\Delta(\lambda t)}{\xi_2-\lambda\left(\frac{\xi_1+\xi_2}{2}\right)}\right]\psi'\left(\lambda\left(\frac{\xi_1+\xi_2}{2}\right)t+(1-t)\xi_2\right)dt. \quad (10)$$

Integrating by parts (10) and changing the variables of integration, we have

$$\begin{aligned}
&P_{\psi,\Lambda,\Delta}(\lambda,\alpha,\beta,\xi_1,\xi_2)\\
&=\left(\frac{\lambda\left(\frac{\xi_1+\xi_2}{2}\right)-\xi_1}{2}\right)\left[\left(\frac{\lambda}{\lambda\left(\frac{\xi_1+\xi_2}{2}\right)-\xi_1}\right)\int_0^1\Lambda\big(\lambda(1-t)\big)\psi'\left(\xi_1 t+\lambda(1-t)\left(\frac{\xi_1+\xi_2}{2}\right)\right)dt\right.\\
&\qquad\left.-\alpha\int_0^1\psi'\left(\xi_1 t+\lambda(1-t)\left(\frac{\xi_1+\xi_2}{2}\right)\right)dt\right]\\
&\quad+\left(\frac{\xi_2-\lambda\left(\frac{\xi_1+\xi_2}{2}\right)}{2}\right)\left[\beta\int_0^1\psi'\left(\lambda\left(\frac{\xi_1+\xi_2}{2}\right)t+(1-t)\xi_2\right)dt\right.\\
&\qquad\left.-\left(\frac{\lambda}{\xi_2-\lambda\left(\frac{\xi_1+\xi_2}{2}\right)}\right)\int_0^1\Delta(\lambda t)\,\psi'\left(\lambda\left(\frac{\xi_1+\xi_2}{2}\right)t+(1-t)\xi_2\right)dt\right]\\
&=\left(\frac{\lambda\left(\frac{\xi_1+\xi_2}{2}\right)-\xi_1}{2}\right)\left\{\left(\frac{\lambda}{\lambda\left(\frac{\xi_1+\xi_2}{2}\right)-\xi_1}\right)\left[\Lambda\big(\lambda(1-t)\big)\frac{\psi(\xi_1 t+\lambda(1-t)\left(\frac{\xi_1+\xi_2}{2}\right))}{\xi_1-\lambda\left(\frac{\xi_1+\xi_2}{2}\right)}\Bigg|_0^1\right.\right.\\
&\qquad\left.-\frac{1}{\lambda\left(\frac{\xi_1+\xi_2}{2}\right)-\xi_1}\int_0^1\frac{\phi\left(\left(\lambda\left(\frac{\xi_1+\xi_2}{2}\right)-\xi_1\right)(1-t)\right)}{1-t}\psi\left(\xi_1 t+\lambda(1-t)\left(\frac{\xi_1+\xi_2}{2}\right)\right)dt\right]\\
&\qquad\left.-\frac{\alpha}{\lambda\left(\frac{\xi_1+\xi_2}{2}\right)-\xi_1}\left[\psi\left(\lambda\left(\frac{\xi_1+\xi_2}{2}\right)\right)-\psi(\xi_1)\right]\right\}
\end{aligned}$$

$$+\left(\frac{\xi_2-\lambda\left(\frac{\xi_1+\xi_2}{2}\right)}{2}\right)\left\{\left(\frac{\lambda}{\xi_2-\lambda\left(\frac{\xi_1+\xi_2}{2}\right)}\right)\left[-\Delta(\lambda t)\frac{\psi\left(\lambda\left(\frac{\xi_1+\xi_2}{2}\right)t+(1-t)\xi_2\right)}{\lambda\left(\frac{\xi_1+\xi_2}{2}\right)-\xi_2}\Bigg|_0^1\right.\right.$$

$$\left.+\frac{1}{\lambda\left(\frac{\xi_1+\xi_2}{2}\right)-\xi_2}\int_0^1\frac{\phi\left(\left(\xi_2-\lambda\left(\frac{\xi_1+\xi_2}{2}\right)\right)t\right)}{t}\psi\left(\lambda\left(\frac{\xi_1+\xi_2}{2}\right)t+(1-t)\xi_2\right)dt\right]$$

$$\left.+\frac{\beta}{\xi_2-\lambda\left(\frac{\xi_1+\xi_2}{2}\right)}\left[\psi(\xi_2)-\psi\left(\lambda\left(\frac{\xi_1+\xi_2}{2}\right)\right)\right]\right\}=$$

$$=\frac{\alpha\psi(\xi_1)+\beta\psi(\xi_2)}{2}++\frac{\lambda}{2}\left[\frac{\Lambda(\lambda)}{\lambda\left(\frac{\xi_1+\xi_2}{2}\right)-\xi_1}+\frac{\Delta(\lambda)}{\xi_2-\lambda\left(\frac{\xi_1+\xi_2}{2}\right)}-\frac{\alpha+\beta}{\lambda}\right]\psi\left(\lambda\left(\frac{\xi_1+\xi_2}{2}\right)\right)$$

$$-\frac{\lambda}{2}\left[\frac{1}{\lambda\left(\frac{\xi_1+\xi_2}{2}\right)-\xi_1}\cdot{}_{\lambda\left(\frac{\xi_1+\xi_2}{2}\right)^-}I_\phi\psi(\xi_1)+\frac{1}{\xi_2-\lambda\left(\frac{\xi_1+\xi_2}{2}\right)}\cdot{}_{\lambda\left(\frac{\xi_1+\xi_2}{2}\right)^+}I_\phi\psi(\xi_2)\right].$$

The proof of Lemma 1 is completed.

Remark 2. Taking $\lambda=1$ and $\phi(t)=t$ in Lemma 1, we get [8, Lemma 2.1].

Theorem 2. Let $\psi: J\to\mathbb{R}$ be a differentiable function on J° and $\alpha,\beta\in[0,1]$. If $|\psi'|^q$ is a uniformly exponentially $(\omega_1,\omega_2,h_1,h_2)-$ convex function with modulus Φ on J for $q>1$ and $p^{-1}+q^{-1}=1$, then the following inequality for generalized fractional integrals holds:

$$\left|P_{\psi,\Lambda,\Delta}(\lambda,\alpha,\beta,\xi_1,\xi_2)\right|$$

$$\leq\left(\frac{\lambda\left(\frac{\xi_1+\xi_2}{2}\right)-\xi_1}{2}\right)\sqrt[p]{A_\Lambda(\alpha,\lambda,p)}$$

$$\cdot\left[\frac{\left|\psi'\left(\lambda\left(\frac{\xi_1+\xi_2}{2}\right)\right)\right|^q}{e^{\omega_1\lambda\left(\frac{\xi_1+\xi_2}{2}\right)}}\cdot H_1+\frac{|\psi'(\xi_1)|^q}{e^{\omega_2\xi_1}}\cdot H_2-G_{h_1h_2}\cdot\Phi\left(\left|\lambda\left(\frac{\xi_1+\xi_2}{2}\right)-\xi_1\right|\right)\right]^{\frac{1}{q}}\qquad(11)$$

$$+\left(\frac{\xi_2-\lambda\left(\frac{\xi_1+\xi_2}{2}\right)}{2}\right)\cdot\sqrt[p]{B_\Delta(\beta,\lambda,p)}$$

$$\cdot\left[\frac{|\psi'(\xi_2)|^q}{e^{\omega_1\xi_2}}\cdot H_1+\frac{\left|\psi'\left(\lambda\left(\frac{\xi_1+\xi_2}{2}\right)\right)\right|^q}{e^{\omega_2\lambda\left(\frac{\xi_1+\xi_2}{2}\right)}}\cdot H_2-G_{h_1h_2}\cdot\Phi\left(\left|\xi_2-\lambda\left(\frac{\xi_1+\xi_2}{2}\right)\right|\right)\right]^{\frac{1}{q}},$$

where

$$A_\Lambda(\alpha,\lambda,p):=\int_0^1\left|\frac{\lambda\Lambda\big((1-t)\lambda\big)}{\lambda\left(\frac{\xi_1+\xi_2}{2}\right)-\xi_1}-\alpha\right|^p dt,\quad B_\Delta(\beta,\lambda,p):=\int_0^1\left|\beta-\frac{\lambda\Delta(t)}{\xi_2-\lambda\left(\frac{\xi_1+\xi_2}{2}\right)}\right|^p dt,\tag{12}$$

$$H_1:=\int_0^1 h_1(t)dt,\qquad H_2:=\int_0^1 h_2(t)dt,\qquad G_{h_1h_2}:=\int_0^1 h_1(t)h_2(t)dt.\tag{13}$$

Proof. From Lemma 1, uniformly exponentially $(\omega_1,\omega_2,h_1,h_2)$- convexity of $|\psi'|^q$, Hölder's inequality and properties of the modulus, we have

$$\begin{aligned}
&\left|P_{\psi,\Lambda,\Delta}(\lambda,\alpha,\beta,\xi_1,\xi_2)\right|\\
&\le\left(\frac{\lambda\left(\frac{\xi_1+\xi_2}{2}\right)-\xi_1}{2}\right)\int_0^1\left|\frac{\lambda\Lambda\big((1-t)\lambda\big)}{\lambda\left(\frac{\xi_1+\xi_2}{2}\right)-\xi_1}-\alpha\right|\cdot\left|\psi'\left(\xi_1t+\lambda(1-t)\left(\frac{\xi_1+\xi_2}{2}\right)\right)\right|dt\\
&+\left(\frac{\xi_2-\lambda\left(\frac{\xi_1+\xi_2}{2}\right)}{2}\right)\int_0^1\left|\beta-\frac{\lambda\Delta(t)}{\xi_2-\lambda\left(\frac{\xi_1+\xi_2}{2}\right)}\right|\left|\psi'\left(\lambda\left(\frac{\xi_1+\xi_2}{2}\right)t+(1-t)\xi_2\right)\right|dt\\
&\le\left(\frac{\lambda\left(\frac{\xi_1+\xi_2}{2}\right)-\xi_1}{2}\right)\left(\int_0^1\left|\frac{\lambda\Lambda\big((1-t)\lambda\big)}{\lambda\left(\frac{\xi_1+\xi_2}{2}\right)-\xi_1}-\alpha\right|^p dt\right)^{\frac{1}{p}}\\
&\cdot\left(\int_0^1\left|\psi'\left(\xi_1t+\lambda(1-t)\left(\frac{\xi_1+\xi_2}{2}\right)\right)\right|^q dt\right)^{\frac{1}{q}}\\
&+\left(\frac{\xi_2-\lambda\left(\frac{\xi_1+\xi_2}{2}\right)}{2}\right)\left(\int_0^1\left|\beta-\frac{\lambda\Delta(t)}{\xi_2-\lambda\left(\frac{\xi_1+\xi_2}{2}\right)}\right|^p dt\right)^{\frac{1}{p}}\\
&\cdot\left(\int_0^1\left|\psi'\left(\lambda\left(\frac{\xi_1+\xi_2}{2}\right)t+(1-t)\xi_2\right)\right|^q dt\right)^{\frac{1}{q}}
\end{aligned}$$

$$\leq \left(\frac{\lambda\left(\frac{\xi_1+\xi_2}{2}\right)-\xi_1}{2}\right)\sqrt[p]{A_\Lambda(\alpha,\lambda,p)}\cdot\left[\int_0^1\left\{\frac{h_1(t)\left|\psi'\left(\lambda\left(\frac{\xi_1+\xi_2}{2}\right)\right)\right|^q}{e^{\omega_1\lambda\left(\frac{\xi_1+\xi_2}{2}\right)}}+\frac{h_2(t)|\psi'(\xi_1)|^q}{e^{\omega_2\xi_1}}\right.\right.$$

$$\left.\left.-h_1(t)h_2(t)\Phi\left(\left|\lambda\left(\frac{\xi_1+\xi_2}{2}\right)-\xi_1\right|\right)\right\}dt\right]^{\frac{1}{q}}+\left(\frac{\xi_2-\lambda\left(\frac{\xi_1+\xi_2}{2}\right)}{2}\right)\sqrt[p]{B_\Delta(\beta,\lambda,p)}$$

$$\cdot\left[\int_0^1\left\{\frac{h_1(t)|\psi'(\xi_2)|^q}{e^{\omega_1\xi_2}}+\frac{h_2(t)\left|\psi'\left(\lambda\left(\frac{\xi_1+\xi_2}{2}\right)\right)\right|^q}{e^{\omega_2\lambda\left(\frac{\xi_1+\xi_2}{2}\right)}}-h_1(t)h_2(t)\Phi\left(\left|\xi_2-\lambda\left(\frac{\xi_1+\xi_2}{2}\right)\right|\right)\right\}dt\right]^{\frac{1}{q}}$$

$$=\left(\frac{\lambda\left(\frac{\xi_1+\xi_2}{2}\right)-\xi_1}{2}\right)\sqrt[p]{A_\Lambda(\alpha,\lambda,p)}$$

$$\cdot\left[\frac{\left|\psi'\left(\lambda\left(\frac{\xi_1+\xi_2}{2}\right)\right)\right|^q}{e^{\omega_1\lambda\left(\frac{\xi_1+\xi_2}{2}\right)}}\cdot H_1+\frac{|\psi'(\xi_1)|^q}{e^{\omega_2\xi_1}}\cdot H_2-G_{h_1h_2}\cdot\Phi\left(\left|\lambda\left(\frac{\xi_1+\xi_2}{2}\right)-\xi_1\right|\right)\right]^{\frac{1}{q}}$$

$$+\left(\frac{\xi_2-\lambda\left(\frac{\xi_1+\xi_2}{2}\right)}{2}\right)\cdot\sqrt[p]{B_\Delta(\beta,\lambda,p)}$$

$$\cdot\left[\frac{|\psi'(\xi_2)|^q}{e^{\omega_1\xi_2}}\cdot H_1+\frac{\left|\psi'\left(\lambda\left(\frac{\xi_1+\xi_2}{2}\right)\right)\right|^q}{e^{\omega_2\lambda\left(\frac{\xi_1+\xi_2}{2}\right)}}\cdot H_2-G_{h_1h_2}\cdot\Phi\left(\left|\xi_2-\lambda\left(\frac{\xi_1+\xi_2}{2}\right)\right|\right)\right]^{\frac{1}{q}}.$$

The proof of Theorem 2 is completed.

Corollary 1. Choosing $\lambda = 1$ and $\phi(t) = t$ in Theorem 2, we have

$$\left|\frac{\alpha}{2}\psi(\xi_1)+\frac{2-\alpha-\beta}{2}\psi\left(\frac{\xi_1+\xi_2}{2}\right)+\frac{\beta}{2}\psi(\xi_2)-\frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\psi(t)dt\right|$$
$$\leq\left(\frac{\xi_2-\xi_1}{4}\right)\cdot\frac{1}{\sqrt[p]{p+1}}\cdot\sqrt[p]{\alpha^{p+1}+(1-\alpha)^{p+1}}$$
$$\cdot\left[\frac{\left|\psi'\left(\frac{\xi_1+\xi_2}{2}\right)\right|^q}{e^{\omega_1\left(\frac{\xi_1+\xi_2}{2}\right)}}\cdot H_1+\frac{|\psi'(\xi_1)|^q}{e^{\omega_2\xi_1}}\cdot H_2-G_{h_1h_2}\cdot\Phi\left(\left|\frac{\xi_2-\xi_1}{2}\right|\right)\right]^{\frac{1}{q}}$$
$$+\left(\frac{\xi_2-\xi_1}{4}\right)\frac{1}{\sqrt[p]{p+1}}\sqrt[p]{\beta^{p+1}+(1-\beta)^{p+1}}$$
$$\cdot\left[\frac{|\psi'(\xi_2)|^q}{e^{\omega_1\xi_2}}\cdot H_1+\frac{\left|\psi'\left(\frac{\xi_1+\xi_2}{2}\right)\right|^q}{e^{\omega_2\left(\frac{\xi_1+\xi_2}{2}\right)}}\cdot H_2-G_{h_1h_2}\cdot\Phi\left(\left|\frac{\xi_2-\xi_1}{2}\right|\right)\right]^{\frac{1}{q}}. \qquad (14)$$

Corollary 2. Taking $\alpha=\beta=1$ in Corollary 1, we get

$$\left|\frac{\psi(\xi_1)+\psi(\xi_2)}{2}-\frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\psi(t)dt\right|$$
$$\leq\left(\frac{\xi_2-\xi_1}{4}\right)\cdot\frac{1}{\sqrt[p]{p+1}}\cdot\left[\frac{\left|\psi'\left(\frac{\xi_1+\xi_2}{2}\right)\right|^q}{e^{\omega_1\left(\frac{\xi_1+\xi_2}{2}\right)}}\cdot H_1+\frac{|\psi'(\xi_1)|^q}{e^{\omega_2\xi_1}}\cdot H_2-G_{h_1h_2}\cdot\Phi\left(\left|\frac{\xi_2-\xi_1}{2}\right|\right)\right]^{\frac{1}{q}}$$
$$+\left(\frac{\xi_2-\xi_1}{4}\right)\frac{1}{\sqrt[p]{p+1}}\left[\frac{|\psi'(\xi_2)|^q}{e^{\omega_1\xi_2}}\cdot H_1+\frac{\left|\psi'\left(\frac{\xi_1+\xi_2}{2}\right)\right|^q}{e^{\omega_2\left(\frac{\xi_1+\xi_2}{2}\right)}}\cdot H_2-G_{h_1h_2}\cdot\Phi\left(\left|\frac{\xi_2-\xi_1}{2}\right|\right)\right]^{\frac{1}{q}}. \qquad (15)$$

Corollary 3. Choosing $\alpha=\beta=0$ in Corollary 1, we obtain

$$\left|\psi\left(\frac{\xi_1+\xi_2}{2}\right)-\frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\psi(t)dt\right|$$
$$\leq\left(\frac{\xi_2-\xi_1}{4}\right)\cdot\frac{1}{\sqrt[p]{p+1}}\cdot\left[\frac{\left|\psi'\left(\frac{\xi_1+\xi_2}{2}\right)\right|^q}{e^{\omega_1\left(\frac{\xi_1+\xi_2}{2}\right)}}\cdot H_1+\frac{|\psi'(\xi_1)|^q}{e^{\omega_2\xi_1}}\cdot H_2-G_{h_1h_2}\cdot\Phi\left(\left|\frac{\xi_2-\xi_1}{2}\right|\right)\right]^{\frac{1}{q}}$$
$$+\left(\frac{\xi_2-\xi_1}{4}\right)\frac{1}{\sqrt[p]{p+1}}$$
$$\cdot\left[\frac{|\psi'(\xi_2)|^q}{e^{\omega_1\xi_2}}\cdot H_1+\frac{\left|\psi'\left(\frac{\xi_1+\xi_2}{2}\right)\right|^q}{e^{\omega_2\left(\frac{\xi_1+\xi_2}{2}\right)}}\cdot H_2-G_{h_1h_2}\cdot\Phi\left(\left|\frac{\xi_2-\xi_1}{2}\right|\right)\right]^{\frac{1}{q}}. \tag{16}$$

Corollary 4. Taking $|\psi'|\leq K$ in Theorem 2, we have

$$\left|P_{\psi,\Lambda,\Delta}(\lambda,\alpha,\beta,\xi_1,\xi_2)\right|$$
$$\leq\left(\frac{\lambda\left(\frac{\xi_1+\xi_2}{2}\right)-\xi_1}{2}\right)\sqrt[p]{A_\Lambda(\alpha,\lambda,p)}$$
$$\cdot\left[K^q\left(\frac{H_1}{e^{\omega_1\lambda\left(\frac{\xi_1+\xi_2}{2}\right)}}\cdot+\frac{H_2}{e^{\omega_2\xi_1}}\right)-G_{h_1h_2}\cdot\Phi\left(\left|\lambda\left(\frac{\xi_1+\xi_2}{2}\right)-\xi_1\right|\right)\right]^{\frac{1}{q}}$$
$$+\left(\frac{\xi_2-\lambda\left(\frac{\xi_1+\xi_2}{2}\right)}{2}\right)\sqrt[p]{B_\Delta(\beta,\lambda,p)}$$
$$\cdot\left[K^q\left(\frac{H_1}{e^{\omega_1\xi_2}}+\frac{H_2}{e^{\omega_2\lambda\left(\frac{\xi_1+\xi_2}{2}\right)}}\right)-G_{h_1h_2}\cdot\Phi\left(\left|\xi_2-\lambda\left(\frac{\xi_1+\xi_2}{2}\right)\right|\right)\right]^{\frac{1}{q}}. \tag{17}$$

Theorem 3. Let $\psi: J\to\mathbb{R}$ be a differentiable function on $\mathring{J}$ and $\alpha,\beta\in[0,1]$. If $|\psi'|^q$ is a uniformly exponentially $(\omega_1,\omega_2,h_1,h_2)-$ convex function with modulus Φ on J for $q\geq 1$, then the following inequality for generalized fractional integrals holds:

$$
\begin{aligned}
&\left|P_{\psi,\Lambda,\Delta}\,(\lambda,\alpha,\beta,\xi_1,\xi_2)\right| \\
&\quad\le \left(\frac{\lambda\left(\frac{\xi_1+\xi_2}{2}\right)-\xi_1}{2}\right)\left(A_\Lambda(\alpha,\lambda,1)\right)^{1-\frac{1}{q}} \\
&\quad\cdot\left[\frac{\left|\psi'\left(\lambda\left(\frac{\xi_1+\xi_2}{2}\right)\right)\right|^q}{e^{\omega_1\lambda\left(\frac{\xi_1+\xi_2}{2}\right)}}\cdot C_\Lambda(\alpha,\lambda,h_1)+\frac{|\psi'(\xi_1)|^q}{e^{\omega_2\xi_1}}\cdot D_\Lambda(\alpha,\lambda,h_2)-E_\Lambda(\alpha,\lambda,h_1,h_2\,)\right. \\
&\quad\left.\cdot\,\Phi\left(\left|\lambda\left(\frac{\xi_1+\xi_2}{2}\right)-\xi_1\right|\right)\right]^{\frac{1}{q}}+\left(\frac{\xi_2-\lambda\left(\frac{\xi_1+\xi_2}{2}\right)}{2}\right)\cdot\left(B_\Delta(\beta,\lambda,1)\right)^{1-\frac{1}{q}} \\
&\quad\cdot\left[\frac{|\psi'(\xi_2)|^q}{e^{\omega_1\xi_2}}\cdot K_\Delta(\beta,\lambda,h_1)+\frac{\left|\psi'\left(\lambda\left(\frac{\xi_1+\xi_2}{2}\right)\right)\right|^q}{e^{\omega_2\lambda\left(\frac{\xi_1+\xi_2}{2}\right)}}\cdot L_\Delta(\beta,\lambda,h_2)-M_\Delta(\beta,\lambda,h_1,h_2\,)\right. \\
&\quad\left.\cdot\,\Phi\left(\left|\xi_2-\lambda\left(\frac{\xi_1+\xi_2}{2}\right)\right|\right)\right]^{\frac{1}{q}}, \qquad (18)
\end{aligned}
$$

where

$$C_\Lambda(\alpha,\lambda,h_1):=\int_0^1\left|\frac{\lambda\Lambda\big((1-t)\lambda\big)}{\lambda\left(\frac{\xi_1+\xi_2}{2}\right)-\xi_1}-\alpha\right|h_1(t)dt, \tag{19}$$

$$D_\Lambda(\alpha,\lambda,h_2):=\int_0^1\left|\frac{\lambda\Lambda\big((1-t)\lambda\big)}{\lambda\left(\frac{\xi_1+\xi_2}{2}\right)-\xi_1}-\alpha\right|h_2(t)dt, \tag{20}$$

$$E_\Lambda(\alpha,\lambda,h_1,h_2\,):=\int_0^1\left|\frac{\lambda\Lambda\big((1-t)\lambda\big)}{\lambda\left(\frac{\xi_1+\xi_2}{2}\right)-\xi_1}-\alpha\right|h_1(t)h_2(t)dt, \tag{21}$$

$$K_\Delta(\beta,\lambda,h_1):=\int_0^1\left|\beta-\frac{\lambda\Delta(t)}{\xi_2-\lambda\left(\frac{\xi_1+\xi_2}{2}\right)}\right|h_1(t)dt, \tag{22}$$

$$L_\Delta(\beta,\lambda,h_2):=\int_0^1\left|\beta-\frac{\lambda\Delta(t)}{\xi_2-\lambda\left(\frac{\xi_1+\xi_2}{2}\right)}\right|h_2(t)dt, \tag{23}$$

$$M_{\Delta}(\beta,\lambda,h_1,h_2) := \int_0^1 \left|\beta - \frac{\lambda\Delta(t)}{\xi_2 - \lambda\left(\frac{\xi_1+\xi_2}{2}\right)}\right| h_1(t)h_2(t)dt. \tag{24}$$

Proof. From Lemma 1, uniformly exponentially $(\omega_1,\omega_2,h_1,h_2)$- convexity of $|\psi'|^q$, the well-known power mean inequality and properties of the modulus, we have

$$\begin{aligned}
&\left|P_{\psi,\Lambda,\Delta}(\lambda,\alpha,\beta,\xi_1,\xi_2)\right| \\
&\le \left(\frac{\lambda\left(\frac{\xi_1+\xi_2}{2}\right)-\xi_1}{2}\right)\int_0^1 \left|\frac{\lambda\Lambda\big((1-t)\lambda\big)}{\lambda\left(\frac{\xi_1+\xi_2}{2}\right)-\xi_1}-\alpha\right|\cdot\left|\psi'\left(\xi_1 t+\lambda(1-t)\left(\frac{\xi_1+\xi_2}{2}\right)\right)\right|dt \\
&+\left(\frac{\xi_2-\lambda\left(\frac{\xi_1+\xi_2}{2}\right)}{2}\right)\int_0^1\left|\beta-\frac{\lambda\Delta(t)}{\xi_2-\lambda\left(\frac{\xi_1+\xi_2}{2}\right)}\right|\left|\psi'\left(\lambda\left(\frac{\xi_1+\xi_2}{2}\right)t+(1-t)\xi_2\right)\right|dt \\
&\le \left(\frac{\lambda\left(\frac{\xi_1+\xi_2}{2}\right)-\xi_1}{2}\right)\left(\int_0^1\left|\frac{\lambda\Lambda\big((1-t)\lambda\big)}{\lambda\left(\frac{\xi_1+\xi_2}{2}\right)-\xi_1}-\alpha\right|dt\right)^{1-\frac{1}{q}} \\
&\cdot\left(\int_0^1\left|\frac{\lambda\Lambda\big((1-t)\lambda\big)}{\lambda\left(\frac{\xi_1+\xi_2}{2}\right)-\xi_1}-\alpha\right|\left|\psi'\left(\xi_1 t+\lambda(1-t)\left(\frac{\xi_1+\xi_2}{2}\right)\right)\right|^q dt\right)^{\frac{1}{q}} \\
&+\left(\frac{\xi_2-\lambda\left(\frac{\xi_1+\xi_2}{2}\right)}{2}\right)\left(\int_0^1\left|\beta-\frac{\lambda\Delta(t)}{\xi_2-\lambda\left(\frac{\xi_1+\xi_2}{2}\right)}\right|dt\right)^{1-\frac{1}{q}} \\
&\cdot\left(\int_0^1\left|\beta-\frac{\lambda\Delta(t)}{\xi_2-\lambda\left(\frac{\xi_1+\xi_2}{2}\right)}\right|\left|\psi'\left(\lambda\left(\frac{\xi_1+\xi_2}{2}\right)t+(1-t)\xi_2\right)\right|^q dt\right)^{\frac{1}{q}}
\end{aligned}$$

$$\leq \left(\frac{\lambda\left(\frac{\xi_1+\xi_2}{2}\right)-\xi_1}{2}\right)[A_\Lambda(\alpha,\lambda,1)]^{1-\frac{1}{q}} \cdot \left[\int_0^1 \left|\frac{\lambda\Lambda((1-t)\lambda)}{\lambda\left(\frac{\xi_1+\xi_2}{2}\right)-\xi_1}-\alpha\right| \left\{\frac{h_1(t)\left|\psi'\left(\lambda\left(\frac{\xi_1+\xi_2}{2}\right)\right)\right|^q}{e^{\omega_1\lambda\left(\frac{\xi_1+\xi_2}{2}\right)}}\right.\right.$$

$$\left.\left.+\frac{h_2(t)|\psi'(\xi_1)|^q}{e^{\omega_2\xi_1}}-h_1(t)h_2(t)\Phi\left(\left|\lambda\left(\frac{\xi_1+\xi_2}{2}\right)-\xi_1\right|\right)\right\}dt\right]^{\frac{1}{q}}$$

$$+\left(\frac{\xi_2-\lambda\left(\frac{\xi_1+\xi_2}{2}\right)}{2}\right)[B_\Delta(\beta,\lambda,p)]^{1-\frac{1}{q}}$$

$$\cdot\left[\int_0^1\left|\beta-\frac{\lambda\Delta(t)}{\xi_2-\lambda\left(\frac{\xi_1+\xi_2}{2}\right)}\right|\left\{\frac{h_1(t)|\psi'(\xi_2)|^q}{e^{\omega_1\xi_2}}+\frac{h_2(t)\left|\psi'\left(\lambda\left(\frac{\xi_1+\xi_2}{2}\right)\right)\right|^q}{e^{\omega_2\lambda\left(\frac{\xi_1+\xi_2}{2}\right)}}\right.\right.$$

$$\left.\left.-h_1(t)h_2(t)\Phi\left(\left|\xi_2-\lambda\left(\frac{\xi_1+\xi_2}{2}\right)\right|\right)\right\}dt\right]^{\frac{1}{q}}$$

$$=\left(\frac{\lambda\left(\frac{\xi_1+\xi_2}{2}\right)-\xi_1}{2}\right)\left(A_\Lambda(\alpha,\lambda,1)\right)^{1-\frac{1}{q}}$$

$$\cdot\left[\frac{\left|\psi'\left(\lambda\left(\frac{\xi_1+\xi_2}{2}\right)\right)\right|^q}{e^{\omega_1\lambda\left(\frac{\xi_1+\xi_2}{2}\right)}}\cdot C_\Lambda(\alpha,\lambda,h_1)+\frac{|\psi'(\xi_1)|^q}{e^{\omega_2\xi_1}}\cdot D_\Lambda(\alpha,\lambda,h_2)-E_\Lambda(\alpha,\lambda,h_1,h_2)\right.$$

$$\left.\cdot\Phi\left(\left|\lambda\left(\frac{\xi_1+\xi_2}{2}\right)-\xi_1\right|\right)\right]^{\frac{1}{q}}+\left(\frac{\xi_2-\lambda\left(\frac{\xi_1+\xi_2}{2}\right)}{2}\right)\cdot\left(B_\Delta(\beta,\lambda,1)\right)^{1-\frac{1}{q}}$$

$$\cdot\left[\frac{|\psi'(\xi_2)|^q}{e^{\omega_1\xi_2}}\cdot K_\Delta(\beta,\lambda,h_1)+\frac{\left|\psi'\left(\lambda\left(\frac{\xi_1+\xi_2}{2}\right)\right)\right|^q}{e^{\omega_2\lambda\left(\frac{\xi_1+\xi_2}{2}\right)}}\cdot L_\Delta(\beta,\lambda,h_2)-M_\Delta(\beta,\lambda,h_1,h_2)\right.$$

$$\left.\cdot\Phi\left(\left|\xi_2-\lambda\left(\frac{\xi_1+\xi_2}{2}\right)\right|\right)\right]^{\frac{1}{q}}.$$

The proof of Theorem 3 is completed.

Corollary 5. Taking $\lambda = 1$ and $\phi(t) = t$ in Theorem 3, we have

$$\left|\frac{\alpha}{2}\psi(\xi_1)+\frac{2-\alpha-\beta}{2}\psi\left(\frac{\xi_1+\xi_2}{2}\right)+\frac{\beta}{2}\psi(\xi_2)-\frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\psi(t)dt\right|$$

$$\leq\left(\frac{\xi_2-\xi_1}{4}\right)\left(\frac{1-2\alpha+2\alpha^2}{2}\right)^{1-\frac{1}{q}} \tag{25}$$

$$\cdot\left[\frac{\left|\psi'\left(\frac{\xi_1+\xi_2}{2}\right)\right|^q}{e^{\omega_1\left(\frac{\xi_1+\xi_2}{2}\right)}}C_\Lambda(\alpha,1,h_1)+\frac{|\psi'(\xi_1)|^q}{e^{\omega_2\xi_1}}D_\Lambda(\alpha,1,h_2)-E_\Lambda(\alpha,1,h_1,h_2\,)\Phi\left(\left|\frac{\xi_2-\xi_1}{2}\right|\right)\right]^{\frac{1}{q}}$$

$$+\left(\frac{\xi_2-\xi_1}{4}\right)\cdot\left(\frac{1-2\beta+2\beta^2}{2}\right)^{1-\frac{1}{q}}$$

$$\cdot\left[\frac{|\psi'(\xi_2)|^q}{e^{\omega_1\xi_2}}K_\Delta(\beta,1,h_1)+\frac{\left|\psi'\left(\frac{\xi_1+\xi_2}{2}\right)\right|^q}{e^{\omega_2\left(\frac{\xi_1+\xi_2}{2}\right)}}L_\Delta(\beta,1,h_2)-M_\Delta(\beta,1,h_1,h_2\,)\Phi\left(\left|\frac{\xi_2-\xi_1}{2}\right|\right)\right]^{\frac{1}{q}}.$$

Corollary 6. Choosing $q=1$ in Theorem 3, we get

$$\left|P_{\psi,\Lambda,\Delta}\,(\lambda,\alpha,\beta,\xi_1,\xi_2)\right|$$

$$\leq\left(\frac{\lambda\left(\frac{\xi_1+\xi_2}{2}\right)-\xi_1}{2}\right) \tag{26}$$

$$\cdot\left[\frac{\left|\psi'\left(\lambda\left(\frac{\xi_1+\xi_2}{2}\right)\right)\right|}{e^{\omega_1\lambda\left(\frac{\xi_1+\xi_2}{2}\right)}}\cdot C_\Lambda(\alpha,\lambda,h_1)+\frac{|\psi'(\xi_1)|}{e^{\omega_2\xi_1}}\cdot D_\Lambda(\alpha,\lambda,h_2)-E_\Lambda(\alpha,\lambda,h_1,h_2\,)\cdot\Phi\left(\left|\lambda\left(\frac{\xi_1+\xi_2}{2}\right)-\xi_1\right|\right)\right]$$

$$+\left(\frac{\xi_2-\lambda\left(\frac{\xi_1+\xi_2}{2}\right)}{2}\right)$$

$$\cdot\left[\frac{|\psi'(\xi_2)|}{e^{\omega_1\xi_2}}\cdot K_\Delta(\beta,\lambda,h_1)+\frac{\left|\psi'\left(\lambda\left(\frac{\xi_1+\xi_2}{2}\right)\right)\right|}{e^{\omega_2\lambda\left(\frac{\xi_1+\xi_2}{2}\right)}}\cdot L_\Delta(\beta,\lambda,h_2)-M_\Delta(\beta,\lambda,h_1,h_2\,)\right.$$

$$\left.\cdot\,\Phi\left(\left|\xi_2-\lambda\left(\frac{\xi_1+\xi_2}{2}\right)\right|\right)\right].$$

Corollary 7. Taking $|\psi'|\leq K$ in Theorem 3, we obtain

$$
\begin{aligned}
&\left|P_{\psi,\Lambda,\Delta}(\lambda,\alpha,\beta,\xi_1,\xi_2)\right| \\
&\le \left(\frac{\lambda\left(\frac{\xi_1+\xi_2}{2}\right)-\xi_1}{2}\right)\left(A_\Lambda(\alpha,\lambda,1)\right)^{1-\frac{1}{q}} \qquad (27)\\
&\cdot\left[\frac{K^q}{e^{\omega_1\lambda\left(\frac{\xi_1+\xi_2}{2}\right)}}C_\Lambda(\alpha,\lambda,h_1)+\frac{K^q}{e^{\omega_2\xi_1}}D_\Lambda(\alpha,\lambda,h_2)-E_\Lambda(\alpha,\lambda,h_1,h_2)\Phi\left(\left|\lambda\left(\frac{\xi_1+\xi_2}{2}\right)-\xi_1\right|\right)\right]^{\frac{1}{q}}\\
&+\left(\frac{\xi_2-\lambda\left(\frac{\xi_1+\xi_2}{2}\right)}{2}\right)\cdot\left(B_\Delta(\beta,\lambda,1)\right)^{1-\frac{1}{q}}\\
&\cdot\left[\frac{K^q}{e^{\omega_1\xi_2}}K_\Delta(\beta,\lambda,h_1)+\frac{K^q}{e^{\omega_2\lambda\left(\frac{\xi_1+\xi_2}{2}\right)}}L_\Delta(\beta,\lambda,h_2)-M_\Delta(\beta,\lambda,h_1,h_2)\Phi\left(\left|\xi_2-\lambda\left(\frac{\xi_1+\xi_2}{2}\right)\right|\right)\right]^{\frac{1}{q}}.
\end{aligned}
$$

3. Applications

Consider the following special means for different positive real numbers $\xi_1 < \xi_2$:

- Arithmetic mean: $A(\xi_1,\xi_2)=\frac{\xi_1+\xi_2}{2}$;
- The harmonic mean: $H(\xi_1,\xi_2)=\frac{2}{\frac{1}{\xi_1}+\frac{1}{\xi_2}}$;
- The logarithmic mean: $L(\xi_1,\xi_2)=\frac{\xi_2-\xi_1}{\ln\xi_2-\ln\xi_1}$;
- The k–generalized log–mean: $L_k(\xi_1,\xi_2)=\left[\frac{\xi_2^{k+1}-\xi_1^{k+1}}{(k+1)(\xi_2-\xi_1)}\right]^{\frac{1}{k}}$, $k\in\mathbb{Z}\backslash\{-1,0\}$.

Using the results obtained in the previous section, we give some applications to special means.

Proposition 1. Let $0<\xi_1<\xi_2$, $\omega_1,\omega_2\in\mathbb{R}^-$, $n\ge 2$, $n\in\mathbb{N}$ and $\alpha,\beta\in[0,1]$. Then for $q>1$ and $p^{-1}+q^{-1}=1$, the following inequality holds:

$$\begin{aligned}&\left|A(\alpha\xi_1^n,\beta\xi_2^n)+\frac{2-\alpha-\beta}{2}A^n(\xi_1,\xi_2)-L_n^n(\xi_1,\xi_2)\right|\\&\quad\le\frac{\xi_2-\xi_1}{4\sqrt[q]{2}\sqrt[p]{p+1}}\\&\quad\cdot\left\{\sqrt[p]{\alpha^{p+1}+(1-\alpha)^{p+1}}\left[\frac{n^qA^{(n-1)q}(\xi_1,\xi_2)}{e^{\omega_1\left(\frac{\xi_1+\xi_2}{2}\right)}}+\frac{n^q\xi_1^{(n-1)q}}{e^{\omega_2\xi_1}}-\frac{\xi_2-\xi_1}{6}\right]^{\frac{1}{q}}\right.\\&\quad\left.+\sqrt[p]{\beta^{p+1}+(1-\beta)^{p+1}}\left[\frac{n^q\xi_2^{(n-1)q}}{e^{\omega_1\xi_2}}+\frac{n^qA^{(n-1)q}(\xi_1,\xi_2)}{e^{\omega_2\left(\frac{\xi_1+\xi_2}{2}\right)}}-\frac{\xi_2-\xi_1}{6}\right]^{\frac{1}{q}}\right\}.\end{aligned}$$

Proof. Taking $\psi(t)=t^n, n\ge 2,\ n\in\mathbb{N}$, $h_1(t)=t,\ h_2(t)=(1-t)$, and $\Phi(t)=t$ in Corollary 1, we obtain the result.

Proposition 2. Let $0<\xi_1<\xi_2, \omega_1,\omega_2\ \in\mathbb{R}^-$ and $\alpha,\beta\ \in[0,1]$. Then for $q>1$ and $p^{-1}+q^{-1}=1$, the following inequality holds:

$$\begin{aligned}&\left|\frac{1}{H\left(\frac{\xi_1}{\alpha},\frac{\xi_2}{\beta}\right)}+\frac{2-\alpha-\beta}{2}\frac{1}{A(\xi_1,\xi_2)}-\frac{1}{L(\xi_1,\xi_2)}\right|\\&\quad\le\frac{\xi_2-\xi_1}{4\sqrt[p]{p+1}}\\&\quad\cdot\left\{\sqrt[p]{\alpha^{p+1}+(1-\alpha)^{p+1}}\left[\frac{1}{H\left(A^{2q}(\xi_1,\xi_2)e^{-\omega_1A(\xi_1,\xi_2)},\ \xi_1^{2q}e^{-\omega_2\xi_1}\right)}-\frac{\xi_2-\xi_1}{12}\right]^{\frac{1}{q}}\right.\\&\quad\left.+\sqrt[p]{\beta^{p+1}+(1-\beta)^{p+1}}\left[\frac{1}{H\left(\xi_2^{2q}e^{-\omega_1\xi_2},\ A^{2q}(\xi_1,\xi_2)e^{-\omega_2A(\xi_1,\xi_2)}\right)}-\frac{\xi_2-\xi_1}{12}\right]^{\frac{1}{q}}\right\}.\end{aligned}$$

Proof. Choosing $\psi(t)=\frac{1}{t}$, $h_1(t)=t,\ h_2(t)=(1-t)$, and $\Phi(t)=t$ in Corollary 1, we obtain the result.

Next, we provide some new error estimates for the mixed trapezium and midpoint formula. Let $\mathcal{P}$ be the partition of the points $\xi_1=x_0<x_1<\ldots<x_k=\xi_2$ of the interval J. Let consider the following quadrature formula:

$$\int_{\xi_1}^{\xi_2}\psi(t)dt=TM(\psi,\mathcal{P};\alpha,\beta)+E(\psi,\mathcal{P};\alpha,\beta),$$

where

$$TM(\psi,\mathcal{P};\alpha,\beta):=\sum_{i=0}^{k}\left[\frac{\alpha}{2}\psi(x_i)+\frac{2-\alpha-\beta}{2}\psi\left(\frac{x_i+x_{i+1}}{2}\right)+\frac{\beta}{2}\psi(x_{i+1})\right](x_{i+1}-x_i)$$

is the mixed trapezium and midpoint version and $E(\psi,\mathcal{P};\alpha,\beta)$ is denote their associated approximation error.

Proposition 3. Let $\psi : J \to \mathbb{R}$ be a differentiable function on J° and $\alpha,\beta \in [0,1]$. If $|\psi'|^q$ is a uniformly exponentially $(\omega_1,\omega_2,h_1,h_2)-$ convex function with modulus Φ on J for $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$, then the following inequality holds:

$$\begin{aligned}
|E(\psi,\mathcal{P};\alpha,\beta)| \leq{}& \frac{1}{4\sqrt[q]{2}\cdot\sqrt[p]{p+1}} \\
&\cdot\sum_{i=1}^{k}(x_{i+1}-x_i)^2\Bigg\{\sqrt[p]{\alpha^{p+1}+(1-\alpha)^{p+1}} \\
&\cdot\left[\frac{\left|\psi'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q}{e^{\omega_1\left(\frac{x_i+x_{i+1}}{2}\right)}}+\frac{|\psi'(x_i)|^q}{e^{\omega_2 x_i}}-\frac{1}{3}\cdot\Phi\left(\left|\frac{x_{i+1}-x_i}{2}\right|\right)\right]^{\frac{1}{q}}+\sqrt[p]{\beta^{p+1}+(1-\beta)^{p+1}} \\
&\cdot\left[\frac{|\psi'(x_{i+1})|^q}{e^{\omega_1 x_{i+1}}}+\frac{\left|\psi'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q}{e^{\omega_2\left(\frac{x_i+x_{i+1}}{2}\right)}}-\frac{1}{3}\cdot\Phi\left(\left|\frac{x_{i+1}-x_i}{2}\right|\right)\right]^{\frac{1}{q}}\Bigg\}.
\end{aligned}$$

Proof. Applying Corollary 1 for $h_1(t)=t,\ h_2(t)=(1-t)$, on the subintervals (x_i,x_{i+1}) of the partition $\mathcal{P}$, we have

$$\begin{aligned}
&\left|\frac{\alpha}{2}\psi(x_i)+\frac{2-\alpha-\beta}{2}\psi\left(\frac{x_i+x_{i+1}}{2}\right)+\frac{\beta}{2}\psi(x_{i+1})-\frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}}\psi(t)dt\right| \\
&\leq\left(\frac{x_{i+1}-x_i}{4}\right)\cdot\frac{1}{\sqrt[q]{2}\cdot\sqrt[p]{p+1}} \\
&\cdot\Bigg\{\sqrt[p]{\alpha^{p+1}+(1-\alpha)^{p+1}}\cdot\left[\frac{\left|\psi'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q}{e^{\omega_1\left(\frac{x_i+x_{i+1}}{2}\right)}}+\frac{|\psi'(x_i)|^q}{e^{\omega_2 x_i}}-\frac{1}{3}\cdot\Phi\left(\left|\frac{x_{i+1}-x_i}{2}\right|\right)\right]^{\frac{1}{q}} \\
&+\sqrt[p]{\beta^{p+1}+(1-\beta)^{p+1}}\cdot\left[\frac{|\psi'(x_{i+1})|^q}{e^{\omega_1 x_{i+1}}}+\frac{\left|\psi'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q}{e^{\omega_2\left(\frac{x_i+x_{i+1}}{2}\right)}}-\frac{1}{3}\cdot\Phi\left(\left|\frac{x_{i+1}-x_i}{2}\right|\right)\right]^{\frac{1}{q}}\Bigg\}.
\end{aligned}$$

Hence,

$$
\begin{aligned}
|\mathrm{E}(\psi,\mathcal{P};\alpha,\beta)| &= \left|\int_{\xi_1}^{\xi_2}\psi(t)dt - \mathrm{TM}(\psi,\mathcal{P};\alpha,\beta)\right| \\
&\leq \sum_{i=0}^{k}\left|\left\{\int_{x_i}^{x_{i+1}}\psi(t)dt - \left(\frac{\alpha}{2}\psi(x_i) + \frac{2-\alpha-\beta}{2}\psi\left(\frac{x_i+x_{i+1}}{2}\right) + \frac{\beta}{2}\psi(x_{i+1})\right)(x_{i+1}-x_i)\right\}\right| \\
&\leq \frac{1}{4\sqrt[q]{2}\cdot\sqrt[p]{p+1}} \\
&\cdot\sum_{i=1}^{k}(x_{i+1}-x_i)^2\left\{\sqrt[p]{\alpha^{p+1}+(1-\alpha)^{p+1}}\right. \\
&\cdot\left[\frac{\left|\psi'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q}{e^{\omega_1\left(\frac{x_i+x_{i+1}}{2}\right)}} + \frac{|\psi'(x_i)|^q}{e^{\omega_2 x_i}} - \frac{1}{3}\cdot\Phi\left(\left|\frac{x_{i+1}-x_i}{2}\right|\right)\right]^{\frac{1}{q}} + \sqrt[p]{\beta^{p+1}+(1-\beta)^{p+1}} \\
&\cdot\left.\left[\frac{|\psi'(x_{i+1})|^q}{e^{\omega_1 x_{i+1}}} + \frac{\left|\psi'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q}{e^{\omega_2\left(\frac{x_i+x_{i+1}}{2}\right)}} - \frac{1}{3}\cdot\Phi\left(\left|\frac{x_{i+1}-x_i}{2}\right|\right)\right]^{\frac{1}{q}}\right\}.
\end{aligned}
$$

4. Conclusion

In this paper, we established an integral identity with three parameters via generalized integral operators. Using this as an auxiliary result, we derived some new bounds on Hermite-Hadamard type integral inequality for differentiable functions that are in absolute value at certain powers uniformly exponentially $(\omega_1, \omega_2, h_1, h_2)$-convex. Our results included several new and known results as particular cases. Moreover, some applications of presented results for special means and error estimates for the mixed trapezium and midpoint formula are found. To the best of our knowledge, these results are new in the literature. Studies relating convexity may have useful applications, such as maximizing the likelihood from multiple linear regressions involving Gauss-Laplace distribution. Finally, we can observe that the new defined class is a new powerful type of convex functions to investigate various inequalities in the field of real analysis.

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Some Properties of ⊕-g-Rad-Supplemented Modules

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Abstract

In this work, some properties of ⊕-g-Rad-supplemented modules are studied. Every ring has an unity and every module is an unitary left module, in this work. It is proved that a direct sum of two ⊕-g-Rad-supplemented modules is ⊕-g-Rad-supplemented.

Keywords: Essential Submodules, g-Small Submodules, g-Supplemented Modules, ⊕-g-Supplemented Modules.

2020 Mathematics Subject Classification: 16D10, 16D80.

1. INTRODUCTION

Throughout this paper all rings will be associative with identity and all modules will be unital left modules.

Let R be a ring and M be an R-module. We will denote a submodule N of M by $N \leq M$. Let M be an R-module and $N \leq M$. If $L=M$ for every submodule L of M such that $M=N+L$, then N is called a *small* submodule of M and denoted by $N \ll M$. Let M be an R-module and $N \leq M$. If there exists a submodule K of M such that $M=N+K$ and $N \cap K=0$, then N is called a *direct summand* of M and it is denoted by $M=N \oplus K$. For any R-module M, we have $M=M \oplus 0$. The intersection of all maximal submodules of M is called the *radical* of M and denoted by $RadM$. If M have no maximal submodules, then it is defined $RadM=M$. M is said to be *semilocal* if $M/RadM$ is semisimple. A submodule N of an R-module M is called an *essential* submodule of M and denoted by $N \trianglelefteq M$ in case $K \cap N \neq 0$ for every submodule $K \neq 0$, or equivalently, $K=0$ for every $K \leq M$ with $N \cap K=0$. Let M be an R-module and K be a submodule of M. K is called a *generalized small* (or briefly, *g-small*) submodule of M if for every essential submodule T of M with the property $M=K+T$ implies that $T=M$, then we write $K \ll_g M$. It is clear that every small submodule is a generalized small submodule but the converse is not true in general. Let M be an R-module. M is called a *hollow* module if every proper submodule of M is small in M. M is called a *generalized hollow* (or briefly, *g-hollow*) module if every proper submodule of M is g-small in M. Here it is clear that every hollow module is generalized hollow. The converse of this statement is not always true. M is called a *local* module if M has the largest submodule, i.e. a proper submodule which contains all other proper submodules. M is said to be *generalized local* (briefly, *g-local*) if M has a large proper essential submodule which contain all proper essential submodules of M or M have no proper essential submodules. Let U and V be submodules of M. If $M=U+V$ and V is minimal with respect to this property, or equivalently, $M=U+V$ and $U\cap V\ll V$, then V is called a *supplement* of U in M. M is said to be *supplemented* if every submodule of M has a supplement in M. If every submodule of M has a supplement that is a direct summand in M, then M is called a *⊕-supplemented* module. Let M be an R-module and $U;V\leq M$. If $M=U+V$ and $M=U+T$ with $T\trianglelefteq V$ implies that $T=V$, or equivalently, $M=U+V$ and $U\cap V\ll_g V$, then V is called a *g-supplement* of U in M. M is said to be *g-supplemented* if every submodule of M has a g-supplement in M. M is said to be *⊕-g-supplemented* if every submodule of M has a g-supplement that is a direct summand in M. Let M be an R-module and $U,V\leq M$. If $M=U+V$ and $U\cap V\leq RadV$, then V is called a *generalized* (*radical*) *supplement* (briefly, *Rad-supplement*) of U in M. M is said to be *generalized* (*radical*) *supplemented* (briefly, *Rad-supplemented*) if every submodule of M has a Rad-supplement in M. M is said to be *generalized* (*radical*) *⊕-supplemented* (briefly, *Rad-⊕-supplemented*) if every submodule of M has a Rad-supplement that is a direct summand in M. The intersection of all essential maximal submodules of an R-module M is called the *generalized radical* of M and denoted by Rad_gM. If M have no essential maximal submodules, then we denote $Rad_gM=M$. An R-module M is said to be *g-semilocal* if M/Rad_gM is semisimple. Let M be an R-module and $U,V\leq M$. If $M=U+V$ and $U\cap V\leq Rad_gV$, then V is called a *generalized radical supplement* (or briefly, *g-radical supplement*) of U in M. M is said to be *generalized radical supplemented* (briefly, g-*radical supplemented*) if every submodule of M has a g-radical supplement in M.

More informations about supplemented modules are in [1] and [11]. More results about ⊕-supplemented modules are in [3]. Generalized (radical) supplemented modules are studied in [10]. Rad-⊕-supplemented modules are studied in [2] and [9]. G-small submodules, g-supplemented modules and g-radical supplemented modules are studied in [4], [5] and [6]. More informations about ⊕-g-supplemented modules are in [7].

2. ⊕-g-RAD-SUPPLEMENTED MODULES

Definition 2.1. Let M be an R-module. If every submodule of M has a g-radical supplement that is a direct summand in M, then M is called a *⊕-g-Rad-supplemented* module. (See [8])

Proposition 2.2. Every ⊕-g-supplemented module is ⊕-g-Rad-supplemented.
Proof. Let M be a ⊕-g-supplemented module and $U\leq M$. Since M is ⊕-g-supplemented, U has a g-supplement V that is a direct summand in M. Here $M=U+V$ and $U\cap V\ll_g V$. Since $U\cap V\ll_g V$, $U\cap V\leq Rad_gV$. Hence V is a g-radical supplement of U in M. Thus M is ⊕-g-Rad-supplemented.

Proposition 2.3. Let $M=U\oplus V$. If U and V are ⊕-g-supplemented, then M is ⊕-g-supplemented.
Proof. Since U and V is ⊕-g-supplemented, by [7], M is ⊕-g-supplemented. Then by Proposition 2.2, M is ⊕-g-Rad-supplemented.

Corollary 2.4. Let $M=M_1\oplus M_2\oplus...\oplus M_n$. If M_i is ⊕-g-supplemented for every i=1,2,...,n, then M is ⊕-g-Rad-supplemented.
Proof. Clear from Proposition 2.3.

Proposition 2.5. Let M be ⊕-g-supplemented module and K be a submodule of M. If $(X+K)/K$ is a direct summand of M/K for every direct summand X of M, then M is ⊕-g-Rad-supplemented.
Proof. By [7], M/K is ⊕-g-supplemented. Then by Prposition 2.3, M is ⊕-g-Rad-supplemented, as desired.

Proposition 2.6. Let M be a distributive ⊕-g-supplemented R-module. Then every factor module of M is ⊕-g-Rad-supplemented.
Proof. By [7], every factor module of M is ⊕-g-supplemented. Then by Prposition 2.3, every factor module of M is ⊕-g-Rad-supplemented, as desired.

Corollary 2.7. Let M be a distributive ⊕-g-supplemented R-module. Then every direct summand of M is ⊕-g-Rad-supplemented.
Proof. Clear from Proposition 2.6, since every direct summand of M is isomorphic to a factor module of M.

Corollary 2.7. Let M be a distributive ⊕-g-supplemented R-module. Then every homomorphic image of M is ⊕-g-Rad-supplemented.
Proof. Clear from Proposition 2.6, since every homomorphic image of M is isomorphic to a factor module of M.

Proposition 2.8. Let M be a ⨁−g−supplemented R−module with (D3) property. Then every direct summand of M is ⨁−g−Rad-supplemented.
Proof. By [7], every direct summand of M is ⊕-g-supplemented. Then by Prposition 2.3, every direct summand of M is ⊕-g-Rad-supplemented, as desired.

Proposition 2.9. Let M be a ⨁−g−supplemented R−module with (D3) property. Then M/K is ⨁−g−Rad-supplemented for every direct summand K of M.
Proof. By [7], M/K is ⊕-g-supplemented. Then by Prposition 2.3, M/K is ⊕-g-Rad-supplemented, as desired.

Proposition 2.10. Let M be a ⨁−g−supplemented R−module with SSP property. Then M/K is ⨁−g−Rad-supplemented for every direct summand K of M.

Proof. By [7], M/K is $\oplus$-g-supplemented. Then by Prposition 2.3, M/K is $\oplus$-g-Rad-supplemented, as desired.

Corollary 2.11. Let M be a $\oplus$−g−supplemented R−module with SSP property. Then every direct summand of M is $\oplus$-g-Rad-supplemented.

Proof. Let U be a direct summand of M and $M=U\oplus V$ with $V\leq M$. By proposition M/V is $\oplus$-g-Rad-supplemented. Then by $M/V=(U+V)/V\cong U/(U\cap V)=U/0\cong U$, U is $\oplus$-g-Rad-supplemented, as desired.

Proposition 2.12. Let M be an R-module and $M=M_1\oplus M_2$. If M_1 and M_2 are $\oplus$-g-Rad-supplemented, then M is also $\oplus$-g-Rad-supplemented.

Proof. Let U be any submodule of M. Since M_2 is $\oplus$-g-Rad-supplemented, $(M_1+U)\cap M_2$ has a g-radical supplement X that is a direct summand of M_2. Since X is a g-radical supplement of $(M_1+U)\cap M_2$ in M_2, $M_2=(M_1+U)\cap M_2+X$ and $(M_1+U)\cap X=(M_1+U)\cap M_2\cap X\leq Rad_gX$. By $M_2=(M_1+U)\cap M_2+X$, $M=M_1+M_2=M_1+(M_1+U)\cap M_2+X=M_1+U+X$. Since M_1 is $\oplus$-g-Rad-supplemented, $(U+X)\cap M_1$ has a g-radical supplement Y that is a direct summand of M_1. Since Y is a g-radical supplement of $(U+X)\cap M_1$ in M_1, $M_1=(U+X)\cap M_1+Y$ and $(U+X)\cap Y=(U+X)\cap M_1\cap Y\leq Rad_gY$. By $M_1=(U+X)\cap M_1+Y$, $M=M_1+U+X=(U+X)\cap M_1+Y+U+X=U+X+Y$. Since $(M_1+U)\cap X\leq Rad_gX$ and $(U+X)\cap Y\leq Rad_gY$, $U\cap(X+Y)\leq(U+Y)\cap X+(U+X)\cap Y\leq(M_1+U)\cap X+(U+X)\cap Y\leq Rad_gX+Rad_gY\leq Rad_g(X+Y)$. Hence $X+Y$ is a g-radical supplement of U in M. Since X is a direct summand of M_2 and Y is a direct summand of M_1, $X\oplus Y$ is a direct summand of $M=M_1\oplus M_2$. Hence M is $\oplus$-g-Rad-supplemented.

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Some Theorems on Absolute Matrix Summability

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Abstract

In this paper, two theorems on absolute Riesz summability factors of infinite series are generalized to the $\varphi-\left|A, p_n\right|_k$ summability method by using almost increasing sequences.

Keywords: Riesz mean, almost increasing sequences, infinite series, absolute matrix summability.

1. Introduction

Let $\sum a_n$ be a given infinite series with the partial sums (s_n) and $\left(p_n\right)$ be a sequence of positive numbers such that

$$P_n=\sum_{v=0}^{n} p_v \rightarrow \infty \quad \text{as} \quad \left(n\rightarrow\infty\right),\ (P_{-i}=p_{-i}=0,\ i\geq 1).$$

The series $\sum a_n$ is said to be summable $\left|\bar{N}, p_n\right|_k$, $k\geq 1$, if [1]

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}\left|W_n-W_{n-1}\right|^k<\infty$$

where

$$W_n=\frac{1}{P_n}\sum_{v=0}^{n} p_v s_v.$$

Let $A=(a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries and let (φ_n) be any sequence of positive real numbers. The series $\sum a_n$ is said to be summable $\varphi-\left|A, p_n\right|_k$ $k\geq 1$, if [2]

$$\sum_{n=1}^{\infty}\varphi_n^{k-1}\left|A_n(s)-A_{n-1}(s)\right|^k<\infty$$

where

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v \ , \quad n = 0, 1, \ldots$$

A positive sequence (b_n) is said to be almost increasing if there exist a positive increasing sequence (c_n) and two positive constants K and L such that $Kc_n \leq b_n \leq Lc_n$ [3].

Theorem 1.1 ([4]). Let (X_n) be an almost increasing sequence, and let there be sequence $(\ _n)$ and (a_n) such that

$$|\Delta \mathrm{a}_n| \leq \ _n, \tag{1}$$

$$_n \to \infty \quad \text{as} \quad n \to \infty, \tag{2}$$

$$\sum_{n=1}^{\infty} n |\Delta \ _n| X_n < \infty, \tag{3}$$

$$|\mathrm{a}_n| X_n = O(1) \quad \text{as} \quad n \to \infty, \tag{4}$$

$$\sum_{n=1}^{m} \frac{|s_n|^k}{n} = O(X_m) \ \text{as} \ m \to \infty, \tag{5}$$

$$\sum_{n=1}^{m} \frac{p_n}{P_n} |s_n|^k = O(1) \quad \text{as} \quad m \to \infty \tag{6}$$

then, the series $\sum a_n \mathrm{a}_n$ is summable $\left|\overline{N}, p_n\right|_k$, $k \geq 1$.

Theorem 1.2 ([4]). Let (X_n) be an almost increasing sequence, and let the conditions (1)-(4) , (6) be satisfied. If the conditions

$$\sum_{n=1}^{\infty} P_n X_n |\Delta \ _n| < \infty, \tag{7}$$

$$\sum_{n=1}^{m} \frac{|s_n|^k}{P_n} = O(X_m) \ \ as \ \ m \to \infty \tag{8}$$

are satisfied, then the series $\sum a_n \mathrm{a}_n$ is summable $\left|\overline{N}, p_n\right|_k$, $k \geq 1$.

2. Main Results

Many works on almost increasing sequences have been done, see ([5-16]). The aim of this paper is to generalize Theorem 1.1 and Theorem 1.2 to the $\ddot{} -\left|A, p_n\right|_k$ summability method. Before stating our theorems, we must first introduce some further notations. Given a normal matrix $A=(a_{nv})$, two lower semimatrices $\bar{A}=(\bar{a}_{nv})$ and $\hat{A}=(\hat{a}_{nv})$ are defined as follows:

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni} , \qquad n,v = 0,1,2... \tag{9}$$

$$\hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v} , \quad \bar{a}_{00} = \hat{a}_{00} = a_{00} , \qquad n = 1,2... \tag{10}$$

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v = \sum_{v=0}^{n} \bar{a}_{nv} a_v \tag{11}$$

$$\bar{\Delta} A_n(s) = \sum_{v=0}^{n} \hat{a}_{nv} a_v . \tag{12}$$

Theorem 2.1. Let $A=(a_{nv})$ be a positive normal matrix such that

$$\bar{a}_{n0} = 1 , \qquad n = 0,1... \tag{13}$$

$$a_{n-1,v} \geq a_{nv} \quad \text{for} \quad n \geq v+1, \tag{14}$$

$$a_{nn} = O\left(\frac{p_n}{P_n}\right). \tag{15}$$

Let (X_n) be an almost increasing sequence and $\left(\frac{\ddot{}_n p_n}{P_n}\right)$ be a non-increasing sequence. If the conditions (1)-(4) and

$$\sum_{n=1}^{m} \left(\frac{\ddot{}_n p_n}{P_n}\right)^{k-1} \frac{|s_n|^k}{n} = O(X_m) \quad \text{as} \quad m \to \infty, \tag{16}$$

$$\sum_{n=1}^{m} \ddot{}_n^{\,k-1} \left(\frac{p_n}{P_n}\right)^{k} |s_n|^k = O(X_m) \quad \text{as} \quad m \to \infty \tag{17}$$

are satisfied, then the series $\sum a_n ª_n$ is summable $\ddot{} -\left|A, p_n\right|_k$, $k \geq 1$.

Theorem 2.2. Let (X_n) be an almost increasing sequence and $A=(a_{nv})$ be a positive normal matrix as in Theorem 2.1. Let $\left(\frac{¨_n p_n}{P_n}\right)$ be a non- increasing sequence. If the conditions (1)-(4) , (7), (17) and

$$\sum_{n=1}^{m}\left(\frac{¨_n p_n}{P_n}\right)^{k-1}\frac{|s_n|^k}{P_n}=O(X_m) \quad \text{as} \quad m\to\infty \tag{18}$$

are satisfied, then the series $\sum a_n ª_n$ is summable $¨ - |A, p_n|_k,\ k\geq 1$.

In order to prove the theorems, we require the following lemmas.

Lemma 2.1 ([4]). Let (X_n) be an almost increasing sequence. If the conditions (2) and (3) are satisfied, then we have

$$nX_n \quad_n = O(1) \quad \text{as} \quad n\to\infty, \tag{19}$$

$$\sum_{n=1}^{\infty} X_n \quad_n < \infty. \tag{20}$$

Lemma 2.2 ([4]). Let (X_n) be an almost increasing sequence. If the conditions (2) and (7) are satisfied, then we have

$$P_n X_n \quad_n = O(1) \quad \text{as} \quad n\to\infty, \tag{21}$$

$$\sum_{n=1}^{\infty} p_n X_n \quad_n < \infty. \tag{22}$$

3. Proof of Theorem 2.1

Let (M_n) denotes A-transform of the series $\sum a_n ª_n$. Then, by (11) and (12), we have

$$\bar{\Delta} M_n = \sum_{v=1}^{n} \hat{a}_{nv} ª_v a_v .$$

By Abel's transformation, we get three part as in the following

$$\bar{\Delta} M_n = \sum_{v=1}^{n-1} \Delta_v (\hat{a}_{nv} \mathsf{a}_v) \sum_{k=1}^{v} a_k + \hat{a}_{nn} \mathsf{a}_n \sum_{v=1}^{n} a_v$$

$$= \sum_{v=1}^{n-1} \Delta_v (\hat{a}_{nv}) \mathsf{a}_v s_v + \sum_{v=1}^{n-1} \hat{a}_{n,v+1} \Delta \mathsf{a}_v s_v + a_{nn} \mathsf{a}_n s_n = M_{n,1} + M_{n,2} + M_{n,3}.$$

To prove Theorem 2.1, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \ddot{\ }_n^{k-1} \left| M_{n,r} \right|^k < \infty \quad \text{for} \quad r = 1, 2, 3.$$

From Hölders's inequality, we obtain

$$\sum_{n=2}^{m+1} \ddot{\ }_n^{k-1} \left| M_{n,1} \right|^k \le \sum_{n=2}^{m+1} \ddot{\ }_n^{k-1} \left(\sum_{v=1}^{n-1} \left| \Delta_v (\hat{a}_{nv}) \right| \left| \mathsf{a}_v \right| \left| s_v \right| \right)^k$$

$$\le \sum_{n=2}^{m+1} \ddot{\ }_n^{k-1} \left(\sum_{v=1}^{n-1} \left| \Delta_v (\hat{a}_{nv}) \right| \left| \mathsf{a}_v \right|^k \left| s_v \right|^k \right) \left(\sum_{v=1}^{n-1} \left| \Delta_v (\hat{a}_{nv}) \right| \right)^{k-1}.$$

Here, we have

$$\Delta_v (\hat{a}_{nv}) = \hat{a}_{nv} - \hat{a}_{n,v+1} = \bar{a}_{nv} - \bar{a}_{n-1,v} - \bar{a}_{n,v+1} + \bar{a}_{n-1,v+1} = a_{nv} - a_{n-1,v} \tag{23}$$

by (10) and (9) . Then, by using (14), (13) , (9), we get

$$\sum_{v=1}^{n-1} \left| \Delta_v (\hat{a}_{nv}) \right| = \sum_{v=1}^{n-1} (a_{n-1,v} - a_{nv}) \le a_{nn}. \tag{24}$$

Thus, from (24), (15), we have

$$\sum_{n=2}^{m+1} \ddot{\ }_n^{k-1} \left| M_{n,1} \right|^k = O(1) \sum_{n=2}^{m+1} \left(\frac{\ddot{\ }_n p_n}{P_n} \right)^{k-1} \left(\sum_{v=1}^{n-1} \left| \Delta_v (\hat{a}_{nv}) \right| \left| \mathsf{a}_v \right|^k \left| s_v \right|^k \right)$$

$$= O(1) \sum_{v=1}^{m} \left| \mathsf{a}_v \right| \left| s_v \right|^k \sum_{n=v+1}^{m+1} \left(\frac{\ddot{\ }_n p_n}{P_n} \right)^k \left| \Delta_v (\hat{a}_{nv}) \right|$$

$$= O(1) \sum_{v=1}^{m} \left(\frac{\ddot{\ }_v p_v}{P_v} \right)^{k-1} \left| \mathsf{a}_v \right| \left| s_v \right|^k \sum_{n=v+1}^{m+1} \left| \Delta_v \left(\hat{a}_{nv} \right) \right|.$$

Here, we know that $\sum_{n=v+1}^{m+1} \left|\Delta_v(\hat{a}_{nv})\right| \le a_{vv}$ by (23) and (14), then

$$\sum_{n=2}^{m+1} \varphi_n^{k-1} \left|M_{n,1}\right|^k = O(1)\sum_{v=1}^{m} \varphi_v^{k-1}\left(\frac{p_v}{P_v}\right)^k \left|\lambda_v\right|\left|s_v\right|^k$$

$$= O(1)\sum_{v=1}^{m-1}\Delta\left|\lambda_v\right|\sum_{r=1}^{v}\varphi_r^{k-1}\left(\frac{p_r}{P_r}\right)^k\left|s_r\right|^k + O(1)\left|\lambda_m\right|\sum_{n=1}^{m}\varphi_n^{k-1}\left(\frac{p_n}{P_n}\right)^k\left|s_n\right|^k$$

$$= O(1)\sum_{v=1}^{m-1} {}_v X_v + O(1)\left|\lambda_m\right| X_m$$

$$= O(1) \quad \text{as} \quad m \to \infty$$

by Abel's transformation, and using the conditions (1) , (17) , (20) and (4).

Again, we obtain

$$\sum_{n=2}^{m+1} \varphi_n^{k-1}\left|M_{n,2}\right|^k \le \sum_{n=2}^{m+1}\varphi_n^{k-1}\left(\sum_{v=1}^{n-1}\left|\hat{a}_{n,v+1}\right|\left|\Delta\lambda_v\right|\left|s_v\right|\right)^k$$

$$\le \sum_{n=2}^{m+1}\varphi_n^{k-1}\left(\sum_{v=1}^{n-1}\left|\hat{a}_{n,v+1}\right|\left|\Delta\lambda_v\right|\left|s_v\right|^k\right)\left(\sum_{v=1}^{n-1}\left|\hat{a}_{n,v+1}\right|\left|\Delta\lambda_v\right|\right)^{k-1}.$$

Here by (10), (9), (14) , we get $\left|\hat{a}_{n,v+1}\right| \le a_{nn}$. Then, by (15) and (1), we obtain

$$\sum_{n=2}^{m+1}\varphi_n^{k-1}\left|M_{n,2}\right|^k = O(1)\sum_{n=2}^{m+1}\left(\frac{\varphi_n p_n}{P_n}\right)^{k-1}\left(\sum_{v=1}^{n-1}\left|\hat{a}_{n,v+1}\right| {}_v\left|s_v\right|^k\right)\left(\sum_{v=1}^{n-1} {}_v\right)^{k-1}$$

$$= O(1)\sum_{n=2}^{m+1}\left(\frac{\varphi_n p_n}{P_n}\right)^{k-1}\left(\sum_{v=1}^{n-1}\left|\hat{a}_{n,v+1}\right| {}_v\left|s_v\right|^k\right)$$

$$= O(1)\sum_{v=1}^{m} {}_v\left|s_v\right|^k \sum_{n=v+1}^{m+1}\left(\frac{\varphi_n p_n}{P_n}\right)^{k-1}\left|\hat{a}_{n,v+1}\right|.$$

Using the fact that $\sum_{n=v+1}^{m+1}\left|\hat{a}_{n,v+1}\right| \leq 1$ by (10), (9), (13) and (14) , we have

$$\sum_{n=2}^{m+1} \ddot{}_n^{\,k-1}\left|M_{n,2}\right|^k = O(1)\sum_{v=1}^{m}\left(\frac{\ddot{}_v\, p_v}{P_v}\right)^{k-1} v \quad _v \frac{|s_v|^k}{v}$$

$$= O(1)\sum_{v=1}^{m-1}\Delta(v \quad _v)\sum_{r=1}^{v}\left(\frac{\ddot{}_r\, p_r}{P_r}\right)^{k-1}\frac{|s_r|^k}{r} + O(1)m \quad _m\sum_{r=1}^{m}\left(\frac{\ddot{}_r\, p_r}{P_r}\right)^{k-1}\frac{|s_r|^k}{r}$$

$$= O(1)\sum_{v=1}^{m-1} v\left|\Delta \quad _v\right| X_v + O(1)\sum_{v=1}^{m-1} \quad _{v+1}X_{v+1} + O(1)m \quad _m X_m$$

$$= O(1) \quad \text{as} \quad m \to \infty$$

by Abels's transformation, and using the conditions (16), (3) , (20), (19) .

Finally, we have

$$\sum_{n=1}^{m} \ddot{}_n^{\,k-1}\left|M_{n,3}\right|^k = \sum_{n=1}^{m} \ddot{}_n^{\,k-1} a_{nn}^k \left|\text{a}_n\right|^k \left|s_n\right|^k = O(1)\sum_{n=1}^{m} \ddot{}_n^{\,k-1}\left(\frac{p_n}{P_n}\right)^k \left|\text{a}_n\right|\left|s_n\right|^k$$

$$= O(1) \;\; as \;\; m \to \infty$$

as in $M_{n,1}$. This completes the proof of Theorem 2.1.

4. Proof of Theorem 2.2

Using Lemma 2.2 and proceeding as in the proof of Theorem 2.1, replacing $\sum_{v=1}^{m}\left(\frac{\ddot{}_v\, p_v}{P_v}\right)^{k-1} \quad _v|s_v|^k$ by $\sum_{v=1}^{m}\left(\frac{\ddot{}_v\, p_v}{P_v}\right)^{k-1} \quad _v P_v \frac{|s_v|^k}{P_v}$, Theorem 2.2 can be proved easily.

If we take $\ddot{}_n = \frac{P_n}{p_n}$ and $a_{nv} = \frac{p_v}{P_n}$, then Theorem 2.1 and Theorem 2.2 reduce to Theorem 1.1 and Theorem 1.2, respectively.

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Statistical Analysis of Earthquake Occurrences in and around the Çaldıran Fault Zone (Turkey)

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Abstract

In the scope of this study, a statistical assessment of earthquake behaviors in the Çaldıran Fault Zone and its surroundings is achieved by using the seismotectonic *b*-value of Gutenberg-Richter relation, probability, and return periods of the earthquakes. Multiple parameters evaluation between these variables is considered for the possible future earthquake forecasting and current hazard assessment. For the analyses, a homogeneous database including 6169 earthquakes with $1.0 \leq M_d \leq 5.6$ between July 12, 1975 and December 29, 2021 was used. Completeness magnitude is estimated as 2.6 for the study region and the *b*-value of magnitude-frequency distribution is calculated as 1.07±0.09. This result shows that the *b*-value of the study area is well represented by the Gutenberg-Richter scaling law. Regional distribution of *b*-value indicates that the areas having the lower *b*-values (<1.0) are generally observed in all parts of the study region including the Çaldıran fault consisting of Alaçayır, Hidirmenteş and Gülderen segments, Hasan Timur fault, Dorutay fault, and Saray fault zone. The probabilities earthquake occurrences with M_d=6.0 in Tr = 10, 20, 50, 70 and 100 years are estimated as about 4 %, 7 %, 18 %, 23 % and 32 %, respectively. Also, the return periods of M_d=5.0, 5.5, and 6.0 earthquakes are calculated as about 29, 90, and 300 years, respectively. The results of probabilities and return periods suggest that earthquake occurrences ranging from 3.0-4.5 magnitude levels are more likely than those of the other occurrences in the short term. Çaldıran Fault zone and its adjacent region are very active seismically and tectonically, and many strong/destructive earthquakes occurred in the historical and instrumental periods; the last of these major earthquakes occurred on November 24, 1976 (M_S=7.3). As a remarkable fact, the results of the present study can be used as a promising guide for earthquake forecasting and further hazard potential in this part of Turkey in the intermediate and long terms.

Keywords: Çaldıran Fault zone, *b*-value, earthquake probability, return period, seismic hazard.

1. Introduction

Earthquakes are one of the most destructive and dangerous natural disasters. It is well known that region-time occurrences are not random and they generally occur without any indicators. Hence, statistical studies on the region-time distributions of earthquake activity are one of the most significant processes for future earthquake potential. It is assumed that Earth's structure is very complex and earthquake occurrences show chaotic process and therefore, forecasting of earthquakes can be attributed to a statistical basis. Earthquake forecasting techniques are generally considered in two classes. The first one considers the empirical measurements of precursory changes, while the second approach uses statistical patterns of seismic activity. For this purpose, many authors have used different algorithms and parameters for seismicity analyses of different parts of the world. Some sample parameters are seismotectonic *b*-value, earthquake probability, return period, and Coulomb failure. Taking into account the statistical basis, statistical behaviors of seismic parameters become very important in the earthquake hazard and for the forecasting of possible future earthquakes. Thus, these types of studies indicate that considering the fractal behaviors of earthquake occurrences may be correlated with the seismogenic stress situations during the earthquake activity and may be related to the next possible earthquake occurrences.

It is well known that the East Anatolian Plateau (EAP) is one of the most active seismotectonic regions of Turkey and has an important seismotectonic potential concerning occurrences of strong/large earthquakes in the intermediate and long terms. EAP is a part of the Alpine-Himalayan orogenic system and the N-S compressional tectonic regime which originated from the northward motion of the Arabian plate and southward motion of the Eurasian plate causes high seismic activity in the region. Main tectonics in and around the EAP are shown in Fig. 1. The Çaldıran fault (CF) is one of the strike-slip faults systems in EAP and is located in the east of the Karlıova triple-junction (KTJ). This fault system has a right-lateral strike-slip mechanism and extends eastward to the North Tabriz fault and westward to Erciş and Tutak faults, considered the southern boundary of the Caucasus block. According to detailed mapping, The CF is divided into three individual segments called Gülderen, Hıdırmenteş, and Alaçayır sections. The GPS-derived slip rate for this fault system has been calculated as 11.9 mm yr^{-1} [1], 8 mm yr^{-1} [2], 7.2 - 8.3 mm yr^{-1} [3], and 3.27 mm yr^{-1} [4]. The CF caused many destructive earthquakes in the instrumental and historical periods. According to the earthquake catalog of the Disaster and Emergency Management Authority (AFAD), the CF and its surroundings experienced many destructive earthquakes in the past and recent years such as 1664, 1779, and 1872. The last of these devastating earthquakes occurred on 24 November 1976 Çaldıran (M_S=7.3, surface wave magnitude) with an approximately 52 km-long surface rupture [5], [6]. The previous destructive earthquakes in the past give an opinion for future seismicity. Using these past events, the main aim of this study is to explore statistically for possible future earthquake forecasting and hazard evaluation by performing seismotectonic *b*-value of Gutenberg-Richter relation, probability, and return periods of the earthquakes around the CF.

2. Earthquake Catalog and Analysis Methods

A part of the data is obtained from [7] for the time period from 1970 to 2006. This catalog is homogeneous for duration magnitude, M_d, and contains 83 events. Also, the earthquakes between 2006 and 2022 are provided from KOERI, and there are 6086 events in this time period. The shallow earthquakes (depth<70 km) are used to achieve the statistical analyses since the seismogenic depth is given between 40 and 45 km for this part of the East Anatolian Region (EAR) [8]. Thus, a database consisting of 6169 earthquakes from November 29, 1970, to December 29, 2021, with a magnitude range of $1.0 \leq M_d \leq 5.6$ is obtained. Epicenter distributions of the catalog are plotted in Fig. 2.

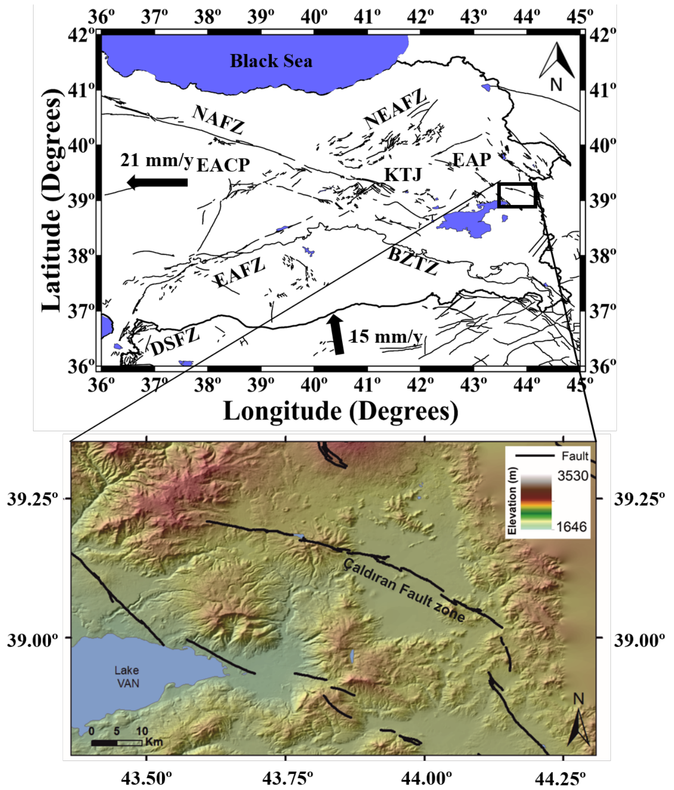

Figure 1. (**Top**) Main tectonic structures in Eastern Turkey [7]. The black rectangular frame depicts the study area. (**Down**) The morphological structure of the Çaldıran Fault Zone [6]. Abbreviations: EACP: Eastern Anatolian Contractional Province; NAFZ: North Anatolian Fault Zone; KTJ: Karlıova Triple Junction; EAP: East Anatolian Plateau; EAFZ: East Anatolian Fault Zone; NEAFZ, Northeast Anatolian Fault Zone; BZTZ: Bitlis-Zagros Thrust Zone; DSFZ: Dead Sea Fault Zone.

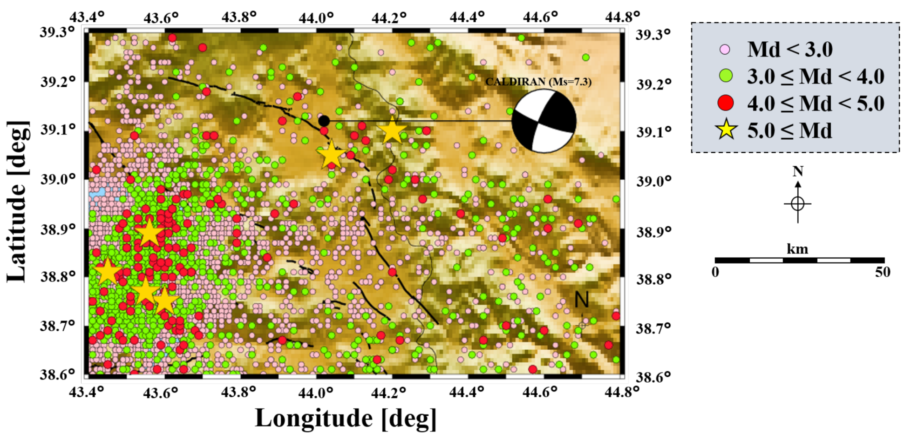

Figure 2. Epicenter distributions of the 6169 earthquakes with $1.0 \leq M_d \leq 5.6$ between 1970 and 2021. Also, the fault plane solution shows the 1976 Çaldıran earthquake (Ms=7.3).

Seismotectonic b-value of Gutenberg-Richter Relation, Probability and Return Periods of the Earthquakes

[9] defines a relation that gives the magnitude-frequency variations of earthquake distributions as follows:

$$\log_{10} N(M) = a - bM \tag{1}$$

According to this equation, *N(M)* is the cumulative number of earthquakes during a specific time period with magnitudes larger than or equal to *M*, and the *a*-value and *b*-value are positive constants. *a*-value is related to the time interval of the catalog, the size of the study region, and the number of earthquakes. On the other hand, the seismotectonic *b*-value is calculated from the slope of the Gutenberg-Richter relation and associated with the relative numbers of small and large events. Detailed region-time analyses of seismicity show that earthquake distributions display chaotic properties, and these properties are complex statistical tools for the description of the earthquake occurrences and their randomness. *b*-value is one of the best-known and most frequently used tools in earthquake statistics. *b*-value changes affect some important factors that as the tectonic features, anisotropic structure, and stress heterogeneities. It is very important to note that the *b*-value and stress distribution show a negative correlation. This negative correlation stems from crack density, geological complexity, material properties, thermal gradient, fault length, strain circumstances, seismic wave velocity changes and attenuation, and slip distribution [10], [11]. In the different parts of the world, generally *b*-value changes from 0.3 to 2.0 with a 0.1 average [12], [13].

The other earthquake statical parameter is the probability which corresponded to the different magnitude sizes within any period and is calculated from the following equation [14]:

$$P(M) = 1 - e^{-N(M)*Tr} \tag{2}$$

Here, *P(M)* is the probability that shows occurred specific event in the future in specific *Tr* years. *N*(*M*) and *M* are taken from Gutenberg-Richter relation. On the other hand, return periods of any earthquakes for different magnitudes can be calculated from the following formula [14]:

$$Q = 1 / N(M) \tag{3}$$

For high-quality and reliable results, magnitude completeness (*Mc*-value) is the other important parameter in the statistical seismicity analyses. *Mc*-value is the minimum magnitude of complete recording and can be estimated from the magnitude-frequency distribution [15]. This magnitude level contains 90 % of the earthquakes in the catalog and temporal variations in *Mc*-value can affect the results of the seismicity parameters, especially in *b*. Therefore, the maximum number of earthquakes in the catalog was aimed to be used for correct results for the analysis of all statistical parameters. Therefore, this type of estimation must be considered in the first stage of the analysis.

3. Results and Discussions

The magnitude completeness value and the seismotectonic b-value of the Gutenberg-Richter relation are shown in Fig. 3a. The average Mc-value for all catalogs is taken as 2.6 and the b-value is calculated as 1.07 ± 0.09. On the other hand, the regional variation of the b-value is also given in Fig 3b. The b-value map is plotted by using a moving window technique in *ZMAP* with a sample of 1100 earthquakes per window and prepared by grid cell spacing of 0.02° in longitude and latitude. The regional change of seismotectonic b-value is calculated between 0.75 and 1.21. According to [13], the average value of the b-value is 1.0 for different seismic regions. Considering this criterion, the b-value map is evaluated. Firstly, the larger b-values (>1.0) are calculated and estimated in and around the eastern part of the Lake Van including the Erciş fault zone. Secondly, the regions with the lower b-values (< 1.0) are generally observed in all parts consisting of the Çaldıran fault, Hasan Timur Lake fault, Dorutay fault, and Saray fault zone. Finally, the b-value variation presents a good correlation with regional seismic activity.

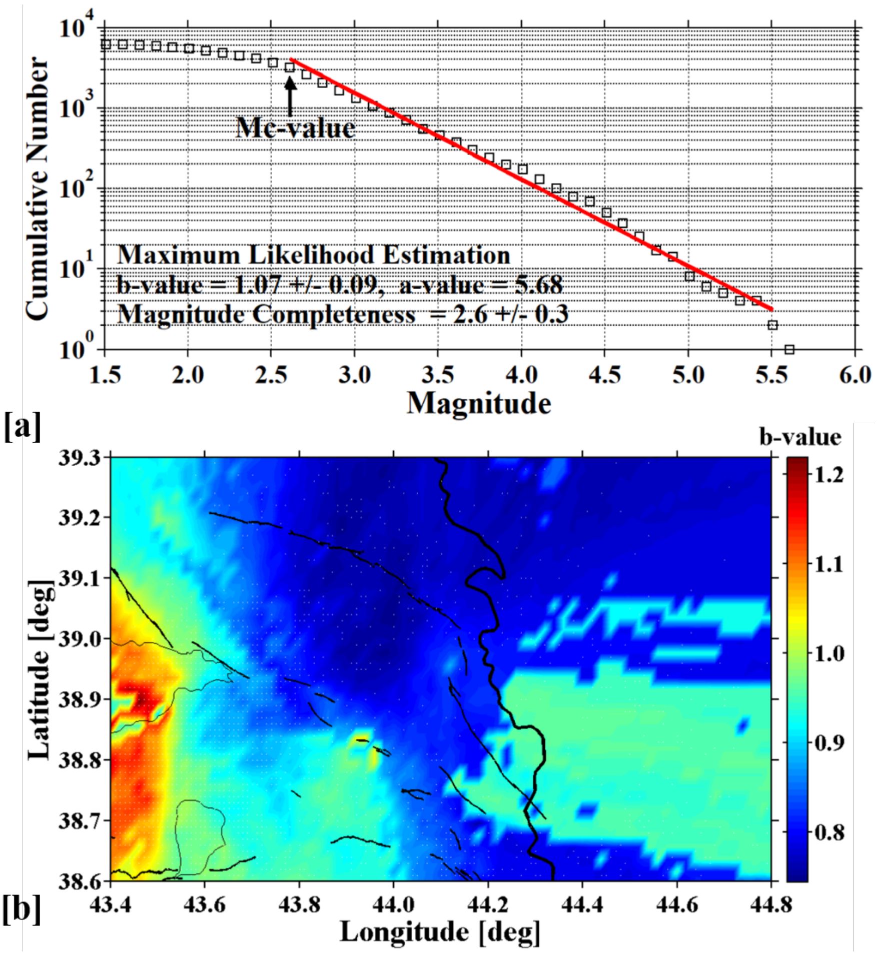

Figure 3. (a) Magnitude Completeness-value and seismotectonic b-value with standard deviations and a-value. (b) The variation in b-value for the study region.

The probability and return periods (or recurrence time) of earthquakes for different magnitudes are shown in Fig. 4. These parameters are very important for the definition of the statistical behaviors of earthquake occurrences. Probabilities of the earthquakes for different magnitude levels indicate great values ranging from 70-100 % for the earthquakes of 1.0 to 4.5 (Fig. 4a). The probabilities of earthquakes with M_d = 5.0, 5.5 and 6.0 for Tr = 10 are calculated as approximately 30 %, 11 %, and 4 %, respectively. On the contrary, M_d = 5.0, 5.5 and 6.0 for Tr = 100 are determined as about 97 %, 68 %, and 32 %, respectively. In Fig. 4b, the small return periods (<1.0 years) are calculated for the magnitude levels between 1.0 and 3.5, while the great return periods than 90 years can be considered for the magnitude levels bigger than 5.5. These outputs can provide valuable results to describe the statistical behaviors of strong earthquake occurrences in the study area.

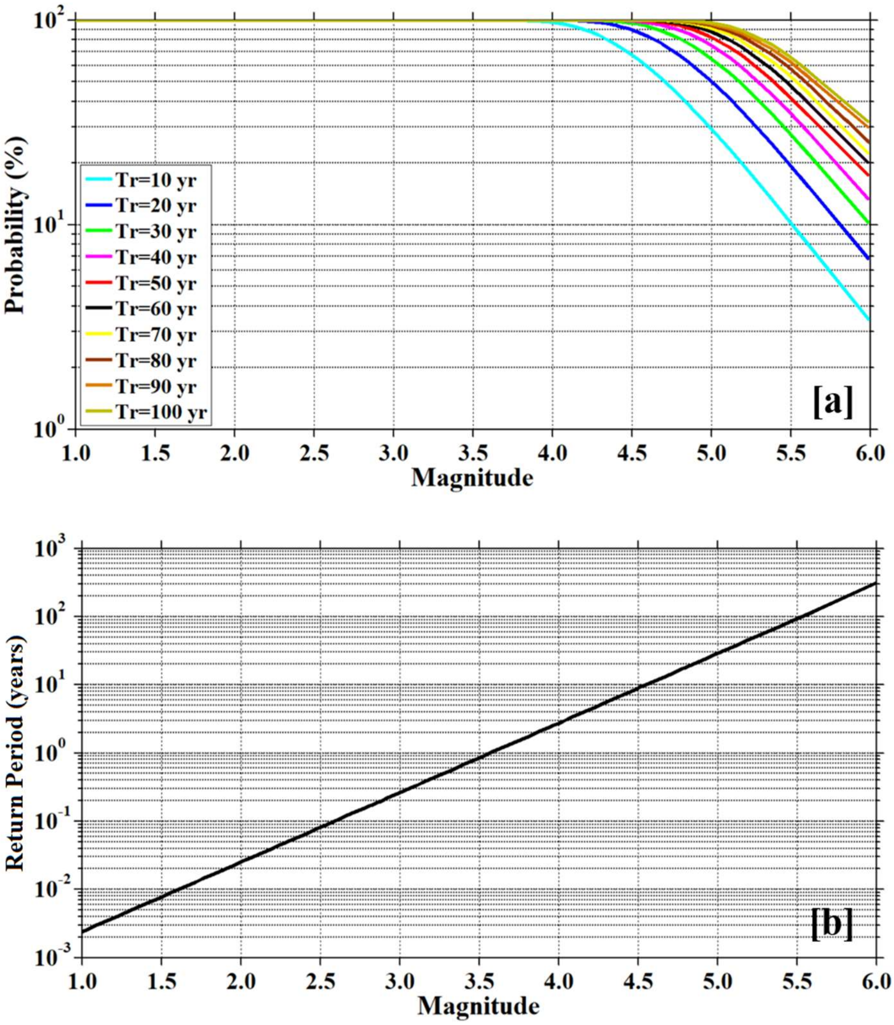

Figure 4. (a) Probability of earthquakes for different magnitudes. *Tr* values show the specific times (10, 20, 30, 40, 50, 60, 70, 80, 90 and 100 years). (b) Return periods for different magnitudes.

4. Conclusions

In this study, seismotectonic *b*-value, earthquake probability, and return periods are investigated statistically for the seismic forecasting of the earthquakes around the Çaldıran Fault zone. For this purpose, we use a homogeneous catalog including 6169 shallow events with $1.0 \leq M_d \leq 6.6$ from 1975 to 2021. We perform our analyses in a rectangular region covered by coordinates 38.6°N and 39.3°N in latitude and 43.4°E and 44.8°E in longitude. The areas with small *b*-values cover the Çaldıran fault, Hasan Timur Lake fault, Dorutay fault, and Saray fault zone, whilst high *b*-values are especially observed around the Erciş fault zone. On the other hand, the regional changes of the return periods for the earthquakes are estimated as relatively smaller, generally changing between 5 and 35 years around the eastern part of the Lake Van including the Erciş fault for the moderate magnitude values. Return periods for $M_d = 6.0$ magnitude size show about 300 years. Finally, these detailed statistical analyses based on the seismotectonic *b*-value, probability, and return periods for earthquakes indicate that there exists an earthquake potential in the near future.

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Statistical Convergence with Riesz Valued Density

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Abstract

The statistical convergence is handled with the natural density of subsets on the natural numbers. Recently, the statistical convergence on Riesz spaces with the order convergence have been attracted the attention of many authors. In this work, we aim to introduce a concept of statistical convergence on Riesz spaces with respect to the Riesz valued density. Also, we give some result between this concept and the statistically order convergence. In the settings of Riesz spaces will shed light on the case of convergences on Riesz spaces.

Keywords: Riesz space, Riesz valued density, weak order unit.

1. Introduction

Riesz space and statistical convergence are the natural and efficient tools in the theory of functional analysis. Riesz space that was introduced by F. Riesz in [18] is an ordered vector space having many applications in measure theory, operator theory, and applications in economics (cf. [1,2,24]). On the other hand, the statistical convergence is a generalization of the ordinary convergence of a real sequence, and the idea of statistical convergence was firstly introduced by Zygmund [25]. After then, Fast [10] and Steinhaus [21] independently improved that idea. Several applications and generalizations of the statistical convergence of sequences have been investigated by several authors (cf. [10,12,16,21,23]). In general, the statistical convergence is handled with the natural density of subsets on the natural numbers $\mathbb{N}$. Natural density of sets value over the closed interval $[0,1]$. In this work, we aim to introduce a concept of vector valued density on Riesz spaces. In the settings of Riesz spaces will shed light on the case of convergences on Riesz spaces and Banach lattices (cf. [4,5,17,23]). The study related to this papers are done by Schmidt in [14,19], where vector measures are introduced in Riesz spaces, and done by Tan in [22], where some properties of Riesz space valued measures are obtained.

The generalized asymptotic density was investigated by Buke [7], and Freedman and Sember [11] introduce a general concept of density. We remind that the natural (or, asymptotic) density of a subset K of natural numbers is defined by

$$\delta(K) := \lim_{n\to\infty} \frac{1}{n} |\{k \leq n : k \in A\}|$$

where the vertical bar of sets will stand for the cardinality of the given sets. We refer the reader to an exposition on the natural density of sets to [7,10,11,12,16]. In the same way, a sequence $x = (x_k)$ is called statistical convergent to L provided that

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0$$

for each $\varepsilon > 0$. Then it is written by $S - \lim x_k = L$.

Recall that a real vector space E with an order relation "≤" is called an ordered vector space if, for each $x, y \in E$ with $x \leq y, x + z \leq y + z$ and $\alpha x \leq \alpha y$ hold for all $z \in E$ and $\alpha \in \mathbb{R}_+$. An ordered vector space E is called a Riesz space or a vector lattice if, for any two vectors $x, y \in E$, the infimum and the supremum

$$x \wedge y = \inf\{x, y\} \text{ and } x \vee y = \sup\{x, y\}$$

exist in E, respectively. The order convergence is the basic tool of Riesz spaces.

Definition 1.1. A sequence (x_n) of a Riesz space is said to be order convergent to a vector x (in symbols $x_n \xrightarrow{o} x$) whenever there exists another sequence (y_n) with $y_n \downarrow 0$ and $|x_n - x| \leq y_n$ for all indices n.

A subset I of a Riesz space E is said to be a solid set if, for each $x \in E$ and $y \in I$ with $|x| \leq |y|$, it follows that $x \in I$. A solid vector subspace is called an order ideal. A positive element e in a Riesz space E is called order unit (or, strong order unit) if the principal ideal $I_e := \{x \in E : \exists \lambda > 0 \text{ with } |x| \leq \lambda e\}$ generated by e is the whole space E, i.e., if, for every $x \in E$, there exists some positive scalar $\lambda > 0$, depending upon x, such that $|x| \leq \lambda e$(cf. [15, Def.21.4])). We refer the reader for an exposition on the order unit to [1,2,3,15,17].

Definition 1.2. Let E be a Riesz space with an order unit e and $\mathcal{F}$ be an subfield of $\mathcal{P}(\mathbb{N})$ which contains all the finite subsets of $\mathbb{N}$. Then a Riesz space valued measure $\mu: \mathcal{F} \to [0, e]$ is called Riesz valued density if it holds the following properties:

a) $\mu(k) = 0$ for all $k \in \mathbb{N}$;
b) *(b)* $\mu(A) + \mu(B) - \mu(A \cap B) \leq e$ *for all* $A, B \subseteq \mathbb{N}$
c) *(c)* $\mu(A) = e - \mu(\mathbb{N} \setminus A)$.

2. Riesz valued statistical convergence

Definition 2.1. Let μ be a Riesz valued density on a Riesz space E with an order unit e. Then, a real valued sequence (x_n) is said to be Riesz valued statistical convergent to x if, for every $\varepsilon > 0$, we have

$$\inf\{\lambda > 0 : \mu(A_\varepsilon) \leq \lambda e\} = 0$$

where $A_\varepsilon = \{n \in \mathbb{N} : |x_n - x| \geq \varepsilon\}$. In this case, we write $x_n \overset{\mathrm{R_V\text{-}st}}{\longrightarrow} x$.

In order to simplify the presentation, we take $d(A_\varepsilon) := \inf\{\lambda > 0 : \mu(A_\varepsilon) \leq \lambda e\}$. Naturally, one can wonder that $d(A_\varepsilon) = 0$ implies $\mu(A_\varepsilon) = 0$, or not.

Remark 2.2. It can be seen that $d(A_\varepsilon) = 0$ does not implies $\inf_\lambda \lambda e = 0$, and so, $\mu(A_\varepsilon)$ need not be 0 in arbitrary Riesz spaces. However, it is clear that $d(A_\varepsilon) = 0$ implies $\mu(A_\varepsilon) = 0$ in Archimedean Riesz spaces. In that case, the Riesz valued statistical convergence and measure statistical convergence are coincide.

Example 2.3. Let E be a Banach lattice with order unit e. Then it follows from [2, Cor.4.4] that the formula

$$\| x \|_\infty = \mathrm{in}\, f\{\lambda > 0 : |x| \leq \lambda e\}$$

defines a norm on E, and it is equivalent to the original norm of E. A real valued sequence (x_n) is Riesz valued statistical convergent to x whenever $\|\mu(A_\varepsilon)\|_\infty = 0$ for every $\varepsilon > 0$, where $A_\varepsilon = \{n \in \mathbb{N} : |x_n - x| \geq \varepsilon\}$.

Theorem 2.4. Let μ be a Riesz valued density on a Riesz space E with an order unit e. If a sequence (x_n) is Riesz valued statistical convergent with respect to the Riezs valued density, then its limit is unique.

Proof: Suppose that $x_n \overset{\mathrm{R_V-st}}{\longrightarrow} x, x_n \overset{\mathrm{R_V\text{-}st}}{\longrightarrow} y$ and $x \neq y$. Therefore, from the definition of Riesz valued statistical convergence, we have

$$A_{\frac{\varepsilon}{2}} = \left\{n \in \mathbb{N} : |x_n - x| \geq \frac{\varepsilon}{2}\right\}$$

and

$$B_{\frac{\varepsilon}{2}} = \left\{n \in \mathbb{N} : |x_n - y| \geq \frac{\varepsilon}{2}\right\}$$

such that $\mu\left(A_{\frac{\varepsilon}{2}}\right) = \mu\left(B_{\frac{\varepsilon}{2}}\right) = 0$ for every $\varepsilon > 0$. Let be $|x - y| = \varepsilon > 0$. From the triangle inequility, we can write

$$|x_n - x| + |x_n - y| \geq |x - y| = \varepsilon$$

If we take $n \notin A_{\frac{\varepsilon}{2}}$, then $|x_n - x| < \frac{\varepsilon}{2}$ for every $\varepsilon > 0$ and also $n \in B_{\frac{\varepsilon}{2}}$. Therefore, $|x_n - x| \geq \frac{\varepsilon}{2}$ for every $\varepsilon > 0$. Since $A_{\frac{\varepsilon}{2}} \cup B_{\frac{\varepsilon}{2}} = \mathbb{N}$, we have $\mu\left(A_{\frac{\varepsilon}{2}} \cup B_{\frac{\varepsilon}{2}}\right) = e$. From the equility of definition 2.1 (c), considering of $\mu\left(A^{\frac{\varepsilon}{2}}_{\frac{\varepsilon}{2}}\right) = 0$, we can conclude $\mu\left(\mathbb{N} \setminus A_{\frac{\varepsilon}{2}}\right) = e$. Since $A^c_{\frac{\varepsilon}{2}} = \mathbb{N} \setminus A_{\frac{\varepsilon}{2}} \subseteq B_{\frac{\varepsilon}{2}}$, obviously $B_{\frac{\varepsilon}{2}} = e$ which is a contradiction and this implies $x = y$. Hence, we conclude that Riesz valued statistical convergence limit is unique. This completes the proof of theorem.

Theorem 2.5. Let μ be a Riesz valued density on a Riesz space E with an order unit e. If a sequence (x_n) in E converges in order to x, then Riesz valued statistical limit of (x_n) is x.

Proof: Suppose that $x_n \to^o x$. Then, there exists a sequence $y_n \downarrow \theta$ such that $|x_n - x| \leq y_n$ holds for all n. That means, since $y_n \downarrow \theta$, the set

$$A_\varepsilon = \{n \in \mathbb{N}: |x_n - x| \geq \varepsilon\}$$

has finite number of terms. Since every finite subset of $\mathbb{N}$ has μ Riesz valued density zero and therefore $\mu(A_\varepsilon) = 0$, that is $x_n \xrightarrow{\mathrm{R_V\text{-}st}} x$.

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Strophoidal Hypersurfaces in Four-Dimensional Euclidean Space

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Abstract

We introduce the strophoidal hypersurfaces in four-dimensional Euclidean space $\mathbb{E}^4$. We give some notions of a Euclidean space. Indicating a rotational hypersurface, we obtain the strophoidal hypersurface, and compute its geometric elements, such as Gauss map, Gaussian curvature, mean curvature. Then, we serve some relations between the curvatures of the hypersurfaces.

Keywords: four-space, rotational hypersurface, strophoidal hypersurface, Gauss map, Gaussian curvature, mean curvature.

1. Introduction

In the literature we meet some papers such as [1,3,5,6,7], and also books such as [2,4,8,9] about hypersurfaces.

We introduce the strophoid-rotational (we called it as strophoidal) hypersurface in Euclidean 4-space $\mathbb{E}^4$. We indicate the notions of four-dimensional Euclidean geometry in Section 2. In Section 3, we recall rotational hypersurface. We construct the stropho-rotational hypersurface, and calculate its curvatures in Section 4. Finally, we give the conclusion.

2. Preliminaries

In $\mathbb{E}^{n+1}$, to find the curvature formulas $\mathfrak{C}_i$, $i = 0,1,\dots,n,$ we take the characteristic polynomial $P_{\mathbf{S}}(\lambda) = 0$ of shape operator $\mathbf{S}$:

$$\det(\mathbf{S} - \lambda I_n) = \sum_{k=0}^{n} (-1)^k s_k \lambda^{n-k} = 0,$$

where I_n indicates the identity matrix. Therefore, we obtain the following curvature formulas

$$\binom{n}{i} \mathfrak{C}_i = s_i.$$

Here, $\binom{n}{0}\mathfrak{C}_0 = s_0 = 1$. k-th fundamental form of hypersurface M^n is given by

$$\mathrm{I}(\mathbf{S}^{k-1}(X), Y) = \langle \mathbf{S}^{k-1}(X), Y\rangle.$$

Therefore, we present the following

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} \mathfrak{C}_i \, \mathrm{I}(\mathbf{S}^{k-1}(X), Y) = 0.$$

In this work, with its transpose, we identify a vector (a, b, c, d).

Let $\boldsymbol{x} = \boldsymbol{x}(u, v, w)$ be an immersion of a hypersurface M^3 of $\mathbb{E}^4$. Dot product of the vectors $\vec{x} = (x_1, x_2, x_3, x_4)$ and $\vec{y} = (y_1, y_2, y_3, y_4)$ in $\mathbb{E}^4$ is given by as follows

$$\vec{x} \cdot \vec{y} = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4.$$

Vector product of $\vec{x} = (x_1, x_2, x_3, x_4)$, $\vec{y} = (y_1, y_2, y_3, y_4)$, $\vec{z} = (z_1, z_2, z_3, z_4)$ in $\mathbb{E}^4$ is defined by the following determinant

$$\vec{x} \times \vec{y} \times \vec{z} = \begin{vmatrix} e_1 & e_2 & e_3 & e_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix}.$$

The Gauss map of $\boldsymbol{x}$ is obtained by

$$\mathcal{G} = \frac{\boldsymbol{x}_u \times \boldsymbol{x}_v \times \boldsymbol{x}_w}{\|\boldsymbol{x}_u \times \boldsymbol{x}_v \times \boldsymbol{x}_w\|}, \tag{2.1}$$

where $\boldsymbol{x}_u = d\boldsymbol{x}/du$. We construct the following

$$\det(\mathrm{I}) = \begin{vmatrix} E & F & A \\ F & G & B \\ A & B & C \end{vmatrix} = (EG - F^2)C - EB^2 + 2FAB - GA^2,$$

$$\det(\mathrm{II}) = \begin{vmatrix} L & M & P \\ M & N & T \\ P & T & V \end{vmatrix} = (LN - M^2)V - LT^2 + 2MPT - NP^2,$$

$$\det(\mathrm{III}) = \begin{vmatrix} X & Y & O \\ Y & Z & R \\ O & R & S \end{vmatrix} = (XZ - Y^2)S - XR^2 + 2YOR - ZO^2.$$

Here, the components are obtained by the following

$$E = \boldsymbol{x}_u \cdot \boldsymbol{x}_u, \quad F = \boldsymbol{x}_u \cdot \boldsymbol{x}_v, \quad G = \boldsymbol{x}_v \cdot \boldsymbol{x}_v, \quad A = \boldsymbol{x}_u \cdot \boldsymbol{x}_w, \quad B = \boldsymbol{x}_v \cdot \boldsymbol{x}_w, \quad C = \boldsymbol{x}_w \cdot \boldsymbol{x}_w,$$

$$L = \boldsymbol{x}_{uu} \cdot \mathcal{G}, \quad M = \boldsymbol{x}_{uv} \cdot \mathcal{G}, \quad N = \boldsymbol{x}_{vv} \cdot \mathcal{G}, \quad P = \boldsymbol{x}_{uw} \cdot \mathcal{G}, \quad T = \boldsymbol{x}_{vw} \cdot \mathcal{G}, \quad V = \boldsymbol{x}_{ww} \cdot \mathcal{G},$$

$$X = \mathcal{G}_u \cdot \mathcal{G}_u, \quad Y = \mathcal{G}_u \cdot \mathcal{G}_{\boldsymbol{v}}, \quad Z = \mathcal{G}_v \cdot \mathcal{G}_{\boldsymbol{v}}, \quad O = \mathcal{G}_u \cdot \mathcal{G}_{\boldsymbol{w}}, \quad R = \mathcal{G}_v \cdot \mathcal{G}_{\boldsymbol{w}}, \quad S = \mathcal{G}_w \cdot \mathcal{G}_{\boldsymbol{w}}.$$

Next, we obtain the curvature formulas for a hypersurface $\boldsymbol{x}(u,v,w)$ in $\mathbb{E}^4$. Using the polynomial of characteristic $P_{\mathbf{S}}(\lambda) = a\lambda^3 + b\lambda^2 + c\lambda + d = 0$, we find the curvature formulas: $\mathfrak{C}_0 = 1$ (by definition),

$$\mathfrak{C}_1 = -\frac{b}{\binom{3}{1}a}, \quad \mathfrak{C}_2 = \frac{c}{\binom{3}{2}a}, \quad \mathfrak{C}_3 = -\frac{d}{\binom{3}{3}a}. \tag{2.2}$$

Theorem 3.1. *Any hypersurface* M^3 *in* $\mathbb{E}^4$, *the fundamental forms and the curvatures are related by the following*

$$\mathfrak{C}_0\mathrm{IV} - 3\mathfrak{C}_1\mathrm{III} + 3\mathfrak{C}_2\mathrm{II} - \mathfrak{C}_3\mathrm{I} = 0.$$

Proof. See [6] for details.

3. Rotational Hypersurfaces

We introduce a kind of rotational hypersurface having generating strophoid curve in $\mathbb{E}^4$.

Assume that $\gamma: I \longrightarrow \Pi$ be a space curve for $I \subset \mathbb{R}$, ℓ be a line in Π. A rotational hypersurface is served by a rotation of the generating curve γ about axis ℓ in $\mathbb{E}^4$.

Supposing ℓ is the line spanned by axis x_4, we give the following orthogonal matrix

$$\mathcal{A}(v,w) = \begin{pmatrix} \cos v \cos w & -\sin v & -\cos v \sin w & 0 \\ \sin v \cos w & \cos v & -\sin v \sin w & 0 \\ \sin w & 0 & \cos w & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $v, w \in \mathbb{R}$.

$\mathcal{A}$ is found by the following

$$\mathcal{A}\ell = \ell, \quad \mathcal{A}^t\mathcal{A} = \mathcal{A}\mathcal{A}^t = \mathcal{I}_4, \quad det\,\mathcal{A} = 1.$$

There exists a Euclidean transformation so that the axis ℓ transformes to the axis x_4 of $\mathbb{E}^4$, while the axis of rotation is ℓ. The generating curve γ is given by the following

$$\gamma(u) = \big(f(u), 0, 0, g(u)\big), \quad \forall u \in I$$

where $f(u), g(u): I \subset \mathbb{R} \longrightarrow \mathbb{R}$ are the derivativable functions. Then, the rotational hypersurface is served by the following

$$\mathcal{R}(u, v, w) = A(v, w).\gamma(u)^t,$$

where $0 \leq v, w < 2\pi$.

Therefore, we re-write rotational hypersurface as follows

$$\mathcal{R}(u, v, w) = (f(u)\cos v \cos w, f(u)\sin v \cos w, f(u)\sin w, g(u)).$$

4. Strophoidal Hypersurfaces

In $\mathbb{E}^4$, with the helps of the matrix $\mathcal{A}$, and the generating curve γ and the translation on x_4, we reveal strophoidal hypersurface $\mathcal{S}(u, v, w)$ having strophoid curve.

Considering the strophoid curve in $\mathbb{E}^4$

$$\mathscr{s}(u) = \left(a\frac{\mathrm{u}^2 - 1}{\mathrm{u}^2 + 1}, 0, 0, au\left(\frac{\mathrm{u}^2 - 1}{\mathrm{u}^2 + 1}\right)\right), \quad a \in \mathbb{R},$$

we compute the Gauss map, and also find the curvatures $\mathfrak{C}_{i=1,2,3}$ of the strophoidal hypersurface.

Drawing its graphics with projection from four-space to three-space, we reveal the Gauss map of the strophoidal hypersurface.

In $\mathbb{E}^4$, rotating the strophoid curve $\mathscr{s}(u)$ about axis $\ell = (0, 0, 0, 1)$ by using orthogonal matrix $\mathcal{A}(v, w)$, and taking $a = 1$ on $\mathscr{s}(u)$, we obtain the strophoidal hypersurface given by as follows

$$\mathcal{S}(u, v, w) = \left(\frac{\mathrm{u}^2 - 1}{\mathrm{u}^2 + 1}\cos v \cos w, \frac{\mathrm{u}^2 - 1}{\mathrm{u}^2 + 1}\sin v \cos w, \frac{\mathrm{u}^2 - 1}{\mathrm{u}^2 + 1}\sin w, \frac{u(\mathrm{u}^2 - 1)}{\mathrm{u}^2 + 1}\right). \tag{4.1}$$

Choosing $w = \pi/4$ in hypersurface (4.1), we projected the surfaces into 3-space. See Figure 1.

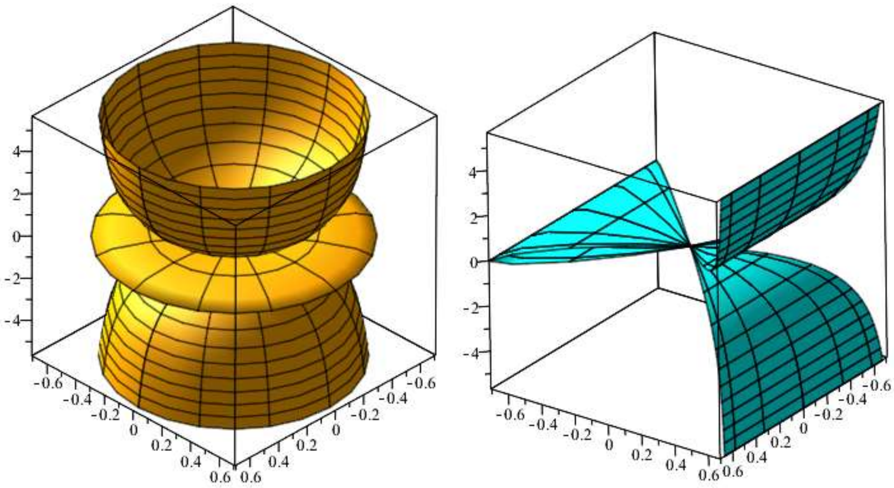

Figure 1. Projections of $\mathcal{S}(u, v, w)$
Left: into $x_1x_2x_4$-space, **Right**: into $x_2x_3x_4$-space

We get the following first quantities

$$\mathrm{I} = \begin{pmatrix} 1 + \left(\frac{2u}{u^2+1}\right)^2 & 0 & 0 \\ 0 & \left(\frac{u^2-1}{u^2+1}\cos w\right)^2 & 0 \\ 0 & 0 & \left(\frac{u^2-1}{u^2+1}\right)^2 \end{pmatrix},$$

by using the first differentials with respect to u, v, w of (4.1).

Using the Gauss map formula (2.1) on (4.1), we have the Gauss map of the stropho-rotational hypersurface (4.1), as follows

$$\mathcal{G} = -\frac{1}{(u^2+1)(u^4+6u^2+1)^{1/2}} \begin{pmatrix} (u^4+4u^2-1)\cos v \cos w \\ (u^4+4u^2-1)\sin v \cos w \\ (u^4+4u^2-1)\sin w \\ -4u \end{pmatrix}.$$

Computing the second differentials of (4.1), we have the following second quantities

$$\mathrm{II} = \begin{pmatrix} L & 0 & 0 \\ 0 & \not{f}\cos^2 w & 0 \\ 0 & 0 & \not{f} \end{pmatrix},$$

where

$$L = -\frac{4(3u^2+1)}{(u^4+6u^2+1)^{1/2}(u^2+1)^2},$$

$$\not{f} = -\frac{(u^2-1)(u^4+4u^2-1)}{(u^4+6u^2+1)^{1/2}(u^2+1)^2}.$$

Theorem 4.1. *The strophoidal hypersurface* (4.1) *in* $\mathbb{E}^4$ *has the following curvatures*

$$\mathfrak{C}_1 = -\frac{2(u^8+10u^6+30u^4-6u^2-3)}{3(u^4+6u^2+1)^{3/2}(u^2-1)},$$

$$\mathfrak{C}_2 = \frac{(u^4+4u^2-1)(u^8+10u^6+48u^4-18u^2-9)}{3(u^4+6u^2+1)^2(u^2-1)^2},$$

$$\mathfrak{C}_3 = -\frac{4(3u^2+1)(u^4+4u^2-1)^2}{(u^4+6u^2+1)^{5/2}(u^2-1)^2}.$$

Proof. Computing eqs. (2.2) on (4.1), we reveal the curvatures.

Corollary 4.1. *The strophoidal hypersurface* (4.1) *is 1-minimal iff the real roots of* $\mathfrak{C}_1 = 0$ *are given by*

$$u = \pm 0.6301900660364116 \ldots$$

Corollary 4.2. *The strophoidal hypersurface* (4.1) *is 2-minimal iff the real roots of* $\mathfrak{C}_2 = 0$ *are given by*

$$u = \pm 0.4858682717566457 \text{ or } u = \pm 0.7774752857703741 \ldots$$

Corollary 4.3. *The strophoidal hypersurface* (4.1) *is 3-minimal iff the real roots of* $\mathfrak{C}_3 = 0$ *are given by*

$$u = \pm 0.4858682717566457 \ldots$$

5. Conclusion

Strophoidal hypersurfaces have never been worked up till now. We have revealed some results of the strophoidal hypersurfaces as in [6].

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SVIR epidemic model with Caputo-Fabrizio derivative

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Abstract

Infectious diseases are diseases caused by small organisms such as bacteria, viruses, fungi or parasites that enter the body. There are many different types of infectious diseases and ways of transmission. Moreover, these diseases have important negative impacts on community health. In this study, we aim to deal with contagious SVIR model with Caputo-Fabrizio derivative,then we give special solution and the stability analysis.

Keywords: Caputo-Fabrizio derivative, SVIR model, stability analysis.

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1. Introduction

Vaccination is a fundamental component of an individual's right of health and is one of the most successful measures of preventive medicine. All over the world, vaccine avoidable diseases have been greatly reduced by routine vaccination programs, and approximately 2-3 million deaths are prevented each year through vaccination. Taking into consideration these facts, we will present a fractional vaccination in a fundamental SIR model given in [1].

Systems with complicated dynamics can be efficaciously represented with fractional derivatives because of their the memory and hereditary belongings. These operators have large spectrum of implementations such as defining many real world problems such as diabetes, polio, covid19 diseases, financial problems, heat and mass transport models [2], [3], [4], [5], [6], [7]. Other applications of fractional derivatives, among others, in [8], [9], [10], [11], [12], [13], [14], [15] and references therein. In this work, we study a SVIR epidemic model in term of the Caputo-Fabrizio (CF) fractional derivative.

Here, we tackle a SVIR model equipped with vaccination strategies established by [1] of the following integer order form:

$$\begin{aligned}
\frac{dS}{dt} &= b - bS - cSI - ds, \\
\frac{dV}{dt} &= dS - eVI - kV - bV, \\
\frac{dI}{dt} &= bSI + eVI - mI - bI, \\
\frac{dR}{dt} &= kV + mI - bR.
\end{aligned} \tag{1}$$

where all parameters are positive. S, V, I, R stand for susceptible, vaccinated, infected and recovered individuals, respectively. If $d = 0$, there are no vaccination in model [1]. m is recovering ratio of infected persons, b is the human death ratio of the inhabitants, d is the ratio of susceptible people who dispose of vaccination procedure, e is the spread ratio when contact with infected individual and susceptible individual once inoculation, β is the mean ratio for susceptible peoples who gain immunity and transport recovering people, c is spread ratio of when contact with infected persons and susceptible persons.

2. Basic definitions

We give some necessary definitions:

Definition 1: [16] Let $a < b$, $g \in H^1(a,b)$ and $\alpha \in [0,1]$, the Caputo-Fabrizio derivative is given by

$${}_0^{CF}D_t^{\alpha} g(t) = \frac{M(\alpha)}{1-\alpha} \int_a^t g'(x) \exp\left[-\alpha \frac{t-x}{1-\alpha}\right] dx, \tag{2}$$

where $M(\alpha)$ is a normalization function satisfying $M(0) = M(1) = 1$. If $g \notin H^1(a,b)$ this derivative can be rearranged as below:

$${}_0^{CF}D_t^{\alpha} g(t) = \frac{\alpha M(\alpha)}{1-\alpha} \int_a^t \left(g(t) - g(x)\right) \exp\left[-\alpha \frac{t-x}{1-\alpha}\right] dx. \tag{3}$$

Remark 1: If $\sigma = \dfrac{1-\alpha}{\alpha} \in [0,\infty)$, $\alpha = \dfrac{1}{1+\sigma} \in [0,1]$, then Eq. (3) is of the form:

$${}_0^{CF}D_t^{\alpha} g(t) = \frac{N(\alpha)}{\alpha} \int_a^t g'(x) \exp\left[-\frac{t-x}{\alpha}\right] dx,$$

with $N(0) = N(\infty) = 1$. Additionally,

$$\lim_{\alpha \to 0} \frac{1}{\alpha} \exp\left[-\frac{t-x}{\alpha}\right] = \delta(x-t).$$

The related integral of the new derivative was given by Nieto and Losada [17].

Definition 2: Let $0 < \alpha < 1$ and g be a function. The fractional integral of order α is defined by [17]:

$$I_t^\alpha g(t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} g(t) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t g(s)\,ds,\ t \ge 0. \tag{4}$$

Moreover, the below result holds

$$\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} + \frac{2\alpha}{(2-\alpha)M(\alpha)} = 1,$$

then $M(\alpha) = \dfrac{2}{2-\alpha}$ for $0 < \alpha < 1$. Using the above results, another form of the new Caputo derivative of order given as [17]:

$${}_0^{CF}D_t^\alpha g(t) = \frac{1}{1-\alpha} \int_a^t g'(x) \exp\left[-\alpha \frac{t-x}{1-\alpha}\right] dx. \tag{5}$$

3. SVIR model with Caputo-Fabrizio derivative

Now, we rearrange Eq. (1) using CF derivative:

$$\begin{aligned}
{}_0^{CF}D_t^\alpha S(t) &= b - bS - cSI - dS, \\
{}_0^{CF}D_t^\alpha V(t) &= dS - eVI - kV - bV, \\
{}_0^{CF}D_t^\alpha I(t) &= bSI + eVI - mI - bI, \\
{}_0^{CF}D_t^\alpha R(t) &= kV + mI - bR.
\end{aligned} \tag{6}$$

3.1 Derivation of special solution

In this part, benefiting from Sumudu transform with an iterative method, we provide a special solution of the Eq. (6). First, we give Sumudu transform:

Theorem 1: [18] Suppose that for the function $f(t)$, CF derivative exists, then Sumudu transform of the CF derivative for the function $f(t)$ defined by

$$ST\left({}_{0}^{CF}D_{t}^{\alpha}\right)\left(f(t)\right)=M(\alpha)\frac{SF\left(f(t)\right)-f(0)}{1-\alpha+\alpha u}.$$

In order to get an iterative solution of the model (6), we apply Sumudu transform on both sides of the Eq.(6), then we have

$$\begin{aligned}
M(\alpha)\frac{SF\left(S(s)\right)-S(0)}{1-\alpha+\alpha s}&=SL\left\{b-S(b+cI+d)\right\},\\
M(\alpha)\frac{SF\left(V(s)\right)-V(0)}{1-\alpha+\alpha s}&=SL\left\{dS-V(eI+k+b)\right\},\\
M(\alpha)\frac{SF\left(I(s)\right)-I(0)}{1-\alpha+\alpha s}&=SL\left\{I(cS+eV-m-b)\right\},\\
M(\alpha)\frac{SF\left(R(s)\right)-R(0)}{1-\alpha+\alpha s}&=SL\left\{kV+mI-bR\right\}.
\end{aligned}\tag{7}$$

Rearranging, we find

$$\begin{aligned}
SF\left(S(t)\right)&=S(0)+\frac{1-\alpha+\alpha s}{M(\alpha)}SL\left\{b-S(b+cI+d)\right\},\\
SF\left(V(t)\right)&=V(0)+\frac{1-\alpha+\alpha s}{M(\alpha)}SL\left\{dS-V(eI+k+b)\right\},\\
SF\left(I(t)\right)&=I(0)+\frac{1-\alpha+\alpha s}{M(\alpha)}SL\left\{I(cS+eV-m-b)\right\},\\
SF\left(R(t)\right)&=R(0)+\frac{1-\alpha+\alpha s}{M(\alpha)}SL\left\{kV+mI-bR\right\}.
\end{aligned}\tag{8}$$

Taking the inverse Sumudu transform on both sides of the Eq. (8), we find

$$
\begin{aligned}
S(t) &= S(0) + SL^{-1}\left\{\frac{1-\alpha+\alpha s}{M(\alpha)} SL\{b - S(b + cI + d)\}\right\},\\
V(t) &= V(0) + SL^{-1}\left\{\frac{1-\alpha+\alpha s}{M(\alpha)} SL\{dS - V(eI + k + b)\}\right\},\\
I(t) &= I(0) + SL^{-1}\left\{\frac{1-\alpha+\alpha s}{M(\alpha)} SL\{I(cS + eV - m - b)\}\right\},\\
R(t) &= R(0) + SL^{-1}\left\{\frac{1-\alpha+\alpha s}{M(\alpha)} SL\{kV + mI - bR\}\right\}.
\end{aligned} \tag{9}
$$

The recursive formula can be obtained as

$$
\begin{aligned}
S_{n+1}(t) &= S_n(0) + SL^{-1}\left\{\frac{1-\alpha+\alpha s}{M(\alpha)} SL\{b - S_n(b + cI_n + d)\}\right\},\\
V_{n+1}(t) &= V_n(0) + SL^{-1}\left\{\frac{1-\alpha+\alpha s}{M(\alpha)} SL\{dS_n - V_n(eI_n + k + b)\}\right\},\\
I_{n+1}(t) &= I_n(0) + SL^{-1}\left\{\frac{1-\alpha+\alpha s}{M(\alpha)} SL\{I_n(cS_n + eV_n - m - b)\}\right\},\\
R_{n+1}(t) &= R_n(0) + SL^{-1}\left\{\frac{1-\alpha+\alpha s}{M(\alpha)} SL\{kV_n + mI_n - bR_n\}\right\}.
\end{aligned} \tag{10}
$$

Thus, the solution of (10) is given by

$$
\begin{aligned}
S(t) &= \lim_{n\to\infty} S_n(t),\\
V(t) &= \lim_{n\to\infty} V_n(t),\\
I(t) &= \lim_{n\to\infty} I_n(t),\\
R(t) &= \lim_{n\to\infty} R_n(t).
\end{aligned}
$$

3.2 Stability analysis and iterative solutions for the proposed model

Suppose that a Banach space X endowed with a norm defined by $\|x\| = \max_{t\in[a,b]}\{|x(t)| : x \in X\}$ and K be a self-map of X. Let $y_{n+1} = g(K, y_n)$ be particular recursive procedure. Assume that $F(K)$ be the fixed-point set of K such that this is nonempty and y_n converges to a point $p \in F(K)$. Let

$\{x_n\} \subset X$ and define $e_n = \left\| x_{n+1} - g\left(K, x_n\right) \right\|$. The iterative method $y_{n+1} = g\left(K, y_n\right)$ is said to be K- stable if $\lim_{n\to\infty} e_n = 0$, which implies that $\lim_{n\to\infty} x^n = p$. Without any loss of generality, the sequence $\{x_n\}$ is bounded, otherwise the sequence will diverge. If all these conditions are satisfied for $y_{n+1} = Ky_n$ known as Picard's iteration, hence the iteration is K-stable. Now, we will state the theorem below.

Theorem 2: [19] Let $\left(X, \|.\|\right)$ be a Banach space and K be a self-map on X such that

$$\left\| K_x - K_y \right\| \le A \left\| x - K_x \right\| + a \left\| x - y \right\|$$

for all x, y in X where $0 \le A, 0 \le a \le 1$. Suppose that K is Picard K-stable.

Now, we consider following recursive formula Eq. (10) connected to the Eq. (6)

$$\begin{aligned}
S_{n+1}(t) &= S_n(0) + SL^{-1}\left\{ \frac{1-\alpha+\alpha s}{M(\alpha)} SL\left\{ b - S_n(b + cI_n + d) \right\} \right\}, \\
V_{n+1}(t) &= V_n(0) + SL^{-1}\left\{ \frac{1-\alpha+\alpha s}{M(\alpha)} SL\left\{ dS_n - V_n(eI_n + k + b) \right\} \right\}, \\
I_{n+1}(t) &= I_n(0) + SL^{-1}\left\{ \frac{1-\alpha+\alpha s}{M(\alpha)} SL\left\{ I_n(cS_n + eV_n - m - b) \right\} \right\}, \\
R_{n+1}(t) &= R_n(0) + SL^{-1}\left\{ \frac{1-\alpha+\alpha s}{M(\alpha)} SL\left\{ kV_n + mI_n - bR_n \right\} \right\}.
\end{aligned}$$

where $\dfrac{1-\alpha+\alpha s}{M(\alpha)}$ is the fractional Lagrange multiplier.

Theorem 3: Let T be a self-map defined by

$$\begin{aligned}
T\left(S_n\left(t\right)\right) &= S_{n+1}\left(t\right) = S_n\left(t\right) + SL^{-1}\left\{\frac{1-\alpha+\alpha s}{M\left(\alpha\right)} SL\left\{b - S_n(b + cI_n + d)\right\}\right\},\\
T\left(V_n\left(t\right)\right) &= V_{n+1}\left(t\right) = V_n\left(t\right) + SL^{-1}\left\{\frac{1-\alpha+\alpha s}{M\left(\alpha\right)} SL\left\{dS_n - V_n(eI_n + k + b)\right\}\right\},\\
T\left(I_n\left(t\right)\right) &= I_{n+1}\left(t\right) = I_n\left(t\right) + SL^{-1}\left\{\frac{1-\alpha+\alpha s}{M\left(\alpha\right)} SL\left\{I_n(cS_n + eV_n - m - b)\right\}\right\},\\
T\left(R_n\left(t\right)\right) &= R_{n+1}\left(t\right) = R_n\left(t\right) + SL^{-1}\left\{\frac{1-\alpha+\alpha s}{M\left(\alpha\right)} SL\left\{kV_n + mI_n - bR_n\right\}\right\}.
\end{aligned}$$

is T - stable in $L^1\left(a,b\right)$ if

$$\begin{aligned}
&1-\left(\left(b+d\right) f_1 - cM_1 f_2 - cM_2 f_3\right) < 1,\\
&1-\left(dg_1 - \left(b+k\right) g_2 - eM_3 g_3 - eM_4 g_4\right) < 1,\\
&1-\begin{pmatrix} \left(-b+m\right) h_1 + cM_1 h_2 + cM_2 h_3 \\ eM_5 h_4 + eM_2 h_5 \end{pmatrix} < 1,\\
&1-\left(bk_1 + kk_2 + mk_3\right) < 1,
\end{aligned}$$

where $f_1, f_2, f_3, g_1, g_2, g_3, g_4, h_1, h_2, h_3, h_4, h_5, k_1, k_2, k_3$ are functions $SL^{-1}\left(\dfrac{1-\alpha+\alpha s}{M\left(\alpha\right)}\right)$.

Proof: In the beginning step, we show that T has fixed point. To get this, we evaluate the following iterations for all $\left(m,n\right) \in \mathbb{N}\times\mathbb{N}$.

$$\begin{aligned}
&T\left(S_n(t)\right)-T\left(S_m(t)\right)=S_n(t)-S_m(t)+SL^{-1}\left\{\frac{1-\alpha+\alpha s}{M(\alpha)}SL\{b-S_n(b+cI_n+d)\}\right\}\\
&\quad -SL^{-1}\left\{\frac{1-\alpha+\alpha s}{M(\alpha)}SL\{b-S_m(b+cI_m+d)\}\right\},\\
&T\left(V_n(t)\right)-T\left(V_m(t)\right)=V_n(t)-V_m(t)+SL^{-1}\left\{\frac{1-\alpha+\alpha s}{M(\alpha)}SL\{dS_n-V_n(eI_n+k+b)\}\right\}\\
&\quad -SL^{-1}\left\{\frac{1-\alpha+\alpha s}{M(\alpha)}SL\{dS_m-V_m(eI_m+k+b)\}\right\},\\
&T\left(I_n(t)\right)-T\left(I_m(t)\right)=I(t)-I_m(t)+SL^{-1}\left\{\frac{1-\alpha+\alpha s}{M(\alpha)}SL\{I_n(cS_n+eV_n-m-b)\}\right\}\\
&\quad -SL^{-1}\left\{\frac{1-\alpha+\alpha s}{M(\alpha)}SL\{I_m(cS_m+eV_m-m-b)\}\right\},\\
&T\left(R_n(t)\right)-T\left(R_m(t)\right)=R_n(t)-R_m(t)+SL^{-1}\left\{\frac{1-\alpha+\alpha s}{M(\alpha)}SL\{kV_n+mI_n-bR_n\}\right\}\\
&\quad -SL^{-1}\left\{\frac{1-\alpha+\alpha s}{M(\alpha)}SL\{kV_m+mI_m-bR_m\}\right\}.
\end{aligned}\tag{11}$$

Without loss of generality, applying the norm of the first Eq. (11), we find

$$\begin{aligned}
\left\|T\left(S_n(t)\right)-T\left(S_m(t)\right)\right\|=\Bigg\|&S_n(t)-S_m(t)+SL^{-1}\left\{\frac{1-\alpha+\alpha s}{M(\alpha)}SL\{b-S_n(b+cI_n+d)\}\right\}\\
&-SL^{-1}\left\{\frac{1-\alpha+\alpha s}{M(\alpha)}SL\{b-S_m(b+cI_m+d)\}\right\}\Bigg\|
\end{aligned}\tag{12}$$

By the application of triangular identity, Eq. (12) gives

$$\begin{aligned}
\left\|T\left(S_n(t)\right)-T\left(S_m(t)\right)\right\|\le\left\|S_n(t)-S_m(t)\right\|+\Bigg\|&SL^{-1}\left\{\frac{1-\alpha+\alpha s}{M(\alpha)}SL\{b-S_n(b+cI_n+d)\}\right\}\\
&-SL^{-1}\left\{\frac{1-\alpha+\alpha s}{M(\alpha)}SL\{b-S_m(b+cI_m+d)\}\right\}\Bigg\|.
\end{aligned}\tag{13}$$

After some calculations, Eq. (13) is converted to

$$\left\|T\left(S_n(t)\right)-T\left(S_m(t)\right)\right\| \le \left\|S_n(t)-S_m(t)\right\| + SL^{-1}\left\{SL\left\{\frac{1-\alpha+\alpha s}{M(\alpha)}\left\{\begin{array}{c}-a\left(S_n-S_m\right)-\theta\left(S_n-S_m\right)\\ -bS_n\left(I_n-I_m\right)-bI_m\left(S_n-S_m\right)\end{array}\right\}\right\}\right\} \tag{14}$$

Because of similar behavior of solutions, we get

$$\begin{aligned}\left\|S_n(t)-S_m(t)\right\| &= \left\|V_n(t)-V_m(t)\right\|,\\ \left\|S_n(t)-S_m(t)\right\| &= \left\|I(t)-I_m(t)\right\|,\\ \left\|S_n(t)-S_m(t)\right\| &= \left\|R_n(t)-R_m(t)\right\|.\end{aligned}$$

Replacing this in the Eq. (14), we find

$$\begin{aligned}&\left\|T\left(S_n(t)\right)-T\left(S_m(t)\right)\right\| \le \left\|S_n(t)-S_m(t)\right\|\\ &+SL^{-1}\left\{SL\left\{\frac{1-\alpha+\alpha s}{M(\alpha)}\left\{-(b+d)\left(S_n-S_m\right)-cS_n\left(S_n-S_m\right)-cI_m\left(S_n-S_m\right)\right\}\right\}\right\}.\end{aligned} \tag{15}$$

Because S_n, I_m are bounded, we describe two different positive constants M_1, M_2, M_3, M_4, M_5 for all t such that

$$\begin{aligned}\left\|S_n\right\| &< M_1,\\ \left\|I_m\right\| &< M_2,\\ \left\|I_n\right\| &< M_3,\\ \left\|V_n\right\| &< M_4.\end{aligned} \tag{16}$$

Using the Eqs. (15) and (16), we get

$$\left\|T\left(S_n(t)\right)-T\left(S_m(t)\right)\right\| \le \left\{1-\left((b+d)\,f_1-cM_1 f_2-cM_2 f_3\right)\right\}\left\|S_n(t)-S_m(t)\right\| \tag{17}$$

where f_1, f_2, f_3 are functions from $SL^{-1}\left\{SL\left\{\frac{1-\alpha+\alpha s}{M(\alpha)}\right\}\right\}$. Repeating the same procedure, we have

$$\begin{aligned}
&\left\|T\left(V_n(t)\right)-T\left(V_m(t)\right)\right\| \le \left\{1-\left(dg_1-(b+k)g_2-eM_3g_3-eM_4g_4\right)\right\}\left\|V_n(t)-V_m(t)\right\|,\\
&\left\|T\left(I_n(t)\right)-T\left(I_m(t)\right)\right\| \le \left\{1-\begin{pmatrix}(-b+m)h_1+cM_1h_2+cM_2h_3\\ +eM_5h_4-eM_2h_5\end{pmatrix}\right\}\left\|I_n(t)-I_m(t)\right\|,\\
&\left\|T\left(R_n(t)\right)-T\left(R_m(t)\right)\right\| \le \left\{1-\left(bk_1+kk_2+mk_3\right)\right\}\left\|R_n(t)-R_m(t)\right\|,
\end{aligned} \tag{18}$$

where

$$\begin{aligned}
&1-\left((b+d)f_1-cM_1f_2-cM_2f_3\right)<1,\\
&1-\left(dg_1-(b+k)g_2-eM_3g_3-eM_4g_4\right)<1,\\
&1-\begin{pmatrix}(-b+m)h_1+cM_1h_2+cM_2h_3\\ eM_5h_4+eM_2h_5\end{pmatrix}<1,\\
&1-\left(bk_1+kk_2+mk_3\right)<1.
\end{aligned}$$

Thus, T has a fixed point. In order to show that the conditions in Theorem 2 is satisfied by T, suppose that Eqs. (17) and (18) hold, let

$$a=(0,0,0,0)$$

and

$$A=\begin{cases}1-\left((b+d)f_1-cM_1f_2-cM_2f_3\right)<1,\\ 1-\left(dg_1-(b+k)g_2-eM_3g_3-eM_4g_4\right)<1,\\ 1-\begin{pmatrix}(-b+m)h_1+cM_1h_2+cM_2h_3\\ eM_5h_4+eM_2h_5\end{pmatrix}<1,\\ 1-\left(bk_1+kk_2+mk_3\right)<1.\end{cases}$$

This shows that the conditions Theorem 2 holds for the mapping T, then T is Picard T-stable.

3.3 Uniqueness of the special solution

In this part, we give the conditions for the uniqueness of special solution to Eq. (6). First, we suppose that the Eq. (6) has an exact solution due to which the special solution converges for a large number *m*.

We consider a Hilbert space $H=L^2\left((a,b)\times(0,T)\right)$ which can be given as the set of those functions.

$$v:(a,b)\times[0,T]\to\mathbb{R},\iint uv\,du\,dv<\infty.$$

Now, we take into consideration the following operator:

$$T(S,V,I,R)=\begin{cases} b-S(b+cI+d), \\ dS-V(eI+k+b), \\ I(cS+eV-\alpha-b), \\ kV+mI-bR. \end{cases}$$

We aim to prove the inner product of

$$\left(T\left(W_{11}-W_{12},W_{21}-W_{22},W_{31}-W_{32},W_{41}-W_{42}\right),\left(w_1,w_2,w_3,w_4\right)\right)$$

where $\left(W_{11},W_{12}\right)$, $\left(W_{21},W_{22}\right)$, $\left(W_{31},W_{32}\right)$ and $\left(W_{41},W_{42}\right)$ are special solution of the model. However

$$\begin{aligned}&\left(T\left(W_{11}-W_{12},W_{21}-W_{22},W_{31}-W_{32},W_{41}-W_{42}\right),\left(w_1,w_2,w_3,w_4\right)\right)\\ &=\begin{cases} \left(-\left(W_{11}-W_{12}\right)\left(b+c\left(W_{31}-W_{32}\right)d\right),w_1\right), \\ \left(d\left(W_{11}-W_{12}\right)-\left(W_{21}-W_{22}\right)\left(e\left(W_{31}-W_{32}\right)+k+b\right),w_2\right), \\ \left(\left(W_{31}-W_{32}\right)\left(c\left(W_{11}-W_{12}\right)+e\left(W_{21}-W_{22}\right)-m-b\right),w_3\right), \\ \left(k\left(W_{21}-W_{22}\right)+m\left(W_{31}-W_{32}\right)-b\left(W_{41}-W_{42}\right),w_4\right). \end{cases}\end{aligned} \tag{19}$$

We consider the first Eq. of (19) without loss of generality

$$\begin{aligned}&\left(-\left(W_{11}-W_{12}\right)\left(b+c\left(W_{31}-W_{32}\right)+d\right),w_1\right)\\ &=\left(-\left(W_{11}-W_{12}\right)b,w_1\right)+\left(-\left(W_{11}-W_{12}\right)c\left(W_{31}-W_{32}\right),w_1\right)\\ &+\left(-\left(W_{11}-W_{12}\right)d,w_1\right).\end{aligned} \tag{20}$$

Because the same role is played both the solution, suppose that

$$W_{11}-W_{12}\cong W_{21}-W_{22}\cong W_{31}-W_{32}\cong W_{41}-W_{42}.$$

Then the Eq. (20) becomes

$$\begin{aligned}&\left(-\left(W_{11}-W_{12}\right)\left(b+c\left(W_{31}-W_{32}\right)+d\right),w_1\right)\\ &\cong\left(-\left(W_{11}-W_{12}\right)b,w_1\right)+\left(-c\left(W_{11}-W_{12}\right)^2,w_1\right)+\left(-\left(W_{11}-W_{12}\right)d,w_1\right).\end{aligned} \tag{21}$$

In the light of norm and inner product relation, we obtain

$$\begin{aligned}&\left(-\left(W_{11}-W_{12}\right)\left(b+c\left(W_{31}-W_{32}\right)+d\right),w_1\right)\\&\cong\left(-b\left(W_{11}-W_{12}\right),w_1\right)+\left(-c\left(W_{11}-W_{12}\right)^2,w_1\right)+\left(-d\left(W_{11}-W_{12}\right),w_1\right)\\&\leq b\left\|W_{11}-W_{12}\right\|\left\|w_1\right\|+c\left\|W_{11}-W_{12}\right\|^2\left\|w_1\right\|+d\left\|W_{11}-W_{12}\right\|\left\|w_1\right\|\\&=\left(b+c\overline{w}_1+d\right)\left\|W_{11}-W_{12}\right\|\left\|w_1\right\|\end{aligned}\tag{22}$$

where $\overline{w}_1=\left\|W_{11}-W_{12}\right\|$. By the similar way, we have

$$\begin{aligned}&\left(d\left(W_{11}-W_{12}\right)-\left(W_{21}-W_{22}\right)\left(e\left(W_{31}-W_{32}\right)+k+b\right),w_2\right)\\&\leq d\left\|W_{21}-W_{22}\right\|\left\|w_2\right\|+e\left\|W_{21}-W_{22}\right\|^2\left\|w_2\right\|\\&+k\left\|W_{21}-W_{22}\right\|\left\|w_2\right\|+b\left\|W_{21}-W_{22}\right\|\left\|w_2\right\|\\&=\left(d+e\overline{w}_2+k+b\right)\left\|W_{21}-W_{22}\right\|\left\|w_2\right\|,\end{aligned}\tag{23}$$

$$\begin{aligned}&\left(\left(W_{31}-W_{32}\right)\left(c\left(W_{11}-W_{12}\right)+e\left(W_{21}-W_{22}\right)-\alpha-b\right),w_3\right)\\&\leq c\left\|W_{31}-W_{32}\right\|^2\left\|w_3\right\|+e\left\|W_{31}-W_{32}\right\|^2\left\|w_3\right\|\\&+\alpha\left\|W_{31}-W_{32}\right\|\left\|w_3\right\|+b\left\|W_{31}-W_{32}\right\|\left\|w_3\right\|\\&=\left(c\overline{w}_3+e\overline{w}_3+m+b\right)\left\|W_{31}-W_{32}\right\|\left\|w_3\right\|,\end{aligned}\tag{24}$$

$$\begin{aligned}&\left(k\left(W_{21}-W_{22}\right)+\alpha\left(W_{31}-W_{32}\right)-b\left(W_{41}-W_{42}\right),w_4\right)\\&\leq k\left\|W_{41}-W_{42}\right\|\left\|w_4\right\|+\alpha\left\|W_{41}-W_{42}\right\|\left\|w_4\right\|+b\left\|W_{41}-W_{42}\right\|\left\|w_4\right\|\\&=\left(k+m+b\right)\left\|W_{41}-W_{42}\right\|\left\|w_4\right\|.\end{aligned}\tag{25}$$

Considering the Eqs. (22), (23), (24), (25) with the Eq. (19) we have

$$\begin{aligned}&\left(T\left(W_{11}-W_{12},W_{21}-W_{22},W_{31}-W_{32},W_{41}-W_{42}\right),\left(w_1,w_2,w_3,w_4\right)\right)\\&\leq\begin{cases}\left(b+c\overline{w}_1+d\right)\left\|W_{11}-W_{12}\right\|\left\|w_1\right\|,\\\left(d+e\overline{w}_2+k+b\right)\left\|W_{21}-W_{22}\right\|\left\|w_2\right\|,\\\left(c\overline{w}_3+e\overline{w}_3+m+b\right)\left\|W_{31}-W_{32}\right\|\left\|w_3\right\|,\\\left(k+m+b\right)\left\|W_{41}-W_{42}\right\|\left\|w_4\right\|.\end{cases}\end{aligned}$$

Using large number k_i for $i=1,2,3,4$ all of solutions converge to exact solution, benefiting from the topology concept, we can find for almost small parameters l_i for $i=1,2,3,4$.

$$\begin{aligned}
&\|S-W_{11}\|,\|S-W_{12}\|<\frac{l_1}{4\left(b+c\overline{w}_1+d\right)\|w_1\|},\\
&\|V-W_{21}\|,\|V-W_{22}\|<\frac{l_2}{4\left(d+e\overline{w}_2+k+b\right)\|w_2\|},\\
&\|I-W_{31}\|,\|I-W_{32}\|<\frac{l_3}{4\left(c\overline{w}_3+e\overline{w}_3+m+b\right)\|w_3\|},\\
&\|R-W_{41}\|,\|R-W_{42}\|<\frac{l_4}{4\left(k+m+b\right)\|w_4\|}.
\end{aligned} \tag{26}$$

Using triangular identity and taking $L=\max\left(l_1,l_2,l_3,l_4\right)$, we have

$$\begin{aligned}
&\left(a+b\overline{w}_1+\theta\right)\|W_{11}-W_{12}\|\|w_1\|<L,\\
&\left(\theta+\xi\overline{w}_2+\beta+a\right)\|W_{21}-W_{22}\|\|w_2\|<L,\\
&\left(b\overline{w}_3+\xi\overline{w}_3+\eta+a\right)\|W_{31}-W_{32}\|\|w_3\|<L,\\
&\left(\beta+\eta+a\right)\|W_{41}-W_{42}\|\|w_4\|<L.
\end{aligned} \tag{27}$$

Because L is very small positive parameter, we conclude on the based of the topology idea that

$$\begin{aligned}
&\left(b+c\overline{w}_1+d\right)\|W_{11}-W_{12}\|\|w_1\|=0,\\
&\left(d+e\overline{w}_2+k+b\right)\|W_{21}-W_{22}\|\|w_2\|=0,\\
&\left(c\overline{w}_3+e\overline{w}_3+m+b\right)\|W_{31}-W_{32}\|\|w_3\|=0,\\
&\left(k+m+b\right)\|W_{41}-W_{42}\|\|w_4\|=0.
\end{aligned}$$

But it is clear that

$$\begin{aligned}
&b+c\overline{w}_1+\theta\neq 0,\\
&d+e\overline{w}_2+k+b\neq 0,\\
&c\overline{w}_3+e\overline{w}_3+m+b\neq 0,\\
&k+m+b\neq 0
\end{aligned} \tag{28}$$

and $\|w_1\| \neq 0, \|w_2\| \neq 0, \|w_3\| \neq 0, \|w_4\| \neq 0.$ So, we have

$$\|W_{11} - W_{12}\| = 0,$$
$$\|W_{21} - W_{22}\| = 0,$$
$$\|W_{31} - W_{32}\| = 0,$$
$$\|W_{41} - W_{42}\| = 0,$$

which yields $W_{11} = W_{12}$, $W_{21} = W_{22}$, $W_{31} = W_{32}$, $W_{41} = W_{42}$. This completes the proof.

4. Conclusion

In this study, we firstly rearrange communicable SVIR model using Caputo-Fabrizo derivative. It is clear that generally it is a difficult to obtain the solution of non-linear equations, here we give a special solution and stability analysis for this extended model.

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Textural Products via Inverse Systems of Ditopological Spaces

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Abstract

In this study, by taking into consideration the suitable inverse system theory established for the topological category whose objects are ditopological plain texture spaces and morphisms are bicontinuous point functions satisfying a compatibility condition, we proved a representation theorem for the infinite ditopological products of the objects in that category via inverse limits.

Keywords : Inverse system, ditopology, plain texture, directed set, inverse limit, category, covariant functor

1. Introduction

An effective method in order to derive a new space from two or more spaces is the theory of inverse systems which was presented in [2] for some classical categories.

Recently, the theory of inverse systems and inverse limits is handled in [7] first-time, in the categories of some special textures. It is seen in [8] that a method used to construct a new ditopological space is the theory of *ditopological inverse systems* and their limit spaces under the name *ditopological inverse limits* as the subspaces of ditopological product spaces described in [4,6].

Particularly, in [8] we gave a detailed analysis of the theory of ditopological inverse systems and inverse limits insofar as the category **ifPDitop** whose objects are the ditopological texture spaces which have plain texturing and morphisms are the bicontinuous, special (w-preserving) point functions, is concerned.

Our main aim in this work is to give a characterization theorem which states that any cartesian product of the objects of **ifPDitop** can be written in terms of the finite cartesian product of those objects, via inverse limit operation.

2. Background

Texture: If S is a set, a *texturing* $\mathcal{S}$ on S is a subset of $\mathcal{P}(S)$ which is a point-separating, complete, completely distributive lattice containing S and $\emptyset$, and for which meet coincides with intersection and finite joins with union. The pair $(S, \mathcal{S})$ is then called a *texture*.

For a texture $(S, \mathcal{S})$, most properties are conveniently defined in terms of the *p-sets* and *q-sets*:

$$P_s = \bigcap\{A \in \mathcal{S} \mid s \in A\}, \quad Q_s = \bigvee\{A \in \mathcal{S} \mid s \notin A\}$$

Plain Texture: The texture $(S, \mathcal{S})$ is called *plain* if $\mathcal{S}$ is closed under arbitrary unions, equivalently if $P_s \not\subseteq Q_s$ for all $s \in S$.

Point Function Between Textures: If $(S, \mathcal{S})$, $(T, \mathcal{T})$ are textures and $\varphi : S \to T$ a point function between the base sets of textures satisfying the compatibility condition $P_s \not\subseteq Q_{s'} \implies P_{\varphi(s)} \not\subseteq Q_{\varphi(s')}$, that is φ is called *ω-preserving.* In addition, the following equalities define the *inverse image* with respect to φ for each $B \in \mathcal{T}$.

$$\varphi^{\leftarrow} B = \bigvee \{P_u \mid \varphi(u) \in B\} = \bigcap \{Q_v \mid \varphi(v) \notin B\}$$

Ditopological Space: Since a texturing $\mathcal{S}$ need not be closed under the operation of taking the set-complement, the notion of topology is replaced by that of *dichotomous topology* or *ditopology* [4,6], namely a pair (τ, κ) of subsets of $\mathcal{S}$, where the set of *open sets* τ satisfies

(1) $S, \emptyset \in \tau$,
(2) $G_1, G_2 \in \tau \implies G_1 \cap G_2 \in \tau$ and
(3) $G_i \in \tau,\, i \in I \implies \bigvee_i G_i \in \tau$,

and the set of *closed sets* κ satisfies the dual conditions

A ditopological texture space with respect to a ditopology (τ, κ) on the texture $(S, \mathcal{S})$ is denoted by $(S, \mathcal{S}, \tau, \kappa)$.

An adequate introduction to the theory of ditopological spaces and the motivation for its study may be obtained from [4,6].

Product Ditopology: Let $(S_j, \mathcal{S}_j, \tau_j, \kappa_j)_{j \in J}$ be ditopological spaces and $(S, \mathcal{S})$ the product [4] of the textures $(S_j, \mathcal{S}_j)_{j \in J}$. Then the ditopology (τ, κ) on $(S, \mathcal{S})$ with subbase $\{\Pi_j^{\leftarrow} G = E(j, G) \mid G \in \tau_j, j \in J\}$ and cosubbase $\{\pi_j^{\leftarrow} K = E(j, K) \mid K \in \kappa_j, j \in J\}$ is called the *product ditopology* on $(S, \mathcal{S})$. In this case, the ditopological space $(S, \mathcal{S}, \tau, \kappa)$ is called the *product of the family* $(S_j, \mathcal{S}_j, \tau_j, \kappa_j)_{j \in J}$, and is denoted by $(\prod\limits_{j \in J} S_j, \bigotimes\limits_{j \in J} \mathcal{S}_j, \bigotimes\limits_{j \in J} \tau_j, \bigotimes\limits_{j \in J} \kappa_j)$.

Bicontinuity: An ω-preserving point function between the ditopological texture spaces, is called *bicontinuous* if the inverse image of every open set is open and the inverse image of every closed set is closed.

The Category **ifPDitop** *:* Objects are ditopological plain texture spaces and morphisms are ω-preserving, bicontinuous point functions.

Specifically, the reader may consult [3] for terms from lattice theory not mentioned here. In addition, we follow the terminology of [1] for all the general concepts relating to category theory.

3. The Theory of Inverse Systems For Ditopological Spaces

Let us recall the notions of *inverse system* and their *inverse limits* constructed in **ifPDitop**, in the light of [8]. Before everything, note that the following:

Remark 3.1. The inverse systems constructed by the objects and morphisms of the category **ifPDitop**, which are the bonding maps satisfying some conditions, have an inverse limit space described as in [8, Definition 4.1], since **ifPDitop** has products and equalizers. Also, the *uniqueness* of the limit space in the category **ifPDitop** was mentioned in [8]. Hence, the operation $\varprojlim$ will be meaningful for the inverse systems mentioned in the context of that category.

Notation: According to the major theorem given as [8, Theorem 4.6], for the directed set Λ take the inverse system $\{(S_\alpha, \mathcal{S}_\alpha, \tau_\alpha, \kappa_\alpha), \varphi_{\alpha\beta}\}_{\alpha\geq\beta}$ constructed in **ifPDitop**, over Λ. In this case, the notations $(\tau_\infty, \kappa_\infty)$ and $(S_\infty, \mathcal{S}_\infty, \tau_\infty, \kappa_\infty)$ will be used as *inverse limit ditopology* and *(ditopological) inverse limit space*, respectively, where $S_\infty = \varprojlim\{S_\alpha\}$, in the remainder of paper.

Now let 's take a glimpse of the *mappings between inverse systems*: Let two inverse systems $\mathcal{A} = \{(S_\alpha, \mathcal{S}_\alpha, \tau_\alpha, \kappa_\alpha), \varphi_{\alpha\beta}\}_{\alpha\geq\beta}$ and $\mathcal{B} = \{(T_\alpha, \mathcal{T}_\alpha, \tau'_\alpha, \kappa'_\alpha), \psi_{\alpha\beta}\}_{\alpha\geq\beta}$ described in **ifPDitop**. Take into consideration [7, Definition 3.9] which introduces the notion *inverse system of mappings* or *mapping of inverse systems* denoted by $\{t_\alpha\} : \mathcal{A} \to \mathcal{B}$, consisting of the components $t_\alpha \in \mathrm{Mor}\,$**ifPDitop**, satisfying the equality $\psi_{\beta\alpha} \circ t_\beta = t_\alpha \circ \varphi_{\beta\alpha}$. Hence, by recalling the notion *inverse limit space* with the notation S_∞ defined as in [7, Definition 4.1] and the map $t_\infty = \varprojlim\{t_\alpha\}_{\alpha\in\Lambda}$ defined in [7, Theorem 4.14], called *inverse limit map* of the inverse system $\{t_\alpha\}$ of mappings, now let 's focus on the following crucial theorem proved in [8, Theorem 4.24]:

Theorem 3.2. *Let* $\{t_\alpha\} : \{(S_\alpha, \mathcal{S}_\alpha, \tau_\alpha, \kappa_\alpha), \varphi_{\beta\alpha}\}_{\beta\geq\alpha} \to \{(T_\alpha, \mathcal{T}_\alpha, \tau'_\alpha, \kappa'_\alpha), \psi_{\beta\alpha}\}_{\beta\geq\alpha}$ *be an inverse system of mappings in* **ifPDitop***, over a directed set* Λ*. Then there exists a unique map* $t_\infty \in \mathrm{Mor}\,$**ifPDitop** *between the spaces* $(S_\infty, \mathcal{S}_\infty, \tau_\infty, \kappa_\infty)$ *and* $(T_\infty, \mathcal{T}_\infty, \tau'_\infty, \kappa'_\infty)$ *having the property that for each* $\alpha \in \Lambda$*,* $t_\alpha \circ \mu_\alpha = \eta_\alpha \circ t_\infty$*.*
In this case,

i) *If each* t_α *is an* **ifPDitop***-isomorphism,* t_∞ *is an* **ifPDitop***-isomorphism.*
ii) *If each* $t_\alpha \circ \mu_\alpha$ *is surjective,* $t_\infty(S_\infty)$ *is jointly dense in* T_∞*.*

Notation: In this study, $\mathbf{Inv_{ifPDitop}}$ denotes the category whose objects are the inverse systems constructed by the objects of category **ifPDitop** and morphisms are the mappings of inverse systems, described as in Theorem 3.2, namely, the inverse systems of **ifPDitop**-morphisms defined between the objects of **ifPDitop**.

Now, let 's recall the notion of *inverse limit map* peculiar to texture theory, introduced in [7, Theorem 4.14].

Note from that Remark 3.1, the inverse systems which are the objects of $\mathbf{Inv_{ifPDitop}}$ have a *unique* inverse limit space as an object of **ifPDitop**. With the reference to this fact, we have the following immediately from [8].

Theorem 3.3. *The limit operation* $\varprojlim$ *of assigning an inverse limit in* **ifPDitop** *to each object in* $\mathbf{Inv_{ifPDitop}}$ *and an inverse limit map* $t_\infty \in \mathrm{Mor}\,$**ifPDitop** *to each inverse system* $\{t_\alpha\}_\alpha \in \mathrm{Mor}\,\mathbf{Inv_{ifPDitop}}$ *of maps* $t_\alpha \in \mathrm{Mor}\,$**ifPDitop***, forms the covariant functor* $\varprojlim : \mathbf{Inv_{ifPDitop}} \to$ **ifPDitop**.

4. Ditopological Textural Product As an Inverse Limit in ifPDitop

It is clear that the notion of inverse limit as an object of **ifPDitop** for any inverse system which is the object of $\mathbf{Inv_{ifPDitop}}$ is derived from the products as the objects of **ifPDitop**.

Conversely, by applying the limit operation to the objects of $\mathbf{Inv_{ifPDitop}}$, one can express as in [5] infinite ditopological cartesian products [4], of the spaces which are the objects of $\mathrm{Ob}\,$**ifPDitop** in terms of the finite cartesian products of those objects in **ifPDitop**, as follows:

Theorem 4.1. *Any arbitrary textural product of the objects in* **ifPDitop** *is exactly the inverse limit space of the inverse system consisting of finite products of those objects.*

Proof. Let $(X_s, \mathcal{S}_s, \tau_s, \kappa_s) \in \mathrm{Ob}\,\mathbf{ifPDitop}$, $s \in \Lambda$ and Γ be directed by the set inclusion, that is $J \leq I \iff J \subseteq I$ for every $I, J \in \Gamma$. Now assume $J \leq I$, for any $J \in \Gamma$. If $x = \{x_s\}_{s\in I} \in \prod_{s\in I} X_s = X_I$ then $x_s \in X_s$ for all $s \in I$. In this case, $\{x_s\}_{s\in J} \in \prod_{s\in J} X_s = X_J$ by the facts that if $s \in J$ then $s \in I$ and $x_s \in X_s$ for all $s \in I$. Therefore, for $J \leq I$, describe the mapping $\varphi_{IJ} : X_I \to X_J$, $\{x_s\}_{s\in I} \mapsto \{x_s\}_{s\in J}$.

It is easy to show that φ_{IJ} is ω-preserving and bicontinuous for $J \leq I$.

Furthermore, note that the mappings φ_{IJ} for $J \leq I$ are the bonding maps: Indeed, for the mapping $\varphi_{II} : X_I \to X_I$, the equality $\varphi_{II}(\{x_s\}_{s\in I}) = \{x_s\}_{s\in I}$ is clear and so φ_{II} is the identity id_{X_I}. In addition, for $K \geq I \geq J$, let 's prove $\varphi_{IJ} \circ \varphi_{KI} = \varphi_{KJ}$. If $\{x_s\}_{s\in K} \in X_K$ then $(\varphi_{IJ} \circ \varphi_{KI})(\{x_s\}_{s\in K}) = \varphi_{IJ}(\varphi_{KI}(\{x_s\}_{s\in K})) = \varphi_{IJ}(\{x_s\}_{s\in I}) = \{x_s\}_{s\in J} = \varphi_{KJ}(\{x_s\}_{s\in K})$.

Consequently, it is clear that $\{(X_I, \mathcal{S}_I, \tau_I, \kappa_I), \varphi_{IJ}\}_{I\geq J} \in \mathrm{Ob}\,\mathbf{Inv}_{\mathbf{ifPDitop}}$.

Let us prove that the inverse limit space of $\{(X_I, \mathcal{S}_I, \tau_I, \kappa_I), \varphi_{IJ}\}_{I\geq J}$ over Γ is **ifPDitop**-isomorphic to the arbitrary ditopological product space constructed on the set $\prod_{s\in\Lambda} X_s$:

Consider a mapping between $\varprojlim\{X_I\}_{I\in\Gamma}$ and $\prod_{s\in\Lambda} X_s$:

If $\{x_I\} \in \varprojlim\{X_I\}_{I\in\Gamma}$ then $\{x_I\} \in \prod_{I\in\Gamma} X_I$ and so $x_I \in X_I$ for every $I \in \Gamma$. Now, for any $s \in \Lambda$ let $I_s = \{s\} \in \Gamma$, so by the fact $X_{I_s} = \prod_{z\in I_s} X_z = \prod_{z\in\{s\}} X_z = X_s$, we have $x_{\{s\}} = x_{I_s} \in X_s$, $s \in \Lambda$. Thus $\{x_{I_s}\} \in \prod_{s\in\Lambda} X_s$ and finally, we can define the mapping $\psi : \lim_{\leftarrow}\{X_I\}_{I\in\Gamma} \to \prod_{s\in\Lambda} X_s$, $\{x_I\}_{I\in\Gamma} \mapsto \{x_{I_s}\}_{s\in\Lambda}$ It is easy to verify that ψ is well-defined. In addition, it can be showed that ψ is an **ifPDitop**-isomorphism, that is ω-preserving and bijective.

Now, if consider the product ditopological spaces $(X_I, \mathcal{S}_I, \tau_I, \kappa_I)$ for $I \in \Gamma$, with the plain texturings then the product texturing $\bigotimes_{I\in\Gamma} \mathcal{S}_I$ and product ditopology $(\bigotimes_{I\in\Gamma} \tau_I, \bigotimes_{I\in\Gamma} \kappa_I)$ can be constructed over the product set $\prod_{I\in\Gamma} X_I$ in a suitable way. Therefore, the restricted texturing and ditopology will be taken over the subset $\varprojlim\{X_I\}_{I\in\Gamma}$ of $\prod_{I\in\Gamma} X_I$. Shortly, if we use the notations $\mathcal{T} = (\bigotimes_{I\in\Gamma} \mathcal{S}_I)|_{\varprojlim\{X_I\}_{I\in\Gamma}}$, $\mathcal{V} = (\bigotimes_{I\in\Gamma} \tau_I)|_{\varprojlim\{X_I\}_{I\in\Gamma}}$ and $\mathcal{Z} = (\bigotimes_{I\in\Gamma} \kappa_I)|_{\varprojlim\{X_I\}_{I\in\Gamma}}$ for the induced texturing, topology and cotopology, respectively, then now we will prove that ψ is bicontinuous with respect to the ditopologies $(\bigotimes_{s\in\Lambda} \tau_s, \bigotimes_{s\in\Lambda} \kappa_s)$ and $(\mathcal{V}, \mathcal{Z})$:

Let $G \in \bigotimes_{s\in\Lambda} \tau_s = \tau_\Lambda$ and $\psi^{-1}[G] \not\subseteq Q_{\{x_I\}_{I\in\Gamma}}$. In this case, $G \not\subseteq Q_{\psi(\{x_I\}_{I\in\Gamma})}$ and so $G \not\subseteq Q_{\{x_{I_s}\}_{s\in\Lambda}}$. Thus, there exists $B \in \mathcal{B}_{\tau_\Lambda}$, the base for the product topology τ_Λ such that $B \subseteq G$ and $B \not\subseteq Q_{\psi(\{x_I\}_{I\in\Gamma})}$. Note here that $B = \bigcap_{j\in J_0\subseteq\Lambda} \pi_j^{-1}[G_j]$, where

$G_j \in \tau_j$ and $j \in J_0$ for the finite set $J_0 \subseteq \Lambda$. Thus, we have $\psi^{-1}(\bigcap_{j\in J_0\subseteq\Lambda} \pi_j^{-1}[G_j]) \subseteq \psi^{-1}[G]$ and so $\bigcap_{j\in J_0} (\pi_j \circ \psi)^{-1}[G_j] \subseteq \psi^{-1}[G]$.

On the other hand, the equality $\pi_j \circ \psi = \pi_{I_j}|_{\underleftarrow{\lim}\{X_I\}_{I\in\Gamma}}$ is obvious by the definition of projection map $\pi_{I_j} : \prod_{I\in\Gamma} X_I \to X_{I_j} = X_j$ and by the facts that $j \in \Lambda$ and $I_j = \{j\} \subseteq \Lambda$ which means that $I_j \in \Gamma$ for $j \in J_0$.

Additionally, if take φ as the inverse of ψ, then we have $\pi_{I_j}|_{\underleftarrow{\lim}\{X_I\}_{I\in\Gamma}} \circ \varphi = \pi_j$. Here, the restriction $\pi_{I_j}|_{\underleftarrow{\lim}\{X_I\}_{I\in\Gamma}}$ is bicontinuous since I_j. projection map π_{I_j} is bicontinuous.

Hence, if $A = \bigcap_{j\in J_0} (\pi_j \circ \psi)^{-1}[G_j] = \bigcap_{j\in J_0} (\pi_{I_j}|_{\underleftarrow{\lim}\{X_I\}_{I\in\Gamma}})^{-1}[G_j]$ then $A \in \mathcal{B}_{\mathcal{V}}$. Here, $\mathcal{B}_{\mathcal{V}}$ denotes the base for the topology $\mathcal{V}$. In this case, $A \subseteq \psi^{-1}[G]$.

In addition, because of the fact $A \not\subseteq Q_{\{x_I\}_{I\in\Gamma}}$ that we have $\psi^{-1}[G] \in \mathcal{V}$ and so ψ is continuous.

Dually, it is easy to verify that ψ is cocontinuous by dealing with the closed sets. Then ψ is bicontinuous. As the final step, that the map φ as the inverse of ψ is bicontinuous can be shown similarly. □

The above theorem could be also stated for the subcategory **ifPDicomp**$_2$ consisting of dicompact [4] and bi-T_2 (bi-Hausdorff) objects of the category **ifPDitop**. Hence, with the above arguments we have:

Corollary 4.2. *The infinite ditopological products of the objects which belong to* **ifPDicomp**$_2$ *can be expressed via inverse limits, in terms of the finite ditopological products in* **ifPDicomp**$_2$ *of those objects.*

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Using Special Functions on Grünwald-Letnikov and Riemann Liouville Fractional Derivative and Fractional Integral

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Abstract

In this study, we define special functions on fractional calculus because, gamma function [1]-[3]-[4]-[7], beta function [2]-[6], Integral function [5], Mittag-Leffler and Integration of the Mittag-Leffler function [3]-[7] are related with fractional derivatives and fractional Integrals [7]-[8]. Also, matlab program is used in some parts. As a reason of, we can describe geometrical meaning of special functions and fractional-order derivative. After we use definition of Grünwald-Letnikov fractional-order derivative and fractional integral [7]-[8], we establish a connection definition of Grünwald-Letnikov with Riemann-Liouville definition [7]-[8].

Keywords: Gamma function, beta function, integral function, Mittag-Leffler function, Integration of the Mittag-Leffler function Grünwald-Letnikov and Riemann-Liouville definitions.

1. Introduction

Gamma Function

In this here, special functions will be defined. Especially, gamma function and other special function will used with definition of the fractional-derivative and fractional- integral [7,8]. However, only we aim using special functions on Grünwald-Letnikov and Riemann-Liouville definitions. Previously, the gamma function is generally shown as [7,8]:

$$\Gamma(k) = \int_0^\infty e^{-t} t^{k-1} dt \qquad (1.1)$$

or we can define the gamma function from (1.1) [1,3,4]:

$$\Gamma(k) = \lim_{a\to\infty} \int_0^a e^{-t} t^{k-1} dt \qquad (1.2)$$

from (1.1) converging in the right half of the complex plane $Re(z) > 0$. So, we say that;
let k is defined complex-coordinate system in (1.1) then we put $x + iy$ instead of k in (1.1). Hence, we write as:

$$\Gamma(x+iy)=\int_0^\infty e^{-t}t^{x+iy-1}dt=\int_0^\infty e^{-t}t^{x-1}e^{iylog(t)}dt \qquad (1.3)$$

and from the rule that: Euler equation: $e^{iylog(t)}=\cos\big(ylog(t)\big)+isin(ylog(t))$

$$\Gamma(x+iy)=\int_0^\infty e^{-t}t^{x-1}(\cos\,(ylog(t)+isin\big(ylog(t)\big)dt$$

$$\Gamma(x+iy)=\int_0^\infty [e^{-t}t^{x-1}cosylog(t)+ie^{-t}t^{x-1}sinylog(t)]dt$$

$$=\int_0^\infty e^{-t}t^{x-1}cosylog(t)dt+i\int_0^\infty e^{-t}t^{x-1}sinylog(t)dt$$

$$=cosy\int_0^\infty e^{-t}t^{x-1}\log(t)\,dt+isiny\int_0^\infty e^{-t}t^{x-1}\log(t)\,dt$$

and Eqn(1.1) convergence at infinity e^{-t}. Result of this operation, when $t\neq 0$ we say that;

$$x=Re(k)>1.$$

Theorem: The gamma function of $\Gamma(k)$;

$$\Gamma(k)=\int_0^\infty e^{-t}t^{k-1}dt$$

is convergent when k is positive variable (1.4). Let we show that (1.4):

$$\Gamma(k)=\int_0^1 e^{-t}t^{k-1}dt+\lim_{a\to\infty}\int_1^a e^{-t}t^{k-1}dt$$

$$\Gamma(k)=\int_0^1 e^{-t}t^{k-1}dt+\int_1^\infty e^{-t}t^{k-1}dt=I_1+I_2$$

and

$$I=\int_0^1 e^{-t}t^{k-1}dt \qquad (1.5)$$

(1.5) is convergent.

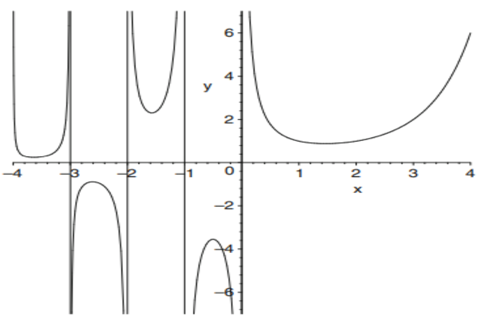

Figure 1: the plot of the gamma function diagram on the coordinate system

Then the exponential function of e^{-t} is decreasing on the closed interval $a \in [0,1]$ and we define this function on this interval but we need to know the function is continuous on definitely intervals. Now, from Eqn (1.5) we consider that;

$$\int_0^1 e^{-t} t^{p-1} dt < \int_0^1 t^{p-1} dt$$

$$\int_0^1 e^{-t} t^{p-1} dt < \frac{1}{p}.$$

To prove, Eqn (1.5):

$$I_2 = \int_1^\infty e^{-t} t^{p-1} dt$$

is convergent and, generally from this formula satisfy:

$$1 \le t \Longrightarrow t^{p-1} e^{-t} \le e^{-t/2}$$

if and only if $t^{p-1} \le e^{t/2}$ if and only if

$$\frac{t^{p-1}}{e^{t/2}} \le 1 \qquad (1.6)$$

As a reason of:

$$\lim_{t\to\infty} \frac{t^{p-1}}{e^{t/2}} = 0$$

we have:

$$\int_1^\infty e^{-t} t^{p-1} dt \le \int_1^\infty e^{-t/2} dt$$

$$\int_1^\infty e^{-t} t^{p-1} dt = \lim_{n\to\infty} \sum_{i=1}^{n} f(t_n) t^{p-1} \Delta t \le 2e^{1/2}$$

Therefore,

$$\int_1^\infty e^{-t} t^{p-1} dt = 2e^{1/2} \qquad (1.7).$$

2. Preliminaries

Beta Function

Generally, we define Beta function as [2,6]:

$$B(\Psi, \phi) = \int_0^1 \tau^{\Psi-1} (1-\tau)^{\Phi-1} d\tau \quad (Re(\Psi) > 0,\ \ Re(\Phi) > 0) \quad (1.8)$$

Using (1.8) this formula related with the Laplace Transform method: Initially, in the first-step we show that the function:

$$h_{\Psi,\Phi}(t) = \int_0^t \tau^{\Psi-1}(1-\tau)^{\Phi-1}d\tau \qquad (1.9)$$

After, we obtain the convolution of the functions $t^{\Psi-1}$ and $t^{\Phi-1}$ and $h_{\Psi,\Phi}(1) = B(\Psi,\Phi)$. Using Laplace transform with Eqn(1.9) then its satisfy:

$$L\left\{h_{\Psi,\Phi}(t) = \int_0^t \tau^{\Psi-1}(1-\tau)^{\Phi-1}d\tau\right\} = \frac{\Gamma(\Psi)}{s^{\Psi}}\frac{\Gamma(\Phi)}{s^{\Phi}} = \frac{\Gamma(\Psi)\Gamma(\Phi)}{s^{\Psi+\Phi}} \qquad (1.10)$$

$H_{\Psi,\Phi}(s)$ defined on frequence domain and so $H_{\Psi,\Phi}(s)$ is the Laplace transform of $h_{\Psi,\Phi}(t)$.
Also, $\Gamma(\Psi)$ and $\Gamma(\Phi)$ are constant, and it is possible to restore the original function $h_{\Psi,\Phi}(t)$ by the inverse Laplace transform of the right-hand-side of [8]:

$$H_{\Psi,\Phi}(s) = \frac{\Gamma(\Psi)}{s^{\Psi}}\frac{\Gamma(\Phi)}{s^{\Phi}} = \frac{\Gamma(\Psi)\Gamma(\Phi)}{s^{\Psi+\Phi}}$$

$$L^{-1}\left\{H_{\Psi,\Phi}(s) = \frac{\Gamma(\Psi)\Gamma(\Phi)}{s^{\Psi+\Phi}}\right\} = \frac{\Gamma(\Psi)\Gamma(\Phi)}{\Gamma(\Psi+\Phi)}t^{\Psi+\Phi-1} \qquad (1.11)$$

when we consider that $t = 1$, the Beta function's value is given:

$$B(\Psi,\Phi) = \frac{\Gamma(\Psi)\Gamma(\Phi)}{\Gamma(\Psi+\Phi)}, \qquad (1.12)$$

from which it follows that: [7],[8]:

$$B(\Psi,\Phi) = B(\Phi,\Psi) \qquad (1.13)$$

3. Integral Functions and Mittag-Leffler Functions

There are several different symbol is defined for the error, imaginary error, complementary error, and exponential integral functions, in Integral functions of Fractional Calculus. Especially, the error function is shown form of (erf) symbol and then it is defined as [5]:

$$\mathrm{erf}(\beta k) = \frac{2\beta}{\sqrt{\pi}}\int_0^k e^{-\beta^2k^2}dk \qquad (1.14)$$

the imaginary error function is defined as (erfi):

$$\mathrm{erfi}(\beta k) = \frac{2\beta}{\sqrt{\pi}}\int_0^z e^{\beta^2k^2}dk = -ierf(i\beta k) \qquad (1.15)$$

complementary error function is demonstrated (erfc):

$$\mathrm{erfc}(\beta k) = \frac{2\beta}{\sqrt{\pi}}\int_k^\infty e^{-\beta^2k^2dk} \qquad (1.16)$$

finally, we indicate the definition of exponential integral function (Ei):

$$\mathrm{Ei}(\beta k) = \int_{-\infty}^{k} \frac{e^{\beta k}}{k} dk \qquad (1.17)$$

Now, if we add extra information on specific topics of exponential function, there are 4 different definitions of integral functions can be established a connection with result of the not only one-parameter but also more than one parameter Mittag-Leffler functions. In this section, we say that the exponential function plays an important role in the theory of integer-order differential equations I. Podlubny [7,8]. However, when we define the one-parameter generalization, the function has generally the form [4,7,8]:

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} = \sum_{k=0}^{\infty} \frac{z^k}{(\alpha k)!} \qquad (1.18)$$

a- Definition and Relation to Some Other Functions:

Generally, a two-parameter function of the Mittag-Leffler function is defined as I. Podlubny [7]-[8]:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \qquad (1.19)$$

and we need to know that α and β are positive. From (1.19) when $\alpha = 1, \beta = 1$ then the two-parameter Mittag-Leffler function is:

$$E_{1,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = \frac{z^0}{0!} + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

$$E_{1,1}(z) = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots = e^z$$

Divergence series and it is defined as exponential number form.

$$E_{1,2}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+2)} = \sum_{k=0}^{\infty} \frac{z^k}{(k+1)!} = \frac{1}{z}\sum_{k=0}^{\infty} \frac{z^{k+1}}{(k+1)!} = \frac{e^z - 1}{z},$$

Generally form of this two-parameter of Mittag-Leffler function is [7]-[8]:

$$E_{1,m}(z) = \frac{1}{z^{m-1}} \left\{ e^z - \sum_{k=0}^{m-2} \frac{z^k}{k!} \right\} \qquad (1.20)$$

The hyperbolic functions formulation of the two-parameter Mittag-Leffler function is shown below:

$$E_{2,1}(z^2) = \sum_{k=0}^{\infty} \frac{z^{2k}}{\Gamma(2k+1)} = \sum_{k=0}^{\infty} \frac{k^{2k}}{(2k)!} = \cosh(z), \qquad (1.21)$$

and,

$$E_{2,2}(z^2) = \sum_{k=0}^{\infty} \frac{z^{2k}}{\Gamma(2\mathrm{k}+2)} = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k+1)!} = \sum_{k=0}^{\infty} \frac{1}{z} \frac{z^{2k+1}}{(2k+1)!} = \frac{1}{z} \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!} = \frac{\sinh(z)}{z}, \qquad (1.22)$$

If the hyperbolic functions of order n [4]:

$$h_r(z,n) = \sum_{k=0}^{\infty} \frac{z^{nk+r-1}}{(nk+r-1)!} = z^{r-1} E_{n,r}(z^n), \quad (r = 1,2, \ldots . n) \qquad (1.23)$$

We analyze of the sine and cosine functions of order n:

$$k_r(z,n) = \sum_{k=0}^{\infty} \frac{(-1)^j z^{nj+r-1}}{(nj+r-1)!} = z^{r-1} E_{n,r}(-z^n), \quad (r = 1,2,\dots,n) \qquad (1.24)$$

Using [18, formulas 7.1.3 and 7.1.8] then the formula satisfy,

$$w(z) = e^{-z^2}\left(1 + \frac{2i}{\sqrt{\pi}}\int_0^z e^{t^2}dt\right) = e^{-z^2} erfc(-iz) \qquad (1.25)$$

and

$$w(z) = \sum_{k=0}^{\infty} \frac{(iz)^k}{\Gamma\left(\frac{k}{2}+1\right)} \qquad (1.26)$$

$$E_{\frac{1}{2},1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma\left(\frac{k}{2}+1\right)} = e^{z^2} erfc(-z) \qquad (1.27)$$

and we mentioned that the definition of the error function is

$$erfc(z) = \frac{2}{\sqrt{\pi}}\int_z^{\infty} e^{-t^2}dt \qquad (1.28)$$

From (1.27) and (1.28), we satisfy the two-parameter Mittag-Leffler function is:

$$E_{\frac{1}{2},1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma\left(\frac{k}{2}+1\right)} = \sum_{k=0}^{\infty} \frac{z^k}{\left(\frac{k}{2}\right)!} = e^{z^2}\left(\frac{2}{\sqrt{\pi}}\int_z^{\infty} e^{-t^2}dt\right) \qquad (1.29)$$

For $\beta = 1$, then we obtain the Mittag-Leffler function is defined as:

$$E_{\alpha,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)} \equiv E_\alpha(z) \qquad (1.30)$$

The function $\varepsilon_t(v,a)$ introduced in [9] to solve differential equations of rational order, is a particular case of the Mittag-Leffler function (1.19)

$$\varepsilon_t(v,a) = t^v \sum_{k=0}^{\infty} \frac{(at)^k}{\Gamma(v+k+1)} = t^v E_{1,v+1}(at) \qquad (1.31)$$

$\ni_\alpha (\beta,t)$ is a particular case of the Mittag-Leffler function [10] in two parameter form (1.19)

$$\ni_\alpha (\beta,t) = t^\alpha \sum_{k=0}^{\infty} \frac{\beta^k t^{k(\alpha+1)}}{\Gamma((k+1)(1+\alpha))} = t^\alpha E_{\alpha+1,\alpha+1}(\beta t^{\alpha+1}) \qquad (1.32)$$

From Eqn (1.29) and (1.30) then the properties of the Miller-Ross function and Rabotnov's function can be deduced from the properties of the Mittag-Leffler function in two parameters (1.19) I. Podlubny [7]-[8]. Plotnikov [11, cf [12]] and Tseytlin [12] used two different functions $Sc_\alpha(z)$ and $Cs_\alpha(z)$ that they define the fractional sine and cosine functions. Those functions has two different parameters I. Podlubny [7]-[8]:

$$Sc_\alpha(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{(2-\alpha)n+1}}{\Gamma((2-\alpha)n+2)} = zE_{2-\alpha,2}(-z^{2-\alpha}) \qquad (1.33)$$

$$Cs_\alpha(z) = \sum_{n=0}^{\infty} \frac{(-1)^n k^{(2-\alpha)n}}{\Gamma((2-\alpha)n+1)} = E_{2-\alpha,1}(-z^{2-\alpha}) \qquad (1.34)$$

Another "fractionalization" of the sine and cosine functions, which can also be expressed in terms of the Mittag-Leffler function [3]-[7]-[8] is recommended by Luchko and Srivastava [13]:

$$sin_{\lambda,\mu}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k k^{2k+1}}{\Gamma(2\mu k + 2\mu - \lambda + 1)} = zE_{2\mu,2\mu-\lambda+1}(-z^2) \qquad (1.34)$$

$$cos_{\lambda,\mu}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k k^{2k}}{\Gamma(2\mu k + \mu - \lambda + 1)} = E_{2\mu,\mu-\lambda+1}(-z^2) \qquad (1.35)$$

Sine and cosine functions were followed from the properties of the Mittag-Leffler function equation (1.19). Generalizations of the Mittag-Leffler function (1.19) to two variables suggested by P. Humbert and P. Delerue [14] and by A. M. Chak [15], were further extended by H.M. Srivastava [16] to the following symmetric form [7]:

$$\xi_{\alpha,\beta,\lambda,\mu}^{\nu,\sigma}(x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{x^{m+\frac{\beta(\nu n+1)-1}{\alpha}} y^{n+\frac{\mu(\sigma m+1)-1}{\lambda}}}{\Gamma(m\alpha + (\nu n+1)\beta)\Gamma(n\lambda + (\sigma m+1)\mu)} \qquad (1.36)$$

Now, several variables have been recommended by S. B. Hadid and Yu. Luchko [17] and operation method will used to solve linear fractional differential equations and also we need to use constant coefficients to show that this equation (1.35):

$$E_{(\alpha_1,\dots,\alpha_m),\beta(z_1,\dots,z_m)}, \qquad (1.37)$$

where $(k;\ I_1,\dots,I_m)$ can be defined as multinomial coefficients [18].

4. Integration of the Mittag-Leffler Function

Eqn (1.19), step-by-step the formulation we satisfy the Mittag-Leffler function in the integration part is in [3]-[5]:

$$\int_0^z E_{\alpha,\beta}(\lambda t^\alpha) t^{\beta-1} dt = z^\beta E_{\alpha,\beta+1}(\lambda z^\alpha) \qquad (\beta > 0) \qquad (1.38)$$

we use special function to obtain fractional-order-term-by-term integration of the series (1.19):

$$\frac{1}{\Gamma(\mathrm{v})}\int_0^z (z-t)^{v-1} E_{\alpha,\beta}(\lambda t^\alpha) t^{\beta-1} dt = z^{\beta+v-1} E_{\alpha,\beta+v}(\lambda z^\alpha) \quad (\beta > 0, v > 0) \quad (1.39)$$

Initially in this Mittag-Leffler function, we gave many basic formulas of series expansion in definition of one-parameter and two-parameter Mittag-Leffler function. After, if these formulas are used for defining integration of the Mittag-Leffler function then:

$$\frac{1}{\Gamma(\alpha)}\int_0^z (z-t)^{\alpha-1}e^{\lambda t}dt = z^\alpha E_{1,\alpha+1}(\lambda z), \quad (\alpha>0) \quad (1.40)$$

$$\frac{1}{\Gamma(\alpha)}\int_0^z (z-t)^{\alpha-1}\cosh\left(\sqrt{\lambda}t\right)dt = z^\alpha E_{2,\alpha+1}(\lambda z^2), \quad (\alpha>0) \quad (1.41)$$

$$\frac{1}{\Gamma(\alpha)}\int_0^z (z-t)^{\alpha-1}\frac{\sinh\left(\sqrt{\lambda}t\right)}{\sqrt{\lambda}t}dt = z^{\alpha+1} E_{2,\alpha+2}(\lambda z^2), \quad (\alpha>0) \quad (1.42)$$

Now, we show that for the fractional integration of the Mittag-Leffler function:

$$\frac{1}{\Gamma(\alpha)}\int_0^z (z-t)^{\alpha-1}E_{2\alpha,\beta}(t^{2\alpha})t^{\beta-1}dt = -z^{\beta-1}E_{2\alpha,\beta}(z^{2\alpha}) + z^{\beta-1}E_{\alpha,\beta}(z^\alpha) \quad (1.43)$$

Proof: When we prove that (1.43) then the fractional integral is:

$$\int_0^z E_{2\alpha,\beta}(t)^{2\alpha}t^{\beta-1}\left\{1+\frac{(z-t)^\alpha}{\Gamma(1+\alpha)}\right\}dt$$

$$= \sum_{k=0}^{\infty}\frac{1}{\Gamma(2k\alpha+\beta)}\int_0^z t^{2k\alpha+\beta-1}\left\{1+\frac{(z-t)^\alpha}{\Gamma(1+\alpha)}\right\}dt$$

$$= z^\beta\sum_{k=0}^{\infty}\frac{z^{2k\alpha}}{\Gamma(2k\alpha+\beta+1)} + z^\beta\sum_{k=0}^{\infty}\frac{z^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+\beta+1)}$$

$$= z^\beta\sum_{k=0}^{\infty}\frac{z^{k\alpha}}{\Gamma(k\alpha+\beta+1)} = z^\beta E_{\alpha,\beta+1}(z^\alpha) \quad (1.44)$$

When we compare Eqn(1.44) and Eqn(1.38) then we have:

$$\int_0^z E_{2\alpha,\beta}(t)^{2\alpha}t^{\beta-1}\left\{1+\frac{(z-t)^\alpha}{\Gamma(1+\alpha)}\right\}dt = \int_0^z E_{\alpha,\beta}(\lambda t^\alpha)t^{\beta-1}dt, \quad (\beta>0) \quad (1.45)$$

Take the differentiate Eqn(1.45) with respect to z variable, then

$$= z^{\beta}\sum_{k=0}^{\infty}\frac{z^{k\alpha}}{\Gamma(k\alpha+\beta+1)} = z^{\beta}E_{\alpha,\beta+1}(z^{\alpha}) \qquad (1.46)$$

is satisfied. Mittag-Leffler function that is similar to the Cristoffel-Darboux formula for describing orthogonal polynomials; definitely,

$$\int_0^t \tau^{\gamma-1}E_{\alpha,\beta}(y\tau^{\alpha})(t-\tau)^{\beta-1}E_{\alpha,\beta}(z(t-\tau)^{\alpha})d\tau = \frac{yE_{\alpha,\gamma+\beta}(yt^{\alpha}) - zE_{\alpha,\gamma+\beta}(zt^{\alpha})}{y-z}t^{\gamma+\beta-1}$$

$$(\gamma > 0, \beta > 0) \qquad (1.47)$$

where y and z $(y \neq z)$ are arbitrary complex numbers.

Actually, definition of the Mittag-Leffler function was used in Eqn (1.19). Hence, we have;

$$\int_0^t \tau^{\gamma-1}E_{\alpha,\beta}(y\tau^{\alpha})(t-\tau)^{\beta-1}E_{\alpha,\beta}(z(t-\tau)^{\alpha})d\tau$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{y^n z^m}{\Gamma(\alpha n+\gamma)\Gamma(\alpha m+\beta)}\int_0^t \tau^{\alpha n+\gamma-1}(t-\tau)^{\alpha m+\beta-1}d\tau$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{y^n z^m t^{\alpha(n+m)+\beta+\gamma-1}}{\Gamma(\alpha(n+m)+\beta+\gamma)}$$

$$= t^{\beta+\gamma-1}\sum_{n=0}^{\infty}\sum_{k=n}^{\infty}\frac{y^n z^{k-n} t^{\alpha k}}{\Gamma(\alpha k+\beta+\gamma)}$$

$$= t^{\beta+\gamma-1}\sum_{k=0}^{\infty}\frac{z^k t^{\alpha k}}{\Gamma(\alpha k+\beta+\gamma)}\sum_{n=0}^{k}\left(\frac{y}{z}\right)^n$$

$$= \frac{t^{\beta+\gamma-1}}{y-z}\sum_{k=0}^{\infty}\frac{t^{\alpha k}(y^{k+1}-z^{k+1})}{\Gamma(\alpha k+\beta+\gamma)} \qquad (1.48)$$

we used Eqn (1.19) and satisfying Eqn (1.48). Another interesting formula establishes the relationship between the Mittag-Leffler function and the function $e^{-\frac{x^2}{4t}}$ [7]. Also, we can apply relationship between the

Mittag-Leffler function and $e^{-\frac{x^2}{4t}}$. In the solution of the diffusion (heat conduction, mass transfer and another physical problems). As a result of this information the equation is satisfied:

$$\int_0^\infty e^{-x^2/4t} E_{\alpha,\beta}(x^\alpha) x^{\beta-1} dx = \sqrt{\pi} t^{\beta/2} E_{\alpha/2,(\beta+1)/2}\left(t^{\alpha/2}\right), \qquad (\beta > 0, \qquad t > 0) \qquad (1.49)$$

To proof of the Eqn (1.49) mention about every fixed value of t the below the series in (1.50):

$$e^{-x^2/4t} E_{\alpha,\beta}(x^\alpha) x^{\beta-1} = \sum_{k=0}^{\infty} \frac{x^{\alpha k+\beta-1}}{\Gamma(\alpha k+\beta)} e^{-x^2/4t}, \qquad (\beta > 0) \qquad (1.50)$$

From the (1.50) improper integral is used to describe the Legendre formula:

$$\int_0^\infty e^{-x^2/4t} E_{\alpha,\beta}(x^\alpha) x^{\beta-1} dx = \int_0^\infty \left(\sum_{k=0}^{\infty} \frac{x^{\alpha k+\beta-1}}{\Gamma(\alpha k+\beta)} e^{-x^2/4t} \right) dx$$

$$= \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k+\beta)} \int_0^\infty x^{\alpha k+\beta-1} e^{-x^2/4t} dx = \sum_{k=0}^{\infty} \frac{\Gamma(\frac{\alpha k+\beta}{2})}{2\Gamma(\alpha k+\beta)} \left(2\sqrt{t}\right)^{\alpha k+\beta} \qquad (1.51)$$

$$\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = \sqrt{\pi} 2^{1-2z} \Gamma(2z)$$

is obtained from Eqn (1.51) and it gives formula (1.49).

The use of the Laplace transform of the Mittag-Leffler function and $t^{\alpha k+\beta-1} E_{\alpha,\beta}^{(k)}(\pm z t^\alpha)$ and

$$E_{\alpha,\beta}^{(k)} \equiv \frac{d^k}{dy^k} E_{\alpha,\beta}(y)):$$

$$\int_0^\infty e^{-pt} t^{\alpha k+\beta-1} E_{\alpha,\beta}^{(k)}(\pm \alpha t^\alpha) dt = \frac{k!\, p^{\alpha-\beta}}{(p^\alpha \mp \alpha)^{k+1}}, \qquad (Re(p) > |\alpha|^{1/\alpha}) \qquad (1.52)$$

Is also a convenient way for satisfying various useful relationships for the Mittag-Leffler function [7]. To illustrate, s defined Laplace transform parameter and s is a frequence function to solve ordinary differential equation and their applications. Now, below the equation comes from Laplace transformation:

$$\frac{1}{s^2} = \frac{s^{\alpha-\beta}}{s^\alpha - 1}\left[s^{\beta-2} - s^{\beta-\alpha-2}\right] \qquad (1.53)$$

t^v is defined as Laplace transform of the function and this function will used after Eqn (1.53) for describing the Mittag-Leffler function. [3, formula 4.3(1)]

$$f(t) = t^v, \qquad Re(v) > -1, \qquad \Gamma(v+1)p^{-v-1}, \qquad Re(p) > 0$$

$$L\{t^v; s\} = \Gamma(v+1)s^{-v-1}, \qquad (Res(s) > 0) \qquad (1.54)$$

then

$$\int_0^t \tau^{\beta-1} E_{\alpha,\beta}(\tau^\alpha) \left[\frac{(t-\tau)^{1-\beta}}{\Gamma(2-\beta)} - \frac{(t-\tau)^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)}\right] d\tau = t, \qquad (0 < \beta < 2, \alpha > 0) \qquad (1.55)$$

Since the fractional integration of the Mittag-Leffler function the formula (1.43) be able to satisfied directly by the inverse Laplace transform of the identity

$$\frac{s^{2\alpha-\beta}}{s^{2\alpha}-1} s^{-\alpha} = -\frac{s^{2\alpha-\beta}}{s^{2\alpha}-1} + \frac{s^{\alpha-\beta}}{s^{\alpha}-1} \qquad (1.56)$$

The formula (1.49) is able to satisfied under favour of the Laplace transform technique.Actually, if $F(s)$ indicates the Laplace transform of a function $f(t)$, i.e.

$$F(s) = L\{f(t); s\} = \int_0^\infty e^{-st} f(t) dt,$$

then by the [3, formula 4.1(33)]

$$f(t) = t^{-1/2} \int_0^\infty e^{-1/4 u^2/t} f(u) du$$

and

$$g(p) = \int_0^\infty e^{-pt} f(t) dt$$

then the integral means that $\pi^{1/2} p^{-1/2} g\left(p^{1/2}\right)$

$$L\left\{\frac{1}{\sqrt{\pi t}} \int_0^\infty e^{-x^2/4t} f(x) dx; s\right\} = s^{-1/2} F\left(s^{1/2}\right) \qquad (1.57)$$

Now, put in (1.57)

$$f(x) = x^{\beta-1}E_{\alpha,\beta}(x^{\alpha}) \qquad (1.58)$$

According to Eqn.(1.52) we have:

$$F(s) = \frac{s^{\alpha-\beta}}{s^{\alpha}-1}$$

Hence,

$$s^{-1/2}F\left(s^{1/2}\right) = \frac{s^{\alpha/2-(\beta+1)/2}}{s^{\alpha/2}-1} = L\left\{t^{\frac{\beta+1}{2}-1}E_{\alpha,\beta}\left(t^{\alpha/2}\right); s\right\} \qquad (1.59)$$

when Eqn.(1.57) and (1.59) are compared then we attain at the relationship (1.49). At the same time, after Laplace transform of the Mittag-Leffler function (1.52) are used, then

$$\frac{s^{\alpha-\beta}}{s^{\alpha}-\alpha}.\frac{s^{\alpha-\gamma}}{s^{\alpha}+\alpha} = \frac{s^{2\alpha-(\beta+\gamma)}}{s^{2\alpha}-\alpha^{2}} \qquad (1.60)$$

we satisfy the convolution of two Mittag-Leffler functions:

$$\int_0^t \tau^{\beta-1}E_{\alpha,\beta}(\alpha\tau^{\alpha})(t-\tau)^{\gamma-1}E_{\alpha,\gamma}(-\alpha(t-\tau)^{\alpha})d\tau = t^{\beta+\gamma-1}E_{2\alpha,\beta+\gamma}(\alpha^2 t^{2\alpha}) \quad (\beta > 0,\ \gamma > 0) \quad (1.61)$$

The relationship (1.61) can also be obtained from (1.47), where we can take $z = -y$ and then utilize the relationship [7[-[8]:

$$E_{\alpha,\beta}(z) + E_{\alpha,\beta}(-z) = 2E_{\alpha,\beta}(z^2)$$

5. Using Special Functions on Grünwald-Letnikov and Riemann-Liouville fractional derivatives and fractional integrals

Initially, this definition comes from the normal definition of derivative: we consider $y = q(t)$ function and q is continuous every point. Also, q is differentiable function in order to it is continuous at $t \in R$. Now, we use limit definition previously:

$$q'(t) = \frac{dq}{dt} = \lim_{h\to 0}\frac{q(t+h)-q(t)}{h} \qquad (1.37)$$

Take the derivative of $q'(t)$

$$q''(t) = \frac{d^2q}{dt^2} = \lim_{h\to 0}\frac{1}{h}(q(t+h) - q(t))'$$

$$q''(t) = \lim_{h\to 0}\frac{1}{h}\left(\frac{\big(q(t+2h) - q(t+h)\big) - (q(t+h) - q(t))}{h}\right)$$

$$q''(t) = \lim_{h\to 0}\frac{1}{h}\left(\frac{q(t+2h) - q(t+h) - q(t+h) + q(t)}{h}\right)$$

$$q''(t) = \lim_{h\to 0}\frac{q(t+2h) - 2q(t+h) + q(t)}{h^2} \qquad (1.38)$$

$$q'''(t) = \lim_{h\to 0}\frac{1}{h^2}(q(t+2h) - 2q(t+h) + q(t))'$$

$$q'''(t) = \lim_{h\to 0}\frac{1}{h^2}\left[\left(\frac{q(t+3h) - q(t+2h)}{h}\right) - 2\left(\frac{q(t+2h) - q(t+h)}{h}\right) + \left(\frac{q(t+h) - q(t)}{h}\right)\right]$$

$$q'''(t) = \lim_{h\to 0}\frac{1}{h^2}\left(\frac{q(t+3h) - q(t+2h) - 2q(t+2h) + 2q(t+h) + q(t+h) - q(t)}{h}\right)$$

$$q'''(t) = \lim_{h\to 0}\frac{q(t+3h) - 3q(t+2h) + 3q(t+h) - q(t)}{h^3} \qquad (1.39)$$

and, use by induction result of $q'(t), q''(t)$ and $q'''(t)$:

$$q^n(t) = \frac{d^nq}{dt^n} = \lim_{h\to 0}\frac{1}{h^n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}q(t-kh) \qquad (1.40)$$

and we define as; fractional-order derivative (1.40), we write as;

$$\binom{n}{k} = \frac{n(n-1)(n-2)\ldots(n-k+1)}{k!}$$

and therefore,

$$q^{(n)}(t) = \frac{d^nq}{dt^n} = \lim_{h\to 0}\frac{1}{h^n}\sum_{k=0}^{n}(-1)^k\frac{n(n-1)(n-2)\ldots(n-k+1)}{k!}q(t-kh) \qquad (1.41)$$

if and only if the fractional order derivative of n is positive. However, when we define this order is negative: then; we define as;

$$q^{(-n)}(t) = \lim_{h \to 0} \frac{1}{h^{-n}} \sum_{k=0}^{n} (-1)^k \binom{-n}{k} q(t - kh) \qquad (1.42)$$

and

$$\binom{-n}{k} = (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}$$

so we say that,

$$q^{(-n)}(t) = \lim_{h \to 0} h^n \sum_{k=0}^{n} (-1)^{2k} \begin{bmatrix} n \\ k \end{bmatrix} q(t - kh) = \lim_{h \to 0} h^n \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q(t - kh) \qquad (1.43)$$

By the Grünwald-Letnikov fractional-order derivatives equation:

$${}_{\alpha}D_t^n q(t) = \lim_{h \to 0 \text{ } and \text{ } nh = t - \alpha} q_h^n(t) = \lim_{h \to 0} \frac{1}{h^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} q(t - kh) = \frac{1}{\Gamma(n)} \int_{\alpha}^{t} (t - \tau)^{n-1} q(\tau) d\tau \qquad (1.44)$$

Especially, special function is used in the Grünwald-Letnikov definition;

Eqn (1.44) can be shown another form:

$${}_{\alpha}D_t^n q(t) = \frac{1}{(n-1)!} \int_{\alpha}^{t} (t - \tau)^{n-1} q(\tau) d\tau$$

since $\Gamma(n) = (n-1)!$ and $n > 0$.

Also, we can give an example of fractional order derivative using with Matlab.

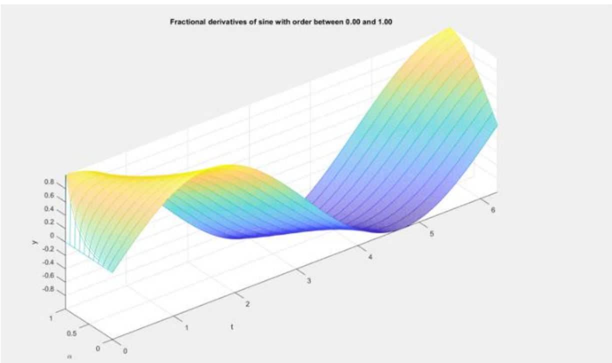

Figure 2: $D^{\alpha}\sin(t) = f(t)$ and we consider that order between 0.00 and 1.00 is an example of fractional-order Grünwald-Letnikov derivative:

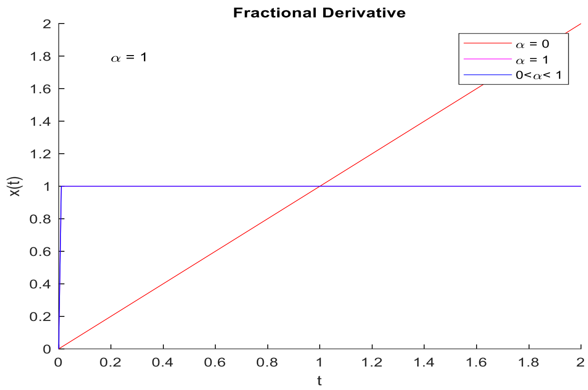

Figure 3: $D^{\alpha}x(t) = f(t)$ where $0 < \alpha < 1$. However, blue diagram is defined when $0 < \alpha < 1$, red diagram is defined when $\alpha = 0$ and pink line is defined as vertical line when $\alpha = 1$. We say that, curves position's are changing according to the fractional derivative's values.

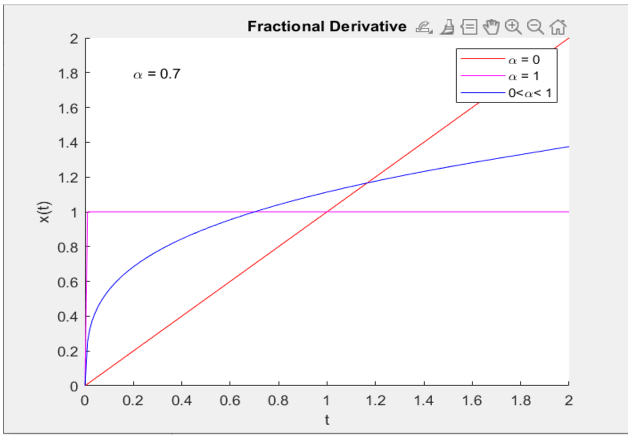

Figure 4: $D^{\alpha}x(t) = f(t)$ where α' values decrease from 1 to 0.7.

Integrating the relationship:

If we use Grünwald-Letnikov definition with special functions:

$$\frac{d}{dt}\left({}_{\alpha}D_t^{-n}q(t)\right) = \frac{1}{(n-2)!}\int_{\alpha}^{t}(t-\tau)^{n-2}q(\tau)d\tau = {}_{\alpha}D_t^{-n+1}q(t) \qquad (1.45)$$

from α to t we satisfy in the general form:

$$ {}_{\alpha}D_t^{-n}q(t) = \int_{\alpha}^{t} {}_{\alpha}D_t^{-n+1}q(t)dt, $$

$$ {}_{\alpha}D_t^{-n+1}q(t) = \int_{\alpha}^{t} \left({}_{\alpha}D_t^{-n+2} \right) dt, $$

and so,

$$ {}_{\alpha}D_t^{-n}q(t) = \int_{\alpha}^{t} dt \int_{\alpha}^{t} \left({}_{\alpha}D_t^{-n+2}q(t) \right) dt $$

$$ = \int_{\alpha}^{t} dt \int_{\alpha}^{t} dt \int_{\alpha}^{t} {}_{\alpha}D_t^{-n+3}q(t)dt $$

$$ \int_{\alpha}^{t} dt \int_{\alpha}^{t} dt \dots \int_{\alpha}^{t} q(t)dt \quad n - times \quad (1.46) $$

In brief, we say that, the function $q(t)$ is continuous and if the fractional-order derivative is n so it is the general formulation:

$$ {}_{\alpha}D_t^{n}q(t) = \lim_{h\to 0 \; and \; nh = t-\alpha} h^{-n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} q(t - kh) \quad n > 0 \quad (1.47) $$

Also, we apply special functions on Grünwald-Letnikov definition in composition with integer-order derivatives:

$$ {}_{\alpha}D_t^{n}q(t) = \sum_{k=0}^{s} \frac{q^{(k)}(\alpha)(t-\alpha)^{-n+k}}{\Gamma(-n+k+1)} + \frac{1}{\Gamma(-n+s+1)} \int_{\alpha}^{t} (t-\tau)^{s-n} q^{(s+1)}(\tau)d\tau \qquad (1.48) $$

Similarly, special function is applied in this section. Otherwise, Eqn (1.48) is written another formulation on this;

$$ {}_{\alpha}D_t^{n}q(t) = \sum_{k=0}^{s} \frac{q^{(k)}(\alpha)(t-\alpha)^{-n+k}}{(k-n)!} + \frac{1}{(s-n)!} \int_{\alpha}^{t} (t-\tau)^{s-n} q^{(s+1)}(\tau)d\tau \qquad (1.49) $$

Now, when we look at Riemann-Liouville definition then we say it is the most important information that function is continuous and we know that $(m+1)$ exponensial derivative operator will used in Riemann-Liouville definition: Similarly, we need to use special functions to determine series expansion form. Therefore, the gamma function has play important role to show step-by-step. General definition is:

$$ {}_{\alpha}D_t^{n}q(t) = \left(\frac{d}{dt} \right)^{c+1} \int_{\alpha}^{t} (t-\tau)^{c-n} q(\tau)d\tau \quad (c \le n < c+1) \quad (1.50) $$

Particularly,

$$ {}_{\alpha}D_t^{n}q(t) = \lim_{h\to\infty} q_h^{(n)}(t) = \sum_{k=0}^{m} \frac{q^{(k)}(\alpha)(t-\alpha)^{-n+k}}{\Gamma(-n+k+1)} + \frac{1}{\Gamma(-n+m+1)} \int_{\alpha}^{t} (t-\tau)^{m-n} q^{(m+1)}(\tau)d\tau \qquad (1.51) $$

and we say that, when h approaches ∞ then $nh = t - \alpha$ in Eqn (1.51). Also, this definition is satisfying to the Grünwald-Letnikov fractional derivative under the estimate, $q(t)$ function must be $c + 1$ times continuously differentiable [7]-[8]: so Eqn (1.51) was applied to use integration by parts and differentiation in this definition.

6. Conclusion

In this thesis, initially, many special functions (gamma [1]-[3]-[4]-[7], beta [2]-[6],integral function [5], Mittag-Leffler functions and Integration of the Mittag-Leffler function [3])and their properties are mentioned. As a reason of , special function must be used to apply on definition of Riemann-Liouville and Grünwald-Letnikov fractional derivatives and fractional integrals [7]-[8]. Specifically, gamma function diagram is identified on cartesian coordinate system. Second of all, limit definitions of fractional derivative are mentioned on our thesis. Also, in some parts, Matlab is used for interpreting geometrical meaning of fractional order derivatives equation for some values of order and we compare curves' positions according to order of fractional derivatives' values. Finally, Riemann-Liouville and Grünwald Letnikov fractional derivatives and fractional integrals [7]-[8] are given and gamma function is applied both of two definitions.

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Wavelet Spectral method for Fractional Jaulent–Miodek equation associated with energy-dependent Schrödinger potential

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Abstract

The coupled J-M equations are arising in different fields of science and engineering such as fluid mechanics, condense matter physics, optics, and plasma physics. In this study, a numerical approach is introduced for investigating the numerical solutions of the fractional order of coupled coupled J-M system. For this purpose, the fractional Bernoulli wavelets via spectral collocation method are employed to reducing the system of fractional order differential equations to some nonlinear system of algebraic equations. Finally, we utilize Newton iterative approach for solving the achieved nonlinear system. For showing the accuracy and efficiency of the purposed method, the solutions of some cases of coupled J-M system are provided.

Keywords: Jaulent–Miodek system**,** Bernoulli wavelets, collocation method, Newton iterative method.

1. Introduction

In 1979, Jaulent and Miodek derived the system of equations as an extension to energy-dependent potentials called Jaulent-Miodek (J-M) system [1-2]. The coupled J-M equations are arising in different fields of science and engineering such as fluid mechanics [3], condense matter physics [4], optics [5], and plasma physics [6]. In this research the fractional Bernoulli wavelets are employed for investigating the numerical solution of the fractional order of coupled J-M system as;

$$\begin{cases} D_t^{\gamma_1} u + u_{xxx} + \dfrac{3}{2} v\, v_{xxx} + \dfrac{9}{2} v_x v_{xx} - 6uu_x - 6uvv_x - \dfrac{3}{2} u_x v^2 v = 0, \\ D_t^{\gamma_2} v + v_{xxx} - 6u_x v - 6uv_x - \dfrac{15}{2} v_x v^2 = 0, \end{cases} \tag{1}$$

subject to the initial conditions

$$u(x,0) = \tfrac{1}{8}\mu^2 \left(1 - 4\,\mathrm{sech}^2(\tfrac{\mu x}{2})\right), \qquad v(x,0) = \mu\,\mathrm{sech}\left(\tfrac{\mu x}{2}\right), \tag{2}$$

where $\gamma_i, i = 1,2$ are the order of fractional derivatives in Caputo sense and μ is an arbitrary constant [7]. For $\gamma_1 = \gamma_2 = 1$, the coupled system (1) has the exact solution in the form:

$$u(x,t)=\frac{1}{8}\mu^2\left(1-4\,sech^2\left(\frac{\mu}{2}(x+\frac{1}{2}\mu^2 t)\right)\right),$$
$$v(x,t)=\mu\,\mathrm{sech}\left(\frac{\mu}{2}(x+\frac{1}{2}\mu^2 t)\right).$$

In recent years, many numerical methods have been purposed for solving the classical Jaulent–Miodek equation; such as unified algebraic method [8], Adomian decomposition method [9], tanh-sech method [10], homotopy perturbation method [11], Exp-function method [12], and homotopy analysis method [13]. But in keeping with the available information, there are a few papers which deals with the nonlinear fractional order coupled Jaulent-Miodek equation.

In this paper, the fractional Bernoulli wavelets and their operational matrix of derivative are employed for solving nonlinear system (1). For this purpose first, the unknown functions and all the nonlinear terms of (1) are expanded in fractional Bernoulli wavelets terms, then by applying the fractional Riemann-Liouville integration operator (of orders $\gamma_i, i=1,2$), we derive a nonlinear algebraic system of equations. This system is discretized via spectral collocation method, and Newton iterative method is utilized for solving the achieved nonlinear system.

The organization of the paper is the following. Initially, preliminaries about farctional derivetive and integration operators are given. Then the definitions of Bernoulli and fractional Bernoulli wavelets, their opeartional matrix of derivative are brefly presented. Afterward, the numerical implementation is introduced. Accordingly, numerical results of the problem are given by figures. The paper finalizes with the concluding remarks and brief discussion of results.

2. Preliminaries on fractional calculus

In this section, we present some basic definitions and concepts of fractional calculus, that are essential for subsequent discussion. There are various definitions for fractional integration and derivative operators. However, the fractional Riemann-Liouville integration and fractional Caputo derivative operators have been used in this study [14].

Definition 2.1. The Riemann-Liouville fractional integral operator of nonnegative order α is defined as:

$$J^\alpha f(x)=\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\ f(t)dt,\qquad x>0,$$

where $J^0 f(x)=f(x)$. The Riemann-Liouville fractional integrals for the polynomials are defined as

$$J^{\alpha}\{x^{\beta}\} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} x^{\beta+\alpha}, \qquad \beta > -1. \tag{3}$$

Definition *2.2*. The Caputo fractional derivative operator of nonnegative order α is defined as:

$$_{c}D^{\alpha}\{f(x)\} = \frac{1}{\Gamma(n-\alpha)} \int_{0}^{x} \frac{f^{n}(t)}{(x-t)^{\alpha+1-n}} dt, \qquad n-1 < \alpha \le n, \quad n \in \mathbb{N}.$$

For the Caputo derivative we have

$$_{c}D^{\alpha}\{x^{\beta}\} = 0, \qquad \beta \in n \in \mathbb{N}, \quad \beta < \lceil \alpha \rceil,$$

and

$$_{c}D^{\alpha}\{x^{\beta}\} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} x^{\beta-\alpha}, \qquad \beta \in \mathbb{N}, \qquad \beta \ge \lceil \alpha \rceil \;\; or \;\; \beta \in \mathbb{R}-\mathbb{N}, \quad \beta > \lfloor \alpha \rfloor. \tag{4}$$

The relations between Reimann-Liouville fractional integral and Caputo fractional derivative operators can be addressed by the following identities [15]:

$$_{c}D^{\alpha}\{J^{\alpha}f(x)\} = f(x), \qquad J^{\alpha}\{\ {}_{c}D^{\alpha}f(x)\} = f(x) - \sum_{j=0}^{n-1} \frac{f^{(j)}(0)}{j!} x^{j}. \tag{5}$$

3. Review on Bernoulli wavelets

In this section, definitions of Bernoulli Wavelets (BWs) and Fractional Bernoulli Wavelets (FBWs) and their operational matrix of derivative are described.

Definition 3.1. BWs of order m, which are denoted by $\psi_{nm}(t) = \psi(k,\hat{n},m,t)$, consist of four arguments, k: a positive integer, $n = 1,2,\dots,2^{k-1}$, $\hat{n} = n-1$, and t is the normalized time. These wavelets are defined on the interval [0,1) as [16]:

$$\psi_{n,m}(t) = 2^{\frac{k-1}{2}} \tilde{B}_{m}(2^{k-1}t-\hat{n}) \chi_{\left[\frac{\hat{n}}{2^{k-1}},\frac{\hat{n}+1}{2^{k-1}}\right)},$$

where $\tilde{B}_{m}(0) = 1$ and

$$\tilde{B}_{m}(2^{k-1}t-\hat{n}) = \frac{B_{m}(t)}{\Lambda_{m}}, \qquad \Lambda_{m} = \sqrt{\frac{(-1)^{m-1}(m!)^{2}}{(2m)!} \vartheta_{2m}},$$

Also the functions $B_{m},\ m = 0,1,\dots,M-1$ are known Bernoulli polynomials, defined as

$$B_m(t)=\sum_{j=0}^{m}\binom{m}{j}\vartheta_{m-j}\,t^j.$$

where $\vartheta_j := B_j(0)$ are the Bernoulli numbers. Therefore Bernoulli wavelets for $m>0$ can be rewritten as

$$\psi_{n,m}(t)=\Theta_m\sum_{j=0}^{m}\binom{m}{j}\vartheta_{m-j}\,2^{j(k-1)}\left(t-\frac{\hat{n}}{2^{k-1}}\right)^j\chi_{\left[\frac{\hat{n}}{2^{k-1}},\frac{\hat{n}+1}{2^{k-1}}\right)},$$

where $\Theta_m=\sqrt{\frac{2m)!\,2^{k-1}}{(-1)^{m-1}\,(m!)^2\vartheta_m}}$ and $\psi_{n,0}(t)=2^{\frac{k-1}{2}}\chi_{\left[\frac{\hat{n}}{2^{k-1}},\frac{\hat{n}+1}{2^{k-1}}\right)}$.

Definition 3.2. Fractional Bernoulli Wavelets are denoted by $\psi_{n,m}^{\alpha}$ and constructed by changing the variable t to x^{α}, $(\alpha>0)$ on the BWs [16], that is:

$$\psi_{n,m}^{\alpha}(x)\coloneqq\psi_{n,m}(x^{\alpha})=\Theta_m\sum_{j=0}^{m}\binom{m}{j}\vartheta_{m-j}\,2^{j(k-1)}\left(x^{\alpha}-\frac{\hat{n}}{2^{k-1}}\right)^j\chi_{\left[\left(\frac{\hat{n}}{2^{k-1}}\right)^{1/\alpha},\left(\frac{\hat{n}+1}{2^{k-1}}\right)^{1/\alpha}\right)}.$$

3.1. Function approximation by FBW

A function $f\in L^2[0,1]$ could be approximated by FBWs, as

$$f(x)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}c_{n,m}\psi_{n,m}^{\alpha}(x), \tag{6}$$

by truncating the infinite series (6) in some suitable k and M, we get

$$f(x)\simeq\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{n,m}\psi_{n,m}^{\alpha}(x)=C_M^T\Psi_{k,M}^{\alpha}(x), \tag{7}$$

where C_M and $\Psi_{k,M}^{\alpha}$ are $2^{k-1}\times M$-dimensional column vectors and defined as

$$C_M=\left(c_{1,0},\dots,c_{1,M-1},\dots,c_{2^{k-1},0},\dots,c_{2^{k-1},M-1}\right)^T, \tag{8}$$

$$\Psi_{k,M}^{\alpha}=\left(\psi_{1,0}^{\alpha},\dots,\psi_{1,M-1}^{\alpha},\dots,\psi_{2^{k-1},0}^{\alpha},\dots,\psi_{2^{k-1},M-1}^{\alpha}\right)^T. \tag{9}$$

In order to determine the coefficients in (7), we put

$$\eta_{ij}\coloneqq\int_{\sqrt[\alpha]{\frac{\hat{n}}{2^{k-1}}}}^{\sqrt[\alpha]{\frac{\hat{n}+1}{2^{k-1}}}}f(x)\psi_{i,j}^{\alpha}(x)x^{\alpha-1}dx, \tag{10}$$

and

$$\lambda_{n,m}^{i,j} := \int_{\sqrt[\alpha]{\frac{\hat{n}}{2^{k-1}}}}^{\sqrt[\alpha]{\frac{\hat{n}+1}{2^{k-1}}}} \psi_{n,m}^{\alpha}(x)\psi_{i,j}^{\alpha}(x)x^{\alpha-1}dx. \tag{11}$$

Now substituting (7) in (10), we get

$$\eta_{ij} \simeq \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{n,m} \int_{\sqrt[\alpha]{\frac{\hat{n}}{2^{k-1}}}}^{\sqrt[\alpha]{\frac{\hat{n}+1}{2^{k-1}}}} \psi_{n,m}^{\alpha}(x)\psi_{i,j}^{\alpha}(x)x^{\alpha-1}dx = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{n,m}\ \lambda_{n,m}^{i,j} = C_M^T\Lambda_M^{i,j},$$

where

$$\Lambda_M^{i,j} = \left(\lambda_{1,0}^{i,j}, \dots, \lambda_{1,M-1}^{i,j}, \dots, \lambda_{2^{k-1},0}^{i,j}, \dots, \lambda_{2^{k-1},M-1}^{i,j}\right)^T,$$

so putting

$$T_M = \left(\eta_{1,0}, \dots, \eta_{1,M-1}, \dots, \eta_{2^{k-1},0}, \dots, \eta_{2^{k-1},M-1}\right)^T,$$

and

$$\Lambda_M = \left(\Lambda_M^{1,0}, \dots, \Lambda_M^{1,M-1}, \dots, \Lambda_M^{2^{k-1},0}, \dots, \Lambda_M^{2^{k-1},M-1}\right)_{(2^{k-1}\times M)\times(2^{k-1}\times M)},$$

the vector C_M is evaluated by

$$C_M^T = T_M\Lambda_M^{-1}. \tag{12}$$

$$v_{r,s,n,m} = \alpha^2\langle\langle v(x,t), \psi_{r,s}^{\alpha}(t)\rangle_{t^{\alpha-1}}, \psi_{n,m}^{\alpha}(x)\rangle_{x^{\alpha-1}}, \tag{13}$$

$$n = 1,2,\dots,2^{k_1-1}, \qquad r = 1,2,\dots,2^{k_2-1}, \qquad m = 0,1,\dots,M_1-1, \qquad s = 0,1,\dots,M_2-1.$$

The two variable function $v(x,t)$ could be approximated by two dimensional FBWs as

$$v(x,t) = \sum_{n=1}^{2^{k_2-1}}\sum_{m=0}^{M_2-1}\sum_{r=1}^{2^{k_1-1}}\sum_{s=0}^{M_1-1} v_{r,s,n,m}\ \psi_{r,s}^{\alpha}(x)\psi_{n,m}^{\alpha}(t) = \Psi_{k_1,M_1}^{\alpha}(x)V\left(\Psi_{k_1,M_1}^{\alpha}(x)\right)^T, \tag{14}$$

where V is dimensional $(2^{k_1-1}\times M_1)\times(2^{k_2-1}\times M_2)$ coefficient matrix.

It is clear that for $k_1 = k_2 = k$, $M_1 = M_2 = M$ and V is $(2^{k-1}\times M)$-dimensional square coefficient matrix.

Theorem 3.1. ([16]) Let $u(x,t) \in C^{M_1,M_2}(D)$ be approximated by two dimensional FBWs as

$$u(x,t) \simeq u_{k_1,M_1,k_2,M_2}(x,t) = \left(\Psi_{k_2,M_2}^{\alpha}\right)^T(t)V\Psi_{k_1,M_1}^{\alpha}(x),$$

there exist constants $C_i \in \mathbb{R}^+,\ i = 1,2,3$ such that

$$\left\|u(x,t)-u_{k_1,M_1,k_2,M_2}(x,t)\right\|_2 \leq \frac{C_1}{A_1}+\frac{C_2}{A_2}+\frac{C_3}{A_1A_2},$$

where $A_i = M_i!\,2^{M_i(k_i+1)-1},\ i=1,2.$

3.2. Operational matrix of derivative for FBWs

The derivative of Ψ^α can be obtained as

$$\frac{d}{dx}\Psi^\alpha(x)=\mathcal{D}\Psi^\alpha(x), \tag{20}$$

where $\mathcal{D}$ is relative operational square matrix of dimension $2^{k-1}\times M$ and could be evaluated as follows

$$\frac{d}{dx}\psi^\alpha_{n,m}(x)=\Theta_m\sum_{j=0}^{m}\binom{m}{j}\vartheta_{m-j}\,2^{j(k-1)}\frac{d}{dx}\left(x^\alpha-\frac{\hat{n}}{2^{k-1}}\right)^j, \tag{21}$$

$$\left(\frac{\hat{n}}{2^{k-1}}\right)^{1/\alpha}\leq x\leq\left(\frac{\hat{n}+1}{2^{k-1}}\right)^{1/\alpha}.$$

On the other hand

$$\frac{d}{dx}\left(x^\alpha-\frac{\hat{n}}{2^{k-1}}\right)^j=\sum_{i=0}^{j}\binom{j}{i}\left(-\frac{\hat{n}}{2^{k-1}}\right)^{j-i}\alpha i x^{\alpha i-1}. \tag{22}$$

Therefore, by using equations (21)-(22), we can write

$$\frac{d}{dx}\psi^\alpha_{n,m}(x)=\Theta_m\sum_{j=0}^{m}\sum_{i=0}^{j}B_{j,i}x^{\alpha i-1}, \tag{23}$$

where $B_{j,i}=\binom{m}{j}\vartheta_{m-j}\,2^{j(k-1)}\binom{j}{i}\left(-\frac{\hat{n}}{2^{k-1}}\right)^{j-i}\alpha i.$ Now we expand $x^{\alpha i-1}$ in terms of FBWs:

$$x^{\alpha i-1}\simeq\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}g_{n,m}\psi^\alpha_{n,m}(x), \tag{24}$$

by (23)-(24), we get

$$\frac{d}{dx}\psi^\alpha_{n,m}(x)=\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}\mu_{n,m,j,i}\psi^\alpha_{n,m}(x),$$

where $\mu_{n,m,j,i}=\Theta_m\sum_{j=0}^{m}\sum_{i=0}^{j}B_{j,i}\,g_{n,m}$. Therefore, we have

$$\mathcal{D} = \begin{pmatrix} \mathcal{D}_1 & 0 & \cdots & 0 \\ 0 & \mathcal{D}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{D}_{2^{k-1}} \end{pmatrix}_{(2^{k-1}\times M)\times(2^{k-1}\times M)}$$

where

$$\mathcal{D}_l = \left(\mu_{l,0,0,0}, \mu_{l,1,0,0}, \dots, \mu_{l,M-1,j,i}\right)_{1\times M}.$$

4. Numerical implementation

In this section we employ the fractional Bernoulli wavelets and their operational matrix of derivative for solving coupled J-M system. For this purpose, we expand the unknowns of system (1) in FBWs with unknown coefficients as:

$$u(x,t) = \left(\Psi^{\alpha}_{k_2,M_2}(t)\right)^T U\, \Psi^{\alpha}_{k_1,M_1}(x), \qquad v(x,t) = \left(\Psi^{\alpha}_{k_2,M_2}(t)\right)^T V\, \Psi^{\alpha}_{k_1,M_1}(x),$$

also the nonlinear terms of (1) are approximated by FBWS as:

$$vv_{xxx} = \Psi^{\alpha}_{k_2,M_2}(t)^T A\, \Psi^{\alpha}_{k_1,M_1}(x), \qquad v_x v_{xx} = \Psi^{\alpha}_{k_2,M_2}(t)^T B\, \Psi^{\alpha}_{k_1,M_1}(x),$$

$$uu_x = \Psi^{\alpha}_{k_2,M_2}(t)^T C\, \Psi^{\alpha}_{k_1,M_1}(x), \qquad uvv_{xx} = \Psi^{\alpha}_{k_2,M_2}(t)^T D\, \Psi^{\alpha}_{k_1,M_1}(x),$$

$$u_x v^2 v = \Psi^{\alpha}_{k_2,M_2}(t)^T E\, \Psi^{\alpha}_{k_1,M_1}(x), \qquad u_x v = \Psi^{\alpha}_{k_2,M_2}(t)^T F\, \Psi^{\alpha}_{k_1,M_1}(x),$$

$$uv_x = \Psi^{\alpha}_{k_2,M_2}(t)^T G\, \Psi^{\alpha}_{k_1,M_1}(x), \qquad v_x v^2 = \Psi^{\alpha}_{k_2,M_2}(t)^T H\, \Psi^{\alpha}_{k_1,M_1}(x), \tag{25}$$

so we have $10 \times \left(2^{k_1-1} \times M_1\right) \times \left(\left(2^{k_2-1} \times M_2\right)\right.$ unknowns in all. On the other hand, equations (25) imply that:

$$\left(\Psi^{\alpha}_{k_2,M_2}(t)^T V\, \Psi^{\alpha}_{k_1,M_1}(x)\right)\left(\Psi^{\alpha}_{k_2,M_2}(t)^T V\, \mathcal{D}^3 \Psi^{\alpha}_{k_1,M_1}(x)\right) = \Psi^{\alpha}_{k_2,M_2}(t)^T A\, \Psi^{\alpha}_{k_1,M_1}(x), \tag{26}$$

$$\left(\Psi^{\alpha}_{k_2,M_2}(t)^T V\, \mathcal{D}\Psi^{\alpha}_{k_1,M_1}(x)\right)\left(\Psi^{\alpha}_{k_2,M_2}(t)^T V\, \mathcal{D}^2 \Psi^{\alpha}_{k_1,M_1}(x)\right) = \Psi^{\alpha}_{k_2,M_2}(t)^T B\, \Psi^{\alpha}_{k_1,M_1}(x), \tag{27}$$

$$\left(\Psi^{\alpha}_{k_2,M_2}(t)^T U\Psi^{\alpha}_{k_1,M_1}(x)\right)\left(\Psi^{\alpha}_{k_2,M_2}(t)^T U\, \mathcal{D}\Psi^{\alpha}_{k_1,M_1}(x)\right) = \Psi^{\alpha}_{k_2,M_2}(t)^T C\, \Psi^{\alpha}_{k_1,M_1}(x), \tag{28}$$

$$\begin{aligned}&\left(\Psi^{\alpha}_{k_2,M_2}(t)^T U\Psi^{\alpha}_{k_1,M_1}(x)\right)\left(\Psi^{\alpha}_{k_2,M_2}(t)^T V\, \Psi^{\alpha}_{k_1,M_1}(x)\right)\left(\Psi^{\alpha}_{k_2,M_2}(t)^T V\, \mathcal{D}^2\, \Psi^{\alpha}_{k_1,M_1}(x)\right) \\ &= \Psi^{\alpha}_{k_2,M_2}(t)^T D\, \Psi^{\alpha}_{k_1,M_1}(x),\end{aligned} \tag{29}$$

$$\left(\Psi_{k_2,M_2}^{\alpha}(t)^T\, U\mathcal{D}\Psi_{k_1,M_1}^{\alpha}(x)\right)\left(\Psi_{k_2,M_2}^{\alpha}(t)^T\, V\,\Psi_{k_1,M_1}^{\alpha}(x)\right)^2\left(\Psi_{k_2,M_2}^{\alpha}(t)^T V\ \Psi_{k_1,M_1}^{\alpha}(x)\right)$$
$$= \Psi_{k_2,M_2}^{\alpha}(t)^T\, E\,\Psi_{k_1,M_1}^{\alpha}(x), \tag{30}$$

$$\left(\Psi_{k_2,M_2}^{\alpha}(t)^T\, U\,\mathcal{D}\Psi_{k_1,M_1}^{\alpha}(x)\right)\left(\Psi_{k_2,M_2}^{\alpha}(t)^T\, V\,\Psi_{k_1,M_1}^{\alpha}(x)\right) = \Psi_{k_2,M_2}^{\alpha}(t)^T\, F\,\Psi_{k_1,M_1}^{\alpha}(x), \tag{31}$$

$$\left(\Psi_{k_2,M_2}^{\alpha}(t)^T\, U\,\Psi_{k_1,M_1}^{\alpha}(x)\right)\left(\Psi_{k_2,M_2}^{\alpha}(t)^T\, V\mathcal{D}\,\Psi_{k_1,M_1}^{\alpha}(x)\right) = \Psi_{k_2,M_2}^{\alpha}(t)^T\, G\,\Psi_{k_1,M_1}^{\alpha}(x), \tag{32}$$

$$\left(\Psi_{k_2,M_2}^{\alpha}(t)^T\, V\,\mathcal{D}\Psi_{k_1,M_1}^{\alpha}(x)\right)\left(\Psi_{k_2,M_2}^{\alpha}(t)^T\, V\,\Psi_{k_1,M_1}^{\alpha}(x)\right)^2 = \Psi_{k_2,M_2}^{\alpha}(t)^T\, H\,\Psi_{k_1,M_1}^{\alpha}(x), \tag{33}$$

Now, the Reimann - Liuville fractional integral operator, $J_t^{\gamma_i}, i = 1,2,$ with respect to variable t is applied on the system (1), that is:

$$u(x,t) - u(x,0) = J^{\gamma_1}_{\ t}\left\{-u_{xxx} - \frac{3}{2}v\,v_{xxx} - \frac{9}{2}v_x v_{xx} + 6uu_x + 6uvv_x + \frac{3}{2}u_x v^2 v\right\}, \tag{34}$$
$$v(x,t) - v(x,0) = J^{\gamma_2}_{\ t}\left\{v_{xxx} - 6u_x v - 6uv_x - \frac{15}{2}v_x v^2\right\}. \tag{35}$$

By combining the equations (26)-(35), we achieve a system of 10 algebraic equations. First, we collocate the obtained system in the following meshes:

$$(x_i, t_j) = \left(a + \frac{i(b-a)}{2^{k_1-1}\times\ M_1}, \frac{jT}{2^{k_2-1}\times\ M_2}\right),$$

$$i = 1,2,\ldots,2^{k_1-1}\times\ M_1, \qquad\qquad j = 1,\ldots,2^{k_2-1}\times\ M_2.$$

Therefore we have a nonlinear system of $(10\times 2^{k_1-1}\times\ M_1)\times(2^{k_2-1}\times\ M_2)$ equations with the same number of unknowns. For solving the current system we apply Newton iterative method.

4. Results and discussion

In this section, we solve the coupled J-M system by introduced method for some fractional orders γ_1, γ_2 and parameter μ. We solved the J-M system for $\gamma_1 = \gamma_2 = 0.5, 0.7, 0.9$ and $\mu = 0.25, 0.5, 0.7$, $k_1 = k_2 = 2$ and $M_1 = M_2 = 3$ and the plots of numerical solutions are shown in figures 1-3. For solving the proposed system, Newton iterative method was used. For stopping the iterations of Newton method, N, we considered the following criteria

$$for\ \varepsilon > 0, \quad \|U_N - U_{N-1}\|_2 < \varepsilon.$$

6. Conclusion

In this paper, the fractional Bernoulli wavelets were defined in new settings and applied based on the wavelet spectral collocation approach for solving the Caputo fractional order coupled J-M system. First, the operational matrix of ordinary derivative was constructed and then employed for reducing the time fractional J-M system to an algebraic nonlinear system. The method is simple, attractive, applicable and can be extended for high-order fractional nonlinear partial differential equations.

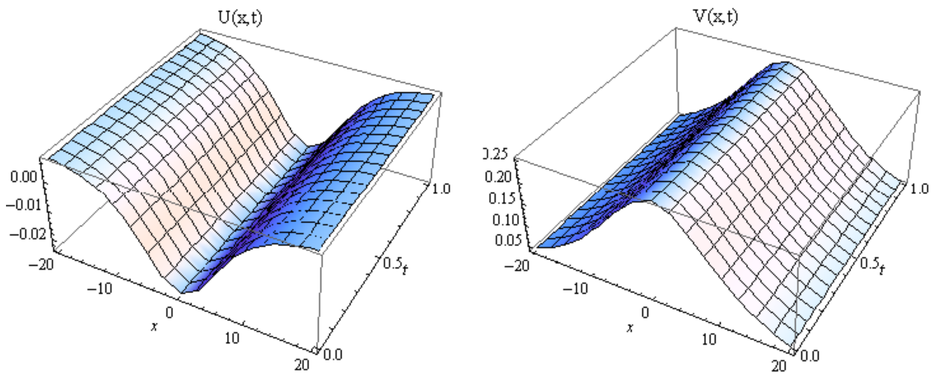

Figure 1. Numerical solution of the coupled J-M system for $\gamma_1 = \gamma_2 = 0.5$ and $\mu = 0.25$.

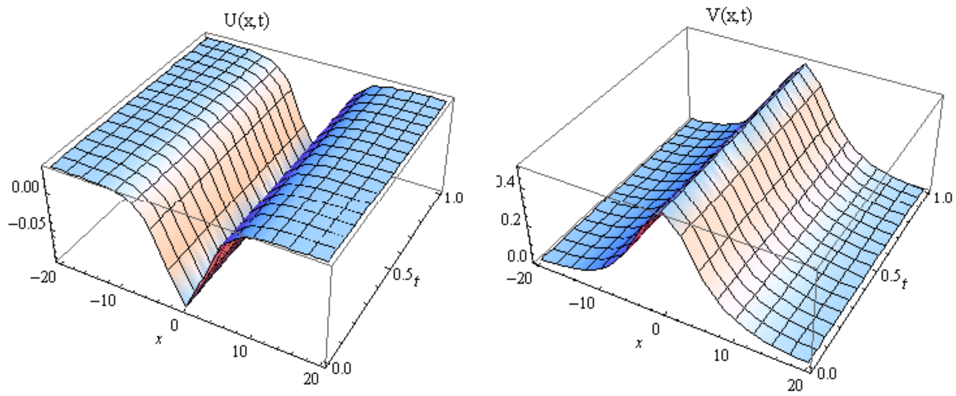

Figure 2. Numerical solution of the coupled J-M system for $\gamma_1 = \gamma_2 = 0.7$ and $\mu = 0.5$.

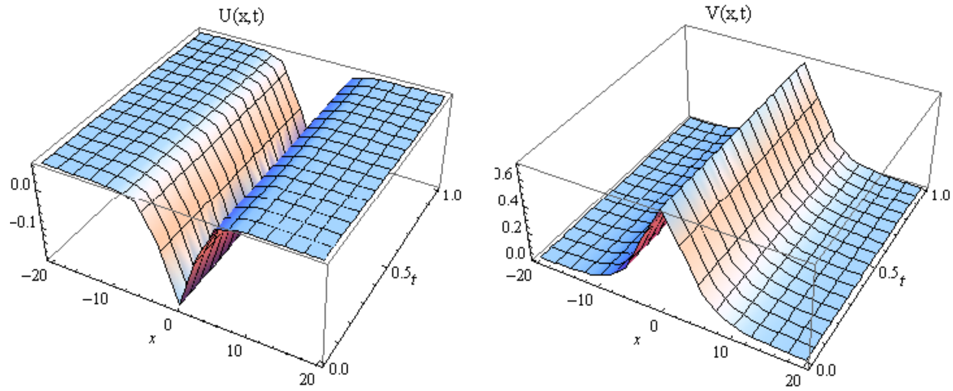

Figure 3. Numerical solution of the coupled J-M system for $\gamma_1 = \gamma_2 = 0.9$ and $\mu = 0.7$.

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Weakly g-Supplemented Lattices

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Abstract

In this work, all lattices are complete modular lattices with the greatest element 1 and the smallest element 0. Let L be a lattice and $a,b\in L$. If $a\vee b=1$ and $a\wedge b\ll_g L$, then b is called a weak g-supplement of a in L. If every element of L has a weak g-supplement in L, then L is called a weakly g-supplemented lattice. In this work, some properties of these lattices are investigated.

Keywords: Lattices, Small Elements, g-Small Elements, g-Supplemented Lattices.

2020 Mathematics Subject Classification: 06C05, 06C15.

1. INTRODUCTION

In this paper, every lattice is complete modular lattice with the smallest element 0 and the greatest element 1. Let L be a lattice, $x,y\in L$ and $x\leq y$. A sublattice $\{a\in L|x\leq a\leq y\}$ is called a *quotient sublattice* and denoted by y/x. An element y of a lattice L is called a *complement* of x in L if $x\wedge y=0$ and $x\vee y=1$, this case we denote $1=x\oplus y$ (in this case we call x and y are *direct summands* of L). L is said to be *complemented* if each element of L has at least one complement in L. An element x of L is said to be *small* or *superfluous* and denoted by $x\ll L$ if $y=1$ for every $y\in L$ such that $x\vee y=1$. The meet of all maximal ($\neq 1$) elements of a lattice L is called the *radical* of L and denoted by $r(L)$. An element a of L is called a *supplement* of b in L if it is minimal for $a\vee b=1$. a is a supplement of b in a lattice L if and only if $a\vee b=1$ and $a\wedge b\ll a/0$. A lattice L is called a *supplemented* lattice if every element of L has a supplement in L. Let L be a lattice and $a,b\in L$. If $a\vee b=1$ and $a\wedge b\ll$L, then a is called a *weak supplement* of b in L. L is said to be *weakly supplemented* if every element of L has a weak supplement in L. We say that an element y of L *lies above* an element x of L if $x\leq y$ and $y\ll 1/x$. L is said to be *hollow* if every element distinct from 1 is superfluous in L, and L is said to be *local* if L has the greatest element ($\neq 1$). We say an element x∈L has *ample supplements* in L if for every $y\in L$ with $x\vee y=1$, x has a supplement z in L with $z\leq y$. L is said to be *amply supplemented* if every element of L has ample supplements in L. It is clear that every amply supplemented lattice is supplemented. Let L be a lattice and $k\in L$. If $t=0$ for very $t\in$L with $k\wedge t=0$, then k is called an *essential element* of L and denoted by $k\trianglelefteq L$. Let L be a lattice and $a\in L$. If $b=1$ for every $b\trianglelefteq L$ with $a\vee b=1$, then a is called a *generalized small* (briefly, *g-small*) element of L and denoted by $a\ll_g L$. Let L be a lattice and $a,b\in L$. If $1=a\vee b$ and $1=a\vee t$ with $t\trianglelefteq b/0$ implies that $t=b$, then b is called a *g-supplement* of a in L. b is a g-supplement of a in L if and only if $1=a\vee b$ and $a\wedge b\ll_g b/0$. If every element of L has a g-supplement in L, then L is called a *g-supplemented* lattice. Let L be a lattice and t be a maximal ($\neq 1$) element of L. If $t\trianglelefteq L$, then t is called a *g-maximal* element of L. The meet of all g-maximal elements of L is called the *g-radical* of L and denoted by $r_g(L)$. If L have not any g-maximal elements, then we call $r_g(L)=1$. Let L be a lattice. If every element of L with distinct from 1 is g-small in L, then L is called a *g-hollow* lattice.

More informations about (amply) supplemented lattices are in [1], [2] and [3]. More results about (amply) supplemented modules are in [4] and [8]. More informations about weakly supplemented lattices are in [1]. More informations about g-small elements and g-supplemented lattices are in [7]. More informations about g-small submodules and g-supplemented modules are in [5] and [6].

Lemma 1.1. Let L be a lattice and $a,b,c,d\in L$. Then the followings are hold.
(i) If $a\leq b$ and $b\ll_g L$, then $a\ll_g L$.
(ii) If $a\ll_g b/0$, then $a\ll_g t/0$ for every $t\in L$ with $b\leq t$.
(iii) If $a\ll_g L$, then $a\vee b\ll_g 1/b$.
(iv) If $a\ll_g b/0$ and $c\ll_g d/0$, then $a\vee c\ll_g (b\vee d)/0$.
(v) If $a_i\ll_g b_i/0$ for $a_i,b_i\in L$ ($i=1,2,...,n$), then $a_1\vee a_2\vee ...\vee a_n\ll_g (b_1\vee b_2\vee ...\vee b_n)/0$.
(vi) If $a\leq b$ and $b\ll_g L$, then $b\ll_g 1/a$.
(vii) If $a\ll_g L$, then $a\leq r_g(L)$.
(viii) $r_g(a/0)\leq r_g(L)$.
Proof. See [7, Lemma 1, Lemma 6 and Lemma 7].

2. WEAKLY g-SUPPLEMENTED LATTICES

Definition 2.1. Let L be a lattice and $a,b\in L$. If $a\vee b=1$ and $a\wedge b\ll_g L$, then b is called a *weak g-supplement* of a in L. If every element of L has a weak g-supplement in L, then L is called a *weakly g-supplemented* lattice.

Proposition 2.2. Let L be a lattice and $a,b\in L$. If b is weak g-supplement of a in L, then a is a weak g-supplement of b in L.
Proof. Clear from definition.

Proposition 2.3. Let L be a weakly g-supplemented lattice. If every nonzero element of L is essential in L, then L is weakly supplemented.

Proof. Let $a \in L$. Since L is weakly g-supplemented, a has a weak g-supplement b in L. Here $a \vee b=1$ and $a \wedge b \ll_g L$. Since every nonzero element of L is essential in L, $a \wedge b \ll L$. Hence b is a weak supplement of a in L and L is weakly supplemented.

Proposition 2.4. Let L be a lattice and $a,b \in L$. If b is a g-supplement of a in L, then b is a weak g-supplement of a in L.
Proof. Since b is a g-supplement of a in L, $a \vee b=1$ and $a \wedge b \ll_g b/0$. Since $a \wedge b \ll_g b/0$, by Lemma 1.1, $a \wedge b \ll_g L$. Hence b is a weak g-supplement of a in L.

Proposition 2.5. Every g-supplemented lattice is weakly g-supplemented.
Proof. Let L be a g-supplemented lattice and $a \in L$. Since L is g-supplemented, a has a g-supplement b in L. Since b is a g-supplement of a in L, by Proposition 2.4, b is a weak g-supplement of a in L. Hence L is weakly g-supplemented.

Proposition 2.6. Let L be a lattice and $a,b \in L$. If $a \vee b$ has a g-supplement x in L and $(a \vee x) \wedge b$ has a g-supplement y in $b/0$, then $x \vee y$ is a weak g-supplement of a in L.
Proof. By [7, Lemma 3], $x \vee y$ is a g-supplement of a in L. Then by Proposition 2.4, $x \vee y$ is a weak g-supplement of a in L.

Corollary 2.7. Let L be a lattice and $a,b \in L$. If $a \vee b$ has a g-supplement in L and $b/0$ is g-supplemented, then a has a weak g-supplement in L.
Proof. Clear from Proposition 2.6.

Proposition 2.8. Let L be a lattice and $a,b \in L$. If $a/0$ and $b/0$ are g-supplemented, then $(a \vee b)/0$ is weakly g-supplemented.
Proof. Since $a/0$ and $b/0$ are g-supplemented, by [7, Lemma 4], $(a \vee b)/0$ is g-supplemented. Then by Proposition 2.5, $(a \vee b)/0$ is weakly g-supplemented.

Proposition 2.9. Let L be a lattice, $a,b \in L$ and $1=a \vee b$. If $a/0$ and $b/0$ are g-supplemented, then L is weakly g-supplemented.
Proof. Since $a/0$ and $b/0$ are g-supplemented and $1=a \vee b$, by [7, Lemma 4], L is g-supplemented. Then by Proposition 2.5, L is weakly g-supplemented.

Corollary 2.10. Let L be a lattice, $a_1,a_2,\ldots,a_n \in L$ and $1=a_1 \vee a_2 \vee \ldots \vee a_n$. If $a_i/0$ is g-supplemented for every $i=1,2,\ldots,n$, then L is weakly g-supplemented.
Proof. Clear from Proposition 2.9.

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Weakly Prime Submodules

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Abstract

Let R be a commutative ring with non-zero identity. We define a proper submodule N of an R-module M to be weakly prime if $0 \neq rm \in N$ $(r \in R, m \in M)$ implies $m \in N$ or $rM \subseteq N$.

In this study, various properties and results concerning weakly prime submodules are given.

Keywords: Prime, prime submodules, weakly prime submodules

1. Introduction

Throughout this study, we consider that R represents a commutative ring with non-zero identity and M is a module over the ring R. The definition and properties of prime ideal have been extended to modules by several authors, see, for example [2,3] . A proper ideal Q of R is called weakly prime ideal if $0 \neq ab \in Q$ implies $a \in Q$ or $b \in Q$. DD Anderson and E. Smith studied weakly prime ideals for a commutative ring with identity in [1]. Some results given in [1] have been proven also for weakly prime submodules. For example, if P is a submodule of a finitely generated multiplicative R-module M, then P is weakly prime if and only if for submodules N and K of M with $0 \neq NK \subseteq P$, either $N \subseteq P$ or $K \subseteq P$ [4].

Let we define some concepts that we will use in this study.

Definition 1.1. Let R be a ring and N be a submodule of an R-module M. The ideal $\{r \in R : rM \subseteq N\}$ is denoted by $(N : M)$. Then $(0 : M)$ is the annihilator of M and denoted by $Ann(M)$.

Definition 1.2. An R-module M is called a multiplicative module if there exists an ideal I of R such that $N = IM$ for each submodule N of M.

2. Prime Submodules

Definition 2.1. A proper submodule N of M is called a prime submodule of M, if $rm \in N$, for some $r \in R$ and $m \in M$, implies that either $m \in N$ or $rM \subseteq N$.

Example 2.1. Let R be a commutative ring with identity. Each prime ideal of R is a prime submodule of R-module R.

Example 2.2. Each proper submodule of a vector space is a prime submodule.

Definition 2.2. M is called a prime module if the zero submodule of M is prime submodule of M.

Lemma 2.1. If N is a prime submodule of an R-module M, then $(N : M)$ is a prime ideal of R.

Proof 2.1. Let N be a submodule of R-module M and let $ab \in (N : M)$ and $b \notin (N : M)$ for some $a, b \in R$. So, $abM \subseteq N$ but $bM \nsubseteq N$. Hence there exists an element $t \in M$ such that $abt \in N$ and $bt \notin N$. Since is a prime submodule of an R-module M, then $a \in (N : M)$. Therefore, $(N : M)$ is a prime ideal of R.

3. Weakly Prime Submodules

Definition 3.1. A proper submodule N of M is called a weakly prime submodule of M, if $0 \neq rm \in N$, for some $r \in R$ and $m \in M$, implies that $m \in N$ or $rM \subseteq N$.

Example 3.1. Let R be a commutative ring with identity. Each weakly prime ideal of R is a prime submodule of R-module R.

We have said that $(N : M)$ is a prime ideal of R for any prime submodule N of R-module M. That can not be generalized for weakly prime submodules. For instance, let M denote the cyclic $\mathbb{Z}$-module $\mathbb{Z}/8\mathbb{Z}$. Take $N = \{0\}$. Certainly, N is a weakly prime submodule of M, but $(N : M) = 8\mathbb{Z}$ is not a weakly prime ideal of R.

By the definitions of prime and weakly prime submodule, we can say that every prime submodule is weakly prime submodule.

However, we know that 0 is always weakly prime, so this proves that a weakly prime submodule is not always prime submodule.

Example 3.2. Given $\mathbb{Z}_n$ as module over itself with n is a composite number. Let $N = \{0\}$ is submodule of M, so N is weakly prime but it is not prime.

Example 3.3. Given M is $\mathbb{Z}_{12}$ as module over ring $\mathbb{Z}$ and $N = \{0, 3, 6, 9\}$ is proper submodule of M with ideal $(N : M) = \{r \in \mathbb{Z} | r\mathbb{Z}_{12} \subseteq N\} = 3\mathbb{Z}$.Therefore, for $rm \in N$ we get $m = 0$, $m = 3$, $m = 6$ and $m = 9$ which are elements in N or $r \in 3\mathbb{Z}$. So, N is prime submodule of M. It is clear that N is also weakly prime submodule.

Proposition 3.1. Let R be a commutative ring and M be an R-module whose annihilator is, $(0 : M) = P$, a prime ideal. If N is a weakly prime submodule of M, then $(N : M)$ is a weakly prime ideal of R.

Proof 3.1. Let $0 \neq ab \in (N : M)$ with $a \notin (N : M)$. Then there exists $m \in M \setminus N$ such that $am \notin N$.
Since $0 \neq abM \subseteq N$, then $abm \in N$.
If $abm = 0$, then $ab \in (0: m) = (0: M) = P$. Since $a \notin (N : M)$, then $b \in (0: M) \subseteq (N : M)$.
If $abm \neq 0$, then $b \in (N: M)$ since $am \notin N$ and N is a weakly prime submodule.

Now, let we give some equivalent conditions to categorize weakly prime submodules.

Theorem 3.2. Let R be a commutative ring, M be an R-module, and N be a proper submodule of M. Then the following statements are equivalent.
$\boldsymbol{i}$) For ideal I of R and a submodule D of M with $0 \neq ID \subseteq N$,either $IM \subseteq N$ or $D \subseteq N$.
$\boldsymbol{ii}$) N is a weakly prime submodule of M.
$\boldsymbol{iii}$) For $m \in M \setminus N$, $(N : Rm) = (N : M) \cup (0 : Rm)$.
$\boldsymbol{iv}$) For $m \in M \setminus N$, $(N : Rm) = (N : M)$ or $(N : Rm) = (0 : Rm)$.

Proof 3.2. $(i \Rightarrow ii)$ Let $0 \neq am \in N$ with $a \in R, m \in M$ and let $I = Ra$, $D = Rm$. Since $0 \neq ID \subseteq N$, then $I \subseteq (N: M)$ or $D \subseteq N$. Hence $a \in (N : M)$ or $m \in N$.

$(ii \Rightarrow i)$ Let N be a weakly prime submodule of M. Let N is a prime submodule. Suppose that $ID \subseteq N$ but $D \nsubseteq N$ for an ideal I of R and a submodule D of M. Let $a \in I$. Then there exists $d \in M$ such that $ad \in N$ with $d \notin N$. Since N is a prime submodule, then $a \in (N : M)$. Hence $I \subseteq (N : M)$.
Assume that N is a weakly prime submodule of M which is not prime. Let $0 \neq ID \subseteq N$ and $D \nsubseteq N$. Then there exist an element $x \in D \setminus N$. We will show that $I \subseteq (N : M)$. Let $r \in I$.
If $0 \neq rx$, then $r \in (N : M)$ because N is weakly prime.
So assume that $rx = 0$. First suppose that $rD \neq 0$, say $rd \neq 0$ where $d \in D$. If $d \notin N$, then $r \in (N : M)$. If $d \in N$, then $r(d + x) = rd + rx = rd \in N$. So $r \in (N : M)$ or $d + x \in N$. Thus $r \in (N : M)$, hence $I \subseteq (N : M)$.
So we can assume that $rD = 0$. Suppose that $Ix \neq 0$, say $ax \neq 0$ where $a \in I$. So, $a \in (N : M)$ since N is weakly prime. As $(r + a)x = rx + ax = ax \in N$, we get $r \in (N : M)$, so $I \subseteq (N : M)$. Therefore, we can assume that $Ix = 0$.
Since $ID \neq 0$, there exits $b \in I$ and $d' \in D$ such that $bd' \neq 0$.
As $(N : M)N = 0$ and $0 \neq b(d' + x) = bd' \in N$ we can divide the proof into the following two cases:

Case 1. $b \in (N : M)$ and $d' + x \notin N$.
Since $0 \neq (r + b)((d' + x) = bd' \in N$, we obtain $r + b \in (N : M)$, so $r \in (N : M)$.
Hence $I \subseteq (N : M)$.

Case 2. $b \notin (N : M)$ and $d' + x \in N$.
As $0 \neq bd' \in N$, so $x \in N$ which is a contradiction. Thus $I \subseteq (N : M)$.

$(ii \Rightarrow iii)$ Clearly, if $m \in M \setminus N$, then $H = (N : M) \cup (0 : Rm) \subseteq (N : Rm)$.
Let $a \in (N : Rm)$ where $m \in M \setminus N$. Then $am \in N$. If $am \neq 0$, then $a \in (0 : Rm)$, so we have equality.

$(iii \Rightarrow iv)$ This is obvious.

$(iv \Rightarrow ii)$ Suppose that $0 \neq rm \in N$ with $r \in R$ and $m \in M \setminus N$. Then $r \in (N : Rm)$ and $r \notin (0 : Rm)$. It follows from (iv) that $r \in (N : Rm) = (N : M)$, as required.

Lemma 3.3. Let M be a multiplicative R-module and P be a prime submodule of M. For the submodules N_1 and N_2 of M such that $N_1 \cap N_2 \subseteq P$, either $N_1 \subseteq P$ or $N_2 \subseteq P$.

Proof 3.3. If $N_1 \cap N_2 \subseteq P$, then $(N_1 \cap N_2 : M) \subseteq (P : M)$.
So, $(N_1 : M) \cap (N_2 : M) = (N_1 \cap N_2 : M) \subseteq (P : M)$.
Hence, $(N_1 : M)(N_2 : M) \subseteq (N_1 : M) \cap (N_2 : M) \subseteq (P : M)$.
Since $(P : M)$ is a prime ideal of the ring R, then $(N_1 : M) \subseteq (P : M)$ or $(N_2 : M) \subseteq (P : M)$.
And since M is multiplicative, then
$N_1 = (N_1 : M)M \subseteq (P : M)M = P$ or $N_2 = (N_2 : M)M \subseteq (P : M)M = P$.
Therefore, $N_1 \subseteq P$ or $N_2 \subseteq P$. ■

This lemma is not always true for the modules which are not multiplicative.
For example, let $R = \mathbb{Z}$ and $M = \mathbb{Z} \oplus \mathbb{Z}$. If we consider the submodules $N_1 = \mathbb{Z} \oplus (0)$ and $N_2 = (0) \oplus \mathbb{Z}$, we see that $N_1 \cap N_2 = \{(0,0)\}$. Although $P = (0) \oplus (0)$ is a prime submodule of M and $N_1 \cap N_2 \subseteq P$, $N_1 \nsubseteq P$ and $N_2 \nsubseteq P$.

Theorem 3.4. Let R be a commutative ring, M be a finitely generated multiplicative R-module and let P be a proper submodule of M. Then the following statements are equivalent for the submodule P [4].
i) P is a weakly prime submodule of M.
ii) $N \subseteq P$ or $K \subseteq P$ for submodules N and K of M such that $0 \neq NK \subseteq P$.

Conclusion

In this study, the definitions of prime and weakly prime submodules and some examples are given. The same and different properties between prime submodules and weak prime submodules are examined. Then, several propositions and theorems are given for categorizing weakly prime submodules.

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Tangent developable surfaces of timelike biharmonic general helices in E(1,1)

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Abstract

In this paper, we obtain parametric equation of tangent developable surfaces of timelike biharmonic general helices in the Lorentzian group of rigid motions E(1,1). Finally, we obtain some figures.

1.Introduction

Paper, sheet metal, and many other materials are approximately unstretchable. The surfaces obtained by bending these materials can be flattened onto a plane without stretching or tearing. More precisely, there exists a transformation that maps the surface onto the plane, after which the length of any curve drawn on the surface remains the same. Such surfaces, when sufficiently regular, are well known to mathematicians as developable surfaces. While developable surfaces have been widely used in engineering, design and manufacture, they have been less popular in computer graphics, despite the fact that their isometric properties make them ideal primitives for texture mapping, some kinds of surface modelling, and computer animation.

On the other hand, the notions of harmonic and biharmonic maps between Riemannian manifolds have been introduced by J. Eells and J.H. Sampson (see [4]).

A smooth map $\phi : N \to M$ is said to be biharmonic if it is a critical point of the bienergy functional:

$$E_2(\phi) = \int_N \frac{1}{2} \left| \mathrm{T}(\phi) \right|^2 dv_h,$$

where $\mathrm{T}(\phi) := \mathrm{tr}\nabla^{\phi} d\phi$ is the tension field of ϕ

The Euler--Lagrange equation of the bienergy is given by $\mathrm{T}_2(\phi) = 0$. Here the section $\mathrm{T}_2(\phi)$ is defined by

$$\mathrm{T}_2(\phi) = -\Delta_{\phi} \mathrm{T}(\phi) + \mathrm{tr}R(\mathrm{T}(\phi), d\phi)d\phi, \qquad (1.1)$$

and called the bitension field of ϕ. Non-harmonic biharmonic maps are called proper biharmonic maps.

2 .Preliminaries

Let $\mathrm{E}(1,1)$ be the group of rigid motions of Euclidean 2-space. This consists of all matrices of the form

$$\begin{pmatrix} \cosh x & \sinh x & y \\ \sinh x & \cosh x & z \\ 0 & 0 & 1 \end{pmatrix}.$$

Topologically, $\mathrm{E}(1,1)$ is diffeomorphic to R^3 under the map

$$\mathrm{E}(1,1) \to \mathrm{R}^3 : \begin{pmatrix} \cosh x & \sinh x & y \\ \sinh x & \cosh x & z \\ 0 & 0 & 1 \end{pmatrix} \to (x, y, z),$$

It's Lie algebra has a basis consisting of

$$\mathbf{e}_1 = \frac{\partial}{\partial x}, \mathbf{e}_2 = \cosh x \frac{\partial}{\partial y} + \sinh x \frac{\partial}{\partial z}, \mathbf{e}_3 = \sinh x \frac{\partial}{\partial y} + \cosh x \frac{\partial}{\partial z},$$

for which

$$[\mathbf{e}_1, \mathbf{e}_2] = \mathbf{e}_3, [\mathbf{e}_2, \mathbf{e}_3] = 0, [\mathbf{e}_1, \mathbf{e}_3] = \mathbf{e}_2.$$

Put

$$x^1 = x, x^2 = \frac{1}{2}(y + z), x^3 = \frac{1}{2}(y - z).$$

Then, we get

$$\mathbf{e}_1=\frac{\partial}{\partial x^1},\mathbf{e}_2=\frac{1}{2}\left(e^{x^1}\frac{\partial}{\partial x^2}+e^{-x^1}\frac{\partial}{\partial x^3}\right),\mathbf{e}_3=\frac{1}{2}\left(e^{x^1}\frac{\partial}{\partial x^2}-e^{-x^1}\frac{\partial}{\partial x^3}\right). \tag{2.1}$$

The bracket relations are

$$[\mathbf{e}_1,\mathbf{e}_2]=\mathbf{e}_3,[\mathbf{e}_2,\mathbf{e}_3]=0,[\mathbf{e}_1,\mathbf{e}_3]=\mathbf{e}_2. \tag{2.2}$$

We consider left-invariant Lorentzian metrics which has a pseudo-orthonormal basis $\{\mathbf{X}_1,\mathbf{X}_2,\mathbf{X}_3\}$. We consider left-invariant Lorentzian metric [9], given by

$$g=-(dx^1)^2+\left(e^{-x^1}dx^2+e^{x^1}dx^3\right)^2+\left(e^{-x^1}dx^2-e^{x^1}dx^3\right)^2, \tag{2.3}$$

where

$$g(\mathbf{e}_1,\mathbf{e}_1)=-1, g(\mathbf{e}_2,\mathbf{e}_2)=g(\mathbf{e}_3,\mathbf{e}_3)=1. \tag{2.4}$$

Let coframe of our frame be defined by

$$\boldsymbol{\theta}^1=dx^1,\boldsymbol{\theta}^2=e^{-x^1}dx^2+e^{x^1}dx^3,\boldsymbol{\theta}^3=e^{-x^1}dx^2-e^{x^1}dx^3.$$

Proposition 2.1. *For the covariant derivatives of the Levi-Civita connection of the left-invariant metric g, defined above the following is true:*

$$\nabla=\begin{pmatrix}0&0&0\\-\mathbf{e}_3&0&-\mathbf{e}_1\\-\mathbf{e}_2&-\mathbf{e}_1&0\end{pmatrix}, \tag{2.5}$$

where the (i,j)-element in the table above equals $\nabla_{\mathbf{e}_i}\mathbf{e}_j$ for our basis

$$\{\mathbf{e}_k,k=1,2,3\}=\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}.$$

3 .Timelike Biharmonic General Helices in the Lorentzian Group of Rigid Motions $\mathsf{E}(1,1)$

Let $\gamma : I \to \mathsf{E}(1,1)$ be a non geodesics timelike curve in the group of rigid motions $\mathsf{E}(1,1)$ parametrized by arc length. Let $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ be the Frenet frame fields tangent to the group of rigid motions $\mathsf{E}(1,1)$ along γ defined as follows:

$\mathbf{T}$ is the unit vector field γ' tangent to γ, $\mathbf{N}$ is the unit vector field in the direction of $\nabla_{\mathbf{T}}\mathbf{T}$ (normal to γ) and $\mathbf{B}$ is chosen so that $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= \kappa\mathbf{N}, \\ \nabla_{\mathbf{T}}\mathbf{N} &= \kappa\mathbf{T} + \tau\mathbf{B}, \\ \nabla_{\mathbf{T}}\mathbf{B} &= -\tau\mathbf{N}, \end{aligned} \tag{3.1}$$

where κ is the curvature of γ, τ is its torsion and

$$\begin{aligned} & g(\mathbf{T},\mathbf{T}) = -1,\ g(\mathbf{N},\mathbf{N}) = 1,\ g(\mathbf{B},\mathbf{B}) = 1, \\ & g(\mathbf{T},\mathbf{N}) = g(\mathbf{T},\mathbf{B}) = g(\mathbf{N},\mathbf{B}) = 0. \end{aligned} \tag{3.2}$$

With respect to the orthonormal basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ we can write

$$\begin{aligned} \mathbf{T} &= T_1\mathbf{e}_1 + T_2\mathbf{e}_2 + T_3\mathbf{e}_3, \\ \mathbf{N} &= N_1\mathbf{e}_1 + N_2\mathbf{e}_2 + N_3\mathbf{e}_3, \\ \mathbf{B} &= \mathbf{T}\times\mathbf{N} = B_1\mathbf{e}_1 + B_2\mathbf{e}_2 + B_3\mathbf{e}_3. \end{aligned} \tag{3.3}$$

Theorem 3.1. ([8]) $\gamma : I \to \mathsf{E}(1,1)$ *is a non geodesic timelike biharmonic curve in the Lorentzian group of rigid motions* $\mathsf{E}(1,1)$ *if and only if*

$$\kappa = \text{constant} \neq 0,$$
$$\kappa^2 - \tau^2 = 1 + 2B_1^2, \tag{3.4}$$
$$\tau' = -2N_1B_1.$$

Theorem 3.2. ([8]) *Let* $\gamma : I \to \mathsf{E}(1,1)$ *is a non geodesic timelike biharmonic general helix in the Lorentzian group of rigid motions* $\mathsf{E}(1,1)$. *Then, the parametric equations of* γ *are*

$$x^1(s) = \cosh \text{ö}\kappa s + \wp_3,$$
$$x^2(s) = \frac{\sinh \text{ö} e^{\cosh \text{ö}\kappa s + \wp_3}}{2\left(\wp_1^2 + \cosh^2 \text{ö}\right)}\{(\cosh \text{ö} - \wp_1)\cos(\wp_1 \kappa s + \wp_2)$$
$$+ (\cosh \text{ö} + \wp_1)\sin(\wp_1 \kappa s + \wp_2)\} + \wp_4, \tag{3.5}$$
$$x^3(s) = \frac{\sinh \text{ö} e^{-\cosh \text{ö}\kappa s - \wp_3}}{2\left(\wp_1^2 + \sinh^2 \text{ö}\right)}\{-(\cosh \text{ö} - \wp_1)\cos(\wp_1 \kappa s + \wp_2)$$
$$+ (\cosh \text{ö} + \wp_1)\sin(\wp_1 \kappa s + \wp_2)\} + \wp_5,$$

where $\wp_1$, $\wp_2$, $\wp_3$, $\wp_4$, $\wp_5$ *are constants of integration.*

4 Tangent Developable Surfaces of Timelike Biharmonic General Helices in the Lorentzian Group of Rigid Motions $\mathsf{E}(1,1)$

Developable surfaces are defined as the surfaces on which the Gaussian curvature is 0 everywhere. The developable surfaces are useful since they can be made out of sheet metal or paper by rolling a flat sheet of material without stretching it. Most large-scale objects such as airplanes or ships are constructed using un-stretched sheet metals, since sheet metals are easy to model and they have good stability and vibration properties. Moreover, sheet metals provide good fluid dynamic properties. In ship or airplane design, the problems usually stem from engineering concerns and in engineering design there has been a strong interest in developable surfaces.

The tangent developable of γ is a ruled surface

$$\Pi_{\left(\gamma,\gamma'\right)}(s,u)=\gamma(s)+u\gamma'(s). \tag{4.1}$$

Theorem 4.1. *Let* $\gamma: I \rightarrow \mathrm{E}(1,1)$ *is a non geodesic timelike biharmonic general helix in the Lorentzian group of rigid motions* $\mathrm{E}(1,1)$. *Then, the parametric equations of tangent developable of* γ *are*

$$x_{\Pi}^{1}(s,u)=\cosh \ddot{o}\kappa s+u\cosh \ddot{o}+\wp_{3},$$

$$x_{\Pi}^{2}(s,u)=\frac{\sinh \ddot{o}e^{\cosh \ddot{o}\kappa s+\wp_{3}}}{2\left(\wp_{1}^{2}+\cosh^{2}\ddot{o}\right)}\{(\cosh \ddot{o}-\wp_{1})\cos(\wp_{1}\kappa s+\wp_{2}) \tag{4.2}$$

$$+(\cosh \ddot{o}+\wp_{1})\sin(\wp_{1}\kappa s+\wp_{2})\}$$

$$+\frac{u\sinh \ddot{o}e^{\cosh \ddot{o}\kappa s+\wp_{3}}}{2}[\cos(\wp_{1}\kappa s+\wp_{2})+\sin(\wp_{1}\kappa s+\wp_{2})]+\wp_{4},$$

$$x_{\Pi}^{3}(s,u)=\frac{\sinh \ddot{o}e^{-\cosh \ddot{o}\kappa s-\wp_{3}}}{2\left(\wp_{1}^{2}+\cosh^{2}\ddot{o}\right)}\{-(\cosh \ddot{o}-\wp_{1})\cos(\wp_{1}\kappa s+\wp_{2})$$

$$+(\cosh \ddot{o}+\wp_{1})\sin(\wp_{1}\kappa s+\wp_{2})\}$$

$$+\frac{u\sinh \ddot{o}e^{-\cosh \ddot{o}\kappa s-\wp_{3}}}{2}[\cos(\wp_{1}\kappa s+\wp_{2})-\sin(\wp_{1}\kappa s+\wp_{2})]+\wp_{5},$$

where $\wp_1,\ \wp_2,\ \wp_3,\ \wp_4,\ \wp_5$ *are constants of integration.*

Proof. From Theorem 3.2, we have

$$\mathbf{T} = \cosh \text{ö}\mathbf{e}_1 + \sinh \text{ö} \cos(\wp_1 \kappa s + \wp_2)\mathbf{e}_2$$

$$+ \sinh \text{ö} \sin(\wp_1 \kappa s + \wp_2)\mathbf{e}_3. \tag{4.3}$$

By equation (2.1) in above equation, we immediately arrive at

$$\mathbf{T} = (\cosh \text{ö}, \frac{1}{2} e^{\cosh \text{ö} \theta + \wp_3} \sinh \text{ö} \cos(\wp_1 \theta + \wp_2) + \sin(\wp_1 \theta + \wp_2)],$$

$$\frac{1}{2} e^{-\cosh \text{ö} \theta - \wp_3} \sinh \text{ö} \cos(\wp_1 \theta + \wp_2) - \sin(\wp_1 \theta + \wp_2)]), \tag{4.4}$$

where $\wp_3$ is constant of integration.

From (4.1) and (4.4), by direct calculation we have (4.2), which proves the theorem.

Using Theorem 4.1 we can give the following result.

Corollary 4.2. *Let* $\gamma : I \rightarrow \mathrm{E}(1,1)$ *is a non geodesic timelike biharmonic general helix in the Lorentzian group of rigid motions* $\mathrm{E}(1,1)$. *Then, the parametric equations of tangent developable of* γ *in terms of torsion are*

$$x_{\Pi}^1(s,u) = \cosh \text{ö} \sqrt{\tau^2 + 1 + 2B_1^2}\, s + u \cosh \text{ö} + \wp_3,$$

$$x_{\Pi}^2(s,u) = \frac{\sinh \text{ö} e^{\cosh \text{ö} \sqrt{\tau^2 + 1 + 2B_1^2}\, s + \wp_3}}{2(\wp_1^2 + \cosh^2 \text{ö})} \{(\cosh \text{ö} - \wp_1) \cos\left(\wp_1 \sqrt{\tau^2 + 1 + 2B_1^2}\, s + \wp_2\right)$$

$$+ (\cosh \text{ö} + \wp_1) \sin\left(\wp_1 \sqrt{\tau^2 + 1 + 2B_1^2}\, s + \wp_2\right)\}$$

$$+ \frac{u \sinh \text{ö} e^{\cosh \text{ö} \sqrt{\tau^2 + 1 + 2B_1^2}\, s + \wp_3}}{2} [\cos\left(\wp_1 \sqrt{\tau^2 + 1 + 2B_1^2}\, s + \wp_2\right)$$

$$+\sin\left(\wp_1\sqrt{\tau^2+1+2B_1^2}\,s+\wp_2\right)]+\wp_4,$$

$$x_{\Pi}^3(s,u)=\frac{\sinh\text{ö}\, e^{-\cosh\text{ö}\sqrt{\tau^2+1+2B_1^2}\,s-\wp_3}}{2\left(\wp_1^2+\cosh^2\text{ö}\right)}\{-(\cosh\text{ö}-\wp_1)\cos\left(\wp_1\sqrt{\tau^2+1+2B_1^2}\,s+\wp_2\right)$$

$$+(\cosh\text{ö}+\wp_1)\sin\left(\wp_1\sqrt{\tau^2+1+2B_1^2}\,s+\wp_2\right)\}$$

$$+\frac{u\sinh\text{ö}\, e^{-\cosh\text{ö}\sqrt{\tau^2+1+2B_1^2}\,s-\wp_3}}{2}[\cos\left(\wp_1\sqrt{\tau^2+1+2B_1^2}\,s+\wp_2\right)$$

$$-\sin\left(\wp_1\sqrt{\tau^2+1+2B_1^2}\,s+\wp_2\right)]+\wp_5,$$

where $\wp_1,\ \wp_2,\ \wp_3,\ \wp_4,\ \wp_5$ *are constants of integration.*

We can use Mathematica in Theorem 4.1, yields

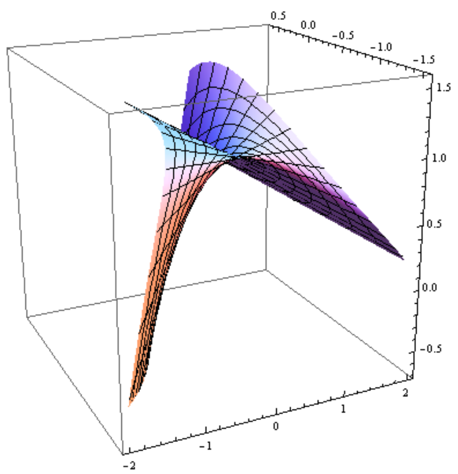

Fig.1

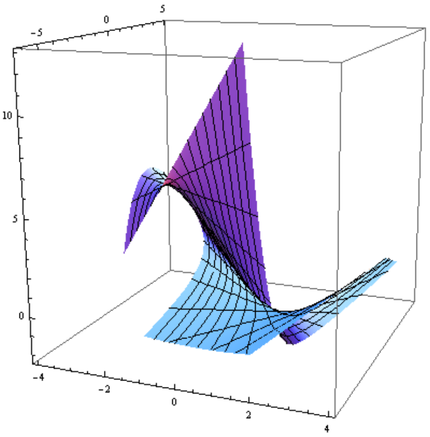

Fig.2

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Spacelike biharmonic general helices in the Lorentzian group of rigid motions E(2)

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Abstract

In this paper, we study spacelike biharmonic general helices in the Lorentzian group of rigid motions E(2). We characterize the spacelike biharmonic general helices in terms of their curvature and torsion in the Lorentzian group of rigid motions E(2).

1. Introduction

The theory of biharmonic functions is an old and rich subject. Biharmonic functions have been studied since 1862 by Maxwell and Airy to describe a mathematical model of elasticity. The theory of polyharmonic functions was developed later on, for example, by Almansi, Levi-Civita and Nicolescu.

Firstly, harmonic maps are given as follows:

Harmonic maps $f:(M,g)\to(N,h)$ between Riemannian manifolds are the critical points of the energy

$$E(f)=\frac{1}{2}\int_M |df|^2 v_g, \tag{1.1}$$

and they are therefore the solutions of the corresponding Euler--Lagrange equation. This equation is given by the vanishing of the tension field

$$\tau(f)=\mathrm{trace}\nabla df. \tag{1.2}$$

Secondly, biharmonic maps are given as follows:

As suggested by Eells and Sampson in [4], we can define the bienergy of a map f by

$$E_2(f)=\frac{1}{2}\int_M |\tau(f)|^2 v_g, \tag{1.3}$$

and say that is biharmonic if it is a critical point of the bienergy.

Jiang derived the first and the second variation formula for the bienergy in [5], showing that the Euler--Lagrange equation associated to E_2 is

$$\tau_2(f) = -\mathrm{J}^f(\tau(f)) = -\Delta\tau(f) - \mathrm{trace}R^N(df,\tau(f))df \tag{1.4}$$
$$= 0,$$

where J^f is the Jacobi operator of f . The equation $\tau_2(f)=0$ is called the biharmonic equation. Since J^f is linear, any harmonic map is biharmonic.

In this paper, we study spacelike biharmonic general helices in the Lorentzian group of rigid motions $\mathrm{E}(2)$. We characterize the spacelike biharmonic general helices in terms of their curvature and torsion in the Lorentzian group of rigid motions $\mathrm{E}(2)$.

2. The Group of Rigid Motions $\mathrm{E}(2)$

Let $E(2)$ be the group of rigid motions of Euclidean 2-space. This consists of all matrices of the form

$$\begin{pmatrix} \cos x & -\sin x & y \\ \sin x & \cos x & z \\ 0 & 0 & 1 \end{pmatrix}.$$

Topologically, $E(2)$ is diffeomorphic to $\mathrm{S}^1 \times \mathrm{R}^2$ under the map

$$E(2) \to \mathrm{S}^1 \times \mathrm{R}^2 : \begin{pmatrix} \cos[x] & -\sin[x] & y \\ \sin[x] & \cos[x] & z \\ 0 & 0 & 1 \end{pmatrix} \to ([x], y, z),$$

where $[x]$ means x modulo $2\pi z$. It's Lie algebra has a basis consisting of

$$\mathbf{e}_1 = \frac{\partial}{\partial x}, \mathbf{e}_2 = \cos x\frac{\partial}{\partial y} + \sin x\frac{\partial}{\partial z}, \mathbf{e}_3 = -\sin x\frac{\partial}{\partial y} + \cos x\frac{\partial}{\partial z}, \tag{2.1}$$

[9] and coframe

$$\boldsymbol{\theta}^1 = dx, \boldsymbol{\theta}^2 = \cos x dy + \sin x dz, \boldsymbol{\theta}^3 = -\sin x dy + \cos x dz.$$

It is easy to check that the metric g is given by

$$g = \left(\boldsymbol{\theta}^1\right)^2 + \left(\boldsymbol{\theta}^2\right)^2 - \left(\boldsymbol{\theta}^3\right)^2. \tag{2.2}$$

The bracket relations are

$$[\mathbf{e}_1, \mathbf{e}_2] = \mathbf{e}_3, [\mathbf{e}_2, \mathbf{e}_3] = 0, [\mathbf{e}_3, \mathbf{e}_1] = \mathbf{e}_2.$$

Proposition 2.1. *For the covariant derivatives of the Levi-Civita connection of the left-invariant metric* g*, defined above the following is true:*

$$\nabla = \begin{pmatrix} 0 & 0 & 0 \\ -\mathbf{e}_3 & 0 & -\mathbf{e}_1 \\ \mathbf{e}_2 & -\mathbf{e}_1 & 0 \end{pmatrix}, \tag{2.3}$$

where the (i, j)*-element in the table above equals* $\nabla_{\mathbf{e}_i}\mathbf{e}_j$ *for our basis*

$$\{\mathbf{e}_k, k = 1,2,3\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}.$$

3. Spacelike Biharmonic General Helices with Timelike Normal in the Lorentzian Group of Rigid Motions $\mathsf{E}(2)$

Let $\gamma : I \to \mathsf{E}(2)$ be a non geodesic spacelike curve with timelike normal in the group of rigid motions $\mathsf{E}(2)$ parametrized by arc length. Let $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ be the Frenet frame fields tangent to the group of rigid motions $\mathsf{E}(2)$. along γ defined as follows:

$\mathbf{t}$ is the unit vector field γ' tangent to γ, $\mathbf{n}$ is the unit vector field in the direction of $\nabla_{\mathbf{t}}\mathbf{t}$ (normal to γ) and $\mathbf{b}$ is chosen so that $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\begin{aligned} &\nabla_{\mathbf{t}(s)}\mathbf{t}(s) = \kappa(s)\mathbf{n}(s), \\ &\nabla_{\mathbf{t}(s)}\mathbf{n}(s) = \kappa(s)\mathbf{t}(s) + \tau(s)\mathbf{b}(s), \\ &\nabla_{\mathbf{t}(s)}\mathbf{b}(s) = \tau(s)\mathbf{n}(s), \end{aligned} \tag{3.1}$$

where $\kappa(s) = |\tau(\gamma)| = |\nabla_{\mathbf{t}(s)}\mathbf{t}(s)|$ is the curvature of γ, $\tau(s)$ is its torsion and

$$g(\mathbf{t}(s), \mathbf{t}(s)) = 1, g(\mathbf{n}(s), \mathbf{n}(s)) = -1, g(\mathbf{b}(s), \mathbf{b}(s)) = 1, \tag{3.2}$$

$$g(\mathbf{t}(s),\mathbf{n}(s)) = g(\mathbf{t}(s),\mathbf{b}(s)) = g(\mathbf{n}(s),\mathbf{b}(s)) = 0.$$

With respect to the orthonormal basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ we can write

$$\mathbf{t}(s) = t_1(s)\mathbf{e}_1 + t_2(s)\mathbf{e}_2 + t_3(s)\mathbf{e}_3, \tag{3.3}$$

$$\mathbf{n}(s) = n_1(s)\mathbf{e}_1 + n_2(s)\mathbf{e}_2 + n_3(s)\mathbf{e}_3,$$

$$\mathbf{b}(s) = \mathbf{t}(s)\times\mathbf{n}(s) = b_1(s)\mathbf{e}_1 + b_2(s)\mathbf{e}_2 + b_3(s)\mathbf{e}_3.$$

Theorem 3.1. $\gamma : I \to \mathsf{E}(2)$ *is a non geodesic spacelike biharmonic curve with timelike normal in the Lorentzian group of rigid motions* $\mathsf{E}(2)$ *if and only if*

$$\kappa(s) = \text{constant} \neq 0,$$

$$\kappa^2(s) + \tau^2(s) = 1 - 2b_1^2(s), \tag{3.4}$$

$$\tau'(s) = 2n_1(s)b_1(s).$$

Proof. Using (3.1), we have

$$\tau_2(\gamma) = \nabla_{\mathbf{t}}^3\mathbf{t}(s) + \kappa(s)R(\mathbf{t}(s),\mathbf{n}(s))\mathbf{t}(s)$$

$$= (3\kappa_1'(s)\kappa(s))\mathbf{t}(s) + (\kappa''(s) + \kappa^3(s) + \kappa(s)\tau^2(s))\mathbf{n}(s)$$

$$+ (2\tau(s)\kappa'(s) + \kappa(s)\tau'(s))\mathbf{b}(s) + \kappa(s)R(\mathbf{t}(s),\mathbf{n}(s))\mathbf{t}(s).$$

By (1.1), we see that γ is a unit speed spacelike biharmonic curve with timelike normal if and only if

$$\kappa(s)\kappa'(s) = 0,$$

$$\kappa''(s)+\kappa^3(s)+\kappa(s)\tau^2(s)=-\kappa(s)R(\mathbf{t}(s),\mathbf{n}(s),\mathbf{t}(s),\mathbf{n}(s)), \tag{3.5}$$

$$2\tau(s)\kappa'(s)+\tau'(s)\kappa(s)=-\kappa(s)R(\mathbf{t}(s),\mathbf{n}(s),\mathbf{t}(s),\mathbf{b}(s)).$$

Since $\kappa \neq 0$ by the assumption that is non-geodesic

$$\kappa(s)=\text{constant}\neq 0,$$

$$\kappa^2(s)+\tau^2(s)=-R(\mathbf{t}(s),\mathbf{n}(s),\mathbf{t}(s),\mathbf{n}(s)), \tag{3.6}$$

$$\tau'(s)=-R(\mathbf{t}(s),\mathbf{n}(s),\mathbf{t}(s),\mathbf{b}(s)).$$

A direct computation using (2.5), yields

$$R(\mathbf{t}(s),\mathbf{n}(s),\mathbf{t}(s),\mathbf{n}(s)) = -1+2b_1^2(s), \tag{3.7}$$

$$R(\mathbf{t}(s),\mathbf{n}(s),\mathbf{t}(s),\mathbf{b}(s)) = -2n_1(s)b_1(s).$$

These, together with (3.6), complete the proof of the theorem.

If we write this curve in the another parametric representation $\gamma=\gamma(\theta)$, where $\theta=\int_0^s \kappa(s)ds$. We have new Frenet equations as follows:

$$\nabla_{\mathbf{t}(\theta)}\mathbf{t}(\theta)=\mathbf{n}(\theta),$$

$$\nabla_{\mathbf{t}(\theta)}\mathbf{n}(\theta)=\mathbf{t}(\theta)+f(\theta)\mathbf{b}(\theta), \tag{3.8}$$

$$\nabla_{\mathbf{t}(\theta)}\mathbf{b}(\theta)=f(\theta)\mathbf{n}(\theta),$$

where $f(\theta)=\dfrac{\tau(\theta)}{\kappa(\theta)}$.

If we write $\{\mathbf{t}(\theta),\mathbf{n}(\theta),\mathbf{b}(\theta)\}$ with respect to the orthonormal basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ as following:

$$\mathbf{t}(\theta)=t_1(\theta)\mathbf{e}_1+t_2(\theta)\mathbf{e}_2+t_3(\theta)\mathbf{e}_3,$$

$$\mathbf{n}(\theta)=n_1(\theta)\mathbf{e}_1+n_2(\theta)\mathbf{e}_2+n_3(\theta)\mathbf{e}_3, \tag{3.9}$$

$$\mathbf{b}(\theta)=\mathbf{t}(\theta)\times\mathbf{n}(\theta)=b_1(\theta)\mathbf{e}_1+b_2(\theta)\mathbf{e}_2+b_3(\theta)\mathbf{e}_3.$$

Theorem 3.2. *Let $\gamma : I \to \mathsf{E}(2)$ is a non geodesic spacelike biharmonic general helix with timelike normal in the Lorentzian group of rigid motions $\mathsf{E}(2)$. Then, the parametric equations of γ are*

$$x(\theta) = \cos\wp\,\theta + a_1,$$

$$y(\theta) = \frac{\sin\wp}{\cos^2\wp + \Xi_1^2}\big((\cos\wp - \Xi_1)\sin[\cos\wp\,\theta + a_1]\cosh[\Xi_1\theta + \Xi_2]$$

$$+(\cos\wp + \Xi_1)\cos[\cos\wp\,\theta + a_1]\sinh[\Xi_1\theta + \Xi_2]\big) + a_2, \tag{3.10}$$

$$z(\theta) = \frac{\sin\wp}{\cos^2\wp + \Xi_1^2}\big((\Xi_1 - \cos\wp)\cos[\cos\wp\,\theta + a_1]\cosh[\Xi_1\theta + \Xi_2]$$

$$+(\cos\wp + \Xi_1)\sin[\cos\wp\,\theta + a_1]\sinh[\Xi_1\theta + \Xi_2]\big) + a_3,$$

where a_1, a_2, a_3, Ξ_1, Ξ_2 are constants of integration and $\wp$ is constant angle.

Proof. Suppose that γ is a non geodesic spacelike biharmonic curve. Substituting the first equation of the Frenet equations (3.8) in the second equation of (3.8), we obtain

$$\mathbf{b}(\theta) = \frac{1}{f(\theta)}\left[\nabla^2_{\mathbf{t}(s)}\mathbf{t}(\theta) - \mathbf{t}(\theta)\right]. \tag{3.11}$$

Using the last equation of (3.8), we obtain

$$\nabla^3_{\mathbf{t}(s)}\mathbf{t}(\theta) - \left(1 + f^2(\theta)\right)\nabla_{\mathbf{t}(s)}\mathbf{t}(\theta) = 0. \tag{3.12}$$

Since the curve $\gamma(\theta)$ is a spacelike general helix, i.e. the tangent vector $\mathbf{t}(\theta)$ makes a constant angle $\wp$, with the constant spacelike vector called the axis of the general helix. So, without loss of generality, we take the axis of a general helix as being parallel to the spacelike vector $\mathbf{e}_1$. Then, using first equation of (3.9), we get

$$t_1(\theta) = g(\mathbf{t}(\theta), \mathbf{e}_1) = \cos\wp. \tag{3.13}$$

On other hand, the tangent vector $\mathbf{T}(\theta)$ is a unit spacelike vector, so the following condition is satisfied:

$$t_2^2(\theta) - t_3^2(\theta) = 1 - \cos^2\wp. \tag{3.14}$$

The general solution of (3.14) can be written in the following form:

$$t_2(\theta) = \sin\wp\cosh\sigma(\theta), 3.15 \tag{12}$$

$$t_3(\theta) = \sin\wp\sinh\sigma(\theta),$$

where σ is an arbitrary function of θ.

So, substituting the components $t_1(\theta)$, $t_2(\theta)$ and $t_3(\theta)$ in the first equation of (3.9), we have the following equation

$$\mathbf{t} = \cos\wp\,\mathbf{e}_1 + \sin\wp\cosh\sigma(\theta)\mathbf{e}_2 + \sin\wp\sinh\sigma(\theta)\mathbf{e}_3. \tag{3.16}$$

If we substitute (3.5) in (3.12), we have

$$\sigma'(\theta)\sigma''(\theta) = 0. \tag{3.17}$$

The general solution of (3.17) is

$$\sigma(\theta) = \Xi_1\theta + \Xi_2, \tag{3.18}$$

where Ξ_1, Ξ_2 are constants of integration.

Thus (3.16) and (3.18), imply

$$\mathbf{t} = \cos\wp\,\mathbf{e}_1 + \sin\wp\cosh[\Xi_1\theta + \Xi_2]\mathbf{e}_2 + \sin\wp\sinh[\Xi_1\theta + \Xi_2]\mathbf{e}_3. \tag{3.19}$$

Using (2.1) in (3.19), we obtain

$$\begin{aligned}\mathbf{t} = (&\cos\wp, \cos[\cos\wp\,\theta + a_1]\sin\wp\cosh[\Xi_1\theta + \Xi_2] \\ &-\sin[\cos\wp\,\theta + a_1]\sin\wp\sinh[\Xi_1\theta + \Xi_2], \\ &\sin[\cos\wp\,\theta + a_1]\sin\wp\cosh[\Xi_1\theta + \Xi_2] \\ &+\cos[\cos\wp\,\theta + a_1]\sin\wp\sinh[\Xi_1\theta + \Xi_2]),\end{aligned} \tag{3.20}$$

where a_1 is constant of integration.

Also, we have

$$\frac{dx}{d\theta} = \cos\wp,$$

$$\frac{dy}{d\theta} = \cos[\cos\wp\,\theta + a_1]\sin\wp\cosh[\Xi_1\theta + \Xi_2]$$

$$-\sin[\cos\wp\,\theta + a_1]\sin\wp\sinh[\Xi_1\theta + \Xi_2], \tag{3.21}$$

$$\frac{dz}{d\theta} = \sin[\cos\wp\,\theta + a_1]\sin\wp\cosh[\Xi_1\theta + \Xi_2]$$

$$+\cos[\cos\wp\,\theta + a_1]\sin\wp\sinh[\Xi_1\theta + \Xi_2].$$

If we take the integral (3.21), we get (3.10). Thus, the proof is completed.

Theorem 3.3. *Let $\gamma : I \to \mathrm{E}(2)$ is a non geodesic spacelike biharmonic general helix with timelike normal in the Lorentzian group of rigid motions $\mathrm{E}(2)$. Then, the parametric equations of γ are*

$$x^1(s) = \cos\wp\,\kappa s + a_1,$$

$$x^2(s) = \frac{\sin\wp}{\cos^2\wp + \Xi_1^2}((\cos\wp - \Xi_1)\sin[\cos\wp\,\kappa s + a_1]\cosh[\Xi_1\kappa s + \Xi_2]$$

$$+(\cos\wp + \Xi_1)\cos[\cos\wp\,\kappa s + a_1]\sinh[\Xi_1\kappa s + \Xi_2]) + a_2, \tag{3.22}$$

$$x^3(s) = \frac{\sin\wp}{\cos^2\wp + \Xi_1^2}((\Xi_1 - \cos\wp)\cos[\cos\wp\,\kappa s + a_1]\cosh[\Xi_1\kappa s + \Xi_2]$$

$$+(\cos\wp + \Xi_1)\sin[\cos\wp\,\kappa s + a_1]\sinh[\Xi_1\kappa s + \Xi_2]) + a_3,$$

where a_1, a_2, a_3 are constants of integration.

Proof. From first equation of (3.4) and the definition of θ, we have

$$\theta = \kappa s. \tag{3.23}$$

So, substituting (3.23) in the system (3.10), we have (3.22) and the assertion is proved.

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Uniform Motion of Timelike Spherical Magnetic Curves on the De-Sitter Space S_1^2

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Abstract

In this paper, we investigate uniform motion of timelike spherical magnetic curves associated with the given magnetic field G on the De-Sitter 2-space S_1^2. Finally, we also analyze the necessary and sufficient conditions of the uniformity of the timelike magnetic curves lying on the S_1^2.

1. Introduction

A magnetic field on a k-dimensional semi-Riemannian manifold (R,h), which has the Levi-Civita connection ∇, is any closed 2-form G on R such that its Lorentz force is a one-to-one anti-symmetric tensor field Ψ given by $h(\Psi(\mathsf{A}),\mathsf{B})=\mathsf{G}(\mathsf{A},\mathsf{B})$, where A,B are any two vector fields tangent to R.

A charged particle follows a trajectory δ under the influence of G, which meets the Lorentz formula $\nabla_{\delta'}\delta' = \Psi(\delta)$. As seen, the natural generalization of geodesics, which meet the Lorentz formula without the influence of any magnetic field, is given by magnetic curves.

A comprehensive research effort has been conducted to explore the magnetic curves and their flows. For instance, scientists showed that Kirchhoff elastic rods are obtained as one of the solution families of the Lorentz force formula. This constructs an affiliation between two unapparent physical phenomena, that is, the Hall effect and the classical elastic theory. Furthermore, critical points of the Landau-Hall functional are acquired as one of the other solution families of the Lorentz force formula. Thus, it also implies that magnetic curves are used to solve a variational problem $[1]$.

As it can be seen in the literature, the major patterns to be taken into account were the situations of magnetic curves in Riemannian spaces and in Riemannian surfaces of constant sectional curvature consecutively regarding situations of less simple curvature, different signatures, and higher dimensions.

A classical model of magnetic fields is easily developed if one multiplies a scalar p (generally known as magnitude or strength) to the area form on a Riemannian surface (R,h). For instance, on a hyperbolic plane H^2, magnetic trajectories are either open curves or closed curves, on the Euclidean plane magnetic trajectories are circles, and on the sphere $\mathbb{S}^2$, they are tiny circles having a particular radius $[2,3]$.

In this construction, one can observe some exclusive behaviors in the three-dimensional case due to the fact that the volume form dv_h and the Hodge star operator of the manifold identify a $(1-1)$ correspondence between divergence-free vector fields and closed 2-forms. In three-dimensional pseudo-Riemannian manifold, this leads to describe the special class of Killing magnetic curves and Killing magnetic fields [4-7].

2 Uniform Motion of Timelike Spherical Magnetic Curves on the De-Sitter Space $\mathbb{S}_1^2$

The description of a uniformly accelerated motion (UAM) in relativity has always been of great interest for many scientists. For example, Rindler used the relation between Lorentzian circles and uniformly accelerated motion in Minkowski spacetime to determine hyperbolic motion in General Relativity $[14,15]$. Covariant definition of the UAM and its explicit solutions were investigated by Friedman and Scarr $[16,17]$. The notion of the UAM was analyzed in detail by giving its novel geometric characterization by Fuente and Romero $[18]$. The description of the unchanged direction motion (UDM) was presented by extending the UAM by Fuente, Romero, and Torres $[19]$. The intrinsic definition of the uniformly circular motion (UCM) was given by Fuente, Romero, and Torres as a particular case of a planar motion $[20]$. In this section, we investigate the UAM, UDM, and UCM of the moving charged particle corresponding to any unit speed timelike spherical magnetic curve in the associated magnetic field $\mathbb{G}$ on the $\mathbb{S}_1^2$. We determine necessary and sufficient conditions that have to be satisfied by the particle in terms of the Sabban scalars of the worldline of magnetic curves. We use the following definitions of the UAM, UDM, and UCM.

Definition 1. *The particle obeys a UAM if*

$$\nabla_{FW}(\nabla_{\delta'}\delta') = 0,$$

where ∇_{FW} *is a Fermi-Walker derivative connection of the curve* δ $[18]$.

Definition 2. *The particle obeys a UDM if*

$$\left|\nabla_{\delta'}\delta'\right|^2 = \text{constant},$$

and

$$\left|\nabla_{FW}(\nabla_{\delta'}\delta')\right|^2 = \text{constant},$$

where ∇_{FW} *is a Fermi-Walker derivative connection of the curve* δ [19].

Definition 3. *The particle obeys a UCM if*

$$\nabla_{FW}(\left|\nabla_{\delta'}\delta'\right|^{-1}\nabla_{\delta'}\delta') = 0,$$

where ∇_{FW} *is a Fermi-Walker derivative connection of the curve* δ [20].

Definition 4. Let ∇_{FW} be the covariant derivative corresponding to Levi-Civita connection ∇ of h. *Then, we have* [29],

$$\nabla_{FW}\mathsf{R} = \nabla_{\delta'}\mathsf{R} + h(\delta', \mathsf{R})\nabla_{\delta'}\delta' - h(\nabla_{\delta'}\delta', \mathsf{R})\delta',$$

where R *is any vector field along the curve* δ.

In the presence of an electromagnetic field, dynamics of the charged particle is defined by the Lorentz force [18,21]. Now, let β be a moving charged particle such that it corresponds to a unit speed timelike spherical magnetic curves in the associated magnetic field G on the S_1^2.

♦ *In the case of an* $\mathbf{S}\delta-$*magnetic curve,* the magnetic trajectories obey the UAM iff

$$\nabla_{FW}(\Psi(\delta)) = 0.$$

the magnetic trajectories obey the UDM iff

$$\left|\Psi(\delta)\right|^2, \left|\nabla_{FW}(\Psi(\delta))\right|^2 \text{ are both constant;}$$

the magnetic trajectories obey the UCM iff

$$\nabla_{FW}(\left|\Psi(\delta)\right|^{-1}\Psi(\delta)) = 0.$$

Corollary 1. i. If one considers Eqs. $(6,23,24)$, then it is computed that $\nabla_{FW}(\Psi(\delta)) = 0$. Thus, magnetic trajectories of the $\mathbf{S}\delta-$*magnetic curve* obey the UAM.

ii. It is computed that $\left|\Psi(\delta)\right|^2 = 1$ and $\left|\nabla_{FW}(\Psi(\delta))\right|^2 = 0$. Thus, magnetic trajectories of the $\mathbf{S}\delta-$*magnetic curve* obey the UDM.

iii. It is computed that $\nabla_{FW}(\left|\Psi(\delta)\right|^{-1}\Psi(\delta)) = 0$. Thus, magnetic trajectories of the $\mathbf{S}\delta-$*magnetic curve* obey the UCM.

♦ *In the case of an* $\mathbf{ST}-$*magnetic curve,* the magnetic trajectories obey the UAM iff

$$\nabla_{FW}(\Psi(\mathbf{T})) = 0.$$

the magnetic trajectories obey the UDM iff

$$\left|\Psi(\mathbf{T})\right|^2, \left|\nabla_{FW}(\Psi(\mathbf{T}))\right|^2 \text{ are both constant;}$$

the magnetic trajectories obey the UCM iff

$$\nabla_{FW}(\left|\Psi(\mathbf{T})\right|^{-1}\Psi(\mathbf{T})) = 0.$$

Corollary 2. i. It is computed that $\nabla_{FW}(\Psi(\delta)) = \mu'\mathbf{N}$. Thus, magnetic trajectories of the $\mathbf{ST}-$*magnetic curve* obey the UAM iff the geodesic curvature is a constant i.e. trajectories follow a pseudo-circle whose center Ω lied on the S_1^2. Here, the center is defined by $\Omega = (\mu\delta - \mathbf{N})/(1+\mu^2)^{\frac{1}{2}}$ [22].

ii. Then it is computed that $|\Psi(\delta)|^2 = 1+\mu^2$ and $|\nabla_{FW}(\Psi(\delta))|^2 = (\mu')^2$. Thus, magnetic trajectories of the $\mathbf{ST}-$*magnetic curve* obey the UDM iff the geodesic curvature is a constant i.e. trajectories follow a pseudo-circle whose center Ω lied on the S_1^2.

iii. Then it is computed that $\nabla_{FW}(|\Psi(\delta)|^{-1}\Psi(\delta)) = (\frac{1}{1+\mu^2})'\delta + (\frac{\mu}{1+\mu^2})'\mathbf{N}$. Thus, magnetic trajectories of the $\mathbf{ST}-$*magnetic curve* obey the UCM iff $\frac{1}{1+\mu^2}$ and $\frac{\mu}{1+\mu^2}$ are both constant, which implies that the geodesic curvature is a constant i.e. trajectories follow a pseudo-circle whose center Ω lied on the S_1^2.

♦ *In the case of an* $\mathbf{SN}-$*magnetic curve,* the magnetic trajectories obey the AUM iff

$$\nabla_{FW}(\Psi(\mathbf{N})) = 0.$$

the magnetic trajectories obey the UDM iff

$$|\Psi(\mathbf{N})|^2, |\nabla_{FW}(\Psi(\mathbf{N}))|^2 \text{ arebothconstant;}$$

the magnetic trajectories obey the UCM iff

$$\nabla_{FW}(|\Psi(\mathbf{N})|^{-1}\Psi(\mathbf{N})) = 0.$$

Corollary 3. i. It is computed that $\nabla_{FW}(\Psi(\delta)) = \mu'\mathbf{T}$. Thus, magnetic trajectories of the $\mathbf{SN}-$*magnetic curve* obey the UAM iff the geodesic curvature is a constant i.e. trajectories follow a pseudo-circle whose center Ω lied on the S_1^2. Here, the center is defined by $\Omega = (\mu\delta - \mathbf{N})/(1+\mu^2)^{\frac{1}{2}}$ [22].

ii. It is computed that $|\Psi(\delta)|^2 = -\mu^2$ and $|\nabla_{FW}(\Psi(\delta))|^2 = -(\mu')^2$. Thus, magnetic trajectories of the $\mathbf{SN}-$*magnetic curve* obey the UDM iff the geodesic curvature is a constant i.e. trajectories follow a pseudo-circle whose center Ω lied on the S_1^2.

iii. It is computed that $\nabla_{FW}(|\Psi(\delta)|^{-1}\Psi(\delta))=0.$ Thus, magnetic trajectories of the $\mathbf{SN}-$*magnetic curve* obey the UCM.

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Smarandache sα curves according to Sabban frame in the Heisenberg group Heis³

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Abstract

In this paper, we study Smarandache sα curves according to Sabban frame in the Heisenberg group Heis³. Finally, we find explicit parametric equations of Smarandache sα curves according to Sabban Frame.

1. Introduction

A smooth map $\phi : N \to M$ is said to be biharmonic if it is a critical point of the bienergy functional:

$$E_2(\phi) = \int_N \frac{1}{2} \left| \mathrm{T}(\phi) \right|^2 dv_h,$$

where $\mathrm{T}(\phi) := \mathrm{tr}\nabla^{\phi} d\phi$ is the tension field of ϕ.

The Euler--Lagrange equation of the bienergy is given by $\mathrm{T}_2(\phi) = 0$. Here the section $\mathrm{T}_2(\phi)$ is defined by

$$\mathrm{T}_2(\phi) = -\Delta_\phi \mathrm{T}(\phi) + \mathrm{tr}R(\mathrm{T}(\phi), d\phi)d\phi, \tag{1.1}$$

and called the bitension field of ϕ. Non-harmonic biharmonic maps are called proper biharmonic maps.

This study is organised as follows: Firstly, we study Smarandache **sα** curves according to Sabban frame in the Heisenberg group Heis^3. Finally, we find explicit parametric equations of Smarandache **sα** curves according to Sabban Frame.

2. The Heisenberg Group Heis^3

Heisenberg group Heis^3 can be seen as the space $\mathbb{R}^3$ endowed with the following multipilcation:

$$(\bar{x}, \bar{y}, \bar{z})(x, y, z) = (\bar{x} + x, \bar{y} + y, \bar{z} + z - \frac{1}{2}\bar{x}y + \frac{1}{2}x\bar{y}) \tag{2.1}$$

Heis^3 is a three-dimensional, connected, simply connected and 2-step nilpotent Lie group.

The Riemannian metric g is given by

$$g = dx^2 + dy^2 + (dz - xdy)^2.$$

The Lie algebra of Heis^3 has an orthonormal basis

$$\mathbf{e}_1 = \frac{\partial}{\partial x}, \mathbf{e}_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \mathbf{e}_3 = \frac{\partial}{\partial z}, \tag{2.2}$$

for which we have the Lie products

$$[\mathbf{e}_1, \mathbf{e}_2] = \mathbf{e}_3, [\mathbf{e}_2, \mathbf{e}_3] = [\mathbf{e}_3, \mathbf{e}_1] = 0$$

with

$$g(\mathbf{e}_1, \mathbf{e}_1) = g(\mathbf{e}_2, \mathbf{e}_2) = g(\mathbf{e}_3, \mathbf{e}_3) = 1.$$

3. Biharmonic $\mathbb{S}$ – Helices According To Sabban Frame In The Heisenberg Group Heis^3

Lemma 3.1. $\alpha : I \to \mathbb{S}^2_{Heis^3}$ *is a biharmonic* $\mathbb{S}$*-curve if and only if*

$$\kappa_g = \text{constant} \neq 0,$$

$$1 + \kappa_g^2 = -[\frac{1}{4} - s_3^2] + \kappa_g[-\alpha_3 s_3],$$

$$\kappa_g^3 = -\alpha_3 s_3 - \kappa_g[\frac{1}{4} - \alpha_3^2].$$

Then the following result holds.

Theorem 3.2. ([9]) *All of biharmonic* $\mathbb{S}$*-curves in* $\mathbb{S}^2_{Heis^3}$ *are helices.*

Theorem 3.3. ([9]) *Let* $\alpha : I \to \mathbb{S}^2_{Heis^3}$ *be a unit speed non-geodesic biharmonic* $\mathbb{S}$*-curve. Then, the position vector of* α *is*

$$\alpha(\sigma)=[-\frac{\sin^2\mathsf{E}}{\mathsf{V}}\cos[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_2]\mathbf{e}_1+[\frac{\sin^2\mathsf{E}}{\mathsf{V}}\sin[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_3]\mathbf{e}_2$$

$$+[\cos\mathsf{E}\sigma-\frac{\mathsf{V}\sigma+\mathsf{M}_1}{2\mathsf{V}^2}\sin^4\mathsf{E}-\frac{\sin 2[\mathsf{M}\sigma+\mathsf{M}_1]}{4\mathsf{V}^2}\sin^4\mathsf{E}$$

$$-[\frac{\sin^2\mathsf{E}}{\mathsf{V}}\sin[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_3][-\frac{\sin^2\mathsf{E}}{\mathsf{V}}\cos[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_2]$$

$$+\frac{\mathsf{M}_2}{\mathsf{V}}\sin^3\mathsf{E}\sin[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_4]\mathbf{e}_3,$$

where $\mathsf{M}_1,\mathsf{M}_2,\mathsf{M}_3,\mathsf{M}_4$ *are constants of integration and*

$$\mathsf{M}=(\frac{\sqrt{1+\kappa_g^2}}{\sin\mathsf{E}}-\cos\mathsf{E})\text{and}\mathsf{V}=\sqrt{1+\kappa_g^2}-\frac{1}{2}\sin 2\mathsf{E}.$$

4 Smarandache sα Curves Of Biharmonic S-Curves According To Sabban Frame In The Heisenberg Group Heis^3

Definition 4.1. *Let* $\alpha:I\to\mathsf{S}^2_{Heis^3}$ *be a unit speed regular curve in the Heisenberg group Heis*^{3} *and* $\{\alpha,\mathbf{t},\mathbf{s}\}$ *be its moving Bishop frame. Smarandache* **sα** *curves are defined by*

$$\gamma_{\mathbf{s\alpha}}=\frac{1}{\sqrt{1+\kappa_g^2}}(\mathbf{s}+\boldsymbol{\alpha}). \tag{4.1}$$

Theorem 4.2. *Let* $\alpha:I\to\mathsf{S}^2_{Heis^3}$ *be a unit speed non-geodesic biharmonic* S*-curve* $\gamma_{\mathbf{ts}}$ *its Smarandache* **sα** *curve. Then, the position vector of Smarandache* **sα** *curve is*

$$\gamma_{\mathbf{s\alpha}}(\sigma)=\frac{1}{\sqrt{1+\kappa_g^2}}[-\frac{\sin^2\mathsf{E}}{\mathsf{V}}\cos[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_2+\frac{1}{\kappa_g}[\sin\mathsf{E}\cos[\mathsf{M}\sigma+\mathsf{M}_1](\mathsf{M}$$

$$+\cos\mathsf{E})-\frac{\sin^2\mathsf{E}}{\mathsf{V}}\cos[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_2]]\mathbf{e}_1$$

$$+\frac{1}{\sqrt{1+\kappa_g^2}}[\frac{\sin^2\mathsf{E}}{\mathsf{V}}\sin[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_3+\frac{1}{\kappa_g}[-\sin\mathsf{E}\sin[\mathsf{M}\sigma+\mathsf{M}_1](\mathsf{M}$$

$$+\cos\mathsf{E})+\frac{\sin^2\mathsf{E}}{\mathsf{V}}\sin[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_3]]\mathbf{e}_2$$

$$+\frac{1}{\sqrt{1+\kappa_g^2}}[\cos\mathsf{E}\sigma-\frac{\mathsf{V}\sigma+\mathsf{M}_1}{2\mathsf{V}^2}\sin^4\mathsf{E}-\frac{\sin 2[\mathsf{M}\sigma+\mathsf{M}_1]}{4\mathsf{V}^2}\sin^4\mathsf{E}$$

$$-[\frac{\sin^2\mathsf{E}}{\mathsf{V}}\sin[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_3][-\frac{\sin^2\mathsf{E}}{\mathsf{V}}\cos[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_2]$$

$$+\frac{\mathsf{M}_2}{\mathsf{V}}\sin^3\mathsf{E}\sin[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_4$$

$$+\frac{1}{\kappa_g}[\cos\mathsf{E}\sigma-\frac{\mathsf{V}\sigma+\mathsf{M}_1}{2\mathsf{V}^2}\sin^4\mathsf{E}-\frac{\sin 2[\mathsf{M}\sigma+\mathsf{M}_1]}{4\mathsf{V}^2}\sin^4\mathsf{E}$$

$$-[\frac{\sin^2\mathsf{E}}{\mathsf{V}}\sin[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_3][-\frac{\sin^2\mathsf{E}}{\mathsf{V}}\cos[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_2]$$

$$+\frac{\mathsf{M}_2}{\mathsf{V}}\sin^3\mathsf{E}\sin[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_4]\mathbf{e}_3,$$

where $\mathsf{M}_1,\mathsf{M}_2,\mathsf{M}_3,\mathsf{M}_4$ *are constants of integration and*

$$\mathsf{M}=(\frac{\sqrt{1+\kappa_g^2}}{\sin\mathsf{E}}-\cos\mathsf{E})\text{and}\mathsf{V}=\sqrt{1+\kappa_g^2}-\frac{1}{2}\sin 2\mathsf{E}.$$

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Frenet ribbons of timelike biharmonic curves according to flat metric in the Lorentzian Heisenberg group Heis³

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Abstract

In this paper, we study Frenet ribbons of timelike biharmonic curves according to flat metric in the Lorentzian Heisenberg group Heis³. We give some characterizations for Frenet ribbons of timelike biharmonic curves. Moreover, we obtain Tchebyshef net on Frenet ribbons in the Lorentzian Heisenberg group Heis³.

1. Introduction

In its classical form web geometry studies local configurations of finitely many smooth foliations in general position. Germs of webs defined by few foliations in general position are far from being interesting. Basic results from differential calculus imply that the theory is locally trivial. As soon as the number of foliations surpasses the dimension of the ambient manifold this is no longer true. The discovery in the last years of the 1920 decade of the curvature for 3-webs on surfaces is considered as the birth of web geometry.

On the other hand, as suggested by Eells and Sampson in [8], we can define the bienergy of a map f by

$$E_2(f)=\frac{1}{2}\int_M |\tau(f)|^2 v_g,$$

where $\tau(f)=\operatorname{trace}\nabla df$ is tension field and say that is biharmonic if it is a critical point of the bienergy.

Jiang derived the first and the second variation formula for the bienergy in [11], showing that the Euler--Lagrange equation associated to E_2 is

$$\tau_2(f)=-\mathrm{J}^f(\tau(f))=-\Delta\tau(f)-\operatorname{trace}R^N(df,\tau(f))df \tag{1.1}$$

$$=0,$$

where J^f is the Jacobi operator of f. The equation $\tau_2(f)=0$ is called the biharmonic equation. Since J^f is linear, any harmonic map is biharmonic. Therefore, we are interested in proper biharmonic maps, that is non-harmonic biharmonic maps.

In this paper, we study Frenet ribbons of timelike biharmonic curves according to flat metric in the Lorentzian Heisenberg group Heis^3. We give some characterizations for Frenet ribbons of timelike biharmonic curves. Moreover, we obtain Tchebyshef net on Frenet ribbons in the Lorentzian Heisenberg group Heis^3.

2. Preliminaries

The Heisenberg group Heis^3 is a Lie group which is diffeomorphic to $\mathbb{R}^3$ and the group operation is defined as

$$(x,y,z)*(\overline{x},\overline{y},\overline{z})=(x+\overline{x},y+\overline{y},z+\overline{z}-\overline{x}y+x\overline{y}).$$

The identity of the group is $(0,0,0)$ and the inverse of (x,y,z) is given by $(-x,-y,-z)$. The left-invariant Lorentz metric on Heis^3 is

$$g=dx^2+(xdy+dz)^2-((1-x)dy-dz)^2.$$

The following set of left-invariant vector fields forms an orthonormal basis for the corresponding Lie algebra:

$$\left\{\mathbf{e}_1=\frac{\partial}{\partial x},\mathbf{e}_2=\frac{\partial}{\partial y}+(1-x)\frac{\partial}{\partial z},\mathbf{e}_3=\frac{\partial}{\partial y}-x\frac{\partial}{\partial z}\right\}. \tag{2.1}$$

The characterising properties of this algebra are the following commutation relations:

$$[\mathbf{e}_2,\mathbf{e}_3]=0,[\mathbf{e}_3,\mathbf{e}_1]=\mathbf{e}_2-\mathbf{e}_3,[\mathbf{e}_2,\mathbf{e}_1]=\mathbf{e}_2-\mathbf{e}_3,$$

with

$$g(\mathbf{e}_1,\mathbf{e}_1)=g(\mathbf{e}_2,\mathbf{e}_2)=1,\ g(\mathbf{e}_3,\mathbf{e}_3)=-1. \tag{2.2}$$

Proposition 2.1. *For the covariant derivatives of the Levi-Civita connection of the left-invariant metric g, defined above the following is true:*

$$\nabla = \begin{pmatrix} 0 & 0 & 0 \\ \mathbf{e}_2 - \mathbf{e}_3 & -\mathbf{e}_1 & -\mathbf{e}_1 \\ \mathbf{e}_2 - \mathbf{e}_3 & -\mathbf{e}_1 & -\mathbf{e}_1 \end{pmatrix}, \tag{2.3}$$

where the (i, j)-element in the table above equals $\nabla_{e_i} e_j$ for our basis

$$\{\mathbf{e}_k, k = 1,2,3\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}.$$

So, we obtain that

$$R(\mathbf{e}_1, \mathbf{e}_3) = R(\mathbf{e}_1, \mathbf{e}_2) = R(\mathbf{e}_2, \mathbf{e}_3) = 0. \tag{2.4}$$

Then, the Lorentz metric g is flat.

3. Timelike Biharmonic Curves According to Flat Metric in the Lorentzian Heisenberg Group $\mathbf{Heis}^3$

Let $\gamma : I \to Heis^3$ be a unit speed timelike curve and $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ are Frenet vector fields, then Frenet formulas are as follows

$$\begin{aligned} &\nabla_{\mathbf{T}}\mathbf{T} = \kappa_1 \mathbf{N}, \\ &\nabla_{\mathbf{T}}\mathbf{N} = \kappa_1 \mathbf{T} + \kappa_2 \mathbf{B}, \\ &\nabla_{\mathbf{T}}\mathbf{B} = -\kappa_2 \mathbf{N}, \end{aligned} \tag{3.1}$$

where κ_1, κ_2 are curvature function and torsion function, respectively.

Theorem 3.1. *(see [16]) If $\gamma : I \to Heis^3$ is a unit speed timelike biharmonic curve according to flat metric, then*

$$\begin{aligned} &\kappa_1 = \text{constant} \neq 0, \\ &\kappa_1^2 - \kappa_2^2 = 0, \\ &\kappa_2 = \text{constant}. \end{aligned} \tag{3.2}$$

4. Frenet Ribbons of Timelike Biharmonic Curve in the Lorentzian Heisenberg Group

The Frenet-Serret apparatus allows one to define certain optimal ribbons and tubes centered around a curve. These have diverse applications in materials science and elasticity theory, as well as to computer graphics.

A Frenet ribbon along a curve γ is the surface traced out by sweeping the line segment $[-a,a]$ generated by the unit normal along the curve. Geometrically, a ribbon is a piece of the envelope of the osculating planes of the curve. Symbolically, the ribbon R has the following parametrization:

$$\mathsf{R}(s,t) = \gamma(s) + t\mathbf{N}, -1 \leq t \leq 1. \tag{4.1}$$

In particular, the binormal $\mathbf{B}$ is a unit vector normal to the ribbon. Moreover, the ribbon is a ruled surface whose reguli are the line segments spanned by $\mathbf{N}$. Thus each of the frame vectors $\mathbf{T}, \mathbf{N},$ and $\mathbf{B}$ can be visualized entirely in terms of the Frenet ribbon.

Theorem 4.1. *Let $\gamma : I \to Heis^3$ be a unit speed timelike biharmonic curve and $\mathsf{R}(s,t)$ its Frenet ribbon in $Heis^3$. Then, the parametric equations of $\mathsf{R}(s,t)$ are*

$$x_{\mathsf{R}}(s,t) = \sinh\varphi s + \ell_1$$

$$-\frac{t}{\kappa_1}\sinh[\frac{\kappa_1 s}{\cosh\varphi} + \ell]\cosh^2\varphi(\sinh[\frac{\kappa_1 s}{\cosh\varphi} + \ell] + \cosh[\frac{\kappa_1 s}{\cosh\varphi} + \ell])$$

$$y_{\mathsf{R}}(s,t) = \frac{1}{\kappa_1}\cosh^2\varphi\cosh[\frac{\kappa_1 s}{\cosh\varphi} + \ell] + \sinh[\frac{\kappa_1 s}{\cosh\varphi} + \ell]]$$

$$+\frac{t}{\kappa_1}(\kappa_1\cosh[\frac{\kappa_1 s}{\cosh\varphi} + \ell] + \sinh\varphi\cosh\varphi\sinh[\frac{\kappa_1 s}{\cosh\varphi} + \ell]$$

$$+\sinh\varphi\cosh\varphi\cosh[\frac{\kappa_1 s}{\cosh\varphi} + \ell])$$

$$+\frac{t}{\kappa_1}(\kappa_1 \sinh[\frac{\kappa_1 s}{\cosh\varphi}+\ell]-\sinh\varphi\cosh\varphi\sinh[\frac{\kappa_1 s}{\cosh\varphi}+\ell]$$

$$-\sinh\varphi\cosh\varphi\cosh[\frac{\kappa_1 s}{\cosh\varphi}+\ell])+\ell_2,$$

$$z_R(s,t)=-\frac{(-1+\ell_1+\sinh\varphi s)\cosh\varphi}{\kappa_1}\cosh[\frac{\kappa_1 s}{\cosh\varphi}+\ell] \tag{4.2}$$

$$+\frac{\cosh^2\varphi\sinh\varphi}{\kappa_1^2}[\sinh[\frac{\kappa_1 s}{\cosh\varphi}+\ell]+\cosh[\frac{\kappa_1 s}{\cosh\varphi}+\ell]]$$

$$-\frac{\cosh\varphi(\sinh\varphi s+\ell_1)}{\kappa_1}\sinh[\frac{\kappa_1 s}{\cosh\varphi}+\ell]]$$

$$+\frac{t}{\kappa_1}(\kappa_1 \cosh[\frac{\kappa_1 s}{\cosh\varphi}+\ell]+\sinh\varphi\cosh\varphi\sinh[\frac{\kappa_1 s}{\cosh\varphi}+\ell]$$

$$+\sinh\varphi\cosh\varphi\cosh[\frac{\kappa_1 s}{\cosh\varphi}+\ell])$$

$$.(1+\cosh^2\varphi-\frac{\ell\cosh\varphi}{2\kappa_1}-\frac{s^2}{4}+(\frac{\kappa_1 s}{\cosh\varphi}+\ell)$$

$$+\frac{\cosh^2\varphi}{8\kappa_1^2}(\cosh 2[\frac{\kappa_1 s}{\cosh\varphi}+\ell]+\sinh[\frac{\kappa_1 s}{\cosh\varphi}+\ell]+2(\frac{\kappa_1 s}{\cosh\varphi}+\ell))])$$

$$+\frac{t}{\kappa_1}(\kappa_1 \sinh[\frac{\kappa_1 s}{\cosh\varphi}+\ell]-\sinh\varphi\cosh\varphi\sinh[\frac{\kappa_1 s}{\cosh\varphi}+\ell]$$

$$-\sinh\varphi\cosh\varphi\cosh[\frac{\kappa_1 s}{\cosh\varphi}+\ell])$$

$$\cdot\cosh^2\varphi-\frac{\ell\cosh\varphi}{2\kappa_1}-\frac{s^2}{4}+(\frac{\kappa_1 s}{\cosh\varphi}+\ell)+\frac{\cosh^2\varphi}{8\kappa_1^2}(\cosh 2[\frac{\kappa_1 s}{\cosh\varphi}+\ell]$$

$$+\sinh[\frac{\kappa_1 s}{\cosh\varphi}+\ell]+2(\frac{\kappa_1 s}{\cosh\varphi}+\ell))])+\ell_3,$$

where $\ell, \ell_1, \ell_2, \ell_3$ *are constants of integration.*

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Galilean transformations of moving particle in Euclidean space R^3

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Abstract

In this paper, we use geometrical descriptions of the curvature and torsion functions of the curve for the calculation. Then we consider a second reference system K^*, which moves relative to the K for an arbitrary direction with a uniform velocity under the Galilean transformation. Finally, we compute the relativistic energy on the moving particle considering relative reference system K^*.

1. Introduction

In physics the Lorentz transformation can be thought as coordinate transformations for any given two coordinate frames, which moves relatively to each other at constant velocity. One of the advantages of this transformation is that it explores the correlation between reference frames provided that both frames are exposed to the same velocity of light. This transformation is used as a base to comprehend some developed concepts such as dilation of time, contraction of length and relativity of simultaneity.

Physical systems and processes either indirectly or directly include the dynamics of fields and/or particles propagating or moving through time and space. Thus, nearly all the major physics laws contain time and position in some way or another. The most fascinating property of this transformation is that it combines the time and space coordinates. As a result of this fact 3-dimensional lengths and time intervals do not have absolute quantities, since 3-dimensional lengths and time intervals are not invariant under this transformation [1].

Galilean transformation, however, is an approximation that we can use only at relative velocity, which is much smaller than the velocity of light. In contrast to Lorentz transformation, Galilean transformation explains the following notions:

i) Time and space are absolute.

ii) Mass, time, and length are not dependent of observer's relative motion.

iii) Velocity of light depends on observer's relative motion [2].

Euclidean geometry is considered to describe idealized space that surrounds us and it can be felt by our tactile and visual perfection. Considering a moving particle as space curves in Euclidean space, it is already investigated various kinematic or geometric properties of them thanks to Frenet frame [5].

In this paper, we deal with the concept of the energy on these space curves, which corresponds to moving particles, in Euclidean space under the Galilean transformation. We observe variation on the energy of the particle in the rest system of inertial frame and its energy in the moving system of inertial frame, which we suppose to move for an arbitrary direction with much smaller velocity than velocity of light. Thus, we call this newly defined energy as a relativistic energy of the particle.

Some researches that motivate us working on this topic can be listed as the following. Energy of the unit vector fields was studied by Wood [6]. Gil-Medrano [7], worked on the relation between energy and volume of vector fields. [8,9], investigated energy distributions and corrected energy distributions on Riemannian manifolds. Altin [10], computed energy of Frenet vector fields for given non-lightlike curves in semi-Euclidean space. Körp nar [11], discussed timelike biharmonic energy of curves in Heisenberg spacetime. Kumar and Srivastava [12], presented difference between curvature and torsion under the relative motion.

2 Preliminaries

2.1 Frenet Frame in Euclidean Space $\mathbb{R}^3$

The characteristics of the intrinsic geometric feautes of a curve Γ can most subtly be determined by using Serret-Frenet equations. Frenet tetrad frame consists of three orthonormal vectors $\mathbf{e}^{\mu}_{(\alpha)}$, assuming the curve Γ is sufficiently smooth at each point. The index within the parenthesis is the tetrad index that describes particular member of the tetrad. In particular, $\mathbf{e}^{\mu}_{(0)}$ is the unit tangent vector, $\mathbf{e}^{\mu}_{(1)}$ is the unit normal, $\mathbf{e}^{\mu}_{(2)}$ is the unit binormal vector of the curve Γ, respectively. Orthonormality conditions are summarized by $\mathbf{e}^{\mu}_{(\alpha)}\mathbf{e}^{\mu}_{(\beta)} = \eta_{\alpha\beta}$, where $\eta_{\alpha\beta}$ is Euclidean metric such that: $\operatorname{diag}(1,1,1)$. For non-negative coefficients κ, τ, and vectors $\mathbf{e}^{\mu}_{(i)}(i = 0,1,2)$ following equations and properties satisfy [5].

$$\frac{\nabla \mathbf{e}^{\mu}_{(0)}}{ds} = \kappa \mathbf{e}^{\mu}_{(1)},$$

$$\frac{\nabla \mathbf{e}^{\mu}_{(1)}}{ds} = -\kappa \mathbf{e}^{\mu}_{(0)} + \tau \mathbf{e}^{\mu}_{(2)}, \tag{2.1}$$

$$\frac{\nabla e^{\mu}_{(2)}}{ds} = -\tau \mathbf{e}^{\mu}_{(1)}.$$

2.2 Energy on the Unit Vector Field in $\mathbb{R}^3$

We first give fundamental definitions and propositions, which are used to compute the energy of the unit vector field.

Definition 2.1 *Let* (M,ρ) *and* $(N,\tilde{h})$ be *two Riemannian manifolds, then the energy of a differentiable map* $f:(M,\rho)\to(N,\tilde{h})$ *can be defined as*

$$\varepsilon nergy(f) = \frac{1}{2}\int_M \sum_{a=1}^{n} \tilde{h}(df(e_a), df(e_a))v, \qquad (2.2)$$

where $\{e_a\}$ *is a local basis of the tangent space and* v *is the canonical volume form in* M [6].

Proposition 2.2 *Let* $Q:T(T^1M)\to T^1M$ *be the connection map. Then following two conditions hold:*

i) $\omega\circ Q=\omega\circ d\omega$ *and* $\omega\circ Q=\omega\circ\tilde{\omega}$, *where* $\tilde{\omega}:T(T^1M)\to T^1M$ *is the tangent bundle projection;*

ii) *for* $\rho\in T_xM$ *and a section* $\xi:M\to T^1M$; *we have*

$$Q(d\xi(\rho))=\nabla_{\rho}\xi, \qquad (2.3)$$

where ∇ *is the Levi-Civita covariant derivative* [6,10].

Definition 2.3 *Let* $\varsigma_1,\varsigma_2\in T_{\xi}(T^1M)$, *then we define*

$$\rho_S(\varsigma_1,\varsigma_2)=\rho(d\omega(\varsigma_1),d\omega(\varsigma_2))+\rho(Q(\varsigma_1),Q(\varsigma_2)). \qquad (2.4)$$

This yields a Riemannian metric on TM. *As known* ρ_S *is called the Sasaki metric that also makes the projection* ω: $T^1M\to M$ *a Riemannian submersion.*

Theorem 2.4 Let Γ be a unit speed curve defined on $\mathbb{R}^3$, we can derive following relations on the energy of tangent, normal, and binormal vectors respectively;

$$\varepsilon nergy\mathbf{e}^{\mu}_{(0)} = \frac{1}{2}\int_0^s \left(1+\kappa^2\right) ds,$$

$$\varepsilon nergy\mathbf{e}^{\mu}_{(1)} = \frac{1}{2}\int_0^s \left(1+\kappa^2+\tau^2\right) ds,$$

$$\varepsilon nergy\mathbf{e}^{\mu}_{(2)} = \frac{1}{2}\int_0^s \left(1+\tau^2\right) ds,$$

where κ, τ are curvature and torsion of the curve Γ, $[6,10]$.

Proof. From (2.2) and (2.3) we know

$$\varepsilon nergy\mathbf{e}^{\mu}_{(0)} = \frac{1}{2}\int_0^s \rho_S \left(d\mathbf{e}^{\mu}_{(0)}(\mathbf{e}^{\mu}_{(0)}), d\mathbf{e}^{\mu}_{(0)}(\mathbf{e}^{\mu}_{(0)})\right) ds.$$

Using Eq. (2.4) we have

$$\rho_S \left(d\mathbf{e}^{\mu}_{(0)}(\mathbf{e}^{\mu}_{(0)}), d\mathbf{e}^{\mu}_{(0)}(\mathbf{e}^{\mu}_{(0)})\right) = \rho(d\omega(\mathbf{e}^{\mu}_{(0)}(\mathbf{e}^{\mu}_{(0)})), d\omega(\mathbf{e}^{\mu}_{(0)}(\mathbf{e}^{\mu}_{(0)})))$$

$$+ \rho(Q(\mathbf{e}^{\mu}_{(0)}(\mathbf{e}^{\mu}_{(0)})), Q(\mathbf{e}^{\mu}_{(0)}(\mathbf{e}^{\mu}_{(0)}))).$$

Since $\mathbf{e}^{\mu}_{(0)}$ is a section, we get

$$d(\omega)\circ d(\mathbf{e}^{\mu}_{(0)}) = d(\omega\circ\mathbf{e}^{\mu}_{(0)}) = d(id_C) = id_{TC}.$$

We also know

$$Q(\mathbf{e}^{\mu}_{(0)}(\mathbf{e}^{\mu}_{(0)})) = \nabla_{\mathbf{e}^{\mu}_{(0)}}\mathbf{e}^{\mu}_{(0)} = \kappa\mathbf{e}^{\mu}_{(1)}.$$

Thus, we find from (2.1)

$$\rho_S \left(d\mathbf{e}^{\mu}_{(0)}(\mathbf{e}^{\mu}_{(0)}), d\mathbf{e}^{\mu}_{(0)}(\mathbf{e}^{\mu}_{(0)})\right) = \rho\left(\mathbf{e}^{\mu}_{(0)}, \mathbf{e}^{\mu}_{(0)}\right) + \rho\left(\nabla_{\mathbf{e}^{\mu}_{(0)}}\mathbf{e}^{\mu}_{(0)}, \nabla_{\mathbf{e}^{\mu}_{(0)}}\mathbf{e}^{\mu}_{(0)}\right)$$

$$= 1+\kappa^2.$$

So we can easily obtain

$$\varepsilon nergy\mathbf{e}^{\mu}_{(0)} = \frac{1}{2}(s + \int_0^s \kappa^2 ds).$$

3. Relativistic Energy on the Moving Inertial Frame

Let K be the first system that contains a moving particle in Euclidean space R^3 such that it has three position vectors with a time parameter. Then we can define another moving relative system K^* corresponding to K such that it moves with uniform velocity v relatively to the K on the direction of $\vec{\mathbf{v}} = (a\mathbf{e}^{\mu}_{(0)} + b\mathbf{e}^{\mu}_{(1)} + c\mathbf{e}^{\mu}_{(2)})v,$ where $a,b,c \in \mathsf{R}$ and $\left\|\vec{\mathbf{v}}\right\| = v.$

Let assume that motion of the moving particle in Euclidean space R^3 corresponds to a space curve Γ, whose Frenet frame characterization is defined in (2.1). Under the Galilean transformation we can transfer this construction to the moving relative inertial system K^* as the following.

$$\Gamma^*(s^*) = \Gamma(s) - \vec{\mathbf{v}}\, s,\ s = s^*, \tag{3.1}$$

where s, s^* are time parameters for the moving particle in the inertial sytem of K and relative inertial system K^*, respectively. We will use s in the moving relative inertial system K^* instead of s^* due to the equality given in (3.1). Thus, we obtain the observed curve $\Gamma^*(s)$ for the relative system as given below.

$$\Gamma^*(s) = \Gamma(s) - v(a\mathbf{e}^{\mu}_{(0)} + b\mathbf{e}^{\mu}_{(1)} + c\mathbf{e}^{\mu}_{(2)})s. \tag{3.2}$$

Theorem 3.1 *The Frenet frame apparatus for the observed curve* $\Gamma^*(s)$ *are stated in terms of the Frenet elements of the first system as following.*

$$\mathbf{e}^{\mu^*}_{(0)} = \left(1+v\left(b\kappa s - a\right)\right)\mathbf{e}^{\mu}_{(0)} + \left(v\left(-a\kappa s + c\tau s - b\right)\right)\mathbf{e}^{\mu}_{(1)} + \left(v\left(-b\tau s - c\right)\right)\mathbf{e}^{\mu}_{(2)},$$

$$\mathbf{e}^{\mu^*}_{(1)} = \frac{1}{\kappa^*} v\left(b\kappa' s + 2b\kappa + a\kappa^2 s - c\kappa\tau s\right)\mathbf{e}^{\mu}_{(0)}$$

$$+\frac{1}{\kappa^*}\left(\kappa + v\left(b\kappa^2 s - 2a\kappa - a\kappa' s + c\tau' s + 2c\tau + b\tau^2 s\right)\right)\mathbf{e}^{\mu}_{(1)}$$

$$+\frac{1}{\kappa^*} v\left(c\tau^2 s - a\kappa\tau s - 2b\tau - b\tau' s\right)\mathbf{e}^{\mu}_{(2)},$$

$$\mathbf{e}^{\mu^*}_{(2)} = \frac{1}{\kappa^*}((v^2\left(-a\kappa s + c\tau s - b\right)\left(c\tau^2 s - a\kappa\tau s - 2b\tau - b\tau' s\right) \tag{3.3}$$

$$+v\left(b\tau s + c\right)\left(\kappa + v\left(b\kappa^2 s - 2a\kappa - a\kappa' s + c\tau' s + 2c\tau + b\tau^2 s\right)\right)\mathbf{e}^{\mu}_{(0)}$$

$$-\frac{1}{\kappa^*}(v\left(1 + vb\kappa s - va\right)\left(c\tau^2 s - a\kappa\tau s - 2b\tau - b\tau' s\right)$$

$$+v^2\left(b\tau s + c\right)\left(b\kappa' s + 2b\kappa + a\kappa^2 s - c\kappa\tau s\right)\mathbf{e}^{\mu}_{(1)}$$

$$+\frac{1}{\kappa^*}(((1+v(b\kappa s - a))(\kappa + v(b\kappa^2 s - 2a\kappa - a\kappa' s + c\tau' s + 2c\tau + b\tau^2 s))$$

$$-v^2(-a\kappa s + c\tau s - b)(b\kappa' s + 2b\kappa + a\kappa^2 s - c\kappa\tau s))\mathbf{e}^{\mu}_{(2)},$$

where κ, τ are curvature and torsion of the Γ and κ^* is the curvature of the observed curve Γ^*. We can also express curvature and torsion of the observed curve Γ^* as the following.

$$\kappa^* = ((bv(\kappa' s + 2\kappa) + vs(a\kappa^2 - \kappa c\tau))^2 + (av(-2\kappa - \kappa' s) + cv(\tau' s + 2\tau)$$

$$+bvs(\kappa^2 + \tau^2) + \kappa)^2 + (bv(-2\tau - \tau' s) + \tau vs(c\tau - a\kappa))^2)^{\frac{1}{2}},$$

$$\tau^* = \frac{1}{u}(v^2\left(-a\kappa s + c\tau s - b\right)\left(c\tau^2 s - a\kappa\tau s - 2b\tau - b\tau' s\right)$$

$$+v\left(b\tau s + c\right)\left(\kappa + v\left(b\kappa^2 s - 2a\kappa - a\kappa' s + c\tau' s + 2c\tau + b\tau^2 s\right)\right)\left(bv\kappa'' s + 3bv\kappa'\right.$$

$$\left.+3av\kappa^2 + 3av\kappa\kappa' s - 2cv\kappa\tau' s - cv\kappa'\tau s - 3cv\kappa\tau - \kappa^2 - b\kappa^3 vs - bv\kappa\tau^2 s\right)$$

$$
\begin{aligned}
&-((v(1+vb\kappa s-va)(c\tau^2 s-a\kappa\tau s-2b\tau-b\tau' s)\\
&+v^2(b\tau s+c)(b\kappa' s+2b\kappa+a\kappa^2 s-c\kappa\tau s))(\kappa'+3bv\kappa\kappa' s+3bv\kappa^2-3av\kappa'\\
&-av\kappa'' s+cv\tau'' s+3cv\tau'+3bv\tau\tau' s+3bv\tau^2+v(a\kappa^3-c\tau^3)s+v\kappa\tau(a\tau-c\kappa)s))\\
&+(((1+v(b\kappa s-a))(\kappa+v(b\kappa^2 s-2a\kappa-a\kappa' s+c\tau' s+2c\tau+b\tau^2 s))\\
&-v^2(-a\kappa s+c\tau s-b)(b\kappa' s+2b\kappa+a\kappa^2 s-c\kappa\tau s))(-2av\kappa'\tau s-av\kappa\tau' s\\
&-3av\kappa\tau+3cv\tau^2+3cv\tau\tau' s-3bv\tau'-bv\tau'' s+\tau\kappa+bv\kappa^2\tau s+bv\tau^3 s)),
\end{aligned}
$$

where

$$
\begin{aligned}
u=&(v^2(-a\kappa s+c\tau s-b)(c\tau^2 s-a\kappa\tau s-2b\tau-b\tau' s)\\
&+v(b\tau s+c)(\kappa+v(b\kappa^2 s-2a\kappa-a\kappa' s+c\tau' s+2c\tau+b\tau^2 s)))^2\\
&-(v(1+vb\kappa s-va)(c\tau^2 s-a\kappa\tau s-2b\tau-b\tau' s)\\
&+v^2(b\tau s+c)(b\kappa' s+2b\kappa+a\kappa^2 s-c\kappa\tau s))^2\\
&+(((1+v(b\kappa s-a))(\kappa+v(b\kappa^2 s-2a\kappa-a\kappa' s+c\tau' s+2c\tau+b\tau^2 s))\\
&-v^2(-a\kappa s+c\tau s-b)(b\kappa' s+2b\kappa+a\kappa^2 s-c\kappa\tau s))^2.
\end{aligned}
$$

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