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Editor Kenan Yildirim

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"*An Istanbul Meeting for World Mathematicians*"
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Chair Opening Speech

Dear Participants,

First of all, thank you very much for your interest in International Online Conference on Mathematics, 1- 3 December 2021, Istanbul, Turkey. This is the fifth edition of the our conference and we are happy due to high interest to conference.

This year, our 150 participants are from 25 different countries which are Pakistan, Algeria, Morocco, China, Greece, Kuwait, Albania, Serbia, Bulgaria, Croatia, Nigeria, India, Italy, Iran, United Kingdom, United States, United Arab Emirates, Tunisia, Azerbaijan, Uganda, Congo, Spain, Romania, Poland, Russia, Oman and France.

Also, on behalf of organizing committee, I present our deepest and special thanks to our Keynote Speakers Guiseppe Conte, Mehmet Emir Köksal and Sakthivel Rathinasamy due to their contributions to conference.

The sixth edition of this conference will be organized as face to face and online in the end of the June in Istanbul. I hope that you will enjoy. Thank you very much for your participation and interest.

Kenan Yıldırım, PhD

Chair on ICOM'21

01 December 2021

5th International Online Conference on Mathematics
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Hadamard Products of Uniformly Starlike and Convex Functions Associated with Deniz-Özkan Differential Operator

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Abstract

In this paper we introduce the subclasses $\beta-\mathcal{TSP}_{\lambda}^{m}(\alpha)$ and $\beta-\mathcal{TUCV}_{\lambda}^{m}(\alpha)$ of analytic functions defined by Deniz- Özkan Differential operator $\mathcal{D}_{\lambda}^{m}f(z)$. We obtain modified Hadamard products of functions belonging to the subclasses $\beta-\mathcal{TSP}_{\lambda}^{m}(\alpha)$ and $\beta-\mathcal{TUCV}_{\lambda}^{m}(\alpha)$.

Keywords: Univalent function, Uniformly starlike, Uniformly convex, Differential Operator, Modified Hadamard product.

1. Introduction and Definitions

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n \tag{1}$$

which are analytic in the open disc $\mathcal{U}=\{z : z\in C : \ |z|<1\}$. Suppose that $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of functions that are the univalent in $\mathcal{U}$. Also denote by $\mathcal{T}$ the subclass of $\mathcal{A}$ consisting of functions of the form

$$f(z)=z-\sum_{n=2}^{\infty}a_n z^n, \ (a_n\geq 0). \tag{2}$$

A function $f\in\mathcal{A}$ is said to be in the class of uniformly convex functions of order α and type β, denoted by $\beta-\mathcal{UCV}(\alpha)$ (see [1]) if

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}-\alpha\right\}>\beta\left|\frac{zf''(z)}{f'(z)}\right|, \quad z\in\mathcal{U}, \tag{3}$$

where $-1\leq\alpha<1$ and $\beta\geq 0$.

A function $f\in\mathcal{A}$ is said to be in the class of uniformly starlike functions of order α and type β, denoted by $\beta-\mathcal{SP}(\alpha)$ (see [1]) if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}-\alpha\right\}>\beta\left|\frac{zf'(z)}{f(z)}-1\right|, \quad z\in\mathcal{U}, \tag{4}$$

where and $-1\leq\alpha<1$ and $\beta\geq 0$.

These classes generalize various other classes which are worth mentioning here. The class $\beta-\mathcal{UCV}(0)=\beta-\mathcal{UCV}$ is the class of $\beta-$uniformly convex functions [4].
Indeed it follows from (3) and (4) that

$$\mathcal{D}_{\lambda}^{m} f(z)\in\beta-\mathcal{UCV}(\alpha)\Leftrightarrow z\left(\mathcal{D}_{\lambda}^{m} f(z)\right)'\in\beta-\mathcal{SP}(\alpha).$$

Especially the classes $1-\mathcal{UCV}(0)=\mathcal{UCV}$ and $1-\mathcal{SP}(0)=\mathcal{SP},$ defined by Goodman 3 and Ronning $[6]$, respectivelly.
For a functions f in $\mathcal{A},$ Deniz and Özkan [2] (see also [5]) introduced the following differential operatör $\mathcal{D}_{\lambda}^{m}$ as follows:

Definition 1. Let $f\in\mathcal{A}$. For the parametres $\lambda\geq 0$ and $m\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$ the differential operatör $\mathcal{D}_{\lambda}^{m}$ on $\mathcal{A}$ defined by

$$\mathcal{D}_{\lambda}^{0} f(z)=f(z)$$
$$\mathcal{D}_{\lambda}^{1} f(z)=\lambda z^3 f'''(z)+(2\lambda+1)z^2 f''(z)+zf'(z)$$
$$\mathcal{D}_{\lambda}^{m} f(z)=\mathcal{D}(\mathcal{D}_{\lambda}^{m-1} f(z)$$

for $z\in\mathcal{U}$.
For a function f in $\mathcal{A},$ from the definition of the differential operatör $\mathcal{D}_{\lambda}^{m}$, we can easily see that

$$\mathcal{D}_{\lambda}^{m} f(z)=z+\sum_{n=2}^{\infty} n^{2m}(\lambda(n-1)+1)^m a_n z^n. \tag{5}$$

Also, $\mathcal{D}_{\lambda}^{m} f(z)\in\mathcal{A}.$

For $f\in\mathcal{A}$ given by (1) and $g(z)\in\mathcal{A}$ given by $g(z)=z+\sum_{n=2}^{\infty} b_n z^n,$ the Hadamard product (or Convolution) of f and g defined by

$$(f*g)(z)=z+\sum_{n=2}^{\infty} a_n b_n z^n=(g*f)(z),\quad z\in\mathcal{U}.$$

Special cases of this operator include the Salagean derivative operator $\mathcal{S}^m$ (see [7]) as follows:

$$\mathcal{D}_{0}^{m} f(z)=\mathcal{S}^m f(z)*\mathcal{S}^m f(z)=\mathcal{S}^{2m} f(z)$$

and

$$\mathcal{D}_{1}^{m} f(z)=\mathcal{S}^m f(z)*\mathcal{S}^m f(z)*\mathcal{S}^m f(z)=\mathcal{S}^{3m} f(z).$$

For $-1\leq\alpha<1,$ $\lambda\geq 0$, $m\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$ and $\beta\geq 0$, let $\beta-\mathcal{SP}_{\lambda}^{m}(\alpha)$ be the subclass of $\mathcal{A},$ consising of functions of the form (1) and satisfying the analytic criterion

$$\operatorname{Re}\left\{\frac{z\left(D_\lambda^m f(z)\right)'}{D_\lambda^m f(z)}-\alpha\right\}>\beta\left|\frac{z\left(D_\lambda^m f(z)\right)'}{D_\lambda^m f(z)}-1\right|,$$

where $\mathcal{D}_\lambda^m f(z)$ is given by (5). We also let $\beta-\mathcal{TSP}_\lambda^m(\alpha)=\beta-\mathcal{SP}_\lambda^m(\alpha)\cap\mathcal{T}$.

Not that $f\in\beta-\mathcal{SP}_\lambda^m(\alpha)$ if and only if $\mathcal{D}_\lambda^m f(z)\in\beta-\mathcal{SP}(\alpha)$. Using the Alexander type relation, we define the class $\beta-\mathcal{UCV}_\lambda^m(\alpha)$ as follows

$$D_\lambda^m f(z)\in\beta-\mathcal{UCV}_\lambda^m(\alpha)\Leftrightarrow z\left(D_\lambda^m f(z)\right)'\in\beta-\mathcal{SP}_\lambda^m(\alpha).$$

We also let $\beta-\mathcal{TUCV}_\lambda^m(\alpha)=\beta-\mathcal{UCV}_\lambda^m(\alpha)\cap\mathcal{T}$.

We note that by specializing the parameters α,β,λ and m, the subclasses $\beta-\mathcal{SP}_\lambda^m(\alpha)$ and $\beta-\mathcal{UCV}_\lambda^m(\alpha)$ reduces to several well-known subclasses of analytic functions. This subclasses are:

i. $\beta-SP_\lambda^0(\alpha)=\beta-\mathcal{SP}(\alpha),\ \beta-\mathcal{UCV}_\lambda^0(\alpha)=\beta-\mathcal{UCV}(\alpha)$

ii. $\beta-\mathcal{SP}_\lambda^0(0)=\beta-\mathcal{SP},\ \beta-\mathcal{UCV}_\lambda^0(0)=\beta-\mathcal{UCV}$

iii. $1-\mathcal{SP}_\lambda^0(0)=\mathcal{SP},\ 1-\mathcal{UCV}_\lambda^0(0)=\mathcal{UCV}$.

In [5], authors obtained the following results:

Theorem 1. A function $f(z)$ of the form (1) is in $\beta-\mathcal{SP}_\lambda^m(\alpha)$ if

$$\sum_{n=2}^{\infty}\left[n(1+\beta)-(\alpha+\beta)\right]n^{2m}\left(\lambda(n-1)+1\right)^m|a_n|\le 1-\alpha \tag{6}$$

where $-1\le\alpha<1,\ \lambda\ge 0,\ m\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$ and $\beta\ge 0$.

Theorem 2. A necessary and sufficient condition for $f(z)$ of the form (2) to be in the class $\beta-\mathcal{TSP}_\lambda^m(\alpha)$ for $-1\le\alpha<1,\ \lambda\ge 0,\ m\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$ and $\beta\ge 0$ is that

$$\sum_{n=2}^{\infty}\left[n(1+\beta)-(\alpha+\beta)\right]n^{2m}\left(\lambda(n-1)+1\right)^m a_n\le 1-\alpha. \tag{7}$$

Theorem 3. A function $f(z)$ of the form (1) is in $\beta-\mathcal{UCV}_\lambda^m(\alpha)$ if

$$\sum_{n=2}^{\infty}\left[n(1+\beta)-(\alpha+\beta)\right]n^{2m+1}\left(\lambda(n-1)+1\right)^m|a_n|\le 1-\alpha \tag{8}$$

where $-1\le\alpha<1,\ \lambda\ge 0,\ m\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$ and $\beta\ge 0$.

Theorem 4. A necessary and sufficient condition for $f(z)$ of the form (2) to be in the class $\beta-\mathcal{TUCV}_\lambda^m(\alpha)$ for $-1\le\alpha<1,\ \lambda\ge 0,\ m\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$ and $\beta\ge 0$ is that

$$\sum_{n=2}^{\infty}\left[n(1+\beta)-(\alpha+\beta)\right]n^{2m+1}\left(\lambda(n-1)+1\right)^{m} a_n \le 1-\alpha. \tag{9}$$

2. Main Results

Hadamard Products of the functions classes $\beta-\mathcal{TSP}_{\lambda}^{m}(\alpha)$ and $\beta-\mathcal{TUCV}_{\lambda}^{m}(\alpha)$

Let the functions $f_j(z)\quad (j=1,2)$ be defined by

$$f_j(z)=z-\sum_{n=2}^{\infty}a_{n,j}z^n \text{ for } a_{n,j}\ge 0,\quad z\in\mathcal{U}. \tag{10}$$

The modified Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$\left(f_1 * f_2\right)(z)=z-\sum_{n=2}^{\infty}a_{n,1}a_{n,2}z^n.$$

Using the techniques of Schild and Siverman [8], we prove the following results.

Theorem 5. For functions $f_j(z)\quad (j=1,2)$ defined by (10), let $f_1(z)\in\beta-\mathcal{TSP}_{\lambda}^{m}(\alpha)$ and $f_2(z)\in\beta-\mathcal{TSP}_{\lambda}^{m}(\delta)$. Then $\left(f_1 * f_2\right)\in\beta-\mathcal{TSP}_{\lambda}^{m}(\gamma)$ where

$$\gamma=\gamma\left(\alpha,\beta,\delta\right)=1-\frac{(1-\alpha)(1-\delta)(1+\beta)}{(2+\beta-\alpha)(2+\beta-\delta)2^{2m}\left[\lambda+1\right]^{m}-(1-\alpha)(1-\delta)}, \tag{11}$$

and $-1\le\delta<1,\ m\in\mathbb{N}_0=\mathbb{N}\cup\{0\},\ -1\le\gamma<1;\ z\in\mathcal{U}.$ The result is best possible for

$$f_1(z)=z-\frac{(1-\alpha)}{(2+\beta-\alpha)2^{2m}\left[\lambda+1\right]^{m}}z^2$$

$$f_2(z)=z-\frac{(1-\delta)}{(2+\beta-\delta)2^{2m}\left[\lambda+1\right]^{m}}z^2$$

Proof: In view of Theorem 2, it suffice to prove that

$$\sum_{n=2}^{\infty}\frac{\left[n(1+\beta)-(\gamma+\beta)\right]n^{2m}\left(\left[\lambda(n-1)+1\right]\right)^{m}}{1-\gamma}a_{n,1}a_{n,2}\le 1,\ \ (-1\le\gamma<1)$$

where γ is defined by (11). On the other hand, under the hypothesis, it follows from (7) and the Cauchy's-Schwarz inequality that

$$\sum_{n=2}^{\infty}\frac{\left[n(1+\beta)-(\alpha+\beta)\right]^{1/2}\left[n(1+\beta)-(\delta+\beta)\right]^{1/2}n^{2m}\left(\left[\lambda(n-1)+1\right]\right)^{m}}{\sqrt{(1-\alpha)(1-\delta)}}\sqrt{a_{n,1}a_{n,2}}\le 1. \tag{12}$$

Thus we need to find the largest γ such that

$$\sum_{n=2}^{\infty}\frac{\left[n(1+\beta)-(\gamma+\beta)\right]^{1/2}n^{2m}\left([\lambda(n-1)+1]\right)^{m}}{1-\gamma}a_{n,1}a_{n,2}$$
$$\leq\sum_{n=2}^{\infty}\frac{\left[n(1+\beta)-(\alpha+\beta)\right]^{1/2}\left[n(1+\beta)-(\delta+\beta)\right]^{1/2}n^{2m}\left([\lambda(n-1)+1]\right)^{m}}{\sqrt{(1-\alpha)(1-\delta)}}\sqrt{a_{n,1}a_{n,2}}$$

or, equivalently that

$$\sqrt{a_{n,1}a_{n,2}}\leq\frac{1-\gamma}{\sqrt{(1-\alpha)(1-\delta)}}\frac{\left[n(1+\beta)-(\alpha+\beta)\right]^{1/2}\left[n(1+\beta)-(\delta+\beta)\right]^{1/2}}{\left[n(1+\beta)-(\gamma+\beta)\right]^{1/2}},\quad (n\geq 2).$$

By view of (12) it is sufficient to find largest γ such that

$$\frac{\sqrt{(1-\alpha)(1-\delta)}}{\left[n(1+\beta)-(\alpha+\beta)\right]^{1/2}\left[n(1+\beta)-(\delta+\beta)\right]^{1/2}n^{2m}\left[\lambda(n-1)+1\right]^{m}}$$
$$\leq\frac{1-\gamma}{\sqrt{(1-\alpha)(1-\delta)}}\frac{\left[n(1+\beta)-(\alpha+\beta)\right]^{1/2}\left[n(1+\beta)-(\delta+\beta)\right]^{1/2}}{\left[n(1+\beta)-(\gamma+\beta)\right]^{1/2}}$$

which yields

$$\gamma=1-\frac{(n-1)(1-\alpha)(1-\delta)(1+\beta)}{\left[n(1+\beta)-(\alpha+\beta)\right]\left[n((1+\beta)-(\delta+\beta)\right]n^{2m}\left[\lambda(n-1)+1\right]^{m}-(1-\alpha)(1-\delta)}\quad (n\geq 2).$$

Since

$$\Phi(n)=1-\frac{(n-1)(1-\alpha)(1-\delta)(1+\beta)}{\left[n(1+\beta)-(\alpha+\beta)\right]\left[n(1+\beta)-(\delta+\beta)\right]n^{2m}\left[\lambda(n-1)+1\right]^{m}-(1-\alpha)(1-\delta)},\quad (n\geq 2)\ (13)$$

is an increasing function of n, for $-1\leq\alpha<1, -1\leq\delta<1$ and $\beta\geq 0,$ letting $n=2$ in (13), we have

$$\gamma\leq\Phi(2)=1-\frac{(1-\alpha)(1-\delta)(1+\beta)}{\left[2+\beta-\alpha\right]\left[2+\beta-\delta\right]2^{2m}\left[\lambda+1\right]^{m}-(1-\alpha)(1-\delta)}$$

which completes the proof.

Theorem 6. Let the functions $f_j(z)\ \ (j=1,2)$ defined by (10), be in the class $\beta-\mathcal{TSP}_{\lambda}^{m}(\alpha)$ with $-1\leq\alpha<1,\ \beta\geq 0.$ Then $\left(f_1*f_2\right)\in\beta-\mathcal{TSP}_{\lambda}^{m}(\eta)$ where

$$\eta=1-\frac{(1-\alpha)^{2}(1+\beta)}{\left[2+\beta-\alpha\right]^{2}2^{2m}\left[\lambda+1\right]^{m}-(1-\alpha)^{2}}.$$

Proof: By taking $\delta=\alpha$ in the Theorem 5, the result follows.

Theorem 7. Let the functions $f(z)$ defined by (2) be in the class $\beta-\mathcal{TSP}_{\lambda}^{m}(\alpha)$. Also let $g(z)=z-\sum_{n=2}^{\infty}b_n z^n$ for $|b_n|\le 1$. Then $(f*g)(z)\in\beta-\mathcal{TSP}_{\lambda}^{m}(\alpha)$.

Proof: Since

$$\begin{aligned}&\sum_{n=2}^{\infty}\left[n(1+\beta)-(\alpha+\beta)\right]n^{2m}\left[\lambda(n-1)+1\right]^{m}|a_n b_n|\\ &\le\sum_{n=2}^{\infty}\left[n(1+\beta)-(\alpha+\beta)\right]n^{2m}\left[\lambda(n-1)+1\right]^{m}a_n|b_n|\\ &\le\sum_{n=2}^{\infty}\left[n(1+\beta)-(\alpha+\beta)\right]n^{2m}\left[\lambda(n-1)+1\right]^{m}a_n\\ &\le 1-\alpha.\end{aligned}$$

It follows that $(f*g)(z)\in\beta-\mathcal{TSP}_{\lambda}^{m}(\alpha)$, by the view of Theorem 1.

Theorem 8. Let the functions $f_j(z)$ $(j=1,2)$ defined by (14) be in the class $\beta-\mathcal{TSP}_{\lambda}^{m}(\alpha)$. Then the function $h(z)$ defined by $h(z)=z-\sum_{n=2}^{\infty}\left(a_{n,1}^2+a_{n,2}^2\right)z^n$ is in the class $\beta-\mathcal{TSP}_{\lambda}^{m}(\xi)$, where

$$\xi=1-\frac{2(1-\alpha)^2(1+\beta)}{(2+\beta-\alpha)^2\,2^{2m}\left[\lambda+1\right]^m-2(1-\alpha)^2}.$$

Proof. By virtue of Theorem 2, it is sufficient to prove that

$$\sum_{n=2}^{\infty}\frac{\left[n(1+\beta)-(\xi+\beta)\right]n^{2m}\left[\lambda(n-1)+1\right]^{m}}{(1-\xi)}\left(a_{n,1}^2+a_{n,2}^2\right)\le 1 \tag{14}$$

where $f_j(z)\in\beta-\mathcal{TSP}_{\lambda}^{m}(\alpha)$ $(j=1,2)$ we find from (14) and Theorem 1, that

$$\sum_{n=2}^{\infty}\left[\frac{\left[n(1+\beta)-(\alpha+\beta)\right]n^{2m}\left[\lambda(n-1)+1\right]^{m}}{(1-\alpha)}\right]^2 a_{n,j}^2\le\left\{\sum_{n=2}^{\infty}\frac{\left[n(1+\beta)-(\alpha+\beta)\right]n^{2m}\left[\lambda(n-1)+1\right]^{m}}{(1-\alpha)}a_{n,j}\right\}^2 \tag{15}$$

which yields

$$\sum_{n=2}^{\infty}\frac{1}{2}\left\{\frac{\left[n(1+\beta)-(\alpha+\beta)\right]n^{2m}\left[\lambda(n-1)+1\right]^{m}}{(1-\alpha)}\right\}^2\left(a_{n,1}^2+a_{n,2}^2\right)\le 1. \tag{16}$$

On comparing (15) and (16), it is easily seen that the inequality (14) will be satisfied if

$$\frac{\left[n(1+\beta)-(\xi+\beta)\right]n^{2m}\left[\lambda(n-1)+1\right]^m}{(1-\xi)}\le\frac{1}{2}\left\{\frac{\left[n(1+\beta)-(\alpha+\beta)\right]n^{2m}\left[\lambda(n-1)+1\right]^m}{(1-\alpha)}\right\}^2$$

for $n\ge 2$. That is,

$$\xi\le 1-\frac{2(n-1)(1-\alpha)^2(1+\beta)}{\left[n(1+\beta)-(\alpha+\beta)\right]^2 n^{2m}\left[\lambda(n-1)+1\right]^m-2(1-\alpha)^2}. \tag{17}$$

The function

$$\psi(n)=1-\frac{2(n-1)(1-\alpha)^2(1+\beta)}{\left[n(1+\beta)-(\alpha+\beta)\right]^2 n^{2m}\left[\lambda(n-1)+1\right]^m-2(1-\alpha)^2}$$

is an increasing function of n $(n\ge 2)$. Therefore $n=2$ in (17), we have,

$$\xi\le\psi(2)=1-\frac{2(1-\alpha)^2(1+\beta)}{(2+\beta-\alpha)^2 2^{2m}\left[\lambda+1\right]^m-2(1-\alpha)^2},$$

which completes the proof.

Corollary 1. For functions $f_j(z)$ $(j=1,2)$ defined by (10), let $f_1(z)\in\beta-\mathcal{TUCV}_\lambda^m(\alpha)$ and $f_2(z)\in\beta-\mathcal{TUCV}_\lambda^m(\delta)$. Then $(f_1*f_2)\in\beta-\mathcal{TUCV}_\lambda^m(\gamma)$ where

$$\gamma=\gamma(\alpha,\beta,\delta)=1-\frac{(1-\alpha)(1-\delta)(1+\beta)}{(2+\beta-\alpha)(2+\beta-\delta)2^{2m+1}\left[\lambda+1\right]^m-(1-\alpha)(1-\delta)},$$

and $-1\le\delta<1$, $m\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$, $-1\le\gamma<1$; $z\in\mathbb{N}$.

The result is best possible for

$$f_1(z)=z-\frac{(1-\alpha)}{(2+\beta-\alpha)2^{2m+1}\left[\lambda+1\right]^m}z^2$$

$$f_2(z)=z-\frac{(1-\delta)}{(2+\beta-\delta)2^{2m+1}\left[\lambda+1\right]^m}z^2$$

Corollary 2. Let the functions $f_j(z)$ $(j=1,2)$ defined by (10), be in the class $\beta-\mathcal{TUCV}_\lambda^m(\alpha)$ with $-1\le\alpha<1$, $\beta\ge 0$. Then $(f_1*f_2)\in\beta-\mathcal{TUCV}_\lambda^m(\eta)$ where

$$\eta=1-\frac{(1-\alpha)^2(1+\beta)}{\left[2+\beta-\alpha\right]^2 2^{2m+1}\left[\lambda+1\right]^m-(1-\alpha)^2}.$$

Corollary 3. Let the functions $f(z)$ defined by (2) be in the class $\beta-\mathcal{TUCV}_\lambda^m(\alpha)$ Also let $g(z)=z-\sum_{n=2}^{\infty}b_nz^n$ for $|b_n|\le 1$. Then $(f*g)(z)\in\beta-\mathcal{TUCV}_\lambda^m(\alpha)$.

Corollary 4. Let the functions $f_j(z)$ $(j=1,2)$ defined by (10) be in the class $\beta-\mathcal{TUCV}_\lambda^m(\alpha)$ Then the function $h(z)$ defined by $h(z)=z-\sum_{n=2}^{\infty}\left(a_{n,1}^2+a_{n,2}^2\right)z^n$ is in the class $\beta-\mathcal{TUCV}_\lambda^m(\xi),$ where

$$\xi=1-\frac{2(1-\alpha)^2(1+\beta)}{(2+\beta-\alpha)^2\,2^{2m+1}[\lambda+1]^m-2(1-\alpha)^2}.$$

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Fekete-Szegö Problem For Some Subclasses of Bi-Univalent Functions Defined By Deniz-Özkan Differential Operator

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Abstract

In this study, we solve Fekete-Szegö problem for a new subclass $\mathcal{B}_{\Sigma}^{m}(\lambda,\beta;\varphi)$ of bi-univalent functions in the open unit disk U defined by Deniz-Özkan differential operator.

Keywords: Analytic function, Univalent function, bi-univalent function, Differential operatör, Fekete- Szegö problem.

1. Introduction and Preliminaries

Let A denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. Further, by S we shall denote the class of all functions in A which are univalent in U. It is well known that every function $f \in S$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \qquad (z \in U)$$

and

$$f(f^{-1}(w)) = w \qquad \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4}\right)$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \ldots$$

A function $f \in A$ is said to be in Σ, the class of bi-univalent functions in U, if both $f(z)$ and $f^{-1}(z)$ are univalent in U. Lewin [11] showed that $|a_2| < 1.51$ for every function $f \in \Sigma$ given by (1). Posteriorly, Brannan and Clunie [1] improved Lewin's result and conjectured that $|a_2| \le \sqrt{2}$ for every function $f \in \Sigma$ given by (1.1). Later, Netanyahu [13] showed that $\max_{f\in\Sigma} |a_2| = \frac{4}{3}$. The coefficient estimate problem for each of the following Taylor-Maclaurin coefficients:

$$|a_n| \quad \left(n \in N = \{1,2,\ldots\}; n \ge 4\right)$$

is still an open problem (see, for details, [15]). Since then, many researchers (see [2,4,8,9,16]) investigated several interesting subclasses of the class Σ and found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. One of the most important problem on coefficients of univalent functions as known Fekete-Szegö problem. Very recently, some results have obtained by [3,7,9,10,14] for this problem.

Let P denote the class of function of p analytic in U such that $p(0)=1$ and $\operatorname{Re}\{p(z)\} > 0$, where

$$p(z) = 1 + p_1 z + p_2 z^2 + \ldots \quad (z \in U).$$

If f and g are analytic in U, we say that f is subordinate to g, written symbolically as

$$f \prec g \qquad \text{or} \qquad f(z) \prec g(z) \quad (z \in U),$$

if there exists a Schwarz function $w(z)$, which (by definition) is analytic in U with $w(0)=0$ and $|w(z)|<1$ in U such that $f(z) = g(w(z)),\ z \in U.$

In particular, if the function $g(z)$ is univalent in U, then we have that:

$$f(z) \prec g(z) \quad (z \in U) \quad \text{if and only if} \quad f(0) = g(0) \ \text{ and } \ f(U) \subseteq g(U).$$

Let φ be an analytic function with positive real part in the unit disk U such that

$$\varphi(0) = 1, \varphi'(0) > 0$$

and $\varphi(U)$ is symmetric with respect to the real axis and has a series expansion of the form (see [12]):

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \ldots \quad (B_1 > 0).$$

Let $u(z)$ and $v(z)$ be two analytic functions in the unit disk U with $u(0) = v(0) = 0$ $|u(z)| < 1, |v(z)| < 1$, and suppose that

$$u(z) = 1 + b_1 z + b_2 z^2 + b_3 z^3 + \ldots \text{ and } \mathrm{v}(w) = 1 + c_1 w + c_2 w^2 + c_3 w^3 + \ldots \cdot \tag{2}$$

For above functions, well-known inequalities are

$$|b_1| \le 1, |b_2| \le 1 - |b_1|^2, |c_1| < 1 \text{ and } |c_2| \le 1 - |c_1|^2. \tag{3}$$

Further we have

$$\varphi(u(z)) = 1 + B_1 b_1 z + \left(B_1 b_2 + B_2 b_1^2\right) z^2 + \ldots \quad (|z| < 1) \tag{4}$$

and

$$\varphi(v(w)) = 1 + B_1 c_1 w + \left(B_1 c_2 + B_2 c_1^2\right) w^2 + \ldots \quad (|w| < 1). \tag{5}$$

In [5] (see also [6]), Deniz and Özkan defined the differential operator D_λ^m (say: Deniz-Özkan differential operator) as follows:

For the parametres $\lambda \ge 0$ and $m \in N_0 = N \cup \{0\}$ the differential operatör D_λ^m on A defined by

$$D_\lambda^0 f(z) = f(z)$$

$$D_\lambda^1 f(z) = \lambda z^3 f'''(z) + (2\lambda + 1) z^2 f''(z) + z f'(z)$$

$$D_\lambda^m f(z) = D(D_\lambda^{m-1} f(z)$$

for $z \in U$.

For a function f in A, from the definition of the differential operatör D_λ^m, we can easily see that

$$D_\lambda^m f(z) = z + \sum_{n=2}^{\infty} n^{2m} (\lambda(n-1)+1)^m a_n z^n. \tag{6}$$

Also, $D_\lambda^m f(z) \in A.$

The main object of this paper is to introduce the following new subclass of bi-univalent functions involving Deniz-Özkan differential operator D_λ^m and discuss Fekete–Szegö functional problem for functions in this new class (see [7]).

2. Fekete-Szegö problem for the functions class $B_\Sigma^m(\lambda, \beta; \varphi)$

Definition 1. A function $f(z)\in\Sigma$ is said to be in the class $B_{\Sigma}^{m}(\lambda,\beta;\varphi)$ if and only if

$$(1-\beta)\frac{D_{\lambda}^{m}f(z)}{z}+\beta\left(D_{\lambda}^{m}f(z)\right)'\prec\varphi(z)$$

and

$$(1-\beta)\frac{D_{\lambda}^{m}g(w)}{w}+\beta\left(D_{\lambda}^{m}g(w)\right)'\prec\varphi(w)$$

where $0\leq\beta\leq1,\ z,w\in U$ and $g(w)=f^{-1}(w)$.

Now, we are ready to find the sharp bounds of Fekete–Szegö functional $a_3-\delta a_2^2$ defined for $f\in B_{\Sigma}^{m}(\lambda,\beta;\varphi)$ given by (1).

Theorem 1. Let $f(z)$ given by (1) be in the class $B_{\Sigma}^{m}(\lambda,\beta;\varphi)$. Then

$$\left|a_3-\delta a_2^2\right|\leq\begin{cases}\dfrac{B_1}{(1+2\beta)\left[9(2\lambda+1)\right]^{m}} & \text{for}\quad 0\leq\left|h(\delta)\right|<\dfrac{1}{2(1+2\beta)\left[9(2\lambda+1)\right]^{m}}\\ 2\mathrm{B}_1\left|h(\delta)\right| & \text{for}\quad \left|h(\delta)\right|\geq\dfrac{1}{2(1+2\beta)\left[9(2\lambda+1)\right]^{m}}\end{cases}\tag{7}$$

where

$$h(\delta)=\frac{B_1^2}{2B_1^2(1+2\beta)\left[9(2\lambda+1)\right]^{m}-2B_2(1+\beta)^2\left[4(\lambda+1)\right]^{2m}}.$$

Proof. Let $f(z)\in B_{\Sigma}^{m}(\lambda,\beta;\varphi)$. By the definition of subordination, there are analytic functions u and v with $u(0)=v(0)=0,\ \left|u(z)\right|<1,\ \left|v(w)\right|<1$, given by (2) and satisfying the following conditions:

$$(1-\beta)\frac{D_{\lambda}^{m}f(z)}{z}+\lambda\left(D_{\lambda}^{m}f(z)\right)'=\varphi\left(u(z)\right)$$

and

$$(1-\beta)\frac{D_{\lambda}^{m}g(w)}{w}+\beta\left(D_{\lambda}^{m}g(w)\right)'=\varphi\left(v(w)\right),$$

where $g(w)=f^{-1}(w)$. Since

$$\begin{aligned}&(1-\beta)\frac{D_\lambda^m f(z)}{z}+\beta\left(D_\lambda^m f(z)\right)' \\ &=1+(1+\beta)\left[4(\lambda+1)\right]^m a_2 z+(1+2\beta)\left[9(2\lambda+1)\right]^m a_3 z^2+\ldots\end{aligned} \tag{8}$$

and

$$\begin{aligned}&(1-\beta)\frac{D_\lambda^m g(w)}{w}+\beta\left(D_\lambda^m g(w)\right)' \\ &=1-(1+\beta)\left[4(\lambda+1)\right]^m a_2 w+(1+2\beta)\left[9(2(\lambda+1))\right]^m \left(2a_2^2-a_3\right)w^2+\ldots,\end{aligned} \tag{9}$$

it follows from (4), (5), (8) and (9) that

$$(1+\beta)\left[4(\lambda+1)\right]^m a_2 = B_1 b_1, \tag{10}$$

$$(1+2\beta)\left[9(2\lambda+1)\right]^m a_3 = B_1 b_2 + B_2 b_1^2, \tag{11}$$

$$-(1+\beta)\left[4(\lambda+1)\right]^m a_2 = B_1 c_1, \tag{12}$$

and

$$(1+2\beta)\left[9(2\lambda+1)\right]^m \left(2a_2^2-a_3\right) = B_1 c_2 + B_2 c_1^2. \tag{13}$$

From (10) and (12), we get

$$c_1 = -b_1 \tag{14}$$

$$2\left[\left(4(\lambda+1)\right)^m (1+\beta)\right]^2 a_2^2 = B_1^2\left(b_1^2+c_1^2\right). \tag{15}$$

By adding (10) to (13), we have

$$2\left[9(2\lambda+1)\right]^m (1+2\beta) a_2^2 = B_1\left(b_2+c_2\right)+B_2\left(b_1^2+c_1^2\right). \tag{16}$$

Therefore, from equalities (15) and (16) we find that

$$\left[2\left[9(2\lambda+1)\right]^m (1+2\beta) B_1^2 - 2B_2\left(\left[4(\lambda+1)\right]^m (1+\beta)\right)^2\right] a_2^2 = B_1^3\left(b_2+c_2\right). \tag{17}$$

We conclude that, from (17)

$$a_2^2=\frac{B_1^3\left(b_2+c_2\right)}{2\left[\left[9\left(2\lambda+1\right)\right]^m\left(1+2\beta\right)B_1^2-B_2\left(\left[4\left(\lambda+1\right)\right]^m\left(1+\beta\right)\right)^2\right]} \tag{18}$$

and subtracting (13) from (11) and using (14)

$$a_3=\frac{2\left(1+2\beta\right)\left[9\left(2\lambda+1\right)\right]^m a_2^2+B_1\left(b_2-c_2\right)}{2\left(1+2\beta\right)\left[9\left(2\lambda+1\right)\right]^m}. \tag{19}$$

From the Eqs. (18) and (19), it follows that

$$a_3-\delta a_2^2=B_1\left[\left(h\left(\delta\right)+\frac{1}{2\left(1+2\beta\right)\left[9\left(2\lambda+1\right)\right]^m}\right)b_2+\left(h\left(\delta\right)-\left[9\left(2\lambda+1\right)\right]^m\right)c_2\right],$$

where

$$h\left(\delta\right)=\frac{B_1^2\left(1-\delta\right)}{2B_1^2\left(1+2\beta\right)\left[9\left(2\lambda+1\right)\right]^m-2B_2\left(1+\beta\right)^2\left[4\left(\lambda+1\right)\right]^{2m}}.$$

Since all B_i are real and $B_1>0$, which implies the assertion (7). This completes the proof of Theorem 1.

By taking $\beta=1$ in Theorem 1, we have

Corollary 1. Let $f(z)$ given by (1) be in the class $B_\Sigma^m\left(\lambda,1;\varphi\right)$. Then

$$\left|a_3-\delta a_2^2\right|\le\begin{cases}\dfrac{B_1}{3\left[9\left(2\lambda+1\right)\right]^m} & \text{for} \quad 0\le\left|h\left(\delta\right)\right|<\dfrac{1}{6\left[9\left(2\lambda+1\right)\right]^m},\\[2ex] 2\mathrm{B}_1\left|h\left(\delta\right)\right| & \text{for} \quad \left|h\left(\delta\right)\right|\ge\dfrac{1}{6\left[9\left(2\lambda+1\right)\right]^m}\end{cases}$$

where

$$h\left(\delta\right)=\frac{B_1^2\left(1-\delta\right)}{6B_1^2\left[9\left(2\lambda+1\right)\right]^m-8B_2\left[4\left(\lambda+1\right)\right]^{2m}}.$$

Remark 1. Putting $m=0$ in Corollary 1, we get Corollary 4 in [17].

Putting $m=0$ in Theorem 1, we have

Corollary 2. Let $f(z)$ given by (1) be in the class $B_\Sigma^0\left(\lambda,\beta;\varphi\right)=B_\Sigma\left(\beta;\varphi\right)$. Then

$$\left|a_3-\delta a_2^2\right|\le\begin{cases}\dfrac{B_1}{1+2\beta} & \text{for} \quad 0\le\left|h(\delta)\right|<\dfrac{1}{2(1+2\beta)}\\ 2\mathrm{B}_1\left|h(\delta)\right| & \text{for} \quad \left|h(\delta)\right|\ge\dfrac{1}{2(1+2\beta)}\end{cases}$$

where

$$h(\delta)=\frac{B_1^2(1-\delta)}{2\left[B_1^2(1+2\beta)-B_2(1+\beta)^2\right]}.$$

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Quasi Focal Curves of Timelike Curves in Minkowski Space

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Abstract

In this study, we firstly characterize focal curves by considering quasi frame in the ordinary space. Then, we obtain the relation of each quasi curvatures of curve in terms of focal curvatures. Finally, we give some new conditions with constant quasi curvatures in the ordinary space.

Keywords: Quasi frame, focal curve, focal curvatures.

1. Backround on Quasi Frame

By way of design and style, this is model to kind of a moving frame with regards to a particle. In the quick stages of regular differential geometry, the Frenet-Serret frame was applied to create a curve in location. After that, Frenet-Serret frame is established by way of subsequent equations for a presented framework,

$$\begin{bmatrix} \nabla_{\mathbf{t}}\mathbf{t} \\ \nabla_{\mathbf{t}}\mathbf{n} \\ \nabla_{\mathbf{t}}\mathbf{b} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix},$$

where $\kappa = \|\mathbf{t}\|$ and τ are the curvature and torsion of γ, respectively.

The quasi frame of a regular spacelike curve γ is given by,

$$\mathbf{t_q} = \mathbf{t}, \mathbf{n_q} = \frac{\mathbf{t} \wedge \mathbf{k}}{\|\mathbf{t} \wedge \mathbf{k}\|}, \mathbf{b_q} = -\mathbf{t_q} \wedge \mathbf{n_q},$$

where $\mathbf{k}$ is the projection vector.

For simplicity, we have chosen the projection vector $\mathbf{k}=(0,1,0)$ in this paper. However, the q-frame is singular in all cases where $\mathbf{t}$ and $\mathbf{k}$ are parallel. Thus, in those cases where $\mathbf{t}$ and $\mathbf{k}$ are parallel the projection vector $\mathbf{k}$ can be chosen as $\mathbf{k}=(0,1,0)$ or $\mathbf{k}=(1,0,0)$.

If the pseudo-angle between the quasi normal vector $\mathbf{n_q}$ and the normal vector $\mathbf{n}$ is choosen as β, then following relation is obtained between the quasi and FS frame.

$$\begin{aligned} \mathbf{t_q} &= \mathbf{t}, \\ \mathbf{n_q} &= \cos\beta\mathbf{n}-\sin\beta\mathbf{b}, \\ \mathbf{b_q} &= \sin\beta\mathbf{n}+\cos\beta\mathbf{b}, \end{aligned}$$

such that short computation by using above equations yields that the variation of parallel adapted quasi frame is given by

$$\begin{aligned} \nabla_{\mathbf{t_q}}\mathbf{t_q} &= \varpi_1\mathbf{n_q}+\varpi_2\mathbf{b_q}, \\ \nabla_{\mathbf{t_q}}\mathbf{n_q} &= \varpi_1\mathbf{t_q}+\varpi_3\mathbf{b_q}, \\ \nabla_{\mathbf{t_q}}\mathbf{b_q} &= \varpi_2\mathbf{t_q}-\varpi_3\mathbf{n_q}, \end{aligned} \tag{1.1}$$

where

$$\varpi_1=\kappa\cos\beta, \quad \varpi_2=-\kappa\sin\beta, \quad \varpi_3=\beta'+\tau,$$

$$\mathbf{t_q}\times\mathbf{n_q}=-\mathbf{b_q}, \mathbf{n_q}\times\mathbf{b_q}=\mathbf{t_q}, \mathbf{b_q}\times\mathbf{t_q}=-\mathbf{n_q}.$$

In this paper, we study quasi focal curves in the Euclidean 3-space. We characterize quasi focal curves in terms of their focal curvatures.

2. Quasi Focal Curves with Quasi Frame In Minkowski Space

The focal curve of α is given by

$$\beta=\alpha+\phi_1\mathbf{n_q}+\phi_2\mathbf{b_q}, \tag{2.1}$$

where the coefficients ϕ_1, ϕ_2 are smooth functions of the parameter of the curve γ, called the first and second focal curvatures of γ, respectively.

Theorem 2.1. *Let $\gamma : I \to \mathsf{E}_1^3$ be a unit speed timelike curve and β its focal curve on E_1^3. Then,*

$$\beta=\alpha+e^{-\int\frac{\varpi_1\varpi_3}{\varpi_2}ds}\left(\int e^{\int\frac{\varpi_1\varpi_3}{\varpi_2}ds}\frac{\varpi_3}{\varpi_2}ds+C\right)\mathbf{n_q}+\left(\frac{1}{\varpi_2}-\frac{\varpi_1}{\varpi_2}e^{-\int\frac{\varpi_1\varpi_3}{\varpi_2}ds}\left(\int e^{\int\frac{\varpi_1\varpi_3}{\varpi_2}ds}\frac{\varpi_3}{\varpi_2}ds+C\right)\right)\mathbf{b_q}, \tag{2.2}$$

where C is a constant of integration.

Proof. Assume that α is a unit speed curve and β its focal curve in E_1^3.

So, by differentiating of the formula (2.1), we get

$$\beta' = (1+\varpi_1\phi_1+\varpi_2\phi_2)\mathbf{t_q}+(\phi_1'-\varpi_3\phi_2)\mathbf{n_q}+(\phi_2'+\varpi_3\phi_1)\mathbf{b_q}$$

From above equation, the first 2 components vanish, we get

$$\begin{aligned}1+\varpi_1\phi_1+\varpi_2\phi_2 &= 0,\\ \phi_1'-\varpi_3\phi_2 &= 0.\end{aligned}$$

Using the above equations, we obtain

$$\phi_1'-\frac{\varpi_3}{\varpi_2}(-1-\varpi_1\phi_1)=0,$$

$$\phi_1'+\frac{\varpi_1\varpi_3}{\varpi_2}\phi_1=-\frac{\varpi_3}{\varpi_2}.$$

By integrating this equation, we find

$$\phi_1=e^{-\int\frac{\varpi_1\varpi_3}{\varpi_2}ds}\left(C-\int e^{\int\frac{\varpi_1\varpi_3}{\varpi_2}ds}\frac{\varpi_3}{\varpi_2}ds\right),$$

$$\phi_2 = -\frac{1}{\varpi_2} - \frac{\varpi_1}{\varpi_2} e^{-\int \frac{\varpi_1 \varpi_3}{\varpi_2} ds} \left(C - \int e^{\int \frac{\varpi_1 \varpi_3}{\varpi_2} ds} \frac{\varpi_3}{\varpi_2} ds\right).$$

By means of obtained equations, we express (2.2). This completes the proof of the theorem.

As an immediate consequence of the above theorem, we have:

Corollary 2.2. *Let $\alpha : I \to E_1^3$ be a unit timelike speed curve and β its focal curve on E_1^3. Then, the focal curvatures of β are*

$$\phi_1 = e^{-\int \frac{\varpi_1 \varpi_3}{\varpi_2} ds} \left(C - \int e^{\int \frac{\varpi_1 \varpi_3}{\varpi_2} ds} \frac{\varpi_3}{\varpi_2} ds\right),$$

$$\phi_2 = -\frac{1}{\varpi_2} - \frac{\varpi_1}{\varpi_2} e^{-\int \frac{\varpi_1 \varpi_3}{\varpi_2} ds} \left(C - \int e^{\int \frac{\varpi_1 \varpi_3}{\varpi_2} ds} \frac{\varpi_3}{\varpi_2} ds\right).$$

Proof. From above theorem, we have above system, which completes the proof.

In the light of Theorem 2.1, we express the following corollary without proof:

Corollary 2.3. *Let $\gamma : I \to E_1^3$ be a unit speed timelike curve and β its focal curve on E^3. If $\varpi_1, \varpi_2, \varpi_3$ are constant then, the focal curvatures of β are*

$$\phi_1 = \left(-\frac{1}{\varpi_1} + e^{\frac{\varpi_1 \varpi_3}{\varpi_2} s} C\right),$$

$$\phi_2 = -\frac{1}{\varpi_2} - \frac{\varpi_1}{\varpi_2}\left(-\frac{1}{\varpi_1} + C e^{\frac{\varpi_1 \varpi_3}{\varpi_2} s}\right).$$

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Some Families of Meromorphic Functions Involving a Differential Operator

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Abstract

Let Σ denote the class of functions of the form $f(z)=\frac{1}{z}+\sum_{k=0}^{\infty}a_k z^k$ which are analytic in the punctured disc $\mathbb{D}=\{z:0<|z|<1\}$. We introduce and study some new families of meromorphic functions defined by a differential operator. A number of useful characteristics of functions in these families are obtained.

Keywords: Meromorphic,neighborhood,operator,partial sum.

1. Introduction

Let Σ denote the class of functions of the form

$$f(z)=z^{-1}+\sum_{n=0}^{\infty}a_n z^n \tag{1}$$

which are analytic in the punctured disc $\mathbb{D}=\{z\in\mathbb{C}:0<|z|<1\}$.

Let $f\in\Sigma$ be of the form (1) and let α,β be real numbers with $\alpha\geq\beta\geq 0$. Raducanu, Orhan and Deniz [10] defined the analogue of the differential operator given in as follows

$$D^0_{\alpha,\beta}f(z)=f(z)$$

$$D^1_{\alpha,\beta}f(z)=D_{\alpha,\beta}f(z)=\alpha\beta(z^2f(z))''+(\alpha-\beta)\frac{\left(z^2f(z)\right)'}{z}+(1-\alpha+\beta)f(z)$$

$$D^m_{\alpha,\beta}f(z)=D_{\alpha,\beta}\left(D^{m-1}_{\alpha,\beta}f(z)\right),\quad z\in\mathbb{D},\quad m\in\mathbb{N}=\{1,2,3,\ldots\}.$$

If $f \in \Sigma$ is given by (1), then from the definition of $D^m_{\alpha,\beta}$ we get

$$D^m_{\alpha,\beta} f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} A(\alpha,\beta,n)^m a_n z^n, \quad z \in \mathbb{D}$$

where

$$A(\alpha,\beta,n) = \left[(n+2)\alpha\beta + \alpha - \beta\right](n+1) + 1.$$

When $\alpha = 1$ and $\beta = 0$, Uralegaddi and Somanatha [13] investigated certain properties of the operator $D^m_{\alpha,\beta}$.

Let $-1 \le B < A \le 1$. A function $f(z) = z^{-1} + \sum_{n=0}^{\infty} a_n z^n \in \Sigma$ is said to be in the class $T_m(\alpha,\beta,A,B)$ if it satisfies the condition

$$\left| \frac{z(D^m_{\alpha,\beta} f(z))' + D^m_{\alpha,\beta} f(z)}{Bz(D^m_{\alpha,\beta} f(z))' + AD^m_{\alpha,\beta} f(z)} \right| < 1 \tag{2}$$

for all $z \in E = \{z : |z| < 1\}$.

Furthermore, a function $f(z) = z^{-1} + \sum_{n=1}^{\infty} |a_n| z^n \in \Sigma$ is said to be in the class $T_m^*(\alpha,\beta,A,B)$ if it satisfies the condition (2).

It should be remarked in passing that the definition (2) is motivated essentially by the recent work of Morga [9] and Srivastava and co-authors [11].

In recent years, many important properties and characteristics of various interesting subclasses of the class Σ of meromorphically functions were inverstigated extensively by (among others) Aouf et al [1], Chen et al. [2], Cho and Owa [3], Dziok et al. [4], El-Ashwah and Aouf [5], He et al. [6], Liu and Srivastava [7], Joshi and Srivastava [8], Raducanu et al. [10], Uralegaddi and Somanatha [13] and also [12].

The main object of this paper is to present neighborhoods and partial sums of functions in the classes $T_m\left(\alpha,\beta,A,B\right)$ and $T_m^{\ *}\left(\alpha,\beta,A,B\right)$ which we introduced here.

2. Properties of the class $T_m^{\ *}(\alpha,\beta,A,B)$

Theorem 1. Let $f(z)=z^{-1}+\sum_{n=1}^{\infty}\left|a_n\right|z^n$ be analytic in $\mathbb{D}=\left\{z:0<\left|z\right|<1\right\}$. Then $f(z)\in T_m^{\ *}(\alpha,\beta,A,B)$ if and only if

$$\sum_{n=1}^{\infty}\left[(1-A)+n(1-B)\right]A\left(\alpha,\beta,n\right)^m\left|a_n\right|\le A-B \tag{3}$$

The result is sharp for the function $f(z)$ given by

$$f(z)=z^{-1}+\frac{(A-B)}{A\left(\alpha,\beta,n\right)^m\left[\left(1-A\right)+n(1-B)\right]}z^n \qquad (n\ge 1). \tag{4}$$

Proof. Let $f(z)=z^{-1}+\sum_{k=n}^{\infty}\left|a_n\right|z^n\in T_m^{*}(\alpha,\beta,A,B).$ Then

$$\left|\frac{z(D_{\alpha,\beta}^m f(z))'+D_{\alpha,\beta}^m f(z)}{Bz(D_{\alpha,\beta}^m f(z))'+AD_{\alpha,\beta}^m f(z)}\right|=\left|\frac{\sum_{n=1}^{\infty}(1+n)A\left(\alpha,\beta,n\right)^m\left|a_n\right|z^{n+1}}{(A-B)+\sum_{n=1}^{\infty}(A+Bn)A\left(\alpha,\beta,n\right)^m\left|a_n\right|z^{n+1}}\right|. \tag{5}$$

Since $\left|\operatorname{Re}z\right|\le\left|z\right|$ for any z, choosing z to be real letting $z\to 1^-$ throuh real values (5) yields

$$\sum_{n=1}^{\infty}(1+n)A\left(\alpha,\beta,n\right)^m\left|a_n\right|\le(A-B)+\sum_{n=1}^{\infty}(A+Bn)A\left(\alpha,\beta,n\right)^m\left|a_n\right|,$$

which gives (3).

On the other hand,we have that

$$\left|\frac{z(D^m_{\alpha,\beta}f(z))'+D^m_{\alpha,\beta}f(z)}{Bz(D^m_{\alpha,\beta}f(z))'+AD^m_{\alpha,\beta}f(z)}\right|\leq\frac{\sum_{n=1}^{\infty}(1+n)A(\alpha,\beta,n)^m|a_n|}{(A-B)+\sum_{n=1}^{\infty}(A+Bn)A(\alpha,\beta,n)^m|a_n|}<1.$$

This shows that $f(z)\in T^*_m(\alpha,\beta,A,B)$.

Next, we prove the following growth and distortion property for the class $T^*_m(\alpha,\beta,A,B)$.

Theorem 2. If $f(z)\in T^*_m(\alpha,\beta,A,B),$ then for $0<|z|=r<1$

$$\frac{1}{r}-\frac{A-B}{(2-(A+B))A(\alpha,\beta,1)^m}r\leq|f(z)|\leq\frac{1}{r}+\frac{A-B}{(2-(A+B))A(\alpha,\beta,1)^m}r \tag{6}$$

and

$$\frac{1}{r^2}-\frac{A-B}{(1-B)A(\alpha,\beta,1)^m}\leq|f'(z)|\leq\frac{1}{r^2}+\frac{A-B}{(1-B)A(\alpha,\beta,1)^m} \tag{7}$$

Proof. Let $f(z)\in T^*_m(\alpha,\beta,A,B)$. Then, we find from Theorem 1. that

$$(2-(A+B))A(\alpha,\beta,1)^m\sum_{n=1}^{\infty}|a_n|\leq\sum_{n=1}^{\infty}[(1-A)+n(1-B)]A(\alpha,\beta,n)^m|a_n|\leq A-B$$

which yields

$$\sum_{n=1}^{\infty}|a_n|\leq\frac{A-B}{(2-(A+B))A(\alpha,\beta,1)^m}. \tag{8}$$

Also, by applying the triangle inequality, we have

$$|f(z)|=\left|z^{-1}+\sum_{n=0}^{\infty}a_nz^n\right|\leq\frac{1}{|z|}+\sum_{n=0}^{\infty}|a_n||z|^n.$$

Since $|z|=r<1,$ we can see that $r^n\leq r.$ Thus, we have

$$|f(z)| \le \frac{1}{r} + r\sum_{n=0}^{\infty} |a_n|$$

and

$$|f(z)| \ge \frac{1}{r} - r\sum_{n=0}^{\infty} |a_n|.$$

From the inequality (8), we obtain the result of (6).

On the other hand, we get

$$(1-A)A(\alpha,\beta,1)^m \sum_{n=1}^{\infty} |a_n| + (1-B)A(\alpha,\beta,1)^m \sum_{n=1}^{\infty} n|a_n| \le \sum_{n=1}^{\infty} [(1-A)+n(1-B)]A(\alpha,\beta,n)^m |a_n| \le A-B$$

and, so from $(1-A)A(\alpha,\beta,1)^m \ge 0$

$$(1-B)A(\alpha,\beta,1)^m \sum_{n=1}^{\infty} n|a_n| \le A-B-(1-A)A(\alpha,\beta,1)^m \sum_{n=1}^{\infty} |a_n|.$$
$$\le A-B$$

Thus, we have

$$\sum_{n=1}^{\infty} n|a_n| \le \frac{A-B}{(1-B)A(\alpha,\beta,1)^m}. \tag{9}$$

By applying the triangle inequality, we obtain

$$|f'(z)| \le \frac{1}{r^2} + \sum_{n=0}^{\infty} n|a_n|$$

and

$$|f'(z)| \le \frac{1}{r^2} - \sum_{n=0}^{\infty} n|a_n|.$$

From the inequality (9), we obtain the result of (7).

Finally, we determine the radius of meromorphically starlikeness and convexity for functions in the class $T_m^*(\alpha,\beta,A,B)$.

Theorem 3. Let $f(z)\in T_m^*(\alpha,\beta,A,B)$. Then

(i) $f(z)$ is meromorphically starlike of order δ in $|z|<r_1$, that is

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}<-\delta \qquad \left(|z|<r_1\right) \tag{10}$$

where $0\le\delta<1$ and

$$r_1=\inf_{n\ge1}\left\{\frac{(1-\delta)[(1-A)+n(1-B)]A\left(\alpha,\beta,n\right)^m}{(A-B)(n+\delta)}\right\}^{\frac{1}{n+1}}$$

(ii) $f(z)$ is meromorphically convex of order δ in $|z|<r_2$, that is

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}<-\delta \qquad \left(|z|<r_2\right) \tag{11}$$

where $0\le\delta<1$ and

$$r_2=\inf_{n\ge1}\left\{\frac{(1-\delta)[(1-A)+n(1-B)]A\left(\alpha,\beta,n\right)^m}{n(A-B)(n+\delta)}\right\}^{\frac{1}{n+1}}.$$

Each of these results is sharp for the function $f(z)$ given by (4).

Proof. (i) From Theorem 1. we have

$$\sum_{n=1}^{\infty}\frac{n+\delta}{(1-\delta)}|a_n||z|^{n+1}<\sum_{k=1}^{\infty}\frac{((1-A)+n(1-B)]A\left(\alpha,\beta,n\right)^m}{(A-B)}|a_n|\le1 \quad \left(|z|<r_1\right).$$

Therefore for $|z|<r_1$ we have

$$\left|\frac{zf'(z)/f(z)+1}{zf'(z)/f(z)-(1-2\delta)}\right| \le \frac{\sum_{n=1}^{\infty}(n+1)|a_n||z|^{n+1}}{2(1-\delta)-\sum_{n=1}^{\infty}[n-(1-2\delta)]|a_n||z|^{n+1}} < 1$$

which shows that (10) is true

(ii) It follows from Theorem 1. that

$$\sum_{n=1}^{\infty}\frac{n(n+\delta)}{(1-\delta)}|a_n||z|^{n+1} < \sum_{n=1}^{\infty}\frac{((1-A)+n(1-B)]A(\alpha,\beta,n)^m}{(A-B)}|a_n| \le 1 \quad (|z|<r_2)$$

Thus for $|z|<r_2$, we obtain

$$\left|\frac{1+zf''(z)/f'(z)+1}{1+zf''(z)/f'(z)-(1-2\delta)}\right| \le \frac{\sum_{n=1}^{\infty}n(n+1)|a_n||z|^{n+1}}{2(1-\delta)-\sum_{n=1}^{\infty}n[n-(1-2\delta)]|a_n||z|^{n+1}} < 1$$

which shows that (11) is true.

Sharpness can be verified easily.

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Energy of Timelike Spherical Magnetic Curves on the De-Sitter Space $\mathbb{S}_1^2$

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Abstract

In this paper, we invesitigate the energy of timelike spherical magnetic curves associated with the given magnetic field $\mathcal{G}$ on the De-Sitter 2-space $\mathbb{S}_1^2$. We use completely geometric approach for this computationsuch that the energy of each timelike spherical magnetic curve is stated by using the geodesic curvature of each magnetic curve.

Keywords: De-Sitter space, magnetic field, timelike magnetic curve, energy, magnetic force, uniform motion.

1. Introduction

A magnetic field on a k-dimensional semi-Riemannian manifold $(\mathcal{R}, h)$, which has the Levi-Civita connection ∇, is any closed 2-form $\mathcal{G}$ on $\mathcal{R}$ such that its Lorentz force is a one-to-one anti-symmetric tensor field Ψ given by $h(\Psi(\mathcal{A}),\mathcal{B}) = \mathcal{G}(\mathcal{A},\mathcal{B})$, where $\mathcal{A},\mathcal{B}$ are any two vector fields tangent to $\mathcal{R}$.

A charged particle follows a trajectory δ under the influence of $\mathcal{G}$, which meets the Lorentz formula $\nabla_{\delta'} \delta' = \Psi(\delta)$. As seen, the natural generalization of geodesics, which meet the Lorentz formula without the influence of any magnetic field, is given by magnetic curves.

A detailed research investigation has been performed to understand the magnetic curves and their associated flows. For example, it is proved that Kirchhoff elastic thin rod is classified as one of the solution classes of the Lorentz force action. This establishes a correlation among two unrelated physical subjects known as the Hall effect and the elastica. Moreover, critical points and the harmonicity of the Landau-Hall functional are computed as one of the other solution classes of the Lorentz force action. As a result, the subject of magnetic curves is interrelated to many other physical and geometric subjects and it has various applications [1-7].

2. Preliminaries

Magnetic Curves

The trajectories of a charged particle moving under the influence of a magnetic field on any manifold are represented by a magnetic curve. A magnetic field on a k-dimensional semi-Riemannian manifold $(\mathcal{R},h)$ is any closed 2-form $\mathcal{G}$. The Lorentz force of the magnetic field $\mathcal{G}$ is an antisymmetric one-to-one tensor field Ψ such that it is defined by

$$h(\Psi(\mathcal{A}),\mathcal{B})=\mathcal{G}(\mathcal{A},\mathcal{B}), \text{where} \forall \mathcal{A},\mathcal{B}\in\mathcal{X}(\mathcal{R}).$$

The magnetic trajectories of the magnetic field $\mathcal{G}$ correspond to magnetic curves δ on R. These curves satisfy the following Lorentz formula

$$\nabla_{\delta'}\delta' = \Psi(\delta'). \tag{1}$$

Evidently, magnetic curves generalize geodesics due to the following equation, which is satisfied by geodesics

$$\nabla_{\delta'}\delta' = 0.$$

This formula obviously represents the Lorentz formula in the nonappearance of the magnetic field. Consequently, a geodesic corresponds to a trajectory of the moving charged particle when it is free from the magnetic field $(\mathcal{G}=0)$ [4].

A significant feature of magnetic curves is that they have a constant kinetic energy since their speed is a constant. This is an obvious result of the antisymmetric property of the Lorentz force.

In the case of a $3D$ pseudo-Riemannian manifold $(\mathcal{R},h)$, vector fields and 2-forms may be described thanks to the volume form dv_h and the Hodge star operator $\star$ of the manifold. Hence, divergence free vector fields and magnetic fields are in $(1-1)$ correspondence. Therefore, Lorentz formula is given for any vector field S on the $3D$ pseudo-Riemannian manifold as follows

$$\Psi(\mathcal{A})=\mathcal{G}\times\mathcal{A}, \tag{2}$$

where G is a magnetic field such that $\forall A\in X(R)$ with $div(\mathcal{G})=0$ [4]. As a consequence, the magnetic flow reduced by the Lorentz formula is written as the following form

$$\nabla_{\delta'}\delta' = \mathcal{G}\times\delta'. \tag{3}$$

The Geometry of the De-Sitter Space $\mathbb{S}_1^2$

In this subsection, we present fundamental definitions of the spherical geometry of the Lorentzian space form, which corresponds to a De-Sitter space $\mathbb{S}_1^2$. Here, we generalize the geometrical understanding of the De-Sitter space in order to comprehend the mathematical method that we improve to define magnetic curves in the $\mathbb{S}_1^2$.

Let $\mathbb{R}_1^{k+1}$ be a $(k+1)$-dimensional vector space equipped with the Lorentzian metric

$$h(,) = -da_1^2 + da_2^2 + \ldots + da_{k+1}^2.$$

In this case, $(\mathbb{R}_1^{k+1}, h(,))$ is named by Minkowski $(k+1)-$ space. The pseudo vector product of $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k \in \mathbb{R}_1^{k+1}$ is described to be

$$\mathbf{a}_1 \times \mathbf{a}_2 \times \ldots \times \mathbf{a}_k = \begin{bmatrix} -u_1 & u_2 & \ldots & u_{k+1} \\ a_1^1 & a_1^2 & \ldots & a_1^{k+1} \\ a_2^1 & a_2^2 & \ldots & a_2^{k+1} \\ \vdots & \vdots & \vdots & \vdots \\ a_k^1 & a_k^2 & \ldots & a_k^{k+1} \end{bmatrix},$$

where $\mathbf{a_i} = \{a_i^1, a_i^2, \ldots, a_i^{k+1}\}$ and $\{u_1, u_2, \ldots, u_{k+1}\}$ is the canonical basis of $\mathbb{R}_1^{k+1}$. A non-zero vector $\mathbf{a} \in \mathbb{R}_1^{k+1}$ is called timelike, lightlike or spacelike if $h(\mathbf{a},\mathbf{a}) < 0,$ $h(\mathbf{a},\mathbf{a}) = 0$ or $h(\mathbf{a},\mathbf{a}) > 0.$ Thus, one can give the norm function of the $\mathbf{a} \in \mathbb{R}_1^{k+1}$ by using the $sign$ function as follows.

$$\|\mathbf{a}\| = \sqrt{sign(\mathbf{a})(\mathbf{a},\mathbf{a})},$$

where

$$sign(\mathbf{a}) = \begin{cases} 1, \mathbf{a}\,\text{is spacelike}, \\ 0, \mathbf{a}\,\text{is lightlike}, \\ -1, \mathbf{a}\,\text{is timelike}. \end{cases}$$

Let I be an open interval and $\delta : I \to \mathbb{R}_1^{k+1}$ be a curve defined on I with the condition of $\delta^{'}(p) \neq 0$ for any $p \in I.$ The curve δ is said to be timelike, lightlike or spacelike if $h\left(\delta^{'}(p), \delta^{'}(p)\right) < 0,$ $h\left(\delta^{'}(p), \delta^{'}(p)\right) = 0,$ and $h\left(\delta^{'}(p), \delta^{'}(p)\right) > 0$ $(\forall p \in I)$ respectively.

In the previous subsection, we state the advantage of studying in three-dimensional space while characterizing magnetic curves. For this reason, we define the De-Sitter 2-space S_1^2 (pseudo spherical 3-space) by using the argument discussed above as the following manner

$$\mathbb{S}_1^2 = \{\mathbf{a} \in \mathbb{R}_1^3 : -a_1^2 + a_2^2 + a_3^2 = 1\}.$$

Hereafter, we assume to have a unit speed timelike regular curve lying fully on the S_1^2. In the theory of curves, the most effective method using to investigate the intrinsic feature of the curve is to consider its orthonormal frame, which is constructed by a number of orthonormal vectors and associated curvatures depending on the dimension of the space. For the case of the $\mathbb{S}_1^2$, this orthonormal frame was introduced by Sabban [8,9]. The curve satisfying the Sabban frame equation is called a Sabban or spherical curve. Finally, we are ready to establish the orthonormal frame of timelike spherical curves lying on the $\mathbb{S}_1^2$.

Let $\delta : I \to \mathbb{S}_1^2$ be a unit speed timelike regular spherical curve, that is it is an arc-length parametrized and sufficiently smooth. Then, Sabban frame is defined along the curve δ as follows

$$\begin{aligned} \nabla_{\delta'} \delta &= \mathbf{T}, \\ \nabla_{\delta'} \mathbf{T} &= \delta + \mu \mathbf{N}, \\ \nabla_{\delta'} \mathbf{N} &= \mu \mathbf{T}, \end{aligned} \tag{4}$$

where ∇ is a Levi-Civita connection and $\mu = \det(\delta, \mathbf{T}, \mathbf{T}')$ is the geodesic curvature of δ. The following identities including pseudo vector product also hold $[10]$.

$$\delta = \mathbf{T} \times \mathbf{N}, \quad \mathbf{T} = \delta \times \mathbf{N}, \quad \mathbf{N} = \delta \times \mathbf{T}.$$

3. Timelike Spherical Magnetic Curves of the De-Sitter Space $\mathbb{S}_1^2$

Let β be a moving charged particle under the influence of a magnetic field $\mathcal{G}$ on the $\mathbb{S}_1^2$. We assume for the rest of the manuscript that the worldline of this particle corresponds to a unit speed regular timelike spherical curve $\delta : I \to \mathbb{S}_1^2$ to define and investigate timelike spherical magnetic curves lying fully on the Lorentzian sphere by using the Sabban orthonormal frame described along with the worldline. From Eqs. $(1-3)$, we are able to find three kinds of distinct magnetic trajectories of the β on the $\mathbb{S}_1^2$, if it is considered the orthonormal vector fields of the curve δ as in the Sabban frame.

Definition: Let $\delta : I \to \mathbb{S}_1^2$ be a unit speed regular timelike spherical curve on the De-Sitter 2-space and $\mathcal{G}$ be the magnetic field on $\mathbb{S}_1^2$. Timelike spherical magnetic curves of β are defined via the Lorentz force formula given by Eqs. $(1-3)$ as follows

$$\begin{aligned} \nabla_{\delta'} \delta &= \Psi(\delta) = \mathcal{G} \times \delta, \\ \nabla_{\delta'} \mathbf{T} &= \Psi(\mathbf{T}) = \mathcal{G} \times \mathbf{T}, \\ \nabla_{\delta'} \mathbf{N} &= \Psi(\mathbf{N}) = \mathcal{G} \times \mathbf{N}. \end{aligned} \quad (5)$$

For further references, we call these timelike spherical magnetic curves as a $\mathbf{S}\delta-$*magnetic curve*, $\mathbf{ST}-$*magnetic curve, and* $\mathbf{SN}-$*magnetic curve*, respectively. In other words, δ is called as an $\mathbf{S}\delta-$*magnetic curve* if the first equation holds; δ is called as an $\mathbf{ST}-$*magnetic curve* if the second equation holds; δ is called as an $\mathbf{SN}-$*magnetic curve* if the third equation holds.

Proposition: Let δ be an arc-length parametrized timelike spherical magnetic curve together with the Sabban frame elements $\{\delta, \mathbf{T}, \mathbf{N}, \mu\}$ on the De-Sitter space $\mathbb{S}_1^2$. Then, Lorentz force Ψ of the magnetic field $\mathcal{G}$ is written in the Sabban frame as follows.

- ♦ In the case of an $\mathbf{S}\delta-$*magnetic curve*, Ψ is defined by

$$\begin{aligned} \Psi(\delta) &= \mathbf{T}, \\ \Psi(\mathbf{T}) &= \delta + c_1 \mathbf{N}, \\ \Psi(\mathbf{N}) &= c_1 \mathbf{T}, \end{aligned} \quad (6)$$

where c_1 is an arbitrary smooth function along with the magnetic curve such that it satisfies $c_1 = h(\Psi(\mathbf{T}), \mathbf{N})$.

- ♦ In the case of an $\mathbf{ST}-$*magnetic curve*, Ψ is defined by

$$\begin{aligned} \Psi(\mathbf{T}) &= \delta + \mu \mathbf{N}, \\ \Psi(\delta) &= \mathbf{T} + c_2 \mathbf{N}, \\ \Psi(\mathbf{N}) &= -c_2 \delta + \mu \mathbf{T}, \end{aligned} \quad (7)$$

where c_2 is an arbitrary smooth function along with the magnetic curve such that it satisfies $c_2 = h(\Psi(\delta), \mathbf{N})$.

- ♦ In the case of an $\mathbf{SN}-$*magnetic curve*, Ψ is defined by

$$\begin{aligned} \Psi(\mathbf{N}) &= \mu\mathbf{T}, \\ \Psi(\delta) &= c_3\mathbf{T}, \\ \Psi(\mathbf{T}) &= c_3\delta + \mu\mathbf{N}, \end{aligned} \tag{8}$$

where c_3 is an arbitrary smooth function along with the magnetic curve such that it satisfies $c_3 = h(\Psi(\mathbf{T}), \delta)$.

Theorem: Let δ be an arc-length parametrized timelike spherical magnetic curve on the De-Sitter space $\mathbb{S}_1^2$.

♦ δ is an $\mathbf{S}\delta$ - *magnetic curve* of the magnetic field $\mathcal{G}$ on the $\mathbb{S}_1^2$ if and only if

$$\mathcal{G}\quad c_1\delta - \mathbf{N}, \tag{9}$$

where $c_1 = h(\Psi(\mathbf{T}), \mathbf{N})$ along with the curve.

♦ δ is an $\mathbf{ST}$- *magnetic curve* of the magnetic field $\mathcal{G}$ on the $\mathbb{S}_1^2$ if and only if

$$\mathcal{G}\quad \mu\delta - c_2\mathbf{T} - \mathbf{N}, \tag{10}$$

where $c_2 = h(\Psi(\delta), \mathbf{N})$ along with the curve.

♦ δ is an $\mathbf{SN}$- *magnetic curve* of the magnetic field $\mathcal{G}$ on the $\mathbb{S}_1^2$ if and only if

$$\mathcal{G}\quad \mu\delta - c_3\mathbf{N}, \tag{11}$$

where $c_3 = h(\Psi(\mathbf{T}), \delta)$ along with the curve.

4. Energy of Timelike spherical Magnetic Curves on the De-Sitter Space $\mathbb{S}_1^2$

In this section, we investigate the energy of timelike spherical magnetic curves associated with the given magnetic field $\mathcal{G}$ on the De-Sitter 2-space $\mathbb{S}_1^2$. We use a completely geometrical approach for this computation such that the energy of each timelike spherical magnetic curve is stated by using the geodesic curvature of each magnetic curve.

A well-known feature of magnetic curves is that they have a constant kinetic energy since their speed is a constant [11]. This is also an obvious result of the antisymmetric property of the Lorentz force.

By considering this fact, we also determine the constant energy condition for each timelike spherical magnetic curves on the $\mathbb{S}_1^2$.

Definition: Let $(\mathcal{R},h)$ and $(\mathcal{R}^*,h^*)$ be two Riemannian manifolds. Then, the energy of a differentiable map $r:(\mathcal{R},h)\to(\mathcal{R}^*,h^*)$ can be defined as

$$\varepsilon nergy(r)=\frac{1}{2}\int_M\sum_{a=1}^{n}h^*\left(df\left(e_a\right),df\left(e_a\right)\right)v, \tag{12}$$

where $\{e_a\}$ is a local basis of the tangent space and v is the canonical volume form in $\mathcal{R}$ [12].

Proposition: Let $Q:T\left(T^1M\right)\to T^1M$ be the connection map. Then following two conditions hold.

i) $c\circ Q=c\circ dc$ and $c\circ Q=c\circ\tilde{c}$, where $\tilde{c}:T\left(T^1M\right)\to T^1M$ is the tangent bundle projection,

ii) for $\rho\in T_xM$ and a section $\varsigma:M\to T^1M$ we have

$$Q(d\varsigma(\rho))=\nabla_\rho\varsigma, \tag{13}$$

where ∇ is the Levi-Civita covariant derivative [12].

Definition: Let $\sigma_1,\sigma_2\in T_\varsigma\left(T^1M\right)$, then we define

$$h_S(\sigma_1,\sigma_2)=\left(dc(\sigma_1),dc(\sigma_2)\right)+\left(Q(\sigma_1),Q(\sigma_2)\right). \tag{14}$$

This yields a Riemannian metric on TM. As known h_S is called the Sasaki metric that also makes the projection $c:\ T^1M\to M$ a Riemannian submersion [12].

Theorem: Let β be a moving charged particle such that it corresponds to a unit speed timelike spherical magnetic curves in the associated magnetic field $\mathcal{G}$ on the $\mathbb{S}_1^2$.

♦ In the case of an $\mathbf{S}\delta-$*magnetic curve*, energy of the particle in the magnetic vector field $\mathcal{G}$ is

$$\varepsilon nergy(\mathcal{G}_{S\delta})=\frac{1}{2}\int_0^s(-1-(\mu-c_1)^2+\left(c_1^{'}\right)^2)ds,$$

where $c_1=h\left(\Psi(\mathbf{T}),\mathbf{N}\right)$ along with the curve.

♦ In the case of an $\mathbf{ST}-$*magnetic curve*, energy of the particle in the magnetic vector field $\mathcal{G}$ is

$$\varepsilon nergy(\mathcal{G}_{\mathbf{ST}})=\frac{1}{2}\int_0^s(-1+(\mu'-c_2)^2-(c_2')^2+c_2^2\mu^2)ds,$$

where $c_2=h(\Psi(\delta),\mathbf{N})$ along with the curve.

♦ In the case of an $\mathbf{SN}-$*magnetic curve*, energy of the particle in the magnetic vector field $\mathcal{G}$ is

$$\varepsilon nergy(\mathcal{G}_{\mathbf{SN}})=\frac{1}{2}\int_0^s(-1+(c_3')^2-\mu^2(c_3-1)^2+(\mu')^2)ds,$$

where $c_3=h(\Psi(\mathbf{T}),\delta)$ along with the curve.

Proof: Let δ be an $\mathbf{S}\delta$-*magnetic curve* of the magnetic field $\mathcal{G}$ on the $\mathbb{S}_1^2$. From the Eq. (12) and the Eq. (13), one gets

$$\varepsilon nergy(\mathcal{G}_{\mathbf{S}\delta})=\frac{1}{2}\int_0^s h_S\left(d\mathcal{G}(\mathbf{T}),d\mathcal{G}(\mathbf{T})\right)ds. \tag{15}$$

By using also the Eq. (14), one also has

$$h_S(d\mathcal{G}(\mathbf{T}),d\mathcal{G}(\mathbf{T}))=h(dc(\mathcal{G}(\mathbf{T})),dc(\mathcal{G}(\mathbf{T})))+h(Q(\mathcal{G}(\mathbf{T})),Q(\mathcal{G}(\mathbf{T}))).$$

Since $\mathbf{T}$ is a section, it is obtained that

$$d(c)\circ d(\mathcal{G})=d(c\circ\mathcal{G})=d(id_C)=id_{TC}.$$

Moreover, it is clear that

$$Q(\mathcal{G}(\mathbf{T}))=\nabla_{\delta'}\mathcal{G}=c_1'\delta-(\mu-c_1)\mathbf{T}.$$

Thus, we find from Eqs. (4,9)

$$\begin{aligned}\rho_S(d\mathcal{G}(\mathbf{T}),d\mathcal{G}(\mathbf{T}))&=h(\mathbf{T},\mathbf{T})+h(\nabla_{\delta'}\mathcal{G},\nabla_{\delta'}\mathcal{G})\\&=-1-(\mu-c_1)^2+(c_1')^2.\end{aligned}$$

This final identity gives the desired result if it is plugged into the Eq. (15). The rest of the proof is completed if one follows similar steps for other cases.

Altough this calculation seems to contain abstract mathematical tools it tells us significant facts about the state of a system. For example, Euler-Lagrange equations determine the dynamics of a system considering its simple motion once one computes the energy of the given system. For a given vector field the following formula is used to describe bending energy of elastica:

$$\mathcal{H} = \frac{1}{2}\int \left\| \nabla_{\delta'} \mathbf{T} \right\|^2 ds, \tag{16}$$

where s is an arclength [12]. Once the elastic features of timelike spherical magnetic curves are determined we can state the energy of each timelike spherical magnetic curve in terms of the bending energy functional. However, this is the topic of another research that we plan to handle later.

At the beginning of the section, we assert that magnetic curves have a constant energy. Now, we give the constant energy condition that has to be satisfied for each timelike spherical magnetic curve in terms of its geodesic curvature.

♦ Constant energy condition of the $\mathbf{S}\delta-$*magnetic curve* in the magnetic vector field $\mathcal{G}$ on the $\mathbb{S}_1^2$;

$$1 = -(\mu - c_1)^2 + \left(c_1'\right)^2.$$

♦ Constant energy condition of the $\mathbf{ST}-$*magnetic curve* in the magnetic vector field $\mathcal{G}$ on the $\mathbb{S}_1^2$;

$$1 = (\mu' - c_2)^2 - \left(c_2'\right)^2 + c_2^2\mu^2.$$

♦ Constant energy condition of the $\mathbf{SN}-$*magnetic curve* in the magnetic vector field $\mathcal{G}$ on the $\mathbb{S}_1^2$;

$$1 = \left(c_3'\right)^2 - \mu^2(c_3 - 1)^2 + \left(\mu'\right)^2.$$

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Commutativity Associated with Euler Second-Order Differential Equation

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Abstract

This work study the commutativity and alongside with the sensitivity of second-order Euler differential equation. The conditions for commutativity of second-order Euler differential equation are investigated. Example will be given to support the results.

Keywords: : Commutativity, Euler Differential Equation and analogue system.

1. Introduction

When the sensitivity, stability, linearity, noise disturbance, robustness effects are considered the change of the order of connection without changing the main function of the total systems (commutativity) may lead positive results. Therefore, the commutativity is very important from the practical point of view.

The first commutativity appeared in 1977 for the first-order time-varying systems [1], and then the results there in are extended to higher-order systems [2-7] the non-zero initial conditions (ICs) and its effects on the sensitivity was studied in [8] while the realization of a fourth-order LTVSs with nonzero ICs by Cascaded two Second-Order Commutative Pairs was introduced in [9].

In this presentation, the commutativity and alongside with the sensitivity of second-order Euler differential equation are studied. The results are illustrated by an example.

.

2. System Description

Considering the cascade connection of second-order systems A and B described as

$$A: a_2(t)y_A''(t) + a_1(t)y_A'(t) + a_0(t)y_A(t) \ = x_A(t), \tag{1}$$

$$\text{B}: b_2(t)y_B''(t) + b_1(t)y_B'(t) + b_0(t)y_B(t) \ = x_B(t), \tag{2}$$

where $a_2(t) \neq 0$ and $b_2(t) \neq 0$. Also, a_i, b_i, x_A, $x_B \in P[t_0, \infty)$. The connections are abbreviated as AB or BA according to their sequence of connection.

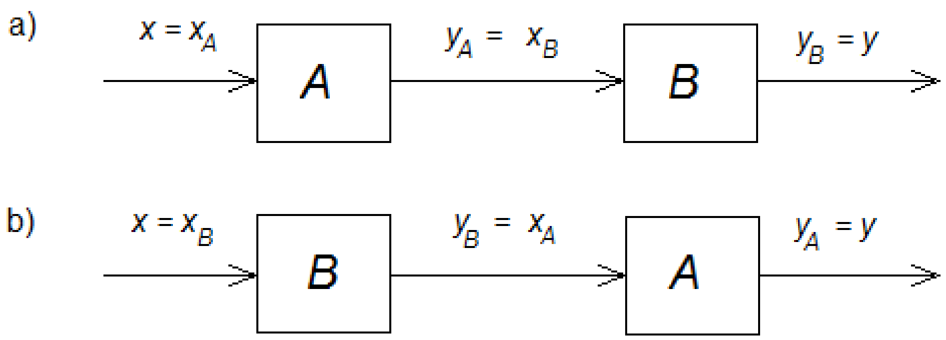

Figure 1. Cascade connection of the differential systems A and B.

The propose is to find the subsystems A and B such that each one of the connections AB and BA are equivalent (the case in which A and B are called commutative subsystems [2]); the found results are expressed by a theorem presented in the next section

3. Main Results

Theorem 1 *(See [2]). The formula for a second-order LTVS A to be commutative with another LTVS B under zero ICs is that the coefficients of B are*

$$\begin{bmatrix} b_2 \\ b_1 \\ b_0 \end{bmatrix} = \begin{bmatrix} a_2 & 0 & 0 \\ a_1 & a_2^{0.5} & 0 \\ a_0 & f_{32} & 1 \end{bmatrix} \begin{bmatrix} k_2 \\ k_1 \\ k_0 \end{bmatrix}, \quad f_{32} = \frac{1}{4}[a_2^{-0.5}(2a_1 - a'_2)]; \tag{3a}$$

$$-a_2^{0.5} \frac{d}{dt}[a_0 - f_{32}^2 - a_2^{0.5} f'_{32}]k_1 = 0, \tag{3b}$$

where k_2, k_1, k_0 are constants and it must satisfy $(3b)$.

Theorem 2 *The commutativity for second-order LTVS A with non-zero ICs with another second or lower-order LTVS B are that:*

i) Eq.(3a) and Eq. (3b) must be satisfied.

ii) The ICs at the initial time (IT) $\boldsymbol{t_0} \leq \boldsymbol{t}$ must hold:

$$\left\{\binom{2}{m}\begin{bmatrix} 1 & 0 \\ -A_2^{-1}A_1 & A_2^{-1} \end{bmatrix} - \binom{m}{2}\begin{bmatrix} 0 & 1 \\ B_2^{-1} & -B_2^{-1}B_1 \end{bmatrix}\right\}\begin{bmatrix} Y_A \\ Y_B \end{bmatrix} = [0], \tag{4}$$

where

$Y_A = [y_A(t),\ y'_A(t)]^T$,

$Y_B = [y_B(t),\ y'_B(t)]^T$ and the matrix A_1 (A_2, B_1, B_2) are described by there entries a'_{ij} $(a''_{ij}, b'_{ij}, b''_{ij})$ respectively:

$$\begin{aligned} a'_{ij} &= \sum_{s=max(0,i-j)}^{i-1} \frac{(i-1)!}{s!\,(i-1-s)!} a^s_{j-i+s}; \quad i = 1, m, \quad j = 1,2, \\ a''_{ij} &= \sum_{s=0}^{i-j} \frac{(i-1)!}{s!\,(i-1-s)!} a^s_{j-i+n+s}; \quad i = 1, m, \quad j = 1, m; \\ &= 0 \ \ for \ \ i = 1, \dots, m-1, \quad j = i+1, \dots, m, \\ b'_{ij} &= \sum_{s=max(0,i-j)}^{i-1} \frac{(i-1)!}{s!\,(i-1-s)!} b^s_{j-i+s}; \quad i = 1,2, \quad j = 1, m, \\ b''_{ij} &= \sum_{s=max(0,i-j-m)}^{i-j} \frac{(i-1)!}{s!\,(i-1-s)!} b^s_{j-i+m+s}; \quad i = 1,2, \quad j = 1, \dots, i; \\ &= 0 \ \ for \ \ i = 1, \quad j = i+1, \dots, 2. \end{aligned} \tag{5}$$

4. Example

In this section, we make use of the formular and conditions obtained from the previous section and illustrate the commutativity of second-order LTVs.

Example 1. Let us first consider the following second-order Euler LTVs

$$A\colon t^2 y''_A(t) + \sqrt{2}\mathrm{t}\, y'_A(t) \ + \frac{17}{11} y_A(t) = x_A(t). \tag{6}$$

By applying the coefficient of Eq. (6) to Eq. (3a), we obtain

$$\begin{bmatrix} b_2 \\ b_1 \\ b_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \sqrt{2}\mathrm{t} & \mathrm{t} & 0 \\ \frac{17}{11} & \frac{1}{4}\left(\frac{-2t + 2\sqrt{2}\mathrm{t}}{t}\right) & 1 \end{bmatrix} \begin{bmatrix} k_2 \\ k_1 \\ k_0 \end{bmatrix}, \tag{7}$$

where k_2, k_1, k_0 are constants.
Considering the matrix in Eq. (7) at $k_1 = 0$, we have

$$B\colon t^2 k_2 y_B''(t) + \sqrt{2}\mathrm{t}\, k_2\, y_B'(t) + (k_0 + \tfrac{17}{11} k_2) y_B(t) = x_B(t). \tag{8}$$

Substituting the coefficient of Eq. (6) in Eq. (3b) lead to

$$k_0 \to 1 - k_2. \tag{9}$$

For commutativity of A and B with non-zero ICs to exist, the equation below must be satisfy

$$y_A' = -\frac{3\sqrt{2}}{11} y_A. \tag{10}$$

Considering a sinusoid of amplitude 200, frequency 25 and phase $\frac{\pi}{30}$rad with ODE 23 [Bogacki - Shampine] as the solver. For $k_2 = k_0 = 0.5$ and $k_1 = 0$. The IT at $t_0 = 1$ and the ICs as $y_A(1) = y_B(1) = 1, y'_A(1) = y'_B(1) = -\frac{3\sqrt{2}}{11}$, AB and BA (solid blue curve) gives the same output. With response to sensitivity toward ICs, by changing $y_A(1) = -1$, $AB1$ (doted-dash red curve) and $BA1$ (dashed-green curve) deviated from each other, this is because Eq. (10) is not obeyed.

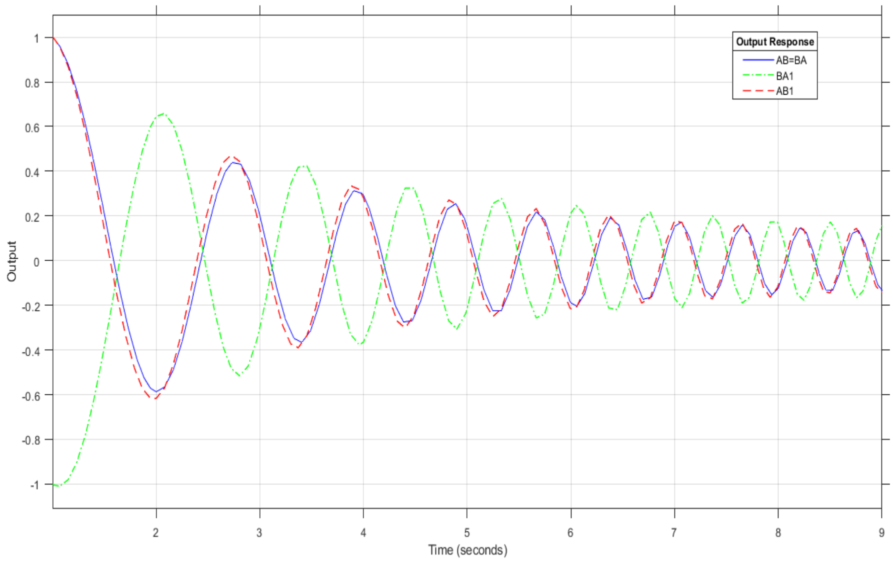

Figure 2. Simulation results for $k_2 = k_0 = 0.5$ and $k_1 = 0$.

6. Conclusion

This presentation shows the results for second-order Euler LTVS A cascaded connected with it commutative pairs of second-order LTVS B. The result obtained shows that the subsystems A and B are said to be commutative under some conditions and are very sensitive toward change in ICs. The results are verified to be correct by an example which is simulated by Simulink toolbox of MATLAB.

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Through Unimodular Matrix on SLE using LaTex:

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Abstract

We deal with the considered results comparing the obtained solutions with the exact ones. Expanding the proposed concepts on generating the system of linear equations through matrix as unimodular or unimodular matrix with compound structures, we give in particular the method of generating SLEs. Our work shows the way for this problem on Determined SLEs. We give the solution for students. In this context, our work gives the method in providing exercises on the topic of SLEs for solutions with integers. For this fact we provide the unimodularity of matrix. We claim that a square matrix is said to be unimodular if it has a determinant value of 1 or -1. The inverse of a unimodular matrix and the product of two unimodular matrices is also unimodular. Those facts are used in our article for determining the solution of system of linear equations (SLEs), because it does not involve fractions at all. In particular, we give the method for generating an SLE via a unimodular matrix with Latex. We consider the result and any example of determined SLE with solution in the form of integers. By using Latex and python programs, we provide and generalize the way for generating a finite SLEs easily in (Arifin and Muktyas[11]-Uka and Hajrulla[12]). Some conclusions help to understand the process.

Keywords: unimodular matrix, latex, python, solution, system of linear equations, transformation,

Introduction

A System of Linear Equation (SLE) that has the same number of equations as the number of variables is called the Determined SLE. Algebra is a branch of mathematics, one of which studies the System of Linear Equation (SLE) and the matrix. A finite collection of linear equations in a variable is called the System of Linear Equation. There are several ways to determine the solution of a System of Linear Equation, such as using the Elementary Row Operation. However, the methods can only be used for a square matrix and nonsingular. On the other hand, there are often problems regarding whether a matrix has an inverse. One of the important theorems concerning states that matrix A has an inverse if only if $\det(A) \neq 0$ (see Anton [1]). Another problem arises if the entries of the inverse matrix are not integers, so it will take longer if done by hand manually. For this, we need a matrix that has a determinant value of 1 or -1 to produce an inverse matrix with all entries being integers. Matrices that have a determinant value of 1 or -1 are called unimodular matrices in Guy[3]. The notion of unimodular matrices can be studied in Born [2] and Guy [3]. The steps in generating a unimodular matrix can also be studied in Hanson [4]. We chose Python to makes the code because it has many advantages. Python is a multipurpose programming language and is easy to learn. Python can also run on various operating system platforms, such as Windows, Linux, Mac OS, Android, and others in (QPythonLab, QPython OL[5]). By using the Python program, a unimodular matrix can be generated so that it can produce a System of Linear Equation with a single solution of all integers. In this paper, we want to provide an application of unimodular matrices in Determined SLE, which is generating a Determined SLE of n variables and n equations with integer solutions. In solving the Determined SLE, it only uses integers, so that it can be a bridge and a first step for students to learn the notion of Determined SLE much deeper and easier. Moreover, we will study the method for generating a System of Linear Equation Determined using Python, in the form AX=B where A is a unimodular matrix, and using LaTex generates a pdf file format containing drill questions with the number of questions and the variable is adjusted to the user's wishes in (Arifin and Muktyas [11]). We will discuss the notion of Determined SLE, unimodular matrix, and Python, and LaTex in session 2 in (Arifin and Muktyas [11]). The results and discussion about the outputs of the resulting program are discussed in session 3. This paper concludes with a conclusion in session 4. METHOD In general, the research methodology used here is a method of exploration and adaptation of pre-existing results, which was examined from the study of literature. The following is a study of the theories used in this research. Systems of Linear Equations In this session, the System of Linear Equation (SLE) will be discussed. For the future, the term system of linear equations is enough to be written with the SLE. Note that the general form of a SLE with n equations and n variables is as follows: AX=B. Looking for solutions to the SLE with n equations and n variables, we can use an elementary row operation. On the other hand, based on the comparison of the values of n (many equations) and n (many variables), one type of SLE is the Determined SLE, which is the SLE with the same variables and equations. In this paper, it is assumed that all SLE provided is a Determined SLE. Consider the following theorem.

Theorem 1. Anton [1]

1. Let A_{nxn} be a matrix. A is invertible if and only if det $(A) \neq 0$. 2. If A is invertible, then $A^{-1} = \frac{adj(A)}{\det(A)}$ 3. A System of Linear Equations with n equations and n variables written in the form $A_{nn}x_{n1} = B_{n1}$ has a single solution if and only if matrix A has an inverse. Unimodular Matrix In this session, we will examine the understanding of the unimodular matrix and how to generate it. In Guy [3], it is stated that a matrix A_{nxn} with each entry being an integer is called unimodular if det (A) = -1 or det (A) = 1. Other terms of the unimodular matrix are stated in Hanson [4], using the term "Nice Matrix". Examples of unimodular matrices are identity matrices, upper triangles, or lower triangles with diagonals of 1 or -1. Consider the following theorem.

Theorem 2. Anton [1]

Let A_{nxn} is a triangle matrix such that $a_{11,}a_{22}, \dots., a_{nn}$ are on the main diagonal. Then det(A)=$a_{11}.a_{22} \dots. a_{nn}$. The following dilemma explains the steps in generating a unimodular matrix or the Nice Matrix. Note that Lemma 3. below which will be used as a reference in making a program to generate a unimodular matrix using Python.

Lemma 3. Hanson [4]

A unimodular matrix A_{nxn} can be constructed in the following way: 1. First, make a diagonal matrix with the diagonal entry a_{ii} = 1 or a_{ii} = -1. 2. Second, fill in any random integers at each entry with i < j. From this, it has formed a top triangular matrix whose determinants are 1 or -1. This is a unimodular matrix. 3. Third, to be a complete matrix, use ERO/ECO downward from the last row/column to the initial row/column.

Lemma 4.Python and LaTeX [11]

In this session, we will examine the Python and TeX / LaTeX programs that we use. We generating an nxnsized unimodular matrix using Python 2.7.14, then produce output in the form of a pdf file format that is ready for the user to use in (https://www.python.org/ [18]). This Python program code is the main result of this paper. The things that become the basis in making the program are Lemma 3 Hanson[4]. above. But before that, it should be noted that the Python program used must be equipped with a "NumPy" plugin. The steps to install NumPy in Windows OS is very easy. On the other hand, the application of Python in group theory and linear algebra can be found in Arifin [6,7], Muktyas [8,9], and Rahman [10], as well as their use in the dimensional theory of a ring and modules, can be studied in Arifin [11,12]. Another application of Python in other fields, that is Data Mining in Demsar [13], Scatterplot matrices in Foreman-Mackey [14], Scientific computing in Oliphant [15], Machine learning in Pedregosa, etc [16], and Image processing in Van der Walt [17]. Python is a popular programming language (https://www.python.org/ [18]). As of September 2018, Python is ranked as the 3rd most popular programming language in the world.

Python is relatively easier to learn and use than other programming languages. The syntax is simple, easy to read, and remember because the philosophy of python itself emphasizes the aspect of readability. Python code is easy to write and easy to read, making it easier to fix if there are errors, and also easy to maintain. Besides being easier to read, python is also more efficient than other languages such as C, C ++, and Java in (Pedregosa [16]). To do something with 5 lines of code in another language, maybe in Python only one line of code is needed. This causes the creation of programs in Python to be more concise and faster than other languages. Python is a multifunctional language. With python you can do a variety of things from text processing, creating websites, creating network programs, robotics, data mining, to artificial intelligence see in (https://www.python.org/[18]). With Python, you can create desktopbased and smartphone-based applications. Python is rich in standard library support. There are a lot of modules and program extensions that you are ready to use to make the program according to your needs. Python community is a community that is very active in developing python so it becomes a very reliable language. Python can interact with other languages. Python code can be called by C, C ++, and vice versa can also be called from other languages(Oliphant [15]). In short, the reason we use Python is that Python is a powerful language and can be run on multiple platforms, but it's also very easy to understand. On the other hand, the reason we use LaTex is that it is freeware, supports writing mathematical formulas, and can run on different OS like Windows and Ubuntu. Another plus is that our program can make many drill questions in a short time. The weakness of the System 0f Linear Equation produced by the program is that there will always be coefficients or matrix entries that are worth one in the first column in a particular row. But actually, this entry of 1 is for the key, which will be 1 main, so that it will be easier for users to use An Elementary Row Operation because this is only as a bridge user to better understand An Elementary Row Operation. Another weakness is that this System of Linear Equation can be generated by first installing Python, the NumPy package, and LaTex. Next, we will discuss LaTeX. MiKTeX is an up-to-date implementation of TeX / LaTeX and related programs [19]. TeX is a typesetting system written by Donald Ervin Knuth who says that it is intended for the creation of beautiful books - and especially for books that contain a lot of mathematics. Moreover, you can learn more about TeX/LaTeX at Kanigoro [20]. Following is the display of the programming used, which is the main result of this paper. This Python program code will close this session.

Now, we will mention that, there are many codes prepared with pyhton programme.In the beginning of the article, mathematical formulas are written not only with LaTex but also phyton programme.

In mathematics, a system of linear equations (or linear system) is a collection of two or more linear equations involving the same set of variables see in (solving systems of linear equations [21]).

Now if we look at many linear equations then we will writes as a figure below:

1.Example: 1.Step: $4x+3y = 20$

$-5x+ 9y= 26$

To solve the above system of linear equations, we need to find the values of the x and y variables. There are multiple ways to solve such a system, such as Elimination of Variables, Cramer's Rule, Row Reduction Technique, and the Matrix Solution. In this article we will cover the matrix solution see in (Arifin and Muktyas[11]) .

In the matrix solution, the system of linear equations to be solved is represented in the form of matrix AX = B. For instance, we can represent *Equation 1* in the form of a matrix as follows:

A=[[4   3]

[-5   9]]

X= [[x]

[y]]

B= [[20]

[26]]

If we use two matrices dot product with together then this code will be below see in(Python, Python TM [18]):

x=inverse(A).B

we need to use inverse function.

Also, we need to add new information in phyton program.

Solving a System of Linear Equations with Numpy

We know that need to do two operations for solving linear equation system such as: matrix inversion and matrix dot products.Numphy library supports both two operations in phyton program see in (https://www.python.org/ [18]):

Exampe code: **$ pip install numpy**

Using the inv() and dot() Methods (https://www.python.org/ [18])

First of all, we will find inverse matrix A that we defined previous part.Firstly, let’s we create A matrix in phyton.The method of the Numphy module can used to constitute A matrix. A matrix can be considered as a list of lists where each list represents a row.Also “(m_list)” code must be used: [4,3] and [-5,9]. This 2 list are rows of matrix A. To create the matrix A with Numpy, the m_list is passed to the array method as shown below:

```
import numpy as np



m_list = [[4,3], [-5,9]]

A = np.array(m_list)
```

Matrix is passed to the linalg.inv() method in Numpy module for finding inverse of a matrix.

```
inv_A = np.linalg.inv(A)

print(inv_A)
```

The next step we find dot product of between inverse matrix A and inverse matrix B.

It is important to mention that matrix dot product is only possible between the matrices **if the inner dimensions of the matrices are equal** i.e. the number of columns of the left matrix must match the number of rows in the right matrix.At the same time, lingalg.dot() function is used to find dot product with Numphy Library. The following script finds the dot product between the inverse of matrix A and the matrix B, which is the solution of the *Equation 1*.

```
B = np.array([20,26])

X = np.linaglg.inv(A).dot(B)



print(x)



OUTPUT: [2. 4.]



This output value x=2 and y=4.
```

If we give new example of system of linear equation with matrix solution in Numphy Library in Phyton code then we will interprete in shown below:

```
 4x+3y+2z = 25
-2x+2y+3z = -10
 3x-5y+2z = -4
```

```
Code:

A = np.array([[4, 3, 2], [-2, 2, 3], [3, -5, 2]])

B = np.array([25, -10, -4])

X = np.linalg.inv(A).dot(B)


print(x)
```

In the script below linalg.inv() and linalg.dot() methods are chained with together. X variable has include solution of equation 2 and is printed as follows:

```
[5. 3. -2.]
```

Therefore, x=5, y=3, z=-2.

Using the solve() Method

In the previous two examples, we used `linalg.inv()` and `linalg.dot()` methods to find the solution of system of equations. However, the Numpy library contains the `linalg.solve()` method, which can be used to directly find the solution of a system of linear equations:

```
A = np.array([[4, 3, 2], [-2, 2, 3], [3, -5, 2]])

B = np.array([25, -10, -4])
```

```
X2 = np.linalg.solve(A,B)

print(X2)

Output:

[5. 3. -2.]
```

Example for Through Unimodular Matrix Using Python see in(QPythonLab, QPython [5]):

```
import numpy as np

import random

import os

import time

print "="*90

print "Generate A Linear Equations System AX=B With A Coefficient Matrix is A
Unimodular Matrix"

print "="*90

 print "Notes: please make sure you have LaTex and pdf reader installed on your
laptop."

print "-"*90

def r_ij(m, baris_i, baris_j, r):

return m[baris_i] + r*m[baris_j]

def tukar(m, baris_i, baris_j):

m[baris_i] = m[baris_i] + m[baris_j]

 m[baris_j] = m[baris_i] - m[baris_j]

 m[baris_i] = m[baris_i] - m[baris_j]

def buat_soal_SPL(n):

 a = np.eye(n)

for i in range(n):

#buat entri diagonalnya 1 atau -1
```

```
 if random.randint(0,1) == 0:
 a[i,i] = -1
else:
 a[i,i] = 1
#acak entri2 di segitiga atasnya (di atas diagonal utama)
for j in range(i+1,n):
a[i,j] = random.randint(-3,3)



#OBE mundur dari baris terakhir
for j in range(n-1,-1,-1):
 for i in range(j+1,n):
if random.randint(0,1) == 0:
a[i] = r_ij(a, i, j, random.randint(-3,1))
else:
 a[i] = r_ij(a, i, j, random.randint(1,4))
 tukar(a, 0, random.randint(1, n-1))
 #~ x = np.zeros((n,1))
 #~ for i in range(n)
: #~ x[i] = random.randint(-5,5)
#~ b = np.dot(a,x)
 b = np.zeros((n,1))
for i in range(n):
 b[i] = random.randint(-5,5)
 a_invers = np.linalg.inv(a)
 x = np.dot(a_invers, b)
 return a.astype(int), x.astype(int), b.astype(int)
```

```
os.system('rm soal_SPL.tex')
 #tulis ke file
 fileku = open('soal_SPL.tex', 'a')
fileku.write('\documentclass{article}\n')
fileku.write('\\begin{document}\n')
fileku.write('Find the solutions of Linear Equations System below:\n')
 fileku.write('\\begin{enumerate}\n')
n = input('Number of variables: ')
 soal = input('Number of questions: ')
waktu_mulai = time.time()
for i in range(soal):
 fileku.write('\item \n')
 # tulis soalnya di sini
aku, xku, bku = buat_soal_SPL(n)
print "\nFor Unimodular Matrix A:"
print aku
print "\nA Column (Solution) Matrix X is Generated:"
print xku
 print "\nand A Column Matrix B is Also Generated:"
print bku fileku.write('$\\begin{array}{') for i in range(n+1):
fileku.write('r@{\ }c@{\ }')
 fileku.write('}\n')
 print "\nTherefore We Will Get A Linear Equations System as Follow:"
for baris in range(n):
 tulisan = ""
for kolom in range(n):
if aku[baris, kolom] < 0:
if kolom == 0:
```

```
tanda = '-'
else:

tanda = ' -& '
 if aku[baris, kolom] == -1:
bil_asli = ""
else:
bil_asli = str(abs(aku[baris, kolom]))
 bil = bil_asli + 'x_{' + str(kolom+1) + '}&'
 elif aku[baris, kolom] == 0:
if kolom == 0:
tanda = ''


else:
tanda = '&'
bil = '&'
else:
 if kolom == 0:
tanda = ''
 else:
 tanda = ' +& '
if aku[baris, kolom] == 1:
 bil_asli = ""
else:
bil_asli = str(abs(aku[baris, kolom]))
bil = bil_asli + 'x_{' + str(kolom+1) + '}&'
 tulisan = tulisan + tanda + bil
```

```
if kolom + 1 == n:

tulisan = tulisan + '=&' + str(bku[baris, 0]) + ' \\\\\n '

#print "\nSPL-SPL yang dihasilkan:"

 print tulisan

 fileku.write(tulisan)

fileku.write('\\end{array}$\n')

fileku.write('\end{enumerate}\n')

fileku.write('\end{document}\n')

 fileku.close() os.system('pdflatex soal_SPL.tex')

 # ~ os.system('defaultpdfviewer soal_SPL.pdf')

lama = time.time() - waktu_mulai

print "-"*90 print "This application runs for: ", lama," seconds (after you input the number of questions)" print "Please open the .pdf file with a name soal_SPL.pdf in the same folder with this program"

print "-"*90

 #os.system('pdfviewer soal_SPL.pdf')
```

Output of code:

Find the solutions of Linear Equations System below:

1.
$$\begin{aligned}
-3x_1 - 4x_2 + 9x_3 - 14x_4 + 6x_5 - 15x_6 - 18x_7 - 8x_8 + 5x_9 - 11x_{10} &= 3\\
-3x_1 - 4x_2 + 6x_3 - 7x_4 - 4x_6 - 3x_7 - 6x_8 - x_9 &= -5\\
-x_1 - 3x_2 + 3x_3 - 2x_4 + 5x_5 - 6x_6 + x_7 - 8x_8 + 5x_9 - 7x_{10} &= 1\\
x_1 - x_2 + 2x_4 + 10x_5 - 7x_6 + 3x_7 - 10x_8 + 10x_9 - 10x_{10} &= 1\\
-4x_1 - 5x_2 + 8x_3 - 9x_4 + x_5 - 3x_6 - 3x_7 - 8x_8 + 2x_9 + 2x_{10} &= -1\\
-x_1 - x_2 + 5x_3 - 14x_4 - 4x_5 - 15x_6 - 13x_7 - 4x_8 - 6x_9 - 7x_{10} &= 0\\
2x_1 - 2x_2 - 7x_3 + 22x_4 + 7x_5 + 15x_6 + 27x_7 + 7x_8 + 15x_9 - 11x_{10} &= 2\\
-x_1 + 2x_2 + 6x_3 - 17x_4 + 2x_5 - 6x_6 - 6x_7 - 10x_8 + 3x_9 + 6x_{10} &= 5\\
-x_1 - 4x_2 + 4x_3 - 3x_4 + 11x_5 - 18x_6 - 8x_7 - 19x_8 + x_9 - 9x_{10} &= -1\\
-x_1 - x_2 + 2x_3 - 3x_4 - x_5 - x_6 - 2x_7 - x_8 - x_9 + x_{10} &= -3
\end{aligned}$$

2.
$$\begin{aligned}
x_1 - 2x_2 + x_3 + x_4 - 3x_5 + 2x_6 + 3x_7 - 5x_8 + 2x_9 - 4x_{10} &= 3\\
-x_1 + x_2 - x_4 + x_5 + x_6 - 2x_7 + 2x_8 - 2x_9 + 3x_{10} &= -2\\
-x_1 - x_2 + x_3 - x_5 + 8x_6 + 2x_7 - 6x_8 &= -1\\
x_1 - 3x_2 + 2x_3 + 2x_4 - 8x_5 + 3x_6 + 2x_7 - 5x_8 + x_9 - 2x_{10} &= 2\\
-x_1 + 4x_2 - x_3 - 5x_4 + 8x_5 - 6x_6 - 8x_7 + 9x_8 - 7x_9 + 3x_{10} &= -1\\
3x_1 - 5x_2 + 5x_3 + 4x_4 - 29x_5 - 7x_6 - 15x_7 + 9x_8 - 19x_9 + 9x_{10} &= -2\\
-x_1 + x_2 - 4x_4 + 13x_5 + 4x_6 + 5x_7 - 15x_8 + 20x_9 - 10x_{10} &= 4\\
- x_2 + 4x_3 - 2x_4 - 13x_5 - 10x_7 + 10x_8 - 20x_9 + x_{10} &= -2\\
3x_1 - 2x_2 - 4x_3 + 8x_4 + x_5 - 3x_6 + 22x_7 + 3x_8 - 4x_9 - 6x_{10} &= 0\\
3x_1 - 4x_2 - x_3 + 9x_4 - 13x_5 - 8x_6 + x_7 - 3x_8 + 19x_9 - 4x_{10} &= -4
\end{aligned}$$

Conclusions

The conclusions of this paper are as follows:

1. If we use the code in this article, all the solutions of Determined SLE are integers (Arifin and Muktyas[11]) and (Uka and Hajrulla[12]).

2. The Python program can be used to generate a Determined SLE with a coefficient matrix in the form of a unimodular matrix, in a short time with the number of equations and variables determined by the user (Pedregosa[16]- https://www.python.org/ [Online][18]) .

3. By using a unimodular matrix, a Determined SLE can be formed that is sure to have a solution and is easy to do because it does not need to involve fractions(Anton and Rorres[1], Guy[3] Arifin and Muktyas[11], https://www.python.org/ [18]) .

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Coefficient Estimates For A Certain Subclass of Bi-Univalent Functions Defined By using Deniz-Özkan Differential Operator

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Abstract

In this paper, we investigate a new subclass $\mathcal{B}_{\Sigma}^{m}(\lambda,\beta;\varphi)$ of bi-univalent functions in the open unit disk U defined by Deniz-Özkan differential operator. We obtain initial coefficients bounds.

Keywords: Analytic function, Univalent function, Bi-univalent function, Coefficient inequality.

1. Introduction

Let A denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. Further, by S we shall denote the class of all functions in A which are univalent in U. It is well known that every function $f \in S$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \qquad (z \in U)$$

and

$$f(f^{-1}(w)) = w \qquad \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4}\right)$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \dots$$

A function $f \in A$ is said to be in Σ, the class of bi-univalent functions in U, if both $f(z)$ and $f^{-1}(z)$ are univalent in U. Lewin [14] showed that $|a_2| < 1.51$ for every function $f \in \Sigma$ given by (1). Posteriorly, Brannan and Clunie [1] improved Lewin's result and conjectured that $|a_2| \leq \sqrt{2}$ for every function $f \in \Sigma$ given by (1). Later, Netanyahu [16] showed that $\max_{f\in\Sigma}|a_2| = \frac{4}{3}$ The coefficient estimate problem for each of the following Taylor-Maclaurin coefficients:

$$|a_n| \quad \left(n \in N = \{1,2,...\}; n \geq 4\right)$$

is still an open problem (see, for details, [21]). Since then, many researchers (see [2,5,8-11,22,23]) investigated several interesting subclasses of the class $\sum$ and found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. Also, many researchers (see [3,4,13,17,18]) investigated the upper bounds of combination of initial coefficients. In fact, its worth to mention that by making use of the Faber polynomial coefficient expansions Jahangiri and Hamidi [12] have obtained estimates for the general coefficients $|a_n|$ for bi-univalent functions subject to certain gap series.

Let P denote the class of function of p analytic in U such that $p(0)=1$ and $\operatorname{Re}\{p(z)\}>0$, where

$$p(z) = 1 + p_1 z + p_2 z^2 + ... \quad (z \in U).$$

If f and g are analytic in U, we say that f is subordinate to g, written symbolically as

$$f \prec g \qquad \text{or} \qquad f(z) \prec g(z) \quad (z \in U),$$

if there exists a Schwarz function $w(z)$, which (by definition) is analytic in U with $w(0)=0$ and $|w(z)|<1$ in U such that $f(z) = g(w(z)),\ z \in U$.

In particular, if the function $g(z)$ is univalent in U, then we have that:

$$f(z) \prec g(z) \quad (z \in U) \quad \text{if and only if} \quad f(0) = g(0) \ \text{ and } \ f(U) \subseteq g(U).$$

Let φ be an analytic function with positive real part in the unit disk U such that

$$\varphi(0) = 1, \varphi'(0) > 0$$

and $\varphi(U)$ is symmetric with respect to the real axis and has a series expansion of the form (see [15]):

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + ... \quad (B_1 > 0).$$

Let $u(z)$ and $v(z)$ be two analytic functions in the unit disk U with $u(0)=v(0)=0$ $|u(z)|<1, |v(z)|<1$, and suppose that

$$u(z) = 1 + b_1 z + b_2 z^2 + b_3 z^3 + ... \text{ and } \mathrm{v}(w) = 1 + c_1 w + c_2 w^2 + c_3 w^3 + ... \cdot \tag{2}$$

For above functions, well-known inequalities are

$$|b_1| \le 1, |b_2| \le 1 - |b_1|^2, |c_1| < 1 \text{ and } |c_2| \le 1 - |c_1|^2. \tag{3}$$

Further we have

$$\varphi(u(z)) = 1 + B_1 b_1 z + \left(B_1 b_2 + B_2 b_1^2\right) z^2 + \ldots (|z| < 1) \tag{4}$$

and

$$\varphi(v(w)) = 1 + B_1 c_1 w + \left(B_1 c_2 + B_2 c_1^2\right) w^2 + \ldots (|w| < 1) \tag{5}$$

In [6] (see, also [7]), Deniz and Özkan defined the differential operator D_λ^m (say: Deniz-Özkan differential operator) as follows:

For the parametres $\lambda \ge 0$ and $m \in N_0 = N \cup \{0\}$ the differential operatör D_λ^m on A defined by

$$D_\lambda^0 f(z) = f(z)$$

$$D_\lambda^1 f(z) = \lambda z^3 f'''(z) + (2\lambda + 1) z^2 f''(z) + z f'(z)$$

$$D_\lambda^m f(z) = D(D_\lambda^{m-1} f(z)$$

for $z \in U$.

For a function f in A, from the definition of the differential operatör D_λ^m, we can easily see that

$$D_\lambda^m f(z) = z + \sum_{n=2}^{\infty} n^{2m} (\lambda(n-1)+1)^m a_n z^n. \tag{6}$$

Also, $D_\lambda^m f(z) \in A.$ For the special cases of $\lambda = 0,1$ we obtain Salagean differential operatör (see [20]).

The main object of this paper is to introduce the following new subclass of bi-univalent functions involving Deniz-Özkan differential operator D_λ^m [6] and to obtain initial bounds for the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ of the functions belonging to this class.

2. Preliminaries and Definitions

The function class $B_{\Sigma}^{m}(\lambda,\beta;\varphi)$ defined as follows:

Definition 1. A function $f(z)\in\Sigma$ is said to be in the class $B_{\Sigma}^{m}(\lambda,\beta;\varphi)$ if and only if

$$(1-\beta)\frac{D_{\lambda}^{m}f(z)}{z}+\beta\left(D_{\lambda}^{m}f(z)\right)'\prec\varphi(z)$$

and

$$(1-\beta)\frac{D_{\lambda}^{m}g(w)}{w}+\beta\left(D_{\lambda}^{m}g(w)\right)'\prec\varphi(w)$$

where $0\le\beta\le1,\ z,w\in U$ and $g(w)=f^{-1}(w)$.

Theorem 1. If $f(z)$ given by (1) is in the class $B_{\Sigma}^{m}(\lambda,\beta;\varphi)$, then

$$|a_2|\le\frac{B_1\sqrt{B_1}}{\sqrt{\chi+B_1(1+\beta)^2\left(4(\lambda+1)\right)^{2m}}} \tag{7}$$

and

$$|a_3|\le\begin{cases}\dfrac{B_1}{(1+2\beta)\left[9(2\lambda+1)\right]^m} & \text{if}\quad B_1<\dfrac{(1+\beta)^2\left[4(\lambda+1)\right]^{2m}}{(1+2\beta)\left[9(2\lambda+1)\right]^m}\\[2ex] \dfrac{xB_1+(1+2\beta)\left[9(2\lambda+1)\right]^m B_1^3}{(1+2\beta)\left[9(2\lambda+1)\right]^m\left(x+(1+\beta)^2\left[4(\lambda+1)\right]^{2m}B_1\right)} & \text{if}\quad B_1\ge\dfrac{(1+\beta)^2\left[4(\lambda+1)\right]^{2m}}{(1+2\beta)\left[9(2\lambda+1)\right]^m}\end{cases} \tag{8}$$

where

$$\chi=\left|B_1^2(1+2\beta)\left[9(2\lambda+1)\right]^m-B_2(1+\beta)^2\left[4(\lambda+1)\right]^{2m}\right|.$$

Proof: Let $f(z)\in B_{\Sigma}^{m}(\lambda,\beta;\varphi)$. Then, there are analytic functions u and v with $u(0)=v(0)=0,\ |u(z)|<1,\ |v(w)|<1$, given by (2) and satisfying the following conditions:

$$(1-\beta)\frac{D_\lambda^m f(z)}{z}+\lambda\left(D_\lambda^m f(z)\right)'=\varphi(u(z)) \tag{9}$$

and

$$(1-\beta)\frac{D_\lambda^m g(w)}{w}+\beta\left(D_\lambda^m g(w)\right)'=\varphi(v(w)), \tag{10}$$

where $g(w)=f^{-1}(w)$. Since

$$\begin{aligned}&(1-\beta)\frac{D_\lambda^m f(z)}{z}+\beta\left(D_\lambda^m f(z)\right)'\\&=1+(1+\beta)\left[4(\lambda+1)\right]^m a_2 z+(1+2\beta)\left[9(2\lambda+1)\right]^m a_3 z^2+\ldots\end{aligned} \tag{11}$$

and

$$\begin{aligned}&(1-\beta)\frac{D_\lambda^m g(w)}{w}+\beta\left(D_\lambda^m g(w)\right)'\\&=1-(1+\beta)\left[4(\lambda+1)\right]^m a_2 w+(1+2\beta)\left[9(2(\lambda+1))\right]^m\left(2a_2^2-a_3\right)w^2+\ldots,\end{aligned} \tag{12}$$

it follows from (4), (5), (11) and (12) that

$$(1+\beta)\left[4(\lambda+1)\right]^m a_2=B_1 b_1, \tag{13}$$

$$(1+2\beta)\left[9(2\lambda+1)\right]^m a_3=B_1 b_2+B_2 b_1^2, \tag{14}$$

$$-(1+\beta)\left[4(\lambda+1)\right]^m a_2=B_1 c_1, \tag{15}$$

and

$$(1+2\beta)\left[9(2\lambda+1)\right]^m\left(2a_2^2-a_3\right)=B_1 c_2+B_2 c_1^2. \tag{16}$$

From (13) and (15), we get

$$c_1=-b_1 \tag{17}$$

$$2\left[\left(4(\lambda+1)\right)^m(1+\beta)\right]^2 a_2^2=B_1^2\left(b_1^2+c_1^2\right). \tag{18}$$

By adding (13) to (16), we have

$$2\left[9(2\lambda+1)\right]^m(1+2\beta)a_2^2=B_1\left(b_2+c_2\right)+B_2\left(b_1^2+c_1^2\right). \tag{19}$$

Therefore, from equalities (18) and (19) we find that

$$\left[2\left[9(2\lambda+1)\right]^m(1+2\beta)B_1^2-2B_2\left(\left[4(\lambda+1)\right]^m(1+\beta)\right)^2\right]a_2^2=B_1^3(b_2+c_2) \tag{20}$$

Then, in view of (13), (17) and (3), we obtain

$$\left|2\left[9(2\lambda+1)\right]^m(1+2\beta)B_1^2-2B_2\left(\left[4(\lambda+1)\right]^m(1+\beta)\right)^2\right||a_2|^2$$
$$\leq B_1^3(|b_2|+|c_2|)\leq 2B_1^3\left(1-|b_1|^2\right)=2B_1^3-2B_1\left(\left[4(\lambda+1)\right]^m(1+\beta)\right)^2|a_2|^2.$$

Thus, we get

$$|a_2|\leq\frac{B_1\sqrt{B_1}}{\sqrt{\chi+B_1(1+\beta)^2\left[4(\lambda+1)\right]^{2m}}},$$

where

$$\chi=\left|B_1^2(1+2\beta)\left[9(2\lambda+1)\right]^m-B_2(1+\beta)^2\left[4(\lambda+1)\right]^{2m}\right|.$$

Next, in order to find the bound on $|a_3|$, subtracting (16) from (14) and using (17), we get

$$2(1+2\beta)\left[9(2\lambda+1)\right]^m a_3=2(1+2\beta)\left[9(2\lambda+1)\right]^m a_2^2+B_1(b_2-c_2). \tag{21}$$

Then in view of (3) and (7), we have

$$2(1+2\beta)\left[9(2\lambda+1)\right]^m|a_3|\leq 2(1+2\beta)\left[9(2\lambda+1)\right]^m|a_2|^2+B_1(|b_2|+|c_2|)$$
$$\leq 2(1+2\beta)\left[9(2\lambda+1)\right]^m|a_2|^2+2B_1\left(1-|b_1|^2\right).$$

From (13), we immediately have

$$B_1(1+2\beta)\left[9(2\lambda+1)\right]^m|a_3|$$
$$\leq\left|B_1(1+2\beta)\left[9(2\lambda+1)\right]^m-(1+\beta)^2\left[4(\lambda+1)\right]^m\right||a_2|^2+B_1^2.$$

Now the assertion (8) follows from (7). This evidently completes the proof of Theorem 1.

By taking $\beta=1$ in Theorem 1, we have

Corollary 1. If $f(z)$ given by (1) is in the class $B_\Sigma^m(\lambda,1;\varphi)$, then

$$|a_2|\leq\frac{B_1\sqrt{B_1}}{\sqrt{\tau+4B_1\left[4(\lambda+1)\right]^{2m}}} \tag{22}$$

and

$$|a_3| \leq \begin{cases} \dfrac{B_1}{3\left[9(2\lambda+1)\right]^m} & \text{if} \quad B_1 < \dfrac{4\left[4(\lambda+1)\right]^{2m}}{3\left[9(2\lambda+1)\right]^m} \\ \dfrac{\tau B_1 + 3\left[9(2\lambda+1)\right]^m B_1^3}{3\left[9(2\lambda+1)\right]^m \left(\tau + 4\left[4(\lambda+1)\right]^{2m} B_1\right)} & \text{if} \quad B_1 \geq \dfrac{4\left[4(\lambda+1)\right]^{2m}}{3\left[9(2\lambda+1)\right]^m} \end{cases} \tag{23}$$

where

$$\tau = \left|3B_1^2\left[9(2\lambda+1)\right]^m - 4B_2\left[4(\lambda+1)\right]^{2m}\right|.$$

Putting $m=0$ in Theorem 1, we have

Corollary 2. [19] If $f(z)$ given by (1) is in the class $B_{\Sigma}^{0}(\lambda,\beta;\varphi) = B_{\Sigma}(\beta;\varphi)$, then

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{\left|B_1^2(1+2\beta) - B_2(1+\beta)^2\right| + B_1(1+\beta)^2}} \tag{24}$$

and

$$|a_3| \leq \begin{cases} \dfrac{B_1}{(1+2\beta)} & \text{if} \quad B_1 < \dfrac{(1+\beta)^2}{1+2\beta} \\ \dfrac{\left|B_1^2(1+2\beta) - B_2(1+\beta)^2\right| B_1 + (1+2\beta)B_1^3}{(1+2\beta)\left(\left|B_1^2(1+2\beta) - B_2(1+\beta)^2\right| + B_1(1+\beta)^2\right)} & \text{if} \quad B_1 \geq \dfrac{(1+\beta)^2}{1+2\beta} \end{cases} \tag{25}$$

Putting $m=0$ in Corollary 1, we have

Corollary 3. [19] If $f(z)$ given by (1) is in the class $B_{\Sigma}^{0}(\lambda,1;\varphi) = H_{\Sigma}(\varphi)$, then

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{\left|3B_1^2 - 4B_2\right| + 4B_1}} \tag{26}$$

and

$$|a_3| \leq \begin{cases} \dfrac{B_1}{3} & \text{if} \quad B_1 < \dfrac{4}{3} \\ \dfrac{|3B_1^2 - 4B_2| B_1 + 3B_1^3}{3\left(|3B_1^2 - 4B_2| + 4B_1\right)} & \text{if} \quad B_1 \geq \dfrac{4}{3} \end{cases} \tag{27}$$

Remark 1. If

$$\varphi(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} = 1 + 2\alpha z + 2\alpha^2 z^2 + \ldots (0 < \alpha \leq 1) \tag{28}$$

in Corollary 2, then we have Theorem 2.2 in [9].

If

$$\varphi(z) = \left(\frac{1+(1-2\alpha)z}{1-z}\right)^{\alpha} = 1 + 2(1-\alpha)z + 2(1-\alpha)^2 z^2 + \ldots (0 < \alpha \leq 1), \tag{29}$$

then we have Theorem 3.2 in [9].

Also, if $\beta = 0$ and $\beta = 1$, we have Theorem 2.1 in [19].

3. References

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On The Spherical Projection of Dual Bézier Curves

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Abstract

In this paper the projection curve of a given Bézier curve in the dual vector space D^3 to the dual unit sphere is studied. The first and second derivatives, norms and cross products of the first and second derivatives of the projection curve taken in this study were studied.

Keywords: Dual Spherial Curves, Derivatives, Ruled surfaces.

1. Introduction:

The dual vector space expresses the duality of a vector space in the literature. However, It is necessary to make the following distinction: the word "dual vector" is defined in this study is an element of a vector space defined as the cartesian set of dual numbers which denoted by D and were introduced in 1873 by William Clifford [1], and developed by Eduard Study [2].

After E.Study, with a dual spherical point corresponding to a directed line in R^3 to study of a ruled surface is reduced to study of a spherical curve, many scientists studied in this area. Especially Hoschek [3] investigated integral invariants for characterization of the closed ruled surfaces. Gürsoy, Gürsoy and Küçük [4-9], Hacisalihoğlu [10,11] were studied the ruled surfaces with integral invariants which are stated as dual quantities.

Bézier curves and ruled surfaces have been studied by [11-19]. Especially dual spherical curves corresponding to a ruled surface were studied before by [20], [21], [22], [23].

2. Materials and method

2.1. Dual Numbers and D-module

Let two dual vectors $\widehat{U}$ and $\widehat{V}$ be given as $\widehat{U} = U + \varepsilon U^*$ and $\widehat{V} = V + \varepsilon V^*$. Then the inner product of two dual vectors $\widehat{U}$ and $\widehat{V}$ is $\langle \widehat{U}, \widehat{V} \rangle = \langle U, V \rangle + \varepsilon(\langle U^*, V \rangle + \langle U, V^* \rangle)$. The norm of a dual vector $\widehat{U} = U + \varepsilon U^*$ is a dual number such that

$$\|\widehat{U}\| = \sqrt{\langle \widehat{U}, \widehat{U} \rangle} = \sqrt{\langle U, U \rangle + 2\varepsilon \langle U, U^* \rangle} = \|U\| + \varepsilon \frac{\langle U, U^* \rangle}{\|U\|} = u + \varepsilon u^* \in D \qquad (1)$$

if the real part of the dual vector is different from zero. i.e. $U \neq 0$. If the norm of a dual vector $\widehat{U}$ is $1 + \varepsilon 0 = 1$ then the vector $\widehat{U}$ is called dual unit vector [9].

Proposition 2.1 [9]:Let a dual vector $\widehat{U} = U + \varepsilon U^*$ be given. If $\|\widehat{U}\| = 1$ then $\|U\| = 1$ and $\langle U, U^* \rangle = 0$.

Proposition 2.2 [9]:Let a dual vector $\widehat{U} = U + \varepsilon U^*$ be given. If $\|\widehat{U}\| \neq 1$ and $U \neq 0$ then

$$\breve{U} = \frac{\widehat{U}}{\|\widehat{U}\|} = \frac{U}{\|U\|} + \varepsilon \frac{U^* - \frac{\langle U, U^* \rangle}{\|U\|^2} U}{\|U\|} = \frac{U}{\|U\|} + \varepsilon \left(\frac{U^*}{\|U\|} - \frac{\langle U, U^* \rangle U}{\|U\|^3} \right) = \dot{U} + \varepsilon \dot{U}^* \tag{2}$$

is a dual unit vector with direction of $\widehat{U}$.

2.2. Bézier Curves

Let $n+1$ control points $b_0, b_1, \ldots, b_n \in \mathbb{R}^3$ be given. The Bézier curve of degree n is defined by

$$B(t) = \sum_{i=0}^{n} b_i B_i^n(t) \tag{3}$$

where $t \in [0,1]$ and the functions $B_i^n(t)$ are called Bersntein polynomials or Bernstein basis functions and defined by such that if $0 \leq i \leq n$ then $B_i^n(t) = \binom{n}{i} (1-t)^{n-i} t^i$; orherwise $B_i^n(t) = 0$ [15].

The first and second derivatives of the Bernstein basis functions $B_i^n(t)$ of degree n satisfy

$$B_i^{n\prime}(t) = \frac{i - nt}{t(1-t)} B_i^n(t) = n\left(B_{i-1}^{n-1}(t) - B_i^{n-1}(t) \right) \tag{4}$$

$$B_i^{n\prime\prime}(t) = n(n-1)\left(\left[B_i^{n-2}(t) - 2B_{i-1}^{n-2}(t) + B_{i-2}^{n-2}(t) \right] \right) \tag{5}$$

$$= \left(\frac{i(i-1) - 2i(n-1)t + n(n-1)t^2}{t^2(1-t)^2} \right) B_i^n(t) \tag{6}$$

[15].

Theorem 2.1: The first derivative of a Bézier curve of degree n given formed by (3) is

$$B'(t) = \sum_{i=0}^{n-1} b_i^{(1)} B_i^{n-1}(t) \tag{7}$$

where $b_i^{(1)} = n(b_{i+1} - b_i)$ [15].

It is clear that the first derivative $B'(t)$ of a Bézier curve $B(t)$ is also a Bézier curve of degree n-1 that its control points are $b_i^{(1)} = n(b_{i+1} - b_i)$.

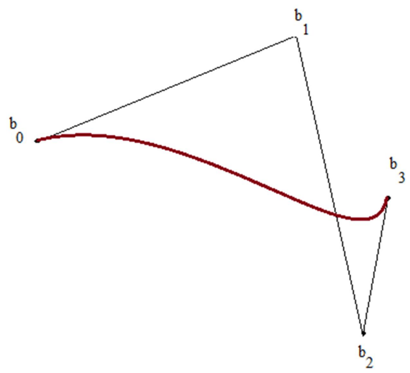

Figure 1: A cubic Bézier curve with control points b_0, b_1, b_2, and b_3.

Corollary 2.1: The second derivative of a Bézier curve of degree *n* is

$$B''(t) = \sum_{i=0}^{n-2} b_i^{(2)} B_i^{n-2}(t) \tag{8}$$

where $b_i^{(2)} = n(n-1)(b_{i+2} - 2b_{i+1} + b_i)$ [15].

Corollary 2.2: The *r*th derivative of a Bézier curve of degree *n* is

$$B^{(r)}(t) = \sum_{i=0}^{n-r} b_i^{(r)} B_i^{n-r}(t) \tag{9}$$

where $b_i^{(r)} = n(n-1)...(n-r+1)\sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j} b_{i+j}$ [15].

Theorem 2.2: A Bézier curve **B**(*t*) of degree *n* with control points **b**0, ..., **b***n* satisfies the following properties.

1- $B(0) = b_0$, $B(1) = b_n$ (Endpoint Interpolation Property) (10)

2- $B'(0) = \frac{dB}{dt}\Big|_{t=0} = n(b_1 - b_0)$ (Endpoint Tangent Property) (11)

$B'(1) = \frac{dB}{dt}\Big|_{t=1} = n(b_n - b_{n-1})$ (12)

3- $\forall t \in [0,1]$ için, $B(t) \in CH(\{b_0, b_1, ..., b_n\})$ (Convex Hull Property (CHP)) (13)

(Thus every point of a Bézier curve lies inside the convex hull of its defining control points.)

4- Let F be an (affine) transformation (for example, a rotation, reflection, translation, or scaling). Then

$$F(B(t)) = F\left(\sum_{i=0}^{n} b_i B_i^n(t)\right) = \sum_{i=0}^{n} F(b_i) B_i^n(t) \tag{14}$$

[15].

Theorem 2.3 : (de Casteljau Algorithm) Let a Bézier curve **B**(t) of degree n with control points b_0, b_1, b_2, ..., b_n be given. Then $B(t_0) = b_0^n$ is satisfied for a specified parameter value $t = t_0 \in [0, 1]$, where b_0^n is obtained by the de Casteljau algorithm as follows:

$$\begin{aligned} b_i^0 &= b_i \\ b_i^j &= (1-t_0) b_i^{j-1} + t_0 b_{i+1}^{j-1} \end{aligned} \tag{15}$$

for $j = 1,...,n$ and $i = 0, 1, ...,n-j$ [15].

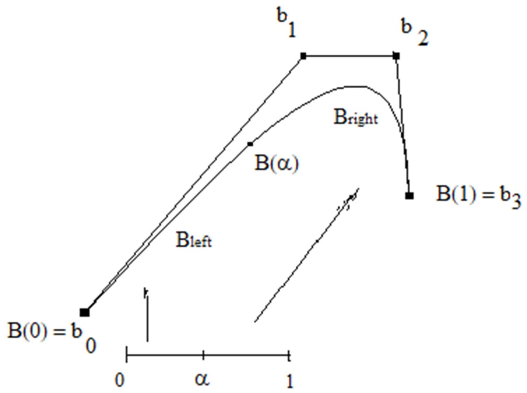

Figure 2: The subdivision of a Bézier curve

Consider a Bézier curve **B**(t) defined on $t \in [0,1]$. Let $\alpha \in [0,1]$ be any parameter. Then the Bézier curve **B**(t) can be divided two curve segments defined over the interval [0, α] and ∈[α,1]. So obtained two curve segments is called **B**left(t) and **B**right(t). Since **B**left(t) and **B**right(t) are polynomial curves they can be represented in Bézier form over the interval [0, 1].

Theorem 2.4 : (Subdivision) Let a Bézier curve $\mathbf{B}(t)$ of degree n with control points $b_0, b_1, b_2, \ldots, b_n$ be given. Two curve segments **B**left(t) and **B**right(t) obtained by subdivision of Bézier curve $\mathbf{B}(t)$ at parameter value $\alpha \in [0,1]$ are also Bézier curves and their control points are $b_0^0, b_0^1, \ldots, b_0^{n-1}, b_0^n$ for $\mathbf{B}_{\text{left}}$; $b_0^n, b_1^{n-1}, \ldots, b_{n-1}^1, b_n^0$ for $\mathbf{B}_{\text{right}}$, where b_i^j are the points computed in the de Casteljau algorithm.[15].

3. Results

Let $\hat{B}(t)$ be a dual Bézier curve with control points $\hat{P}_0, \hat{P}_1, \ldots, \hat{P}_n \in D^3$ where $\hat{P}_i = P_i + \varepsilon P_i^*$, $P_i, P_i^* \in R^3$ for $i = 0,1,\ldots,n$. $\hat{P}_i = P_i + \varepsilon P_i^* \in D^3$ Then for $t \in [0,1]$, the dual Bézier curve can be defined as

$$\hat{B}(t) = \sum_{i=0}^{n} B_i^n(t)\hat{P}_i \tag{16}$$

Since each control point $\hat{P}_i = P_i + \varepsilon P_i^*$ then for $t \in [0,1]$ the dual Bézier curve can be written as

$$\begin{aligned}\hat{B}(t) &= \sum_{i=0}^{n} B_i^n(t)P_i + \varepsilon \sum_{i=0}^{n} B_i^n(t)P_i^* \\ &= B(t) + \varepsilon B^*(t)\end{aligned} \tag{17}$$

where $B(t)$ and $B^*(t)$ are real Bézier curves of degree n with control points $P_0, P_1, \ldots, P_n$ and $P_0^*, P_1^*, \ldots, P_n^*$ respectively.

Let the coordinate frame in R^3 be denoted as $\{e_1, e_2, e_3\}$. Then the j.th coordinat element of any control point $P_i = (P_{i_1}, P_{i_2}, P_{i_3})$ in R^3 for $j = 1,2,3$ is the inner product $P_{ij} = \langle P_i, e_j \rangle$. So any control point P_i is stated as

$$P_i = \sum_{j=1}^{3} \langle P_i, e_j \rangle\, e_j. \tag{18}$$

Similarly since any control point of $\hat{P}_i$ in D^3 is stated as $\hat{P}_i = P_i + \varepsilon P_i^* = (P_{i_1}, P_{i_2}, P_{i_3}) + \varepsilon(P_{i_1}^*, P_{i_2}^*, P_{i_3}^*)$ where $P_i, P_i^* \in R^3$ then

$$\hat{P}_i = \sum_{j=1}^{3} \langle P_i, e_j \rangle\, e_j + \varepsilon \sum_{j=1}^{3} \langle P_i^*, e_j \rangle\, e_j \tag{19}$$

can be stated. The norm of the curve $B(t)$ at any time t is

$$\begin{aligned}\|B(t)\| &= \left\| \sum_{i=0}^{n} B_i^n(t)P_i \right\| = \sqrt{\sum_{j=1}^{3} \left(\sum_{i=0}^{n} B_i^n(t)P_{ij} \right)^2} \\ &= \sqrt{\sum_{j=1}^{3} \left(\sum_{i=0}^{n} B_i^n(t)\langle P_i, e_j \rangle \right)^2} = \sqrt{\sum_{j=1}^{3} \langle \sum_{i=0}^{n} B_i^n(t)P_i\, , e_j \rangle^2} = \sqrt{\sum_{j=1}^{3} \langle B(t), e_j \rangle^2}\end{aligned} \tag{20}$$

Now for $t \in [0,1]$ the dual Bézier curve $\hat{B}(t)$ can be expressed as

$$\begin{aligned}\hat{B}(t) &= \sum_{i=0}^{n} B_i^n(t)P_i + \varepsilon \sum_{i=0}^{n} B_i^n(t)P_i^* \\ &= \sum_{i=0}^{n} B_i^n(t)\left(\sum_{j=1}^{3} \langle P_i, e_j \rangle\, e_j\right) + \varepsilon \sum_{i=0}^{n} B_i^n(t)\left(\sum_{j=1}^{3} \langle P_i^*, e_j \rangle\, e_j\right) \\ &= \sum_{i=0}^{n} \sum_{j=1}^{3} B_i^n(t)\langle P_i, e_j \rangle\, e_j + \varepsilon \sum_{i=0}^{n} \sum_{j=1}^{3} B_i^n(t)\langle P_i^*, e_j \rangle\, e_j\end{aligned} \tag{21}$$

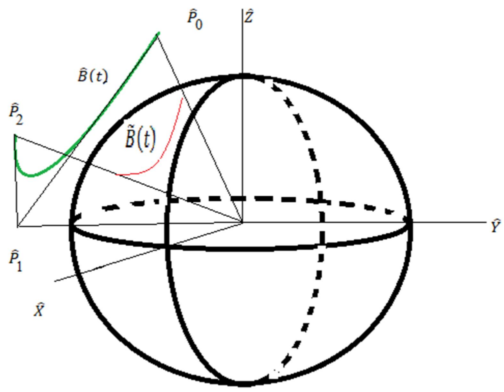

Figure 3: Unit dual Sphere and Projection curve $\tilde{B}(t)$ of the Bézier curve $\hat{B}(t)$ to unit dual sphere

The projection of the dual Bézier curve $\hat{B}(t)$ to unit dual sphere D-module is a curve in Fig. 3 denoted by $\tilde{B}(t)$ and defined by

$$\tilde{B}(t) = \frac{\hat{B}(t)}{\|\hat{B}(t)\|} = \frac{\sum_{i=0}^{n} B_i^n(t)\hat{P}_i}{\left\|\sum_{i=0}^{n} B_i^n(t)\hat{P}_i\right\|} = \frac{\sum_{i=0}^{n} B_i^n(t)(P_i+\varepsilon P_i{}^*)}{\left\|\sum_{i=0}^{n} B_i^n(t)\hat{P}_i\right\|} = \bar{B}(t) + \varepsilon\bar{B}^*(t) \tag{22}$$

Since the norm of the curve $\hat{B}(t)$ is

$$\left\|\hat{B}(t)\right\| = \|B(t) + \varepsilon B^*(t)\| = \|B(t)\| + \varepsilon\frac{\langle B(t), B^*(t)\rangle}{\|B(t)\|} \tag{23}$$

the projection curve $\tilde{B}(t)$ can be stated as

$$\tilde{B}(t) = \frac{\hat{B}(t)}{\|\hat{B}(t)\|} = \frac{\hat{B}(t)}{\|B(t)\| + \varepsilon\frac{\langle B(t), B^*(t)\rangle}{\|B(t)\|}} = \left(\frac{1}{\|B(t)\|} - \varepsilon\frac{\langle B(t), B^*(t)\rangle}{\|B(t)\|^3}\right)\hat{B}(t) \tag{24}$$

When (16) and (19) is also replaced by (24) the projection curve $\tilde{B}(t)$ of the dual Bézier curve $\hat{B}(t)$ to unit dual sphere

$$\tilde{B}(t) = \left(\frac{1}{\|B(t)\|} - \varepsilon\frac{\langle B(t), B^*(t)\rangle}{\|B(t)\|^3}\right)\left(\sum_{i=0}^{n} B_i^n(t)\left(\sum_{j=1}^{3}\langle P_i, e_j\rangle e_j\right) + \varepsilon\sum_{i=0}^{n} B_i^n(t)\left(\sum_{j=1}^{3}\langle P_i{}^*, e_j\rangle e_j\right)\right) \tag{25}$$

$$= \left(\frac{1}{\sqrt{\sum_{j=1}^{3}\left(\sum_{i=0}^{n} B_i^n(t)\langle P_i, e_j\rangle\right)^2}} - \varepsilon\frac{\sum_{i=0}^{n}\sum_{j=0}^{n} B_i^n(t)B_j^n(t)\langle P_i, P_j{}^*\rangle}{\left(\sum_{j=1}^{3}\left(\sum_{i=0}^{n} B_i^n(t)\langle P_i, e_j\rangle\right)^2\right)^{3/2}}\right)\left(\sum_{i=0}^{n}\sum_{j=1}^{3}\langle P_i, e_j\rangle e_j B_i^n(t) + \varepsilon\sum_{i=0}^{n}\sum_{j=1}^{3}\langle P_i{}^*, e_j\rangle e_j B_i^n(t)\right)$$

$$= \frac{\sum_{i=0}^{n}\sum_{j=1}^{3}\langle P_i, e_j\rangle e_j B_i^n(t)}{\sqrt{\sum_{j=1}^{3}\left(\sum_{i=0}^{n} B_i^n(t)\langle P_i, e_j\rangle\right)^2}}$$

$$+ \varepsilon\left(\frac{\sum_{i=0}^{n}\sum_{j=1}^{3}\langle P_i{}^*, e_j\rangle e_j B_i^n(t)}{\sqrt{\sum_{j=1}^{3}\left(\sum_{i=0}^{n} B_i^n(t)\langle P_i, e_j\rangle\right)^2}} - \frac{\sum_{i=0}^{n}\sum_{j=0}^{n} B_i^n(t)B_j^n(t)\langle P_i, P_j{}^*\rangle}{\left(\sum_{j=1}^{3}\left(\sum_{i=0}^{n} B_i^n(t)\langle P_i, e_j\rangle\right)^2\right)^{3/2}}\sum_{i=0}^{n}\sum_{j=1}^{3}\langle P_i, e_j\rangle e_j B_i^n(t)\right)$$

$$= \bar{B}(t) + \varepsilon \bar{B}^*(t)$$

can be written. Therefore this theorem can be stated as follows

Theorem 3.1: Let $\hat{B}(t) = B(t) + \varepsilon B^*(t)$ be a dual Bézier curve with control points $\hat{P}_0, \hat{P}_1, \dots, \hat{P}_n \in D^3$ where $\hat{P}_i = P_i + \varepsilon P_i{}^*$, $P_i, P_i{}^* \in R^3$ for $i = 0,1,\dots,n$. Then the projection curve $\tilde{B}(t)$ of the dual Bézier curve $\hat{B}(t)$ to unit dual sphere is

$$\tilde{B}(t) = \bar{B}(t) + \varepsilon \bar{B}^*(t)$$

where

$$\bar{B}(t) = \frac{\sum_{i=0}^{n} \sum_{j=1}^{3} \langle P_i, e_j \rangle e_j B_i^n(t)}{\sqrt{\sum_{j=1}^{3} \left(\sum_{i=0}^{n} B_i^n(t) \langle P_i, e_j \rangle \right)^2}} \tag{26}$$

and

$$\bar{B}^*(t) = \frac{\sum_{i=0}^{n} \sum_{j=1}^{3} \langle P_i{}^*, e_j \rangle e_j B_i^n(t)}{\sqrt{\sum_{j=1}^{3} \left(\sum_{i=0}^{n} B_i^n(t) \langle P_i, e_j \rangle \right)^2}} - \frac{\sum_{i=0}^{n} \sum_{j=0}^{n} B_i^n(t) B_j^n(t) \langle P_i, P_j{}^* \rangle}{\left(\sum_{j=1}^{3} \left(\sum_{i=0}^{n} B_i^n(t) \langle P_i, e_j \rangle \right)^2 \right)^{3/2}} \sum_{i=0}^{n} \sum_{j=1}^{3} \langle P_i, e_j \rangle \, e_j B_i^n(t) \tag{27}$$

It can be written more simply as

$$\bar{B}(t) = \frac{B(t)}{\|B(t)\|} \quad \text{and} \quad \bar{B}^*(t) = \frac{B^*(t)}{\|B(t)\|} - \frac{\langle B(t), B^*(t) \rangle}{\|B(t)\|^3} B(t) \tag{28}$$

From Proposition 2.1 the inner product these vectors $\langle \bar{B}(t), \bar{B}^*(t) \rangle = 0$ satisfies.

According to E.Study’s theorem any dual unit vector corresponds to a oriented line in R^3. Since for every $t \in [0,1]$ the projection curve $\tilde{B}(t)$ of the dual Bézier curve $\hat{B}(t)$ to unit dual sphere is a dual unit vector, for any $t_0 \in [0,1]$, the projection curve $\tilde{B}(t_0)$ also corresponds to a oriented line in R^3. So the projection curve $\tilde{B}(t)$ corresponds to a ruled surface in R^3. The oriented line corresponding to $\tilde{B}(t_0)$ is a line with direction of the vector $\bar{B}(t_0)$ and its distance from origine is $\|\bar{B}^*(t_0)\|$.

If $\bar{B}^*(t)$ is denoted from (36) as $\bar{B}^* = \frac{B^*}{\|B\|} - \frac{\langle B, B^* \rangle}{\|B\|^3} B$ for shortness, the magnitude $\|\bar{B}^*\|$ of the dual part of the projection curve $\tilde{B}(t)$ of the dual Bézier curve $\hat{B}(t)$ is obtained as follows:

$$\begin{aligned}
\|\bar{B}^*\| &= \left\| \frac{B^*}{\|B\|} - \frac{\langle B, B^* \rangle}{\|B\|^3} B \right\| \\
&= \frac{1}{\|B\|^3} \left\| \left(\|B\|^2 B^* - \langle B, B^* \rangle B \right) \right\| \\
&= \frac{1}{\|B\|^3} \sqrt{\left\langle \left(\|B\|^2 B^* - \langle B, B^* \rangle B \right), \left(\|B\|^2 B^* - \langle B, B^* \rangle B \right) \right\rangle} \\
&= \frac{1}{\|B\|^3} \sqrt{\|B\|^4 \langle B^*, B^* \rangle - 2\|B\|^2 \langle B, B^* \rangle^2 + \langle B, B^* \rangle^2 \langle B, B \rangle}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\|B\|^3}\sqrt{\|B\|^4\langle B^*,B^*\rangle - 2\|B\|^2\langle B,B^*\rangle^2 + \langle B,B^*\rangle^2\|B\|^2} \\
&= \frac{1}{\|B\|^2}\sqrt{\|B\|^2\langle B^*,B^*\rangle - \langle B,B^*\rangle^2} \\
&= \frac{1}{\|B\|^2}\sqrt{\langle B,B\rangle\langle B^*,B^*\rangle - \langle B,B^*\rangle^2} \\
&= \frac{1}{\|B\|^2}\sqrt{\langle B\times B^*, B\times B^*\rangle} = \frac{\|B\times B^*\|}{\|B\|^2} = \frac{\|B^*\| sin\theta}{\|B\|}
\end{aligned}
\tag{29}
$$

where θ is an angle between the vectors B and B^*.

Now the vector $\bar{B}(t)$ and the magnitude of the vector $\bar{B}^*(t)$ for $t=0$ and $t=1$ can be easily stated by end point interpolation property of Bézier curves from (18). In case for $t_0 \neq 0$ or $t_0 \neq 1$ they can be calculated by the de Casteljau algorithm (theorem 2.3) as follows:

Theorem 3.2: From (29) the projection curve $\tilde{B}(t)$ of the dual Bézier curve $\hat{B}(t)$ for $t_0 = 0$ and $t_0 = 1$ are

$$\tilde{B}(t)\big|_{t=t_0=0} = \bar{B}(t)|_{t=t_0=0} + \varepsilon\bar{B}^*(t)|_{t=t_0=0} = \frac{P_0}{\|P_0\|} + \varepsilon\left(\frac{P_0{}^*}{\|P_0\|} - \frac{\langle P_0,P_0{}^*\rangle}{\|P_0\|^3}P_0\right) \tag{30}$$

$$\tilde{B}(t)\big|_{t=t_0=1} = \bar{B}(t)|_{t=t_0=1} + \varepsilon\bar{B}^*(t)|_{t=t_0=1} = \frac{P_n}{\|P_n\|} + \varepsilon\left(\frac{P_n{}^*}{\|P_n\|} - \frac{\langle P_n,P_n{}^*\rangle}{\|P_n\|^3}P_n\right) \tag{31}$$

Theorem 3.3: The projection curve $\tilde{B}(t)$ of the dual Bézier curve $\hat{B}(t)$ for any $t_0 \in (0,1)$ is

$$\tilde{B}(t)\big|_{t=t_0} = \bar{B}(t)|_{t=t_0} + \varepsilon\bar{B}^*(t)|_{t=t_0} = \frac{P_0^n}{\|P_0^n\|} + \varepsilon\left(\frac{P_0^{n*}}{\|P_0^n\|} - \frac{\langle P_0^n,P_0^{n*}\rangle}{\|P_0^n\|^3}P_0^n\right) \tag{32}$$

where P_0^n and P_0^{n*} are the points computed in the de Casteljau algorithm.

Theorem 3.3: Let the dual spherical projection curve $\tilde{B}(t)$ of the given the dual Bézier curve $\hat{B}(t)$ to dual unit sphere for every $t \in [0,1]$ be given. For shortness, if the curves $B(t)$ and $B^*(t)$ are denoted by B and B^* respectively then the derivative of the curve $\tilde{B}$ and magnitude is

$$\tilde{B}' = \frac{B'\|B\|^2 - \langle B,B'\rangle B}{\|B\|^3} + \varepsilon\left[\frac{B^{*\prime}}{\|B\|} - \frac{B^*\langle B,B'\rangle}{\|B\|^3} - \left(\frac{\langle B^*,B'\rangle + \langle B^{*\prime},B\rangle}{\|B\|^3} - \frac{\langle B^*,B\rangle\langle B,B'\rangle}{\|B\|^5}\right)B - \frac{\langle B^*,B\rangle}{\|B\|^3}B'\right]$$

So the magnitude of this derivative is

$$\|\tilde{B}'\| = \frac{\|(B'\|B\|^2 - \langle B,B'\rangle B)\|}{\|B\|^3} + \varepsilon\left[\frac{\|B\|^2\langle B^{*\prime},B'\rangle - \langle B^*,B'\rangle\langle B,B'\rangle - \langle B^*,B\rangle\|B'\|^2 + \frac{2\langle B^*,B\rangle\langle B,B'\rangle^2}{\|B\|^2}}{\|(B'\|B\|^2 - \langle B,B'\rangle B)\|}\right]$$

Theorem 3.4: Let the dual spherical projection curve $\tilde{B}(t)$ of the given the dual Bézier curve $\hat{B}(t)$ to dual unit sphere for every $t \in [0,1]$ be given. For shortness, if the curves $B(t)$ and $B^*(t)$ are denoted by B and B^* respectively then the second order derivative of $\tilde{B}$ is

$$\tilde{B}'' = \frac{B''}{\|B\|} - \frac{2\langle B, B'\rangle B'}{\|B\|^3} - \left(\frac{\|B'\|^2 + \langle B'', B\rangle}{\|B\|^3} + \frac{3\langle B, B'\rangle^2}{\|B\|^5}\right)B +$$

$$+\varepsilon\left[\begin{aligned} &\frac{B^{*''}}{\|B\|} - \frac{2\langle B, B'\rangle B^{*'}}{\|B\|^3} - \left(\frac{\|B'\|^2 + \langle B'', B\rangle}{\|B\|^3} + \frac{3\langle B, B'\rangle^2}{\|B\|^5}\right)B^* - \frac{\langle B^*, B\rangle}{\|B\|^3}B'' \\ &+\left(-\frac{\langle B^*, B''\rangle + 2\langle B^{*'}, B'\rangle + \langle B, B^{*''}\rangle}{\|B\|^3} + \frac{4\langle B^*, B'\rangle\langle B, B'\rangle}{\|B\|^5} + \frac{4\langle B^{*'}, B\rangle\langle B, B'\rangle}{\|B\|^5}\right)B \\ &+\left(\frac{\langle B^*, B\rangle\|B'\|^2 + \langle B^*, B\rangle\langle B'', B\rangle}{\|B\|^5} - \frac{5\langle B^*, B\rangle\langle B, B'\rangle^2}{\|B\|^7}\right)B - \left(2\frac{\langle B^*, B'\rangle + \langle B^{*'}, B\rangle}{\|B\|^3} - 4\frac{\langle B^*, B\rangle\langle B, B'\rangle}{\|B\|^5}\right)B' \\ &+ \end{aligned}\right]$$

The cross product of the vectors $\tilde{B}' \times \tilde{B}''$ is as follows

Theorem 3.5: Let $\hat{B}(t) = B(t) + \varepsilon B^*(t)$ be a dual Bézier curve with control points $\hat{P}_0, \hat{P}_1, \dots, \hat{P}_n \in D^3$ where $\hat{P}_i = P_i + \varepsilon P_i^*$, $P_i, P_i^* \in R^3$ for $i = 0,1,\dots,n$. Then The cross product of the vectors $\tilde{B}' \times \tilde{B}''$ of the projection curve $\tilde{B}(t)$ of the dual Bézier curve $\hat{B}(t)$ to unit dual sphere is

$$\tilde{B}' \times \tilde{B}'' = \frac{1}{\|B\|^2}B' \times B'' + \left(\frac{\|B'\|^2 + \langle B'', B\rangle}{\|B\|^4} + \frac{5\langle B, B'\rangle^2}{\|B\|^6}\right)B \times B' - \frac{\langle B, B'\rangle}{\|B\|^4}B \times B''$$

$$+\varepsilon\left[\begin{aligned} &\frac{1}{\|B\|^2}B' \times B^{*''} - \frac{2\langle B, B'\rangle}{\|B\|^4}B' \times B^{*'} - \left(\frac{\|B'\|^2 + \langle B'', B\rangle}{\|B\|^4} + \frac{3\langle B, B'\rangle^2}{\|B\|^6}\right)B' \times B^* - \frac{\langle B^*, B\rangle}{\|B\|^4}B' \times B'' \\ &+\left(-\frac{\langle B^*, B''\rangle + 2\langle B^{*'}, B'\rangle + \langle B, B^{*''}\rangle}{\|B\|^4} + \frac{4\langle B^*, B'\rangle\langle B, B'\rangle}{\|B\|^6} + \frac{4\langle B^{*'}, B\rangle\langle B, B'\rangle}{\|B\|^6}\right)B' \times B \\ &+\left(\frac{\langle B^*, B\rangle\|B'\|^2 + \langle B^*, B\rangle\langle B'', B\rangle}{\|B\|^6} - \frac{5\langle B^*, B\rangle\langle B, B'\rangle^2}{\|B\|^8}\right)B' \times B + \\ &-\frac{\langle B, B'\rangle}{\|B\|^4}B \times B^{*''} + \frac{2\langle B, B'\rangle^2}{\|B\|^6}B \times B^{*'} - \frac{\langle B, B'\rangle}{\|B\|^4}B^* \times B'' + \frac{2\langle B, B'\rangle^2}{\|B\|^6}B^* \times B' \\ &+\frac{2\langle B, B'\rangle\langle B^*, B\rangle}{\|B\|^6}B \times B'' + \left(2\frac{\langle B, B'\rangle\langle B^*, B'\rangle + \langle B, B'\rangle\langle B^{*'}, B\rangle}{\|B\|^6} - 7\frac{\langle B^*, B\rangle\langle B, B'\rangle^2}{\|B\|^8}\right)B \times B' \\ &+\frac{1}{\|B\|^2}B^{*'} \times B'' - \frac{2\langle B, B'\rangle}{\|B\|^4}B^{*'} \times B' - \left(\frac{\|B'\|^2 + \langle B'', B\rangle}{\|B\|^4} + \frac{3\langle B, B'\rangle^2}{\|B\|^6}\right)B^{*'} \times B \\ &-\frac{\langle B^*, B\rangle}{\|B\|^4}B' \times B'' + \left(\frac{\langle B^*, B\rangle\|B'\|^2 + \langle B^*, B\rangle\langle B'', B\rangle}{\|B\|^6}\right)B' \times B \\ &-\left(\frac{\langle B^*, B'\rangle + \langle B^{*'}, B\rangle}{\|B\|^4}\right)B \times B'' + \left(2\frac{\langle B, B'\rangle\langle B^*, B'\rangle + \langle B, B'\rangle\langle B^{*'}, B\rangle}{\|B\|^6} - \frac{2\langle B^*, B\rangle\langle B, B'\rangle^2}{\|B\|^8}\right)B \times B' \end{aligned}\right]$$

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Cofinitely ⊕-g-Rad-Supplemented Modules

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Abstract

In this work, all rings have unity and all modules are unitary left modules. Let M be an R-module. If every cofinite submodule of M has a g-radical supplement that is a direct summand in M, then M is called a cofinitely ⊕-g-radical supplemented (briefly, cofinitely ⊕-g-Rad-supplemented) module. In this work, some properties of these modules are investigated.

Keywords: Small Submodules, g-Small Submodules, Supplemented Modules, g-Supplemented Modules.

2020 Mathematics Subject Classification: 16D10, 16D80.

1. INTRODUCTION

Throughout this paper all rings will be associative with identity and all modules will be unital left modules.

Let M be an R-module and $N\leq M$. If M/N is finitely generated, then N is called a *cofinite* submodule of M. Let M be an R-module and $N\leq M$. If $L=M$ for every submodule L of M such that $M=N+L$, then N is called a *small* submodule of M and denoted by $N\ll M$. Let M be an R-module and $N\leq M$. If there exists a submodule K of M such that $M=N+K$ and $N\cap K=0$, then N is called a *direct summand* of M and it is denoted by $M=N\oplus K$. The intersection of all maximal submodules of M is called the *radical* of M and denoted by $RadM$. If M have no maximal submodules, then it is defined by $RadM=M$. A submodule N of an R-module M is called an *essential* submodule of M and denoted by $N\trianglelefteq M$ in case $K\cap N\neq 0$ for every submodule $K\neq 0$, or equivalently, $K=0$ for every $K\leq M$ with $N\cap K=0$. Let M be an R-module and K be a submodule of M. K is called a *generalized small* (or briefly, *g-small*) submodule of M if for every essential submodule T of M with the property $M=K+T$ implies that $T=M$, then we write $K\ll_g M$ (in [16], it is called an *e-small* submodule of M and denoted by $K\ll_e M$). Let M be an R-module. M is called a *hollow* module if every proper submodule of M is small in M. M is called a *generalized hollow* (or briefly, *g-hollow*) module if every proper submodule of M is g-small in M. Here it is clear that every hollow module is generalized hollow. The converse of this statement is not always true. M is called a *local* module if M has the largest submodule, i.e. a proper submodule which contains all other proper submodules. M is called a *generalized local* (briefly, *g-local*) if M has the large proper essential submodule which contain all proper essential submodules of M or M have no proper essential submodules. Let U and V be submodules of M. If $M=U+V$ and V is minimal with respect to this property, or equivalently, $M=U+V$ and $U\cap V\ll V$, then V is called a *supplement* of U in M. M is said to be *supplemented* if every submodule of M has a supplement in M. M is said to be *cofinitely supplemented* if every cofinite submodule of M has a supplement in M. If every submodule of M has a supplement that is a direct summand in M, then M is called a $\oplus$-*supplemented* module. M is said to be $\oplus$-*cofinitely supplemented* if every cofinite submodule of M has a supplement that is a direct summand in M. Let M be an R-module and $U,V\leq M$. If $M=U+V$ and $M=U+T$ with $T\trianglelefteq V$ implies that $T=V$, or equivalently, $M=U+V$ and $U\cap V\ll_g V$, then V is called a *g-supplement* of U in M. M is said to be *g-supplemented* if every submodule of M has a g-supplement in M. M is said to be *cofinitely g-supplemented* if every cofinite submodule of M has a g-supplement in M. M is said to be $\oplus$-*g-supplemented* if every submodule of M has a g-supplement that is a direct summand in M (see [11]). M is said to be *cofinitely* $\oplus$-*g-supplemented* if every cofinite submodule of M has a g-supplement that is a direct summand in M (see [12]). A module M is said to have the *Summand Sum Property* (*SSP*) if the sum of two direct summands of M is again a direct summand of M (see [15, Exercise 39.17 (3)]). We say that a module M has ($D3$) property if $M_1\cap M_2$ is a direct summand of M for every direct summandsM_1 and M_2 of M with $M=M_1+M_2$ (see [3]). Let M be an R-module and $U,V\leq M$. If $M=U+V$ and $U\cap V\leq RadV$, then V is called a *generalized* (*radical*) *supplement* (briefly, *Rad-supplement*) of U in M. M is said to be *generalized* (*radical*) *supplemented* (briefly, *Rad-supplemented*) if every submodule of M has a Rad-supplement in M. M is said to be *cofinitely Rad-supplemented* if every cofinite submodule of M has a Rad-supplement in M. M is said to be *generalized* (*radical*) $\oplus$-*supplemented* (briefly, *Rad*-$\oplus$-*supplemented*) if every submodule of M has a Rad-supplement that is a direct summand in M. M is said to be *cofinitely Rad*-$\oplus$-*supplemented* if every cofinite submodule of M has a Rad-supplement that is a direct summand in M. The intersection of all essential maximal submodules of an R-module M is called the *generalized radical* of M and denoted by Rad_gM (in [16], it is denoted by Rad_eM). If M have no essential maximal submodules, then we denote $Rad_gM=M$. Let M be an R-module and $U,V\leq M$. If $M=U+V$ and $U\cap V\leq Rad_gV$, then V is called a *generalized radical supplement* (or briefly, *g-radical supplement*) of U in M. M is said to be *generalized radical supplemented* (briefly, *g-radical supplemented*) if every submodule of M has a g-radical supplement in M. M is said to be *cofinitely g-radical supplemented* if every cofinite submodule of M has a g-radical supplement in M. M is said to be $\oplus$-*g-Rad-supplemented* if every submodule of M has a g-radical supplement that is a direct summand in M (see [13]).

More informations about supplemented modules are in [2] and [15]. More results about $\oplus$-supplemented modules are in [6]. More details about cofinitely supplemented modules are in [1]. More informations about $\oplus$-cofinitely supplemented modules are in [3]. More details about generalized (radical) supplemented modules are in [14]. More details about generalized (radical) $\oplus$-supplemented modules are in [4] and [5]. More results about cofinitely Rad-$\oplus$-supplemented modules are in [5]. More informations about g-supplemented modules are in [8]. More details about cofinitely g-supplemented modules are in [7]. More informations about g-radical supplemented modules are in [9]. More details about cofinitely g-radical supplemented modules are in [10].

2. COFINITELY ⊕-G-RADICAL SUPPLEMENTED MODULES

Definition 1.1. Let M be an R-module. If every cofinite submodule of M has a g-radical supplement that is a direct summand in M, then M is called a *cofinitely ⊕-g-radical supplemented* (briefly, *cofinitely ⊕-g-Rad-supplemented*) module.

Proposition 2.2. Every ⊕-g-Rad-supplemented module is cofinitely ⊕-g-Rad-supplemented.
Proof. Clear from definitions.

Proposition 2.3. Every fintely generated cofinitely ⊕-g-Rad-supplemented module is ⊕-g-Rad-supplemented.
Proof. Let M be a finitely generated cofinitely ⊕-g-Rad-supplemented module and $U\leq M$. Since M is finitely generated, M/U is also finitely generated and U is a cofinite submodule of M. Since M is cofinitely ⊕-g-Rad-supplemented, U has a g-radical supplement that is a direct summand in M. Hence M is ⊕-g-Rad-supplemented, as desired.

Proposition 2.4. Every cofinitely ⊕-g-supplemented module is cofinitely ⊕-g-Rad-supplemented.
Proof. Let M be a cofinitely ⊕-g-supplemented module and U be a cofinite submodule of M. Then U has a g-supplement V that is a direct summand in M. Since V is a g-supplement of U in M, $M=U+V$ and $U\cap V\ll_g V$. Then $U\cap V\ll Rad_g V$ and V is a g-radical supplement of U in M. Hence M is cofinitely ⊕-g-Rad-supplemented, as desired.

Proposition 2.5. Every ⊕-cofinitely supplemented module is cofinitely ⊕-g-Rad-supplemented.
Proof. Let M be a ⊕-cofinitely supplemented module and U be a cofinite submodule of M. Then U has a supplement V that is a direct summand in M. Here $M=U+V$ and $U\cap V\ll V$. Since $U\cap V\ll V$, $U\cap V\ll_g V$. Then V is a g-supplement of U in M. Then M is cofinitely ⊕-g-supplemented and by Proposition 2.4, M is ⊕-g-Rad-supplemented.

Proposition 2.6. Every ⊕-g-supplemented module is cofinitely ⊕-g-Rad-supplemented.
Proof. Since every ⊕-g-supplemented module is cofinitely ⊕-g-supplemented, by Proposition 2.4, every ⊕-g-supplemented module is cofinitely ⊕-g-Rad-supplemented.

Proposition 2.7. Every ⊕-supplemented module is cofinitely ⊕-g-Rad-supplemented.
Proof. Clear from Proposition 2.6, since every ⊕-supplemented module is ⊕-g-supplemented.

Proposition 2.8. Hollow and local modules are cofinitely ⊕-g-Rad-supplemented.
Proof. Clear from Proposition 2.7, since hollow and local modules are ⊕-supplemented.

Proposition 2.9. Every (D1) module is cofinitely ⊕-g-Rad-supplemented.
Proof. Clear from Proposition 2.7, since every (D1) module is ⊕-supplemented.

Proposition 2.10. Every g-hollow module is cofinitely ⊕-g-Rad-supplemented.
Proof. Let M be a g-hollow module. Then M is ⊕-g-supplemented and by Proposition 2.6, M is cofinitely ⊕-g-Rad-supplemented, as desired.

3. CONCLUSION

Cofinitely ⊕-g-Rad-supplemented modules are special parts of ⊕-g-Rad-supplemented modules.

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On The Spectral Properties of a Boundary Value Problem

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Abstract

In this study, the properties of the eigenvalues of the boundary value problem, the oscillation properties of the eigenfunctions, and the asymptotic formulas of the eigenvalues and eigenfunctions are obtained. For these conditions, definitions suitable for the boundary value problem and auxiliary hypotheses are used.

Keywords: Eigenvalue, Eigenfunction, Oscillation Theorem, Asymptotic formula, Boundary Value Problem.

1. Introduction

We consider the following boundary value problem with a spectral parameter in the equation and the boundary conditions.

$$-u'' + q(x)u = \lambda^2 u,\ 0 < x < 1, \tag{1.1}$$

$$(\alpha_0 + \alpha_1 \lambda)u(0) + u'(0) = 0, \tag{1.2}$$

$$(\beta_0 + \beta_1 \lambda)u(1) + u'(1) = 0. \tag{1.3}$$

Here λ is a spectral parameter, $q(x)$ is a nonnegative continuous function on the interval [0,1] and α_i and β_i reel constants (i=0,1). The present article is devoted to studying the properties of the eigenvalues and eigenfunctions of the boundary value (1.1)-(1.3). The boundary value problem involving parameters in boundary conditions is considered in [1,11] and other statements are studied in [7,9].

In this article, the eigenvalues and eigenfunctions for the boundary value problem (1.1)-(1.3) are examined and the oscillation theorem about the zeros of eigenfunction is proved.

Throughout the article, we assume that the following conditions are met.

$$\alpha_0 < 0,\ \beta_0 > 0,\ |\alpha_1| + |\beta_1| \neq 0. \tag{1.4}$$

2. Some Properties of the Eigenvalues of the Boundary Value Problem

Lemma 2.1. The eigenvalues of the boundary value problem (1.1)-(1.3) are real.

Proof: $u(x,\lambda)$ is the eigenfunction of the boundary value problem (1.1)-(1.3) and λ is the eigenvalue satisfying this function. Multiply both sides of equation (1.1) by $\overline{u(x,\lambda)}$ and take the integral from 0 to 1 corcerning for to x.

$$-\int_0^1 u''(x,\lambda)\,\overline{u(x,\lambda)}dx + \int_0^1 q(x)u(x,\lambda)\,\overline{u(x,\lambda)}dx = \lambda^2 \int_0^1 u(x,\lambda)\,\overline{u(x,\lambda)}dx. \tag{2.1}$$

By using the formula of integration by part,

$$-u'(x,\lambda)\overline{u(x,\lambda)} + \int_0^1 |u'(x,\lambda)|^2\,dx + q(x)\int_0^1 |u(x,\lambda)|^2 dx = \lambda^2 \int_0^1 |u(x,\lambda)|^2 dx\,,$$

$$-u'(1,\lambda)\overline{u(1,\lambda)} + u'(0,\lambda)\overline{u(0,\lambda)} + \int_0^1 |u'(x,\lambda)|^2\,dx + q(x)\int_0^1 |u(x,\lambda)|^2 dx = \lambda^2 \int_0^1 |u(x,\lambda)|^2 dx\,,$$

Using the boundary conditions ,

$$\overline{u(1,\lambda)}\,(\beta_0 + \beta_1\lambda)u(1,\lambda) - \overline{u(0,\lambda)}(\alpha_0 + \alpha_1\lambda)u(0,\lambda) + \int_0^1 |u'(x,\lambda)|^2\,dx$$
$$+\,q(x)\int_0^1 |u(x,\lambda)|^2 dx = \lambda^2 \int_0^1 |u(x,\lambda)|^2 dx,$$

$$(\beta_0 + \beta_1\lambda)|u(1,\lambda)|^2 - (\alpha_0 + \alpha_1\lambda)|u(0,\lambda)|^2 + \int_0^1 \left|u'(x,\lambda)\right|^2 dx + q(x)\int_0^1 |u(x,\lambda)|^2 dx$$
$$= \lambda^2 \int_0^1 |u(x,\lambda)|^2 dx,$$

Put it in appropriate parentheses as the quadratic equation connected to the λ:

$$\lambda\left(-\beta_1|u(1,\lambda)|^2 + \alpha_1|u(0,\lambda)|^2\right) - \beta_0|u(1,\lambda)|^2 + \alpha_0|u(0,\lambda)|^2 - \int_0^1 |u'(x,\lambda)|^2$$
$$-\,q(x)\int_0^1 |u(x,\lambda)|^2 dx + \lambda^2 \int_0^1 |u(x,\lambda)|^2 dx\,.$$

$$A(\lambda) = \int_0^1 |u(x,\lambda)|^2 dx,$$

$$B(\lambda) = -\beta_1 |u(1,\lambda)|^2 + \alpha_1 |u(0,\lambda)|^2,$$

$$C(\lambda) = -\beta_0 |u(1,\lambda)|^2 + \alpha_0 |u(0,\lambda)|^2 - \int_0^1 |u'(x,\lambda)|^2 - q(x) \int_0^1 |u(x,\lambda)|^2 dx\,.$$

Hence, the eigenvalue λ is a root of the quadratic equation

$$A(\lambda)z^2 + B(\lambda)z + C(\lambda) = 0. \tag{2.2}$$

By (1.4), $A(\lambda) > 0$ and $C(\lambda) < 0$; so, $B^2(\lambda) - 4A(\lambda)C(\lambda)$>0. The equation (2.2) has only real roots. Lemma 2.1 is proved.

Lemma 2.2. The eigenvalues of the boundary value problem (1.1)-(1.3) constitute an at most countable set without finite limit points. All eigenvalues of the boundary value problem (1.1)-(1.3) are simple.

2. Oscillatory Properties of the Eigenfunctions of Problem (1.1)-(1.3)

Lemma 3.1. Let u(x) is a solution to the equation

$$u'' + g(x)u = 0, \tag{3.1}$$

where the initial conditions

$$u(0) = 1,\ u'(0) = -\alpha_0 - \alpha_1\lambda, \tag{3.2}$$

and $v(x)$ is a solution to the equation

$$v'' + h(x)v = 0, \tag{3.3}$$

where th initial conditions

$$v(0) = 1,\ v'(0) = -\beta_0 - \beta_1\lambda. \tag{3.4}$$

Assume that

$$g(x) < h(x)\ (0 \le x \le 1).$$

$u(x)$ has m zeros while $v(x)$ has at least m zeros in some interval. Additionally, the kth zero of v(x) is less than one kth zero of $u(x)$.

Lemma 3.2 The number $\lambda = 0$ is not an eigenvalue of the boundary value problem (1.1)-(1.3).

Proof: Denote by $\varphi(x)$ a solution to the initial problem

$$-\varphi'' + q(x)\varphi = 0, \tag{3.5}$$

$$\varphi(0) = 1\,, \varphi'(0) = -\alpha_0, \tag{3.6}$$

Prove that $\beta_0\varphi(1) + \varphi(1) \neq 0$.

Assume the opposite

$$\beta_0\varphi(1) + \varphi(1) = 0.$$

Then we hold

$$-\beta_0 = \frac{\varphi'(1)}{\varphi(1)} < 0. \tag{3.7}$$

Conversely (3.5) multiply by $\varphi(x)$ and integrate from 0 to 1,

$$\int_0^1 \varphi''(x)\varphi(x)dx = \int_0^1 q(x)\varphi^2(x)dx,$$

$$\varphi(1)\varphi'(1) - \varphi(0)\varphi'(0) = \int_0^1 \varphi'^2(x)dx + \int_0^1 q(x)\varphi^2(x)dx.$$

Using the conditions $\alpha_0 < 0$ and $q(x) \geq 0$, the right-hand side of the equation is positive. From (3.6) and (3.7) the left side becomes negative. Since the acceptance is wrong, the proof is completed.

Lemma 3.3. $[13, p. 14]$ If $x_0(0<x_0<1)$ is a zero of the function $\varphi(x, \lambda_0)$ then for every sufficiently small $\varepsilon > 0$ there is $\delta > 0$ such that, for $|\lambda - \lambda_0| < \delta$, the function $\varphi(x, \lambda)$ has exactly are zero in the interval $|x - x_0| < \varepsilon$.

Corollary 3.1. The solution $\varphi(x, \lambda)$ may lose a zero or acquire a new zero under the variation of λ if and only if the zero enters or exists the interval through the boundary points 0 and 1.

The following oscillatory theorem proves the existence of a countable set of eigenvalues of the boundary value problem (1.1)-(1.3).

Theorem 3.1. (Oscillation Theorem) There are an unboundedly decreasing sequence of negative eigenvalues $\{\lambda_{-a}\}_{n=1}^{\infty}$ and an unboundedly increasing sequence of positive eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ of the boundary value problem (1.1)-(1.3)

$$\ldots < \lambda_{-n} < \lambda_{-n+1} < \cdots < \lambda_{-1} < \lambda_1 < \cdots < \lambda_{n-1} < \lambda_n < \cdots$$

There are exist number n_*, $n^* \in \mathbb{N}$ and $k_*, k^* \in \mathbb{N} \cup \{0\}$ such that the eigenfunctions corresponding to the eigenvalues $\lambda_{-n}\ (n \geq n_*)$ and $\lambda_n (n \geq n^*)$ have $n + k_* - n_*$ and $n + k^* - n^*$simple zeros in the interval (1,1) respectively.

4. The Definition and Properties of the Function $\theta_m(x)(m \in \mathbb{Z}\backslash\{0\})$. Some Auxiliary Assertions

Let $u_m(x)$ be an eigenfunction of the boundary value problem (1.1)-(1.3) which corresponds to an eigenvalue λ_m, where $m \in \mathbb{Z}\backslash\{0\}$.

$$\Delta= \max\left\{\frac{|\alpha_0|}{|\alpha_1|}, \frac{|\beta_0|}{|\beta_1|}\right\} + 1. \tag{4.1}$$

The following inequality is clear for $|\lambda| \geq 0$,

$$\alpha_1 > 0 \ \ \alpha_0 + \alpha_1\lambda > 0, \alpha_1 > 0 \ \ \alpha_0 + \alpha_1\lambda < 0, \tag{4.2}$$

and

$$\beta_1 > 0 \ \ \beta_0 + \beta_1\lambda > 0, \beta_1 < 0 \ \ \beta_0 + \beta_1\lambda < 0. \tag{4.3}$$

Let $\mathbb{N}_0 \in \mathbb{N}$ be a natural number such that

$$\lambda_m^2 \geq \Delta_0, \Delta_0= \max\{\Delta, 2C_0 + 1\} \text{ and } C_0 = \max_{0\leq x\leq 1} q(x).$$

Since it is $m \in \mathbb{Z}\backslash\{0\}$, such that $|m| \geq \mathbb{N}_0$.

$$\theta_m(x) = \arctan\frac{u_m(x)}{u'_m(x)}.$$

Let's define below the equality

$$\theta_m(x) = \arg\big(u'_m(x) + iu_m(x)\big), \tag{4.4}$$

and from the conditions (1.2)-(1.3) we get

$$\theta_m(0) = \arctan\left[\frac{-1}{\alpha_0 + \alpha_1\lambda}\right] + \pi,$$

$$\theta_m(1) = \arctan\left[\frac{-1}{\beta_0+\beta_1\lambda}\right] + \pi k.$$

The function is obtained by adding 2π multiples of (4.2)-(4.3) for other x's. The $u_m(x)$ and $u'_m(x)$ cannot vanish simultaneously.

Lemma 4.1. The function $\theta_m(x)$ satisfies the differential equation

$$\theta'_m(x) = cos^2\theta_m(x) + \big(\lambda_m^2 - q(x)\big)sin^2\theta_m(x),$$

and increases on the interval [0,1].

Denote by $x_{m.k}(k = \overline{1, k_m})$ the zeros of the eigenfunction $u_m(x)$ in the interval (0,1). The oscillatory Theorem 3.1 implies that the equalities $k_{|m|} = |m| + k^* - n^*$ and $k_{-|m|} = |m| + k_* - n_*$ are valid for all $m \in \mathbb{Z}\backslash\{0\}$ large enough.

Lemma 4.2. The following estimates are valid for the eigenvalues $\lambda_m(|m| \geq \mathbb{N}_0)$

$$C_1|m| \leq |\lambda_m| \leq C_2|m|, \tag{4.5}$$

where C_1 and C_2 are some positive constants.

Lemma 4.3 Suppose that $q(x) \in C[0,1]$ and $0 = t_0 < t_1 < \cdots < t_v < t_{v+1} = 1$.

Then we get

$$\int_0^1 q(x)dx - \sum_{k=1}^{v-1} q(t_k)\Delta t_k = O\big(\omega(\delta)\big), \tag{4.6}$$

where $\Delta t_k = t_{k+1} - t_k = max_{1\le k\le v}\Delta t_k$, $\omega(\delta) = \delta + \omega_1(\delta)$, and $\omega_1(\delta)$ is the modulus of continuity of the function $q(x)$ on the interval $[0,1]$. Moreover, if $q(x) \not\equiv$const, then we can replace, $\omega(\delta)$ in (4.6) by $\omega_1(\delta)$.

It is easy to verify that

$$\theta_m(1) = -tan^{-1}\frac{1}{\beta_0 + \beta_1\lambda_m} + \pi k_m\,.$$

5. Asymptotic Formulas for the Eigenvalues and Eigenfunctions of the Boundary Value Problem (1.1)-(1.3)

Throughout this section, we suppose that $m \in \mathbb{Z}\backslash\{0\}$, $|m| \ge \mathbb{N}_0$, where $\mathbb{N}_0$ is a natural number.

Suppose that $v_m(x)$ s an eigenfunction of the boundary value problem (1.1)-(1.3) with $|m|$ zeros in interval (1.1). Denote by μ_m the eigenvalue that corresponds to $v_m(x)$. Theorem 3.1 of oscillation implies that $\mu_m = \lambda_{m-k^*+n^*}$ for $m > 0$ and $\mu_m = \lambda_{m+k_*-n_*}$ for $m < 0$.

Denote zeros of $v_m(x)$ by $x_{m,k}$: $0 < x_{m,1} < x_{m,2} < \cdots < x_{m\,|m|,} < 1$.

Theorem 5.1. The following asymptotic formulas are valid:

$$\mu_m(x) = \left(\frac{\alpha_1\beta_1 m\pi}{\alpha_1\beta_1 m\pi + \beta_1 - \alpha_1}\right)\pi(m - sgnm) + \frac{\alpha_1\beta_1}{(\alpha_1\beta_1 m\pi + \beta_1 - \alpha_1}\left(\int_0^1 q(x)dx\right)$$
$$+ O(m^{-1}\omega_1(m^{-1}),$$

$$v(x) = sgnm\left[sin\left(\left(\frac{\alpha_1\beta_1 m\pi}{\alpha_1\beta_1 m\pi + \beta_1 - \alpha_1}\right)\pi(m - sgnm) + \frac{\alpha_1\beta_1}{(\alpha_1\beta_1 m\pi + \beta_1 - \alpha_1}\left(\int_0^1 q(x)dx\right)\right.\right.$$
$$\left.\left. + O\big(m^{-1}\omega_1(m^{-1})\big)\right)x + O(m)\right].$$

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Hardy Space of Rabotnov Function

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Abstract

In this paper, we obtain conditions for the normalized Rabotnov function to belong to the Hardy space $\mathcal{H}^\infty$.

Keywords: Analytic function, starlike and convex functions, Rabotnov function, Hardy space.

1. Introduction

Denote by $\mathbb{D}=\left\{z\in\mathbb{C}:\left|z\right|<1\right\}$ the open unit disk and $\mathcal{H}$ be set of all analytic functions on $\mathbb{D}$. Let $\mathcal{A}$ be a class of functions f in $\mathbb{D}$ which satisfy the usual normalization conditions $f(0)=f'(0)-1=0$. Traditionally, the subclass of $\mathcal{A}$ consisting of univalent functions is denoted by $\mathcal{S}$. The classes of starlike and convex functions in $\mathbb{D}$ are two important $\kappa\left(\kappa\in[0,1)\right)$ in $\mathbb{D}$ are defined by $\mathcal{S}^*(\kappa):=\left\{f: f\in\mathcal{S} \text{ and } \Re\left(zf'(z)/f(z)>\kappa\right)\right\}$ and $\mathcal{C}(\kappa):=\left\{f: f\in\mathcal{S} \text{ and } 1+\Re\left(zf''(z)/f'(z)>\kappa\right)\right\}$, respectively. The familiar classes $\mathcal{S}^*:=\mathcal{S}^*(0)$ and $\mathcal{C}:=\mathcal{C}(0)$ are known, respectively, as the classes of starlike and convex functions in $\mathbb{D}$. In [1], for $\gamma<1$, the author introduced the classes:

$$\mathcal{P}(\gamma):=\left\{p\in\mathcal{H}:\exists\,\eta\in\mathbb{R}\text{ such that } p(0)=1,\ \Re\left[e^{i\eta}p(z)\right]>\gamma,\ z\in\mathbb{D}\right\}$$

and $\mathcal{R}(\gamma):=\left\{g\in\mathcal{A}: g'\in\mathcal{P}(\gamma)\right\}$.

When $\eta=0$, the classes $\mathcal{P}(\gamma)$ and $\mathcal{R}(\gamma)$ will be denoted by $\mathcal{P}_0(\gamma)$ and $\mathcal{R}_0(\gamma)$, respectively. Also, for $\gamma=0$ we denote $\mathcal{P}_0(\gamma)$ and $\mathcal{R}_0(\gamma)$ simply $\mathcal{P}$ and $\mathcal{R}$, respectively. Moreover, the Hadamard product (or convolution) of two power series belongs to the class $\mathcal{A}$ given by $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ and $g(z)=z+\sum_{n=2}^{\infty}b_nz^n$ defined as

$$(f*g)(z):=z+\sum_{n\geq 2}a_nb_nz^n=:(g*f)(z),\quad (z\in\mathbb{D}).$$

Let $\mathcal{H}^p\left(0<p\leq\infty\right)$ denote the Hardy space of all analytic functions $f(z)$ in $\mathbb{D}$ and define the integral means $M_p(r,f)$ by

$$M_p(r,f)=\begin{cases}\left(\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\left|f\left(re^{i\theta}\right)\right|^p d\theta\right)^{\frac{1}{p}} & (0<p<\infty)\\ \sup\limits_{0\le\theta\le 2\pi}\left|f\left(re^{i\theta}\right)\right| & (p=\infty)\end{cases}.$$

An analytic function $f(z)$ in $\mathbb{D}$, is said to belong to the Hardy space $\mathcal{H}^p(0<p\le\infty)$, if the set $\{M_p(r,f): r\in[0,1)\}$ is bounded. It is important to remind here that $\mathcal{H}^p$ is Banach space with the norm defined by (see [2, p. 23])

$$\|f\|_p=\lim_{r\to 1^-}M_p(r,f)$$

for $1\le p\le\infty$. On the other hand, we known that $\mathcal{H}^\infty$ is the class of bounded analytic functions in $\mathbb{D}$, while $\mathcal{H}^2$ is the class of power series $\sum a_n z^n$ such that $\sum|a_n|^2<\infty$. In addition, it is known from [2] that $\mathcal{H}^q$ is a subset of $\mathcal{H}^p$ for $0<p\le q\le\infty$. Also, two well-known results about the Hardy space $\mathcal{H}^p$ are the following (see [2]):

$$\Re\{f'(z)\}>0\Rightarrow\begin{cases}f'\in\mathcal{H}^q & (q<1)\\ f^{\frac{q}{1-q}}\in\mathcal{H}^q & (q\in(0,1))\end{cases}. \tag{1}$$

2. Preliminaries

The Rabotnov [9] function $R_{\alpha,\beta}(z)$, defined by

$$R_{\alpha,\beta}(z)=z^\alpha\sum_{n=0}^{\infty}\frac{\beta^n}{\Gamma((n+1)(1+\alpha))}z^{n(1+\alpha)},\quad (\alpha>-1,\ \beta\in\mathbb{C},\ z\in\mathbb{D}). \tag{2}$$

The Rabotnov function $R_{\alpha,\beta}(z)$ does not belong to the class $\mathcal{A}$. Therefore, we consider the following normalization for the function $R_{\alpha,\beta}(z)$:

$$\mathbb{R}_{\alpha,\beta}(z)=\Gamma(1+\alpha)z^{1/(1+\alpha)}R_{\alpha,\beta}\left(z^{1/(1+\alpha)}\right)=z+\sum_{n=2}^{\infty}\frac{\beta^{n-1}\Gamma(1+\alpha)}{\Gamma(n(1+\alpha))}z^n,\quad (z\in\mathbb{D}). \tag{3}$$

In this recent years, the authors in [1,4,5,7-8,10,13,14] studied the Hardy space of some special functions as normalized; Hypergeometric, Bessel, Struve, Lommel, Wright and Mittag-Leffler. Motivated by above studies, our main aim is to determine some conditions on the parameters such that the Rabotnov function $\mathbb{R}_{\alpha,\beta}(z)$ is convex of order κ, Also, we find some conditions for the Hadamard products $\mathbb{R}_{\alpha,\beta}(z)*f(z)$ to belong to $\mathcal{H}^\infty\cap\mathcal{R}$, where f is an analytic function in $\mathcal{R}$. Moreover, we investigate the Hardy space of the mentioned the normalized Rabotnov function $\mathbb{R}_{\alpha,\beta}(z)$.

In order to prove the main results we need the following preliminary results.

Lemma 1. (Silverman [11]) Let $f(z)=z+\sum_{n=2}^{\infty}a_n z^n \in \mathcal{A}.$ If

$$\sum_{n=2}^{\infty}(n-\kappa)|a_n|\leq 1-\kappa,$$

then the function $f(z)$ is in the class $\mathcal{C}(\kappa).$

Lemma 2. (Eenigenburg and Keogh, [3]) Let $\kappa\in[0,1).$ If the function $f\in\mathcal{C}$ is not of the form

$$\begin{cases} f(z)=k+lz\left(1-ze^{i\theta}\right)^{2\kappa-1} & \left(\kappa\neq\frac{1}{2}\right) \\ f(z)=k+l\log\left(1-ze^{i\theta}\right) & \left(\kappa=\frac{1}{2}\right) \end{cases} \tag{4}$$

for some $k,l\in\mathbb{C}$ and $\theta\in\mathbb{R},$ then the following statements hold:

a: There exist $\delta=\delta(f)>0$ such that $f'\in\mathcal{H}^{\delta+\frac{1}{2(1-\kappa)}}.$

b: If $\kappa\in\left[0,\frac{1}{2}\right),$ then there exist $\tau=\tau(f)>0$ such that $f'\in\mathcal{H}^{\tau+\frac{1}{1-2\kappa}}.$

c: If $\kappa\geq\frac{1}{2},$ then $f\in\mathcal{H}^{\infty}.$

Lemma 3. (Stankiewich and Stankiewich, [12]) $\mathcal{P}_0(\lambda)*\mathcal{P}_0(\mu)\subset\mathcal{P}_0(\gamma),$ where $\gamma=1-2(1-\lambda)(1-\mu).$ The value of γ is the best possible.

2. Main Results

In this section, we present our main results related to the some geometric properties and Hardy class of normalized Rabotnov function $\mathbb{R}_{\alpha,\beta}(z).$

Theorem 1. Let $\kappa\in[0,1),$ $\alpha>-1,$ $\beta\in\mathbb{C}$ and $\left(1+\frac{|\beta|}{1+\alpha}\right)e^{\frac{|\beta|}{1+\alpha}}>2.$ The following inequality is true:

$$\frac{\left[1+\frac{3|\beta|}{1+\alpha}+\frac{|\beta|^2}{(1+\alpha)^2}\right]e^{\frac{|\beta|}{1+\alpha}}-2}{\left(1+\frac{|\beta|}{1+\alpha}\right)e^{\frac{|\beta|}{1+\alpha}}-2}\leq\kappa \tag{5}$$

holds, then the normalized Rabotnov function $\mathbb{R}_{\alpha,\beta}(z)$ is convex of order κ in $\mathbb{D}.$

Proof. By virtue of the Silverman's result which is given in Lemma 1, in order to prove the convex of order κ of the function, $\mathbb{R}_{\alpha,\beta}(z)$ it is enough to show that the following inequality

$$\sum_{n=2}^{\infty} n(n-\kappa)\left|\frac{\beta^{n-1}\Gamma(1+\alpha)}{\Gamma(n(1+\alpha))}\right| \leq 1-\kappa \tag{6}$$

is satisfied under our assumptions. According to the hypothesis of theorem, by using the inequality

$$(1+\alpha)^{n-1}(n-1)!\Gamma(1+\alpha) \leq \Gamma(n(1+\alpha))$$

and thus

$$\frac{\Gamma(1+\alpha)}{\Gamma(n(1+\alpha))} \leq \frac{1}{(1+\alpha)^{n-1}(n-1)!}, \quad n \in \mathbb{N}, \tag{7}$$

we have

$$\begin{aligned}
\sum_{n=2}^{\infty} n(n-\kappa)\left|\frac{\beta^{n-1}\Gamma(1+\alpha)}{\Gamma(n(1+\alpha))}\right| &= \sum_{n=2}^{\infty} n(n-\kappa)\frac{|\beta|^{n-1}\Gamma(1+\alpha)}{\Gamma(n(1+\alpha))} \\
&\leq \sum_{n=2}^{\infty} n(n-\kappa)\frac{|\beta|^{n-1}}{(1+\alpha)^{n-1}(n-1)!} \\
&= \sum_{n=2}^{\infty} n^2 \frac{|\beta|^{n-1}}{(1+\alpha)^{n-1}(n-1)!} - \sum_{n=2}^{\infty} \kappa n \frac{|\beta|^{n-1}}{(1+\alpha)^{n-1}(n-1)!} \\
&= \left(1+\frac{3|\beta|}{1+\alpha}+\frac{|\beta|^2}{(1+\alpha)^2}\right)e^{\frac{|\beta|}{1+\alpha}} - \kappa\left(1+\frac{|\beta|}{1+\alpha}\right)e^{\frac{|\beta|}{1+\alpha}} + \kappa - 1.
\end{aligned}$$

The inequality (5) implies that the last sum is bounded above by $1-\kappa$. Therefore the inequality (6) is satisfied, that is, $\mathbb{R}_{\alpha,\beta}(z)$ is convex of order κ in $\mathbb{D}$.

Theorem 2. Let $\kappa \in [0,1)$, $\alpha > -1$, $\beta \in \mathbb{C}$. If inequality

$$\kappa < 2 - e^{\frac{|\beta|}{1+\alpha}} \tag{8}$$

holds, then $\frac{\mathbb{R}_{\alpha,\beta}(z)}{z} \in \mathcal{P}_0(\kappa)$.

Proof. In order to prove $\frac{\mathbb{R}_{\alpha,\beta}(z)}{z} \in \mathcal{P}_0(\kappa)$, it is enough to show that $|p(z)-1| < 1$, where $p(z) = \frac{1}{1-\kappa}\left(\frac{\mathbb{R}_{\alpha,\beta}(z)}{z} - \kappa\right)$. Now, using the inequalities (7), we have

$$
\begin{aligned}
|p(z)-1| &= \left|\frac{1}{1-\kappa}\left(1+\sum_{n=2}^{\infty}\frac{\beta^{n-1}\Gamma(1+\alpha)}{\Gamma(n(1+\alpha))}z^{n-1}-\kappa\right)-1\right| \\
&\leq \frac{1}{1-\kappa}\sum_{n=2}^{\infty}\frac{|\beta|^{n-1}\Gamma(1+\alpha)}{\Gamma(n(1+\alpha))} \\
&\leq \frac{1}{1-\kappa}\sum_{n=2}^{\infty}\frac{|\beta|^{n-1}}{(1+\alpha)^{n-1}(n-1)!}=\frac{1}{1-\kappa}\left(e^{\frac{|\beta|}{1+\alpha}}-1\right).
\end{aligned}
$$

Consequently, from (8) $\frac{\mathbb{R}_{\alpha,\beta}(z)}{z}$ is in the class $\mathcal{P}_0(\kappa)$, and the proof is completed.

Theorem 3. Let $\kappa\in[0,1)$, $\alpha>-1$, $\beta\in\mathbb{C}$. If the inequality (5) is satisfied, then

$$
\mathbb{R}_{\alpha,\beta}(z)\in\begin{cases}\mathcal{H}^{\frac{1}{1-2\kappa}}, & \kappa\in\left[0,\frac{1}{2}\right) \\ \mathcal{H}^{\infty}, & \kappa\in\left[\frac{1}{2},1\right).\end{cases}
$$

Proof. It is known that Gauss hypergeometric function is defined by

$$
{}_2F_1(a,b,c;z)=\sum_{n\geq 0}\frac{(a)_n(b)_n}{(c)}\frac{z^n}{n!} \quad (z\in\mathbb{C}). \tag{9}
$$

Now, using the equality (9) it is possible to show that the function $\mathbb{R}_{\alpha,\beta}(z)$ can not be written in the forms which are given by (4) for corresponding values of κ. More precisely, we can write that the following equalities:

$$
k+lz\left(1-ze^{i\theta}\right)^{2\kappa-1}=k+l\sum_{n\geq 0}\frac{(1-2\kappa)_n}{n!}e^{in\theta}z^{n+1} \tag{10}
$$

and

$$
k+l\log\left(1-ze^{i\theta}\right)=k-l\sum_{n\geq 0}\frac{1}{n+1}e^{in\theta}z^{n+1} \tag{11}
$$

hold true for $k,l\in\mathbb{C}$ and $\theta\in\mathbb{R}$. If we consider the series representation of the function $\mathbb{R}_{\alpha,\beta}(z)$ which is given by (3), then we see that the function $\mathbb{R}_{\alpha,\beta}(z)$ is not of the forms (10) for $\kappa\neq\frac{1}{2}$ and (11) for $\kappa=\frac{1}{2}$, respectively. On the other hand, Theorem 1, states that the function $\mathbb{R}_{\alpha,\beta}(z)$ is convex of order under hypothesis. Therefore, the proof is completed by applying Lemma 2.

Theorem 4. Let $\alpha>-1$, $\beta\in\mathbb{C}$ and $f(z)\in\mathcal{R}$. If the inequality

$$e^{\frac{|\beta|}{1+\alpha}} < \frac{3}{2}. \tag{12}$$

holds, then the Hadamard product $u(z) = \mathbb{R}_{\alpha,\beta}(z) * f(z) \in \mathcal{H}^\infty \cap \mathcal{R}.$

Proof. Since $f(z) \in \mathcal{R},$ then we can say that $f'(z) \in \mathcal{P}.$ Also, from properties of Hadamard product we can write $u'(z) = \frac{\mathbb{R}_{\alpha,\beta}(z)}{z} * f'(z).$ It is known from Theorem 2 that the function $\frac{\mathbb{R}_{\alpha,\beta}(z)}{z} \in \mathcal{P}_0\left(\frac{1}{2}\right)$ under the condition (12). So, it follows from Lemma 3 that $u'(z) \in \mathcal{P}.$ This means that $u(z) \in \mathcal{R}.$ If we consider the result which is given by (1), then we have $u'(z) \in \mathcal{H}^q$ for $q < 1$ and $u(z) \in \mathcal{H}^{\frac{q}{1-q}}$ for $0 < q < 1,$ or equivalently, $u(z) \in \mathcal{H}^q$ for all $0 < q < \infty.$

Now, from the known upper bound for the Caratheodory functions (see [6, Theorem 1, p.533]), we have that, if the function $f(z) \in \mathcal{R},$ then $|a_n| \le \frac{2}{n}$ for $n \ge 2.$ Using this fact together with the inequality (7), we get

$$\begin{aligned} |u(z)| &= \left|\mathbb{R}_{\alpha,\beta}(z) * f(z)\right| = \left| z + \sum_{n=2}^{\infty} \frac{\beta^{n-1}\Gamma(1+\alpha)}{\Gamma(n(1+\alpha))} a_n z^n \right| \\ &\le 1 + \sum_{n=2}^{\infty} \frac{|\beta|^{n-1}}{(1+\alpha)^{n-1}(n)!} z^n = \left(\frac{1+\alpha}{|\beta|}\right)\left(e^{\frac{|\beta|}{1+\alpha}} - 1\right). \end{aligned}$$

This means that the function $u(z)$ is convergent absolutely for $|z| = 1$ under the hypothesis. On the other hand, we known from [2, Theorem 3.11, p.42] that $u'(z) \in \mathcal{H}^q$ implies the function $u(z)$ is continuous in $\overline{\mathbb{D}},$ where $\overline{\mathbb{D}}$ is closure of $\overline{\mathbb{D}}.$ Since $\overline{\mathbb{D}}$ is a compact set, $u(z)$ is bounded in $\overline{\mathbb{D}},$ that is, $u(z) \in \mathcal{H}^\infty.$ Thus, the proof is completed.

Theorem 5. Let $\alpha > -1,$ $\beta \in \mathbb{C},$ $\lambda \in [0,1),$ $\mu < 1$ and $\gamma = 1 - 2(1-\lambda)(1-\mu).$ Suppose that the function $f(z) \in \mathcal{P}_0(\mu).$ If the inequality

$$\lambda < 2 - e^{\frac{|\beta|}{1+\alpha}}, \tag{13}$$

holds, then $\mathbb{R}_{\alpha,\beta}(z) * f(z) \in \mathcal{R}_0(\gamma).$

Proof. If $f(z) \in \mathcal{R}_0(\mu),$ then this implies that $f'(z) \in \mathcal{P}_0(\mu).$ We know from the Theorem 2 that the function $\frac{\mathbb{R}_{\alpha,\beta}(z)}{z} \in \mathcal{P}_0(\lambda).$ Since $u'(z) = \frac{\mathbb{R}_{\alpha,\beta}(z)}{z} * f'(z),$ taking into account the Lemma 3 we may write that $u'(z) \in \mathcal{P}_0(\gamma).$ This implies that $u(z) \in \mathcal{R}_0(\gamma).$

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Several New Bounds of Gauss-Jacobi Type Quadrature Formula Pertaining to s-Convex Functions

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Abstract

In this paper, authors found a new interesting integral identity regarding Gauss-Jacobi type quadrature formula using generalized fractional integral operators. By using this identity as an auxiliary result, some new bounds with respect to Gauss-Jacobi type quadrature formula pertaining to s-convex functions are established. It is pointed out that several special cases are deduced from the main results for suitable choices of function inside the generalized fractional integral operators. Some basic fractional integral operators of important interest that we investigated in details are Riemann-Liouville fractional integral operator, k-Riemann-Liouville fractional integral operator and conformable fractional integral operator. The Gauss-Jacobi type quadrature formula has remained an area of great interest due to its wide applications in the field of mathematical analysis. We believe that this new results are crucial and of great interest for interested readers working in the fields of inequalities, fractional calculus, quantum calculus, numerical analysis and applied mathematics. These ideas and techniques of this paper may stimulate further research in these directions for different class of functions.

Keywords: Gauss-Jacobi type quadrature formula, s-convex functions, generalized fractional integral operators, Hölder's inequality, power-mean inequality.

1. Introduction and Preliminaries

Definition A function $\psi: I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be convex on I, if

$$\psi(\tau x + (1-\tau)y) \leq \tau\psi(x) + (1-\tau)\psi(y) \tag{1}$$

holds for all $x, y \in I$ and $\tau \in [0,1]$.

In their paper [1], Hudzik and Maligranda considered, among others, the class of functions which are s-convex in the second sense. This class is defined in the following way:

Definition A function $\psi: [0, +\infty[\longrightarrow \mathbb{R}$ is said to be s-convex in the second sense, if

$$\psi(\tau x + (1-\tau)y) \leq \tau^s\psi(x) + (1-\tau)^s\psi(y) \tag{2}$$

holds for all $x, y \in [0, +\infty[, \tau \in [0,1]$ and for some fixed $s \in]0,1]$.

Authors of recent decades have studied convex and s-convex in the second sense function, see [2] - [8].

The Gauss-Jacobi type quadrature formula has the following representation:

$$\int_{a_1}^{a_2} (x-a_1)^p (a_2-x)^q \psi(x)dx = \sum_{k=0}^{+\infty} B_{m,k}\psi(\gamma_k) + R_m^*|\psi|, \tag{3}$$

for certain $B_{m,k}$, γ_k and rest $R_m^*|\psi|$, see [9, 10].

The Gauss-Jacobi type quadrature formula has remained an area of great interest due to its wide applications in the field of mathematical analysis. Recently in [11], Liu obtained several integral inequalities for the left-hand side of (3). Also in [12], Özdemir *et al.* established several integral inequalities concerning the left-hand side of (3) via some kinds of convexity.

Let us recall some special functions and evoke some basic definitions as follows:

Definition For $a_1, a_2 > 0$ the beta function is defined by

$$\beta(a_1, a_2) = \int_0^1 \tau^{a_1-1}(1-\tau)^{a_2-1}d\tau. \tag{4}$$

Definition For $k \in \mathbb{R}^+$ and $\alpha > 0$, the integral representation of k-gamma function is given as

$$\Gamma_k(\alpha) = \int_0^{+\infty} \tau^{\alpha-1}e^{-\frac{\tau^k}{k}}d\tau. \tag{5}$$

One can note that

$$\Gamma_k(\alpha+k) = \alpha\Gamma_k(\alpha) \tag{6}$$

For $k = 1$, (5) gives integral representation of the well-known gamma function.

Remark The function $\varsigma: [0, +\infty[\longrightarrow [0, +\infty[$, which is constructed from the work of Sarikaya *et al.* [13], has the following four conditions:

$$\int_0^1 \frac{\varsigma(\tau)}{\tau}d\tau < +\infty,$$

$$\frac{1}{A_1} \le \frac{\varsigma(\tau_1)}{\varsigma(\tau_2)} \le A_1 \text{ for } \frac{1}{2} \le \frac{\tau_1}{\tau_2} \le 2, \tag{7}$$

$$\frac{\varsigma(\tau_2)}{\tau_2^2} \leq A_2 \frac{\varsigma(\tau_1)}{\tau_1^2} \text{ for } \tau_1 \leq \tau_2$$

and

$$\left|\frac{\varsigma(\tau_2)}{\tau_2^2} - \frac{\varsigma(\tau_1)}{\tau_1^2}\right| \leq A_3|\tau_2 - \tau_1| \frac{\varsigma(\tau_2)}{\tau_2^2} \text{ for } \frac{1}{2} \leq \frac{\tau_1}{\tau_2} \leq 2,$$

where A_1, A_2 and $A_3 > 0$ are independent of $\tau_1, \tau_2 > 0$.

Moreover, Sarikaya *et al.* [13] used the above function ς in order to define the following fractional integral operators.

Definition The generalized left-side and right-side fractional integrals are given as follows:

$$_{a_1^+}I_\varsigma\psi(x) = \int_{a_1}^{x} \frac{\varsigma(x-\tau)}{x-\tau}\psi(\tau)d\tau \qquad (x > a_1) \tag{8}$$

and

$$_{a_2^-}I_\varsigma\psi(x) = \int_{x}^{a_2} \frac{\varsigma(\tau-x)}{\tau-x}\psi(\tau)d\tau \qquad (x < a_2), \tag{9}$$

respectively.

Furthermore, Sarikaya *et al.* [13] noticed that the generalized fractional integrals given by Definition may contain some types of fractional integrals such as the Riemann-Liouville and other fractional integrals for some special choices of function ς.

Motivated by the above results and literature, the aim of this paper is to establish in the next section, a new interesting integral identity regarding Gauss-Jacobi type quadrature formula using generalized fractional integral operators. By using this identity as an auxiliary result, some new bounds with respect to Gauss-Jacobi type quadrature formula pertaining to s-convex functions will be obtain. It is pointed out that several special cases will be given from the main results for suitable choices of function inside the generalized fractional integral operators. Some basic fractional integral operators of important interest that we will investigate in details are Riemann-Liouville fractional integral operator, k-Riemann-Liouville fractional integral operator and conformable fractional integral operator.

2. Main Results

Throughout this remaining study for $a_1 < a_2$, let us denote $P = [a_1, a_2]$ with the interior $P^\circ = (a_1, a_2)$, and

$$\Phi(\tau) := \int_0^\tau \frac{\varsigma\big((a_2 - a_1)u\big)}{u} du < \infty, \quad \forall\, \tau \in [0, 1], \tag{10}$$

where ς is the function defined as in above Remark.

For establishing some new bounds integral inequalities for Gauss-Jacobi type quadrature formula, we need the following basic lemma.

Lemma Let $P \subseteq \mathbb{R}$ and assume that $\psi: P \longrightarrow \mathbb{R}$ be a continuous function on P°. Then for any fixed $p, q > 0$, we have

$$\int_{a_1}^{a_2} \left[\Phi\left(\frac{x - a_1}{a_2 - a_1}\right)\right]^p \left[\Phi\left(\frac{a_2 - x}{a_2 - a_1}\right)\right]^q \psi(x)dx$$
$$= (a_2 - a_1)\int_0^1 [\Phi(\tau)]^p\, [\Phi(1-\tau)]^q \psi\big((1-\tau)a_1 + \tau a_2\big)d\tau. \tag{11}$$

We denote

$$T_{\psi,\Phi}^{p,q}(a_1, a_2) := (a_2 - a_1)\int_0^1 [\Phi(\tau)]^p\, [\Phi(1-\tau)]^q \psi\big((1-\tau)a_1 + \tau a_2\big)d\tau. \tag{12}$$

Proof. By using (12) and changing the variable $x = (1-\tau)a_1 + \tau a_2$, we have

$$T_{\psi,\Phi}^{p,q}(a_1, a_2) = (a_2 - a_1)\int_{a_1}^{a_2} \left[\Phi\left(\frac{x - a_1}{a_2 - a_1}\right)\right]^p \left[\Phi\left(1 - \frac{x - a_1}{a_2 - a_1}\right)\right]^q \psi(x)\frac{dx}{a_2 - a_1}$$
$$= \int_{a_1}^{a_2} \left[\Phi\left(\frac{x - a_1}{a_2 - a_1}\right)\right]^p \left[\Phi\left(\frac{a_2 - x}{a_2 - a_1}\right)\right]^q \psi(x)dx,$$

which completes the proof.

Remark Taking $\varsigma(u) = u$ in above Lemma, we get the following identity:

$$\int_{a_1}^{a_2} (x - a_1)^p\, (a_2 - x)^q \psi(x)dx = (a_2 - a_1)^{p+q+1} \int_0^1 \tau^p (1-\tau)^q\, \psi\big((1-\tau)a_1 + \tau a_2\big)d\tau. \tag{13}$$

With the help of above Lemma, we have the following new results via the frame of s-convexity of function in the second sense.

Theorem Let $P \subseteq \mathbb{R}$ and assume that $\psi: P \longrightarrow \mathbb{R}$ be a continuous function on P°. If $|\psi|^{\frac{k}{k-1}}$ is s-convex function in the second sense on P for $k > 1$, then for any fixed $p, q > 0$, we have

$$\left|T_{\psi,\Phi}^{p,q}(a_1, a_2)\right| \leq (a_2 - a_1)\left[A_{\Phi}^{p,q}(k)\right]^{\frac{1}{k}}\left[\frac{|\psi(a_1)|^{\frac{k}{k-1}} + |\psi(a_2)|^{\frac{k}{k-1}}}{s+1}\right]^{\frac{k-1}{k}}, \tag{14}$$

where

$$A_{\Phi}^{p,q}(k) := \int_0^1 [\Phi(\tau)]^{kp}\,[\Phi(1-\tau)]^{kq}d\tau. \tag{15}$$

Proof. Since $|\psi|^{\frac{k}{k-1}}$ is s-convex function in the second sense on P, combining with above Lemma, Hölder's inequality and properties of the modulus, we get

$$\begin{aligned}\left|T_{\psi,\Phi}^{p,q}(a_1, a_2)\right| &\leq (a_2 - a_1)\int_0^1 [\Phi(\tau)]^{p}\,[\Phi(1-\tau)]^{q}\left|\psi\big((1-\tau)a_1 + \tau a_2\big)\right|d\tau \\ &\leq (a_2 - a_1)\left[\int_0^1 [\Phi(\tau)]^{kp}\,[\Phi(1-\tau)]^{kq}d\tau\right]^{\frac{1}{k}}\left[\int_0^1 \left|\psi\big((1-\tau)a_1 + \tau a_2\big)\right|^{\frac{k}{k-1}}d\tau\right]^{\frac{k-1}{k}} \\ &\leq (a_2 - a_1)\left[A_{\Phi}^{p,q}(k)\right]^{\frac{1}{k}}\left[\int_0^1 \left((1-\tau)^s|\psi(a_1)|^{\frac{k}{k-1}} + \tau^s|\psi(a_2)|^{\frac{k}{k-1}}\right)d\tau\right]^{\frac{k-1}{k}} \\ &= (a_2 - a_1)\left[A_{\Phi}^{p,q}(k)\right]^{\frac{1}{k}}\left[\frac{|\psi(a_1)|^{\frac{k}{k-1}} + |\psi(a_2)|^{\frac{k}{k-1}}}{s+1}\right]^{\frac{k-1}{k}},\end{aligned}$$

which completes the proof.

We point out some interesting special cases of above Theorem.

Corollary Under the assumptions of above Theorem with $\varsigma(u) = u$, we have

$$\left|T_{\psi}^{p,q}(a_1, a_2)\right| \leq (a_2 - a_1)^{p+q+1}[\beta(kp+1, kq+1)]^{\frac{1}{k}}\left[\frac{|\psi(a_1)|^{\frac{k}{k-1}} + |\psi(a_2)|^{\frac{k}{k-1}}}{s+1}\right]^{\frac{k-1}{k}}. \tag{16}$$

Corollary Under the assumptions of above Theorem with $\varsigma(u) = \frac{u^{\alpha}}{\Gamma(\alpha)}$ and $\alpha > 0$, we get

$$\left|T_{\psi}^{p,q}(a_1,a_2)\right| \leq \frac{(a_2-a_1)^{\alpha(p+q)+1}}{\Gamma^{p+q}(\alpha+1)}[\beta(k\alpha p+1,k\alpha q+1)]^{\frac{1}{k}}\left[\frac{|\psi(a_1)|^{\frac{k}{k-1}}+|\psi(a_2)|^{\frac{k}{k-1}}}{s+1}\right]^{\frac{k-1}{k}}. \quad (17)$$

Corollary Under the assumptions of above Theorem with $\varsigma(u) = \frac{u^{\frac{\alpha}{k_1}}}{k_1\Gamma_{k_1}(\alpha)}$ and $\alpha, k_1 > 0$, we obtain

$$\left|T_{\psi}^{p,q}(a_1,a_2)\right| \leq \frac{(a_2-a_1)^{\frac{\alpha(p+q)}{k_1}+1}}{\left[k_1\Gamma_{k_1}(\alpha+k_1)\right]^{p+q}}\left[\beta\left(\frac{k\alpha p}{k_1}+1,\frac{k\alpha q}{k_1}+1\right)\right]^{\frac{1}{k}}\left[\frac{|\psi(a_1)|^{\frac{k}{k-1}}+|\psi(a_2)|^{\frac{k}{k-1}}}{s+1}\right]^{\frac{k-1}{k}}. \quad (18)$$

Corollary Under the assumptions of above Theorem with $\varsigma(u) = u(a_2-u)^{\alpha-1}$, $\alpha > 0$ and $\psi(x)$ is symmetric to $x = \frac{a_1+a_2}{2}$, we have

$$\left|T_{\psi}^{p,q}(a_1,a_2)\right| \leq \frac{(a_2-a_1)^{\frac{k-1}{k}(p+q)+1}}{\alpha^{p+q}}[C^{p,q}(\alpha,k)]^{\frac{1}{k}}\left[\frac{|\psi(a_1)|^{\frac{k}{k-1}}+|\psi(a_2)|^{\frac{k}{k-1}}}{s+1}\right]^{\frac{k-1}{k}}, \quad (19)$$

where

$$C^{p,q}(\alpha,k) \coloneqq \int_{a_1}^{a_2}\left([{a_2}^{\alpha}-\tau^{\alpha}]^{kp}[{a_2}^{\alpha}-(a_1+a_2-\tau)^{\alpha}]^{kq}\right)d\tau. \quad (20)$$

Theorem Let $P \subseteq \mathbb{R}$ and assume that $\psi\colon P \longrightarrow \mathbb{R}$ be a continuous function on P°. If $|\psi|^{l}$ is s-convex function in the second sense on P for $l \geq 1$, then for any fixed $p, q > 0$, we have

$$\left|T_{\psi,\Phi}^{p,q}(a_1,a_2)\right| \leq (a_2-a_1)\left[A_{\Phi}^{p,q}(1)\right]^{\frac{l-1}{l}}\left[B_{\Phi}^{p,q}(s)|\psi(a_1)|^{l}+B_{\Phi}^{q,p}(s)|\psi(a_2)|^{l}\right]^{\frac{1}{l}}, \quad (21)$$

where

$$B_{\Phi}^{p,q}(s) \coloneqq \int_0^1 (1-\tau)^s[\Phi(\tau)]^p\,[\Phi(1-\tau)]^q d\tau \quad (22)$$

and $A_{\Phi}^{p,q}(1)$ is defined as in first Theorem for value $k = 1$.

Proof. Since $|\psi|^{l}$ is s-convex function in the second sense on P, combining with above Lemma, the well-known power mean inequality and properties of the modulus, we get

$$\left|T_{\psi,\Phi}^{p,q}(a_1,a_2)\right| \leq (a_2-a_1)\int_0^1 [\Phi(\tau)]^p\,[\Phi(1-\tau)]^q\left|\psi\big((1-\tau)a_1+\tau a_2\big)\right|d\tau$$

$$\leq (a_2-a_1)\left[\int_0^1 [\Phi(\tau)]^p\,[\Phi(1-\tau)]^q d\tau\right]^{\frac{l-1}{l}}$$

$$\times\left[\int_0^1 [\Phi(\tau)]^p[\Phi(1-\tau)]^q\left|\psi\big((1-\tau)a_1+\tau a_2\big)\right|^l d\tau\right]^{\frac{1}{l}}$$

$$\leq (a_2-a_1)\left[A_\Phi^{p,q}(1)\right]^{\frac{l-1}{l}}\left[\int_0^1 [\Phi(\tau)]^p[\Phi(1-\tau)]^q\big((1-\tau)^s|\psi(a_1)|^l+\tau^s|\psi(a_2)|^l\big)\,d\tau\right]^{\frac{1}{l}}$$

$$= (a_2-a_1)\left[A_\Phi^{p,q}(1)\right]^{\frac{l-1}{l}}\left[B_\Phi^{p,q}(s)|\psi(a_1)|^l+B_\Phi^{q,p}(s)|\psi(a_2)|^l\right]^{\frac{1}{l}},$$

which completes the proof.

We point out some interesting special cases of above Theorem.

Corollary Under the assumptions of above Theorem with $\varsigma(u)=u$, we have

$$\left|T_\psi^{p,q}(a_1,a_2)\right| \leq (a_2-a_1)^{\frac{p+q}{l}+1}[\beta(p+1,q+1)]^{\frac{l-1}{l}} \times\left[\beta(p+1,s+q+1)|\psi(a_1)|^l+\beta(q+1,s+p+1)|\psi(a_2)|^l\right]^{\frac{1}{l}}. \tag{23}$$

Corollary Under the assumptions of above Theorem with $\varsigma(u)=\frac{u^\alpha}{\Gamma(\alpha)}$ and $\alpha>0$, we get

$$\left|T_\psi^{p,q}(a_1,a_2)\right| \leq \frac{(a_2-a_1)^{\frac{\alpha(p+q)}{l}+1}}{\Gamma^{\frac{p+q}{l}}(\alpha+1)}[\beta(\alpha p+1,\alpha q+1)]^{\frac{l-1}{l}} \times\left[\beta(\alpha p+1,s+\alpha q+1)|\psi(a_1)|^l+\beta(\alpha q+1,s+\alpha p+1)|\psi(a_2)|^l\right]^{\frac{1}{l}}. \tag{24}$$

Corollary Under the assumptions of above Theorem with $\varsigma(u)=\frac{u^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ and $\alpha,k>0$, we obtain

$$\left|T_\psi^{p,q}(a_1,a_2)\right| \leq \frac{(a_2-a_1)^{\frac{\alpha(p+q)}{kl}+1}}{[k\Gamma_k(\alpha+k)]^{\frac{p+q}{l}}}\left[\beta\left(\frac{\alpha p}{k}+1,\frac{\alpha q}{k}+1\right)\right]^{\frac{l-1}{l}} \times\left[\beta\left(\frac{\alpha p}{k}+1,s+\frac{\alpha q}{k}+1\right)|\psi(a_1)|^l+\beta\left(\frac{\alpha q}{k}+1,s+\frac{\alpha p}{k}+1\right)|\psi(a_2)|^l\right]^{\frac{1}{l}}. \tag{25}$$

Corollary Under the assumptions of above Theorem with $\varsigma(u) = u(a_2 - u)^{\alpha-1}$, $\alpha > 0$ and $\psi(x)$ is symmetric to $x = \frac{a_1+a_2}{2}$, we have

$$\left|T_{\psi}^{p,q}(a_1, a_2)\right| \leq (a_2 - a_1)\left[A_{\Phi}^{p,q}(1)\right]^{\frac{l-1}{l}} \times \left[D^{p,q}(s, \alpha;\ a_1, a_2)|\psi(a_1)|^l + D^{q,p}(s, \alpha;\ a_1, a_2)|\psi(a_2)|^l\right]^{\frac{1}{l}}, \tag{26}$$

where

$$D^{p,q}(s, \alpha;\ a_1, a_2) := \frac{1}{\alpha^{p+q}} \times \int_0^1 (1-\tau)^s \left[{a_2}^{\alpha} - (\tau(a_1 - a_2) + a_2)^{\alpha}\right]^p \left[{a_2}^{\alpha} - \left((1-\tau)(a_1 - a_2) + a_2\right)^{\alpha}\right]^q d\tau. \tag{27}$$

3. Conclusion

In this paper, we found a new interesting integral identity regarding Gauss-Jacobi type quadrature formula using generalized fractional integral operators. By applied this identity as an auxiliary result, some new bounds with respect to Gauss-Jacobi type quadrature formula pertaining to s-convex functions are established. Furthermore, several special cases are deduced from the main results for suitable choices of function inside the generalized fractional integral operators, like Riemann-Liouville fractional integral operator, k-Riemann-Liouville fractional integral operator and conformable fractional integral operator. These ideas and techniques of this paper may stimulate further research in these directions for different class of convex functions for interested readers.

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Dynamics of a Biological System with Discontinuous Effects

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Abstract

Differential equations theory gives the facility needed to understand and analyze a wide range of real-world events in a number of fields. This theory has been significantly explored and extended to new kinds of differential equations, such as differential equations with piecewise constant arguments. The purpose of this study is to use differential equations with piecewise constant argument of generalized form to simulate a nonautonomous prey-predator system with Beddington-DeAngelis functional response. We also discuss some of the system's properties, such as positive invariance.

Keywords*:* Differential Equations with Piecewise Constant Argument of Generalized Form, Nonautonomous Prey-Predator Model, Positive Invariance, Functional Response.

1. Introduction and Preliminaries

Since differential equations are so closely linked to so many disciplines of mathematics, they have evolved quickly. Solving differential equations can be done in a variety of ways. Several analytical and numerical methodologies are used in these procedures. Differential equations can predict the real-world situations around us. They are used in many different fields from biology, economics, physics, chemistry and engineering. Moreover, differential equations have a great advantage to perceive many complex phenomena such as electromagnetic waves, motion of celestial bodies, absorption of vitamins by the human body, sound produced by musical instruments, movement of vehicles over land and water, traffic jams and many other problems.

The piecewise constant argument was first proposed in differential equations theory in 1980s [8, 9]. These differential equations known as differential equations with piecewise constant arguments (EPCA) have since been substantially developed [1, 15, 21, 24, 25]. Because of their wide variety of applications, this class of differential equations has attracted a lot of attention and studied extensively. In most of these studies, the present method for examining EPCA is based on reducing them to discrete equations [1, 24]. However, Akhmet has given a new and alternative method for dealing with these equations having piecewise constant arguments of generalized form (EPCAG) [2]-[4]. The results on EPCA and on EPCAG are widely available in the literature [1]-[12], [14]-[16], [18]-[25].

The generic model for EPCA is defined as follows:

$$x'(s) = f(s, x(s), x([s])) \tag{1}$$

In (1), $s \in \mathbb{R}$ and $x \in \mathbb{R}^n$ and [.] shows the largest integer function. Many exciting discoveries have been made in the EPCA theory, as well as a wide range of applications have been studied. In addition to mathematical analysis, these systems have been used to develop a variety of models in different fields [9, 14, 16, 25]. Later, Akhmet has generalized differential equations with piecewise constant arguments by using arbitrary piecewise constant functions as arguments [2]-[3]. These differential equations have been proposed by the form

$$x'(s) = f(s, x(s), x(\beta(s))) \tag{2}$$

A function x(s) is a solution to the previous equation (2) on the interval $[\theta_k, \theta_{k+1}),\ k \epsilon N$, and with the potential exception of the places $\theta_k,\ k \epsilon N$, the derivative appears everywhere. If there are one-sided derivatives. Note that $\beta(s)$ is not continuous, in fact, the equation (2) is really part of the delay differential or functional differential equations paradigm, where delays are a series of discrete functions. It is also worth mentioning that (2) is a nonautonomous equation since the delays alter with s [24]. Akhmet has developed a new strategy based on forming an analogous integral equation to study EPCAG such as (2). This development is crucial as it allows researchers to analyze systems including piecewise constant arguments, which are nonlinear as well in terms of solution values at discrete points θ_k in time, with greater precision. The momentous book of Akhmet is suggested for more information on more theoretical and practical problems [4].

If a function of x is evaluated at s and in argument [s]...[s-N], it is possible that it will be designated as a delayed type, where N is a positive number. In fact, there exist three categories which can be classified as follows [23]:
(i) If the arguments are s and $[s+1] \dots [s+N]$, the equation is of advanced kind.
(ii) The equation can be called as of mixed kind if it contains both retarded and advanced arguments.
(iii) The equation is said to be of neutral kind if the derivative of largest order occurs at t and another point.

In this paper, we aim to consider a nonautonomous prey-predator system with functional response of Beddington-DeAngelis type and also with generalized piecewise constant argument of retarded type. Before defining the issue model, some information about prey-predator systems will be given. Ordinary differential equations are utilized to tackle a variety of mathematical biology problems. There are three basic types of interactions in an ecological system. The following are the classifications for these types: (i) "Predator-Prey (+, -)", (ii) "Competition (-, -)" and (iii) "Mutualism (+, +)".

Predator-prey population dynamics are one of the most important and exciting issues in mathematical biology, as well as one of the most important topics in mathematical ecology. Lotka and Volterra developed a set of nonlinear differential equations to model the simplest example of a predator-prey system [13, 23]. This two species "prey-predator" model is known as the "Lotka-Volterra prey-predator model". The model is made up of two connected nonlinear differential equations.

The following is a generic description of a prey-predator model. Assume we have two species, whose sizes are given by $x(s)$ and $y(s)$, respectively, at a reference time s. The following differential equations may be used to identify the generic model for defining the dynamics of prey-predator populations in continuous time:

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}s} &= \varepsilon H(x) - \tau H(x, y),\\ \frac{\mathrm{d}y}{\mathrm{d}s} &= \mu H(x, y) - \omega H(y).\end{aligned} \tag{3}$$

In equation (3), x and y are the densities of prey and predators at time s. $\frac{\mathrm{d}x}{\mathrm{d}s}$ and $\frac{\mathrm{d}y}{\mathrm{d}s}$ are the instantaneous growth rates of the two species, respectively. The interaction of the two species is described by the positive real parameters $\varepsilon, \tau, \mu,$ and ω. The Lotka-Volterra equations system is a Kolmogorov model. A functional response describes the intake rate of consumers as a function of the amount of food available in a specific habitat.
In terms of prey or predator reliance, we may divide the functional response into three groups:

1. Prey-dependent functions: The phrase prey-dependent functional response alludes to how only the density of prey influences the consumption rate of predators. The most frequent forms of prey-dependent functional responses are
   (i) Lotka-Volterra type,
   (ii) Holling type I,
   (iii) Holling type II,
   (iv) Holling type III.
2. Predator-dependent functions,
3. Ratio-dependent functions.

In the literature, see, for instance, [15] and the references cited therein, results on predator-prey systems involving two species have been published employing Lotka Volterra models with ratio-dependent functional responses and introducing piecewise constant argument of generalized type into the models. For two biological models with generalized piecewise constant arguments, these conclusions include positive invariance, permanence, and other features [5]. Moreover, local stability and Hopf bifurcation were presented for the autonomous delayed predator prey system with Beddington-DeAngelis functional response [8]. The local stability of the positive equilibrium and local Hopf bifurcation were investigated using a Beddington-DeAngelis functional response predator-prey model with two delays [6]. In the present paper, we will suggest a system that is almost identical but has a generalized piecewise constant delay, and then look at positive invariance for the proposed model. Before we go into detail about our model, consider the following prey-predator scenario:

$$\begin{aligned}x' &= x(r - wx) - \frac{axy}{b + x + cy},\\ y' &= my\left(-d + \frac{ax}{b + x + cy}\right).\end{aligned} \tag{4}$$

In (4), prey and predator densities are represented by the letters x and y, respectively. The model parameters $\mathrm{r} > 0,\ \mathrm{w} > 0,\ \mathrm{a} > 0,\ \mathrm{b} > 0,\ \mathrm{c} > 0,\ \mathrm{m} > 0$ and $\mathrm{d} > 0$ all have positive values. See [17] for further information and specifics on the coefficients of system (4). There is a ratio dependent response function in this model. According to the findings, "ratio dependent models" tend to provide more acceptable dynamics. Because the bulk of these studies assume constant model parameters, the majority of the results are for models with a constant environment. This implies that the models are self contained, with all biological and environmental parameters maintaining constant across time. The majority of biological and environmental characteristics, on the other hand, change throughout time. When this is taken into account, a model must be nonautonomous, which complicates things. The nonautonomous version of (4) is given by

$$\begin{aligned} x' &= x(r(s) - w(s)x) - \frac{a(s)xy}{b(s) + x + c(s)y}, \\ y' &= m(s)y\left(-d(s) + \frac{a(s)x}{b(s) + x + c(s)y}\right). \end{aligned} \tag{5}$$

In above system (5), x, y and factors $\mathrm{r(s)} > 0,\ \mathrm{w(s)} > 0,\ \mathrm{a(s)} > 0,\ \mathrm{b(s)} > 0,\ \mathrm{c(s)} > 0,\ \mathrm{m(s)} > 0$ and $\mathrm{d(s)} > 0$ contain biological connotations that are similar to those in (4) except that the factors are now dependent on time.

We shall propose a separate analog from the Lotka-Volterra system in this work. Behavior of the biological systems over time or in equilibrium is described by the analytical or numerical solutions of the equations. The purpose of theoretical biological organization techniques is to understand how organism components interact. We will suppose that the models average growth rates, as well as other associated characteristics, change over time, and we will account for this in order to create a redesigned system. The reduction of differential equations including piecewise constant arguments to discrete equations is a common technique for examining them. The impact of inserting a generalized piecewise constant delay into a biological model will be examined in this paper.

The following model will be the major emphasis of this paper

$$\begin{aligned} x' &= x\big(r(s) - w(s)x(\beta(s))\big) - \frac{a(s)xy}{b + x + cy}), \\ y' &= m(s)y\left(-d(s) + \frac{a(s)x(\beta(s))}{b(s) + x\,(\beta(s)) + c(s)y\,(\beta(s))}\right), \end{aligned} \tag{6}$$

where $\mathrm{s} \in \mathrm{R}$, $\beta(s) = \theta_i$, if $\theta_i \leq \mathrm{s} < \theta_{i+1}, \mathrm{i} \in \mathrm{Z}$, is a function of identification, θ_i, $\mathrm{i} \in z$ is a well-ordered series of real numbers with the property $|\theta_i| \to \infty$ as $|\mathrm{i}| \to \infty$.

We will need certain hypotheses for this paper, which are listed below.

(B1) variable coefficients of the system, $\mathrm{r(s), w(s), a(s), b(s), c(s), m(s)}$ and $\mathrm{d(s)}$ are continuous and are bounded from below and above by positive constants.

(B2) The requirement $\theta_{i+1} - \theta_i \leq \theta,$, i $\in$ Z, is satisfied by a positive number θ.

2. Main Results

We will use the following definition which is identical to [18] but changed for our case as in [5, 8].

Definition 2.1: If a pair of functions $(x(s), y(s))$ satisfies the following requirements, it is a solution of equation (6) on the interval $[\theta_0, \infty)$:

(i) on $[\theta_0, \infty)$, pair of functions $(x(s), y(s))$ is continuous,
(ii) with the potential exception of the locations $\theta_i, i \geq 0$, where one-sided derivatives exist, the derivatives $x'(s)$ and $y'(s)$ exist for $t \in [\theta_0, \infty)$,
(iii) (6) holds true for $(x(s), y(s))$ on every interval $[\theta_i, \theta_{i+1}), i \geq 0$.

Since we cope with a predator-prey model given by (6), we will only look at solutions $(\mathrm{x(s)}, \mathrm{y(s)})$ where $x(\theta_0) = x_0 > 0$ and $y(\theta_0) = y_0 > 0$. Furthermore, it is assumed that the system (6) has a unique solution in the sense of Definitions 2.1 for any initial value (x_0, y_0).

We explore solutions that begin at θ_0 for the purpose of simplicity.

Lemma 2.1: Assume (B1) is true. The following system of integral equations is equivalent to system (6) with $\mathrm{x}(\boldsymbol{\theta}_0) = \mathrm{x}_0$ and $\mathrm{y}(\boldsymbol{\theta}_0) = \mathrm{y}_0$

$$x(s) = \mathrm{x}_0\, exp\left(\int_{\boldsymbol{\theta}_0}^{s} \zeta\,(t, x(t), y(t), x(\beta(t)))dt\right),$$
$$y(s) = \mathrm{y}_0\, exp\left(\int_{\boldsymbol{\theta}_0}^{s} \eta\,(t, x(\beta(t), y(\beta(t)))dt\right),$$

where $\zeta(s, x, y, z) = r(s) - w(s)z - \frac{a(s)y}{b(s)+x+c(s)y}$ and $\eta(s, x, y) = m(s)(-d(s) + \frac{a(s)x}{b(s)+x+c(s)y})$.

Proof: See Lemma 2.2.1 in [5].

Theorem 2.1: For system (6), the set $int(\mathbb{R}_+^2) = \{(x, y) \in \mathbb{R}^2 |\, \mathrm{x} > 0, \mathrm{y} > 0\}$ is positively invariant.

The result of the Theorem 2.1 is an immediate consequence of Lemma 2.1.

Assuming that requirements (B1) and (B2) are fulfilled, positive invariance for the proposed model (6) is analyzed in the following theorem.

Theorem 2.2: Let $r^l c^l > a^u,\ r^u w^u\theta < w^l, a^u m^u < 4$ and $a^l m^l > d^u m^u$. Then, the set defined by

$$\mu = \{(x,y) \in \mathbb{R}^2 | n_1 \le x \le N_1, n_2 \le y \le N_2\} \tag{7}$$

is positively invariant for system (6), where

$$n_1 = \frac{r^l c^l - a^u}{w^u c^l}, \qquad N_1 = \frac{r^u}{w^l},$$

$$n_2 = \frac{n_1(a^l m^l - d^u m^u) - b^u \mathrm{d}^u m^u}{c^u d^u m^u}, \qquad N_2 = \frac{n_1(m^u a^u - m^l \mathrm{d}^l) - m^l \mathrm{d}^l b^l}{m^l c^l d^l}.$$

Proof: Suppose that (x(s), y(s)) be the solution of (6) starting from the point $(x(\theta_0), y(\theta_0)) = (x_0, y_0)$ with

$$n_1 \le x_o \le N_1,\ n_2 \le y_0 \le N_2.$$

First, we consider the prey equation in (6). It follows from the positivity of the solutions of (6) that

$$x'(s) \le x(s)(r^u(s) - w^l x(\beta(s))), \quad s \ge \theta_0.$$

For $s \in [\theta_0, \theta_1)$,

$$x'(s) \le x(s)(r^u(s) - w^l x_0)) = w^l x(s)(N_1 - x_0),$$

which implies together with (B2) that

$$x(s) \le x_0 \exp(N_1 - x_0)(s - \theta_0) \le x_0 \exp(w^l(N_1 - x_0)\theta) = H(x_0).$$

Now, we want to find the maximum value of the continuous function $H(x_0)$ on the closed interval $[n_1, N_1]$. The supposition $r^u w^u \theta < w^l$ shows that $H'(x_0) > 0$ on the closed interval $[n_1, N_1]$. Hence $H(x_0) \le H(N_1) = N_1$. All of these discussions lead to

$$x(s) \le N_1 \text{ for } s \in [\theta_0, \theta_1) \text{ whenever } n_1 \le x_0 \le N_1. \tag{8}$$

From the prey equation in the other direction, we obtain for $s \in [\theta_0, \theta_1)$

$$x'(\mathrm{s}) \ge x(s)\left(r^l - w^u x_0 - \frac{a^u}{c^l}\right) = w^u x(s)(n_1 - x_0),$$

which clearly implies that

$$x(s) \geq x_0 \exp\big(w^u(n,-x_0)(s-\theta)\big) \geq x_0 \exp(w^u(n_1-x_0)\,\theta) = h(x_0).$$

By using the same way that we have used for $H(x_0)$, we find that the function $h(x_0)$ attains its minimum value at n_1. This means that $h(x_0) \geq h(n_1)= n_1$ on the interval $[n_1, N_1]$. Thus,

$$x(s) \geq n_1 \text{ for } s \in [\theta_0, \theta_1) \text{ whenever } n_1 \leq x_0 \leq N_1. \tag{9}$$

When we combine (8) and (9), we obtain

$$n_1 \leq x_0 \leq N_1 \Rightarrow n_1 \leq x(s) \leq N_1 \text{ for } s \in [\theta_0, \theta_1). \tag{10}$$

This, together with the continuity of $x(s)$ implies that $n_1 \leq x(\theta_1) = x_1 \leq N_1$. Hence, when the same technique used for the interval $[\theta_0, \theta_1)$ is repeated for $s \in [\theta_1, \theta_2)$ it can be easily seen that

$$n_1 \leq x_1 \leq N_1 \Rightarrow n_1 \leq x(s) \leq N_1 \text{ for } s \in [\theta_1, \theta_2).$$

This now implies that $s_1 \leq x(\theta_2) = x \leq N_1$. Continuing the process on each interval $[\theta_i, \theta_{i+1}), i = 1,2,3, ...$, in a similar manner, we can conclude that

$$n_1 \leq x_1 \leq N_1 \Rightarrow n_1 \leq x(s) \leq N_1 \text{ for every } s \geq \theta_0.$$

From the predator equation in (6) and the positiveness of $y(s)$, we have

$$y'(s) < y(s)\left(-m^l d^l + \frac{m^u a^u y\big(\beta(s)\big)}{b^l + x\big(\beta(s)\big) + c^l y\big(\beta(s)\big)}\right), \quad s \geq \theta_0$$

For $s \in [\theta_0, \theta_1)$, this inequality takes the form

$$\begin{aligned} y'(\mathrm{s}) \leq y(s)\left(-m^l d^l + \frac{m^u a^u y_0}{b^l + x_0 + c^l y_0}\right) &\leq \left(-m^l d^l + \frac{m^u a^u N_1}{b^l + N_1 + c^l y_0}\,(N_2 - y_0)\right) \\ = \frac{c^l m^l d^l y(s)}{b^l + N_1 + C^l q_0}(N_2 - y_0), \end{aligned}$$

which produces

$$\begin{aligned} y'(s) \leq y(s) \exp\left(\frac{c^l m^l d^l}{b^l + N_1 + C^l y_0}(N_2 - y_0)\,(\mathrm{s} - \theta_0)\right] &\leq y(s) \exp\left[\frac{c^l m^l d^l}{b^l + N_1 + C^l y_0}(N_2 - y_0)\,\theta\right) \\ = \mathrm{B}\,(y_0). \end{aligned}$$

By using the supposition $a^u m^u \leq 4$ we find that the derivative of the function B (y_0) is always positive. That being the case B $(y_0) \leq B(N_2) = N_2$ on the interval $[n_2\ , N_2]$. Then,

$$y(s) \le N_2 \text{ for } s \in [\theta_1, \theta_2) \text{ whenever } n_2 \le y_0 \le N_2. \tag{11}$$

Next, we continue with the other direction of the predator equation on the interval $[\theta_1, \theta_2)$:

$$y'(s) \ \ge\ y(s)\,[-\,\mathrm{d}^u m^u \ +\ \frac{a^l m^l x_0}{b^u + x_0 + c^u y_0}] \ge\ [-\,\mathrm{d}^u m^u \ +\ \frac{a^l m^l n_1}{b^u + n_1 + c^u y_0}] = \frac{c^u d^u m^u y(s)}{b^u + n_1 + c^u y_0}\,(n_2 - y_0).$$

and through this inequality we get to

$$y'(s) \ \ge\ y_0 \exp\,[\frac{c^u d^u m^u}{b^u + n_1 + c^u y_0}\,(n_2 - y_0)\,(\mathrm{s} -\ \theta_0)]\,) \ \ge\ y_0 \exp\,[\frac{c^u d^u m^u}{b^u + n_1 + c^u y_0}\,(n_2 - y_0)\,\theta] = \mathrm{b}(\mathrm{y}_0).$$

For the reason that $a^L m^L \theta_1 < a^u m^u \theta < 4$, we get $\mathrm{b}(\mathrm{y}_0) \ \ge\ b(n_2) \ =\ n_2$ on the interval $[n_2\ \ , N_2]$ by simply evaluating $\mathrm{b}'(\mathrm{y}_0)$. Consequently, we have

$$y(s) \ \ge\ n_2 \quad for \quad s \in [\theta_0, \theta_1) \quad \text{whenever} \quad n_2 \le y_0 \le N_2. \tag{12}$$

From (11) and (12), it follows that

$$n_2 \le y_0 \le N_2 \ \ \Rightarrow\ \ n_2 \le y(s) \le N_2 \text{ for } s \in [\theta_0, \theta_1).$$

Since y(s) is continuous, we can construct the desired result on each interval $[\theta_i, \theta_{i+1})$, $i = 1,2,3, \ldots$, following the same way discussed previously for x(s). That is to say,

$$n_2 \le y_0 \le N_2 \ \ \Rightarrow\ \ n_2 \le y(s) \le N_2 \text{ for all } s \ \ge\ \theta_0.$$

3. Conclusion

In this paper, we concentrated on the prey and predator systems and proposed a different parallel to the Lotka-Volterra system. We investigated the dynamical behavior of a biological system using differential equations with piecewise constant arguments of generalized form and a functional response of Beddington-DeAngelis type. We address positive invariance for the resultant system by including the piecewise constant argument of generalized type.

4. References

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Fixed point results for (α,μ,φ)-generalized Meir-Keeler contractions on quasi 2-normed spaces

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Abstract

In this paper, there are proved some results which garantee the existence and the uniqueness of a common fixed point for generalized Meir-Keeler contraction, using α-admissible and μ-subadmissible mappings using a comparison function in quasi 2-normed spaces. The main theorem generalizes some known results.

Keywords: Meir-Keeler contraction, quasi 2-normed space, comparison function, fixed point

1. Introduction

The normed spaces are generalized to 2-normed spaces by Gahler [1]. Many authors have worked in these spaces [2], [3]. Furthermore, some topological properties and fixed point theorems are studied by many authors ([4], [5],[7]), in 2-normed spaces.

In 2006, 2-normed spaces are extended to quasi 2-normed spaces by Park [8]. The study of fixed point in quasi two normed space have been the object of some research papers [9,10].
Meir-Keleer [11] in 1969 introduced a new contraction and proved the existence of a fixed point in metric spaces. Samet et al. [12] presented the concept of α-admissible mappings and generalized many known contractions. Many researchers ([13], [14],…[20]) have worked on these contractions creating new ones and they have studied fixed point of them.

Inspired by above mentioned works, in this paper we present a new generalized Meir-Keleer contraction and prove a theorem for common fixed point.

2. Preliminaries

Definition 2.1[1] Let E be a linear space with $\dim E \geq 2$. The function $\|.,.\|: E \to R^+$ is called 2-norm, if it satisfies the following conditions:

(1) $\|x,y\| = 0$ if and only if the vectors $\{x,y\}$ are dependent in E;
(2) For every $(x,y) \in E^2$, $\|x,y\| = \|y,x\|$;
(3) For every $(\alpha,x,y) \in R \times E^2$, $\|\alpha x,y\| = |\alpha|\|x,y\|$;
(4) For all $(x,y,z) \in E^3$, $\|x+y,z\| \leq \|x,z\| + \|y,z\|$.

The couple $(E,\|.,.\|)$ is called a quasi 2-normed space.

Definition 2.2[8] Let E be a linear space with $\dim E \geq 2$. The function $\|.,.\|: E \rightarrow R^{+}$ is called a quasi 2-norm, if it satisfies the following conditions:

(1) $\|x,y\| = 0$ if and only if the vectors $\{x,y\}$ are dependent in E;
(2) For every $(x,y) \in X^2$, $\|x,y\| = \|y,x\|$;
(3) For every $(\alpha,x,y) \in R \times X^2$, $\|\alpha x,y\| = |\alpha|\|x,y\|$;
(4) There exists $s \geq 1$, such that for all $(x,y,z) \in E^3$, $\|x+y,z\| \leq s(\|x,z\| + \|y,z\|)$.

The couple $(E,\|.,.\|)$ is called quasi 2-normed space.

Example 2.3 Let E be a linear space with $\dim E \geq 2$ and $(p_i)_{i=1}^{2}$ are two norms defined in E and $s \geq 1$. The function $\|\cdot,\cdot\|: E^2 \rightarrow R^{+}$ such that $\|x_1,x_2\| = s \cdot p_1(x_1) \cdot p_2(x_2)$ is a quasi 2-norm. The pair $(E,\|.,.\|)$ is called quasi 2-normed space.

Definition 2.4 [8] A sequence $\{x_k\}_{k\in N}$ in a quasi 2-normed space $(E,\|.,.\|)$ is said to be a Cauchy sequence if for every $\varepsilon > 0$, there exists $p \in N$, such that for every $k,l \in N, k.l > p, \|x_k - x_l, e\| < \varepsilon$, where $e \in E$. (It is denoted $\lim\limits_{k,l\rightarrow+\infty}\|x_k - x_l, e\| = 0$.)

Definition 2.5 [8] Let $(E,\|.,.\|)$ be a quasi 2-normed space. The sequence $\{x_k\}_{k\in N}$ in E is called convergent to $x \in E$, if for every $\varepsilon > 0$, there exists $p \in N$, such that for every $k \in N, k > p, \|x_k - x, e\| < \varepsilon$, where $e \in E$.

Definition 2.6 [8] The quasi 2-normed space $(E,\|.,.\|)$ is called complete if every Cauchy sequence in E is convergent in E. It is called quasi 2-Banach space.

Definition 2.7 [12] Let $(E,\|.,.\|)$ be a quasi 2-normed space and $\alpha: E \times E \rightarrow R$ be a map. The function $T: E \rightarrow E$ is called α-admissible if for every $(x,y) \in E^2$, $\alpha(x,y) \geq 1$ then $\alpha(Tx,Ty) \geq 1$.

Definition 2.8 [20] Let $(E,\|.,.\|)$ be a quasi 2-normed space and $\mu: E \times E \rightarrow R$ be a map. The function $T: E \rightarrow E$ is called μ-subadmissible if for every $(x,y) \in E^2$, $\mu(x,y) \leq 1$ then $\mu(Tx,Ty) \leq 1$.

Theorem 2.9 [11] Let (X,d)be a metric space and $T: E \rightarrow E$ a function. If the function Tsatisfies the following implication: *for every* $\varepsilon > 0$, *there exist* $\delta > 0$ *such that* $\varepsilon \leq d(x,y) < \varepsilon + \delta$ *implies* $d(Tx,Ty) < \varepsilon$, then it has a unique fixed point in X.

Definition 2.10 Let $(E,\|.,.\|)$ be a quasi 2-normed space and $\alpha: E \times E \rightarrow R$ be a map. The pair of functions $f: E \rightarrow E$ and $g: E \rightarrow E$ is called α-admissible if for every $(x,y) \in E^2$, $\alpha(x,y) \geq 1$ then $\alpha(fx,gy) \geq 1$.

Example 2.11 Let $E = R^3$ and $\alpha: E \times E \rightarrow R$, $\alpha(x,y) = \begin{cases} 5, & x \neq y \\ 0, & x = y \end{cases}$ and $f: E \rightarrow E$, $f(x) = \frac{x}{3}$, $g: E \rightarrow E$, $g(x) = 2x$. The pair of functions f and g is α-admissible.

Definition 2.12 Let $(E, \|.,.\|)$ be a quasi 2-normed space and $\mu: E \times E \to R$ be a map. The pair of functions $T: E \to E$ and $S: E \to E$ is called μ-subadmissible if for every $(x, y) \in E^2$, $\mu(x, y) \leq 1$ then $\mu(Tx, Sy) \leq 1$.

Example 2.13 Let $E = R^3$ and $\mu: E \times E \to R, \mu(x, y) = \begin{cases} \frac{1}{3}, & x \neq y \\ 0, & x = y \end{cases}$ and $f: E \to E$, $f(x) = \frac{2x}{3}$, $g: E \to E$, $g(x) = -x$. The pair of functions f and g is α-admissible.

Definition 1.14 [21] The function $\varphi: R^+ \to R^+$ is called a comparison function if it satisfies the following conditions:

a) φ is a nondeacreasing function;
b) For every $t > 0$, $0 < \varphi(t) < t$;
c) For each $t > 0$, $\lim_{n \to +\infty} \varphi^n(t) = 0$.

The set of comparison function is denoted by Φ.

3. Main results

Let $(E, \|.,.\|)$ be a quasi 2-normed space and $f: E \to E$ and $g: E \to E$ be two functions.

Denote $M_{f,s} = \max\{\|x - y, e\|, \|x - fx, e\|, \|y - gy, e\|, \frac{\|x - fy, e\| + \|y - fx, e\|}{2s}\}$ and $M_{f,g,s} = \max\{\|x - y, e\|, \|x - fx, e\|, \|y - gy, e\|, \frac{\|x - gy, e\| + \|y - fx, e\|}{2s}\}$, for every $x, y \in E$ and $s > 1$

Definition 3.1 Let $(E, \|.,.\|)$ be a quasi 2-normed space, $\alpha: E \times E \to R$, $\mu: E \times E \to R$ two maps. The function $f: E \to E$ is said that completes the (α, μ, φ)-generalized Meir-Keeler contraction if it satisfies the following implication:

for every $\varepsilon > 0$, *there exist* $\delta > 0$ *such that* $\varepsilon \leq \mu(x, y)\varphi(M_{f,s}(x, y)) < \varepsilon + \delta$ *implies* $\alpha(x, y)\|fx - fy, e\| < \varepsilon$

for every $x, y \in X$ and $\varphi \in \Phi$.

Definition 3.2 Let $(E, \|.,.\|)$ be a quasi 2-normed space, $\alpha: E \times E \to R$, $\mu: E \times E \to R$ two maps. The couple of functions $f: E \to E$ and $g: E \to E$ is said that completes the (α, μ, φ)-generalized Meir-Keeler contraction if it satisfies the following implication:

for every $\varepsilon > 0$, *there exists* $\delta > 0$ *such that* $\varepsilon \leq \mu(x, y)\varphi(M_{f,g,s}(x, y)) < \varepsilon + \delta$ *implies* $\alpha(x, y)\|fx - gy, e\| < \varepsilon$

for every $x, y \in X$ and $\varphi \in \Phi$.

Theorem 3.3 Let $(E, \|.,.\|)$ be a quasi 2-normed space with $s > 1$. If $f, g: E \to E$ two function that satisfies (α, μ, φ)-generalized Meir-Keeler contraction and f is continuous then there exits a unique common fixed point for f and g.

Proof. Let x_0 be an arbitrary point from the quasi 2-normed space E. Denote $x_{2n} = fx_{2n-1}$ and $x_{2n+1} = gx_{2n}$.

If there exists any $p \in N$ such that $x_{2p} = x_{2p+1}$, we have $\|x_{2p+1} - x_{2p+2,}e\| = \|gx_{2p} - fx_{2p+1,}e\| \leq \alpha(x_{2p}, x_{2p+1})\|fx_{2p+1} - gx_{2p,}e\| < \varepsilon \leq \mu(x_{2p}, x_{2p+!})\varphi\left(M_{f,g,s}(x_{2p+1}, x_{2p})\right) \leq \varphi\left(M_{f,g,s}(x_{2p+1}, x_{2p})\right)$

$$\begin{aligned}
M_{f,g,s}&(x_{2p+1}, x_{2p}) \\
&= \max\{\|x_{2p+1} - x_{2p,}e\|, \|x_{2p+1} - fx_{2p+1,}e\|, \|x_{2p} \\
&- gx_{2p,}e\|, \frac{\|x_{2p+1} - gx_{2p,}e\| + \|x_{2p} - fx_{2p+1,}e\|}{2s}\} \\
&= \max\{\|x_{2p+1} - x_{2p,}e\|, \|x_{2p+1} - x_{2p+2,}e\|, \|x_{2p} \\
&- x_{2p+1,}e\|, \frac{\|x_{2p+1} - x_{2p+1,}e\| + \|x_{2p} - x_{2p+2,}e\|}{2s}\} \\
&= \max\{\|x_{2p+1} - x_{2p+1,}e\|, \|x_{2p+2} - x_{2p+1,}e\|, \frac{\|x_{2p} - x_{2p+1,}e\| + \|x_{2p+1} - x_{2p+2,}e\|}{2}\} \\
&= \|x_{2p+2} - x_{2p+1,}e\|
\end{aligned}$$

As a result we take the inequality $\|x_{2p+1} - x_{2p+2,}e\| < \varphi(\|x_{2p+1} - x_{2p+2,}e\|)$, which contradicts the condition 2 of the comparison function. It remains that $\|x_{2p+1} - x_{2p+2,}e\| = 0$ and $x_{2p+1} = x_{2p+2}$. Consequently, the sequence $\{x_n\}_{n\in N}$ is constant and x_0 is the common fixed point of the pair of functions f and g.

Suppose that $x_{2n} \neq x_{2n+1}$ for each $n \in N$.

$$\begin{aligned}
\|x_{2n} - x_{2n+1,}e\| = \|fx_{2n-1} - gx_{2n,}e\| &\leq \alpha(x_{2n-1}, x_{2n})\|fx_{2n-1} - gx_{2n,}e\| < \varepsilon \\
&\leq \mu(x_{2n-1}, x_{2n})\varphi\left(M_{f,g,s}(x_{2n-1}, x_{2n})\right) \leq \varphi\left(M_{f,g,s}(x_{2n-1}, x_{2n})\right)
\end{aligned}$$

$$\begin{aligned} &M_{f,g,s}(x_{2n-1},x_{2n}) \\ &\qquad = \max\{\|x_{2n-1}-x_{2n,}e\|, \|x_{2n-1}-fx_{2n-1,}e\|, \|x_{2n} \\ &\qquad - gx_{2n,}e\|, \frac{\|x_{2n-1}-gx_{2n,}e\| + \|x_{2n}-fx_{2p-1,}e\|}{2s}\} \\ &\qquad = \max\{\|x_{2n-1}-x_{2n,}e\|, \|x_{2n-1}-x_{2n,}e\|, \|x_{2n} \\ &\qquad - x_{2n+1,}e\|, \frac{\|x_{2n-1}-x_{2n+1,}e\| + \|x_{2n}-x_{2n,}e\|}{2s}\} \\ &\qquad = \max\{\|x_{2n-1}-x_{2n,}e\|, \|x_{2n}-x_{2n+1,}e\|, \frac{\|x_{2n-1}-x_{2n+1,}e\|}{2s}\} \end{aligned}$$

Case 1. If we take $M_{f,g,s}(x_{2n-1},x_{2n}) = \|x_{2n-1}-x_{2n,}e\|$, the inequality $\|x_{2n}-x_{2n+1,}e\| \le \varphi(\|x_{2n-1}-x_{2n,}e\|)$ holds. Using the condition 2 of the comparison function φ we have:

$$\|x_{2n}-x_{2n+1,}e\| < \varphi(\|x_{2n-1}-x_{2n,}e\|) \le \varphi\left(\varphi(\|x_{2n-3}-x_{2n-2,}e\|)\right) \le \cdots \le \varphi^{2n}(\|x_1-x_{0,}e\|).$$

Case 2. Taking $M_{f,g,s}(x_{2n-1},x_{2n}) = \|x_{2n}-x_{2n+1,}e\|$, we have $\|x_{2n}-x_{2n+1,}e\| < \varphi(\|x_{2n}-x_{2n+1,}e\|)$ which is not true. As a result Case 2 does not hold.

Case 3.We note that $M_{f,g,s}(x_{2n-1},x_{2n}) =$

$$= \frac{\|x_{2n-1}-x_{2n+1,}e\|}{2s} \le \frac{s(\|x_{2n-1}-x_{2n,}e\| + \|x_{2n}-x_{2n+1,}e\|)}{2s} = \frac{\|x_{2n-1}-x_{2n,}e\| + \|x_{2n}-x_{2n+1,}e\|}{2}$$
$$\le \max\{\|x_{2n-1}-x_{2n,}e\|, \|x_{2n}-x_{2n+1,}e\|\}$$

which results that we are in *Case* 1 and *Case* 2.

From the above relations we have that for every $n \in N$, $\|x_n - x_{n+1,}e\| < \varphi^n(c)$ where $c = \|x_0 - x_{1,}e\|$.

Let prove that the sequence $\{x_n\}_{n\in N}$ is Cauchy. Taking $n,k \in N$, we have

$$\|x_{n+k}-x_{n,}e\| \le s(\|x_{n+k}-x_{n+1,}e\| + \|x_{n+1}-x_{n,}e\|) \le s^2(\|x_{n+k}-x_{n+2,}e\| + \|x_{n+2}-x_{n+1,}e\|) + s\|x_{n+1}-x_{n,}e\| \le s^k\|x_{n+k}-x_{n+k-1,}e\| + s^{k-1}\|x_{n+k-1}-x_{n+k-2,}e\| + \cdots + s\|x_{n+1}-x_{n,}e\| \le s^k\varphi^{n+k-1}(c) + s^{k-1}\varphi^{n+k-2}(c) + \cdots + s\varphi^n(c) = s\varphi^n(c)(s^{k-1}\varphi^{k-1}(c) + \cdots + 1) = s\varphi^n(c)\frac{1-(s\varphi(c))^k}{1-s\varphi(c)}$$

Taking the limit of both sides in $\|x_{n+k}-x_{n,}e\| \le s\varphi^n(c)\frac{1-(s\varphi(c))^k}{1-s\varphi(c)}$ when $n,k \to +\infty$ we have that

$\lim_{n,k\to+\infty}\|x_{n+k}-x_{n,}e\| = 0$. As a result the sequence $\{x_n\}_{n\in N}$ is Cauchy.

Since the 2-normed space $(E, \|.,.\|)$ is complete, the Cauchy sequence $\{x_n\}_{n\in N}$ converges to a point $x^\bullet \in X$.

Firstly, we prove that $x^\bullet$ is a fixed point for the function f, $x^\bullet = fx^\bullet$. Using the continuity of f we obtain

$\lim_{n\to+\infty}\|x^\bullet - x_{2n}, e\| = \lim_{n\to+\infty}\|x^\bullet - fx_{2n-1}, e\| = \|x^\bullet - fx^\bullet, e\|$ and $\lim_{n\to+\infty}\|x^\bullet - x_{2n}, e\| = 0$.

It yields that $\|x^\bullet - fx^\bullet, e\| = 0$ and $x^\bullet = fx^\bullet$.

Secondly, we assure that $x^\bullet$ is also a fixed point for the function g, $x^\bullet = gx^\bullet$.

Let suppose that $x^\bullet \neq gx^\bullet$, which means that $\|x^\bullet - gx^\bullet, e\| \neq 0$

$$\begin{aligned}\|x^\bullet - gx^\bullet, e\| &= \|fx^\bullet - gx^\bullet, e\| \leq \alpha(x^\bullet, x^\bullet)\|fx^\bullet - gx^\bullet, e\| < \varepsilon \leq \mu(x^\bullet, x^\bullet)\varphi(M_{f,g,s}(x^\bullet, x^\bullet)) \\ &\leq \varphi(M_{f,g,s}(x^\bullet, x^\bullet))\end{aligned}$$

$$\begin{aligned}M_{f,g,s}(x^\bullet, x^\bullet) &= \max\left\{\|x^\bullet - x^\bullet, e\|, \|x^\bullet - fx^\bullet, e\|, \|x^\bullet - gx^\bullet, e\|, \frac{\|x^\bullet - fx^\bullet, e\| + \|x^\bullet - gx^\bullet, e\|}{2s}\right\} \\ &= \|x^\bullet - gx^\bullet, e\|\end{aligned}$$

$$\|x^\bullet - gx^\bullet, e\| < \varphi(\|x^\bullet - gx^\bullet, e\|)$$

which contradicts the condition 2 of the comparison function. It yields that $x^\bullet = gx^\bullet$.

Consequently, $x^\bullet$ is a common fixed point of f and g, $fx^\bullet = x^\bullet = gx^\bullet$.

Finally, we prove that the fixed point $x^\bullet$ of f and g, is unique.

Suppose that there exists another point $x^{\bullet\bullet} \in E$ such that $x^\bullet \neq x^{\bullet\bullet}$, and $fx^{\bullet\bullet} = x^{\bullet\bullet} = gx^{\bullet\bullet}$.

$$\begin{aligned}\|x^\bullet - x^{\bullet\bullet}, e\| &= \|fx^\bullet - gx^{\bullet\bullet}, e\| \leq \alpha(x^\bullet, x^{\bullet\bullet})\|fx^\bullet - gx^{\bullet\bullet}, e\| < \varepsilon \leq \mu(x^\bullet, x^{\bullet\bullet})\varphi(M_{f,g,s}(x^\bullet, x^{\bullet\bullet})) \\ &\leq \varphi(M_{f,g,s}(x^\bullet, x^{\bullet\bullet}))\end{aligned}$$

$$\begin{aligned}M_{f,g,s}(x^\bullet, x^{\bullet\bullet}) &= \max\left\{\|x^\bullet - x^{\bullet\bullet}, e\|, \|x^\bullet - fx^{\bullet\bullet}, e\|, \|x^\bullet - gx^{\bullet\bullet}, e\|, \frac{\|x^{\bullet\bullet} - fx^\bullet, e\| + \|x^\bullet - gx^{\bullet\bullet}, e\|}{2s}\right\} \\ &= \|x^\bullet - x^{\bullet\bullet}, e\|\end{aligned}$$

$$\|x^\bullet - x^{\bullet\bullet}, e\| < \varphi(\|x^\bullet - x^{\bullet\bullet}, e\|)$$

which is a contradiction. It remains that $x^\bullet = x^{\bullet\bullet}$.

Theorem 3.4 Let $(E, \|.,.\|)$ be a quasi 2-normed space with $s > 1$. If function $f: E \to E$ satisfies (α, μ, φ)-generalized Meir-Keeler contraction and f is continuous then there exits a unique fixed point for f.

Proof. Taking the function $g: E \to E$, $gx = fx$ for each $x \in X$, we are in the conditions of Theorem 2.1, which garantees that f has a unique fixed point $x^{\bullet} \in X$, $fx^{\bullet} = x^{\bullet}$.

Corollary 3.5 Let $(E, \|.,.\|)$ be a quasi 2-normed space with $s > 1$. If function $f: E \to E$ satisfies the following condition:

for every $\varepsilon > 0$, *there exist* $\delta > 0$ *such that* $\varepsilon \leq \mu(x,y)\varphi(N_{f,s}(x,y)) < \varepsilon + \delta$ *implies* $\alpha(x,y)\|fx - fy, e\| < \varepsilon$, for every $x, y \in X$ and $\varphi \in \Phi$, where $N_{f,s}(x,y) = \max\{\|x - y, e\|, \frac{\|x-fx,e\|+\|y-fy,e\|}{2s}, \frac{\|x-fy,e\|+\|y-fx,e\|}{2s}\}$, then it has a unique fixed point in X.

Proof. Since $N_{f,s}(x,y) \leq M_{f,s}(x,y)$ for every $x, y \in X$ then we are in conditions of Theorem 2.2.

Corollary 3.6 Let $(E, \|.,.\|)$ be a quasi 2-normed space with $s > 1$. If function $f: E \to E$ satisfies the following condition:

for every $\varepsilon > 0$, *there exist* $\delta > 0$ *such that* $\varepsilon \leq \varphi(M_{f,s}(x,y)) < \varepsilon + \delta$ *implies* $\alpha(x,y)\|fx - fy, e\| < \varepsilon$, for every $x, y \in X$ and $\varphi \in \Phi$, then it has a unique fixed point in X.

Proof. Taking $\mu(x,y) = 1$ for every $x, y \in X$, f has a unique fixed point from Theorem 2.2.

Corollary 3.7 Let $(E, \|.,.\|)$ be a quasi 2-normed space with $s > 1$. If function $f: E \to E$ satisfies the following condition

for every $\varepsilon > 0$, *there exist* $\delta > 0$ *such that* $\varepsilon \leq \varphi(\|x - y, e\|) < \varepsilon + \delta$ *implies* $\|fx - fy, e\| < \varepsilon$, for every $x, y \in X$ and $\varphi \in \Phi$,

then it has a unique fixed point in X.

Proof. If we take $\alpha(x,y) = 1$ for every $x, y \in X$ and since $\|x - y, e\| \leq M_{f,s}(x,y)$ then it easy to prove that the function f has a unique fixed point.

Example 3.8 Let $(E, \|.,.\|)$ be the quasi 2-normed space given in Example 2.3. Define $\alpha: E \times E \to R$, $\alpha(x,y) = \begin{cases} 5, & x \neq y \\ 0, & x = y \end{cases}$, $\mu: E \times E \to R$, $\mu(x,y) = \begin{cases} \frac{1}{3}, & x \neq y \\ 0, & x = y \end{cases}$, $\varphi: R^{+} \to R^{+}, \varphi(t) = \frac{t}{2}$. Let take the function $f: E \to E$, $f(x) = \frac{x}{42}$.

Suppose that for every $\varepsilon > 0$, there is $\delta > 0$ such that $\varepsilon \leq \mu(x,y)\varphi(\|x-y,e\|) < \varepsilon+\delta$. Taking $\delta = \frac{\varepsilon}{6}$, we have $\varepsilon \leq \frac{1}{6}\|x-y,e\| < \frac{7}{6}\varepsilon$.

We see that $\alpha(x,y)\|fx-fy,e\| = 6\left\|\frac{x}{42}-\frac{y}{42},e\right\| = \frac{6}{7}\left(\frac{1}{6}\cdot\|x-y,e\|\right) < \frac{6}{7}\cdot\frac{7}{6}\varepsilon = \varepsilon$. Since we are in condition of Corollary 3.7, f has a unique fixed point, the vector $x = 0$.

Conclusions In this paper are presented and proved some new results on generalized Meir-Keleer contractions. Theorem 3.3 is the highlight of the paper where there are proved the existence and uniqueness of a common fixed point for a pair of functions that satisfies a new generalized Meir-Keleer condition in which a comparison function is used. Theorem 3.4 assures that a function under a new generalized Meir-Keller contraction has a unique fixed point. Some important corollaries are obtained.

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Curvatures of the Astro-Rotational Hypersurfaces

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Abstract

In this work, we introduce and examine differential geometric properties of the astro-rotational hypersurface which its profile curve has astroid curve in the four dimensional Euclidean space $\mathbb{E}^4$. We reveal the curvatures $\mathfrak{C}_{i=1,2,3}$ of the astro-rotational hypersurface, that is the first (ie. the mean) curvature $\mathfrak{C}_1$, the second curvature $\mathfrak{C}_2$, the third (ie. the Gaussian) curvature $\mathfrak{C}_3$. Moreover, projecting the astro-rotational hypersurface into 3-spaces, we indicate surfaces, visually.

Keywords: 4-space, curvature, Gauss map, astro-rotational hypersurface.

1. Introduction

We find following papers about hyper-surfaces in the literature: Arslan et al. [1], Ganchev and Milousheva [3], Güler et al. [5,7], Güler and Turgay [8], and also some books: Eisenhart [2], Gray et al. [4], Hacısalihoglu [9], Nitsche [10].

In this paper, we introduce the astro-rotational hypersurface in Euclidean 4-space $\mathbb{E}^4$. We give the fundamental notions of the four dimensional Euclidean geometry in Section 2. In Section 3, we define rotational hypersurface. We obtain astro-rotational hypersurface, and calculate its curvatures in the last section.

2. Preliminaries

In $\mathbb{E}^{n+1}$, to find the i-th curvature formulas $\mathfrak{C}_i$, where $i = 0,1,\ldots,n$, we use characteristic polynomial of shape operator $\mathbf{S}$:

$$P_{\mathbf{S}}(\lambda) = 0 = \det(\mathbf{S} - \lambda I_n) = \sum_{k=0}^{n} (-1)^k \, s_k \lambda^{n-k},$$

where I_n denotes the identity matrix of order n. Then, we get curvature formulas $\binom{n}{i}\mathfrak{C}_i = s_i$. Here,

$\binom{n}{0}\mathfrak{C}_0 = s_0 = 1$ by definition. k-th fundamental form of hypersurface M^n is defined by

$$\mathrm{I}(\mathbf{S}^{k-1}(X), Y) = \langle \mathbf{S}^{k-1}(X), Y \rangle.$$

Therefore, we have

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} \mathfrak{C}_i \, \mathrm{I}(\mathbf{S}^{k-1}(X), Y) = 0.$$

In the rest of this paper, we shall identify a vector $(\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d})$ with its transpose $(\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d})^t$.

Let $\mathbf{x} = \mathbf{x}(u, v, w)$ be an isometric immersion of a hypersurface M^3 in $\mathbb{E}^4$. Inner product of vectors $\vec{x} = (x_1, x_2, x_3, x_4)$ and $\vec{y} = (y_1, y_2, y_3, y_4)$ in $\mathbb{E}^4$ is given by as follows

$$\langle \vec{x}, \vec{y} \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4.$$

Vector product $\vec{x} \times \vec{y} \times \vec{z}$ of $\vec{x} = (x_1, x_2, x_3, x_4)$, $\vec{y} = (y_1, y_2, y_3, y_4)$, $\vec{z} = (z_1, z_2, z_3, z_4)$ in $\mathbb{E}^4$ is defined by as follows

$$\vec{x} \times \vec{y} \times \vec{z} = \det \begin{pmatrix} e_1 \, e_2 \, e_3 \, e_4 \\ x_1 x_2 x_3 x_4 \\ y_1 y_2 y_3 y_4 \\ z_1 \, z_2 \, z_3 \, z_4 \end{pmatrix}.$$

The Gauss map of a hypersurface $\mathbf{x}$ is given by

$$e = \frac{\mathbf{x}_u \times \mathbf{x}_v \times \mathbf{x}_w}{\|\mathbf{x}_u \times \mathbf{x}_v \times \mathbf{x}_w\|}, \tag{2.1}$$

where $\mathbf{x}_u = d\mathbf{x}/du$. For a hypersurface $\mathbf{x}$ in $\mathbb{E}^4$, we have

$$\det\mathrm{I} = \det \begin{pmatrix} E & F & A \\ F & G & B \\ A & B & C \end{pmatrix} = (EG - F^2)C - EB^2 + 2FAB - GA^2,$$

$$\det\mathrm{II} = \det \begin{pmatrix} L & M & P \\ M & N & T \\ P & T & V \end{pmatrix} = (LN - M^2)V - LT^2 + 2MPT - NP^2,$$

$$\det\mathrm{III} = \det \begin{pmatrix} X & Y & O \\ Y & Z & R \\ O & R & S \end{pmatrix} = (XZ - Y^2)S - XR^2 + 2YOR - ZO^2.$$

Here, the coefficients are given by

$$E = \langle \mathbf{x}_u, \mathbf{x}_u \rangle,\ F = \langle \mathbf{x}_u, \mathbf{x}_v \rangle,\ G = \langle \mathbf{x}_v, \mathbf{x}_v \rangle,\ A = \langle \mathbf{x}_u, \mathbf{x}_w \rangle,\ B = \langle \mathbf{x}_v, \mathbf{x}_w \rangle,\ C = \langle \mathbf{x}_w, \mathbf{x}_w \rangle,$$

$$L = \langle \mathbf{x}_{uu}, e \rangle,\quad M = \langle \mathbf{x}_{uv}, e \rangle,\quad N = \langle \mathbf{x}_{vv}, e \rangle,\quad P = \langle \mathbf{x}_{uw}, e \rangle,\quad T = \langle \mathbf{x}_{vw}, e \rangle,\quad V = \langle \mathbf{x}_{ww}, e \rangle,$$

$$X = \langle e_u, e_u \rangle,\quad Y = \langle e_u, e_v \rangle,\quad Z = \langle e_v, e_v \rangle,\quad O = \langle e_u, e_w \rangle,\quad R = \langle e_v, e_w \rangle,\quad S = \langle e_w, e_w \rangle,$$

and e is the Gauss map (i.e. the unit normal vector field).

Next, we will obtain the fourth fundamental form matrix for a hypersurface $\mathbf{x}(u, v, w)$ in $\mathbb{E}^4$. Using characteristic polynomial $P_{\mathbf{S}}(\lambda) = a\lambda^3 + b\lambda^2 + c\lambda + d = 0$, we obtain curvature formulas: $\mathfrak{C}_0 = 1$ (by definition),

$$\mathfrak{C}_1 = -\frac{b}{\binom{3}{1}a},\ \ \mathfrak{C}_2 = \frac{c}{\binom{3}{2}a},\ \ \mathfrak{C}_3 = -\frac{d}{\binom{3}{3}a}. \tag{2.2}$$

Theorem 3.1. For *any hypersurface* M^3 *in* $\mathbb{E}^4$, *the fundamental forms and the curvatures are related by*

$$\mathfrak{C}_0\mathrm{IV} - 3\mathfrak{C}_1\mathrm{III} + 3\mathfrak{C}_2\mathrm{II} - \mathfrak{C}_3\mathrm{I} = 0.$$

Proof. See [6] for details.

3. Rotational Hypersurfaces

We introduce a kind of rotational hypersurface which its profile curve has astroid curve in the four dimensional Euclidean space $\mathbb{E}^4$.

$\gamma: I \longrightarrow \Pi$ be a space curve for an open interval $I \subset \mathbb{R}$, and let ℓ be a line in Π. A rotational hypersurface is defined as a hypersurface rotating a curve γ profile curve around axis ℓ in $\mathbb{E}^4$.

We may suppose that ℓ is the line spanned by the vector $(0, 0, 0, 1)^t$. The orthogonal matrix which fixes the above vector is

$$A(v, w) = \begin{pmatrix} \cos v \cos w & -\sin v & -\cos v \sin w & 0 \\ \sin v \cos w & \cos v & -\sin v \sin w & 0 \\ \sin w & 0 & \cos w & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $v, w \in \mathbb{R}$.

The matrix A can be found by solving the following equations, simultaneously,

$$A\ell = \ell,\ \ A^t A = AA^t = \mathcal{I}_4,\ \ det\ A = 1.$$

When the axis of rotation is ℓ, there is an Euclidean transformation by which the axis ℓ transformed to the axis x_4 of $\mathbb{E}^4$. Parametrization of the profile space curve γ is given by

$$\gamma(u) = \big(f(u), 0, 0, g(u)\big),$$

where $f(u), g(u)\colon I \subset \mathbb{R} \longrightarrow \mathbb{R}$ are differentiable functions for all $u \in I$. Hence, the rotational hypersurface spanned by the vector $(0, 0, 0, 1)$ is given by

$$R(u, v, w) = A(v, w).\gamma(u)^t,$$

where $0 \leq v, w < 2\pi$.

Therefore, we can re-write rotational hypersurface as follows

$$R(u, v, w) = \begin{pmatrix} f(u)\cos v \cos w \\ f(u)\sin v \cos w \\ f(u)\sin w \\ g(u) \end{pmatrix}.$$

4. Astro-Rotational Hypersurfaces

Now, by using rotational matrix in $\mathbb{E}^4$, and profile curve γ with translation vector on axis x_4, we find rotational hypersurface which has astroid curve. Resulting hypersurface that we called it as the astro-rotational hypersurface $\mathcal{R}(u, v, w)$.

Considering the astroid curve in $\mathbb{E}^4$

$$\gamma(u) = (a\cos^3 u, 0, 0, a\sin^3 u),\ \ a \in \mathbb{R},$$

we calculate the Gauss map, and also find the curvatures $\mathfrak{C}_{i=1,2,3}$ of the astro-rotational hypersurface.

We also draw some figures of the astro-rotational hypersurface, and its Gauss map with projection from four dimensional Euclidean space to the three dimensional Euclidean space.

In $\mathbb{E}^4$, astro-rotational hypersurface $\mathcal{R}(u, v, w)$ spanned by the vector $(0, 0, 0, 1)$, is defined by as follows

$$\mathcal{R}(u,v,w)=\begin{pmatrix} a\cos^3 u\cos v\cos w \\ a\cos^3 u\sin v\cos w \\ a\cos^3 u\sin w \\ a\sin^3 u \end{pmatrix}. \tag{4.1}$$

Taking $w = \pi/4$ in (4.1), we have projection surface into 3-space as in Figure 1.

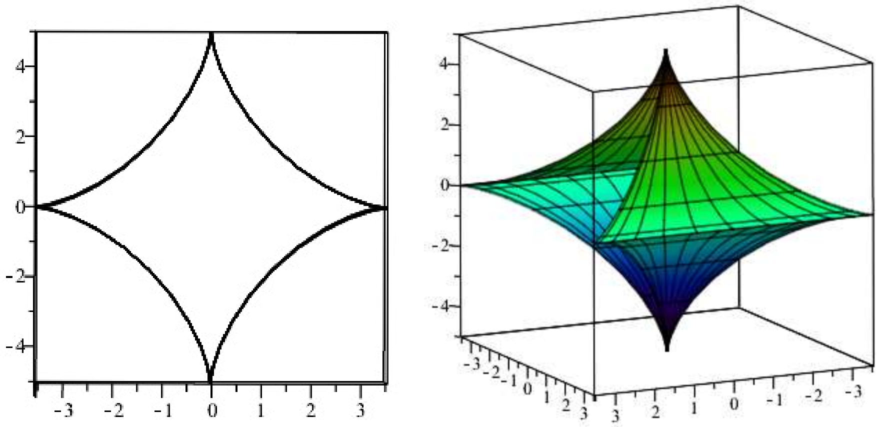

Figure 1. Projection of $\mathcal{R}(u,v,w)$ into $x_1x_2x_4$-space,
Left: front view, **Right**: side view,

Using the first differentials of (4.1) with respect to u, v, w, we get the first quantities as follows

$$\mathrm{I}=\begin{pmatrix} 9a^2\sin^2 u\cos^2 u & 0 & 0 \\ 0 & a^2\cos^6 u\cos^2 w & 0 \\ 0 & 0 & a^2\cos^6 u \end{pmatrix}$$

where

$$\det \mathrm{I} = 9a^6\sin^2 u\cos^{14} u\cos^2 w.$$

Using the Gauss map formula (2.1) on (4.1), we have the Gauss map of the astro-rotational hypersurface (4.1), as follows

$$e=\begin{pmatrix} \sin u\cos v\cos w \\ \sin u\sin v\cos w \\ \sin u\sin w \\ \cos u \end{pmatrix}.$$

Computing the second differentials of (4.1) with respect to u, v, w, we have the second quantities as follows

$$\mathrm{II} = \begin{pmatrix} 3a \sin u \cos u & 0 & 0 \\ 0 & -a \sin u \cos^3 u \cos^2 w & 0 \\ 0 & 0 & -a \sin u \cos^3 u \end{pmatrix}.$$

Theorem 4.1. *The astro-rotational hypersurface* (4.1) *in* $\mathbb{E}^4$ *has the following curvature formulas,* $\mathfrak{C}_0 = 1$ *(by definition),*

$$\mathfrak{C}_1 = \frac{7\cos^2 u - 6}{9a \sin u \cos^3 u},$$

$$\mathfrak{C}_2 = \frac{-5\cos^2 u + 3}{9a^2 \cos^6 u},$$

$$\mathfrak{C}_3 = \frac{\sin u}{3a^3 \cos^7 u}.$$

Proof. Computing eqs. (2.2) on (4.1), we have the curvatures.

5. Conclusion

Astro-rotational hypersurfaces have not been worked, recently. We have extended some well-known results of the astro-helicoidal hypersurfaces with the help of [6].

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Robust parallel solver for computational continuum mechanic problems

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Abstract

This paper discusses possible ways of computational technology development for segregated/coupled solving the systems of nonlinear partial differential equations in black-box software. These systems describe physical and chemical processes in the continuum mechanics approximation (multiphysics) [1]. The following requirements for the black-box numerical methods are formulated:
– robustness (the least number of problem-dependent components);
– efficiency (close-to-optimal algorithmic complexity);
– parallelism (faster than the best sequential algorithm).

Keywords:High-performance and parallel computing, mathematical modelling, black-box software.

1. Introduction

Mathematical modeling of physical and chemical processes has always been an important activity in science and engineering. Now a scientist or engineer cannot understand all details of the mathematical model formulation, numerical algorithms, parallel computing technologies and parallel supercomputer architectures. This fact has been motivated the black-box software development.

To some extent, attempts to automate mathematical modeling have already been exploited in the first black-box computational fluid dynamics (CFD) code. In 1978, great English scientist Brian Spalding conceived the idea of a single CFD code capable of handling all fluid-flow processes. Consequently, Concentration Heat and Momentum Ltd (CHAM) abandoned the policy of developing individual application-specific CFD codes, and during late 1978 the company began creating the world's first general-purpose CFD code PHOENICS, which is an acronym for **P**arabolic, **H**yperbolic **O**r **E**lliptic **N**umerical **I**ntegration **C**ode **S**eries. The initial creation of PHOENICS was largely the work of Brian Spalding and Harvey Rosten, and the code was launched commercially in 1981, and so here for the first time, a single CFD code was to be used for all thermo-fluids problems. Clearly, for reasons of robustness, each general-purpose code should be based on a robust computational technique for solving a wide class of nonlinear (initial-)boundary value problems of continuum mechanics.

CFD software has become an essential modeling tool to study and validate flow problems in engineered systems. Many computer-aided engineering (CAE) programs for CFD are available with different capabilities, making it difficult to select the best program for a specific application. Really modern codes are collections of common building blocks and diagnostic tools, helping users to develop their own application-specific software without having to start from scratch. Users will, therefore, need a basic knowledge of numerical methods.

We define software to be black-box if it does not require any additional input from the user apart from the physical problem specification consisting of the domain geometry, boundary and initial conditions, source terms, the enumeration of equations to be solved (heat conductivity equation, Navier–Stokes equations, Maxwell equations, etc.) and mediums. The user does not need to know anything about numerical methods or high-performance and parallel computing. Aim of this book is to relate our experience in the development of robust (the least number of problem-dependent components), efficient (close-to-optimal algorithmic complexity) and parallel (faster than the fastest sequential algorithm) computational technique in the black-box solution of multidimensional nonlinear (initial-)boundary value problems of computational physics [2].

Goal of this paper is to analyze opportunity of creating a robust computational technology for numerical solution of the nonlinear initial-boundary continuum mechanics problems that we know how to solve.

2. Auxiliary Space Method

The basic idea of the Auxiliary Space Method is to transfer a nonlinear problem to an auxiliary space (grid) where it is simple to solve. The solution in the auxiliary space is then transferred back to the original space. The mismatch between the auxiliary space and the full space is corrected by applying a few smoothing iteration steps [3].

Let us formulate a basic single-grid Gauss-Seidel-type algorithm for the numerical solution of discretized linear problems

$$A_0\varphi_0 = b_0 .$$

The linear two-grid algorithm takes the matrix form

$$b_0 - A_0\varphi_0^{(q+1)} = M\left(b_0 - A_0\varphi_0^{(q)}\right),$$

where the iteration matrix M becomes

$$M = A_0 S_0^{\nu}\left(A_0^{-1} - P_{A\to 0}A_A^{-1}R_{0\to A}\right).$$

Here $P_{A\to 0}$ and $R_{0\to A}$ are the intergrid operators (Fig. 1), S_0 is the smoothing iteration matrix, ν and q are the smoothing and intergrid iteration counters, respectively.

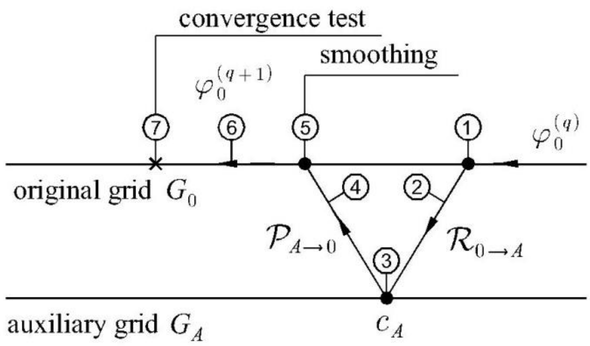

Figure 1. Linear intergrid iteration of the Auxiliary Space Method [5].

Summarize main properties of the two-grid algorithm [5]:
a) If smoothing and approximation property hold, then a fixed (and small) number of appropriate smoothing steps is sufficient to guarantee excellent h-independent convergence

$$\rho_q \le C_A \eta(\nu) < 1,$$

where ρ_q is an average reduction factor of the residual defined by

$$\rho_q = \left(\frac{\left\| b_0 - A_0 \phi_0^{(q)} \right\|}{\left\| b_0 - A_0 \phi_0^{(0)} \right\|} \right)^{1/q}$$

Here C_A is h-independent constant, $\eta(\nu)$ is a monotonically decreasing function.
b) In the two-grid algorithm, the original system $A_0\varphi_0 = b_0$ is replaced by an equivalent equation

$$P^{-1}A_0\varphi_0 = P^{-1}b_0$$

where $P^{-1} = A_0^{-1}(I-M)$ is invertible matrix.
c) If original grid is structured, then the iteration matrix M becomes

$$M = A_0 S_0^{\nu}\left(A_0^{-1} - A_A^{-1}\right).$$

If $A_A = A_0$, then the two-grid algorithm transforms to direct single-grid solver ($S = I$).
d) In general, the two-grid algorithm has three extra problem-dependent components (the number of smoothing iterations and the intergrid operators).

2. Robust Multigrid Technique

We will always choose an auxiliary grid to be structured boundary-(un)fitted grid and choose Robust Multigrid Technique (RMT) to solve the auxiliary discrete problem [2,3,5]. The multigrid schedule of RMT is the V-cycle with no presmoothing (a sawtooth cycle). The sawtooth cycle is a special case of the V-cycle, in which smoothing before the coarse grid correction (pre-smoothing) is deleted (Fig. 2 and 3). The cycle is also directly related to the memory requirements.

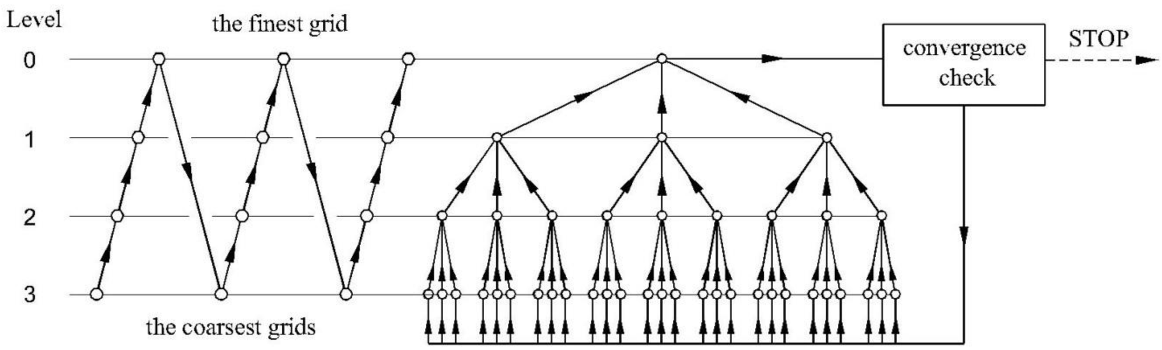

Figure 2. Sequential multigrid cycle of RMT

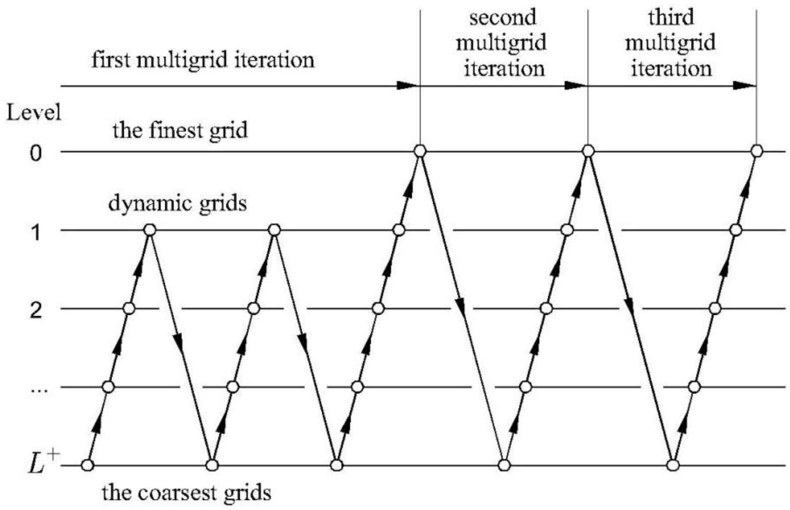

Figure 3. Parallel multigrid cycle of RMT

RMT can be represented as the iterative method

$$b_0 - A_0\varphi_0^{(q+1)} = A_0 Q_0 \left(b_0 - A_0\varphi_0^{(q)}\right),$$

where the matrix Q_0 is defined in recurrence form

$$Q_0 = \begin{cases} S_l^{\nu_l}(d_l R_{0\to l} + P_{l+1\to l} Q_{l+1}), & l = 0,1,2,\dots,L_3^+ - 2 \\ S_l^{\nu_l} d_l R_{0\to l}, & l = L_3^+ - 1 \end{cases},$$

$$d_l = A_l^{-1} - P_{l+1\to l} A_{l+1}^{-1} R_{l\to l+1},$$

where S_l is the smoothin iteration matrix ($||S_l|| \leq 1$), ν_l is the number of smoothing iterations on grids of the level l, L_3^+ is the coarsest grid level obtained by the triple coarsening. If smoothing and approximation property hold, then a fixed (and small) number of appropriate smoothing steps is sufficient to guarantee excellent h-independent convergence

$$\rho_q \leq ||A_0 Q_0|| \leq C_A \eta(\nu_0) + C_A C_R \sum_{l=1}^{L_3^+-1} C^l \eta(\nu_l) < 1$$

Note that the additional amount of work compared to standard geometric multigrid is proportional to the number of grid levels.

3. Conclusion

If the original computational grid is structured, then it is possible to develop RMT-based single-grid algorithm having h-independent convergence and $O(n_b^{-2} N_G^{3+k/d} N_M^3)$ arithmetic operations. This algorithm has extra problem-dependent component: the number of smoothing iterations. If the original computational grid is unstructured, then it is possible to develop RMT-based two-grid algorithm having h-independent convergence and $O(n_b^{-2} N_G^3 N_M^3 \log N_G^{1/d})$ arithmetic operations. This algorithm has three extra problem-dependent components: the number of smoothing iterations and intergrid transfer operators.

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Analyzing the performance of French and German car brands by using a new simultaneous analysis method

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Abstract

We analyze the data of French car brands which constitute the first multi-table and the German car brands, the second multi-table, by using a duale data analysis method : CONCORGS1D. This method is an extension of CONCORGS1 method in the context of two horizontal multi-tables measured on the same set of individuals. Each brand is made up of several models and these car models from two vertical multi-tables measured on the same number of variables.

Keywords : CONCORGS, Dual method, Horizontal multi-table, Vertical multi-table.

1. Introduction

When it comes to establishing the link between two paired multi-arrays in line, the CONCORG method of Kissita et al (2004) [5] which is the extension of the CONCOR method of Lafosse and Hanafi (1997) [10], has made it possible to formalize this link. The idea is to establish the dependency ratios that have certain tables which constitute the first multi-table with other tables which constitute the second multi-table. In addition, another method like CONCORGS1 of Kissita et al. (2009a) [7] has also established this link. These methods have been applied in many fields such as: sensory analysis and ecology (cf. Kissita et al (2004)) [5], genomics, metabolomics and proteomics. However, when we have two groups of partitioned individuals measured on the same set of variables, for example, when we retrace on the same macroeconomic variables the evolution of the macroeconomic situation between two monetary communities, the dual methods of the direct methods that are proposed in this work make it possible to establish the proximity relationships that several tables of the first multi-table have with other tables of the second multi-table while describing the internal structures of each table. These methods are the first to formalize this type of link.

This work is organized as follows: in section (2), after having defined the notations and the data, we will present the different direct methods allowing to establish the links between two multi-arrays having the same number of lines (cf. Figure (1)). In section (3), we will propose the dual methods of the methods presented in section (2). Finally, in section (4), we will apply this method to French and German car brands.

2. The link factor analysis methods between two sets of variables paired in rows

In this section, we will first present the data and notations then the methods of factor analysis of link between two sets of partitioned variables having the same individuals (called direct methods).

2.1 **Data and ratings**

A' denotes the transpose of the matrix A: We consider M + N statistical triples (X_i;Q_{X_i};D) and (Y_j;Q_{Y_j};D) where X_i (i = 1,…,M) of dimension n x P_{X_i} and Y_j (j = 1,….,N) of dimension n x P_{Y_j} (cf. Figure (1)), are the tables defined respectively on the products I x $J_i^X$and I x J_j^Y. We denote by n the cardinal of I, P_i^X the cardinal of J_i^X and P_j^Y that of J_j^Y. Let Q_i^X (respectivly Q_j^Y) of dimension P_i^X x P_i^X (respectively P_j^Y x P_j^Y) be the metric defined on the space of individuals $R^{P_i^X}$(respectively $R^{P_j^Y}$). Q_{bd}^X= diag(Q_i^X/i=1,…,M) (respectively Q_{bd}^Y= diag(Q_j^Y/j = 1,…,N)) of dimension P^X x P^X (respectively P^Y x P^Y) is the block-diagonal metric defined in R^{P^X}(respectively R^{P^Y}) with $P^X = \sum_{i=1}^{M} P_i^X$ (respectively P^Y=$\sum_{j=1}^{N} P_j^Y$) is the total number of columns in the table X (respectively Y) and D of dimension n x n is a metric of the weights of individuals over R^n, with for example D = (1/n)Id_n where Id_n is the identity matrix of order n.

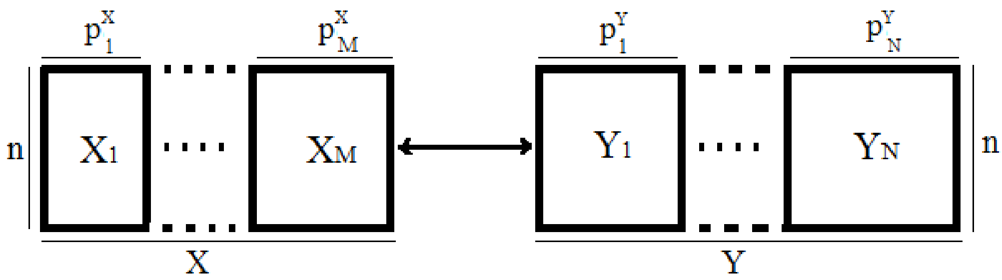

Figure 1 - Two horizontal multi-tables

X = [X_1,…, X_i ,...,X_M] (respectively Y= [Y_1,…,Y_j,...,Y_N]) is a horizontal multi-tables of dimension n x P^X (respectively n x P^Y), we assume that X_i and Y_j are centered with respect to D and possibly reduced. If X is a matrix of dimension n x P^Y, vec(X) is the column vector, formed by the superposition of the columns of X.

$P^{\perp}_{\underline{a}_{X_i}}$=$Id_{P_i^X}$-$P_{\underline{a}_{X_i}}$ is the projector Q_i^X-orthogonal on the subspace generated by the orthogonal of the vector $\underline{a}_{X_i}$ in $R^{P_i^X}$ which is normalized with respect to Q_i^X.

$V_{X_iY_j}$=X_i'DY_j of dimension P_i^X x P_j^Y the matrix inter-covariances between the variables of X_i and those of Y_j.

The methods that are presented in this section are based on the operator of the inter-covariance matrix between two tables. It allows to establish pairs of links between two tables having the same individuals.

2.2 The CONCORGS1 methods

This method was proposed by Kissita et al [7].The CONCORGS1 method consists of searching for orthogonal matrices. $A_X = [A'_{X_1}/\ldots/A'_{X_i}/\ldots/A'_{X_M}]'$ and $A_Y = [A'_{Y_1}/\ldots/A'_{Y_j}/\ldots/A'_{Y_N}]'$ of respective dimensions P^X x r and P^Y x r where A_{X_i}=[$\underline{a}_{X_i}^{(1)}/\ldots/\underline{a}_{X_i}^{(s)}/\ldots/\underline{a}_{X_i}^{(r)}$] (respectively A_{Y_i}=[$\underline{a}_{Y_i}^{(1)}/\ldots/\underline{a}_{Y_i}^{(s)}/\ldots/\underline{a}_{Y_i}^{(r)}$]) a matrix of dimension P_i^X x r (respectively P_i^Y x r), which maximize three criteria which are equal to the optimum and equal to the CONCORG criteria when r = 1, with r ≤ min(P_i^X,P_j^Y), i=1,…,M and j=1,…,N. The first criterion consists in maximizing the function

$$g_1(A_X,A_Y)=||\mathrm{diag}(A'_X Q_{bd}^X V_{XY} Q_{bd}^Y A_Y)||^2 \quad (1)$$

under normalization constraints $A'_X Q_{bd}^X A_X = A'_Y Q_{bd}^Y A_Y = Id_r$. The second criterion maximizes

$$g_2(A_X,A_Y)=\sum_{i=1}^{M} ||\mathrm{diag}(A'_{X_i} Q_i^X V_{X_iY} Q_{bd}^Y A_Y)||^2 \quad (2)$$

under normalization constraints $A'_{X_i} Q_i^X A_{X_i} = A'_Y Q_{bd}^Y A_Y = Id_r$ for all i=1,…,M. And the third maximizes

$$g_3(A_X,A_Y)=\sum_{j=1}^{N} ||\mathrm{diag}(A'_X Q_{bd}^X V_{XY_j} Q_j^Y A_{Y_j})||^2 \quad (3)$$

under normalization constraints $A'_X Q_{bd}^X A_X = A'_{Y_j} Q_j^Y A_{Y_j} = Id_r$, for all j=1,…,N. The solution of this method is given by an iterative and convergent algorithm (cf Kissita (2003)) [7].

3 The duale method

In this section, we propose the dual method of the CONCORGS1 methods presented in section (2) in the context where we measure on two groups of partitioned individuals the same set of variables.

3.1 Data and ratings

In the context of figure (2), we denote (X_i, Q, D_i^X) and (Y_j, Q, D_j^Y), the M + N studies where X_i (i=1,…,M) is an array of dimension n_i^X x p and Y_j (j =1,…,N) is an array of dimension n_j^Y x p defined respectively on the products I_i^X x J and I_j^Y x J. The set of variables being the same for all arrays. We denote by n_i^X the cardinal of I_i^X, by n_j^Y the cardinal of I_j^Y and by p the cardinal of J. Let's also note Q of dimension p x p the metric defined on the space of individuals R^p.
D_i^X of the dimension n_i^X x n_i^X is the metric defined on $R^{n_i^X}$ and D_j^Y of the dimension n_j^Y x n_j^Y is the metric defined on $R^{n_j^Y}$.

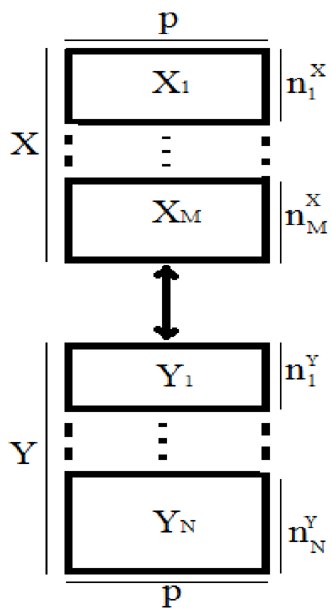

Figure 2- Two vertical multi-tables

D_{bd}^X= diag(D_i^X/i=1,…,M) of dimension n^X x n^X is the block-diagonal metric of individuals defined in R^{n^X} of metrics D_i^X where $n^X = \sum_{i=1}^{M} n_i^X$ is the total number of rows in table X.
D_{bd}^Y= diag(D_j^Y/j=1,…,N) of dimension n^Y x n^Y is the block-diagonal metric of individuals defined in R^{n^Y} of metrics D_j^Y where $n^Y = \sum_{j=1}^{N} n_j^Y$ is the total number of rows in table Y.
The vertical table X is the superposition of the tables X_i which are centered and possibly reduced.
It is the same for the vertical table Y and the tables Y_j.
$P_{\underline{u}_{X_i}}^{\perp}$=$Id_{n_i^X}$-$P_{\underline{u}_{X_i}}$ is the projector D_i^X-orthogonal on the subspace generated by the orthogonal of the vector $\underline{u}_{X_i}$ in $R^{n_i^X}$ which is normalized with respect to D_i^X.

$W_{X_iY_j}$=X_iQY_j' of the dimension n_i^X x n_j^Y is the matrix of inter - scalar products between the individuals of the tables X_i and those of the tables Y_j. This operator is dual of the inter-covariance matrix $V_{X_iY_j}$=$X_i'DY_j$ between tables X_i and Y_j defined above. It is the basis of dual methods.

3.2 The CONCORGS1D method

The analysis of the dual simultaneous Generalized Concordance 1 (CONCORGS1D) allows to formalize the proximities between the individuals of the two vertical multi-tables X and Y (cf. Figure(2)). It consists in determining the orthogonal matrices $U_X = [U'_{X_1}/\ldots/U'_{X_i}/\ldots/U'_{X_M}]'$ and $U_Y = [U'_{Y_1}/\ldots/U'_{Y_j}/\ldots/U'_{Y_N}]'$ of respective dimensions n^X x r and n^Y x r where $U_{X_i} = [\underline{u}_{X_i}^{(1)}/\ldots/\underline{u}_{X_i}^{(s)}/\ldots/\underline{u}_{X_i}^{(r)}]$ respectively U_{Y_i}=$[\underline{u}_{Y_i}^{(1)}/\ldots/\underline{u}_{Y_i}^{(s)}/\ldots/\underline{u}_{Y_i}^{(r)}]$) a matrix of dimension n_i^X x r (respectively n_i^Y x r), which maximize three criteria which are equal to the optimum and equal to the CONCORG criteria when r = 1, with r ≤ min(n_i^X,n_j^Y), i $= 1, \ldots,$ M and j $= 1, \ldots,$ N. The first criterion consists in maximizing the function

$$l_1(U_X,U_Y)=||\mathrm{diag}(U_X' D_{bd}^X W_{XY} D_{bd}^Y D_Y)||^2 \qquad (4)$$

under normalization constraints $D_X' D_{bd}^X U_X$=$U_Y' D_{bd}^Y U_Y$=Id_r. The second criterion maximizes

$$l_2(U,U_Y)=\sum_{i=1}^{M} ||\mathrm{diag}(U_{X_i}' D_i^X W_{X_iY} D_{bd}^Y U_Y)||^2 \qquad (5)$$

under normalization constraints $U_{X_i}' D_i^X U_{X_i}$=$U_Y' D_{bd}^Y U_Y$=Id_r for all i=1,…,M. And the third maximizes

$$l_3(U_X,U_Y)=\sum_{j=1}^{N} ||\mathrm{diag}(U_X' D_{bd}^X W_{XY_j} D_j^Y U_{Y_j})||^2 \qquad (6)$$

under normalization constraints $U_X' D_{bd}^X U_X$=$U_{Y_j}' D_j^Y U_{Y_j}$=Id_r, for all j=1,…,N. With respect to the theorem of Cliff, N (1966) [1] we set T_X=$(D_{bd}^X)^{1/2}U_X$, T_Y=$(D_{bd}^Y)^{1/2}U_Y$ and K_{XY}=$(D_{bd}^X)^{1/2}W_{XY}(D_{bd}^Y)^{1/2}$ with

$$E_{T_XT_Y} = K_{XY}T_Y\mathrm{diag}(T_X' K_{XY} T_Y) \qquad (7)$$

then, the first criterion becomes

$$l_4(U_X,U_Y)=\mathrm{tr}((T_X'K_{XY}T_Y)\mathrm{diag}(T_X'K_{XY}T_Y)), \quad (8)$$

under standardization constraints $T_X'T_X=T_Y'T_Y=Id_r$. The algorithm allowing to solve this problem compared to the first criterion (7) is presented as follows:

1. Randomly choose the matrices T_X and T_Y such that $T_X'T_X=T_Y'T_Y=Id_r$ and a number ξ (we can take ξ=0.00001)

2 Determine the update of T_X by performing the following singular value decomposition:

$$K_{XY}T_Y\mathrm{diag}(T_X'K_{XY}T_Y) = \mathrm{P}\Delta S'$$

where $P'P=S'S = SS' = Id_r$ a diagonal matrix containing singular values classified in descending order and $W_{XY} = \mathrm{X}\mathrm{Q}Y'$. The update is set as follows $T_X^* = PS'$ and $U_X^* = (D_{bd}^X)^{-1/2}T_X^*$

3 Determine the update of T_Y by performing the following singular value decomposition :

$$K_{YX}T_X\mathrm{diag}(T_X'K_{XY}T_Y) = \mathrm{H}\Delta L'$$

where $H'H=L'L = LL' = Id_r$ a diagonal matrix containing singular values classified in descending order and $W_{YX} = \mathrm{Y}\mathrm{Q}X'$. The update is set as follows $T_Y^* = HL'$ and $U_Y^* = (D_{bd}^Y)^{-1/2}T_Y^*$

4 If $l_4(U_X^*, U_Y^*) - l_4(U_X, U_Y) \le$ ξ , then, the algorithm stops, otherwise go to 2.

Proof of monotonicity in relation to the (8):

Note that in this subsection, we will only show the monotonicity of the algorithm with respect to with respect to U_X. For monotony compared to U_Y, the demonstration proceeds in the same way, just exchanging the role of the matrices U_X and U_Y. The aim is to show that:

$$l_4(U_X, U_Y) \le l_4(U_X^*, U_Y) \le l_4(U_X^*, U_Y^*) \quad (9)$$

To establish this monotonicity, it is sufficient to show that

$$l_4(U_X, U_Y) \le l_4(U_X^*, U_Y) \quad (10)$$

Indeed, from relation (7), we can write

$$T_X^*E_{T_XT_Y} = T_X^*K_{XY}T_Y\mathrm{diag}(T_X'K_{XY}T_Y)$$

Considering $\underline{u}_X^{(s)}$=[$\underline{u}_{X_1}^{(s)'}$ / .../$\underline{u}_{X_i}^{(s)'}$ / .../$\underline{u}_{X_M}^{(s)'}$]′ and $\underline{u}_Y^{(s)}$=[$\underline{u}_{Y_1}^{(s)'}$ / .../$\underline{u}_{Y_j}^{(s)'}$ / .../$\underline{u}_{Y_N}^{(s)'}$]′, the s-th vectors column blocks of matrices U_X and U_Y respectively, we can derive the s-th element diagonal, then we write

$$\underline{u}_X^{(s)*} K_{XY} \underline{u}_Y^{(s)} \underline{u}_X^{(s)'} K_{XY} \underline{u}_Y^{(s)} = \underline{u}_X^{(s)*} K_{XY} \underline{u}_Y^{(s)} \underline{u}_Y^{(s)'} K_{YX} \underline{u}_X^{(s)} = \underline{u}_X^{(s)*} G_{XY_s} \underline{u}_X^{(s)} \quad (11)$$

where $G_{XY_s} = K_{XY} \underline{u}_Y^{(s)} \underline{u}_Y^{(s)'} K_{YX}$ is a positive semi definite symmetric matrix. There fore

$$\left\| G_{XY_s}^{1/2} \underline{u}_X^{(s)} - G_{XY_s}^{1/2} \underline{u}_X^{(s)*} \right\|^2 \geq 0 \quad (12)$$

Relationship (12) can be written as

$$\underline{u}_X^{(s)'} G_{XY_s} \underline{u}_X^{(s)} + \underline{u}_X^{(s)*} G_{XY_s} \underline{u}_X^{(s)*} \geq 2 \underline{u}_Y^{(s)*} K_{XY} \underline{u}_Y^{(s)} \underline{u}_X^{(s)'} K_{XY} \underline{u}_Y^{(s)} \quad (13)$$

by replacing $G_{XY_s} = K_{XY} \underline{u}_Y^{(s)} \underline{u}_Y^{(s)'} K_{YX}$ in (13), we find

$$\underline{u}_X^{(s)'*} \underline{u}_X^{(s)'} K_{XY} \underline{u}_Y^{(s)} \underline{u}_Y^{(s)'} K_{YX} \underline{u}_X^{(s)} + \underline{u}_X^{(s)*} K_{XY} \underline{u}_Y^{(s)} \underline{u}_Y^{(s)'} K_{YX} \underline{u}_X^{(s)*}$$
$$\geq 2\, \underline{u}_X^{(s)*} K_{XY} \underline{u}_Y^{(s)} \underline{u}_X^{(s)'} K_{XY} \underline{u}_Y^{(s)} \quad (14)$$

$\underline{u}_Y^{(s)'} K_{YX} \underline{u}_X^{(s)*}$ being symmetrical, we have

$$\underline{u}_X^{(s)'} K_{XY} \underline{u}_Y^{(s)} \underline{u}_X^{(s)'} K_{XY} \underline{u}_Y^{(s)} + \underline{u}_X^{(s)*} K_{XY} \underline{u}_Y^{(s)} \underline{u}_X^{(s)*'} K_{XY} \underline{u}_Y^{(s)} \geq 2\, \underline{u}_X^{(s)*} K_{XY} \underline{u}_Y^{(s)} \underline{u}_X^{(s)'} K_{XY} \underline{u}_Y^{(s)} \quad (15)$$

Summing up the relation (15) with respect to s, we find

$$\operatorname{tr}\big((U_X' K_{XY} U_Y)\operatorname{diag}(U_X' K_{XY} U_Y)\big) + \operatorname{tr}\left(\left(U_X^{*'} K_{XY} U_Y\right)\operatorname{diag}\left(U_X^{*'} K_{XY} U_Y\right)\right)$$
$$\geq 2 \operatorname{tr}\left(\left(U_X^{*'} K_{XY} U_Y\right)\operatorname{diag}(U_X' K_{XY} U_Y)\right) \quad (16)$$

Referring to Cliff (1966) [2], from the update U_X we have the relation

$$\operatorname{tr}\left(\left(U_X^{*'} K_{XY} U_Y\right)\operatorname{diag}\left(U_X^{*'} K_{XY} U_Y\right)\right) \geq \operatorname{tr}\big((U_X' K_{XY} U_Y)\operatorname{diag}(U_X' K_{XY} U_Y)\big) \quad (17)$$

Combining relations (16) and (17), we find

$$\mathrm{tr}\left(\left(U_X^{*\prime} K_{XY} U_Y\right)\mathrm{diag}\left(U_X^{*\prime} K_{XY} U_Y\right)\right) \geq \mathrm{tr}\left((U_X' K_{XY} U_Y)\mathrm{diag}(U_X' K_{XY} U_Y)\right)$$

which gives

$$l_4(U_X^*, U_Y) \geq l_4(U_Y, U_Y)$$

Finally, we have demonstrated the monotonicity of the algorithm with respect to U_X , So, it can be said that the U_X and U_Y monotonically increase the function f_1. Moreover, the function being bounded, continuous and monotonically increasing, in particular on the set formed by the normalized vectors constituted by the columns of the vectors U_X and U_Y the algorithm converges.

4 Example of application

4.1 Presentation of data

The data processed in this article was collected on the website "www.motorlegend.com". They have been grouped into two groups. The first group consists of four German brands, namely : BMW, Mercedes, Opel and Volkswagen. Each brand is composed of a number of models (Table (1)) : this is the first multi-table. The second group consists of three French brands, namely : Citroen, Peugeot and Renault. Each brand is composed of a certain number of models (Table (2)) : this is the second multi-table.

Six characteristics have been selected for all models. This finally gives two vertical multi-tables with the same number of variables as can be seen in figure (6).

4.2 Objectives

Establish the performance proximities between :

French brands, German brands, French and German brands. Know the best German and French models, the most powerful, the most powerful, the fastest, the heaviest, widest and longest.

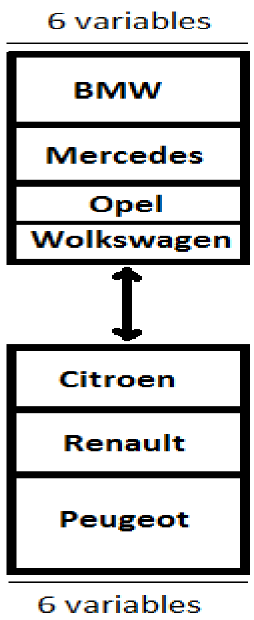

Figure 3 - Two vertical multi-tables of German and French brand data with six variables in common

TABLE 1 – List of models and acronyms of German car brands

Models	acronyms	Models	acronyms
BMW X4	B1	Opel Adam S 1.4	O1
BMW Serie4	B2	Opel Cascado 1.6 Turbo	O2
BMW M2	B3	Opel Astra GTC 2.OT	O3
BMW X5	B4	Opel Corsa 1.6 Turbo	O4
BMW Serie3	B5	Opel Insignia 1.6 Turbo	O5
BMW Serie5	B6	Opel Insignia 2.8V6 Turbo	O6
Mercedes Classe SLC43	M1	Volkswagen Polo R WRC	V1
Mercedes Classe E 220d	M2	Volkswagen Coccinelle Dune 2.OTFSI	V2
Mercedes Classe C 450	M3	Volkswagen Golf R2.OTSI 300	V3
Mercedes Classe C300	M4	Volkswagen Touareg 3.OV6TDI	V4
Mercedes Classe A 45	M5	Volkswagen Passat Alltrack 2.OTDI	V5
		Volkswagen Tiguan 2.OTDO	V6

TABLE 2 – List of models and acronyms of French car brands

Models	acronyms	Models	acronyms
Citroën DS3 1.6THPN	C1	Renault Talisma ETce	R1
Citroën DS5 BlueHDI	C2	Renault Megane ETce	R2
Citroën DS3 CabrioRacing	C3	Renault Megane Trophy	R3
Citroën DS3 1.6THPA	C4	Renault Megane Estate	R4
Citroën C4 1.6THP	C5	Renault Laguna 3.0V6 Dci	R5
Citroën DS3 Racing 1.6	C6	Renault Laguna Estate 2.0	R6
Citroën DS3 1.6 Vti	C7	Renault Spider 2.0	R7
Peugeot 308 Gti	P1		
Peugeot 508 RXH BlueHDi	P2		
Peugeot 208 GTI	P3		
Peugeot 308 1.6THP	P4		

Table 3- The eigenvalues, the inertias of each axis and the cumulative

Axes	EigenV	inertia	Cum. Inertia
1	61.0307870	52.4473163	52.44732
2	39.6300992	34.0564566	86.50377
3	8.3133133	7.1441152	93.64789
4	5.4585862	4.6908816	98.33877
5	1.2629778	1.0853505	99.42412
6	0.6701277	0.5758798	100

Table 3 shows us that the inertia explained by the first two factorial axes is 86.50. This is quite normal.

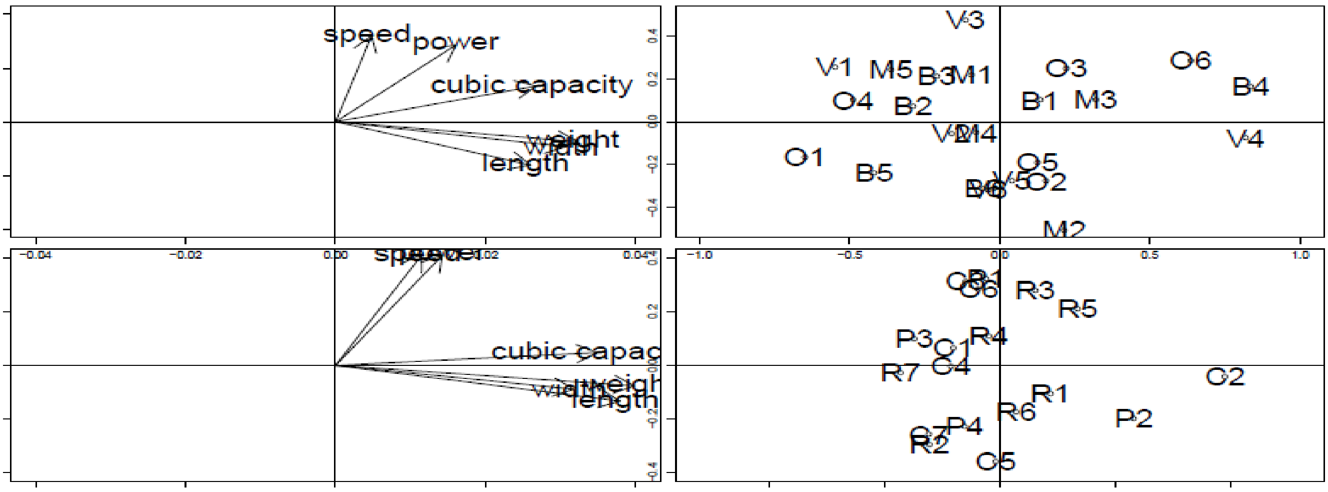

FIGURE 4 – French and German brands

In Figure 4, we have first the German models and then the French models. As far as the German models are concerned, all the models on axis 1, i.e. B4, V4 and O6, are characterised by length, width, weight and engine capacity. On the other hand, we have O1 and O4. In addition, models like V3 are characterised by power and speed. Thus, the model like B6 is much less powerful and less fast. As far as the French models are concerned, we can see that C2 and P2 located on axis 1 are characterised by length, width, weight and displacement. The model like R7 and P3 are less long, less wide and lighter. On the other hand, C3, C6, R3 and P1 located on axis 2 which is the axis of power and speed unlike C5 and P4.

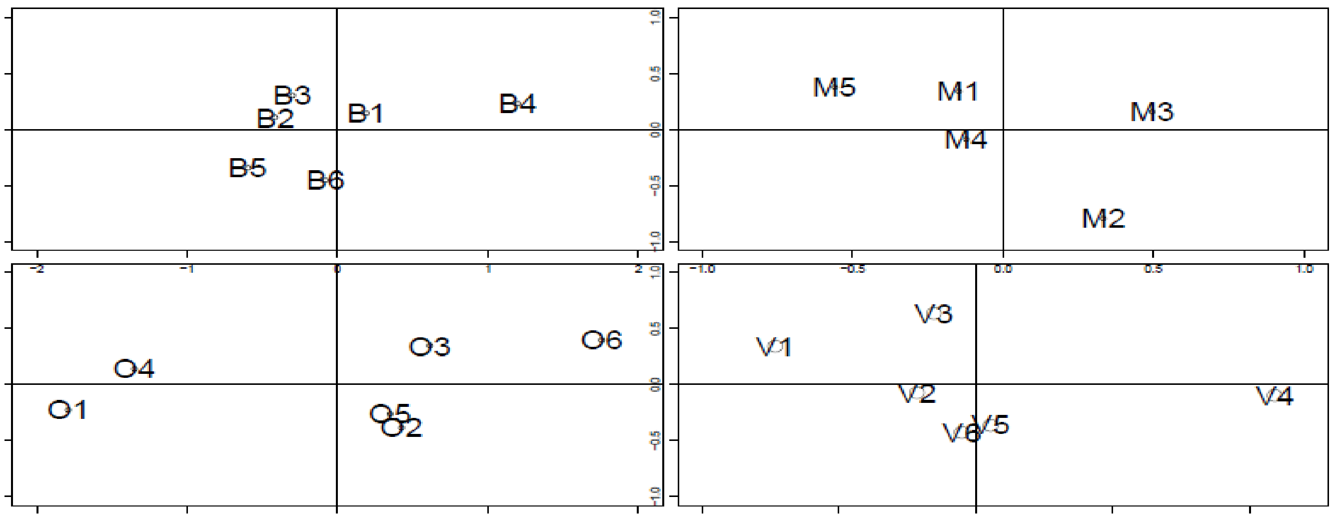

FIGURE 5 – German brands

Figures 5 and 6 show the exact positions of the models of each brand. The German models are shown (see figure 5) and the French models (see figure 6).

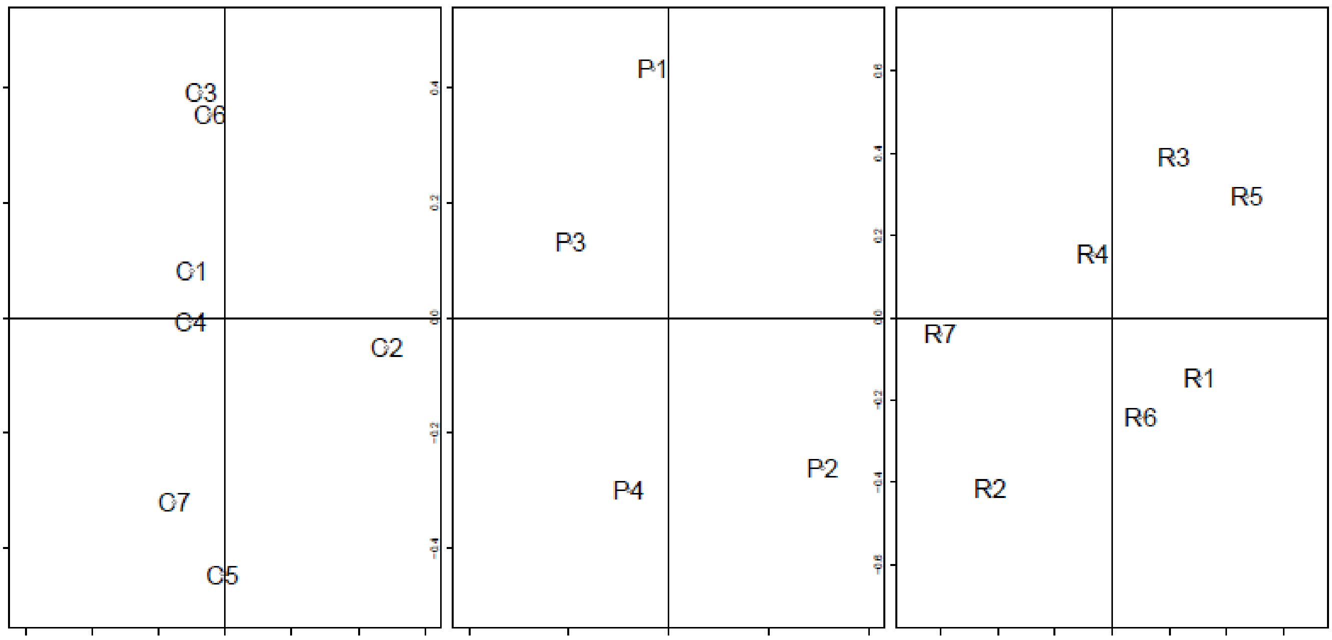

FIGURE 6 – French brands

Conclusion :
We have just proposed the extension of the so-called direct method, based on the inter-covariance matrix: CONCORGS1 called respectively the method : CONCORGS1D called dual, based on the inter-product matrix scalars in the case where on two groups of partitioned individuals are measured the same set of variables. In terms of application, it can be seen that the German (B4, V4 and O6) and French (C2 and P2) models have the same characteristics on the one hand. Furthermore, the German (V3) and French (C3, C6, R3 and P1) models have the same characteristic links.

In other words, from the point of view of power and speed, the choice of models is made between V3, P1, R3, C3 and C6. But in terms of length, width and weight, the choice is B4, V4, O6, C2 and P2.

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Neighborhoods and Partial Sums of Certain Meromorphic Functions

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Abstract

In this study, using a differential operator, we define a new subclass of meromorphic functions. Some properties neighborhoods and partial sums of functions in this subclass are given.

Keywords: Meromorphic, Neighborhood, Deifferential operator, Partial sum.

1. Introduction

Let Σ denote the class of functions of the form

$$f(z) = z^{-1} + \sum_{n=0}^{\infty} a_n z^n \tag{1}$$

which are analytic in the punctured disc $\mathbb{D} = \left\{ z \in \mathbb{C} : 0 < \left| z \right| < 1 \right\}$.

Let $f \in \Sigma$ be of the form (1) and let α, β be real numbers with $\alpha \geq \beta \geq 0$. Raducanu, Orhan and Deniz [8] defined the analogue of the differential operator given in as follows

$$D_{\alpha,\beta}^{0} f(z) = f(z)$$

$$D_{\alpha,\beta}^{1} f(z) = D_{\alpha,\beta} f(z) = \alpha\beta(z^2 f(z))'' + (\alpha - \beta)\frac{\left(z^2 f(z)\right)'}{z} + (1 - \alpha + \beta) f(z)$$

$$D_{\alpha,\beta}^{m} f(z) = D_{\alpha,\beta}\left(D_{\alpha,\beta}^{m-1} f(z)\right), \quad z \in \mathbb{D}, \quad m \in \mathbb{N} = \left\{1, 2, 3, \ldots\right\}.$$

If $f \in \Sigma$ is given by (1), then from the definition of $D_{\alpha,\beta}^{m}$ we get

$$D_{\alpha,\beta}^{m} f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} A(\alpha,\beta,n)^{m} a_n z^n, \quad z \in \mathbb{D}$$

where

$$A(\alpha,\beta,n) = \left[(n+2)\alpha\beta + \alpha - \beta\right](n+1) + 1.$$

When $\alpha = 1$ and $\beta = 0$, Uralegaddi and Somanatha [12] investigated certain properties of the operator $D_{\alpha,\beta}^{m}$.

Let $-1 \le B < A \le 1$. A function $f(z) = z^{-1} + \sum_{n=0}^{\infty} a_n z^n \in \Sigma$ is said to be in the class $T_m(\alpha,\beta,A,B)$ if it satisfies the condition

$$\left| \frac{z(D_{\alpha,\beta}^{m} f(z))' + D_{\alpha,\beta}^{m} f(z)}{Bz(D_{\alpha,\beta}^{m} f(z))' + AD_{\alpha,\beta}^{m} f(z)} \right| < 1 \tag{2}$$

for all $z \in E = \{z : |z| < 1\}$.

Furthermore, a function $f(z) = z^{-1} + \sum_{n=1}^{\infty} |a_n| z^n \in \Sigma$ is said to be in the class $T_m^{*}(\alpha,\beta,A,B)$ if it satisfies the condition (2).

It should be remarked in passing that the definition (2) is motivated essentially by the recent work of Morga [7] and Srivastava and co-authors [10].

In recent years, many important properties and characteristics of various interesting subclasses of the class Σ of meromorphically functions were inverstigated extensively by (among others) Aouf et al. [2], Dziok et al. [3], El-Ashwah and Aouf [4], He et al. [6], Raducanu et al. [8], Uralegaddi and Somanatha [12] and also [11].

The main object of this paper is to present neighborhoods and partial sums of functions in the classes $T_m(\alpha,\beta,A,B)$ and $T_m^{*}(\alpha,\beta,A,B)$ which we introduced here.

2. Neighborhoods and partial sums

Following the earlier works (based upon the familiar concept of neighborhoods of analytic functions) by Goodman [5] and Ruscheweyh [9] and (more recently)by Altıntas and Owa [1] and Srivastava and Owa [11] ,we begin by introducing here the δ -neighborhood of a function $f \in \Sigma$ of the form (1) by means of the definition

$$\mathrm{N}_{\delta}(f)=\left\{g(z)=z^{-1}+\sum_{n=0}^{\infty} b_n z^n \in \Sigma : \sum_{n=0}^{\infty} \frac{[(1-A)+n(1-B)]}{(A-B)} A(\alpha,\beta,n)^m \left|b_n-a_n\right| \le \delta, -1 \le B < A \le 1; \delta \ge 0\right\}$$

where $\alpha \ge \beta \ge 0.$

Making use of this definition, we now prove that:

Theorem 1. Let $\delta > 0$ and $-1 < A \le 0$. If $f(z)=z^{-1}+\sum_{n=0}^{\infty} a_n z^n \in \Sigma$ satisfies the condition

$$\frac{f(z)+\varepsilon z^{-1}}{1+\varepsilon} \in T_m(\alpha,\beta,A,B) \tag{3}$$

for any complex number ε such that $|\varepsilon| < \delta$, then $\mathrm{N}_{\delta}(f) \subset T_m(\alpha,\beta,A,B)$.

Proof. It is obvious from (2) that $g(z) \in T_m(\alpha,\beta,A,B)$ if and only if for any complex number σ with $|\sigma| = 1$

$$\frac{z(D^m_{\alpha,\beta} g(z))' + D^m_{\alpha,\beta} g(z)}{Bz(D^m_{\alpha,\beta} g(z))' + AD^m_{\alpha,\beta} g(z)} \ne \sigma \qquad (z \in E),$$

which is equivalent to

$$\frac{g(z) * h(z)}{z^{-1}} \ne 0 \qquad (z \in E) \tag{4}$$

where

$$h(z) = z^{-1} + \sum_{n=0}^{\infty} c_n z^n$$

$$= z^{-1} + \sum_{n=0}^{\infty} \frac{[(1+n) - \sigma(A+nB)]}{\sigma(B-A)} A(\alpha, \beta, n)^m z^n. \tag{5}$$

From (5), we have

$$|c_n| = \left| \frac{[(1+n) - \sigma(A+nB)}{\sigma(B-A)} A(\alpha, \beta, n)^m \right|$$

$$\leq \frac{(1-A) + n(1-B)}{(A-B)} A(\alpha, \beta, n)^m.$$

If $f(z) = z^{-1} + \sum_{n=0}^{\infty} a_n z^n \in \Sigma$ satisfies the condition (3), then (4) yields

$$\left| \frac{f(z) * h(z)}{z^{-1}} \right| \geq \delta \qquad (z \in E). \tag{6}$$

Now let $p(z) = z^{-1} + \sum_{n=0}^{\infty} b_n z^n \in \mathrm{N}_\delta(f)$, then

$$\left| \frac{(p(z) - f(z)) * h(z)}{z^{-1}} \right| = \left| \sum_{n=0}^{\infty} (b_n - a_n) c_n z^{n+1} \right|$$

$$\leq |z| \sum_{n=0}^{\infty} \frac{[(1-A) + n(1-B)]}{(A-B)} A(\alpha, \beta, n)^m |b_n - a_n| < \delta.$$

Thus for any complex number σ such that $|\sigma| = 1$, we have

$$\frac{p(z) * h(z)}{z^{-1}} \neq 0 \qquad (z \in E),$$

which implies that $p(z) \in T_m(\alpha, \beta, A, B)$.

Theorem 2. Let $-1 < A \leq 0$, $f(z) = z^{-1} + \sum_{n=0}^{\infty} a_n z^n \in \Sigma$, $s_1(z) = z^{-1}$ and $s_k(z) = z^{-1} + \sum_{n=0}^{k-2} a_n z^n \quad (k \geq 2)$.

Suppose that

$$\sum_{n=0}^{\infty} c_n \left| a_n \right| \leq 1 \tag{7}$$

where

$$c_n = \frac{(1-A) + n(1-B)}{(A-B)} A\left(\alpha, \beta, n\right)^m .$$

Then we have

i. $f(z) \in T_m(\alpha, \beta, A, B)$

ii. $$\operatorname{Re}\left\{\frac{f(z)}{s_k(z)}\right\} > 1 - \frac{1}{c_{k-1}} \tag{8}$$

and

$$\operatorname{Re}\left\{\frac{s_k(z)}{f(z)}\right\} > \frac{c_{k-1}}{1 + c_{k-1}}. \tag{9}$$

The estimates are sharp.

Proof. i. It is obvious that $z^{-1} \in T_m(\alpha, \beta, A, B)$. Thus from Theorem 1. and the condition (7), we have $\mathrm{N}_1(z^{-1}) \subset T_m(\alpha, \beta, A, B)$. This gives $f(z) \in T_m(\alpha, \beta, A, B)$.

ii. It is easy to see that $c_{n+1} > c_n > 1$. Thus

$$\sum_{n=0}^{k-2} \left| a_n \right| + c_{k-1} \sum_{n=k-1}^{\infty} \left| a_n \right| \leq \sum_{n=0}^{\infty} c_n \left| a_n \right| \leq 1. \tag{10}$$

Let

$$h_1(z) = c_{k-1}\left\{\frac{f(z)}{s_k(z)} - \left(1 - \frac{1}{c_{k-1}}\right)\right\}$$

$$= 1 + \frac{c_{k-1}\sum_{n=k-1}^{\infty} a_n z^{n+1}}{1 + \sum_{n=0}^{k-2} a_n z^{n+1}}.$$

It follows from (10) that

$$\left|\frac{h_1(z)-1}{h_1(z)+1}\right| \le \frac{c_{k-1}\sum_{n=k-1}^{\infty}|a_n|}{2 - 2\sum_{n=0}^{k-2}|a_n| - c_{k-1}\sum_{n=k-1}^{\infty}|a_n|} \le 1 \qquad (z \in E).$$

From this we obtain the inequality (8).

If we take

$$f(z) = z^{-1} - \frac{z^{k-1}}{c_{k-1}}, \tag{11}$$

then

$$\frac{f(z)}{s_k(z)} = 1 - \frac{z^k}{c_{k-1}} \to 1 - \frac{1}{c_{k-1}} \text{ as } \quad z \to 1^- .$$

This shows that the bound in (8)is best possible for each *k*.

Similarly, if we take

$$h_2(z) = (1 + c_{k-1})\left\{\frac{s_k(z)}{f(z)} - \frac{c_{k-1}}{1 + c_{k-1}}\right\}$$

$$=1-\frac{(1+c_{k-1})\sum\limits_{n=k-1}^{\infty} a_n z^{n+1}}{1+\sum\limits_{n=0}^{\infty} a_n z^{n+1}}$$

then we deduce that

$$\left|\frac{h_2(z)-1}{h_2(z)+1}\right| \le \frac{(1+c_{k-1})\sum\limits_{n=k-1}^{\infty}|a_n|}{2-2\sum\limits_{n=0}^{k-2}|a_n|+(1-c_{k-1})\sum\limits_{n=k-1}^{\infty}|a_n|} \le 1 \qquad (z\in E)$$

which yields (9). The estimate (9) is sharp with the extramal function $f(z)$ given by (11).

Theorem 3. Let $f(z)=z^{-1}+\sum\limits_{n=1}^{\infty}|a_n|z^n$ be analytic in $\mathbb{D}=\{z:0<|z|<1\}$. Then $f(z)\in T_m^*(\alpha,\beta,A,B)$ if and only if

$$\sum_{n=1}^{\infty}\left[(1-A)+n(1-B)\right]A(\alpha,\beta,n)^m|a_n| \le (A-B) \tag{12}$$

The result is sharp for the function $f(z)$ given by

$$f(z)=z^{-1}+\frac{(A-B)}{A(\alpha,\beta,n)^m\left[(1-A)+n(1-B)\right]}z^n \qquad (n\ge 1)$$

Proof. Let $f(z)=z^{-1}+\sum\limits_{n=1}^{\infty}|a_n|z^n\in T_m^*(\alpha,\beta,A,B)$. Then

$$\left|\frac{z(D^m_{\alpha,\beta}f(z))'+D^m_{\alpha,\beta}f(z)}{Bz(D^m_{\alpha,\beta}f(z))'+AD^m_{\alpha,\beta}f(z)}\right| = \left|\frac{\sum\limits_{n=1}^{\infty}(1+n)A(\alpha,\beta,n)^m|a_n|z^{n+1}}{(A-B)+\sum\limits_{n=1}^{\infty}(A+Bn)A(\alpha,\beta,n)^m|a_n|z^{n+1}}\right|. \tag{13}$$

Since $|\operatorname{Re} z|\le|z|$ for any z, choosing z to be real letting $z\to 1^-$ throuh real values (12) yields

$$\sum_{n=1}^{\infty}(1+n)A(\alpha,\beta,n)^m|a_n| \le (A-B)+\sum_{n=1}^{\infty}(A+Bn)A(\alpha,\beta,n)^m|a_n|,$$

which gives (13).

On the other hand, we have that

$$\left|\frac{z(D^m_{\alpha,\beta}f(z))' + D^m_{\alpha,\beta}f(z)}{Bz(D^m_{\alpha,\beta}f(z))' + AD^m_{\alpha,\beta}f(z)}\right| \le \frac{\sum_{n=1}^{\infty}(1+n)A(\alpha,\beta,n)^m|a_n|}{(A-B)+\sum_{n=1}^{\infty}(A+Bn)A(\alpha,\beta,n)^m|a_n|} < 1.$$

This shows that $f(z) \in T^*_m(\alpha,\beta,A,B)$.

For $\delta \ge 0, -1 \le B < A \le 1$ and $f(z) = z^{-1} + \sum_{n=0}^{\infty}|a_n|z^n \in \Sigma$, we define neighborhood of $f(z)$ by

$$\mathrm{N}^*_\delta(f) = \left\{ g(z) = z^{-1} + \sum_{n=0}^{\infty}|b_n|z^n \in \Sigma : \sum_{n=0}^{\infty}\frac{\left[(1-A)+n(1-B)\right]}{(A-B)}A(\alpha,\beta,n)^m \| b_n| - |a_n\| \le \delta \right\}.$$

Theorem 4. Let $A+B \le 0$. If $f(z) = z^{-1} + \sum_{n=1}^{\infty}|a_n|z^n \in T^*_{m+1}(\alpha,\beta,A,B)$, then $N^*_\delta(f) \subset T^*_m(\alpha,\beta,A,B)$, where $\delta = \frac{2}{3}$. The result is sharp.

Proof. Using the same method as in Theorem 1., we would have

$$h(z) = z^{-1} + \sum_{n=1}^{\infty}c_n z^n = z^{-1} + \sum_{n=1}^{\infty}\frac{(1+n)-\sigma(A+nB)}{\sigma(B-A)}A(\alpha,\beta,n)^m z^n.$$

Under the hypothesis $A+B \le 0$, we obtain that

$$\begin{aligned}\left|\frac{f(z)*h(z)}{z^{-1}}\right| &= \left|1+\sum_{n=1}^{\infty}c_n|a_n|z^{n+1}\right| \\ &\ge 1-\frac{1}{3}\sum_{n=1}^{\infty}\frac{[(1-A)+n(1-B)]}{(A-B)}A(\alpha,\beta,n)^{m+1}|a_n|.\end{aligned}$$

From Theorem 3., we get

$$\left|\frac{f(z)*h(z)}{z^{-1}}\right| \geq \frac{2}{3} = \delta.$$

The remaining part of the proof is similar to that of Theorem 1.

To show the sharpness,we consider the function

$$f(z) = z^{-1} + \frac{A-B}{\left(2-(A+B)\right)3^{n+1}} z \in T^{*}_{m+1}(\alpha,\beta,A,B)$$

and

$$g(z) = z^{-1} + \left[\frac{A-B}{(2-(A+B))3^{n+1}} + \frac{(A-B)\delta'}{(2-(A+B))3^{n}}\right] z$$

where $\delta' > \frac{2}{3}$. Then the function $g(z)$ belong to $\mathrm{N}^{*}_{\delta'}(f)$.

On the other hand, we find from Theorem 3. that $g(z)$ is not in $T^{*}_{n}(\alpha,\beta,A,B)$. Now the proof is complete.

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Use of Fractional Calculus in Science and Engineering

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Abstract

In this work, a brief history and development of fractional calculus (FC) in the literature is mentioned. Definitions of some fractional derivatives available in the literature are presented. Moreover, the use of FC in various disciplines is presented by giving examples from the literature.

Keywords: Fractional calculus, fractional differential equation, modeling, science, engineering.

1. Introduction

Modern calculus was founded in the 17th century by an English scientist Isaac Newton (1643-1727) and a German mathematician Gottfried Wilhelm Leibniz (1646-1716). FC is the generalization of the ordinary differentiation and integration to non-integer order. In classical calculus, the derivative has an important meaning connected with the concept of a tangent as opposite to what is the case with FC.

Leibniz invented the notation $\frac{d^n y}{dx^n} = D^n y$ for the n^{th}-order derivative, where n is a non-negative integer number. FC has an origin as in the similar meaning of extension of real numbers to complex numbers or the extension of factorials to the factorials of complex numbers. In 1965, a French mathematician Guillaume François Antoine Marquis de l'Hopital (1661-1704), working under the direction of Swiss mathematician Johann Bernoulli (1667-1748), one of the famous mathematicians of the period, wrote a letter to Leibniz and asked an important question about the order of derivative. In the letter, l'Hopital asked that "What if $n = \frac{1}{2}$?". In 1695, Leibniz replied, "From this apparent paradox, one day will be useful consequences will be drawn." Is it possible to extend the derivative of integer-order (IO) $D^n y$ into the case where n is any (rational, irrational and even complex) number? In 1697, Leibniz used the notation $d^{\frac{1}{2}} y$ and stated that differential calculus might have been used to achieve the same result. After Leibniz, in 1738, Swiss mathematician and physicist Leonhard Euler (1707-1783) introduced the derivative $\frac{d^n(x^k)}{dx^n}$, where n is an arbitrary (non-integer) order. Later, in 1822, a French mathematician and physicist, Jean Baptiste Joseph Fourier (1768-1830) suggested an integral representation in order to define the derivative, and his version can be considered the first definition for the derivative of arbitrary (positive) order. Euler and Fourier made mention of derivatives of arbitrary order, but they gave no application and example. Then, in 1823, a Norwegian mathematician Niels Henrik Abel (1802-1829), a pioneer in the development of several branches of modern mathematics, gave the first application of FC by solving an integral equation. After that, in 1832, a French mathematician and engineer Joseph Liouville (1809-1882) extended the formula of the IO derivative of the exponential function to derivatives of arbitrary order as follows: $D^\alpha e^{ax} = a^\alpha e^{ax}$. Moreover, he derived the formula

$$D^\alpha f(x) = \sum_{n=0}^{\infty} c_n a_n^\alpha e^{a_n x},$$

where

$$f(x) = \sum_{n=0}^{\infty} c_n e^{a_n x}, \qquad Rea_n > 0.$$

The above formula is Liouville's first definition for a fractional derivative. However, it has a drawback such that it is applicable only for the above function. Aware of this advantage, he presented his second definition for fractional derivative and obtained [1]

$$D^\alpha x^{-\beta} = (-1)^\alpha \frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)} x^{-\alpha-\beta}, \qquad \beta > 0$$

where

$$\Gamma(\beta) x^{-\beta} = \int_0^\infty t^{\beta-1} e^{-xt} dt, \qquad \beta > 0.$$

Moreover, he applied these definitions to some problems in potential theory. Nevertheless, his second definition has a disadvantage because it is useful for only rational functions. FC originated the Riemann-Liouville definition of fractional integral in the form

$${}_aD_x^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt.$$

When $a = 0$, the above expression is the Riemann (a German mathematician Georg Friedrich Bernhard Riemann (1826-1866)) definition of fractional integral, and if $a = -\infty$, the expression presents the Liouville definition (see [1]).

In 1867-1868, an Austrian mathematician Anton Karl Grünwald (1838-1920) and a Russian mathematician Aleksey Letnikov (1837-1888) introduced the following fractional derivative

$${}_aD_x^\alpha f(x) = \lim_{h\to 0} \frac{1}{h^\alpha} \sum_{k=0}^{\left[\frac{x-a}{h}\right]} (-1)^k \binom{\alpha}{k} f(x-kh), [x] - integer\ part\ of\ x,$$

that allows one to take the derivative a non-integer number of times.

In the 20th century, many studies on FC were carried out by various scientists and new derivative definitions were proposed. A French mathematician Jacques Salomon Hadamard (1865-1963), a German mathematician and physicist Hermann Klaus Hugo Weyl (1885-1955), a Hungarian mathematician Marcel Riesz (1886-1969), a French mathematician Andre Marchaud (1887-1973), a German mathematician Hermann Kober (1888-1973), a Polish-born American mathematician Antoni Zygmund

(1900-1992), a Hungarian mathematician Arthur Erdelyi (1908-1977), an Italian mathematician Michele Caputo (1927-) are famous known mathematicians studying on FC in the 20th century.

2. Fractional Derivatives

In this Section, we have listed some of the famous fractional derivatives [2] as follows:

Liouville derivative:

$$D^{\alpha}[f(x)] = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{-\infty}^{x}(x-\xi)^{-\alpha}f(\xi)d\xi\,, \qquad -\infty < x < +\infty$$

Liouvelli left-sided derivative:

$$D_{0^+}^{\alpha}[f(x)] = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_{0}^{x}(x-\xi)^{-\alpha+n-1}f(\xi)d\xi\,, \qquad x > 0$$

Liouvelli right-sided derivative:

$$D_{-}^{\alpha}[f(x)] = \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_{x}^{\infty}(x-\xi)^{-\alpha+n-1}f(\xi)d\xi\,, \qquad x < \infty$$

Riemann-Liouvelli left-sided derivative:

$${}^{RL}D_{a^+}^{\alpha}[f(x)] = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_{a}^{x}(x-\xi)^{n-\alpha-1}f(\xi)d\xi\,, \qquad x \geq a$$

Riemann-Liouvelli right-sided derivative:

$${}^{RL}D_{b^-}^{\alpha}[f(x)] = \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_{x}^{b}(\xi-x)^{n-\alpha-1}f(\xi)d\xi\,, \qquad x \leq b$$

Caputo left-sided derivate:

$${}_{*}D_{a^+}^{\alpha}[f(x)] = \frac{1}{\Gamma(n-\alpha)}\int_{a}^{x}(x-\xi)^{n-\alpha-1}\frac{d^n}{d\xi^n}[f(\xi)]d\xi\,, \qquad x \geq a$$

Caputo right-sided derivate:

$${}_{*}D_{b^-}^{\alpha}[f(x)] = \frac{(-1)^n}{\Gamma(n-\alpha)}\int_{x}^{b}(\xi-x)^{n-\alpha-1}\frac{d^n}{d\xi^n}[f(\xi)]d\xi\,, \qquad x \leq b$$

Grünwald-Letnikov left-sided derivative:

$$ {}_{v}^{GL}D_{a^+}^{\alpha}[f(x)] = \lim_{h\to 0}\frac{1}{h^\alpha}\sum_{k=0}^{[n]}(-1)^k\frac{\Gamma(\alpha+1)(x-kh)}{\Gamma(k+1)\Gamma(\alpha-k+1)}\,, \qquad nh = x-a $$

Grünwald-Letnikov right-sided derivative:

$$ {}^{GL}D_{b^-}^{\alpha}[f(x)] = \lim_{h\to 0}\frac{1}{h^\alpha}\sum_{k=0}^{[n]}(-1)^k\frac{\Gamma(\alpha+1)(x+kh)}{\Gamma(k+1)\Gamma(\alpha-k+1)}\,, \qquad nh = b-x $$

Weyl derivative:

$$ {}_{x}D_{\infty}^{\alpha}[f(x)] = D_{-}^{\alpha}[f(x)] = (-1)^m\left(\frac{d}{d\xi}\right)^n\left[{}_{x}W_{\infty}^{\alpha}[f(x)]\right] $$

Marchaud derivative:

$$ D_{+}^{\alpha}[f(x)] = \frac{\alpha}{\Gamma(1-\alpha)}\int_{-\infty}^{x}\frac{f(x)-f(\xi)}{(x-\xi)^{1+\alpha}}d\xi $$

Marchaud left-sided derivative:

$$ D_{+}^{\alpha}[f(x)] = \frac{\alpha}{\Gamma(1-\alpha)}\int_{0}^{\infty}\frac{f(x)-f(x-\xi)}{\xi^{1+\alpha}}d\xi $$

Marchaud right-sided derivative:

$$ D_{-}^{\alpha}[f(x)] = \frac{\alpha}{\Gamma(1-\alpha)}\int_{0}^{\infty}\frac{f(x)-f(x+\xi)}{\xi^{1+\alpha}}d\xi $$

Hadamard derivative:

$$ D_{+}^{\alpha}[f(x)] = \frac{\alpha}{\Gamma(1-\alpha)}\int_{o}^{x}\frac{f(x)-f(\xi)}{[\ln(x/\xi)]^{1+\alpha}}\frac{d\xi}{\xi} $$

Chen left-sided derivative:

$$ D_{c}^{\alpha}[f(x)] = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{c}^{x}(x-\xi)^{-\alpha}f(\xi)d\xi\,, \qquad x > c $$

Chen right-sided derivative:

$$ D_{c}^{\alpha}[f(x)] = -\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x}^{c}(\xi-x)^{-\alpha}f(\xi)d\xi\,, \qquad x < c $$

Davidson-Essex derivative:

$$D_0^{\alpha}[f(x)] = \frac{1}{\Gamma(1-\alpha)} \frac{d^{n+1-k}}{dx^{n+1-k}} \int_0^x (x-\xi)^{-\alpha} \frac{d^k}{d\xi^k}[f(\xi)]d\xi$$

Coimbra derivative:

$$D_0^{\alpha(x)}[f(x)] = -\frac{1}{\Gamma\big(1-\alpha(x)\big)} \left\{ \int_0^x (x-\xi)^{-\alpha x} \frac{d}{d\xi}[f(\xi)]d\xi + f(0)x^{-\alpha(x)} \right\}$$

Canavati derivative:

$${}_aD_x^{\nu}[f(x)] = \frac{1}{\Gamma(1-\mu)} \frac{d}{dx} \int_0^x (x-\xi)^{\mu} \frac{d^n}{d\xi^n}[f(\xi)]d\xi\,, \qquad n = \lfloor \nu \rfloor\,, \qquad \mu = n - \nu$$

Jumarie derivative, $n = 1$:

$$D_x^{\alpha}[f(x)] = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x (x-\xi)^{n-\alpha-1}[f(\xi) - f(0)]d\xi$$

Riesz derivative:

$$D_x^{\alpha}[f(x)] = -\frac{1}{2\cos\left(\frac{\alpha\pi}{2}\right)} \frac{1}{\Gamma(\alpha)} \frac{d^n}{dx^n} \left\{ \int_{-\infty}^x (x-\xi)^{n-\alpha-1} f(\xi)d\xi + \int_x^{\infty} (\xi - x)^{n-\alpha-1} f(\xi)d\xi \right\}$$

Cossar derivative:

$$D_-^{\alpha}[f(x)] = -\frac{1}{\Gamma(1-\alpha)} \lim_{N\to\infty} \frac{d}{dx} \int_x^N (\xi - x)^{-\alpha} f(\xi)d\xi$$

Local fractional Yang derivative:

$$D_-^{\alpha}[f(x)]\big|_{x=x_0} = \lim_{x\to x_0} \frac{\Delta^{\alpha}[f(x) - f(x_0)]}{(x-x_0)^{\alpha}}$$

Left Riemann-Liouville derivative of variable fractional order:

$${}_aD_x^{\alpha(\cdot,\cdot)}[f(x)] = \frac{d}{dx} \int_a^x (x-\xi)^{-\alpha(\xi,x)} f(\xi) \frac{d\xi}{\Gamma[1-\alpha(\xi,x)]}$$

Right Riemann-Liouville derivative of variable fractional order:

$$ {}_{x}D_{b}^{\alpha(\cdot,\cdot)}[f(x)] = \frac{d}{dx}\int_{x}^{b}(\xi - x)^{-\alpha(\xi,x)} f(\xi)\frac{d\xi}{\Gamma[1-\alpha(\xi,x)]} $$

Left Caputo derivative of variable fractional order:

$$ {}_{a}D_{x}^{\alpha(\cdot,\cdot)}[f(x)] = \int_{a}^{x}(x - \xi)^{-\alpha(\xi,x)} \frac{d}{d\xi} f(\xi)\frac{d\xi}{\Gamma[1-\alpha(\xi,x)]} $$

Right Caputo derivative of variable fractional order:

$$ {}_{x}D_{b}^{\alpha(\cdot,\cdot)}[f(x)] = \int_{x}^{b}(\xi - x)^{-\alpha(\xi,x)} \frac{d}{d\xi} f(\xi)\frac{d\xi}{\Gamma[1-\alpha(\xi,x)]} $$

Caputo derivative of variable fractional order:

$$ {}_{*}D_{x}^{\alpha(x)}[f(x)] = \frac{1}{\Gamma\big(1-\alpha(x)\big)}\int_{0}^{x}(x-\xi)^{-\alpha(\xi,x)}\frac{d}{d\xi}f(\xi)d\xi $$

Modified Riemann-Liouville fractional derivative:

$$ D^{\alpha}[f(x)] = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{0}^{x}(x-\xi)^{-\alpha}[f(\xi)-f(0)]d\xi $$

Osler fractional derivative:

$$ {}_{a}D_{z}^{\alpha}f(z) = \frac{\Gamma(\alpha+1)}{2\pi i}\int_{(a,z^{+})}\frac{f(\xi)}{(\xi-z)^{1+\alpha}}d\xi $$

k −fractional Hilfer derivative:

$$ {}^{k}D^{\mu,\nu}f(x) = {I_{k}}^{\nu(1-\mu)}\frac{d}{dx}{I_{k}}^{(1-\mu)(1-\nu)}f(x) $$

3. Applications of FC

Static models were used until 1695; thereafter, dynamic models appeared until the 1960s; Finally, fractional-order (FO) models have also appeared after the 1960s. Geometry and algebra, differential and integral calculus are needed for static and dynamic models, respectively. For FO models, FC is needed for a more advanced but smaller-scale characterization of our more complex world.

FO modeling is more accurate than IO modeling sine it takes the history of the system in to account; therefore it is a suitable and practical tool for investigating, defining, modeling, analyzing and synthesizing electrical, chemical, biological, environmental and many other systems. Further, FO modeling introduces extra freedom in controlling the behavior of the systems; this supplies superior properties over the IO models of complex systems and phenomena.

3.1. Applications in Circuit Theory

Many stability theorems and fundamentals of filters, oscillators, charging circuits are expressed in terms of FO circuits. A FO capacitor impedance is defined by

$$z(s) = \frac{1}{Cs^{\alpha}},$$

where C is a constant, and s is the complex frequency, and α is a non-integer constant; for $\alpha = 1$ an ordinary capacitor results. Attempts for realizing FO elements are constructed by using infinite ladder networks and/or by integrated circuit technology. The first attempt leads to bulky realizations and the second needs developing fractal structures on silicon.

Figure 1 and 2 show the synthesis of the FO capacitor using RC elements and the synthesis by active circuit elements, respectively.

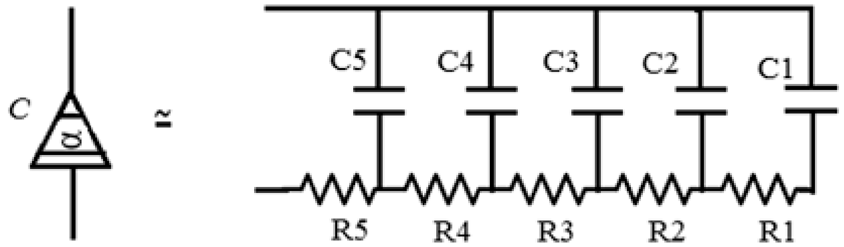

Figure 1. FO capacitor approximated using R-C ladder circuit [3]

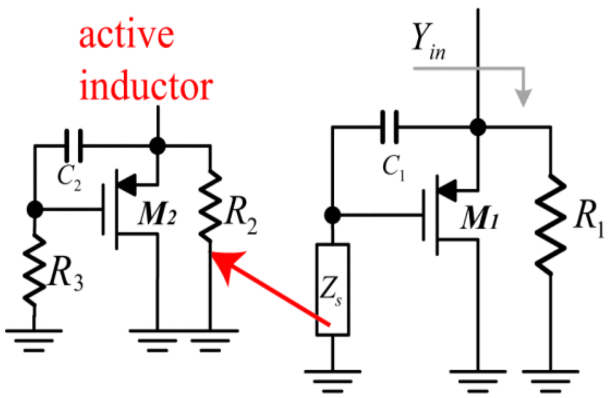

Figure 2. Proposed MOS based second-order FO capacitor emulator with an active inductor [4]

3.1. Application in Control Theory

Stability and control of many engineering processes are achieved by using the well-known conventional proportional-integral-derivative (PID) controller depicted in the following figure. PID controller mainly aims to minimize the difference between the desired output and the actual output of a physical system. The process is actuated by the signal produced by the controller. The actuating signal produced by a PID controller is composed of the error signal itself (P-control), its integral (I-control) and its derivative (D-control).

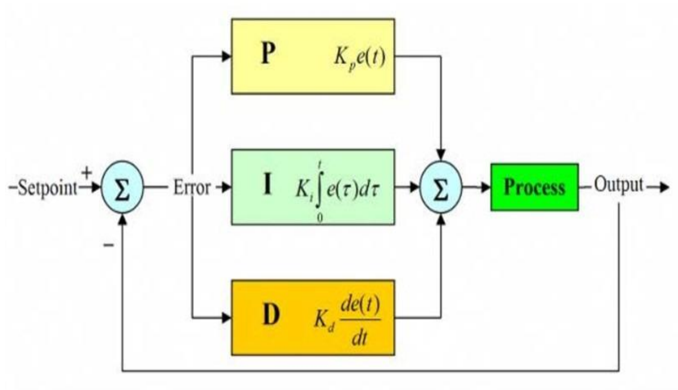

Figure 3. Diagram of PID controller [5]

Similar to the needs of more realistic models by using FO calculus, especially for F behavior of many natural phenomena, use of corresponding FO PID (FOPID) controllers considering the effects of the history has been employed for control purposes. In addition to three control parameters (K_p, K_i, K_d) used by PID controllers, proposed FOPID controllers uses two more parameters (λ, μ; integral and derivative orders); thus, more flexible, suitable and practical controller designs have been achieved.

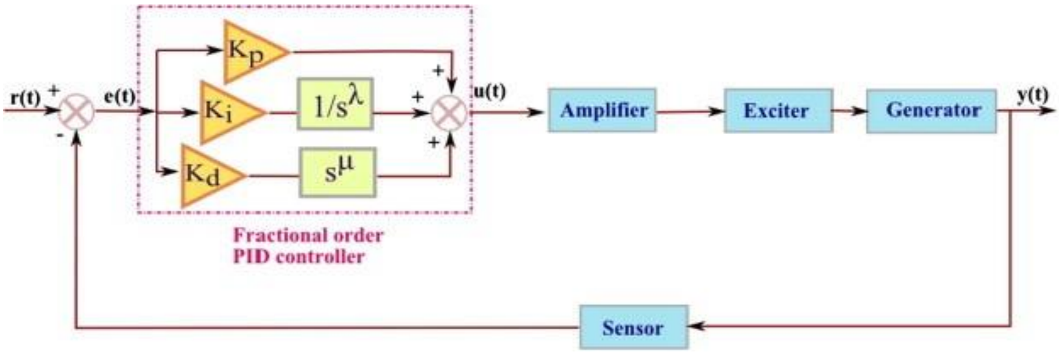

Figure 4. Diagram of FOPID controller [6]

3.3. Applications in Signal and Image Processing

FO models are also used in signal and image processing areas. Fractional differential equations are used for investigation and more accurate representation of speech signals. For example, the speech signal frame is shown in Fig. 5 at the top along with its fractional integrations at the buttom.

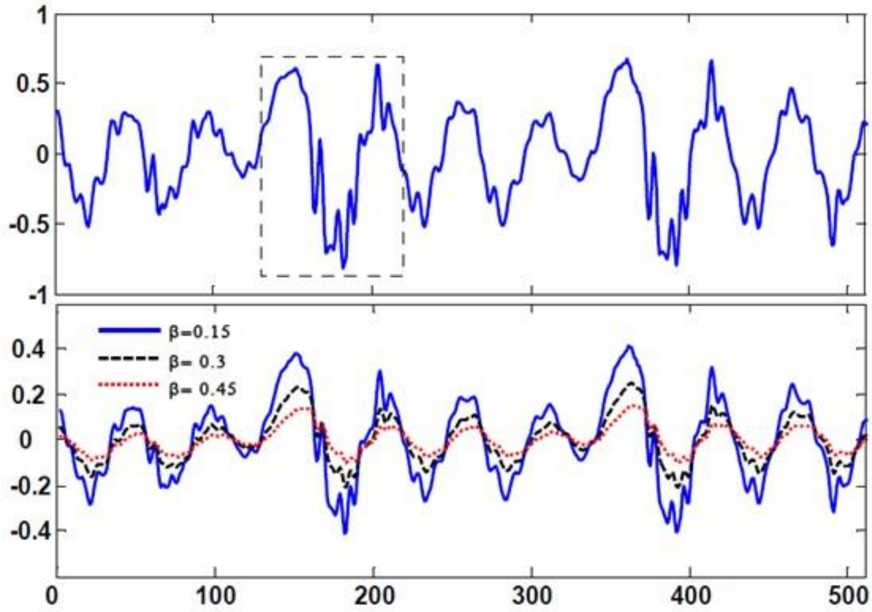

Figure 5. Sample speech frame and its fractional integral basis function [7]

Fractional differentiation methods on image enhancements are capable of preserving high-frequency contours and improve low-frequency details. Fig. 6 shows that the detail texture is enhanced when the original image and image obtained by fractional differentiation using eight direction masks are compared.

(A) Original (B) $\alpha = 0.53$ (C) Original (D) $\alpha = 0.47$

Figure 6. Fractional differentiation based methods are applied in the field of image enhancement [8]

3.4. Application in Biology

As in the above mentioned applications, FC supplies useful mathematical tools for modeling many biological processes as well. The methods of FC are used for solving bio heat transfer problems arising in peripheral tissue regions. On the other hand, magnetic resonance imaging has been fused as an incomparable technology for the judgment of pathological disarrangements in tissues. The FO model grapes the appearance of exponential rates in normal and diseased brain tissue. FO models are also used as a powerful tool to identify the comportment of premotor neurons in the vestibule-ocular reflex system much better than IO models. FO models cover the space in the apprehension of certain patterns, where the IO models are not sufficient for a full explanation of relations between immune system, treatment compliance, age, and other co-morbidities of individuals for the confection of HIV.

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On s-Supplemented Modules

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Abstract

In this work, some new properties of socle-supplemented (briefly, s-supplemented) modules are studied. Every ring has unity and every module is unitary left module, in this work. It is proved that every factor module and every homomorphic image of a s-supplemented module are s-supplemented.

Keywords: Small Submodules, Radical, Socle, Supplemented Modules.

2020 Mathematics Subject Classification: 16D10, 16D70.

1. INTRODUCTION

Throughout this paper all rings will be associative with identity and all modules will be unital left modules.

Let R be a ring and M be an R-module. We will denote a submodule N of M by $N\leq M$. Let M be an R-module and $N\leq M$. If $L=M$ for every submodule L of M such that $M=N+L$, then N is called a *small* submodule of M and denoted by $N\ll M$. . Let M be an R-module. M is called a *hollow* module if every proper submodule of M is small in M. M is called a *local* module if M has the largest submodule, i.e. a proper submodule which contains all other proper submodules. A submodule N of an R-module M is called an *essential* submodule and denoted by $N\trianglelefteq M$ in case $K\cap N\neq 0$ for every submodule $K\neq 0$, or eqiuvalently, $N\cap L=0$ for $L\leq M$ implies that $L=0$. Let M be an R-module and $U,V\leq M$. If $M=U+V$ and V is minimal with respect to this property, or equivalently, $M=U+V$ and $U\cap V\ll V$, then V is called a *supplement* of U in M. M is said to be *supplemented* if every submodule of M has a supplement in M. M is said to be *essential supplemented* (or briefly, *e-supplemented*) if every essential submodule of M has a supplement in M. Let M be an R-module and $U\leq M$. If for every $V\leq M$ such that $M=U+V$, U has a supplement X with $X\leq V$, we say U has *ample supplements* in M. If every submodule of M has ample supplements in M, then M is called an *amply supplemented* module. The intersection of all maximal submodules of an R-module M is called the *radical* of M and denoted by $RadM$. If M have no maximal submodules, then we denote $RadM=M$. The sum of all simple submodules of an R-module M is called the *socle* of M and denoted by $SocM$. Let M be an R-module. It is defined the relation β^* on the set of submodules of an R-module M by $X\beta^*Y$ if and only if $Y+K=M$ for every $K\leq M$ such that $X+K=M$ and $X+T=M$ for every $T\leq M$ such that $Y+T=M$. Let M be an R-module and $K\leq V\leq M$. We say V *lies above* K in M if $V/K\ll M/K$.

More informations about (amply) supplemented modules are in [2], [6], [7] and [8]. More results about essential supplemented modules are in [4] and [5]. The definition of β^* relation and some properties of this relation are in [1].

Lemma 1.1. Let M be an R-module.

(1) If $K\leq L\leq M$, then $K\trianglelefteq M$ if and only if $K\trianglelefteq L\trianglelefteq M$.

(2) Let N be an R-module and $f: M\to N$ be an R-module homomorphism. If $K\trianglelefteq N$, then $f^{-1}(K)\trianglelefteq M$.

(3) For $N\leq K\leq M$, if $K/N\trianglelefteq M/N$, then $K\trianglelefteq M$.

(4) If $K_1\trianglelefteq L_1\leq M$ and $K_2\trianglelefteq L_2\leq M$, then $K_1\cap K_2\trianglelefteq L_1\cap L_2$.

(5) If $K_1\trianglelefteq M$ and $K_2\trianglelefteq M$, then $K_1\cap K_2\trianglelefteq M$.

Proof. See [7, 17.3].

Lemma 1.2. Let M be an R-module. The following assertions hold.

(1) If $K\leq L\leq M$, then $L\ll M$ if and only if $K\ll M$ and $L/K\ll M/K$.

(2) Let N be an R-module and $f: M\to N$ be an R-module homomorphism. If $K\ll M$, then $f(K)\ll N$. The converse is true if f is an epimorphism and $Kef\ll M$.

(3) If $K\ll M$, then $(K+L)/L\ll M/L$ for every $L\leq M$.

(4) If $L\leq M$ and $K\ll L$, then $K\ll M$.

(5) If $K_1,K_2,...,K_n\ll M$, then $K_1+K_2+...+K_n\ll M$.

(6) Let $K_1,K_2,...,K_n,L_1,L_2,...,L_n\leq M$. If $K_i\ll L_i$ for every $i=1,2,...,n$, then $K_1+K_2+...+K_n\ll L_1+L_2+...+L_n$.

Proof. See [2, 2.2] and [7, 19.3].

Lemma 1.3. Let M be an R-module. The following statements hold.

(i) $SocM$ is equal to the intersection of all essential submodules of M.

(ii) For $K\leq M$, $SocK=K\cap SocM$.

(iii) $SocM\trianglelefteq M$ if and only if $SocK\neq 0$ for every nonzero submodule K of M.

(iv) Let N be an R-module and $f: M\to N$ be an R-module homomorphism. Then $f(SocM)\leq Socf(M)$.

(v) For $K \leq M$, $(SocM+K)/K \leq Soc(M/K)$.
(vi) If $M = \oplus_{\Lambda} M_{\lambda}$, then $SocM = \oplus_{\Lambda} SocM_{\lambda}$.
Proof. See [7, 21.2].

2. s-SUPPLEMENTED MODULES

Definition 2.1. Let M be an R-module. If every $U \leq M$ with $SocM \leq U$ has a supplement in M, then M is called a *socle supplemented* (or briefly, *s-supplemented*) module. (See also [3])

Clearly we can see that every supplemented module is s-supplemented.

Definition 2.2. Let M be an R-module and $X \leq M$. If X is a supplement of an essential submodule of M, then X is called an *e-supplement* submodule in M. (See [4])

Proposition 2.3. Every s-supplemented module is essential supplemented.
Proof. Let M be a s-supplemented R-module and $U \trianglelefteq M$. Since $U \trianglelefteq M$, by Lemma 1.3, $SocM \leq U$. Since M is s-supplemented, U has a supplement in M. Hence every essential submodule of M has a supplement in M and M is essential supplemented.

Proposition 2.4. Every factor module of a s-supplemented module is essential supplemented.
Proof. Let M be a s-supplemented module and $K \leq M$. Since M is s-supplemented, by Proposition 2.3, M is essential supplemented. Then M/K is essential supplemented.

Proposition 2.5. Every homomorphic image of a s-supplemented module is essential supplemented.
Proof. Let M be a s-supplemented module and $f : M \rightarrow N$ be an R-module epimorphism with N an R-module. Since M is s-supplemented, by Proposition 2.3, M is essential supplemented. Then N is essential supplemented. Hence every homomorphic image of a s-supplemented module is essential supplemented.

Proposition 2.6. Every direct summand of a s-supplemented module is essential supplemented.
Proof. Clear from Proposition 2.5.

Proposition 2.7. Let $M=M_1+M_2$. If M_1 and M_2 are s-supplemented, then M is essential supplemented.
Proof. Since M_1 and M_2 are s-supplemented, by Proposition 2.3, these modules are essential supplemented. Then N is essential supplemented.

Proposition 2.8. Let $M_i \leq M$ for $i=1,2,...,n$. If M_i is s-supplemented for every $i=1,2,...,n$, then $M_1+M_2+...+M_n$ is essential supplemented.
Proof. Clear from Proposition 2.7.

Proposition 2.9. Let M be an R-module. If M is s-supplemented, then every finitely M-generated module is essential supplemented.
Proof. Let N be a finitely M-generated R-module. Then there exist a finite index set Λ and an R-module epimorphism $f : M^{(\Lambda)} \to N$. Since M is s-supplemented, by Proposition 2.8, $M^{(\Lambda)}$ is essential supplemented. Then N is essential supplemented.

Proposition 2.10. Let R be a ring. If $_RR$ is s-supplemented, then every finitely generated R-module is essential supplemented.
Proof. Clear from Proposition 2.9.

Proposition 2.11. Hollow modules are s-supplemented.
Proof. Clear from definitions.

Proposition 2.12. Every local module is s-supplemented.
Proof. Since every local module is hollow, by Proposition 2.11, every local module is s-supplemented.

Proposition 2.13. Let M be a s-supplemented R-module. Then $M/RadM$ have no proper essential submodules.
Proof. Since M is s-supplemented, by Proposition 2.3, M is essential supplemented. Then $M/RadM$ have no proper essential submodules.

Proposition 2.14. Every supplemented module is s-supplemented.
Proof. Clear from definitions.

Proposition 2.15. Every factor module of a supplemented module is s-supplemented.
Proof. Let M be a supplemented R-module and $K \leq M$. Since M is supplemented, M/K is supplemented. Then by Proposition 2.14, M/K is s-supplemented.

Proposition 2.16. Every homomorphic image of a supplemented module is s-supplemented.
Proof. Let M be a supplemented module and $f : M \to N$ be an R-module epimorphism with N an R-module. Since M is supplemented, then N is supplemented. Then by Proposition 2.14, N is s-supplemented.

Corollary 2.17. Every direct summand of a supplemented module is s-supplemented.
Proof. Clear from Proposition 2.16.

Lemma 2.18. Let M be an R-module. If every submodule of M which contains $SocM$ is β^* equivalent to a e-supplement submodule in M, then M is e-supplemented.

Proof. Let $U \trianglelefteq M$. Then $SocM \leq U$ and by hypothesis, there exists an e-supplement submodule X in M such that $U\beta^{*}X$. Since X is an e-supplement submodule in M, there exists an essential submodule Y of M such that X is a supplement of Y in M. Since $Y \trianglelefteq M$, $SocM \leq Y$ and by hypothesis, there exists an e-supplement submodule V in M such that $Y\beta^{*}V$. Since X is a supplement of Y in M and $Y\beta^{*}V$, by [1, Theorem 2.6(ii)], X is a supplement of V in M. Since V is a supplement submodule in M, we can see that V is a supplement of X in M and since $U\beta^{*}X$, by [1, Theorem 2.6 (ii)], V is a supplement of U in M. Hence M is e-supplemented.

Corollary 2.19. Let M be an R-module. If every submodule of M which contains $SocM$ lies above an e-supplement submodule in M, then M is s-supplemented.
Proof. Clear from Lemma 2.18.

3. CONCLUSION

Supplemented modules are actual subjects in Module Theory and can be studied on these modules.

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$p-$valently Convex of Complex Order for a General Integral Operator

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Abstract

In this study, we defined a new general $p-$valent integral operator in the unit disk $\mathbb{U}$. We obtained some sufficient conditions for the integral operator $\mathcal{F}_{p,m,l,\mu}(z)$ to be $p-$valently convex of complex order.

Keywords: Analytic function; Integral operator; $\beta-$ uniformly $p-$ valent starlike and $\beta-$ uniformly $p-$valent convex function; complex order.

1. Introduction and Preliminaries

Let A_p denote the class of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \quad \left(p \in \mathbb{N} = \{1, 2, ...\}\right) \tag{1}$$

which are analytic in the open disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

A function $f \in S_p^*(\gamma, \alpha)$ is $p-$valently starlike of complex order $\gamma\left(\gamma \in \mathbb{C} - \{0\}\right)$ and type $\alpha\left(0 \le \alpha < p\right)$, that is, $f \in S_p^*(\gamma, \alpha)$, if it is satisfies the following inequality;

$$\operatorname{Re}\left\{p + \frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)} - p\right)\right\} > \alpha, \quad \left(z \in \mathbb{U}\right). \tag{2}$$

Furthermore, a function $f \in C_p(\gamma, \alpha)$ is $p-$valently convex of complex order $\gamma\left(\gamma \in \mathbb{C} - \{0\}\right)$ and type $\alpha\left(0 \le \alpha < p\right)$, that is, $f \in C_p(\gamma, \alpha)$, if it satisfies the following inequality;

$$\operatorname{Re}\left\{p+\frac{1}{\gamma}\left(1+\frac{zf''(z)}{f'(z)}-p\right)\right\}>\alpha, \quad (z\in\mathbb{U}). \tag{3}$$

In particular cases, for $p=1$ in the classes $S_p^*(\gamma,\alpha)$ and $C_p(\gamma,\alpha)$, we obtain the clases $S^*(\gamma,\alpha)$ and $C(\gamma,\alpha)$ of starlike functions of complex order $\gamma\left(\gamma\in\mathbb{C}-\{0\}\right)$ and type $\alpha(0\le\alpha<p)$ and convex functions of complex order $\gamma\left(\gamma\in\mathbb{C}-\{0\}\right)$ and type $\alpha(0\le\alpha<p)$, respectively, which were introduced and studied by Frasin [11]. Also, for $\alpha=0$ in the clases $S_p^*(\gamma,\alpha)$ and $C_p(\gamma,\alpha)$, we obtain the clases $S^*(\gamma,\alpha)$ and $C(\gamma,\alpha)$ which are called $p-$valently starlike of complex order $\gamma\left(\gamma\in\mathbb{C}-\{0\}\right)$ and $p-$valently convex of complex order $\gamma\left(\gamma\in\mathbb{C}-\{0\}\right)$, respectively. Setting $p=1$ and $\alpha=0$, we obtain the classess $S^*(\gamma)$ and $C(\gamma)$. The class $S^*(\gamma)$ of starlike functions of complex order $\gamma\left(\gamma\in\mathbb{C}-\{0\}\right)$ was defined by Nasr and Aouf (see [15]) while the class $C(\gamma)$ of convex functions of complex order $\gamma\left(\gamma\in\mathbb{C}-\{0\}\right)$ was considered earlier by Wiatrowski (see[22]). Note that $S_p^*(1,\alpha)=S_p^*(\alpha)$ and $C_p(1,\alpha)=C_p(\alpha)$ are, respectively, the classes of $p-$valently starlike and $p-$valently convex functions of order $\alpha(0\le\alpha<p)$ in $\mathbb{U}$. Also, we note that $S_1^*(\alpha)=S^*(\alpha)$ and $C_1(\alpha)=C(\alpha)$ are, respectively, the usual clasess of starlike and convex functions of order $\alpha(0\le\alpha<1)$ in $\mathbb{U}$. In special cases $S_1^*(0)=S^*$ and $C_1=C$ are, respectively, the familier classess of starlike and convex functions in $\mathbb{U}$.

A function $f\in\beta-\mathcal{US}_p(\alpha)$ is $\beta-$ uniformly $p-$ valently starlike of order $\alpha(-1\le\alpha<p)$, that is, $f\in\beta-\mathcal{US}_p(\alpha)$, if it is satisfies the following inequality;

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\beta\left|\frac{zf'(z)}{f(z)}-p\right|+\alpha, \quad (\beta\ge0,\ z\in\mathbb{U}). \tag{4}$$

Furthermore a function $f\in\beta-\mathcal{UC}_p(\alpha)$ is $\beta-$uniformly $p-$valently convex of order $\alpha(-1\le\alpha<p)$, that is, $f\in\beta-\mathcal{UC}_p(\alpha)$, if it is satisfies the following inequality;

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\beta\left|1+\frac{zf''(z)}{f'(z)}-p\right|+\alpha, \quad (\beta\ge0,\ z\in\mathbb{U}). \tag{5}$$

These classes generalize various other classes which are worthly to mention here. For example $p=1$, the classes $\beta-\mathcal{US}(\alpha)$ and $\beta-\mathcal{UC}(\alpha)$ introduced by Bharti, Parvatham and Swaminathan (see [2]). Also, the classes $\beta-\mathcal{US}_p(\lambda,\gamma,\alpha)$ and $\beta-\mathcal{UC}_1(0)=\beta-\mathcal{UCV}$ are the known classes of $\beta-$uniformly starlike and convex functions, respectively (see[13]). Using the Alexander type relation, we can obtain the class $\beta-\mathcal{US}_p(\alpha)$ in the following way:

$$f \in \beta-\mathcal{UC}_p(\alpha) \Leftrightarrow \frac{zf'}{p} \in \beta-\mathcal{US}_p(\alpha).$$

The class $1-\mathcal{UC}_1(0)=\mathcal{UCV}$ of uniformly convex functions was defined by Goodman [12] while the class $1-\mathcal{US}_1(0)=S\mathcal{P}$ was considered by Ronning [19].

For $f \in \mathcal{A}_p$ given by (1.1) and $g(z)$ given by

$$g(z)=z^p+\sum_{k=p+1}^{\infty} b_k z^k \tag{6}$$

Their convolution (or Hadamard product), denoted by $(f*g)$, is defined as

$$(f*g)(z)=z^p+\sum_{k=p+1}^{\infty} a_k b_k z^k=(g*f)(z), \ (z\in\mathbb{U}). \tag{7}$$

Shenan et al. [20] introduced the operator $D_p^n:\mathcal{A}_p\to\mathcal{A}_p$ is defined by

$$\begin{aligned} D_p^0 f(z) &= f(z) \\ D_p^1 f(z) &= Df(z)=\frac{zf'(z)}{p} \\ D_p^n f(z) &= D\left(D^{n-1}f(z)\right). \end{aligned} \tag{8}$$

The differential operator D_p^n for analytic and univalent functions was introduced by Salagean ([21]) for $p=1$. It can be easily seen that the operator D_p^n on the function $f(z)$ is given by (1)

$$D_p^n f(z)=z^p+\sum_{k=p+1}^{\infty}\left(\frac{k}{p}\right)^n a_k z^k.$$

By using the operator D_p^n defined by (1,9), we introduce the new classes $\beta-\mathcal{US}_p(n,\gamma,\alpha)$ and $\beta-\mathcal{UC}_P(n,\gamma,\alpha)$ as follows:

Definition 1. Let $-1\le\alpha<p,\ \beta\ge 0$ and $\gamma\in\mathbb{C}-\{0\}$. A function $f\in\mathcal{A}_p$ is in the class $\beta-\mathcal{US}_p(n,\gamma,\alpha)$ if and only if for all $z\in\mathbb{U}$

$$\operatorname{Re}\left\{p+\frac{1}{\gamma}\left(\frac{z\left(D_p^n f(z)\right)'}{D_p^n f(z)}-p\right)\right\}>\beta\left|\frac{1}{\gamma}\left(\frac{z\left(D_p^n f(z)\right)'}{D_p^n f(z)}-p\right)\right|+\alpha. \tag{9}$$

Definition 2. Let $-1\le\alpha<p,\ \ \beta\ge 0$ and $\gamma\in\mathbb{C}-\{0\}$. A function $f\in\mathcal{A}_p$ is in the class $\beta-\mathcal{UC}_p(n,\gamma,\alpha)$ if and only if for all $z\in\mathbb{U}$

$$\operatorname{Re}\left\{p+\frac{1}{\gamma}\left(\frac{z\left(D_p^n f(z)\right)''}{\left(D_p^n f(z)\right)'}+1-p\right)\right\}>\beta\left|\frac{1}{\gamma}\left(\frac{z\left(D_p^n f(z)\right)''}{\left(D_p^n f(z)\right)'}+1-p\right)\right|+\alpha. \tag{10}$$

We note that by specializing the parameters $n,\ p,\ \gamma,\ \beta$ and α in the classes $\beta-\mathcal{US}_p(n,\gamma,\alpha)$ and $\beta-\mathcal{UC}_p(n,\gamma,\alpha)$, these classes reduces to several well-known subclasses of analytic functions. For example, for $p=1$ and $n=0$ the classes $\beta-\mathcal{US}_p(n,\gamma,\alpha)$ and $\beta-\mathcal{UC}_p(n,\gamma,\alpha)$ reduces to the classes $\beta-\mathcal{US}_p(\gamma,\alpha)$ and $\beta-\mathcal{UC}_p(\gamma,\alpha)$, respectively. Someone can find mor information about these classes in Deniz, Orhan and Sokol [9] and Orhan, Deniz and Raducanu [16].

Definition 3. Let $l=(l_1,l_2,...l_m)\in\mathbb{N}_0^m$, $\mu=(\mu_1,\mu_2,...,\mu_m)\in\mathbb{R}_+^m$ for all $i=\overline{1,m},\ \ m\in\mathbb{N}.$ we define the following general integral operators

$$\begin{aligned}&\mathcal{I}_{p,m}^{l,\mu}\left(f_1,f_2,...,f_m\right):\mathcal{A}_p^m\to\mathcal{A}_p\\&\mathcal{I}_{p,m}^{l,\mu}\left(f_1,f_2,...,f_m\right)=\mathcal{F}_{p,m,l,\mu}(z),\\&\mathcal{F}_{p,m,l,\mu}(z)=\int_0^z pt^{p-1}\prod_{i=1}^m\left(\frac{D_p^{l_i}f_i(t)}{t^p}\right)^{\mu_i}dt\end{aligned} \tag{11}$$

where $f_i\in\mathcal{A}_p$ for all $i=\overline{1,m}$ and D_p^n is defined by (8).

Remark 1. We note that if $l_1=l_2=...=l_m=0$ for all $i=\overline{1,m}$, then the integral operator $\mathcal{F}_{p,m,l,\mu}(z)$ reduces to the operator $F_p(z)$ which was studied by Frasin (see [10]). Upon setting $p=1$ in the operator (11), we can obtain the integral operator $\mathbb{F}_m(z)$ which was studied by Oros and Oros (see [17]). For $p=1$ and $l_1=l_2=...=l_m=0$ in (11), the integral operator $\mathcal{F}_{p,m,l,\mu}(z)$ reduces to the operator $\mathbb{F}_m(z)$ which was studied by Breaz and Breaz (see [6]). Observe that when $p=m=1,\ l_1=0$ and $\mu_1=\mu$, we obtain the integral operator $I_\mu(f)(z)$ which was studied by Pescar and Owa (see [18]), for $\mu_1=\mu\in[0,1]$ special case of the operator $I_\mu(f)(z)$ was studied by by Miller, Mocanu and Reade (see [14]). For $p=m=1,\ l_1=0$ and $\mu_1=\mu=1$ in (11), we have Alexander integral operator $I(f)(z)$ in [1].

In this paper, we consider the integral operator $\mathcal{F}_{p,m,l,\mu}(z)$ defined by (11), and study its properties on the classes $\beta-\mathcal{US}_p(n,\gamma,\alpha)$ and $\beta-\mathcal{UC}_p(n,\gamma,\alpha)$. As special cases the order of convexity of the operator $\int_0^z\left(\frac{f(t)}{t}\right)^\mu dt$ are given.

2. Sufficient conditions of the integral operator $\mathcal{F}_{p,m,l,\mu}(z)$

First, in this section we prove a sufficient condition for the integral operator $\mathcal{F}_{p,m,l,\mu}(z)$ to be $p-$valently convex.

Theorem 1. Let $l=(l_1,l_2,...,l_m)\in\mathbb{N}_0^m,\ \mu=(\mu_1,\mu_2,...,\mu_m)\in\mathbb{R}_+^m,\ -1\le\alpha_i<p,\ \beta_i\ge 0,\ \gamma\in\mathbb{C}-\{0\}$ and $f_i\in\beta_i-\mathcal{US}_p(l_i,\gamma,\alpha_i)$ for all $i=\overline{1,m}$. Moreover, suppose that these numbers satisfy the following inequality

$$0\le p+\sum_{i=1}^m\mu_i(\alpha_i-p)<p. \tag{12}$$

Then the integral operator $\mathcal{F}_{p,m,l,\mu}(z)$ defined by (11) is $p-$ valently convex of complex order $\gamma(\gamma\in\mathbb{C}-\{0\})$ and type $p+\sum_{i=1}^m\mu_i(\alpha_i-p)$.

Proof. From the definition (11), we observe that $\mathcal{F}_{p,m,l,\mu}(z)\in\mathcal{A}_p$. On the other hand, it is easy to see that

$$\mathcal{F}'_{p,m,l,\mu}(z)=pz^{p-1}\prod_{i=1}^m\left(\frac{D_p^{l_i}f_i(z)}{z^p}\right)^{\mu_i} \tag{13}$$

Now we differentiate (13) logarithmically and multiply by z, we obtain

$$\frac{z\mathcal{F}''_{p,m,l,\mu}(z)}{\mathcal{F}'_{p,m,l,\mu}(z)}+1-p=\sum_{i=1}^{m}\mu_i\left(\frac{z\left(D_p^{l_i}f_i\right)'(z)}{\left(D_p^{l}f_i\right)(z)}-p\right). \tag{14}$$

Then multiplying the ralation (14) with $\frac{1}{\gamma}$,

$$\frac{1}{\gamma}\left(\frac{z\mathcal{F}''_{p,m,l,\mu}(z)}{\mathcal{F}'_{p,m,l,\mu}(z)}+1-p\right)=\sum_{i=1}^{m}\mu_i\frac{1}{\gamma}\left(\frac{z\left(D_p^{l_i}f_i\right)'(z)}{\left(D_p^{l_i}f_i\right)(z)}-p\right). \tag{15}$$

The relation (15) is equivalent to

$$p+\frac{1}{\gamma}\left(\frac{z\mathcal{F}''_{p,m,l,\mu}(z)}{\mathcal{F}'_{p,m,l,\mu}(z)}+1-p\right)=p+\sum_{i=1}^{m}\mu_i\left(p+\frac{1}{\gamma}\left(\frac{z\left(D_p^{l_i}f_i\right)'(z)}{\left(D_p^{l_i}f_i\right)(z)}-p\right)\right)-p\sum_{i=1}^{m}\mu_i. \tag{16}$$

Lastly, we calculate the real part of both sides of (16) and obtain

$$\operatorname{Re}\left\{p+\frac{1}{\gamma}\left(\frac{z\mathcal{F}''_{p,m,l,\mu}(z)}{\mathcal{F}'_{p,m,l,\mu}(z)}+1-p\right)\right\}=\sum_{i=1}^{m}\mu_i\operatorname{Re}\left\{\left(p+\frac{1}{\gamma}\left(\frac{z\left(D_p^{l_i}f_i\right)'(z)}{\left(D_p^{l_i}f_i\right)(z)}-p\right)\right)\right\}-p\sum_{i=1}^{m}\mu_i+p. \tag{17}$$

Since $f_i\in\beta_i-\mathcal{U}S_p\left(l_i,\gamma,\alpha_i\right)$ for all $i=\overline{1,m}$ from (9) and (17), we have

$$\operatorname{Re}\left\{p+\frac{1}{\gamma}\left(\frac{z\mathcal{F}''_{p,m,l,\mu}(z)}{\mathcal{F}'_{p,m,l,\mu}(z)}+1-p\right)\right\}>\sum_{i=1}^{m}\frac{\mu_i\beta_i}{|\gamma|}\left|\frac{z\left(D_p^{l_i}f_i\right)'(z)}{\left(D_p^{l_i}f_i\right)(z)}-p\right|+p+\sum_{i=1}^{m}\mu_i\left(\alpha_i-p\right). \tag{18}$$

Because $\sum_{i=1}^{m}\frac{\mu_i\beta_i}{|\gamma|}\left|\frac{z\left(D_p^{l_i}f_i\right)'(z)}{\left(D_p^{l_i}f_i\right)(z)}-p\right|>0$, for all $i=\overline{1,m}$, from (18), we obtain

$$\operatorname{Re}\left\{p+\frac{1}{\gamma}\left(\frac{z\mathcal{F}''_{p,m,l,\mu}(z)}{\mathcal{F}'_{p,m,l,\mu}(z)}+1-p\right)\right\}>p+\sum_{i=1}^{m}\mu_i\left(\alpha_i-p\right).$$

Therefore, the operator $\mathcal{F}_{p,m,l,\mu}(z)$ is $p-$valently convex of complex order $\gamma\left(\gamma\in\mathbb{C}-\{0\}\right)$ and type $p+\sum_{i=1}^{m}\mu_i(\alpha_i-p)$. This evidently completes the proof of Theorem 1.

Remark 1.

1. Letting $\gamma=1$ and $l_i=0$ for all $i=\overline{1,m}$ in Theorem 1., we obtain Theorem 2.4 in [10].
2. Letting $p=1,\ \gamma=1$ and $l_i=0$ for all $i=\overline{1,m}$ in Theorem 1., we obtain Theorem 1 in [4].
3. Letting $p=1,\ \gamma=1$ and $\alpha_i=l_i=0$ for all $i=\overline{1,m}$ in Theorem 1., we obtain Theorem 2.8 in [7].
4. Letting $p=1,\ \beta=0$ and $l_i=0$ for all $i=\overline{1,m}$ in Theorem 1., we obtain Theorem 1 in [3].
5. Letting $p=1,\ \beta=0,\ \alpha_i=\mu$ and $l_i=0$ for all $i=\overline{1,m}$ in Theorem 1., we obtain Theorem 1 in [8].
6. Letting $p=1,\ \beta=0,\ \alpha_i=0$ and $l_i=0$ for all $i=\overline{1,m}$ in Theorem 1., we obtain Theorem 1 in [5].

Putting $p=m=1,\ l_1=0,\ \mu_1=\mu,\ \alpha_1=\alpha,\ \beta_1=\beta$ and $f_1=f$ in Theorem 1., we have

Corollary 1. Let $\mu>0,\ -1\le\alpha<1,\ \beta\ge 0,\ \gamma\in\mathbb{C}-\{0\}$ and $f\in\beta-\mathcal{US}(\gamma,\alpha)$. If $0\le 1+\mu(\alpha-1)<1$, then $\int_0^z\left(\frac{f(t)}{t}\right)^{\mu}dt$ is convex of complex order $\gamma\left(\gamma\in\mathbb{C}-\{0\}\right)$ and type $\mu(\alpha-1)+1$ in $\mathbb{U}$.

Theorem 2. Let $l=(l_1,l_2,...,l_m)\in\mathbb{N}_0^m,\ \mu=(\mu_1,\mu_2,...,\mu_m)\in\mathbb{R}_+^m,\ -1\le\alpha_i<p,\ \beta_i>0,\ \gamma\in\mathbb{C}-\{0\}$ for all $i=\overline{1,m}$ and

$$\left|\frac{z\left(D_p^{l_i}f_i\right)'(z)}{\left(D_p^{l_i}f_i\right)(z)}-p\right|>-\frac{p+\sum_{i=1}^{m}\mu_i(\alpha_i-p)}{\sum_{i=1}^{m}\frac{\mu_{i(\alpha_i-p)}}{|\gamma|}}\tag{19}$$

for all $i=\overline{1,m}$, then the integral operator $\mathcal{F}_{p,m,l,\mu}(z)$ defined by (11) is $p-$valenty convex of complex order $\gamma\left(\gamma\in\mathbb{C}-\{0\}\right)$.

Proof. From (18) and (19) we easily get $\mathcal{F}_{p,m,l,\mu}(z)$ is $p-$valenty convex of complex order γ.

From Theorem 2., we easily get

Corollary 2. Let $l=(l_1,l_2,...,l_m)\in\mathbb{N}_0^m,\ \mu=(\mu_1,\mu_2,...,\mu_m)\in\mathbb{R}_+^m,\ -1\le\alpha_i<p,\ \beta_i>0,\ \gamma\in\mathbb{C}-\{0\}$ for all $i=\overline{1,m}$ and

$$\operatorname{Re}\left(\frac{z\left(D_p^{l_i} f_i\right)'}{\left(D_p^{l_i} f_i\right)(z)}\right) > p - \frac{p+\sum_{i=1}^{m}\mu_i\left(\alpha_i - p\right)}{\sum_{i=1}^{m}\frac{\mu_i\beta_i}{|\gamma|}}$$

that is $D_p^{l_i} f_i \in S_P^*(\sigma)$, where $\sigma = p - \left(p+\sum_{i=1}^{m}\mu_i\left(\alpha_i - p\right)\right) / \sum_{i=1}^{m}\frac{\mu_i\beta_i}{|\gamma|}$; $0 \le \sigma < p$ for all $i=\overline{1,m}$, then the integral operator $\mathcal{F}_{p,m,l,\mu}(z)$ is $p-$valently convex of complex order $\gamma\left(\gamma \in \mathbb{C}-\{0\}\right)$.

Putting $p=m=1,\ l_1=0, \mu_1=\mu, \alpha_1=\alpha, \beta_1=\beta$ and $f_1=f$ in Corollary 2., we have

Corollary 3. Let $\mu>0,\ -1\le\alpha<1,\ \beta\ge 0,\ \gamma\in\mathbb{C}-\{0\}$ and $f\in S^*(\sigma)$ where $\sigma=\left[\mu\left(\beta+(1-\alpha)|\gamma|\right)-|\gamma|\right]/\mu\beta$, $0\le\sigma<1$, then the integral operator $\int_0^z\left(\frac{f(t)}{t}\right)^{\mu}dt$ is convex of complex order $\gamma\left(\gamma\in\mathbb{C}-\{0\}\right)$ in $\mathbb{U}$.

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On General Results on Absolute Matrix Summability Factors

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Abstract
In the present paper, two known theorems dealing with absolute Riesz summability with weaker conditions are generalized for $\varphi-\left|A,\beta;\delta\right|_k$ summability of infinite series and Fourier series.

Keywords: Fourier series, infinite series, absolute matrix summability.

1. Introduction

Let $\sum a_n$ be an infinite series with its partial sums (s_n). Let (p_n) be a sequence of positive numbers such that

$$P_n=\sum_{v=0}^{n} p_v \to \infty \ as \ \ n\to\infty, \ (P_{-k}=p_{-k}=0, \ k\geq 1).$$

Let $A=(a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of non-zero diagonal entries. Then A defines the sequence-to sequence transformation, mapping the sequence $s=(s_n)$ to $As=(A_n(s))$, where

$$A_n(s)=\sum_{v=0}^{n} a_{nv}s_v \qquad n=0,1,...$$

Let (φ_n) be any sequence of positive real numbers. The series $\sum a_n$ is said to be summable $\varphi-\left|A,\beta;\delta\right|_k$, $k\geq 1$, $\delta\geq 0$ and β is a real number, if (see [1])

$$\sum_{n=1}^{\infty}\varphi_n^{\beta(\delta k+k-1)}\left|A_n(s)-A_{n-1}(s)\right|^k<\infty. \tag{1}$$

Further, two lower semimatrices $\overline{A}=(\overline{a}_{nv})$ and $\hat{A}=(\hat{a}_{nv})$ are defined as follows:

$$\overline{a}_{nv}=\sum_{i=v}^{n} a_{ni}\,, \ \ n,v=0,1,... \tag{2}$$

$$\hat{a}_{00}=\overline{a}_{00}=a_{00}\,, \ \ \hat{a}_{nv}=\overline{a}_{nv}-\overline{a}_{n-1,v}\,, \ \ n=1,2,... \tag{3}$$

$$\overline{\Delta}A_n(s)=A_n(s)-A_{n-1}(s)=\sum_{v=0}^{n}\hat{a}_{nv}a_v. \tag{4}$$

By taking $\beta=1$, $\delta=0$, $\varphi_n=P_n/p_n$ and $a_{nv}=\dfrac{p_v}{P_n}$ in (1), we get $\left|\bar{N},p_n\right|_k$ summability method (see [2]).

Also by taking $\beta=1$, $\delta=0$, $\varphi_n=n$, $a_{nv}=\dfrac{p_v}{P_n}$ and $p_n=1$ for all values of n, then we obtain $\left|C,1\right|_k$ summability method (see [3]).

For any sequence (λ_n), it should be noted that $\Delta\lambda_n=\lambda_n-\lambda_{n+1}$, $\Delta^0\lambda_n=\lambda_n$ and $\Delta^k\lambda_n=\Delta\Delta^{k-1}\lambda_n$ for $k=1,2,...$ (see [4]). Also, it should be noted that (t_n) is the n-th $(C,1)$ mean of the sequence (na_n), i.e., $t_n=\dfrac{1}{n+1}\sum_{v=1}^{n} v\,a_v$.

2. Known Result

Bor [5] has proved the following theorem on $\left|\bar{N},p_n\right|_k$ summability by using a positive non-decreasing sequence.

Theorem 2.1. Let (X_n) be a positive non-decreasing sequence. If the sequences (X_n), (γ_n), (λ_n) and (p_n) satisfy the conditions

$$\left|\Delta\lambda_n\right|\le\gamma_n, \tag{5}$$

$$\gamma_n\to 0 \quad as \quad n\to\infty, \tag{6}$$

$$\sum_{n=1}^{\infty} n\left|\Delta\gamma_n\right|X_n<\infty, \tag{7}$$

$$|\lambda_n|\,X_n=O(1) \quad as \quad n\to\infty. \tag{8}$$

$$\sum_{v=1}^{n}\frac{\left|t_v\right|^k}{vX_v^{k-1}}=O(X_n) \quad as \quad n\to\infty, \tag{9}$$

$$P_n=O\left(np_n\right), \tag{10}$$

$$P_n\Delta p_n=O\left(p_np_{n+1}\right), \tag{11}$$

then the series $\sum a_n\dfrac{P_n\lambda_n}{np_n}$ is summable $\left|\bar{N},p_n\right|_k$, $k\ge 1$.

3. Main Result

The aim of this paper is to generalize Theorem 2.1 to the $\varphi-\left|A,\beta;\delta\right|_k$ summability method. For further informations, the readers should refer to [6–17] on the subject of this paper.

Now we shall prove the following theorem.

Theorem 3.1. Let $\varphi_n p_n = O(P_n)$ and $P_n = O(\varphi_n p_n)$. Let $A = (a_{nv})$ be a positive normal matrix such that

$$\overline{a}_{n0} = 1\ , \quad n = 0,1,..., \tag{12}$$

$$a_{n-1,v} \geq a_{nv} \ \text{for} \ \ n \geq v+1, \tag{13}$$

$$a_{nn} = O\left(\frac{p_n}{P_n}\right), \tag{14}$$

$$\left|\hat{a}_{n,v+1}\right| = O\left(v\left|\,\Delta_v(\hat{a}_{nv})\right|\right). \tag{15}$$

Let (X_n) be a positive non-decreasing sequence. If the conditions (5)-(8), (10), (11) and

$$\sum_{v=1}^{n} \varphi_v^{\beta(\delta k+k-1)-k+1} \frac{|t_v|^k}{vX_v^{k-1}} = O(X_n) \ \ as \quad n \to \infty, \tag{16}$$

$$\sum_{n=v+1}^{\infty} \varphi_n^{\beta(\delta k+k-1)-k+1} \left|\hat{a}_{n,v+1}\right| = O\left(\varphi_v^{\beta(\delta k+k-1)-k+1}\right), \tag{17}$$

$$\sum_{n=v+1}^{\infty} \varphi_n^{\beta(\delta k+k-1)-k+1} \left|\Delta_v(\hat{a}_{nv})\right| = O\left(\varphi_v^{\beta(\delta k+k-1)-k}\right) \tag{18}$$

are satisfied, then the series $\sum a_n \frac{P_n \lambda_n}{np_n}$ is summable $\varphi - \left|A,\beta;\delta\right|_k$, $k \geq 1$, $\delta \geq 0$ and $-\beta\left(\delta k + k - 1\right) + k > 0.$

We need the following lemmas to prove Theorem 3.1.

Lemma 3.1 ([18]). Let (X_n) be a positive non-decreasing sequence, then under the condition (6) and (7), we have

$$nX_n\gamma_n = O(1) \ \ as \ \ n \to \infty, \tag{19}$$

$$\sum_{n=1}^{\infty} X_n \gamma_n < \infty. \tag{20}$$

Lemma 3.2 ([19]). If the conditions (10) and (11) are satisfied, then

$$\Delta\left(\frac{P_n}{n^2 p_n}\right) = O\left(\frac{1}{n^2}\right). \tag{21}$$

4. Proof of Theorem 3.1

Let (I_n) denotes the A-transform of the series $\sum \frac{a_n \lambda_n P_n}{n p_n}$. Then, we have

$$\bar{\Delta} I_n = \sum_{v=1}^{n} \hat{a}_{nv} \frac{a_v \lambda_v P_v}{v p_v} = \sum_{v=1}^{n} \hat{a}_{nv} \frac{v a_v \lambda_v P_v}{v^2 p_v}$$

by (4). Then, we get

$$\begin{aligned}
\bar{\Delta} I_n &= \sum_{v=1}^{n-1} \Delta_v \left(\frac{\hat{a}_{nv} \lambda_v P_v}{v^2 p_v} \right) \sum_{r=1}^{v} r a_r + \frac{\hat{a}_{nn} P_n \lambda_n}{n^2 p_n} \sum_{v=1}^{n} v a_v \\
&= \frac{a_{nn} P_n \lambda_n}{n^2 p_n} (n+1) t_n + \sum_{v=1}^{n-1} \frac{P_v \lambda_v \Delta_v (\hat{a}_{nv})}{v^2 p_v} (v+1) t_v + \sum_{v=1}^{n-1} \frac{\hat{a}_{n,v+1} \Delta \lambda_v P_v}{v^2 p_v} (v+1) t_v \\
&\quad + \sum_{v=1}^{n-1} \hat{a}_{n,v+1} \lambda_{v+1} \Delta \left(\frac{P_v}{v^2 p_v} \right) (v+1) t_v = I_{n,1} + I_{n,2} + I_{n,3} + I_{n,4}
\end{aligned}$$

by using Abel's transformation, and to complete the proof of Theorem 3.1, it is sufficient to show that

$$\sum_{n=1}^{\infty} \varphi_n^{\beta(\delta k + k - 1)} \left| I_{n,r} \right|^k < \infty, \text{ for } r = 1,2,3,4.$$

First, by using Abel's transformation and the conditions (10), (14), (8), (16), (5) and (20), we have

$$\begin{aligned}
\sum_{n=1}^{m} \varphi_n^{\beta(\delta k + k - 1)} \left| I_{n,1} \right|^k &= \sum_{n=1}^{m} \varphi_n^{\beta(\delta k + k - 1)} \left| \frac{a_{nn} P_n \lambda_n}{n^2 p_n} (n+1) t_n \right|^k \\
&= O(1) \sum_{n=1}^{m} \varphi_n^{\beta(\delta k + k - 1)} \left(\frac{P_n}{n p_n} \right)^k a_{nn}^k \mid \lambda_n \mid^{k-1} \mid \lambda_n \mid \mid t_n \mid^k \\
&= O(1) \sum_{n=1}^{m} \varphi_n^{\beta(\delta k + k - 1) - k + 1} \mid \lambda_n \mid \frac{\mid t_n \mid^k}{n X_n^{k-1}} \\
&= O(1) \sum_{n=1}^{m-1} \Delta \left| \lambda_n \right| \sum_{r=1}^{n} \varphi_r^{\beta(\delta k + k - 1) - k + 1} \frac{\left| t_r \right|^k}{r X_r^{k-1}} + O(1) \left| \lambda_m \right| \sum_{n=1}^{m} \varphi_n^{\beta(\delta k + k - 1) - k + 1} \frac{\left| t_n \right|^k}{n X_n^{k-1}} \\
&= O(1) \sum_{n=1}^{m-1} \left| \Delta \lambda_n \right| X_n + O(1) \left| \lambda_m \right| X_m \\
&= O(1) \sum_{n=1}^{m-1} \gamma_n X_n + O(1) \left| \lambda_m \right| X_m = O(1) \quad as \quad m \to \infty.
\end{aligned}$$

Now, applying Hölder's inequality with indices k and k', where $k>1$ and $1/k+1/k'=1$, we have

$$\sum_{n=2}^{m+1}\varphi_n^{\beta(\delta k+k-1)}\left|I_{n,2}\right|^k=\sum_{n=2}^{m+1}\varphi_n^{\beta(\delta k+k-1)}\left|\sum_{v=1}^{n-1}\frac{P_v\lambda_v\Delta_v(\hat{a}_{nv})}{v^2p_v}(v+1)t_v\right|^k=O(1)\sum_{n=2}^{m+1}\varphi_n^{\beta(\delta k+k-1)}\left\{\sum_{v=1}^{n-1}\frac{P_v}{vp_v}|\Delta_v(\hat{a}_{nv})||\lambda_v||t_v|\right\}^k$$

$$=O(1)\sum_{n=2}^{m+1}\varphi_n^{\beta(\delta k+k-1)}\sum_{v=1}^{n-1}\left(\frac{P_v}{vp_v}\right)^k|\Delta_v(\hat{a}_{nv})||\lambda_v|^k|t_v|^k\left\{\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\right\}^{k-1}.$$

Now using (2), (3), (12) and (13), we get

$$\sum_{v=1}^{n-1}\left|\Delta_v(\hat{a}_{nv})\right|=\sum_{v=1}^{n-1}\left|a_{nv}-a_{n-1,v}\right|=\sum_{v=1}^{n-1}(a_{n-1,v}-a_{nv})=\bar{a}_{n-1,0}-a_{n-1,0}-\bar{a}_{n0}+a_{n0}+a_{nn}\le a_{nn}.$$

Hence, by using (14), (18), (10) and (8), we get

$$\sum_{n=2}^{m+1}\varphi_n^{\beta(\delta k+k-1)}\left|I_{n,2}\right|^k=O(1)\sum_{n=2}^{m+1}\varphi_n^{\beta(\delta k+k-1)}a_{nn}^{k-1}\left\{\sum_{v=1}^{n-1}\left(\frac{P_v}{vp_v}\right)^k|\Delta_v(\hat{a}_{nv})||\lambda_v|^k|t_v|^k\right\}$$

$$=O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^k\frac{1}{v^k}|\lambda_v|^k|t_v|^k\sum_{n=v+1}^{m+1}\varphi_n^{\beta(\delta k+k-1)-k+1}\left|\Delta_v(\hat{a}_{nv})\right|$$

$$=O(1)\sum_{v=1}^{m}\varphi_v^{\beta(\delta k+k-1)-k+1}|\lambda_v|\frac{|t_v|^k}{vX_v^{k-1}}=O(1)\quad as\quad m\to\infty,$$

as in $I_{n,1}$.

By using (15), (5), (14), (18), (10), (19) and applying Hölder's inequality, we get

$$\sum_{n=2}^{m+1}\varphi_n^{\beta(\delta k+k-1)}\left|I_{n,3}\right|^k=\sum_{n=2}^{m+1}\varphi_n^{\beta(\delta k+k-1)}\left|\sum_{v=1}^{n-1}\frac{\hat{a}_{n,v+1}\Delta\lambda_vP_v}{v^2p_v}(v+1)t_v\right|^k=O(1)\sum_{n=2}^{m+1}\varphi_n^{\beta(\delta k+k-1)}\left\{\sum_{v=1}^{n-1}\left(\frac{P_v}{vp_v}\right)|\hat{a}_{n,v+1}||\Delta\lambda_v||t_v|\right\}^k$$

$$=O(1)\sum_{n=2}^{m+1}\varphi_n^{\beta(\delta k+k-1)}\left(\sum_{v=1}^{n-1}\left(\frac{P_v}{vp_v}\right)v\left|\Delta_v(\hat{a}_{nv})\right|\gamma_v\left|t_v\right|\right)^k$$

$$=O(1)\sum_{n=2}^{m+1}\varphi_n^{\beta(\delta k+k-1)}\sum_{v=1}^{n-1}\left(\frac{P_v}{vp_v}\right)^k(v\gamma_v)^k|\Delta_v(\hat{a}_{nv})||t_v|^k\left\{\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\right\}^{k-1}$$

$$=O(1)\sum_{n=2}^{m+1}\varphi_n^{\beta(\delta k+k-1)-k+1}\sum_{v=1}^{n-1}\left(\frac{P_v}{vp_v}\right)^k(v\gamma_v)^k|\Delta_v(\hat{a}_{nv})||t_v|^k$$

$$=O(1)\sum_{v=1}^{m}\left(\frac{P_v}{vp_v}\right)^k (v\gamma_v)^{k-1}(v\gamma_v)|t_v|^k \sum_{n=v+1}^{m+1}\varphi_n^{\beta(\delta k+k-1)-k+1}|\Delta_v(\hat{a}_{nv})|$$

$$=O(1)\sum_{v=1}^{m}\varphi_v^{\beta(\delta k+k-1)-k+1}v\gamma_v\frac{|t_v|^k}{vX_v^{k-1}}.$$

Here, by Abel's transformation and using the conditions (16), (7), (20) and (19), we get

$$\sum_{n=2}^{m+1}\varphi_n^{\beta(\delta k+k-1)}|I_{n,3}|^k=O(1)\sum_{v=1}^{m-1}\Delta(v\gamma_v)\sum_{r=1}^{v}\varphi_r^{\beta(\delta k+k-1)-k+1}\frac{|t_r|^k}{rX_r^{k-1}}+O(1)m\gamma_m\sum_{v=1}^{m}\varphi_v^{\beta(\delta k+k-1)-k+1}\frac{|t_v|^k}{vX_v^{k-1}}$$

$$=O(1)\sum_{v=1}^{m-1}v|\Delta\gamma_v|X_v+O(1)\sum_{v=1}^{m-1}\gamma_vX_v+O(1)m\gamma_mX_m=O(1)\ \ as\ \ m\to\infty.$$

Now, using the fact that $\Delta\left(\frac{P_v}{v^2p_v}\right)=O\left(\frac{1}{v^2}\right)$, and also using (15), (14), (17), (8), we have

$$\sum_{n=2}^{m+1}\varphi_n^{\beta(\delta k+k-1)}|I_{n,4}|^k=\sum_{n=2}^{m+1}\varphi_n^{\beta(\delta k+k-1)}\left|\sum_{v=1}^{n-1}\hat{a}_{n,v+1}\lambda_{v+1}\Delta\left(\frac{P_v}{v^2p_v}\right)(v+1)t_v\right|^k=O(1)\sum_{n=2}^{m+1}\varphi_n^{\beta(\delta k+k-1)}\left\{\sum_{v=1}^{n-1}\frac{1}{v}|\hat{a}_{n,v+1}||\lambda_{v+1}||t_v|\right\}^k$$

$$=O(1)\sum_{n=2}^{m+1}\varphi_n^{\beta(\delta k+k-1)}\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\lambda_{v+1}|^k\frac{|t_v|^k}{v}\left\{\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\right\}^{k-1}$$

$$=O(1)\sum_{n=2}^{m+1}\varphi_n^{\beta(\delta k+k-1)-k+1}\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\lambda_{v+1}|^k\frac{|t_v|^k}{v}$$

$$=O(1)\sum_{v=1}^{m}|\lambda_{v+1}|^{k-1}|\lambda_{v+1}|\frac{|t_v|^k}{v}\sum_{n=v+1}^{m+1}\varphi_n^{\beta(\delta k+k-1)-k+1}|\hat{a}_{n,v+1}|$$

$$=O(1)\sum_{v=1}^{m}\varphi_v^{\beta(\delta k+k-1)-k+1}|\lambda_{v+1}|\frac{|t_v|^k}{vX_v^{k-1}}$$

$$=O(1)\ \text{as}\ m\to\infty$$

as in $I_{n,1}$. This completes the proof of Theorem 3.1.

5. An Application

Let f be a periodic function with period 2π and Lebesque integrable over $(-\pi,\pi)$. The trigonometric Fourier series of f is defined as

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right) = \sum_{n=0}^{\infty} C_n(x)$$

where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx,\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx,\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx.$$

Write $\phi(t) = \frac{1}{2}\{f(x+t)+f(x-t)\}$ and $\phi_1(t) = \frac{1}{t}\int_0^t \phi(u)du.$ If $\phi_1(t) \in BV(0,\pi)$, then $t_n(x) = O(1)$, where $t_n(x)$ is the n-th $(C,1)$ mean of the sequence $(nC_n(x))$ (see [20]). By using this, the following theorem has been obtained in [5].

Theorem 5.1. If $\phi_1(t) \in BV(0,\pi)$, and the sequences (p_n), (λ_n), (γ_n) and (X_n) satisfy the conditions of Theorem 2.1, then the series $\sum C_n(x)\frac{P_n\lambda_n}{np_n}$ is summable $\left|\bar{N}, p_n\right|_k$, $k \ge 1$.

Theorem 5.1 is generalized for $\varphi - \left|A,\beta;\delta\right|_k$ summability method as in the following form.

Theorem 5.2. If $\phi_1(t) \in BV(0,\pi)$, and the sequences (p_n), (λ_n), (γ_n), (φ_n) and (X_n) satisfy the conditions of Theorem 3.1, then the series $\sum C_n(x)\frac{P_n\lambda_n}{np_n}$ is summable $\varphi - \left|A,\beta;\delta\right|_k$, $k \ge 1$, $\delta \ge 0$ and $-\beta(\delta k + k - 1) + k > 0.$

6. Conclusions

If we take $\beta = 1$, $\delta = 0$, $\varphi_n = P_n/p_n$ and $a_{nv} = p_v/P_n$ in Theorem 3.1, then we get Theorem 2.1. Also, if we take $\beta = 1$, $\delta = 0$, $\varphi_n = P_n/p_n$ and $a_{nv} = p_v/P_n$ in Theorem 5.2, then we get Theorem 5.1.

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Novel Convergence Results in Vector Valued Metric Spaces

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Abstract

We enunciate $\Gamma_k -$ Reich type contraction in this study, which is a generalization of Reich type contraction in the setting of vector valued metric spaces. We explore fixed point outcomes and their uniqueness for a single operator, as well as common fixed point findings for two operators. The proposed results enrich and extend a multitude of results in the current context of study.

Keywords: Vector valued metric space, fixed point, $\Gamma_k -$ Reich type contraction, semilinear operator.

1. Introduction and Preliminaries

Fixed point theory combines analysis, topology, and geometry in a satisfying way. It has a wide range of applications in science, including physics, mathematical engineering, economics, biology, and chemistry. Fixed points results on some spaces that satisfies some specific contractive conditions are useful in various research activities, see e.g., [12,13,14,15,16,17].

Recently, in 2012, a new contraction named as F-contraction was given by Wardowski [11] and proved fixed point theorems using the novel concept of F-contraction. After that, many researchers worked on F-contractive mappings see e.g., [1,3,4,13,16], and proved many convergence results within this approach.

Reich [6], on the other hand, proposed a unique generalisation of Banach's fixed point theorem for single and multivalued mappings. For several authors, Reich type mappings have been the focus of intense investigation since then.

Motivated by the work mentioned above, in this paper, we introduce a generalization of Reich type contractive condition [6,7] named as $\Gamma_k -$ Reich type contraction to prove some fixed point and common fixed point results within the frame work of vector valued metric spaces. We also discuss the existence and uniqueness of the results.

In this manuscript, we denote by $\mathbb{R}^m$, the set of $m \times 1$ non-negative real matrices, θ be $m \times m$ zero matrix, $\check{O}$ be $m \times 1$ zero matrix, $\check{I}$ be $m \times m$ identity matrix and $\mathrm{M}_{(m\times m)}(\mathbb{R}_+)$be $m \times m$ matrices with non-negative elements. The Cauchyness, completeness and convergnce in V_{vms} is same as in the usual metric space.

Definition 1.1. [5] Let $\mathbb{Z}$ be nonempty set, $\mathbb{R}^m$ denotes real matrices of $m \times 1$, then $d_v: \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}^m$ then $(\mathbb{Z}, d_v)$ is called a vector valued metric space (V_{vms}), if for all $x_1, x_2, x_3 \in \mathbb{Z}$, the follwing postulates are satisfied:

1. $d_v(x_1, x_2) \succeq \check{O}$ and $d_v(x_1, x_2) = \check{O}$ if and only if $x_1 = x_2$,
2. $d_v(x_1, x_2) = d_v(x_2, x_1)$,
3. $d_v(x_1, x_2) \preceq d_v(x_1, x_3) + d_v(x_3, x_2)$,

where $\check{O}$ represents $m \times 1$ real matrices, $\preceq$ is coordinate wise ordering on $\mathbb{R}^m, \alpha \preceq \beta$ iff $\alpha_j \leq \beta_j$ and $\alpha \prec \beta$ iff $\alpha_j \prec \beta_j$, for all $j \in \{1,2,3 \dots m\}$ respectively.

Theorem 1.2. [10] Suppose $P \in \mathbb{M}_{(m\times m)}(\mathbb{R}_+)$ then:

1. P is convergent to zero.
2. Eigen values of P lies in the open unit disc i.e $|\mu| < 1$, for all $\mu \in \mathbb{C}$ with $\det(P - \mu I) = 0$.
3. $\check{I} - P$ is nonsingular and

$$(\check{I} - P)^{-1} = \check{I} + P + \cdots + P^n + \cdots$$

where $P \in \mathbb{M}_{(m\times m)}(\mathbb{R}_+)$, then P is convergent to zero iff $P^n \to \theta$ as $n \to \infty$.

Definition 1.3. [10] Suppose that $P = [p_{i,j}]$ and $Q = [q_{i,j}]$ (be two real matrices $n \times m$), then $P \geq Q(> Q)$, if $p_{i,j} \geq q_{i,j}(> q_{i,j})$, for all $1 \leq i \leq n, 1 \leq j \leq m$. If O is null matrix then and $P \geq O(> O)$, then P is positive matrix.

Example 1.4. A matrix $P \in \mathbb{M}_{(m\times m)}(\mathbb{R}_+)$of the form:

$$P = \begin{pmatrix} g & h \\ g & h \end{pmatrix} \text{ or } P = \begin{pmatrix} g & g \\ h & h \end{pmatrix}$$

with $g + h < 1$, then P converges to zero.

Example 1.5. A matrix $P \in \mathbb{M}_{(m\times m)}(\mathbb{R}_+)$of the form:

$$P = \begin{pmatrix} \alpha_1 & 0 & \dots & 0 \\ 0 & \alpha_2 & \dots & 0 \\ \vdots & \dots & \ddots & \vdots \\ 0 & 0 & \dots & \alpha_n \end{pmatrix}_{n\times n}$$

if $\max\{\alpha_j : j \in \{1,2,3 \dots n\}\} < 1$, then P is convergent to zero.

Definition 1.6. [2] Suppose $\Gamma: \mathbb{R}_+^m \to \mathbb{R}^m$ such as

1. Γ is strictly increasing i.e for all
   $\alpha = (\alpha_j)_{j=1}^m, \beta = (\beta_j)_{j=1}^m \in \mathbb{R}_+^m$ with $\alpha \prec \beta$ then $\Gamma(\alpha) \prec \Gamma(\beta)$, (1)

2. For all $\{\alpha_n\} = \{\alpha_n^1, \alpha_n^2, \alpha_n^3 \dots \alpha_n^m\} \in \mathbb{R}_+^m$ such that $\lim_{n\to\infty} \alpha_n^j = 0$, iff $\lim_{n\to\infty} \beta_n^j = -\infty$ such that

$$\{\beta_n^1, \beta_n^2, \beta_n^3 \dots \beta_n^m\} = \Gamma\{\alpha_n^1, \alpha_n^2, \alpha_n^3 \dots \alpha_n^m\} \tag{2}$$

for each $j \in \{1,2,3 \dots m\}$.

3. There exists $\omega \in (0,1)$ such as $\lim_{\alpha_j \to 0^+} \alpha_j^\omega \beta_j = 0$ such that

$$\{\beta_n^1, \beta_n^2, \beta_n^3 \dots \beta_n^m\} = \Gamma\{\alpha_n^1, \alpha_n^2, \alpha_n^3 \dots \alpha_n^m\} \tag{3}$$

for each $j \in \{1,2,3 \dots m\}$, where Γ^m denotes set of all mappings satisfying $\{1\} - \{2\}$.

Definition 1.7. [2] Let $\mathbf{\Psi}: \mathbb{Z} \to \mathbb{Z}$ be perov type Γ-operator in V_{vms} denoted as $(\mathbb{Z}, d_v)$. If there exists $\Gamma \in \Gamma^m$ and $k = (k_j)_{j=1}^m \in \mathbb{R}_+^m$ for all $x, y \in \mathbb{Z}$ and $d_v(\mathbf{\Psi}x, \mathbf{\Psi}y) > \breve{O}$ such that

$$k + \Gamma(d_v(\mathbf{\Psi}x, \mathbf{\Psi}y)) \preceq \Gamma(Pd_v(x, y))$$

Theorem 1.8. [2] Let $\mathbf{\Psi}: \mathbb{Z} \to \mathbb{Z}$ be perov type Γ-operator in V_{vms} denoted as $(\mathbb{Z}, d_v)$, then $\mathbf{\Psi}$ has a unique fixed point.

2. New Convergence Results

In this section, we will discuss some fixed point theorems in V_{vms} by using the definition of $\Gamma_k -$ Reich type operator.

Definition 2.1. Let $\mathbf{\Psi}: \mathbb{Z} \to \mathbb{Z}$ be a self-mapping and as $(\mathbb{Z}, d_v)$ be a V_{vms}. Then the mapping $\mathbf{\Psi}$ is called a $\Gamma_k -$ Reich type operator, if there exist $P, Q, S \in \mathbb{M}_{(m\times m)}(\mathbb{R}_+), \Gamma \in \Gamma^m$ and $k = \left(k_j\right)_{j=1}^m \in \mathbb{R}_+^m$ for all $x, y \in \mathbb{Z}$ and $d_v(\mathbf{\Psi}x, \mathbf{\Psi}y) > \breve{O}$ such that

$$k + \Gamma(d_v(\mathbf{\Psi}x, \mathbf{\Psi}y)) \preceq \Gamma(Pd_v(x, y) + Qd_v(x, \mathbf{\Psi}x) + Sd_v(y, \mathbf{\Psi}y)), \qquad (4)$$

where $P + Q + S < \breve{I}$ and $S \neq \breve{I}$.

Theorem 2.2. Let $\psi: \mathbb{Z} \to \mathbb{Z}$ be $\Gamma_k -$ Reich type operator in V_{vms} $(\mathbb{Z}, d_v)$. If there exists $P, Q, S \in \mathbb{M}_{(m\times m)}(\mathbb{R}_+)$such that

1. $(\breve{I} - S)$ is non singular and $(\breve{I} - S)^{-1} \in \mathbb{M}_{(m\times m)}(\mathbb{R}_+)$. (5)
2. $\mathbb{H} = (P + Q)(\breve{I} - S)^{-1}$ such that $\theta < \mathbb{H} \preceq \breve{I}$. (6)

Then there exists a fixed point of ψ.

Proof. Consider $x_0 \in \mathbb{Z}$ defined as $x_n = \mathbf{\Psi}_{n-1}$, where $\{x_n\} \in \mathbb{Z}$ for some $n \in N \cup \{0\}$. Suppose $x_{n_0} = x_{n_0+1}$ then $\mathbf{\Psi}$ has a fixed point n_0 for some $n_0 \in N \cup \{0\}$. Further assume that $x_{n_0} \neq x_{n_0+1}$ then $d_v(x_n, x_{n+1}) = \{\alpha_n^1, \alpha_n^2, \alpha_n^3 \ldots \alpha_n^m\} = \alpha_n$, where $\alpha_n^j > 0$, for all $n \in N \cup \{0\}$ respectively and for each $j \in \{1,2,3 \ldots m\}$. By using (4), we get

$$\begin{aligned}
\Gamma(\alpha_n) &= \Gamma\big(d_v(x_n, x_{n+1})\big) = \Gamma\big(d_v(\mathbf{\Psi}x_{n-1}, \mathbf{\Psi}x_n)\big) \\
&\preceq \Gamma\big(Pd_v(x_{n-1}, x_n) + Qd_v(x_{n-1}, \mathbf{\Psi}x_{n-1}) + Sd_v(x_n, \mathbf{\Psi}x_n)\big) - k \\
&= \Gamma\big(Pd_v(x_{n-1}, x_n) + Qd_v(x_{n-1}, x_n) + Sd_v(x_n, x_{n+1})\big) - k \\
&= \Gamma\big((P + Q)d_v(x_{n-1}, x_n) + Sd_v(x_n, x_{n+1})\big) - k \\
&= \Gamma((P + Q)\alpha_{n-1} + S\alpha_n) - k \\
&\prec \Gamma\big((P + Q)\alpha_{n-1} + S\alpha_n\big).
\end{aligned}$$

Since, we can write the above equation as

$$\Gamma(\alpha_n) \prec \Gamma\big((P + Q)\alpha_{n-1} + S\alpha_n\big),$$

by using equation (1) and equation $(5), (6)$ given in theorem we get

$$\begin{aligned}
\alpha_n &\prec (P + Q)\alpha_{n-1} + S\alpha_n \\
\alpha_n &\prec (\breve{I} - S)^{-1}(P + Q)\alpha_{n-1} = \mathbb{H}\alpha_{n-1} \preceq \alpha_{n-1},
\end{aligned}$$

which shows that

$$\alpha_n \prec \alpha_{n-1}, \text{ for all } n \in N \cup \{0\}.$$

As we know that $\Gamma\{\alpha_n\} = \{\beta_n^1, \beta_n^2, \beta_n^3 \dots \beta_n^m\} = \beta_n$, then by using the above equation we can write

$$\beta_n \prec \beta_{n-1}, \text{ for all } n \in N \cup \{0\}.$$

Consequently, we have

$$\beta_n^j < \beta_{n-1}^j, \text{ for all } n \in N \cup \{0\} \text{ and for each } j \in \{1,2,3 \dots m\}.$$

Now, we choose $k_j > 0$, then

$$\beta_n^j < \beta_{n-1}^j - k_j, \text{ for all } n \in N \cup \{0\} \text{ and for each } j \in \{1,2,3 \dots m\}.$$

Hence, we acquire

$$\beta_n^j < \beta_0^j - nk_j, \text{ for all } n \in N \cup \{0\}. \tag{7}$$

Taking limit $n \to \infty$ in equation (7), we obtain

$$\lim_{n\to\infty} = -\infty.$$

By using equation (2), we obtain

$$\lim_{n\to\infty} \alpha_n^j = 0, \text{ for each } j \in \{1,2,3 \dots m\}.$$

Now by using the equation (3), there exists $\omega \in (0,1)$ such that

$$\lim_{n\to\infty} \left(\alpha_n^j\right)^\omega \beta_n^j = 0, \text{ for each } j \in \{1,2,3 \dots m\}.$$

Utilizing equation (7), we get

$$\left(\alpha_n^j\right)^\omega \beta_n^j - \left(\alpha_n^j\right)^\omega \beta_0^j \le \left(\alpha_n^j\right)^\omega nk_j \le 0, \tag{8}$$

taking limit $n \to \infty$ in equation (8), we get

$$\lim_{n\to\infty} n\left(\alpha_n^j\right)^\omega = 0, \text{ for each } j \in \{1,2,3 \dots m\}. \tag{9}$$

Making use of equation (9), there exists $n^j \in \{1,2,3\ldots\}$ for each $n \geq n^j$ such that $n\left(\alpha_n^j\right)^{\omega} \leq 1$. Furthermore, for $n \geq n_0 = \max\{n^j : j \in \{1,2,3 \ldots m\}\}$, we get

$$\alpha_n^j \leq \frac{1}{n^{\frac{1}{\omega}}}.$$

Now we have to show that $\{x_n\}$ is a Cauchy sequence in V_{vms} denoted as $(\mathbb{Z}, d_v)$ for all $n, m \in \mathbb{N}$ with $n < m$, we have

$$\begin{aligned}
d_v(x_n, x_m) &\leq d_v(x_n, x_{n+1}) + d_v(x_{n+1}, x_{n+2}) + d_v(x_{n+2}, x_{n+3}) \ldots + d_v(x_{m-1}, x_m) \\
&= \alpha_n + \alpha_{n+1} + \alpha_{n+2} + \cdots + \alpha_{m-1} \\
&= \left(\alpha_n^{(j)}\right)_{j=1}^m + \left(\alpha_{n+1}^{(j)}\right)_{j=1}^m + \left(\alpha_{n+2}^{(j)}\right)_{j=1}^m + \cdots + \left(\alpha_{m-1}^{(j)}\right)_{j=1}^m \\
&= \left(\sum_{i=1}^{m-1} \alpha_i^{(j)}\right)_{j=1}^m \\
&\leq \left(\sum_{i=1}^{\infty} \alpha_i^{(j)}\right)_{j=1}^m \\
&\leq \left(\sum_{i=1}^{\infty} \frac{1}{i^{\frac{1}{\omega}}}\right)_{j=1}^m.
\end{aligned}$$

By taking limit $n \to \infty$, series $\sum_{i=1}^{\infty} \frac{1}{i}$ is convergent, and we obtain

$$\lim_{n\to\infty} d_v(x_n, x_m) = 0,$$

which shows that $\{x_n\}$ is Cauchy sequence in a complete V_{vms}. Therefore, there exists $z^* \in \mathbb{Z}$ such that $x_n \to z^*$ as $n \to \infty$.

Furthermore, we have to show that z^* is a fixed point of $\boldsymbol{\Psi}$. For this, let $x = x_n$ and $y = z^*$, then by using equation (4), we acquire

$$\begin{aligned}
\Gamma\big(d_v(\boldsymbol{\Psi}x_n, \boldsymbol{\Psi}z^*)\big) &\leq \Gamma\big(Pd_v(x_n, z^*) + Qd_v(x_n, \boldsymbol{\Psi}x_n) + Sd_v(z^*, \boldsymbol{\Psi}z^*)\big) - k \\
&= \Gamma\big(Pd_v(x_n, z^*) + Qd_v(x_n, x_{n+1}) + Sd_v(z^*, \boldsymbol{\Psi}z^*)\big) - k \\
&\prec \Gamma\big(Pd_v(x_n, z^*) + Qd_v(x_n, x_{n+1}) + Sd_v(z^*, \boldsymbol{\Psi}z^*)\big),
\end{aligned}$$

by using equation (1), we obtain

$$d_v(x_{n+1}, \mathbf{\Psi} z^*) \prec P d_v(x_n, z^*) + Q d_v(x_n, x_{n+1}) + S d_v(z^*, \mathbf{\Psi} z^*). \tag{10}$$

By taking limit $n \to \infty$, in equation (10), we get

$$d_v(z^*, \mathbf{\Psi} z^*) \preceq S d_v(z^*, \mathbf{\Psi} z^*) < d_v(z^*, \mathbf{\Psi} z^*),$$

which is contradiction. Hence $z^* = \mathbf{\Psi} z^*$.

To show that $\mathbf{\Psi}$ has a definite fixed point, we formulate the following theorem.

Theorem 2.3. If the condition of the Theorem (2.2) are satisfied and $P \leq \check{I}$, then $\mathbf{\Psi}$ has a unique fixed point.

Proof. According to the Theorem (2.2), there exist a fixed point of $\mathbf{\Psi}, z^* \in \mathbf{\Psi}$. Now we have to show that $\mathbf{\Psi}$ has a unique fixed point for this let y^* be another fixed point of $\mathbf{\Psi}$ with $y^* \neq z^*$, then by using (4) , we have

$$\begin{aligned} k + \Gamma\big(d_v(y^*, z^*)\big) &= \Gamma\big(d_v(\mathbf{\Psi} y^*, \mathbf{\Psi} z^*)\big) \\ &\preceq \Gamma\big(P d_v(y^*, z^*) + Q d_v(y^*, \mathbf{\Psi} y^*) + S d_v(z^*, \mathbf{\Psi} z^*)\big) \\ &= \Gamma\big(P d_v(y^*, z^*) + Q d_v(y^*, y^*) + S d_v(z^*, z^*)\big) \\ &= \Gamma\big(P d_v(y^*, z^*)\big) \preceq \Gamma d_v(y^*, z^*) \end{aligned}$$

which is a contradiction, hence $y^* = z^*$ then $z^* \in \mathbf{\Psi}$ has a unique fixed point.

Corollary 2.4. Let $\mathbf{\Psi}: \mathbb{Z} \to \mathbb{Z}$ be a complete V_{vms} $(\mathbb{Z}, d_v)$. If $\Gamma \in \Gamma^m$ and $k = \big(k_j\big)_{j=1}^m \in \mathbb{R}_+^m$ for all $x, y \in \mathbb{Z}$ and $d_v(\mathbf{\Psi} x, \mathbf{\Psi} y) > \check{O}$ such that

$$k + \Gamma(d_v(\mathbf{\Psi} x, \mathbf{\Psi} y)) \preceq \Gamma\big(d_v(x, y)\big),$$

then $\mathbf{\Psi}$ has a unique fixed point.

Proof. If we take $P = \check{I}$ and $Q = S = \theta$, in Theorem (2.2) then we get the desired result.

3. Common Fixed Point for Two Self Operators

In this section, we present some common fixed point results corcerning $\Gamma_k -$ Reich type operator for two self operators.

Theorem 3.1. Let $\mathbf{\Psi}, \Phi: \mathbb{Z} \to \mathbb{Z}$ be $\Gamma_k -$ Reich type operator in V_{vms} $(\mathbb{Z}, d_v)$. If there exists $P, Q, S \in \mathrm{M}_{(m \times m)}(\mathbb{R}_+)$such that

1. $(\breve{I} - S)$ is non singular and $(\breve{I} - S)^{-1} \in \mathbb{M}_{(m\times m)}(\mathbb{R}_+)$. (11)
2. $\mathbb{H} = (P + Q)(\breve{I} - S)^{-1}$ such that $\theta < \mathbb{H} \leq \breve{I}$. (12)
3. $S, Q \leq \breve{I}$. (13)

Then $\mathbf{\Psi}$ and Φ has a common fixed point.

Proof. Consider $x_0 \in \mathbb{Z}$ defined as $\{x_n\} \in \mathbb{Z}$

$$x_{2n+1} = \mathbf{\Psi} x_{2n}$$
$$x_{2n+2} = \Phi x_{2n+1}$$

for some $n \in N \cup \{0\}$. Suppose $x_{2n_0} = x_{2n_0+1}$ then $\mathbf{\Psi}$ and Φ has a fixed point x_{2n_0} for some $n_0 \in N \cup \{0\}$. Further assume that $x_{2n_0} \neq x_{2n_0+1}$ then $d_v(x_{2n}, x_{2n+1}) = \{\alpha_{2n}^1, \alpha_{2n}^2, \alpha_{2n}^3 \ldots \alpha_{2n}^m\} = \alpha_{2n}$, where $\alpha_{2n}^j > 0$, for all $n \in N \cup \{0\}$ respectively and for each $j \in \{1,2,3 \ldots m\}$. By using equation (4), we get

$$\begin{aligned}
\Gamma(\alpha_{2n}) &= \Gamma\big(d_v(x_{2n}, x_{2n+1})\big) = \Gamma\big(d_v(\mathbf{\Psi} x_{2n-1}, \Phi x_{2n})\big) \\
&\leq \Gamma\big(P d_v(x_{2n-1}, x_{2n}) + Q d_v(x_{2n-1}, \mathbf{\Psi} x_{2n-1}) + S d_v(x_{2n}, \Phi x_{2n})\big) - k \\
&= \Gamma\big(P d_v(x_{2n-1}, x_{2n}) + Q d_v(x_{2n-1}, x_{2n}) + S d_v(x_{2n}, x_{2n+1})\big) - k \\
&= \Gamma\big((P + Q) d_v(x_{2n-1}, x_{2n}) + S d_v(x_{2n}, x_{2n+1})\big) - k \\
&= \Gamma((P + Q)\alpha_{2n-1} + S\alpha_{2n}) - k \\
&\prec \Gamma\big((P + Q)\alpha_{2n-1} + S\alpha_{2n}\big).
\end{aligned}$$

Since, we can write the above equation as

$$\Gamma(\alpha_{2n}) \prec \Gamma\big((P + Q)\alpha_{2n-1} + S\alpha_{2n}\big),$$

by using the (1) and using the points (11), (12) given in theorem we get

$$\begin{aligned}
\alpha_{2n} &\prec (P + Q)\alpha_{2n-1} + S\alpha_{2n} \\
\alpha_{2n} &\prec (\breve{I} - S)^{-1}(P + Q)\alpha_{2n-1} = \mathbb{H}\alpha_{2n-1} \preceq \alpha_{2n-1},
\end{aligned}$$

which shows that

$$\alpha_{2n} \prec \alpha_{2n-1}, \text{ for all } n \in N \cup \{0\}.$$

Similarly

$$\begin{aligned}\Gamma(\alpha_{2n+1}) &= \Gamma\big(d_v(x_{2n+2}, x_{2n+1})\big) = \Gamma\big(d_v(\boldsymbol{\Psi} x_{2n}, \Phi x_{2n+1})\big)\\ &\preceq \Gamma\big(P d_v(x_{2n}, x_{2n+1}) + Q d_v(x_{2n}, \boldsymbol{\Psi} x_{2n}) + S d_v(x_{2n+1}, \Phi x_{2n+1})\big) - k\\ &= \Gamma\big(P d_v(x_{2n}, x_{2n+1}) + Q d_v(x_{2n}, x_{2n+1}) + S d_v(x_{2n+1}, x_{2n+2})\big) - k\\ &= \Gamma\big((P+Q) d_v(x_{2n}, x_{2n+1}) + S d_v(x_{2n+1}, x_{2n+2})\big) - k\\ &= \Gamma((P+Q)\alpha_{2n} + S\alpha_{2n+1}) - k\\ &\prec \Gamma\big((P+Q)\alpha_{2n} + S\alpha_{2n+1}\big).\end{aligned}$$

Since, we can write the above equation as

$$\Gamma(\alpha_{2n+1}) \prec \Gamma((P+Q)\alpha_{2n-1} + S\alpha_{2n}),$$

by using the equation (1) and using the points (11), (12) given in theorem we get

$$\begin{aligned}\alpha_{2n+1} &\prec (P+Q)\alpha_{2n} + S\alpha_{2n+1}\\ \alpha_{2n+1} &\prec (\check{I} - S)^{-1}(P+Q)\alpha_{2n} = \mathbb{H}\alpha_{2n+1} \preceq \alpha_{2n},\end{aligned}$$

which shows that

$$\alpha_{2n+1} \prec \alpha_{2n}, \text{ for all } n \in N \cup \{0\}.$$

In general, we get

$$\alpha_n \prec \alpha_{n-1}, \text{ for all } n \in N \cup \{0\}.$$

As we know that $\Gamma\{\alpha_n\} = \{\beta_n^1, \beta_n^2, \beta_n^3 \dots \beta_n^m\} = \beta_n$, then by using the above equation we can write

$$\beta_n \prec \beta_{n-1}, \text{ for all } n \in N \cup \{0\}.$$

Accordingly, we have

$$\beta_n^j < \beta_{n-1}^j, \text{ for all } n \in N \cup \{0\} \text{ and for each } j \in \{1,2,3 \dots m\}.$$

Now we choose $k_i > 0$, then

$$\beta_n^j < \beta_{n-1}^j - k_j, \text{ for all } n \in N \cup \{0\} \text{ and for each } j \in \{1,2,3 \dots m\}.$$

Since, we get

$$\beta_n^j < \beta_0^j - nk_j, \text{ for all } n \in N \cup \{0\} \text{ and for each } j \in \{1,2,3 \dots m\}. \tag{14}$$

Taking limit $n \to \infty$ in (14), we get

$$\lim_{n\to\infty} = -\infty.$$

By using equation (2), we get

$$\lim_{n\to\infty} \alpha_n^j = 0, \text{ for each } j \in \{1,2,3 \dots m\}.$$

Now by using the equation (3), there exist $\omega \in (0,1)$ such that

$$\lim_{n\to\infty} \left(\alpha_n^j\right)^{\omega} \beta_n^j = 0, \text{ for each } j \in \{1,2,3 \dots m\}$$

By using the equation (14), we get

$$\left(\alpha_n^j\right)^{\omega} \beta_n^j - \left(\alpha_n^j\right)^{\omega} \beta_0^j \leq \left(\alpha_n^j\right)^{\omega} n k_j \leq 0, \tag{15}$$

taking limit $n \to \infty$ in equation (15), we get

$$\lim_{n\to\infty} n\left(\alpha_n^j\right)^{\omega} = 0, \text{ for each } j \in \{1,2,3 \dots m\}. \tag{16}$$

By using the above equation (16), there exists $n^j \in \{1,2,3 \dots\}$ for each $n \geq n^j$ such that $n\left(\alpha_n^j\right)^{\omega} \leq 1$. Furthermore, for $n \geq n_0 = \max\left\{n^j : j \in \{1,2,3 \dots m\}\right\}$, we get

$$\alpha_n^j \leq \frac{1}{n^{\frac{1}{\omega}}}.$$

Now we have to show that $\{x_n\}$ is a Cauchy sequence in V_{vms} denoted as $(\mathbb{Z}, d_v)$ for all $n, m \in \mathbb{N}$ with $n < m$, we have

$$\begin{aligned} d_v(x_n, x_m) &\preceq d_v(x_n, x_{n+1}) + d_v(x_{n+1}, x_{n+2}) + d_v(x_{n+2}, x_{n+3}) \dots + d_v(x_{m-1}, x_m) \\ &= \alpha_n + \alpha_{n+1} + \alpha_{n+2} + \cdots + \alpha_{m-1} \\ &= \left(\alpha_n^{(j)}\right)_{j=1}^m + \left(\alpha_{n+1}^{(j)}\right)_{j=1}^m + \left(\alpha_{n+2}^{(j)}\right)_{j=1}^m + \cdots + \left(\alpha_{m-1}^{(j)}\right)_{j=1}^m \\ &= \left(\sum_{i=1}^{m-1} \alpha_i^{(j)}\right)_{j=1}^m \\ &\preceq \left(\sum_{i=1}^{\infty} \alpha_i^{(j)}\right)_{j=1}^m \\ &\preceq \left(\sum_{i=1}^{\infty} \frac{1}{i^{\frac{1}{\omega}}}\right)_{j=1}^m . \end{aligned}$$

By taking limit $n \to \infty$, series $\sum_{i=1}^{\infty} \frac{1}{i^1}$ is convergent, we obtain

$$\lim_{n\to\infty} d_v(x_n, x_m) = 0,$$

which shows that $\{x_n\}$ is Cauchy sequence in a complete V_{vms}, then there exist $z^* \in \mathbb{Z}$ such that $x_n \to z^*$ as $n \to \infty$. Furthermore, we have to show that z^* is a fixed point of $\mathbf{\Psi}$, for this $x = x_{2n}$ and $y = z^*$, then by using equation (4) we get

$$\begin{aligned} \Gamma\big(d_v(\mathbf{\Psi} x_{2n}, \Phi z^*)\big) &\preceq \Gamma\big(Pd_v(x_{2n}, z^*) + Qd_v(x_{2n}, \mathbf{\Psi} x_{2n}) + Sd_v(z^*, \Phi z^*)\big) - k \\ &= \Gamma\big(Pd_v(x_{2n}, z^*) + Qd_v(x_{2n}, x_{2n+1}) + Sd_v(z^*, \Phi z^*)\big) - k \\ &\prec \Gamma\big(Pd_v(x_{2n}, z^*) + Qd_v(x_{2n}, x_{2n+1}) + Sd_v(z^*, \Phi z^*)\big) \end{aligned}$$

by using equation (1), we get

$$d_v(x_{2n+1}, \Phi z^*) \prec Pd_v(x_{2n}, z^*) + Qd_v(x_{2n}, x_{2n+1}) + Sd_v(z^*, \Phi z^*), \tag{17}$$

by taking limit $n \to \infty$, in equation (17) we get

$$d_v(z^*, \Phi z^*) \preceq \mathrm{S}\mathrm{d}_v(z^*, \Phi z^*) < d_v(z^*, \Phi z^*),$$

which is contradiction hence $z^* = \Phi z^*$. Similarly

$$\begin{aligned} \Gamma\big(d_v(\mathbf{\Psi} z^*, \Phi x_{2n+1})\big) &\preceq \Gamma\big(Pd_v(z^*, x_{2n+1}) + Qd_v(z^*, \mathbf{\Psi} z^*) + Sd_v(x_{2n+1}, \Phi x_{2n+1})\big) - k \\ &= \Gamma\big(Pd_v(z^*, x_{2n+1}) + Qd_v(z^*, \mathbf{\Psi} z^*) + Sd_v(x_{2n+1}, x_{2n+2})\big) - k \\ &\prec \Gamma\big(Pd_v(z^*, x_{2n+1}) + Qd_v(z^*, \mathbf{\Psi} z^*) + Sd_v(x_{2n+1}, x_{2n+2})\big), \end{aligned}$$

by using equation (1), we get

$$d_v(\mathbf{\Psi}z^*, x_{2n+2}) \prec Pd_v(z^*, x_{2n+1}) + Qd_v(z^*, \mathbf{\Psi}z^*) + Sd_v(x_{2n+1}, x_{2n+2}), \tag{18}$$

by taking limit $n \to \infty$, in equation (18) we get

$$d_v(\mathbf{\Psi}z^*, z^*) \preceq Qd_v(z^*, \mathbf{\Psi}z^*) < d_v(z^*, \mathbf{\Psi}z^*),$$

which is contradiction hence $z^* = \mathbf{\Psi}z^*$. Thus, z^* is a common fixed point of the mappings $\mathbf{\Psi}$ and Φ.

Now, we show that the two operators $\mathbf{\Psi}$ and Φ have a unique common fixed point.

Theorem 3.2. If the condition of the Theorem (2.5) are satisfied and $P \leq \check{I}$, then $\mathbf{\Psi}$ and Φ has a unique common fixed point.

Proof. According to the Theorem (2.5), there exist a common fixed point z^* of $\mathbf{\Psi}$ and Φ. Now we have to show that $\mathbf{\Psi}$ and Φ has a unique common fixed point. For this let y^* be another common fixed point of $\mathbf{\Psi}$ and Φ with $y^* \neq z^*$, then by using (4), we have

$$\begin{aligned} k + \Gamma\big(d_v(y^*, z^*)\big) &= \Gamma\big(d_v(\mathbf{\Psi}y^*, \Phi z^*)\big) \\ &\preceq \Gamma\big(Pd_v(y^*, z^*) + Qd_v(y^*, \mathbf{\Psi}y^*) + Sd_v(z^*, \Phi z^*)\big) \\ &= \Gamma\big(Pd_v(y^*, z^*) + Qd_v(y^*, y^*) + Sd_v(z^*, z^*)\big) \\ &= \Gamma\big(Pd_v(y^*, z^*)\big) \preceq \Gamma d_v(y^*, z^*), \end{aligned}$$

which is a contradiction, hence $y^* = z^*$, and therefore, $z^*$is a unique common fixed point of $\mathbf{\Psi}$ and Φ.

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Smarandache TNB Curves of Helices in Sol Space

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Abstract

In this paper, we define new Smarandache TNB curves of helices in the Sol^3. We obtain parametric and vector equations of Smarandache TNB curves.

Keywords: Helix, Sol Space, Smarandache **TNB** curve, Sol metric.

1. Introduction

A fundamental advance in theory of curves was the advent of analytic geometry in the seventeenth century. This enabled a curve to be described using an equation rather than an elaborate geometrical construction. This not only allowed new curves to be defined and studied, but it enabled a formal distinction to be made between curves that can be defined using algebraic equations, algebraic curves, and those that cannot, transcendental curves. Previously, curves had been described as "geometrical" or "mechanical" according to how they were, or supposedly could be, generated.

In this paper, we study Smarandache **TNB** curves of helices in the Sol^3. We characterize Smarandache **TNB** curves of helices in terms of their curvature and torsion. Finally, we find out their explicit parametric equations.

2. Riemannian Structure of Sol Space Sol^3

Sol space, one of Thurston's eight 3-dimensional geometries, can be viewed as R^3 provided with Riemannian metric

$$g_{\mathrm{Sol}^3} = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2,$$

where (x, y, z) are the standard coordinates in R^3.

Note that the Sol metric can also be written as:

$$g_{\mathrm{Sol}^3} = \sum_{i=1}^{3} \boldsymbol{\omega}^i \otimes \boldsymbol{\omega}^i,$$

where

$$\boldsymbol{\omega}^1 = e^z dx,\ \boldsymbol{\omega}^2 = e^{-z} dy,\ \boldsymbol{\omega}^3 = dz,$$

and the orthonormal basis dual to the 1-forms is

$$\mathbf{e}_1 = e^{-z}\frac{\partial}{\partial x},\ \mathbf{e}_2 = e^{z}\frac{\partial}{\partial y},\ \mathbf{e}_3 = \frac{\partial}{\partial z}.$$

Proposition 2.1. *For the covariant derivatives of the Levi-Civita connection of the left-invariant metric* g_{Sol^3}*, defined above the following is true:*

$$\nabla = \begin{pmatrix} -\mathbf{e}_3 & 0 & \mathbf{e}_1 \\ 0 & \mathbf{e}_3 & -\mathbf{e}_2 \\ 0 & 0 & 0 \end{pmatrix},$$

where the (i, j)*-element in the table above equals* $\nabla_{\mathbf{e}_i}\mathbf{e}_j$ *for our basis*

$$\{\mathbf{e}_k, k = 1,2,3\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}.$$

Lie brackets can be easily computed as:

$$[\mathbf{e}_1, \mathbf{e}_2] = 0,\ [\mathbf{e}_2, \mathbf{e}_3] = -\mathbf{e}_2,\ [\mathbf{e}_1, \mathbf{e}_3] = \mathbf{e}_1.$$

The isometry group of Sol^3 has dimension 3. The connected component of the identity is generated by the following three families of isometries:

$$\begin{aligned} (x, y, z) &\to (x + c, y, z), \\ (x, y, z) &\to (x, y + c, z), \\ (x, y, z) &\to (e^{-c}x, e^{c}y, z + c). \end{aligned}$$

3. General Helices in Sol Space Sol^3

Assume that $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be the Frenet frame field along γ. Then, the Frenet frame satisfies the following Frenet-Serret equations:

$$
\begin{aligned}
\nabla_{\mathbf{T}}\mathbf{T} &= \kappa\mathbf{N}, \\
\nabla_{\mathbf{T}}\mathbf{N} &= -\kappa\mathbf{T}+\tau\mathbf{B}, \\
\nabla_{\mathbf{T}}\mathbf{B} &= -\tau\mathbf{N},
\end{aligned}
$$

where κ is the curvature of γ and τ its torsion and

$$
\begin{aligned}
g_{\mathrm{Sol}^3}(\mathbf{T},\mathbf{T}) &= 1, g_{\mathrm{Sol}^3}(\mathbf{N},\mathbf{N})=1, g_{\mathrm{Sol}^3}(\mathbf{B},\mathbf{B})=1, \\
g_{\mathrm{Sol}^3}(\mathbf{T},\mathbf{N}) &= g_{\mathrm{Sol}^3}(\mathbf{T},\mathbf{B})=g_{\mathrm{Sol}^3}(\mathbf{N},\mathbf{B})=0.
\end{aligned}
$$

With respect to the orthonormal basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$, we can write

$$
\begin{aligned}
\mathbf{T} &= T_1\mathbf{e}_1+T_2\mathbf{e}_2+T_3\mathbf{e}_3, \\
\mathbf{N} &= N_1\mathbf{e}_1+N_2\mathbf{e}_2+N_3\mathbf{e}_3, \\
\mathbf{B} &= \mathbf{T}\times\mathbf{N}=B_1\mathbf{e}_1+B_2\mathbf{e}_2+B_3\mathbf{e}_3.
\end{aligned}
$$

Theorem 3.1. *Let $\gamma : I \to \mathrm{Sol}^3$ be a unit speed non-geodesic general helix. Then, the parametric equations of γ are*

$$
x(s) = \frac{\sin \mathrm{P} e^{-\cos \mathrm{P} s-\mathrm{C}_3}}{\mathrm{C}_1^2+\cos^2 \mathrm{P}}[-\cos\mathrm{P}\cos[\mathrm{C}_1 s+\mathrm{C}_2]+\mathrm{C}_1\sin[\mathrm{C}_1 s+\mathrm{C}_2]]+\mathrm{C}_4,
$$

$$
\begin{aligned}
y(s) &= \frac{\sin \mathrm{P} e^{\cos \mathrm{P} s+\mathrm{C}_3}}{\mathrm{C}_1^2+\cos^2 \mathrm{P}}[-\mathrm{C}_1\cos[\mathrm{C}_1 s+\mathrm{C}_2]+\cos\mathrm{P}\sin[\mathrm{C}_1 s+\mathrm{C}_2]]+\mathrm{C}_5, \\
z(s) &= \cos \mathrm{P} s+\mathrm{C}_3,
\end{aligned}
$$

where $\mathrm{C}_1,\mathrm{C}_2,\mathrm{C}_3,\mathrm{C}_4,\mathrm{C}_5$ are constants of integration.

4. Smarandache TNB Curves of Helices in Sol Space Sol^3

Definition 4.1. *Let $\gamma : I \to \mathrm{Sol}^3$ be a unit speed helix in the Sol Space Sol^3 and $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ be its moving Frenet frame. Smarandache* **TNB** *curves are defined by*

$$
\gamma_{\mathbf{TNB}} = \frac{1}{\sqrt{2\kappa^2-2\kappa\tau+2\tau^2}}(\mathbf{T}+\mathbf{N}+\mathbf{B}).
$$

Theorem 4.2. *Let $\gamma : I \to \mathrm{Sol}^3$ be a unit speed non-geodesic helix in the Sol Space Sol^3. Then, the equation of Smarandache* **TNB** *curve of a unit speed non-geodesic helix is given by*

$$\begin{aligned}
\gamma_{\mathbf{TNB}} \quad = \quad & \mathrm{W}[\sin\mathrm{P}\cos[\mathrm{C}_1 s+\mathrm{C}_2]+\frac{1}{\kappa}[-\frac{1}{\mathrm{C}_1}\sin\mathrm{P}\sin[\mathrm{C}_1 s+\mathrm{C}_2]+\cos\mathrm{P}\sin\mathrm{P}\cos[\mathrm{C}_1 s+\mathrm{C}_2]] \\
& +[\frac{1}{\kappa}\sin\mathrm{P}\sin[\mathrm{C}_1 s+\mathrm{C}_2]][\sin^2\mathrm{P}\sin^2[\mathrm{C}_1 s+\mathrm{C}_2]-\sin^2\mathrm{P}\cos^2[\mathrm{C}_1 s+\mathrm{C}_2] \\
& -\frac{1}{\kappa}\cos\mathrm{P}[\frac{1}{\mathrm{C}_1}\sin\mathrm{P}\cos[\mathrm{C}_1 s+\mathrm{C}_2]-\cos\mathrm{P}\sin\mathrm{P}\sin[\mathrm{C}_1 s+\mathrm{C}_2]]]\mathbf{e}_1 \\
& +\mathrm{W}[\sin\mathrm{P}\sin[\mathrm{C}_1 s+\mathrm{C}_2]+\frac{1}{\kappa}[\frac{1}{\mathrm{C}_1}\sin\mathrm{P}\cos[\mathrm{C}_1 s+\mathrm{C}_2]-\cos\mathrm{P}\sin\mathrm{P}\sin[\mathrm{C}_1 s+\mathrm{C}_2]] \\
& -[\frac{1}{\kappa}\sin\mathrm{P}\cos[\mathrm{C}_1 s+\mathrm{C}_2]][\sin^2\mathrm{P}\sin^2[\mathrm{C}_1 s+\mathrm{C}_2]-\sin^2\mathrm{P}\cos^2[\mathrm{C}_1 s+\mathrm{C}_2] \\
& -\frac{1}{\kappa}\cos\mathrm{P}[-\frac{1}{\mathrm{C}_1}\sin\mathrm{P}\sin[\mathrm{C}_1 s+\mathrm{C}_2]+\cos\mathrm{P}\sin\mathrm{P}\cos[\mathrm{C}_1 s+\mathrm{C}_2]]]]\mathbf{e}_2 \\
& +\mathrm{W}[\cos\mathrm{P}+\frac{1}{\kappa}[\sin^2\mathrm{P}\sin^2[\mathrm{C}_1 s+\mathrm{C}_2]-\sin^2\mathrm{P}\cos^2[\mathrm{C}_1 s+\mathrm{C}_2]] \\
& +\frac{1}{\kappa}\sin\mathrm{P}\cos[\mathrm{C}_1 s+\mathrm{C}_2]][\frac{1}{\mathrm{C}_1}\sin\mathrm{P}\cos[\mathrm{C}_1 s+\mathrm{C}_2]-\cos\mathrm{P}\sin\mathrm{P}\sin[\mathrm{C}_1 s+\mathrm{C}_2] \\
& -\frac{1}{\kappa}\sin\mathrm{P}\sin[\mathrm{C}_1 s+\mathrm{C}_2]][-\frac{1}{\mathrm{C}_1}\sin\mathrm{P}\sin[\mathrm{C}_1 s+\mathrm{C}_2]+\cos\mathrm{P}\sin\mathrm{P}\cos[\mathrm{C}_1 s+\mathrm{C}_2]]]\mathbf{e}_3,
\end{aligned}$$

where $\mathrm{C}_1, \mathrm{C}_2$ are constants of integration and

$$\mathrm{W}=\frac{1}{\sqrt{2\kappa^2-2\kappa\tau+2\tau^2}}.$$

Corollary 4.3. *Let $\gamma : I \to \mathrm{Sol}^3$ be a unit speed non-geodesic helix in the Sol Space Sol^3. Then, the parametric equations of Smarandache* **TNB** *curves of a unit speed non-geodesic helix are given by*

$$
\begin{aligned}
x_{\mathbf{TNB}}(s) \quad = \quad & \exp[-\mathrm{W}[\cos\mathrm{P}+\frac{1}{\kappa}[\sin^2\mathrm{P}\sin^2[\mathrm{C}_1 s+\mathrm{C}_2]-\sin^2\mathrm{P}\cos^2[\mathrm{C}_1 s+\mathrm{C}_2] \\
& +\frac{1}{\kappa}\sin\mathrm{P}\cos[\mathrm{C}_1 s+\mathrm{C}_2][\frac{1}{\mathrm{C}_1}\sin\mathrm{P}\cos[\mathrm{C}_1 s+\mathrm{C}_2]-\cos\mathrm{P}\sin\mathrm{P}\sin[\mathrm{C}_1 s+\mathrm{C}_2] \\
& -\frac{1}{\kappa}\sin\mathrm{P}\sin[\mathrm{C}_1 s+\mathrm{C}_2][-\frac{1}{\mathrm{C}_1}\sin\mathrm{P}\sin[\mathrm{C}_1 s+\mathrm{C}_2]+\cos\mathrm{P}\sin\mathrm{P}\cos[\mathrm{C}_1 s+\mathrm{C}_2]]]] \\
& \mathrm{W}[\sin\mathrm{P}\cos[\mathrm{C}_1 s+\mathrm{C}_2]+\sin\mathrm{P}\cos[\mathrm{C}_1 s+\mathrm{C}_2]) \\
& +[\frac{1}{\kappa}\sin\mathrm{P}\sin[\mathrm{C}_1 s+\mathrm{C}_2][\sin^2\mathrm{P}(1-2\cos^2[\mathrm{C}_1 s+\mathrm{C}_2] \\
& -\frac{1}{\kappa}\cos\mathrm{P}[\frac{1}{\mathrm{C}_1}\sin\mathrm{P}\cos[\mathrm{C}_1 s+\mathrm{C}_2]-\cos\mathrm{P}\sin\mathrm{P}\sin[\mathrm{C}_1 s+\mathrm{C}_2]]], \\
y_{\mathbf{TNB}}(s) \quad = \quad & \exp[\mathrm{W}[\cos\mathrm{P}+\frac{1}{\kappa}[\sin^2\mathrm{P}\sin^2[\mathrm{C}_1 s+\mathrm{C}_2]-\sin^2\mathrm{P}\cos^2[\mathrm{C}_1 s+\mathrm{C}_2]4.3 \\
& +\frac{1}{\kappa}\sin\mathrm{P}\cos[\mathrm{C}_1 s+\mathrm{C}_2][\frac{1}{\mathrm{C}_1}\sin\mathrm{P}\cos[\mathrm{C}_1 s+\mathrm{C}_2]-\cos\mathrm{P}\sin\mathrm{P}\sin[\mathrm{C}_1 s+\mathrm{C}_2] \\
& -\frac{1}{\kappa}\sin\mathrm{P}\sin[\mathrm{C}_1 s+\mathrm{C}_2][-\frac{1}{\mathrm{C}_1}\sin\mathrm{P}\sin[\mathrm{C}_1 s+\mathrm{C}_2]+\cos\mathrm{P}\sin\mathrm{P}\cos[\mathrm{C}_1 s+\mathrm{C}_2]]]] \\
& \mathrm{W}[\sin\mathrm{P}\sin[\mathrm{C}_1 s+\mathrm{C}_2]+\sin\mathrm{P}\sin[\mathrm{C}_1 s+\mathrm{C}_2] \\
& -[\frac{1}{\kappa}\sin\mathrm{P}\cos[\mathrm{C}_1 s+\mathrm{C}_2][\sin^2\mathrm{P}(1-2\cos^2[\mathrm{C}_1 s+\mathrm{C}_2]) \\
& -\frac{1}{\kappa}\cos\mathrm{P}[-\frac{1}{\mathrm{C}_1}\sin\mathrm{P}\sin[\mathrm{C}_1 s+\mathrm{C}_2]+\cos\mathrm{P}\sin\mathrm{P}\cos[\mathrm{C}_1 s+\mathrm{C}_2]]]], \\
z_{\mathbf{TNB}}(s) \quad = \quad & \mathrm{W}[\cos\mathrm{P}+\frac{1}{\kappa}[\sin^2\mathrm{P}\sin^2[\mathrm{C}_1 s+\mathrm{C}_2]-\sin^2\mathrm{P}\cos^2[\mathrm{C}_1 s+\mathrm{C}_2] \\
& +\frac{1}{\kappa}\sin\mathrm{P}\cos[\mathrm{C}_1 s+\mathrm{C}_2][\frac{1}{\mathrm{C}_1}\sin\mathrm{P}\cos[\mathrm{C}_1 s+\mathrm{C}_2]-\cos\mathrm{P}\sin\mathrm{P}\sin[\mathrm{C}_1 s+\mathrm{C}_2] \\
& -\frac{1}{\kappa}\sin\mathrm{P}\sin[\mathrm{C}_1 s+\mathrm{C}_2][-\frac{1}{\mathrm{C}_1}\sin\mathrm{P}\sin[\mathrm{C}_1 s+\mathrm{C}_2]+\cos\mathrm{P}\sin\mathrm{P}\cos[\mathrm{C}_1 s+\mathrm{C}_2]]],
\end{aligned}
$$

where $\mathrm{C}_1,\mathrm{C}_2$ *are constants of integration and*

$$\mathrm{W}=\frac{1}{\sqrt{2\kappa^2-2\kappa\tau+2\tau^2}}.$$

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Fractional Bernoulli wavelets for solving fractional Burger's Equation

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Abstract

Burgers equation is one of the basic and important non-linear partial differential equation including diffusive effects and non-linear propagation effects. In this study, the fractional order Bernoulli wavelets are adopted to acquire the approximate solution of one dimensional time-fractional Burger's equation. For this purpose, the dervetive operational matrices of classic (non fractional) and fractional orders are made and employed to transform the nonlinear Burger equation into a nonlinear algebraic system, which is solved by Newton iterative method. For analyzing the effect of fractional order on the solutions, the problem (1)-(2) has been solved for some different values of α. To validate the proposed method, we have considered some illustrative examples and compared with the exact results.

Keywords: fractional Burger equation, fractional Bernoulli wavelets, collocation method.

1. Introduction

In recent years, most of the practical problems arising in different fields of science like biology, chemistry, physics, engineering, and mathematics lead to nonlinear fractional partial differential equations. Burgers equation is one of the basic and important non-linear partial differential equation including diffusive effects and non-linear propagation effects. Fractional Burgers equation can describe the process of unidirectional propagation of weakly nonlinear acoustic waves through a pipe filled with gas. They are also onnected with applications in acoustic phenomena and have been used to model turbulence and certain steady-state viscous flows. Moreover, Burgers equations are used to model the formation and decay of nonplanar shock waves, where the variable x is a coordinate moving with the wave at the speed of sound and the dependent variable u represents the velocity fluctuations. The Burgers equations occur in various areas of applied sciences and physical applications, such as modeling of fluid mechanics and financial mathematics, and the equation has still interesting applications in physics and astrophysics.

Main problem

In this study, the fractional order Bernoulli wavelets are adopted to acquire the approximate solution of one dimensional time-fractional Burger's equation. Burgers' equation is the diffusive equation:

$$ {}_{C}D_t^{\gamma}u(x,t) + u(x,t)u_x(x,t) - \nu u_{xx}(x,t) = H(x,t), \qquad (x,t) \in [0,1] \times [0,T], \tag{1}$$

subject to the following initial and boundary conditions

$$u(x,0) = f(x), \qquad u(0,t) = p(t), \qquad u(1,t) = q(t). \tag{2}$$

where $\nu > 0$ denotes the coefficient of kinematic viscosity and the prescribed function $H(x,t)$ is sufficiently smooth. Also the fractional order derivative, γ, is considered in the Caputo sense.

2. Preliminaries on fractional calculus

In this section, we present some basic definitions and concepts on fractional calculus that are essential for subsequent discussion. There are various definitions for fractional integration and derivative operators. However, the fractional Riemann-Liouville integration and fractional Caputo derivative operators have been used in this study.

Definition 2.1. The Riemann-Liouville fractional integral operator of nonnegative order α is defined as [1]

$$J^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)\,dt\,, \qquad x > 0, \tag{3}$$

where $J^0 f(x) = f(x)$.

The Riemann-Liouville fractional integrals for the polynomials are defined as

$$J^{\alpha}x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}x^{\beta+\alpha}, \qquad \beta > -1. \tag{4}$$

Also the mentioned operator is linear, that is for real constant λ we have

$$J^{\alpha}\{\lambda f(x) + g(x)\} = \lambda J^{\alpha}\{f(x)\} + J^{\alpha}\{g(x)\}. \tag{5}$$

Definition 2.2. The Caputo fractional derivative operator of nonnegative order α is defined as [1]

$$ {}_cD^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{f^{(n)}(t)}{(x-t)^{\alpha+1-n}}\, dt, \quad n-1<\alpha\leq n, \quad n\in\mathbb{N}. \tag{6} $$

For the Caputo derivative we have] 2]

$$ D^{\alpha} x^{\beta} = 0, \qquad \beta\in\mathbb{N}_0, \qquad \beta<\lceil\alpha\rceil, $$

and

$$ D^{\alpha} x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} x^{\beta-\alpha}, \qquad \beta\in\mathbb{N}_0, \qquad \beta\geq\lceil\alpha\rceil \ \ or \ \ \beta\in\mathbb{R}-\mathbb{N}_0, \qquad \beta>\lfloor\alpha\rfloor, $$

Similar to the Riemann-Liouville fractional integral operator, the Caputo fractional derivative operator is linear, that is, for a real constant λ, we have

$$ {}_cD^{\alpha}\{\lambda f(x)+g(x)\} = \lambda \ \ {}_cD^{\alpha}\{f(x)\} + \ \ {}_cD^{\alpha}\{g(x)\}. $$

The relations between Reimann-Liouville fractional integral and Caputo fractional derivative operators can be addressed by the following identities [2]:

$$ {}_cD^{\alpha}\{J^{\alpha} f(x)\} = f(x), \tag{7} $$

and

$$ J^{\alpha}\{{}_cD^{\alpha} f(x)\} = f(x) - \sum_{j=0}^{n-1} \frac{f^{(j)}(0)}{j!} x^j . \tag{8} $$

3. Review on Bernoulli wavelets

In this section definitions of Fractional Bernoulli Wavelets (FBWs) and their operational matrix of Caputo fractional derivative are described.

3.1. BWs and FBWs

Definition 3.1. BWs of order m, which are denoted by $\psi_{nm}(t)=\psi(k,\hat{n},m,t)$, consist of four arguments, k; a positive integer, $n=1,2,\ldots,2^{k-1}$, $\hat{n}=n-1$, and t is the normalized time. These wavelets are defined on the interval $[0,1)$ as ([3]):

$$ \psi_{nm}(t) = 2^{\frac{k-1}{2}} \tilde{B}_m(2^{k-1}t-\hat{n})\chi_{\left[\frac{\hat{n}}{2^{k-1}},\frac{\hat{n}+1}{2^{k-1}}\right)}, \tag{9} $$

where $\tilde{B}_0(t)=1$ and

$$\tilde{B}_m(t) = \frac{B_m(t)}{\Lambda_m}, \qquad m > 0, \tag{10}$$

and $\Lambda_m = \sqrt{\frac{(-1)^{m-1}(m!)^2}{(2m)!}\vartheta_{2m}}$ is the normality coefficient.

The functions $B_m, m = 0,1,\dots,M-1$ are known Bernoulli polynomials, defined as

$$B_m(t) = \sum_{j=0}^{m} \binom{m}{j} \vartheta_{m-j} t^j, \tag{11}$$

where $\vartheta_j \coloneqq B_j(0)$ are the Bernoulli numbers. Therefore Bernoulli wavelets for $m > 0$ can be rewritten as

$$\psi_{nm}(t) = \Theta_m \sum_{j=0}^{m} \binom{m}{j} \vartheta_{m-j} 2^{j(k-1)} \left(t - \frac{\hat{n}}{2^{k-1}}\right)^j \chi_{\left[\frac{\hat{n}}{2^{k-1}}, \frac{\hat{n}+1}{2^{k-1}}\right)}, \tag{12}$$

where $\Theta_m = \sqrt{\frac{2^{k-1}(2m!)}{(-1)^{m-1}(m!)^2\vartheta_{2m}}}$ and $\psi_{n,0} = 2^{\frac{k-1}{2}} \chi_{\left[\frac{\hat{n}}{2^{k-1}}, \frac{\hat{n}+1}{2^{k-1}}\right)}$.

Definition 3.2. Fractional Bernoulli wavelets are denoted by $\psi_{n,m}^{\alpha}$ and constructed by changing the variable t to x^{α}, $(\alpha > 0)$ on the BWs)[3](, that is

$$\psi_{n,m}^{\alpha} \coloneqq \psi_{n,m}(x^{\alpha}) == \Theta_m \sum_{j=0}^{m} \binom{m}{j} \vartheta_{m-j} 2^{j(k-1)} \left(x^{\alpha} - \frac{\hat{n}}{2^{k-1}}\right)^j \chi_{\left[\left(\frac{\hat{n}}{2^{k-1}}\right)^{1/\alpha}, \left(\frac{\hat{n}+1}{2^{k-1}}\right)^{1/\alpha}\right)}. \tag{13}$$

Remark: The Bernoulli polynomials satisfies the following relation:

$$\int_0^1 B_m(x) B_n(x) dx = \frac{(-1)^{n-1} m!\, n!}{(m+n)!} \vartheta_{m+n}, \qquad m, n > 1. \tag{14}$$

Thus these polynomials are not orthogonal, consequently the FBWs, which are constructed by Bernoulli polynomials, are not orthogonal, too.

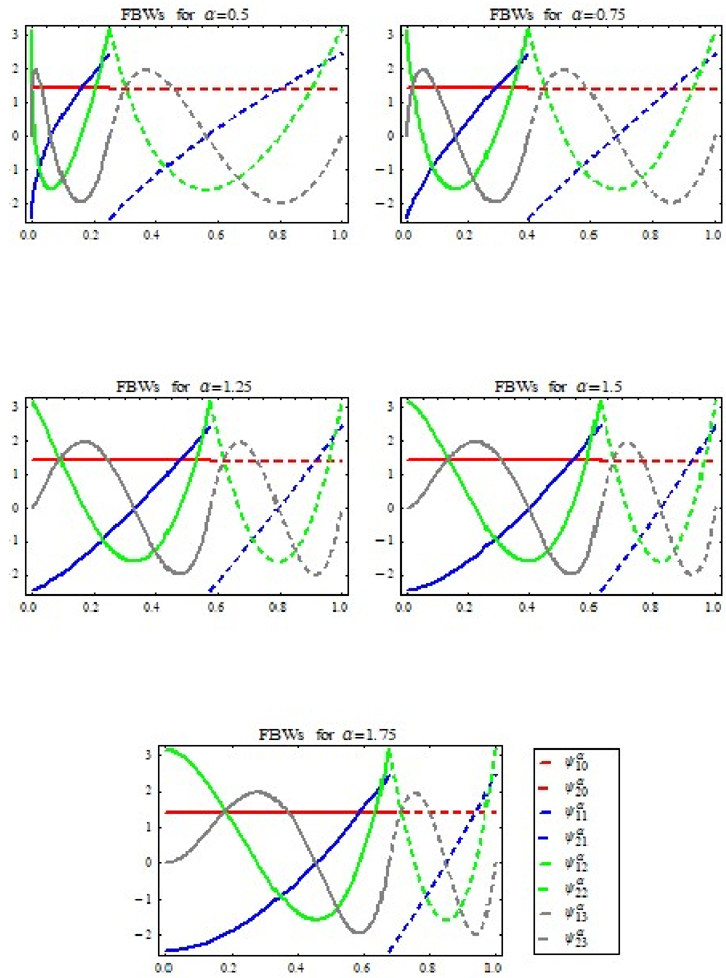

Figure 1. FBWs for $M = 4, k = 2$ and $\alpha = 0.5, 0.75, 1.25, 1.5$ and 1.75.

3.2. Function approximation by FBW

A function $f \in L^2[0,1]$ could be approximated by FBWs, as

$$f(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}^{\alpha}(x), \tag{15}$$

by truncating the infinite series (15) in some suitable k and M, we get

$$f(x) \simeq \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}^{\alpha}(x) = C_M^T \Psi_{k,M}^{\alpha}(x), \tag{16}$$

where C_M and $\Psi_{k,M}^{\alpha}$ are $2^{k-1} \times M$-dimensional column vectors and defined as

$$C_M = \left(c_{1,0}, \dots, c_{1,M-1}, \dots, c_{2^{k-1},0}, \dots, c_{2^{k-1},M-1}\right)^T, \tag{17}$$

$$\Psi_{k,M}^{\alpha} = \left(\psi_{1,0}^{\alpha}, \dots, \psi_{1,M-1}^{\alpha}, \dots, \psi_{2^{k-1},0}^{\alpha}, \dots, \psi_{2^{k-1},M-1}^{\alpha}\right)^T. \tag{18}$$

In order to determine the coefficients in (16), we put

$$\eta_{ij} := \int_{\sqrt[\alpha]{\frac{\hat{n}}{2^{k-1}}}}^{\sqrt[\alpha]{\frac{\hat{n}+1}{2^{k-1}}}} f(x) \psi_{i,j}^{\alpha}(x) x^{\alpha-1} dx, \tag{19}$$

and

$$\lambda_{n,m}^{i,j} := \int_{\sqrt[\alpha]{\frac{\hat{n}}{2^{k-1}}}}^{\sqrt[\alpha]{\frac{\hat{n}+1}{2^{k-1}}}} \psi_{n,m}^{\alpha}(x) \psi_{i,j}^{\alpha}(x) x^{\alpha-1} dx. \tag{20}$$

Now substituting (16) in (19), we get

$$\eta_{ij} \simeq \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \int_{\sqrt[\alpha]{\frac{\hat{n}}{2^{k-1}}}}^{\sqrt[\alpha]{\frac{\hat{n}+1}{2^{k-1}}}} \psi_{n,m}^{\alpha}(x) \psi_{i,j}^{\alpha}(x) x^{\alpha-1} dx = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}\ \lambda_{n,m}^{i,j} = C_M^T \Lambda_M^{i,j},$$

where

$$\Lambda_M^{i,j} = \left(\lambda_{1,0}^{i,j}, \dots, \lambda_{1,M-1}^{i,j}, \dots, \lambda_{2^{k-1},0}^{i,j}, \dots, \lambda_{2^{k-1},M-1}^{i,j}\right)^T,$$

so putting

$$T_M = \left(\eta_{1,0}, \dots, \eta_{1,M-1}, \dots, \eta_{2^{k-1},0}, \dots, \eta_{2^{k-1},M-1}\right)^T,$$

and

$$\Lambda_M = \left(\Lambda_M^{1,0}, \dots, \Lambda_M^{1,M-1}, \dots, \Lambda_M^{2^{k-1},0}, \dots, \Lambda_M^{2^{k-1},M-1}\right)_{(2^{k-1}\times M)\times(2^{k-1}\times M)},$$

the vector C_M is evaluated by

$$C_M^T = T_M \Lambda_M^{-1}. \tag{21}$$

$$v_{r,s,n,m} = \alpha^2 \langle\langle v(x,t), \psi_{r,s}^{\alpha}(t) \rangle_{t^{\alpha-1}}, \psi_{n,m}^{\alpha}(x) \rangle_{x^{\alpha-1}}, \tag{22}$$

$$n = 1,2, \dots, 2^{k_1-1}, \quad r = 1,2, \dots, 2^{k_2-1}, \quad m = 0,1, \dots, M_1 - 1, \quad s = 0,1, \dots, M_2 - 1.$$

The two variable function $v(x,t)$ could be approximated by two dimensional FBWs as

$$v(x,t) = \sum_{n=1}^{2^{k_2-1}} \sum_{m=0}^{M_2-1} \sum_{r=1}^{2^{k_1-1}} \sum_{s=0}^{M_1-1} v_{r,s,n,m}\, \psi_{r,s}^{\alpha}(x)\psi_{n,m}^{\alpha}(t) = \Psi_{k_1,M_1}^{\alpha}(x) V \left(\Psi_{k_1,M_1}^{\alpha}(x)\right)^T, \tag{23}$$

where V is dimensional $(2^{k_1-1} \times M_1) \times (2^{k_2-1} \times M_2)$ coefficient matrix.

It is clear that for $k_1 = k_2 = k$, $M_1 = M_2 = M$ and V is $(2^{k-1} \times M)$-dimensional square coefficient matrix.

Theorem 3.1. ([3]) Let $u(x,t) \in C^{M_1,M_2}(D)$ be approximated by two dimensional FBWs as

$$u(x,t) \simeq u_{k_1,M_1,k_2,M_2}(x,t) = \left(\Psi_{k_2,M_2}^{\alpha}\right)^T (t) V \Psi_{k_1,M_1}^{\alpha}(x),$$

there exist constants $C_i \in \mathbb{R}^+$, $i = 1,2,3$ such that

$$\left\| u(x,t) - u_{k_1,M_1,k_2,M_2}(x,t) \right\|_2 \leq \frac{C_1}{A_1} + \frac{C_2}{A_2} + \frac{C_3}{A_1 A_2}, \tag{24}$$

where $A_i = M_i!\, 2^{M_i(k_i+1)-1}$, $i = 1,2$.

3.3. Operational matrix of Riemann-Liouville fractional integration for FBWs

The Riemann-Liouville fractional integration of Ψ^{α} can be obtained as

$$J^{\xi} \Psi^{\alpha}(x) = \mathcal{F}^{\xi,\alpha} \Psi^{\alpha}(x), \tag{25}$$

where $\mathcal{F}^{\xi,\alpha}$ is relative operational square matrix of dimension $2^{k-1} \times M$ and could be evaluated by using equation (4) and (13) as follows

$$J^{\xi} \psi_{n,m}^{\alpha}(x) = \Theta_m \sum_{j=0}^{m} \binom{m}{j} \vartheta_{m-j} 2^{j(k-1)} J^{\xi} \left(x^{\alpha} - \frac{\hat{n}}{2^{k-1}}\right)^j, \tag{26}$$

$$\left(\frac{\hat{n}}{2^{k-1}}\right)^{1/\alpha} \leq x \leq \left(\frac{\hat{n}+1}{2^{k-1}}\right)^{1/\alpha}.$$

On the other hand

$$J^{\xi}\left(x^{\alpha}-\frac{\hat{n}}{2^{k-1}}\right)^{j}=\sum_{i=0}^{j}\binom{j}{i}\left(-\frac{\hat{n}}{2^{k-1}}\right)^{j-i}\frac{\Gamma(\alpha i+1)}{\Gamma(\alpha i+\xi+1)}x^{\alpha i+\xi}. \tag{27}$$

Thus, using equation (26)-(27), we can write

$$J^{\xi}\psi_{n,m}^{\alpha}(x)=\Theta_{m}\sum_{j=0}^{m}\sum_{i=0}^{j}A_{j,i}x^{\alpha i+\xi}, \tag{28}$$

where

$$A_{j,i}=\binom{m}{j}\vartheta_{m-j}2^{j(k-1)}\binom{j}{i}\left(-\frac{\hat{n}}{2^{k-1}}\right)^{j-i}\frac{\Gamma(\alpha i+1)}{\Gamma(\alpha i+\xi+1)}.$$

Now we expand $x^{\alpha i+\xi}$ in terms of FBWs:

$$x^{\alpha i+\xi}\simeq\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}d_{n,m}^{\xi,\alpha}\,\psi_{n,m}^{\alpha}(x), \tag{29}$$

by (28)-(29), we get

$$J^{\xi}\psi_{n,m}^{\alpha}(x)=\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}\eta_{n,m,j,i}^{\xi,\alpha}\,\psi_{n,m}^{\alpha}(x),$$

where

$$\eta_{n,m,j,i}^{\xi,\alpha}=\sum_{j=0}^{m}\sum_{i=0}^{j}A_{j,i}d_{n,m}^{\xi,\alpha}.$$

Thus we have

$$\mathcal{F}^{\xi,\alpha}=\begin{pmatrix}\mathcal{F}_{1}^{\xi,\alpha} & 0 & \cdots & 0\\ 0 & \mathcal{F}_{2}^{\xi,\alpha} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \mathcal{F}_{2^{k-1}}^{\xi,\alpha}\end{pmatrix}_{(2^{k-1}\times M)\times(2^{k-1}\times M)}, \tag{30}$$

where

$$\mathcal{F}_{l}^{\mathcal{F}^{\xi,\alpha}}=\left(\eta_{l,0,0,0}^{\xi,\alpha},\eta_{l,1,0,0}^{\xi,\alpha},\dots,\eta_{l,M-1,i,j}^{\xi,\alpha}\right)_{1\times M},$$

and $\mathbf{0}$ is a $1\times M$-dimensional row matrix which its all entries are zero.

3.4. Operational matrix of derivative for FBWs

The derivative of Ψ^{α} can be obtained as

$$\frac{d}{dx}\Psi^{\alpha}(x) = \mathcal{D}\Psi^{\alpha}(x), \tag{31}$$

where $\mathcal{D}$ is relative operational square matrix of dimension $2^{k-1} \times M$ and could be evaluated as follows

$$\frac{d}{dx}\psi_{n,m}^{\alpha}(x) = \Theta_m \sum_{j=0}^{m} \binom{m}{j} \vartheta_{m-j} 2^{j(k-1)} \frac{d}{dx}\left(x^{\alpha} - \frac{\hat{n}}{2^{k-1}}\right)^{j}, \tag{32}$$

$$\left(\frac{\hat{n}}{2^{k-1}}\right)^{1/\alpha} \leq x \leq \left(\frac{\hat{n}+1}{2^{k-1}}\right)^{1/\alpha}.$$

On the other hand

$$\frac{d}{dx}\left(x^{\alpha} - \frac{\hat{n}}{2^{k-1}}\right)^{j} = \sum_{i=0}^{j} \binom{j}{i} \left(-\frac{\hat{n}}{2^{k-1}}\right)^{j-i} \alpha i x^{\alpha i-1}. \tag{33}$$

Therefore, by using equations (32)-(33), we can write

$$\frac{d}{dx}\psi_{n,m}^{\alpha}(x) = \Theta_m \sum_{j=0}^{m} \sum_{i=0}^{j} B_{j,i} x^{\alpha i-1}, \tag{34}$$

where $B_{j,i} = \binom{m}{j} \vartheta_{m-j} 2^{j(k-1)} \binom{j}{i} \left(-\frac{\hat{n}}{2^{k-1}}\right)^{j-i} \alpha i$. Now we expand $x^{\alpha i-1}$ in terms of FBWs:

$$x^{\alpha i-1} \simeq \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} g_{n,m} \psi_{n,m}^{\alpha}(x), \tag{35}$$

by (34)-(35), we get

$$\frac{d}{dx}\psi_{n,m}^{\alpha}(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} \mu_{n,m,j,i} \psi_{n,m}^{\alpha}(x),$$

where $\mu_{n,m,j,i} = \Theta_m \sum_{j=0}^{m} \sum_{i=0}^{j} B_{j,i}\, g_{n,m}$. Therefore, we have

$$\mathcal{D} = \begin{pmatrix} \mathcal{D}_1 & 0 & \cdots & 0 \\ 0 & \mathcal{D}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{D}_{2^{k-1}} \end{pmatrix}_{(2^{k-1}\times M)\times(2^{k-1}\times M)} \tag{36}$$

where

$$\mathcal{D}_l = \left(\mu_{l,0,0,0}, \mu_{l,1,0,0}, \dots, \mu_{l,M-1,j,i}\right)_{1\times M}.$$

4. Numerical implementation

By using properties of FBWs, their operational matrices of derivative and Riemann-Liouville fractional integration, spectral collocation and Galerkin methods, a new approach is introduced in this section for solving Burgers equation. For this purpose, we first expand by ${}_cD_t^{\gamma} u(x,t)$ FBWs of order α as

$${}_cD_t^{\gamma} u(x,t) \simeq \Psi_{k_1,M_1}^{\alpha}(x)^T U \Psi_{k_2,M_2}^{\alpha}(t), \tag{37}$$

where U is $2^{k_1-1}M_1 \times 2^{k_2-1}M_2$-dimensional unknown coefficients matrix. By using Riemann-Liouville fractional integral of order γ with respect to variable t ,we get

$$u(x,t) \simeq J_t^{\gamma}\left(\Psi_{k_1,M_1}^{\alpha}(x)^T U \Psi_{k_2,M_2}^{\alpha}(t)\right) + f(x) = \Psi_{k_1,M_1}^{\alpha}(x)^T U \mathcal{F}^{\gamma,\alpha} \Psi_{k_2,M_2}^{\alpha}(t) + f(x). \tag{38}$$

by equation (38) and derivative operational matrix for FBWs, we get

$$\frac{\partial u(x,t)}{\partial x} = \Psi_{k_1,M_1}^{\alpha}(x)^T \mathcal{D}^T U \mathcal{F}^{\gamma,\alpha} \Psi_{k_2,M_2}^{\alpha}(t) + f'(x). \tag{39}$$

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \Psi_{k_1.M_1}^{\alpha}(x)^T (\mathcal{D}^2)^T U \mathcal{F}^{\gamma,\alpha} \Psi_{k_2.M_2}^{\alpha}(t) + f''(x), \tag{40}$$

By substituting equation (37)-(40) in equation (1) we have

$$\begin{aligned} G(x,t) &= \Psi_{k_1.M_1}^{\alpha}(x)^T U \Psi_{k_2.M_2}^{\alpha}(x)^T - \nu \Psi_{k_1.M_1}^{\alpha}(x)^T (\mathcal{D}^2)^T U \mathcal{F}^{\gamma,\alpha} \Psi_{k_2.M_2}^{\alpha}(t) \\ &+ \left(\Psi_{k_1.M_1}^{\alpha}(x)^T (\mathcal{D}^2)^T U \mathcal{F}^{\gamma,\alpha} \Psi_{k_2.M_2}^{\alpha}(t)\right) \times \left(\Psi_{k_1.M_1}^{\alpha}(x)^T \mathcal{D}^T U \mathcal{F}^{\gamma,\alpha} \Psi_{k_2.M_2}^{\alpha}(t)\right) \\ &+ f^{'}(x) \Psi_{k_1.M_1}^{\alpha}(x)^T U \mathcal{F}^{\gamma,\alpha} \Psi_{k_2.M_2}^{\alpha}(t) + f(x) \Psi_{k_1.M_1}^{\alpha}(x)^T \mathcal{D}^T U \mathcal{F}^{\gamma,\alpha} \Psi_{k_2.M_2}^{\alpha}(t), \end{aligned} \tag{41}$$

where

$$G(x,t)=H(x,t)+\nu f''(x)-f(x)f'(x).$$

For solving the obtained equation, the double collocation method is employed. First we define residual error function as

$$\mathbf{Res}(x,t)\coloneqq G(x,t)-\Psi^{\alpha}_{k_1.M_1}(x)^T U\Psi^{\alpha}_{k_2.M_2}(t)+\nu\Psi^{\alpha}_{k_1.M_1}(x)^T(\mathcal{D}^2)^T U\mathcal{F}^{\gamma,\alpha}\Psi^{\alpha}_{k_2.M_2}(t)$$

$$-\left(\Psi^{\alpha}_{k_1.M_1}(x)^T U\Psi^{\alpha}_{k_2.M_2}(t)\right)\times\left(\Psi^{\alpha}_{k_1.M_1}(x)^T\mathcal{D}^T U\mathcal{F}^{\gamma,\alpha}\Psi^{\alpha}_{k_2.M_2}(t)\right)$$

$$-f'(x)\Psi^{\alpha}_{k_1.M_1}(x)^T U\Psi^{\alpha}_{k_2.M_2}(t)-f(x)\Psi^{\alpha}_{k_1.M_1}(x)^T\mathcal{D}^T U\mathcal{F}^{\gamma,\alpha}\Psi^{\alpha}_{k_2.M_2}(t).$$

now we define the collocation points as $x_j=\frac{j}{2^{k_1-1}\times M_1}$ for $j=1,2,\dots,2^{k_1-1}\times M_1-2$ and $t_i=\frac{i}{2^{k_2-1}\times M_2}T$ for $i=1,2\dots,2^{k_2-1}\times M_2$. By substituting the collocation points in (42) and forcing the residual function to be vanished in the collocation meshes, $\mathbf{Res}(x_j,t_i)=0$, we have

$$G(x_j,t_i)-\Psi^{\alpha}_{k_1,M_1}(x_j)^T U\Psi^{\alpha}_{k_1,M_1}(t_i)+\nu\Psi^{\alpha}_{k_1,M_1}(x_j)^T(\mathcal{D}^2)^T U\mathcal{F}^{\gamma,\alpha}\Psi^{\alpha}_{k_2,M_2}(t_i)$$

$$-\left(\Psi^{\alpha}_{k_1,M_1}(x_j)^T U\Psi^{\alpha}_{k_2,M_2}(t_i)\right)\times\left(\Psi^{\alpha}_{k_1,M_1}(x_j)^T\mathcal{D}^T U\mathcal{F}^{\gamma,\alpha}\Psi^{\alpha}_{k_2,M_2}(t_i)\right)$$

Further, using the boundary conditions $u(0,t)=p(t)$ and $u(1,t)=q(t)$, we have

$$\Psi^{\alpha}_{k_1,M_1}(0)^T U\mathcal{F}^{\gamma,\alpha}\Psi^{\alpha}_{k_2,M_2}(t)+f(0)=p(t),\qquad \Psi^{\alpha}_{k_1,M_1}(1)^T U\mathcal{F}^{\gamma,\alpha}\Psi^{\alpha}_{k_2,M_2}(t)+f(1)=q(t).\qquad(43)$$

by collocating the modified boundary conditions in points

$$t_j=\frac{j}{2^{k_2-1}\times M_2},\qquad j=1,2,\dots,2^{k_2-1}\times M_2,\qquad(44)$$

we also get $2\times 2^{k_2-1}\times M_2$ equations. Combining the obtained equations, we achive a $2^{k_1-1}M_1\times 2^{k_2-1}M_2$ nonlinear system of equations. For solving this system, we apply the iterative Newton method. By specifying the unknown matrix U, we find the approximate solution of problem (1)-(2).

5. Numerical example

In this section we solve an example for showing the accuracy of the presented method.

Example 1. Consider the problem

$$ {}_{c}D_t^{\gamma} u(x,t) + u(x.t)u_x(x.t) - \nu u_{xx}(x.t) = \frac{2t^{2-\gamma}e^x}{\Gamma(3-\gamma)} + t^4 e^{2x} - \nu t^2 e^x, $$

subject to the initial and boundary conditions

$$ u(x,0) = 0, \qquad u(0,t) = t^2, \qquad u(1,t) = et^2, \qquad t \geq 0, $$

where the exact solution of thi problem is $u(x,t) = t^2 e^x$. First, we solved this problem for $\gamma = 0.5,\ \nu = 1,\ k = 2$ and $M = 2, 3,4$. The numerical results for $t = 1$ and some different values for M are tabulated in Tables 1, also related ϵ_{L^2} and ϵ_{L^∞} errors are reported in Table 2.

Table 1. Numerical solutions of example 1 for $\gamma = 0.5,\ \nu = 1, t = 1, k = 2$ and $M = 2,3,4$.

x	$M = 2$	$M = 3$	$M = 4$	$Exact$
0	1.000000	1.000000	1.000000	1.000000
0.1	1.105287	1.105216	1.105197	1.105171
0.2	1.221644	1.221493	1.221455	1.221403
0.3	1.350217	1.349992	1.349935	1.349859
0.4	1.492287	1.491996	1.491922	1.491825
0.5	1.649270	1.648922	1.648838	1.648721
0.6	1.822727	1.822342	1.822247	1.822119
0.7	2.014378	2.013979	2.013882	2.013753
0.8	2.226118	2.225747	2.225661	2.225541
0.9	2.460020	2.459745	2.459680	2.459603
1	2.718282	2.718282	2.718282	2.718282

Table 2. ϵ_{L^2} and ϵ_{L^∞} for example 1 for $\gamma = 0.5,\ \nu = 1, t = 1, k = 2$ and $M = 2,3,4$.

	$M = 2$	$M = 3$	$M = 4$
ϵ_{L^2}	4.523×10^{-3}	6.008×10^{-4}	2.946×10^{-5}
ϵ_{L^∞}	5.177×10^{-3}	4.383×10^{-4}	4.016×10^{-5}

For analyzing the effect of the fractional order γ on the numerical solution, we solved the problem for $\nu = 1$, $k = 2$, $M = 3$ and some values of γ and the results for $t = 1$ are given in Table 3.

Table 3. Numerical solutions of example 1 for $\nu = 1$, $t = 1$, $k = 2$, $M = 3$ and some values of γ.

x	$\gamma = 0.25$	$\gamma = 0.75$	$\gamma = 0.95$	$Exact$
0	1.000000	1.000000	1.000000	1.000000
0.1	1.105217	1.105206	1.105180	1.105171
0.2	1.221500	1.221485	1.221415	1.221403
0.3	1.359995	1.349914	1.349866	1.349859
0.4	1.492000	1.491982	1.491796	1.491825
0.5	1.648950	1.648830	1.648735	1.648721
0.6	1.822337	1.822314	1.822108	1.822119
0.7	2.013967	2.013822	2.013777	2.013753
0.8	2.225713	2.225608	2.225570	2.225541
0.9	2.459759	2.459724	2.459619	2.459603
1	2.718282	2.718282	2.718282	2.718282

6. Conslusion

In this paper, the fractional Bernoulli wavelets were defined in new settings and applied by collocation method for solving an important family of time fractional partial differential equations, the Burger's equation. First, the operational matrices of Caputo fractional and ordinary derivatives were constructed and then employed for reducing the time fractional partial equation to an algebraic system. For solving the proposed system, collocation method has been used. The proposed approach has promising applications as it can be extended and applied to the study of the numerical solutions of some other types of fractional integro-differential equations.

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Translation Hypersurfaces and Curvatures in the Four Dimensional Euclidean Space

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Abstract

In this study, we consider and examine the differential geometry of the translation hypersurfaces in the four dimensional Euclidean space $\mathbb{E}^4$. We compute the curvatures $\mathfrak{C}_i$, where $i = 1,2,3$, of the translation hypersurface. In addition, we give some relations for the curvatures of the hypersurface.

Keywords: Euclidean space, 4-space, curvature, Gauss map, translation hypersurface.

1. Introduction

In the literature, we see some papers about the translation surfaces (TS) and the translation hypersurfaces (TH) such as [1-17].

A translation surface in $\mathbb{E}^3$ is a surface generated by translations. For two space curves α, β with a common point P, the curve α is shifted such that point P is moving on β. Then the curve α generates the following TS:

$$\mathbf{x}(u, v) = \alpha(u) + \beta(v).$$

So, a translation hypersurface in the four dimensional Euclidean space $\mathbb{E}^4$ is a hypersurface generated by translations: for three space curves α, β, γ with a common point P, the curve α is shifted such that point P is moving on β and γ, respectively. Therefore, the curve α generates a TH in $\mathbb{E}^4$. TH is parametrized by

$$\mathbf{x}(u, v, w) = \alpha(u) + \beta(v) + \gamma(w) = \big(u, v, w, f(u) + g(v) + h(w)\big).$$

where $f(u), g(v), h(w)$ are differentiable functions for all $u, v, w \in I \subset \mathbb{R}$. Moreover, we can define the following TH, similarly,

$$\mathbf{x}(u, v, w) = \begin{pmatrix} f(u) \\ 0 \\ 0 \\ u \end{pmatrix} + \begin{pmatrix} 0 \\ g(v) \\ 0 \\ \mathrm{v} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ h(w) \\ \mathrm{w} \end{pmatrix} = \begin{pmatrix} f(u) \\ g(v) \\ h(w) \\ u + \mathrm{v} + \mathrm{w} \end{pmatrix}. \tag{1.1}$$

In this work, we reveal the curvatures of any hypersurface in $\mathbb{E}^4$. We indicate some basic elements of the four dimensional Euclidean geometry. In addition, we obtain the curvatures $\mathfrak{C}_{i=1,2,3}$ of the TH in (1.1).

2. Preliminaries

We have the characteristic polynomial of the shape operator **S**, to obtain the i-th curvature formulas $\mathfrak{C}_{i=0,1,\dots,n}$ in $\mathbb{E}^{n+1}$:

$$P_{\mathbf{S}}(\lambda) = 0 = \det(\mathbf{S} - \lambda\mathcal{I}_n) = \sum_{k=0}^{n} (-1)^k \, s_k \lambda^{n-k}, \tag{2.1}$$

where, $\mathcal{I}_n$ describes the identity matrix. Then, we get the curvature formulas $\binom{n}{i}\mathfrak{C}_i = s_i$, where $\binom{n}{0}\mathfrak{C}_0 = s_0 = 1$ by definition. k-th fundamental form of the hypersurface M^n is given by $\mathrm{I}(\mathbf{S}^{k-1}(X), Y) = \langle \mathbf{S}^{k-1}(X), Y\rangle$. Hence, we get the following

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} \mathfrak{C}_i \, \mathrm{I}(\mathbf{S}^{k-1}(X), Y) = 0. \tag{2.2}$$

We identify a vector $(\mathrm{a},\mathrm{b},\mathrm{c},\mathrm{d})$ with its transpose in this work. One can assume $\mathbf{M} = \mathbf{M}(u, v, w)$ be an isometric immersion of the hypersurface M^3 in $\mathbb{E}^4$. The inner product of $\vec{x} = (x_1, x_2, x_3, x_4)$ and $\vec{y} = (y_1, y_2, y_3, y_4)$ in $\mathbb{E}^4$ is defined by

$$\langle \vec{x}, \vec{y}\rangle = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4.$$

The triple vector product in $\mathbb{E}^4$ is given by

$$\vec{x} \times \vec{y} \times \vec{z} = \det\begin{pmatrix} e_1 & e_2 & e_3 & e_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{pmatrix}.$$

The Gauss map of the hypersurface **M** is defined by

$$e = \frac{\mathbf{M}_u \times \mathbf{M}_v \times \mathbf{M}_w}{\|\mathbf{M}_u \times \mathbf{M}_v \times \mathbf{M}_w\|},$$

where $\mathbf{M}_u = d\mathbf{M}/du$. We give the following fundamental form matrices for a hypersurface **M** in $\mathbb{E}^4$, respectively,

$$\mathrm{I} = \begin{pmatrix} E & F & A \\ F & G & B \\ A & B & C \end{pmatrix} = \begin{pmatrix} \langle \mathbf{M}_u, \mathbf{M}_u \rangle & \langle \mathbf{M}_u, \mathbf{M}_v \rangle & \langle \mathbf{M}_u, \mathbf{M}_w \rangle \\ \langle \mathbf{M}_v, \mathbf{M}_u \rangle & \langle \mathbf{M}_v, \mathbf{M}_v \rangle & \langle \mathbf{M}_v, \mathbf{M}_w \rangle \\ \langle \mathbf{M}_w, \mathbf{M}_u \rangle & \langle \mathbf{M}_w, \mathbf{M}_v \rangle & \langle \boldsymbol{M}_w, \boldsymbol{M}_w \rangle \end{pmatrix},$$

$$\mathrm{II} = \begin{pmatrix} L & M & P \\ M & N & T \\ P & T & V \end{pmatrix} = \begin{pmatrix} \langle \mathbf{M}_{uu}, e \rangle & \langle \mathbf{M}_{uv}, e \rangle & \langle \mathbf{M}_{uw}, e \rangle \\ \langle \mathbf{M}_{vu}, e \rangle & \langle \mathbf{M}_{vv}, e \rangle & \langle \mathbf{M}_{vw}, e \rangle \\ \langle \mathbf{M}_{wu}, e \rangle & \langle \mathbf{M}_{wv}, e \rangle & \langle \mathbf{M}_{ww}, e \rangle \end{pmatrix},$$

$$\mathrm{III} = \begin{pmatrix} X & Y & O \\ Y & Z & R \\ O & R & S \end{pmatrix} = \begin{pmatrix} \langle e_u, e_u \rangle & \langle e_u, e_v \rangle & \langle e_u, e_w \rangle \\ \langle e_v, e_u \rangle & \langle e_v, e_v \rangle & \langle e_v, e_w \rangle \\ \langle e_w, e_u \rangle & \langle e_w, e_v \rangle & \langle e_w, e_w \rangle \end{pmatrix}.$$

3. Curvatures

In this section, we compute the curvatures for a hypersurface $\mathbf{M}(u, v, w)$ in the four dimensional Euclidean space $\mathbb{E}^4$.

Taking the characteristic polynomial $P_{\mathbf{S}}(\lambda) = a\lambda^3 + b\lambda^2 + c\lambda + d = 0$, we have the following curvature formulas: $\mathfrak{C}_0 = 1$ (by definition), and

$$\mathfrak{C}_1 = -\frac{b}{\binom{3}{1} a},$$

$$\mathfrak{C}_2 = \frac{c}{\binom{3}{2} a},$$

$$\mathfrak{C}_3 = -\frac{d}{\binom{3}{3} a}.$$

Then, we clearly have the following curvature folmulas:

Theorem 3.1. *Any hypersurface* M^3 *in* $\mathbb{E}^4$ *has the following curvature formulas,* $\mathfrak{C}_0 = 1$ *(by definition),*

$$\mathfrak{C}_1 = \frac{(EN + GL - 2FM)C + (EG - F^2)V - LB^2 - NA^2 - 2(APG - BPF - ATF + BTE - ABM)}{3[(EG - F^2)C - EB^2 + 2FAB - GA^2]}, \tag{3.1}$$

$$\mathfrak{C}_2 = \frac{(EN + GL - 2FM)V + (LN - M^2)C - ET^2 - GP^2 - 2(APN - BPM - ATM + BTL - PTF)}{3[(EG - F^2)C - EB^2 + 2FAB - GA^2]}, \tag{3.2}$$

$$\mathfrak{C}_3 = \frac{(LN - M^2)V - LT^2 + 2MPT - NP^2}{(EG - F^2)C - EB^2 + 2FAB - GA^2}. \tag{3.3}$$

Proof. Solving $\det(\mathbf{S} - \lambda . \mathcal{I}_3) = 0$ with some computations, we have the coefficients of the polynomial $P_{\mathbf{S}}(\lambda)$. Here, $\mathcal{I}_3$ is the identity matrix.

Theorem 3.2. For *any hypersurface* M^3 *in* $\mathbb{E}^4$, *its curvatures are given by the following formula*

$$\mathfrak{C}_0 \mathrm{IV} - 3\mathfrak{C}_1 \mathrm{III} + 3\mathfrak{C}_2 \mathrm{II} - \mathfrak{C}_3 \mathrm{I} = 0. \tag{3.4}$$

4. Curvatures of Translation Hypersurfaces

Using the first differentials of (1.1) depends on u, v, w, we get the first fundamental form matrix of (1.1):

$$\mathrm{I} = \begin{pmatrix} 1 + f'^2 & 1 & 1 \\ 1 & 1 + g'^2 & 1 \\ 1 & 1 & 1 + h'^2 \end{pmatrix}.$$

So, we get the Gauss map of (1.1):

$$e = \frac{1}{(\det \mathrm{I})^{1/2}} \begin{pmatrix} g'h' \\ f'h' \\ f'g' \\ -f'g'h' \end{pmatrix}, \tag{4.1}$$

where

$$\det \mathrm{I} = f'^2 g'^2 + f'^2 h'^2 + g'^2 h'^2 + f'^2 g'^2 h'^2,$$

and $f' = df/du,\ g' = dg/dv,\ h' = dh/dw.$ Then, we obtain the second fundamental form matrix of (1.1):

$$\mathrm{II} = \frac{1}{(\det \mathrm{I})^{1/2}} \begin{pmatrix} f''g'h' & 0 & 0 \\ 0 & f'g''h' & 0 \\ 0 & 0 & f'g'h'' \end{pmatrix}.$$

Hence, we obtain the following third fundamental form matrix of (1.1):

$$\mathrm{III}=\frac{1}{(\det\mathrm{I})^2}\begin{pmatrix} f'^2g'^2h'^2(g'^2h'^2+g'^2+h'^2) & -f'f''g'g''h'^4 & -f'f''h'h''g'^4 \\ -f'f''g'g''h'^4 & f'^2g'^2h'^2(f'^2h'^2+f'^2+h'^2) & -g'g''h'h''f'^4 \\ -f'f''h'h''g'^4 & -g'g''h'h''f'^4 & f'^2g'^2h'^2(f'^2g'^2+f'^2+g'^2) \end{pmatrix}.$$

Next, we give the curvature formulasof the TH (1.1) in $\mathbb{E}^4$.

Theorem 4.1. *The translation hypersurface* (1.1) *in* $\mathbb{E}^4$ *has the following curvature formulas,* $\mathfrak{C}_0=1$ *(by definition),*

$$\mathfrak{C}_1=\frac{g'h'(g'^2h'^2+g'^2+h'^2)f''+f'h'(f'^2h'^2+f'^2+h'^2)g''+f'g'(f'^2g'^2+f'^2+g'^2)h''}{3(\det\mathrm{I})^{3/2}},$$

$$\mathfrak{C}_2=\frac{f'g'h'^2(1+h'^2)f''g''+f'g'^2h'(1+g'^2)f''h''+f'^2g'h'(1+f'^2)g''h''}{3(\det\mathrm{I})^2},$$

$$\mathfrak{C}_3=\frac{f'^2g'^2h'^2f''g''h''}{(\det\mathrm{I})^{5/2}},$$

where

$$\det\mathrm{I}=f'^2g'^2+f'^2h'^2+g'^2h'^2+f'^2g'^2h'^2.$$

Proof. Computing (3.1), (3.2), and (3.3) of (1.1), we have the curvatures, clearly.

Corollary 4.1. *The translation hypersurface* (1.1) *has the following fourth fundamental form matrix*

$$\mathrm{IV}=\frac{1}{(\det\mathrm{I})^{7/2}}\begin{pmatrix} \lambda & \mathcal{L} & \mathcal{E} \\ \mathcal{L} & \mathfrak{H} & \sigma \\ \mathcal{E} & \sigma & \varkappa \end{pmatrix},$$

where

$\det\mathrm{I}=f'^2g'^2+f'^2h'^2+g'^2h'^2+f'^2g'^2h'^2,$

$\lambda=g'^2h'^2\left(f'h'^5g''+\left(f'g'^4h''+h'\left((1+g'^2)h'^2+g'^2\right)^2f''\right)g'\right)f'',$

$\mathcal{L}=-f'g'h'^2f''g''(((f'^3+f')h'^5+f'^3h'^3)g''+(-f'^3g'^2h''+h'^3((1+g'^2)h'^2+g'^2)f')g'),$

$\mathcal{E}=(\left((f'^3+f')g'^5+f'^3g'^3\right)h''+(-f'^3h'^2g''+g'^3((1+g'^2)\,h'^2+g'^2)f')h')f'g'^2h'f''h'',$

$\mathfrak{H}=f'^2h'^2(\,f'g'h'^5+(f'^4g'h''+g''((f'^2+1)h'^2+f'^2)^2h')f')g''^2,$

$\sigma=f'^2g'h'\left((-(h'^3+h')f'^5-f'^3h'^3)g''+g'\left((-(\,g'^2+1)f'^5-f'^3g'^2)h''+g'^2h'^3f''\right)\right)g''h'',$

$$\varkappa = f'^2 g'^2 \big(f'g'(f'^2 + g'^2 + f'^2 g'^2)^2 h'' + h'(f''g'^5 + g''f'^5)\big) h''^2.$$

Corollary 4.2. *The curves* $f(u), g(v)$ *and* $h(w)$ *are the constant functions on the translation hypersurface* (1.1), *then the curvatures of* (1.1) *are equals to zero, ie.* $\mathfrak{C}_i = 0$. *Therefore, hypersurface* (1.1) *is the* i*-minimal translation hypersurface, where* $i = 1,2,3$.

Corollary 4.3. *The curves* $f(u), g(v)$ *and* $h(w)$ *are the constant functions on the translation hypersurface* (1.1), *then the fourth fundamental form matrix* IV *is equal to zero matrix.*

5. Conclusion

Translation surfaces are generated by two space curves. However, translation hypersurfaces are generated by greater than two space curves. We expand the results of the translation hypersurfaces by using its curvatures in $\mathbb{E}^4$. Moreover, we find some minimality conditions of the translation hypersurfaces.

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Some measures of dependence in the case of sub-Gaussian symmetric alpha-stable random vectors

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Abstract

Sub-Gaussian alpha-stable distributions are a particular sub-class of multivariate alpha-stable distributions, which have been used in fields such as finance and signal processing. For these particular distributions, we specify three measures of dependance proposed with the aim to quantify the dependence between the components of a symmetric alpha-stable random vector: the codifference, the generalized association parameter and the signed symmetric covariation coefficient and state a relation between these three measures. We also establish a relation which allows us to estimate the generalized association parameter without a previous estimation of the spectral measure.

Keywords: Codifference, Covariation, Generalized association parameter, Sub-Gaussian alpha-stable random vector.

1. Introduction

Stable distributions are a rich class of probability distributions, which includes the Gaussian, Cauchy and Lévy distributions in a family that allows for skewness and heavy tails. These laws, characterized by Paul Lévy, are the only possible limiting laws for normalized sums of independent, identically distributed random variables. Over the years, the interest in these laws has greatly increased and they are now widely applied in telecommunications and many other fields such as physics, biology, genetics and geology, see Uchaikin and Zolotarev [11]. However, stable non- Gaussian random vectors do not possess moments of second order. As a consequence, the concept of the correlation matrix, which allows us to understand the association between the coordinates of a finite variance random vector, is meaningless. Over the years, several measures of dependence have been proposed with the aim to overcome this drawback. In 1976, Paulauskas [8] proposed the generalized association parameter (g.a.p), applicable to general symmetric alpha-stable random vector. In 1983, Astrauskas [1] introduced another measure of bivariate dependence called the codifference, which is also defined for general symmetric alpha-stable random vector. Based on the covariation introduced by Miller [7] and Cambanis and Miller [2]. Kodia and Garel [4] proposed the signed symmetric covariation coefficient (scov) for symmetric α-stable random vectors with $\alpha > 1$.

Sub-Gaussian stable random distributions are a particular sub-class of the multivariate α-stable distributions. For instance, Kring et al. [5] fitted these distributions to asset returns and Tsihrintzis and Nikias [10] give some algorithms for signal detection in sub-Gaussian impulsive interference. Sub-Gaussian symmetric alpha-stable random vectors inherit their dependence structure from the underlying

Gaussian random vector. In that context, Kodia and Garel [4] established that the matrix of signed symmetric covariation coefficients and the matrix of generalized association parameters, called generalized covariation matrix, reduce to the correlation matrix of the underlying Gaussian random vector.

In this paper, we state a relation between the codifference, the generalized association parameter and the signed symmetric covariation coefficient in the context of sub-Gaussian symmetric alpha-stable random vectors. We also establish a relation which allows us to estimate the generalized association parameter without a previous estimation of the spectral measure.

This paper is organized as follow: Section 2 gives a brief reminder of basic definitions and properties of general stable random variables and vectors and the above-mentioned measures of dependence. We focus on sub-Gaussian symmetric α-stable random vectors in Section 3. In this part, we state a relation between the codifference, the generalized association parameter and the signed symmetric covariation coefficient. We also establish a relation which allows us to estimate the generalized association parameter without a previous estimation of the spectral measure.

2. Alpha-stable random vectors and some measures of dependence

For our purposes, we define stable random variables and vectors by their characteristic functions. Following Samorodnitsky and Taqqu [9], we denote the law of a stable random variable by $S_\alpha(\gamma, \beta, d)$, with $0 < \alpha \leq 2$, $\gamma \geq 0$, $-1 < \beta \leq 1$, and d a real parameter.

A random variable has a stable distribution $S_\alpha(\gamma,\beta,d)$ if its characteristic function has the form

$$\phi_X(t)=E\exp(itX)=\exp\{-\gamma^\alpha|t|^\alpha[1+i\beta\,\mathrm{sign}(t)\omega(t,\alpha)]+itd\}, \tag{1}$$

where

$$\omega(t,\alpha)=\begin{cases} -tan\,\dfrac{\pi\alpha}{2} & if\ \ \alpha\neq 1, \\ \dfrac{\pi}{2}\ln|t| & if\ \ \alpha=1, \end{cases}$$

with t a real number, and $\mathrm{sign}(t) = 1$ if $t > 0$, $\mathrm{sign}(t) = 0$ if $t = 0$ and $\mathrm{sign}(t) = -1$ if $t<0$.

The parameter α is the characteristic exponent or index of stability, β is a measure of skewness, is a scale parameter, and d is a location parameter. The case α = 2 corresponds to the Gaussian distribution, which is the only one having a finite variance. When $\beta = d = 0$, the distribution symmetric (i.e X and –X have the same law) and is denoted $S\alpha S(\gamma)$ or shortly SαS.

Let 0<α<2. The characteristic function of a bivariate random vector $\mathbf{X} = (X_1,X_2)$ is given by

$$\phi_{\boldsymbol{X}}(\mathrm{t})=\exp\left\{-\int_{S_2}|\langle \boldsymbol{t},\boldsymbol{s}\rangle|^{\alpha}[1+i sign(\langle \boldsymbol{t},\boldsymbol{s}\rangle)\omega(\langle \boldsymbol{t},\boldsymbol{s}\rangle,\alpha)]\Gamma(d\boldsymbol{s})+i\langle \boldsymbol{t},\boldsymbol{d}\rangle\right\}, \tag{2}$$

where Γ is a finite symmetric measure on the unit circle $S_2=\{s\in R^2 : \| s \|=1\}$, and d is a vector in R^2. Here ⟨t, s⟩ denotes the inner product of R^2. The measure Γ is called the spectral measure of the bivariate α-stable random vector **X**, and the pair (Γ, d) is unique. The vector **X** is symmetric if and only if **d** = 0 and Γ is symmetric on S_2. In this case, its characteristic function is given by

$$\phi_X(\mathrm{t})=\exp\left\{-\int_{S_2}|\langle t,s\rangle|^{\alpha}\Gamma(ds)\right\}. \tag{3}$$

For any $u\in R^2$, the projection $\langle u,X\rangle=\sum_{k=1}^{2}u_k X_k$ has a univariate SαS distribution. The spectral measure determines the projection parameter function

$$\gamma^{\alpha}(u)=\int_{S_2}|\langle t,s\rangle|^{\alpha}\Gamma(ds). \tag{4}$$

Miller [7] defined the covariation as follows.

Definition 2.1. Let X_1 and X_2 be jointly SαS with α>1 and let Γ be the spectral measure of the random vector (X_1, X_2). The covariation of X_1 on X_2 is the real number defined by

$$[X_1,X_2]_{\alpha}=\int_{S_2}s_1 s_2^{\langle\alpha-1\rangle}\Gamma(ds), \tag{5}$$

where, for real numbers s and a, if a ≠ 0, $s^{\langle a\rangle}=|s|^{a}sign(s)$ and if a=0, $s^{\langle a\rangle}=sign(s)$.

This definition is equivalent to:

$$[X_1,X_2]_{\alpha}=\frac{1}{\alpha}\frac{\partial\gamma^{\alpha}(\theta_1,\theta_2)}{\partial\theta_1}\Big|_{\theta_1=0,\ \theta_2=1,} \tag{6}$$

where θ_1 and θ_2 are real numbers and $\gamma(\theta_1,\theta_2)$ is the scale parameter of the random variable $Y=\theta_1X_1+\theta_2X_2$.

It is well known that although the covariation is linear in its first argument, it is, in general, not linear in its second argument and not symmetric in its arguments. We also have

$$[X_1,X_2]_{\alpha}=\int_{S_2}|s_1|^{\alpha}\Gamma(ds)=\gamma^{\alpha}X_1 \tag{7}$$

where γX_1 is the scale parameter of the $S\alpha S$ random variable X_1.

The covariation norm is defined by

$$\| X_1 \|_\alpha = ([X_1, X_2]_\alpha)^{1/\alpha}. \qquad (8)$$

When X_1 and X_2 are independent, then $[X_1, X_2]_\alpha = 0$. Proofs of these properties and other details are given in Samorodnitsky and Taqqu ([9], pp. 87-97).

The codifference is a measure of bivariate dependence introduced by Astrauskas [1] and defined for all $0<\alpha \leq 2$.

Definition 2.2. The codifference of two jointly $S\alpha S$, $0<\alpha \leq 2.$ random variables X_1 and X_2 quals

$$\gamma X_1,X_2 = \| X_1 \|_\alpha^\alpha + \| X_2 \|_\alpha^\alpha - \| X_1 - X_2 \|_\alpha^\alpha, \qquad (9)$$

where $\| X_1 \|_\alpha$, $\| X_2 \|_\alpha$ and $\| X_1 - X_2 \|_\alpha$ denote, respectively, the scale parameters of X_1, X_2 and $X_1 - X_2$.

Like the covariation, the codifference reduces to the covariance when $\alpha = 2$ and vanishes when the random variables are independent. However, in contrast to the covariation, the codifference is symmetric in all its arguments, namely $\gamma X_1, X_2 = \gamma X_2, X_1$.

The generalized association parameter (g.a.p) is a measure of dependence introduced by Paulauskas [8].

Definition 2.3. Let (X_1, X_2) be $S\alpha S$, $0< \alpha \leq 2$ random vector and Γ its spectral measure on the unit circle S_2. Let (U_1, U_2) be a random vector on S_2 with probability distribution $\tilde{\Gamma}=\Gamma/\Gamma(S_2)$. Due to the symmetry of Γ, one has $EU_1 = EU_2$. The g.a.p. is defined as:

$$\tilde{\rho}(X_1, X_2) = \frac{EU_1 U_2}{(EU_1^2 EU_2^2)^{1/2}}. \qquad (10)$$

Proposition 2.1. Let (X_1, X_2) be $S\alpha S$, $0< \alpha \leq 2$, random vector. Then

a) we always have $-1\leq \tilde{\rho}(X_1, X_2) \leq 1$;

b) if X_1 and X_2 are independent $\tilde{\rho}(X_1, X_2)0$;

c) $|\tilde{\rho}(X_1, X_2)| = 1$, if, and only if, the distribution of (X_1, X_2) is concentrated on a line;

d) for $\alpha= 2$, $\tilde{\rho}$ coincides with the correlation coefficient of the Gaussian random vector;

e) $\tilde{\rho}$ is independent of α and depends only on the spectral measure.

Proof. See Paulauskas [8].

Kodia and Garel [4] proposed a mesure of dependence of bivariate $S\alpha S$ random vectors with 1<$\alpha \leq 2$, based on the covariation and called the signed symmetric covariation coefficient (scov).

Definition 2.4. Let (X_1, X_2) be a bivariate $S\alpha S$ random vector with $\alpha > 1$. The signed symmetric covariation coefficient of X_1 and X_2 is the quantity:

$$scov(X_1, X_2) = k_{(X_1,X_2)} \left| \frac{[X_1,X_2]_\alpha [X_2,X_1]_\alpha}{\|X_1\|_\alpha^\alpha \|X_2\|_\alpha^\alpha} \right|^{1/2}, \qquad (11)$$

where

$$k_{(X_1,X_2)} = \begin{cases} sign([X_1,X_2])_\alpha \;\; if \quad sign([X_1,X_2])_\alpha = sign([X_2,X_1])_\alpha, \\ -1 \qquad\qquad if \quad sign([X_1,X_2])_\alpha = -sign([X_2,X_1])_\alpha . \end{cases} \qquad (12)$$

Proposition 2.2. Let (X_1, X_2) be $S\alpha S$, 0< α ≤2, random vector. Then

a) we always have -1≤ $scov(X_1, X_2)$≤1;

b) if X_1 and X_2 are independent $scov(X_1, X_2)$= 0;

c) $|scov(X_1, X_2)| = 1$, if, and only if, the distribution of (X_1, X_2) is concentrated on a line;

d) for α= 2, $scov$ coincides with the correlation coefficient of the Gaussian random vector.

Proof. See Kodia and Garel [4].

3. Measures of dependence in the sub-Gaussian $S\alpha S$ case

In general, α-stable random vectors have a complex dependence structure defined by the spectral measure. Since this measure is very difficult to estimate even in low dimensions, we have to retract to certain subclasses, where the spectral measure becomes simpler. One of these special classes is the multivariate α-stable sub-Gaussian distributions (see Kring et al. [5]). We recall the definition of a sub-Gaussian $S\alpha S$ random vector in R^2, as given in Samorodnitsky and Taqqu ([9], pp. 77-94).

Definition 3.1. Let 0< α <2, let $\boldsymbol{G} = (G_1, G_2)$ be a zero mean Gaussian random vector in R^2, and let A be a positive random variable such that $A \sim S\alpha_{/2}\left(\left(\cos\frac{\pi\alpha}{4}\right)^{2/\alpha}, 1, 0\right)$, independent of **G**. Then the random vector

$$\boldsymbol{X} = \left(A^{1/2}G_1, A^{1/2}G_2\right) \qquad (3)$$

is called a sub-Gaussian $S\alpha S$ random vector in R^2 with underlying Gaussian vector **G**. It is also said to be subordinated to **G**.

The characteristic function of X has the particular form:

$$\Phi_X(t) = Eexp\left\{i\sum_{m=1}^{2} t_m X_m\right\} = exp\left\{-\left|\frac{1}{2}\sum_{j=1}^{2}\sum_{k=1}^{2} t_j t_k R_{jk}\right|^{\alpha/2}\right\}, \qquad (14)$$

where $R_{jk} = EG_jG_k, j, k = 1,2,$ are the covariances of the underlying Gaussian random vector **G**.

From (14), we note that for sub-Gaussian $S\alpha S$ random vectors, we do not need the spectral measure in the characteristic function. Such vectors inherit their dependence structure from the underlying Gaussian random vector.

In the following proposition, we give a simpler form for the signed symmetric covariation coefficient.

Proposition 3.1. Let **X** be a sub-Gaussian $S\alpha S$ random vector with characteristic function (14), 1 < α < 2, then the signed symmetric covariation coefficient of the components X_1 and X_2, can simply be written as:

$$scov(X_1, X_2) = sign([X_1, X_2]_\alpha)\left(\frac{[X_1, X_2]_\alpha [X_2, X_1]_\alpha}{\|X_1\|_\alpha^\alpha \|X_2\|_\alpha^\alpha}\right)^{1/2}. \qquad (15)$$

Proof. We know that

$$[X_j, X_k]_\alpha = 2^{-\frac{\alpha}{2}} R_{jk} R_{jj}^{\frac{(\alpha-1)}{2}} \; and \; [X_k, X_j]_\alpha = 2^{-\frac{\alpha}{2}} R_{jk} R_{kk}^{\frac{(\alpha-1)}{2}}, \qquad (16)$$

see Samorodnitsky and Taqqu ([9], pp. 89). Then, we always have $sign\left(\left[X_j, X_k\right]_\alpha\right) = sign\left(R_{jk}\right) = sign\left(\left[X_k, X_j\right]_\alpha\right)$.

The next result is due to Kodia and Garel [4].

Proposition 3.2. Let **X** be a sub-Gaussian random vector with characteristic function (14), 0 < α < 2, then the matrix of generalized association parameters of **X**, called Generalized covariation matrix, reduces to the correlation matrix of the underlying Gaussian vector **G**. In particular, when α> 1, the matrix of signed symmetric covariation coefficients of **X** also reduces to the correlation matrix of the underlying Gaussian vector **G**.

Proof. See Kodia and Garel [4].

The following result establishes a relation between the codifference, the g.a.p. and the scov.

Proposition 3.3. Let **X** be a sub-Gaussian $S\alpha S$ random vector with characteristic function (14), 0 < α < 2, then

$$\tilde{\rho}(X_1, X_2) = \frac{\|X_1\|_\alpha^2 + \|X_2\|_\alpha^2 - (\|X_1\|_\alpha^\alpha + \|X_2\|_\alpha^\alpha - \tau_{X_1,X_2})^{\frac{2}{\alpha}}}{2\|X_1\|_\alpha \|X_2\|_\alpha}, \tag{17}$$

where $\|X_1\|_\alpha, \|X_2\|_\alpha$ *and* τ_{X_1,X_2} denote, respectively, the scale parameters of the components X_1 and X_2 and the codifference between X_1 and X_2.

In particular, if α > 1, then

$$scov(X_1, X_2) = \frac{\|X_1\|_\alpha^2 + \|X_2\|_\alpha^2 - (\|X_1\|_\alpha^\alpha + \|X_2\|_\alpha^\alpha - \tau_{X_1,X_2})^{\frac{2}{\alpha}}}{2\|X_1\|_\alpha \|X_2\|_\alpha} \ . \tag{18}$$

Proof. Let **X** = $(X_1, \dots, X_d)$ be a sub-Gaussian $S\alpha S$ random vector with characteristic function (14), 0 < α< 2 and denote $\|X_j\|_\alpha$ the scale parameter of the component X_j, $1 \le j \le d$. Then $\|X_j\|_\alpha = \left\|A^{1/2} G_j\right\|_\alpha = 2^{-1/2} R_{jj}^{1/2}$, where $R_{jj} = Var(G_j)$ is the variance of the Gaussian random variable G_j, see Samorodnitsky and Taqqu, ([9], pp. 20-21). We can write

$$\|X_j - X_k\|_\alpha = 2^{-\frac{1}{2}}[Var(G_j - G_k)]^{\frac{1}{2}} = 2^{-\frac{1}{2}}[R_{jj} + R_{kk} - 2R_{jk}]^{\frac{1}{2}} \tag{19}$$

From (9), we have

$$\|X_j - X_k\|_\alpha = (\|X_j\|_\alpha^\alpha + \|X_k\|_\alpha^\alpha - \tau_{X_j,X_k})^{\frac{1}{\alpha}}. \tag{20}$$

By equating the equations (19) and (20), we have

$$2^{-\frac{1}{2}}\left[R_{jj} - R_{kk} - 2R_{jk}\right]^{\frac{1}{2}} = \left(\left\|X_j\right\|_\alpha^\alpha + \|X_k\|_\alpha^\alpha - \tau_{X_j,X_k}\right)^{\frac{1}{\alpha}}$$

$$\Leftrightarrow \quad \left[R_{jj} - R_{kk} - 2R_{jk}\right]^{\frac{1}{2}} = \left(\left\|X_j\right\|_\alpha^\alpha + \|X_k\|_\alpha^\alpha - \tau_{X_j,X_k}\right)^{\frac{1}{\alpha}}$$

$$\Leftrightarrow \quad \left[R_{jj} - R_{kk} - 2R_{jk}\right]^{\frac{1}{2}} = 2^{-\frac{1}{2}}\left(\left\|X_j\right\|_\alpha^\alpha + \|X_k\|_\alpha^\alpha - \tau_{X_j,X_k}\right)^{\frac{1}{\alpha}}$$

$$\Leftrightarrow \quad R_{jj} - R_{kk} - 2R_{jk} = 2\left(\left\|X_j\right\|_\alpha^\alpha + \|X_k\|_\alpha^\alpha - \tau_{X_j,X_k}\right)^{\frac{1}{\alpha}}$$

$$\Leftrightarrow \quad 2R_{jk} = R_{jj} + R_{kk} - 2\left(\left\|X_j\right\|_\alpha^\alpha + \|X_k\|_\alpha^\alpha - \tau_{X_j,X_k}\right)^{\frac{1}{\alpha}}$$

$$\Leftrightarrow \quad \frac{R_{jk}}{R_{11}^{1/2}R_{22}^{1/2}} = \frac{R_{jj} + R_{kk} - 2\left(\left\|X_j\right\|_\alpha^\alpha + \|X_k\|_\alpha^\alpha - \tau_{X_j,X_k}\right)^{\frac{1}{\alpha}}}{2R_{11}^{1/2}R_{22}^{1/2}}$$

$$\Leftrightarrow \quad r_{G_1,G_2} = \frac{2^{-1}R_{jj} + 2^{-1}R_{kk} - 2\left(\left\|X_j\right\|_\alpha^\alpha + \|X_k\|_\alpha^\alpha - \tau_{X_j,X_k}\right)^{\frac{1}{\alpha}}}{R_{11}^{1/2}R_{22}^{1/2}}$$

$$\Leftrightarrow \quad \tilde{\rho}(X_1, X_2) = \frac{2^{-1}R_{jj} + 2^{-1}R_{kk} - 2\left(\left\|X_j\right\|_\alpha^\alpha + \|X_k\|_\alpha^\alpha - \tau_{X_j,X_k}\right)^{\frac{1}{\alpha}}}{R_{11}^{1/2}R_{22}^{1/2}}$$

$$\Leftrightarrow \quad \tilde{\rho}(X_1, X_2) = \frac{\|X_1\|_\alpha^2 + \|X_2\|_\alpha^2 - 2\left(\left\|X_j\right\|_\alpha^\alpha + \|X_k\|_\alpha^\alpha - \tau_{X_j,X_k}\right)^{\frac{1}{\alpha}}}{2\|X_1\|_\alpha\|X_2\|_\alpha}$$

In particular, if α > 1, the equality (18) is a consequence of **Proposition 3.2**.

Remark 3.1. Proposition 3.3 shows clearly that for a sub-Gaussian $S\alpha S$ random vector, the g.a.p. and the scov of two components of this random vector can be expressed by means of the scale parameters of these components and their codifference.

Lemma 3.1. Let 0 < α ≤2. Let X = (X_1, X_2) be a sub-Gaussian random vector. Then the g.a.p. between the components X_1 and X_2 can be expressed by

$$\tilde{\rho}(X_1, X_2) = \frac{\|X_1\|_\alpha^2 + \|X_2\|_\alpha^2 - \|X_1 - X_2\|_\alpha^2}{2\|X_1\|_\alpha\|X_2\|_\alpha}, \qquad (21)$$

where $\|X_1\|_\alpha$, $\|X_2\|_\alpha$ and $\|X_1 - X_2\|_\alpha$ denote, respectively, the scale parameters of the random variables X_1 and X_2 and $X_1 - X_2$.

In particular, when α> 1, we have

$$scov(X_1, X_2) = \widetilde{\rho}(X_1, X_2) = \frac{\|X_1\|_\alpha^2 + \|X_2\|_\alpha^2 - \left\|X_1 - X_2\right\|_\alpha^2}{2\|X_1\|_\alpha \|X_2\|_\alpha}. \qquad (22)$$

Proof. The relation (21) is deduced from relation (17) in which we replace τ_{X_1,X_2} by its expression, i. e. $\|X_1\|_\alpha^\alpha + \|X_2\|_\alpha^\alpha - \|X_{1-}X_2\|_\alpha^\alpha$. The equation (22) is a consequence of **Proposition 3.2**.

Remark 3.2. This is an important result because it establishes that in the case of sub-Gaussian symmetric α-stable random vector, we can estimate the generalized association parameter without a previous estimation of the spectral measure. We have just to estimate the characteristic exponent α and the scales parameter of X_1, X_2 and $X_1 - X_2$. Nowdays, we have several methods which give good results for estimating these parameters, see for instance Davidov and Paulauskas [3] and McCulloch [6]. They allow us to define several consistent estimators of the generalized association parameter and the signed symmetric covaiation coefficient in the case of sub-Gaussian symmetric alpha-stable random vectors.

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New type constant Π_2- slope curves according to type-2 Bishop frame

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Abstract

In this paper, we study Smarandache $\Pi_1\Pi_2B$ curves of biharmonic new type constant Π_2-slope curves according to type-2 Bishop frame in the SOL^3. Type-2 Bishop equations of Smarandache $\Pi_1\Pi_2B$ curves are obtained in terms of base curve's type-2 Bishop invariants.

Keywords: Type-2 Bishop frame, Sol Space, Smarandache $\mathbf{\Pi}_1\mathbf{\Pi}_2\mathbf{B}$ curve.

1. Introduction

The theory of biharmonic functions is an old and rich subject. The biharmonic functions were first studied by Maxwell and Airy to describe a mathematical model of elasticity in 1862.

Let (N,h) and (M,g) be Riemannian manifolds. A smooth map $\phi : N \to M$ is said to be biharmonic if it is a critical point of the bienergy functional:

$$E_2(\phi) = \int_N \frac{1}{2}|\mathrm{T}(\phi)|^2 dv_h,$$

where the section $\mathrm{T}(\phi) := \mathrm{tr}\nabla^{\phi} d\phi$ is the tension field of ϕ, [5,6].

The Euler--Lagrange equation of the bienergy is given by $\mathrm{T}_2(\phi) = 0$. Here the section $\mathrm{T}_2(\phi)$ is defined by

$$\mathrm{T}_2(\phi) = -\Delta_{\phi}\mathrm{T}(\phi) + \mathrm{tr}R(\mathrm{T}(\phi), d\phi)d\phi,$$

and called the bitension field of ϕ, [8].

This study is organised as follows: Firstly, we study Smarandache $\mathbf{\Pi}_1\mathbf{\Pi}_2\mathbf{B}$ curves of biharmonic new type constant $\mathbf{\Pi}_2 -$ slope curves according to type-2 Bishop frame in the SOL^3. Secondly, type-2 Bishop equations of Smarandache $\mathbf{\Pi}_1\mathbf{\Pi}_2\mathbf{B}$ curves are obtained in terms of base curve's type-2 Bishop invariants. Finally, we express some interesting relations and illustrate some examples of our main results.

2. Riemannian Structure of Sol Space Sol^3

Sol space, one of Thurston's eight 3-dimensional geometries, can be viewed as $\mathbb{R}^3$ provided with Riemannian metric

$$g_{\mathrm{Sol}^3} = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2,$$

where (x, y, z) are the standard coordinates in $\mathbb{R}^3$.

Let γ be a unit speed regular curve in SOL^3 and $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be its Frenet--Serret frame. Let us express a relatively parallel adapted frame:

$$\begin{aligned}
\nabla_{\mathbf{T}}\mathbf{\Pi}_1 &= -\varepsilon_1\mathbf{B}, \\
\nabla_{\mathbf{T}}\mathbf{\Pi}_2 &= -\varepsilon_2\mathbf{B}, \\
\nabla_{\mathbf{T}}\mathbf{B} &= \varepsilon_1\mathbf{\Pi}_1 + \varepsilon_2\mathbf{\Pi}_2,
\end{aligned}$$

where

$$\begin{aligned}
g_{\mathrm{SOL}^3}(\mathbf{B}, \mathbf{B}) &= 1, g_{\mathrm{SOL}^3}(\mathbf{\Pi}_1, \mathbf{\Pi}_1) = 1, g_{\mathrm{SOL}^3}(\mathbf{\Pi}_2, \mathbf{\Pi}_2) = 1, \\
g_{\mathrm{SOL}^3}(\mathbf{B}, \mathbf{\Pi}_1) &= g_{\mathrm{SOL}^3}(\mathbf{B}, \mathbf{\Pi}_2) = g_{\mathrm{SOL}^3}(\mathbf{\Pi}_1, \mathbf{\Pi}_2) = 0.
\end{aligned}$$

We shall call this frame as Type-2 Bishop Frame. In order to investigate this new frame's relation with Frenet--Serret frame, first we write

$$\tau = \sqrt{\varepsilon_1^2 + \varepsilon_2^2}.$$

The relation matrix between Frenet--Serret and type-2 Bishop frames can be expressed

$$\begin{aligned}
\mathbf{T} &= \sin \mathrm{A}(s)\mathbf{\Pi}_1 - \cos \mathrm{A}(s)\mathbf{\Pi}_2, \\
\mathbf{N} &= \cos \mathrm{A}(s)\mathbf{\Pi}_1 + \sin \mathrm{A}(s)\mathbf{\Pi}_2, \\
\mathbf{B} &= \mathbf{B}.
\end{aligned}$$

So by Frenet--Serret frame, we may express

$$\begin{aligned}
\varepsilon_1 &= -\tau \cos \mathrm{A}(s), \\
\varepsilon_2 &= -\tau \sin \mathrm{A}(s).
\end{aligned}$$

The frame $\{\mathbf{\Pi}_1, \mathbf{\Pi}_2, \mathbf{B}\}$ is properly oriented, and τ and $\mathsf{A}(s) = \int_0^s \kappa(s)ds$ are polar coordinates for the curve γ. We shall call the set $\{\mathbf{\Pi}_1, \mathbf{\Pi}_2, \mathbf{B}, \varepsilon_1, \varepsilon_2\}$ as type-2 Bishop invariants of the curve γ, [12].

With respect to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, we can write

$$\begin{aligned} \mathbf{\Pi}_1 &= \pi_1^1\mathbf{e}_1 + \pi_1^2\mathbf{e}_2 + \pi_1^3\mathbf{e}_3, \\ \mathbf{\Pi}_2 &= \pi_2^1\mathbf{e}_1 + \pi_2^2\mathbf{e}_2 + \pi_2^3\mathbf{e}_3. \\ \mathbf{B} &= B^1\mathbf{e}_1 + B^2\mathbf{e}_2 + B^3\mathbf{e}_3, \end{aligned}$$

Theorem 2.1. *Let $\gamma : I \to \mathsf{SOL}^3$ be a unit speed non-geodesic biharmonic new type constant $\mathbf{\Pi}_2$ −slope curves according to type-2 Bishop frame in the SOL^3. Then, the parametric equations of γ are*

$$\begin{aligned} \mathbf{x}(s) &= e^{-\frac{1}{\kappa}\cos[\kappa s]\cos\mathsf{E} + \frac{1}{\kappa}\sin[\kappa s]\sin\mathsf{E} - \mathsf{R}_3}[\sin[\kappa s]\cos\mathsf{E}\cos[\mathsf{R}_1 s + \mathsf{R}_2] \\ &\quad - \cos[\kappa s]\sin\mathsf{E}\cos[\mathsf{R}_1 s + \mathsf{R}_2]]ds, \\ \mathbf{y}(s) &= \int e^{\frac{1}{\kappa}\cos[\kappa s]\cos\mathsf{E} - \frac{1}{\kappa}\sin[\kappa s]\sin\mathsf{E} + \mathsf{R}_3}[\sin[\kappa s]\cos\mathsf{E}\sin[\mathsf{R}_1 s + \mathsf{R}_2] \\ &\quad - \cos[\kappa s]\sin\mathsf{E}\sin[\mathsf{R}_1 s + \mathsf{R}_2]]ds, \\ \mathbf{z}(s) &= \frac{1}{\kappa}\cos[\kappa s]\cos\mathsf{E} - \frac{1}{\kappa}\sin[\kappa s]\sin\mathsf{E} + \mathsf{R}_3, \end{aligned}$$

where $\mathsf{R}_1, \mathsf{R}_2, \mathsf{R}_3$ are constants of integration.

3. Smarandache $\mathbf{\Pi}_1\mathbf{\Pi}_2\mathbf{B}$ Curves of Biharmonic Constant $\mathbf{\Pi}_2$ −Slope

Definition 3.1. *Let $\gamma : I \to \mathsf{SOL}^3$ be a unit speed curve in the Sol Space SOL^3 and $\{\mathbf{\Pi}_1, \mathbf{\Pi}_2, \mathbf{B}\}$ be its moving type-2 Bishop frame. Smarandache $\mathbf{\Pi}_1\mathbf{\Pi}_2\mathbf{B}$ curves are defined by*

$$\bar{\gamma} = \frac{1}{\sqrt{2\varepsilon_1^2 + 2\varepsilon_2^2}}(\mathbf{\Pi}_1 + \mathbf{\Pi}_2 + \mathbf{B}).$$

Then, we have the following theorem.

Theorem 3.2. *Let $\gamma : I \to \mathrm{SOL}^3$ be a unit speed non-geodesic biharmonic constant $\mathbf{\Pi}_2-$slope curves according to type-2 Bishop frame in the SOL^3. Then, the equation of Smarandache $\mathbf{\Pi}_1\mathbf{\Pi}_2\mathbf{B}$ curves of biharmonic constant $\mathbf{\Pi}_2-$slope curves is given by*

$$\begin{aligned}\bar{\gamma}(s) &= \frac{1}{\sqrt{2\varepsilon_1^2+2\varepsilon_2^2}}[\cos\mathsf{E}\cos[\mathsf{R}_1 s+\mathsf{R}_2]-\sin[\mathsf{R}_1 s+\mathsf{R}_2]+\sin\mathsf{E}\cos[\mathsf{R}_1 s+\mathsf{R}_2]]\mathbf{e}_1\\ &+\frac{1}{\sqrt{2\varepsilon_1^2+2\varepsilon_2^2}}[\cos\mathsf{E}\sin[\mathsf{R}_1 s+\mathsf{R}_2]+\cos[\mathsf{R}_1 s+\mathsf{R}_2]+\sin\mathsf{E}\sin[\mathsf{R}_1 s+\mathsf{R}_2]]\mathbf{e}_2\\ &+\frac{1}{\sqrt{2\varepsilon_1^2+2\varepsilon_2^2}}[\cos\mathsf{E}-\sin\mathsf{E}]\mathbf{e}_3,\end{aligned}$$

where $\mathsf{R}_1,\mathsf{R}_2$ are constants of integration.

We have the following corollary of Theorem 3.2.

Corollary 3.3. *Let $\gamma : I \to \mathrm{SOL}^3$ be a unit speed non-geodesic biharmonic constant $\mathbf{\Pi}_2-$slope curve according to type-2 Bishop frame in the SOL^3. Then, the parametric equations of Smarandache $\mathbf{\Pi}_1\mathbf{\Pi}_2\mathbf{B}$ curve of biharmonic constant $\mathbf{\Pi}_2-$slope curve are given by*

$$\begin{aligned}x_{\bar{\gamma}}(s) &= \frac{1}{\sqrt{2\varepsilon_1^2+2\varepsilon_2^2}}e^{\frac{-1}{\sqrt{2\varepsilon_1^2+2\varepsilon_2^2}}[\cos\mathsf{E}-\sin\mathsf{E}]}[\cos\mathsf{E}\cos[\mathsf{R}_1 s+\mathsf{R}_2]-\sin[\mathsf{R}_1 s+\mathsf{R}_2]+\sin\mathsf{E}\cos[\mathsf{R}_1 s+\mathsf{R}_2]],\\ y_{\bar{\gamma}}(s) &= \frac{1}{\sqrt{2\varepsilon_1^2+2\varepsilon_2^2}}e^{\frac{1}{\sqrt{2\varepsilon_1^2+2\varepsilon_2^2}}[\cos\mathsf{E}-\sin\mathsf{E}]}[\cos\mathsf{E}\sin[\mathsf{R}_1 s+\mathsf{R}_2]+\cos[\mathsf{R}_1 s+\mathsf{R}_2]+\sin\mathsf{E}\sin[\mathsf{R}_1 s+\mathsf{R}_2]],\\ z_{\bar{\gamma}}(s) &= \frac{1}{\sqrt{2\varepsilon_1^2+2\varepsilon_2^2}}[\cos\mathsf{E}-\sin\mathsf{E}],\end{aligned}$$

where $\mathsf{R}_1,\mathsf{R}_2$ are constants of integration.

Proof. According to Theorem 3.2, we have system The conclusion holds. This ends the proof.

We can now state the main result of the paper.

In this section, we shall call the set $\{\overline{\mathbf{\Pi}}_1, \overline{\mathbf{\Pi}}_2, \overline{\mathbf{B}}\}$ as type-2 Bishop frame, $\bar{\varepsilon}_1$ and $\bar{\varepsilon}_2$ as Bishop curvatures of Smarandache $\mathbf{\Pi}_1\mathbf{\Pi}_2\mathbf{B}$ curve.

Theorem 3.4. *Let $\gamma : I \to \mathrm{SOL}^3$ be a unit speed non-geodesic biharmonic constant $\mathbf{\Pi}_2$ −slope curve according to type-2 Bishop frame in the SOL^3. Then, type-2 Bishop frame of Smarandache $\mathbf{\Pi}_1\mathbf{\Pi}_2\mathbf{B}$ curve of biharmonic constant $\mathbf{\Pi}_2$ −slope curve are given by*

$$
\begin{aligned}
\overline{\mathbf{\Pi}}_1 \quad = \quad & [[\mathsf{W}\varepsilon_1 \sin[\kappa s] - \frac{\mathsf{W}}{\bar{\kappa}}\varepsilon_1(\varepsilon_1+\varepsilon_2)\cos[\kappa s]]\cos\mathsf{E}\cos[\mathsf{R}_1 s+\mathsf{R}_2] \\
& + [\mathsf{W}\varepsilon_2 \sin[\kappa s] - \frac{\mathsf{W}}{\bar{\kappa}}\varepsilon_2(\varepsilon_1+\varepsilon_2)\cos[\kappa s]]\sin\mathsf{E}\cos[\mathsf{R}_1 s+\mathsf{R}_2] \\
& - [-\mathsf{W}(\varepsilon_1+\varepsilon_2)\sin[\kappa s] - \frac{\mathsf{W}}{\bar{\kappa}}\left(\varepsilon_1^2+\varepsilon_2^2\right)\cos[\kappa s]]\sin[\mathsf{R}_1 s+\mathsf{R}_2]]\mathbf{e}_1 \\
& + [[\mathsf{W}\varepsilon_1 \sin[\kappa s] - \frac{\mathsf{W}}{\bar{\kappa}}\varepsilon_1(\varepsilon_1+\varepsilon_2)\cos[\kappa s]]\cos\mathsf{E}\sin[\mathsf{R}_1 s+\mathsf{R}_2] \\
& + [\mathsf{W}\varepsilon_2 \sin[\kappa s] - \frac{\mathsf{W}}{\bar{\kappa}}\varepsilon_2(\varepsilon_1+\varepsilon_2)\cos[\kappa s]]\sin\mathsf{E}\sin[\mathsf{R}_1 s+\mathsf{R}_2] \\
& + [\mathsf{W}(\varepsilon_1+\varepsilon_2)\cos[\kappa s] - \frac{\mathsf{W}}{\bar{\kappa}}\left(\varepsilon_1^2+\varepsilon_2^2\right)\sin[\kappa s]]\cos[\mathsf{R}_1 s+\mathsf{R}_2]]\mathbf{e}_2 \\
& + [-[\mathsf{W}\varepsilon_1 \sin[\kappa s] - \frac{\mathsf{W}}{\bar{\kappa}}\varepsilon_1(\varepsilon_1+\varepsilon_2)\cos[\kappa s]]\sin\mathsf{E} \\
& + [-\mathsf{W}\varepsilon_2 \cos[\kappa s] - \frac{\mathsf{W}}{\bar{\kappa}}\varepsilon_2(\varepsilon_1+\varepsilon_2)\sin[\kappa s]]\cos\mathsf{E}]\mathbf{e}_3,
\end{aligned}
$$

$$\begin{aligned}
\overline{\mathbf{\Pi}}_2 \;=\; & [[-\mathsf{W}\varepsilon_1\cos[\kappa s]-\frac{\mathsf{W}}{\overline{\kappa}}\varepsilon_1(\varepsilon_1+\varepsilon_2)\sin[\kappa s]]\cos\mathsf{E}\cos[\mathsf{R}_1 s+\mathsf{R}_2] \\
& +[-\mathsf{W}\varepsilon_2\cos[\kappa s]-\frac{\mathsf{W}}{\overline{\kappa}}\varepsilon_2(\varepsilon_1+\varepsilon_2)\sin[\kappa s]]\sin\mathsf{E}\cos[\mathsf{R}_1 s+\mathsf{R}_2] \\
& -[\mathsf{W}(\varepsilon_1+\varepsilon_2)\cos[\kappa s]-\frac{\mathsf{W}}{\overline{\kappa}}\left(\varepsilon_1^2+\varepsilon_2^2\right)\sin[\kappa s]]\sin[\mathsf{R}_1 s+\mathsf{R}_2]]\mathbf{e}_1 \\
& +[[-\mathsf{W}\varepsilon_1\cos[\kappa s]-\frac{\mathsf{W}}{\overline{\kappa}}\varepsilon_1(\varepsilon_1+\varepsilon_2)\sin[\kappa s]]\cos\mathsf{E}\sin[\mathsf{R}_1 s+\mathsf{R}_2]3.5 \\
& +[-\mathsf{W}\varepsilon_2\cos[\kappa s]-\frac{\mathsf{W}}{\overline{\kappa}}\varepsilon_2(\varepsilon_1+\varepsilon_2)\sin[\kappa s]]\sin\mathsf{E}\sin[\mathsf{R}_1 s+\mathsf{R}_2] \\
& +[\mathsf{W}(\varepsilon_1+\varepsilon_2)\cos[\kappa s]-\frac{\mathsf{W}}{\overline{\kappa}}\left(\varepsilon_1^2+\varepsilon_2^2\right)\sin[\kappa s]]\cos[\mathsf{R}_1 s+\mathsf{R}_2]]\mathbf{e}_2 \\
& +[-[-\mathsf{W}\varepsilon_1\cos[\kappa s]-\frac{\mathsf{W}}{\overline{\kappa}}\varepsilon_1(\varepsilon_1+\varepsilon_2)\sin[\kappa s]]\sin\mathsf{E} \\
& +[-\mathsf{W}\varepsilon_2\cos[\kappa s]-\frac{\mathsf{W}}{\overline{\kappa}}\varepsilon_2(\varepsilon_1+\varepsilon_2)\sin[\kappa s]]\cos\mathsf{E}]\mathbf{e}_3,
\end{aligned}$$

$$\begin{aligned}
\overline{\mathbf{B}} \;=\; & [[-\frac{\mathsf{W}^2}{\overline{\kappa}}\varepsilon_2((\varepsilon_1+\varepsilon_2)^2+\left(\varepsilon_1^2+\varepsilon_2^2\right))\cos\mathsf{E}\cos[\mathsf{R}_1 s+\mathsf{R}_2] \\
& +\frac{\mathsf{W}^2}{\overline{\kappa}}\varepsilon_1((\varepsilon_1+\varepsilon_2)^2+(\varepsilon_1+\varepsilon_2)^2)\sin\mathsf{E}\cos[\mathsf{R}_1 s+\mathsf{R}_2]]]\mathbf{e}_1 \\
& +[[-\frac{\mathsf{W}^2}{\overline{\kappa}}\varepsilon_2((\varepsilon_1+\varepsilon_2)^2+\left(\varepsilon_1^2+\varepsilon_2^2\right))\cos\mathsf{E}\sin[\mathsf{R}_1 s+\mathsf{R}_2] \\
& +\frac{\mathsf{W}^2}{\overline{\kappa}}\varepsilon_1((\varepsilon_1+\varepsilon_2)^2+(\varepsilon_1+\varepsilon_2)^2)\sin\mathsf{E}\sin[\mathsf{R}_1 s+\mathsf{R}_2]]]\mathbf{e}_2 \\
& +[[\frac{\mathsf{W}^2}{\overline{\kappa}}\varepsilon_2((\varepsilon_1+\varepsilon_2)^2+\left(\varepsilon_1^2+\varepsilon_2^2\right))\sin\mathsf{E}+\frac{\mathsf{W}^2}{\overline{\kappa}}\varepsilon_1((\varepsilon_1+\varepsilon_2)^2+(\varepsilon_1+\varepsilon_2)^2)\cos\mathsf{E}]]\mathbf{e}_3
\end{aligned}$$

where $\mathsf{R}_1, \mathsf{R}_2$ *are constants of integration and*

$$\mathsf{W}=\frac{1}{\sqrt{\varepsilon_1^2+2\varepsilon_2^2}}.$$

Using the derivative formulae of the type-2 Bishop frame, we get

Theorem 3.5. *Let* $\gamma : I \to \mathsf{SOL}^3$ *be a unit speed non-geodesic biharmonic constant* $\mathbf{\Pi}_2-$*slope curve according to type-2 Bishop frame in the* SOL^3. *Then, type-2 Bishop frame of Smarandache* $\mathbf{\Pi}_1\mathbf{\Pi}_2\mathbf{B}$ *curve of biharmonic constant* $\mathbf{\Pi}_2-$*slope curve are given by*

$$\begin{aligned}
\nabla_{\overline{\mathbf{T}}}\overline{\mathbf{\Pi}}_1 \quad = \quad & -\bar{\varepsilon}_1[[-\frac{\mathsf{W}^2}{\bar{\kappa}}\varepsilon_2((\varepsilon_1+\varepsilon_2)^2+\left(\varepsilon_1^2+\varepsilon_2^2\right))\cos\mathsf{E}\cos\left[\mathsf{R}_1 s+\mathsf{R}_2\right] \\
& +\frac{\mathsf{W}^2}{\bar{\kappa}}\varepsilon_1((\varepsilon_1+\varepsilon_2)^2+(\varepsilon_1+\varepsilon_2)^2)\sin\mathsf{E}\cos\left[\mathsf{R}_1 s+\mathsf{R}_2\right]]]\mathbf{e}_1 \\
& -\bar{\varepsilon}_1[[-\frac{\mathsf{W}^2}{\bar{\kappa}}\varepsilon_2((\varepsilon_1+\varepsilon_2)^2+\left(\varepsilon_1^2+\varepsilon_2^2\right))\cos\mathsf{E}\sin\left[\mathsf{R}_1 s+\mathsf{R}_2\right] \\
& +\frac{\mathsf{W}^2}{\bar{\kappa}}\varepsilon_1((\varepsilon_1+\varepsilon_2)^2+(\varepsilon_1+\varepsilon_2)^2)\sin\mathsf{E}\sin\left[\mathsf{R}_1 s+\mathsf{R}_2\right]]]\mathbf{e}_2 \\
& -\bar{\varepsilon}_1[[\frac{\mathsf{W}^2}{\bar{\kappa}}\varepsilon_2((\varepsilon_1+\varepsilon_2)^2+\left(\varepsilon_1^2+\varepsilon_2^2\right))\sin\mathsf{E} \\
& +\frac{\mathsf{W}^2}{\bar{\kappa}}\varepsilon_1((\varepsilon_1+\varepsilon_2)^2+(\varepsilon_1+\varepsilon_2)^2)\cos\mathsf{E}]]\mathbf{e}_3
\end{aligned}$$

$$\begin{aligned}
\nabla_{\overline{\mathbf{T}}}\overline{\mathbf{\Pi}}_2 \quad = \quad & -\bar{\varepsilon}_2[[-\frac{\mathsf{W}^2}{\bar{\kappa}}\varepsilon_2((\varepsilon_1+\varepsilon_2)^2+\left(\varepsilon_1^2+\varepsilon_2^2\right))\cos\mathsf{E}\cos\left[\mathsf{R}_1 s+\mathsf{R}_2\right] \\
& +\frac{\mathsf{W}^2}{\bar{\kappa}}\varepsilon_1((\varepsilon_1+\varepsilon_2)^2+(\varepsilon_1+\varepsilon_2)^2)\sin\mathsf{E}\cos\left[\mathsf{R}_1 s+\mathsf{R}_2\right]]]\mathbf{e}_1 \\
& -\bar{\varepsilon}_2[[-\frac{\mathsf{W}^2}{\bar{\kappa}}\varepsilon_2((\varepsilon_1+\varepsilon_2)^2+\left(\varepsilon_1^2+\varepsilon_2^2\right))\cos\mathsf{E}\sin\left[\mathsf{R}_1 s+\mathsf{R}_2\right] \\
& +\frac{\mathsf{W}^2}{\bar{\kappa}}\varepsilon_1((\varepsilon_1+\varepsilon_2)^2+(\varepsilon_1+\varepsilon_2)^2)\sin\mathsf{E}\sin\left[\mathsf{R}_1 s+\mathsf{R}_2\right]]]\mathbf{e}_2 \\
& -\bar{\varepsilon}_2[[\frac{\mathsf{W}^2}{\bar{\kappa}}\varepsilon_2((\varepsilon_1+\varepsilon_2)^2+\left(\varepsilon_1^2+\varepsilon_2^2\right))\sin\mathsf{E} \\
& +\frac{\mathsf{W}^2}{\bar{\kappa}}\varepsilon_1((\varepsilon_1+\varepsilon_2)^2+(\varepsilon_1+\varepsilon_2)^2)\cos\mathsf{E}]]\mathbf{e}_3
\end{aligned}$$

$$
\begin{aligned}
\nabla_{\overline{\mathbf{T}}}\overline{\mathbf{B}} \quad = \quad & \bar{\varepsilon}_1[[\mathsf{W}\varepsilon_1\sin[\kappa s]-\frac{\mathsf{W}}{\bar{\kappa}}\varepsilon_1(\varepsilon_1+\varepsilon_2)\cos[\kappa s]]\cos\mathsf{E}\cos[\mathsf{R}_1 s+\mathsf{R}_2] \\
& +[\mathsf{W}\varepsilon_2\sin[\kappa s]-\frac{\mathsf{W}}{\bar{\kappa}}\varepsilon_2(\varepsilon_1+\varepsilon_2)\cos[\kappa s]]\sin\mathsf{E}\cos[\mathsf{R}_1 s+\mathsf{R}_2] \\
& -[-\mathsf{W}(\varepsilon_1+\varepsilon_2)\sin[\kappa s]-\frac{\mathsf{W}}{\bar{\kappa}}\left(\varepsilon_1^2+\varepsilon_2^2\right)\cos[\kappa s]]\sin[\mathsf{R}_1 s+\mathsf{R}_2]]\mathbf{e}_1 \\
& +\bar{\varepsilon}_1[[\mathsf{W}\varepsilon_1\sin[\kappa s]-\frac{\mathsf{W}}{\bar{\kappa}}\varepsilon_1(\varepsilon_1+\varepsilon_2)\cos[\kappa s]]\cos\mathsf{E}\sin[\mathsf{R}_1 s+\mathsf{R}_2] \\
& +[\mathsf{W}\varepsilon_2\sin[\kappa s]-\frac{\mathsf{W}}{\bar{\kappa}}\varepsilon_2(\varepsilon_1+\varepsilon_2)\cos[\kappa s]]\sin\mathsf{E}\sin[\mathsf{R}_1 s+\mathsf{R}_2] \\
& +[\mathsf{W}(\varepsilon_1+\varepsilon_2)\cos[\kappa s]-\frac{\mathsf{W}}{\bar{\kappa}}\left(\varepsilon_1^2+\varepsilon_2^2\right)\sin[\kappa s]]\cos[\mathsf{R}_1 s+\mathsf{R}_2]]\mathbf{e}_2 \\
& +\bar{\varepsilon}_1[-[\mathsf{W}\varepsilon_1\sin[\kappa s]-\frac{\mathsf{W}}{\bar{\kappa}}\varepsilon_1(\varepsilon_1+\varepsilon_2)\cos[\kappa s]]\sin\mathsf{E} \\
& +[-\mathsf{W}\varepsilon_2\cos[\kappa s]-\frac{\mathsf{W}}{\bar{\kappa}}\varepsilon_2(\varepsilon_1+\varepsilon_2)\sin[\kappa s]]\cos\mathsf{E}]\mathbf{e}_3,
\end{aligned}
$$

$$
\begin{aligned}
& +\bar{\varepsilon}_2[[-\mathsf{W}\varepsilon_1\cos[\kappa s]-\frac{\mathsf{W}}{\bar{\kappa}}\varepsilon_1(\varepsilon_1+\varepsilon_2)\sin[\kappa s]]\cos\mathsf{E}\cos[\mathsf{R}_1 s+\mathsf{R}_2] \\
& +[-\mathsf{W}\varepsilon_2\cos[\kappa s]-\frac{\mathsf{W}}{\bar{\kappa}}\varepsilon_2(\varepsilon_1+\varepsilon_2)\sin[\kappa s]]\sin\mathsf{E}\cos[\mathsf{R}_1 s+\mathsf{R}_2] \\
& -[\mathsf{W}(\varepsilon_1+\varepsilon_2)\cos[\kappa s]-\frac{\mathsf{W}}{\bar{\kappa}}\left(\varepsilon_1^2+\varepsilon_2^2\right)\sin[\kappa s]]\sin[\mathsf{R}_1 s+\mathsf{R}_2]]\mathbf{e}_1 \\
& +\bar{\varepsilon}_2[[-\mathsf{W}\varepsilon_1\cos[\kappa s]-\frac{\mathsf{W}}{\bar{\kappa}}\varepsilon_1(\varepsilon_1+\varepsilon_2)\sin[\kappa s]]\cos\mathsf{E}\sin[\mathsf{R}_1 s+\mathsf{R}_2] \\
& +[-\mathsf{W}\varepsilon_2\cos[\kappa s]-\frac{\mathsf{W}}{\bar{\kappa}}\varepsilon_2(\varepsilon_1+\varepsilon_2)\sin[\kappa s]]\sin\mathsf{E}\sin[\mathsf{R}_1 s+\mathsf{R}_2] \\
& +[\mathsf{W}(\varepsilon_1+\varepsilon_2)\cos[\kappa s]-\frac{\mathsf{W}}{\bar{\kappa}}\left(\varepsilon_1^2+\varepsilon_2^2\right)\sin[\kappa s]]\cos[\mathsf{R}_1 s+\mathsf{R}_2]]\mathbf{e}_2 \\
& +\bar{\varepsilon}_2[-[-\mathsf{W}\varepsilon_1\cos[\kappa s]-\frac{\mathsf{W}}{\bar{\kappa}}\varepsilon_1(\varepsilon_1+\varepsilon_2)\sin[\kappa s]]\sin\mathsf{E} \\
& +[-\mathsf{W}\varepsilon_2\cos[\kappa s]-\frac{\mathsf{W}}{\bar{\kappa}}\varepsilon_2(\varepsilon_1+\varepsilon_2)\sin[\kappa s]]\cos\mathsf{E}]\mathbf{e}_3,
\end{aligned}
$$

where $\mathsf{R}_1, \mathsf{R}_2$ *are constants of integration and*

$$\mathrm{W} = \frac{1}{\sqrt{2\varepsilon_1^2 + 2\varepsilon_2^2}}.$$

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Poisson algebras and Poisson prime ideals

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Abstract

The concept of Poisson algebra is one of the most important concepts in mathematics that make a link between commutative and noncommutative algebra. The Poisson algebra can be defined as a Lie algebra that satisfies the Leibniz rule. In this talk, I will give the definition of the Poisson algebra, talk about some properties of Poisson algebras, Poisson prime ideals, Poisson spectra, simple Poisson algebras, Skew polynomial Poisson algebras and Generalized Weyl Poisson algebras.

Keywords: Non-commutative Algebras, Poisson prime ideals, Poisson polynomials algebras.

1. Introduction

A nonempty set S' with binary operator $(\cdot)$ is a *semigroup* $(S', \cdot)$ if for all $g, h, k \in S'$

1. $g \cdot h \in S'$, and
2. $g \cdot (h \cdot k) = (g \cdot h) \cdot k$.

A nonempty set S with binary operator $(+)$ is a *group* $(S, +)$ if for all $g, h, k \in S$

1. $g + h \in S$,
2. $g + (h + k) = (g + h) + k$,
3. $\exists\, e \in S$ s.t. $e + g = g + e = g$, and
4. $\exists g^{-1} \in S$ s.t. $g + g^{-1} = g^{-1} + g = e$.
5. S is an abelian if $g + h = h + g$.

A nonempty set V with two binary operators $(+)$ and $(\times)$ is a *vector space* over a field $\mathbb{C}$ if for all $\lambda_1, \lambda_2 \in \mathbb{C}$ and $v, u \in V$.

1. $(V, +)$ is an abelian group,
2. $\lambda_1 \times v \in V$,
3. $\lambda_1 \times (u + v) = \lambda_1 \times u + \lambda_1 \times v$,
4. $(\lambda_1 + \lambda_2) \times v = \lambda_1 \times v + \lambda_2 \times v$,
5. $\lambda_1 \times (\lambda_2 \times v) = (\lambda_1 \lambda_2) \times v$, and
6. $1 \times v = v$.

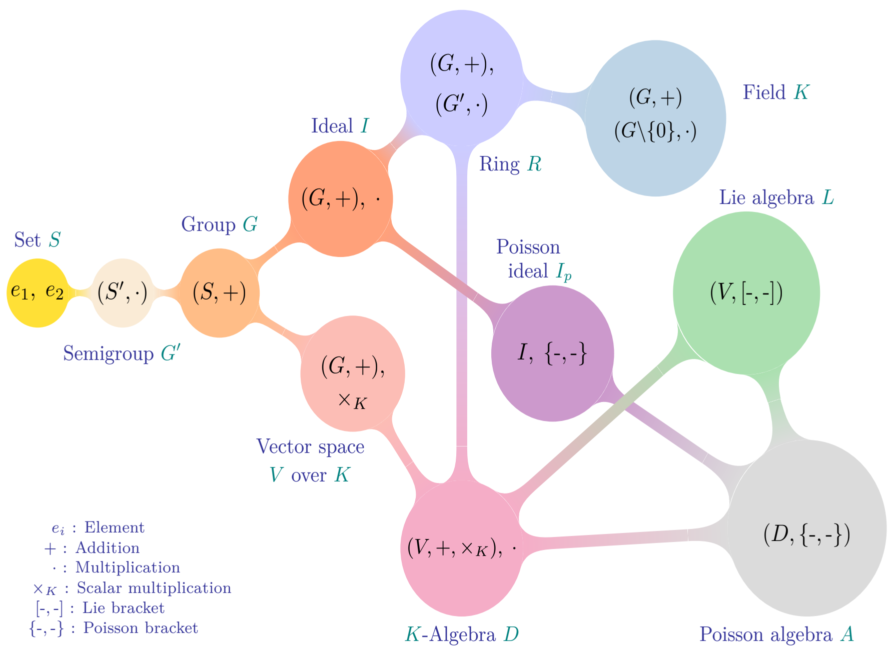

Figure 1: Algebraic structure

2. Poisson Algebras

Definition 1: A (commutative) K-algebra $(D, +, \cdot)$ is called a *Poisson algebra* if there exists bilinear product $\{\cdot,\cdot\}$ on D, called a Poisson bracket, such that $(D, \{\cdot,\cdot\})$ is

1. $\{a, b\} = -\{a, b\}$ for all $a, b \in D$ (anti-commutative),
2. $\{a, \{c, b\}\} + \{b, \{c, a\}\} + \{c, \{b, a\}\} = 0$ for all $a, b, c \in D$ (Jacobi identity), and
3. $\{a \cdot b, c\} = a \cdot \{b, c\} + \{a, c\} \cdot b$ for all $a, b, c \in D$ (Leibniz rule).

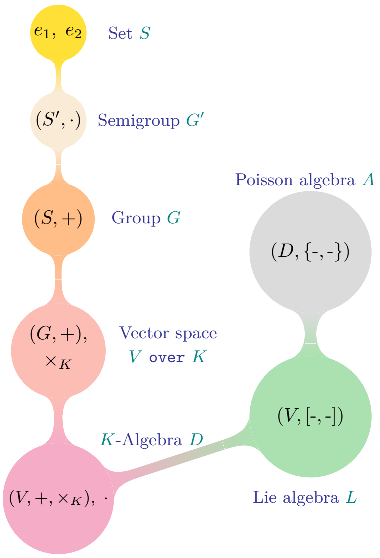

Figure 2: Poisson algebras structure

3. Poisson ideals

Definition 2: Let D be a Poisson algebra. A subset I of D is a *Poisson ideal* of D if

1. I is an ideal of the algebra D, and
2. $\{a, b\} \in I$ for all $d \in D$ and $a \in I$.

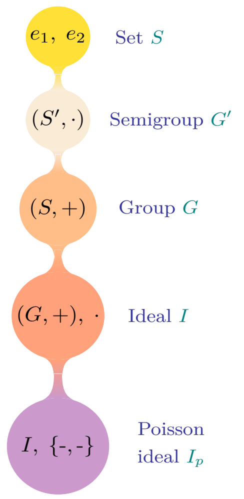

Figure 3: Poisson prime ideals structure

Definition 3: Let D be a Poisson algebra. The algebra D is a *simple Poisson* algebra if the only Poisson ideals of D are D and 0.

4. Poisson prime ideals

Definition 4: Let D be a Poisson algebra. A Poisson ideal P is a *Poisson prime ideal* of D if the following satisfies:

$$IJ \subseteq P \implies I \subseteq P \text{ or } J \subseteq P,$$

where I and J are Poisson ideals of D.

Definition 5: Let D be a Poisson algebra. A set of all Poisson prime ideals of D is called the *Poisson spectrum* of D and is denoted by $\mathrm{PSpec}(D)$.

5. Poisson centre

Definition 6: Let D be a Poisson algebra then

$$\mathrm{PZ}(D) \coloneqq \{a \in D \mid \{a, d\} = 0 \text{ for all } d \in D\}$$

is called the *Poisson centre* of D.

6. Poisson derivations

Definition 7: Let D be an associative Poisson algebra over K. A K-linear map $\alpha: D \to D$ is called a *derivation* (respectively, *Poisson derivation*) on D if α satisfies 1 (respectively, satisfies 1 and 2) of the following conditions:

1. $\alpha(a\,b) = \alpha(a)\,b + a\,\alpha(b)$ for all $a, b \in D$;

2. $\alpha(\{a, b\}) = \{\alpha(a), b\} + \{a, \alpha(b)\}$ for all $a, b \in D$.

A set of all *derivations* (respectively, *Poisson derivations*) on D is denoted by $\mathrm{Der}_K(D)$ (respectively, $\mathrm{PDer}_K(D)$).

7. Poisson polynomial algebras

Theorem 8 [Oh2]: *Let D be a Poisson algebra over K and α, δ be K-linear maps on D. Then the polynomial ring $D[y]$ becomes a Poisson algebra with Poisson bracket:*

$$\{a, y\} = \alpha(a)y + \delta(a) \ \ for\ all\ d \in D \tag{1}$$

if and only if α is a Poisson derivation on D and δ is a derivation on D such that

$$\delta(\{a, b\}) - \{\delta(a), b\} - \{a, \delta(b)\} = \delta(a)\alpha(b) - \alpha(a)\delta(b) \ \ for\ all\ a, b \in D. \tag{2}$$

The Poisson algebra $D[y]$ is denoted by $D[y;\ \alpha, \delta]$ and if δ is zero then it is denoted by $D[y;\ \alpha]$.

Proof.

$$(D, \{\text{-}, \text{-}\}) \underset{(\alpha \in \mathrm{PDer}(D), \delta \in \mathrm{Der}(D))\ (2)}{\overset{\alpha,\ \delta}{\rightleftarrows}} (D[y], (1)) \quad \boxed{D[y; \alpha, \delta]}$$

Example 9 [Oh2]: Let $K[y]$ be a polynomial ring. Notice that, $K[y]$ is a Poisson algebra with trivial Poisson bracket (i.e. $\{a, b\} = 0$ for all $a, b \in K[y]$). For any $f, g \in K[y]$, set

$$\alpha = f\frac{d}{dy} \quad \text{and} \quad \delta = g\frac{d}{dy},$$

then α is a Poisson derivation, δ is a derivation and (α, δ) satisfies (2). Hence, by Theorem 8 the algebra $K[y, x] = K[y][x;\ \alpha, \delta]$ is a Poisson algebra with Poisson bracket defined by the rule

$$\{y, x\} = \alpha(y)x + \delta(y) = fx + g.$$

8. Generalized Weyl Poisson algebras

Definition 10 [Bav2]**:** Let D be a Poisson algebra, $\partial = (\partial_1, \dots, \partial_n) \in \mathrm{PDer}_K(D)^n$ be an n-tuple of commuting Poisson derivations of D, $a = (a_1, \dots, a_n) \in \mathrm{PZ}(D)^n$ be an n-tuple of Poisson central elements of D such that $\partial_i(a_j) = 0$ for all $i \neq j$. The commutative GWA

$$A = D[X, Y; a] = D[X_1, \dots, X_n, Y_1, \dots, Y_n]/(X_1Y_1 - a_1, \dots, X_nY_n - a_n)$$

is a Poisson algebra with Poisson bracket defined by the rule: For all $i, j = 1, \dots, n$ and $d \in D$,

$$\{Y_i, d\} = \partial_i(d)Y_i, \qquad \{X_i, d\} = -\partial_i(d)X_i, \qquad \{Y_i, X_i\} = \partial_i(a_i), \qquad \text{and}$$

$$\{X_i, X_j\} = \{Y_i, Y_j\} = \{X_i, Y_j\} = 0, \text{ for all } i \neq j.$$

This Poisson algebra is denoted by $A = D[X, Y; a, \partial\}$ and is called the *generalized Weyl Poisson algebra* of rank *n* (GWPA) where $X = (X_1, \dots, X_n)$ and $Y = (Y_1, \dots, Y_n)$.

Example 11 [Bav2]**:** The classical Poisson polynomial algebra $P_{2n} = [X_1, \dots, X_n, Y_1, \dots, Y_n]$ with Poisson bracket $\{Y_i, X_j\} = \delta_{ij}$ and $\{X_i, X_j\} = \{Y_i, Y_j\} = 0$ for all i, j where δ_{ij} is the Kronecker delta function, is a GWPA

$$P_{2n} \cong K[H_1, \dots, H_n][X, Y; a, \partial\},$$

Where $K[H_1, \dots, H_n]$ is a Poisson polynomial algebra with trivial Poisson bracket, $a = (H_1, \dots, H_n), \partial = (\partial_1, \dots, \partial_n)$ and $\partial_i = \frac{\partial}{\partial H_i}$ (via the isomorphism of Poisson algebras

$$P_{2n} \to K[H_1, \dots, H_n][X, Y; a, \partial\}, X_i \to X_i, Y_i \to Y_i, X_iY_i \to H_i).$$

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Some Convexity Properties for a New $p-$valent Integral Operator

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Abstract

In this paper, we define a new general $p-$valent integral operator and obtain the properties of convexity of this integral operator of $p-$valent function on some subclasses of analytic functions.

Keywords: Analytic functions; Integral operators; $\beta-$ uniformly $p-$ valent starlike and $\beta-$ uniformly $p-$valent convex functions; of complex order.

1. Introduction and Preliminaries

Let A_p denote the class of the form

$$f(z)=z^p+\sum_{k=p+1}^{\infty}a_kz^k, \quad \left(p\in\mathbb{N}=\{1,2,...\}\right) \tag{1}$$

which are analytic in the open disc $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$. Furthermore, a function $f\in C_p(\gamma,\alpha)$ is $p-$valently convex of complex order $\gamma\left(\gamma\in\mathbb{C}-\{0\}\right)$ and type $\alpha\left(0\leq\alpha<p\right)$, that is, $f\in C_p(\gamma,\alpha)$, if it satisfies the following inequality;

$$\operatorname{Re}\left\{p+\frac{1}{\gamma}\left(1+\frac{zf''(z)}{f'(z)}-p\right)\right\}>\alpha, \quad \left(z\in\mathbb{U}\right). \tag{2}$$

In particular cases, for $p=1$ in the class $C_p(\gamma,\alpha)$, we obtain the clases $C(\gamma,\alpha)$ of convex functions of complex order $\gamma\left(\gamma\in\mathbb{C}-\{0\}\right)$ and type $\alpha\left(0\leq\alpha<p\right)$, which was introduced and studied by Frasin [7]. Also, for $\alpha=0$ in the class $C_p(\gamma,\alpha)$, we obtain the class $C(\gamma,\alpha)$ which is called $p-$valently convex of complex order $\gamma\left(\gamma\in\mathbb{C}-\{0\}\right)$. Setting $p=1$ and $\alpha=0$, we obtain the class $C(\gamma)$. The class $C(\gamma)$ of convex functions of complex order $\gamma\left(\gamma\in\mathbb{C}-\{0\}\right)$ was considered earlier by Wiatrowski [15]. Note that

$C_p(1,\alpha)=C_p(\alpha)$ is the class of $p-$valently convex functions of order $\alpha(0\le\alpha<p)$ in $\mathbb{U}$. Also, we note that $C_1(\alpha)=C(\alpha)$ is the usual class of convex functions of order $\alpha(0\le\alpha<1)$ in $\mathbb{U}$. In special cases $C_1=C$ is the familier class of convex functions in $\mathbb{U}$.

Furthermore a function $f\in\beta-\mathcal{UC}_p(\alpha)$ is $\beta-$uniformly $p-$valently convex of order $\alpha(-1\le\alpha<p)$, that is, $f\in\beta-\mathcal{UC}_p(\alpha)$, if it is satisfies the following inequality;

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\beta\left|1+\frac{zf''(z)}{f'(z)}-p\right|+\alpha,\quad(\beta\ge0,\ z\in\mathbb{U}). \tag{3}$$

These class generalize various other classes which are worthly to mention here. For example $p=1$, the class $\beta-\mathcal{UC}(\alpha)$ introduced by Bharti, Parvatham and Swaminathan [1]. Also, the class $\beta-\mathcal{UC}_1(0)=\beta-\mathcal{UCV}$ is the known class of $\beta-$uniformly convex functions [9].

The class $1-\mathcal{UC}_1(0)=\mathcal{UCV}$ of uniformly convex functions was defined by Goodman [8] (see, also [12]).

For $f\in\mathcal{A}_p$ given by (1) and $g(z)$ given by

$$g(z)=z^p+\sum_{k=p+1}^{\infty}b_kz^k \tag{4}$$

Their convolution (or Hadamard product), denoted by $(f*g)$, is defined as

$$(f*g)(z)=z^p+\sum_{k=p+1}^{\infty}a_kb_kz^k=(g*f)(z),\ (z\in\mathbb{U}). \tag{5}$$

Shenan et al. [14] introduced the operator $D_p^n:\mathcal{A}_p\to\mathcal{A}_p$ is defined by

$$\begin{aligned}D_p^0f(z)&=f(z)\\ D_p^1f(z)&=Df(z)=\frac{zf'(z)}{p}\\ D_p^nf(z)&=D\left(D^{n-1}f(z)\right).\end{aligned} \tag{6}$$

The differential operator D_p^n for analytic and univalent functions was introduced by Salagean ([13]) for $p=1$. It can be easily seen that the operator D_p^n on the function $f(z)$ is given by (1)

$$D_p^n f(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{k}{p}\right)^n a_k z^k.$$

By using the operator D_p^n defined by (1.6), we introduce the new class $\beta-\mathcal{UC}_p(n,\gamma,\alpha)$ as follows:

Definition 1.1 Let $-1 \le \alpha < p$, $\beta \ge 0$ and $\gamma \in \mathbb{C}-\{0\}$. A function $f \in \mathcal{A}_p$ is in the class $\beta-\mathcal{UC}_p(n,\gamma,\alpha)$ if and only if for all $z \in \mathbb{U}$

$$\operatorname{Re}\left\{p+\frac{1}{\gamma}\left(\frac{z\left(D_p^n f(z)\right)''}{\left(D_p^n f(z)\right)'}+1-p\right)\right\} > \beta\left|\frac{1}{\gamma}\left(\frac{z\left(D_p^n f(z)\right)''}{\left(D_p^n f(z)\right)'}+1-p\right)\right|+\alpha. \tag{7}$$

We note that by specializingthe parameters $n,\ p,\ \gamma,\ \beta$ and α in the class $\beta-\mathcal{UC}_p(n,\gamma,\alpha)$, this class reduces to several well-known subclasses of analytic functions. For example, for $p=1$ and $n=0$ the class $\beta-\mathcal{UC}_p(n,\gamma,\alpha)$ reduces to the class $\beta-\mathcal{UC}(\gamma,\alpha)$.

Definition 1.2 Let $l=(l_1,l_2,...l_m)\in\mathbb{N}_0^m$, $\mu=(\mu_1,\mu_2,...,\mu_m)\in\mathbb{R}_+^m$ for all $i=\overline{1,m}$, $m\in\mathbb{N}$. We define the following general integral operators

$$\begin{aligned}
&\mathcal{J}_{p,m}^{l,\mu}(g_1,g_2,...,g_m):\mathcal{A}_p^m \to \mathcal{A}_p\\
&\mathcal{J}_{p,m}^{l,\mu}(g_1,g_2,...,g_m)=\mathcal{G}_{p,m,l,\mu}(z),\\
&\mathcal{G}_{p,m,l,\mu}(z)=\int_0^z pt^{p-1}\prod_{i=1}^{m}\left(\frac{\left(D_p^{l_i}g_i(t)\right)'}{pt^{p-1}}\right)^{\mu_i}dt
\end{aligned} \tag{8}$$

where $g_i \in \mathcal{A}_p$ for all $i=\overline{1,m}$ and D_p^l is defined by (6).

Remark 1.3 For $l_1 = l_2 = \ldots = l_m = 0$ in (8) the integral operator $\mathcal{G}_{p,m,l,\mu}(z)$ reduces to the operator $G_p(z)$ which was studied by Frasin (see [6]). For $p = 1,\ l_1 = l_2 = \ldots = l_m = 0$ in (8) the integral operator $G_{\mu_1,\mu_2,\ldots,\mu_m}(z)$ which was studied by Breaz,Owa and Breaz (see [4]). If $p = m = 1,\ l_1 = 0$ and $\mu_1 = \mu$, we obtain tha integral operator $G(z)$ which was introduced and studied by Pfaltzgraff (see [11]) and Kim and Merkes (see [10]).

In this paper, we consider the integral operator $\mathcal{G}_{p,m,n,l,\mu}(z)$ defined by (8) and study its properties on the class $\beta - \mathcal{UC}_p(n,\gamma,\alpha)$. As special cases the order of convexity of the operator $\int_0^z (g'(t))^{\mu} dt$ are given.

2. Sufficient conditions of the integral operator $\mathcal{G}_{p,m,l,\mu}(z)$

Next, in this section we give a sufficient condition for the integral operator $\mathcal{G}_{p,m,l,\mu}(z)$ to be $p-$valently convex.

Theorem 1. Let $l = (l_1, l_2, \ldots, l_m) \in \mathbb{N}_0^m,\ \mu = (\mu_1, \mu_2, \ldots, \mu_m) \in \mathbb{R}_+^m,\ -1 \le \alpha_i < p,\ \beta_i > 0,\ \gamma \in \mathbb{C} - \{0\}$ and $f_i \in \beta_i - \mathcal{UC}_P(l_i, \gamma, \alpha_i)$ for all $i = \overline{1,m}$. Moreover, suppose that these numbers satisfy the following inequality

$$0 \le p + \sum_{i=1}^{m} \mu_i (\alpha_i - p) < p.$$

Then, the integral operator $\mathcal{G}_{p,m,l,\mu}(z)$ defined by (1.8) is $p-$ valently convex of complex order $\gamma(\gamma \in \mathbb{C} - \{0\})$ and type $0 \le p + \sum_{i=1}^{m} \mu_i (\alpha_i - p) < p.$

Proof. From the definition (1.13), we observe that $\mathcal{G}_{p,m,l,\mu}(z) \in \mathcal{A}_p$. On the other hand, it is easy to see that

$$\mathcal{G}_{p,m,l,\mu}(z) = pz^{p-1} \prod_{i=1}^{m} \left(\frac{\left(D_p^{l_i} g_i(z)\right)'}{pz^{p-1}} \right)^{\mu_i}. \tag{9}$$

Now, we differentiate (9) logarithmically, we have

$$\frac{G''_{p,m,l,\mu}(z)}{G'_{p,m,l,\mu}(z)} = \frac{p-1}{z} + \sum_{i=1}^{m} \mu_i \left(\frac{\left(D_p^{l_i} g_i\right)''(z)}{\left(D_p^{l_i} g_i\right)'(z)} - \frac{p-1}{z} \right). \tag{10}$$

Then multiplying this relation (10) with $\frac{z}{\gamma}$, we obtain

$$\frac{1}{\gamma}\left(\frac{G''_{p,m,l,\mu}(z)}{G'_{p,m,l,\mu}(z)} + 1 - p \right) = \sum_{i=1}^{m} \mu_i \frac{1}{\gamma} \left(\frac{z\left(D_p^{l_i} g_i\right)''(z)}{\left(D_p^{l_i} g_i\right)'(z)} + 1 - p \right)$$

or

$$p + \frac{1}{\gamma}\left(\frac{G''_{p,m,l,\mu}(z)}{G'_{p,m,l,\mu}(z)} + 1 - p \right) = p + \sum_{i=1}^{m} \mu_i \frac{1}{\gamma} \left(\frac{z\left(D_p^{l_i} g_i\right)''(z)}{\left(D_p^{l_i} g_i\right)'(z)} + 1 - p \right). \tag{11}$$

Taking the real part of both sides of (11), we have

$$\begin{aligned} \mathrm{Re}\left\{ p + \frac{1}{\gamma}\left(\frac{zG''_{p,m,l,\mu}(z)}{G'_{p,m,l,\mu}(z)} + 1 - p \right) \right\} &= p + \sum_{i=1}^{m} \mathrm{Re} \frac{1}{\gamma} \left(\frac{z\left(D_p^{l_i} g_i\right)''(z)}{\left(D_p^{l_i} g_i\right)'(z)} + 1 - p \right) \\ &= p - p\sum_{i=1}^{m} \mu_i + \mathrm{Re}\left\{ p + \frac{1}{\gamma}\left(\frac{zG''_{p,m,l,\mu}(z)}{G'_{p,m,l,\mu}(z)} + 1 - p \right) \right\}. \end{aligned} \tag{12}$$

Since $g_i \in \beta_i - \mathcal{UC}_p(l_i, \gamma, \alpha_i)$ for all $i = \overline{1,m}$ from (7) and (12), we have

$$\begin{aligned}\operatorname{Re}\left\{p+\frac{1}{\gamma}\left(\frac{z\mathcal{G}''_{p,m,l,\mu}(z)}{\mathcal{G}'_{p,m,l,\mu}(z)}+1-p\right)\right\} &> p-p\sum_{i=1}^{m}\mu_i\left\{\beta_i\left|\frac{1}{\gamma}\left(\frac{z\left(D_p^{l_i}g_i\right)'(z)}{\left(D_p^{l_i}g_i\right)(z)}+1-p\right)\right|+\alpha_i\right\} \\ &= p+\sum_{i=1}^{m}\mu_i\left(\alpha_i-p\right)+\sum_{i=1}^{m}\frac{\mu_i\beta_i}{|\gamma|}\left|\frac{z\left(D_p^{l_i}g_i\right)''(z)}{\left(D_p^{l_i}g_i\right)'(z)}+1-p\right| > p+\sum_{i=1}^{m}\mu_i\left(\alpha_i-p\right).\end{aligned} \tag{13}$$

Therefore, the operator $\mathcal{G}_{p,m,l,\mu}(z)$ is $p-$valently convex of complex order $\gamma\left(\gamma\in\mathbb{C}-\{0\}\right)$ and type $p+\sum_{i=1}^{m}\mu_i\left(\alpha_i-p\right)$. This evidently completes the proof of Theorem 1.

Remark 2.2

1. Letting $\gamma=1$ and $l_i=0$ for all $i=\overline{1,m}$ in Theorem 1., we obtain Theorem 3.1 in [6].
2. Letting $p=1,\ \beta=0$ and $l_i=0$ for all $i=\overline{1,m}$ in Theorem 1., we obtain Theorem 3 in [2].
3. Letting $p=1,\ \beta=0,\ \alpha_i=\mu$ and $l_i=0$ for all $i=\overline{1,m}$ in Theorem 1., we obtain Theorem 3 in [5].
4. Letting $p=1,\ \beta=0,\ \alpha_i=0$ and $l_i=0$ for all $i=\overline{1,m}$ in Theorem 1., we obtain Theorem 2 in [3].

Putting $p=m=1,\ l_1=0,\ \mu_1=\mu,\ \alpha_1=\alpha,\ \beta_1=\beta$ and $g_1=g$ in Theorem 1., we have

Corollary 1. Let $\mu>0,\ -1\le\alpha<1,\ \beta\ge0,\ \gamma\in\mathbb{C}-\{0\}$ and $g\in\beta-\mathcal{UC}(\gamma,\alpha)$. If $0\le1+\mu(\alpha-1)<1$, then $\int_0^z\left(g'(t)\right)^{\mu}dt$ is convex of complex order $\gamma\left(\gamma\in\mathbb{C}-\{0\}\right)$ and type $\mu(\alpha-1)+1$ in $\mathbb{U}$.

Theorem 2. Let $l=\left(l_1,l_2,...,l_m\right)\in\mathbb{N}_0^m,\ \mu=\left(\mu_1,\mu_2,...,\mu_m\right)\in\mathbb{R}_+^m,\ -1\le\alpha_i<p,\ \beta_i>0,\ \gamma\in\mathbb{C}-\{0\}$ and $f_i\in\beta_i-\mathcal{UC}_p\left(l_i,\gamma,\alpha_i\right)$ for all $i=\overline{1,m}$ and

$$\left|\frac{z\left(D_p^{l_i}f_i\right)''(z)}{\left(D_p^{l_i}f_i\right)'(z)}+1-p\right|>-\frac{p+\sum_{i=1}^{m}\mu_i\left(\alpha_i-p\right)}{\sum_{i=1}^{m}\frac{\mu_i\beta_i}{|\gamma|}} \tag{14}$$

for all $i=\overline{1,m}$, then the integral operator $\mathcal{G}_{p,m,l,\mu}(z)$ defined by (8) is $p-$valently convex of complex order $\gamma(\gamma\in\mathbb{C}-\{0\})$.

Proof. From (13) and (14) we easily get $\mathcal{G}_{p,m,l,\mu}(z)$ is $p-$valently convex of complex order γ.

From Theorem 2., we easily get

Corollary 2. Let $l=(l_1,l_2,...,l_m)\in\mathbb{N}_0^m$, $\mu=(\mu_1,\mu_2,...,\mu_m)\in\mathbb{R}_+^m$, $-1\le\alpha_i<p$, $\beta_i>0$, $\gamma\in\mathbb{C}-\{0\}$ for all $i=\overline{1,m}$ and $D_p^{l_i}g_i\in\mathcal{C}_p(\sigma)$, where $\sigma=p-\left(p+\sum_{i=1}^{m}\mu_i(\alpha_i-p)\right)/\sum_{i=1}^{m}\frac{\mu_i\beta_i}{|\gamma|};0\le\sigma<p$ for all $i=\overline{1,m}$, then the integral operator $\mathcal{G}_{p,m,l,\mu}(z)$ is $p-$valently convex of complex order $\gamma(\gamma\in\mathbb{C}-\{0\})$.

Putting $p=m=1$, $l_1=0, \mu_1=\mu, \alpha_1=\alpha, \beta_1=\beta$ and $g_1=g$ in Corollary 2., we have

Corollary 3. Let $\mu>0$, $-1\le\alpha<1$, $\beta>0$, $\gamma\in\mathbb{C}-\{0\}$ and $g\in\mathcal{C}(\rho)$ where $\rho=\left[\mu(\beta+(1-\alpha)|\gamma|)-|\gamma|\right]/\mu\beta$, $0\le\rho<1$, then the integral operator $\int_0^z(g'(t))^\mu dt$ is convex of complex order $\gamma(\gamma\in\mathbb{C}-\{0\})$ in $\mathbb{U}$.

3.References

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Energy on the $\mathbf{N_f}$ $-$Magnetic Curves

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Abstract

In this paper, we define a normal magnetic curve ($\mathbf{N_f}$ $-$magnetic curve) geometrically, which is associated with the magnetic field $\mathcal{B}$ on the $3D$ Riemannian manifold by considering a normal force on the particle. Moreover, we obtain the energy on the $\mathbf{N_f}$ $-$magnetic curves in the $\mathcal{B}$ magnetic field.

Keywords: Magnetic field, normal force, $\mathbf{N_f}$-magnetic curve, energy, magnetic force, Riemannian manifold.

1. Introduction

Lorentz force action and Maxwell equations are the basis for classical optics, electric circuits, and classical electrodynamics. Gauss law for electromagnetism also includes important relation, i.e. $\nabla \cdot \mathcal{B}$ 0. It shows the diveregence freeness of the $\mathcal{B}$. This feature allows one to determine magnetic field $\mathcal{B}$ in a given space. The trajectory of the magnetic charged particle is influenced by action of the magnetic field. A magnetic field $\mathcal{B}$ defined on a $n-$dimensional Riemannian manifold is a closed 2-form such that its Lorentz force is a one-to-one tensor field ϕ given by

$$\rho(\phi(\mathcal{Y}), \mathcal{X}) = \mathcal{B}(\mathcal{Y}, \mathcal{X}), \quad \forall \mathcal{Y}, \mathcal{X} \in X(M^n). \tag{1}$$

The magnetic trajectories associated with the magnetic field $\mathcal{B}$ are magnetic curves ζ in (M^n, ρ) such that they satisfy

$$\nabla_{\zeta'} \zeta' = \phi(\zeta'). \tag{2}$$

The Lorentz force (ϕ) is associated with a magnetic field $\mathcal{B}$ in the following manner

$$\phi(\mathcal{Y}) = \mathcal{B} \times \mathcal{Y}. \tag{3}$$

Finally, the Lorentz force equations of magnetic curves ζ is calculated as follows

$$\nabla_{\zeta'}\zeta' = \phi(\zeta') = \mathcal{B}\times\zeta'. \tag{4}$$

Details of the derivation of the aforementioned formulae and some important results related to magnetic curves can be foun [1-15].

In electrodynamical perspective, the attitude of the charged particle is investigated with the help of the Lorentz force law and the 2nd law of the Newton, which is given as the following

$$m\mathbf{a} = \mathbf{F} = q(\mathcal{E} + \mathbf{v}\times\mathcal{B}), \tag{5}$$

where $\mathbf{v}$ is velocity, m is mass, and q is electric charge under the magnetic field $\mathcal{B}$ and electric field $\mathcal{E}$, for the particle.

2. Kinematics of the Particle

Let Γ be a moving particle such that its coordinate is showed by $\Gamma = \Gamma(t)$, where t is a time parameter. By changing the time parameter with the arc-lentgh parameter, it is obtained that

$$v = \frac{ds}{dt} = \|\mathbf{v}\|,$$

where $\mathbf{v} = \mathbf{v}(t) = \frac{d\zeta}{dt}$ and $\frac{d\zeta}{dt} \neq 0$. The Frenet-Serret system is established by orthonormal vectors $\mathbf{e}^{\mu}_{(\alpha)}$, provided that the curve is smooth at each point. In particular, $\mathbf{e}^{\mu}_{(0)}$ is the unit tangent vector, $\mathbf{e}^{\mu}_{(1)}$, $\mathbf{e}^{\mu}_{(2)}$ are unit normal and binormal vectors of the curve ζ, respectively. Orthonormality conditions are summarized by $\mathbf{e}^{\mu}_{(\alpha)}\mathbf{e}^{\mu}_{(\beta)} = \eta_{\alpha\beta}$, where $\eta_{\alpha\beta}$ is Euclidean metric such that: $\mathrm{diag}(1,1,1)$. For non-negative coefficients $\kappa, \tau,$ and vectors $\mathbf{e}^{\mu}_{(i)}(i = 0,1,2)$ following equations and properties satisfy [15]:

$$\begin{aligned}
\nabla_{\zeta'}\mathbf{e}^{\mu}_{(0)} &= \kappa\mathbf{e}^{\mu}_{(1)}, \\
\nabla_{\zeta'}\mathbf{e}^{\mu}_{(1)} &= -\kappa\mathbf{e}^{\mu}_{(0)} + \tau\mathbf{e}^{\mu}_{(2)}, \\
\nabla_{\zeta'}\mathbf{e}^{\mu}_{(2)} &= -\tau\mathbf{e}^{\mu}_{(1)}.
\end{aligned} \tag{6}$$

It can also be deduced that

$$\mathbf{v}(s) = \frac{d\zeta}{dt} = \frac{ds}{dt}\mathbf{e}^{\mu}_{(0)}, \tag{7}$$

and

$$\mathbf{a}(s) = \frac{d\mathbf{v}}{dt} = \frac{d^2 s}{dt^2}\mathbf{e}^{\mu}_{(0)} + \kappa\left(\frac{ds}{dt}\right)^2 \mathbf{e}^{\mu}_{(1)}. \tag{8}$$

Finally, for any particle sliding down on a surface with a mass $m,$ the normal force is

$$\mathbf{N_f} = \varkappa N\mathbf{e}^{\mu}_{(1)}, \tag{9}$$

where $N = \|\mathbf{N_f}\|,\ \varkappa = \pm 1;$ the gravitational force is

$$\mathbf{G} = m(g_0\mathbf{e}^{\mu}_{(0)} + g_1\mathbf{e}^{\mu}_{(1)}), \tag{10}$$

where $g_{i=0,1}$ are gravitational coefficient; the frictional force is

$$f = -\alpha N\mathbf{e}^{\mu}_{(0)}, \tag{11}$$

where α is frictional coefficient [16] .

3. $\mathbf{N_f}$ −Magnetic curves in 3D Remannian manifolds

Now, we assume that for a moving charged particle in any magnetic field $\mathcal{B}$ on (M^3,ρ), there exists a normal force acting on the particle. Then, trajectories of the particle of the magnetic field $\mathcal{B}$ on the 3D Riemannian surface give a new kind of magnetic curve.

Definition: Let ζ be an arc-length parameterized magnetic curve in the 3D Riemannian manifold (M^3,ρ) and $\mathcal{B}$ be a magnetic field on M^3. We call the curve ζ as a $\mathbf{N_f}$-magnetic curve if the normal force field of the curve meets the following Lorentz force equation;

$$\nabla_{\zeta'}\mathbf{N_f} = \phi(\mathbf{N_f}) = \mathcal{B}\times\mathbf{N_f}. \tag{12}$$

Proposition: Let ζ be an arc-length parametrized $\mathbf{N_f}$ − magnetic curve of $\mathcal{B}$ with the Frenet frame elements $\{\mathbf{e}^{\mu}_{(0)},\mathbf{e}^{\mu}_{(1)},\mathbf{e}^{\mu}_{(2)},\kappa,\tau\}$. Then, Lorentz force ϕ of a magnetic field $\mathcal{B}$ is written in the Frenet frame as the following way

$$\begin{aligned}
\phi(\mathbf{e}^{\mu}_{(0)}) &= \omega_1\mathbf{e}^{\mu}_{(0)} + \kappa\mathbf{e}^{\mu}_{(1)} + \omega_2\mathbf{e}^{\mu}_{(2)},\\
\phi(\mathbf{e}^{\mu}_{(1)}) &= \omega_3\mathbf{e}^{\mu}_{(0)} + \omega_4\mathbf{e}^{\mu}_{(2)},\\
\phi(\mathbf{e}^{\mu}_{(2)}) &= \omega_5\mathbf{e}^{\mu}_{(0)} - \tau\mathbf{e}^{\mu}_{(1)} + \omega_6\mathbf{e}^{\mu}_{(2)},
\end{aligned} \tag{13}$$

where $\omega_i,\ 1 \le i \le 6$ are smooth functions along the curve ζ.

Proof: Let ζ be an arc-length parametrized $\mathbf{N_f}-$magnetic curve in (M^3,ρ) together with the Frenet frame elements $\{\mathbf{e}^{\mu}_{(0)},\mathbf{e}^{\mu}_{(1)},\mathbf{e}^{\mu}_{(2)},\kappa,\tau\}$. Knowing the fact that $\phi(\mathbf{e}^{\mu}_{(0)}) \in span\{\mathbf{e}^{\mu}_{(0)},\mathbf{e}^{\mu}_{(1)},\mathbf{e}^{\mu}_{(2)}\}$ and

$$-\varkappa N\kappa\mathbf{e}^{\mu}_{(0)} = \rho(\phi(\mathbf{N_f}),\mathbf{e}^{\mu}_{(0)}) = -\rho(\mathbf{N_f},\phi(\mathbf{e}^{\mu}_{(0)})) = -\rho(\varkappa N\mathbf{e}^{\mu}_{(1)},\phi(\mathbf{e}^{\mu}_{(0)})),$$

we get $\phi(\mathbf{e}^{\mu}_{(0)}) = \omega_1\mathbf{e}^{\mu}_{(0)} + \kappa\mathbf{e}^{\mu}_{(1)} + \omega_2\mathbf{e}^{\mu}_{(2)}$; where ω_1 and ω_2 are some smooth functions along the curve ζ. Proof is completed by using similar procedure for $\phi(\mathbf{e}^{\mu}_{(1)})$ and $\phi(\mathbf{e}^{\mu}_{(2)})$.

Theorem: ζ is a unit speed $\mathbf{N_f}-$magnetic curve of the magnetic field $\mathcal{B}$ if and only if

$$\mathcal{B} = \tau\mathbf{e}^{\mu}_{(0)} - (\tau\frac{\omega_1}{\omega_3} + \kappa\frac{\omega_5}{\omega_3})\mathbf{e}^{\mu}_{(1)} + \kappa\mathbf{e}^{\mu}_{(2)}, \quad \omega_3 \neq 0 \tag{14}$$

or equivalently

$$\mathcal{B} = \tau\mathbf{e}^{\mu}_{(0)} - (\tau\frac{\omega_2}{\omega_4} + \kappa\frac{\omega_6}{\omega_4})\mathbf{e}^{\mu}_{(1)} + \kappa\mathbf{e}^{\mu}_{(2)}, \quad \omega_4 \neq 0 \tag{15}$$

along the curve ζ.

Proof: Let us choose $\mathcal{B} = a_0\mathbf{e}^{\mu}_{(0)} + a_1\mathbf{e}^{\mu}_{(1)} + a_2\mathbf{e}^{\mu}_{(2)}$, where $a_i,\ i=0,1,2$ are some functions along ζ. We also suppose that $\mathcal{B}$ does not vanish along the curve. Now, from the definition of $\mathbf{N_f}$-magnetic curve we have

$$\nabla_{\zeta'}\mathbf{N_f} = (a_0\mathbf{e}^{\mu}_{(0)} + a_1\mathbf{e}^{\mu}_{(1)} + a_2\mathbf{e}^{\mu}_{(2)}) \times (\varkappa N\mathbf{e}^{\mu}_{(1)}).$$

Here, we obtain $a_0 = \tau$ and $a_2 = \kappa$. Also, from the definition of Lorentz force ϕ, we get $\phi(\mathcal{B}) = \mathcal{B}\times\mathcal{B} = 0.$ Thus, we have

$$0 = a_0\phi(\mathbf{e}^{\mu}_{(0)}) + a_1\phi(\mathbf{e}^{\mu}_{(1)}) + a_2\phi(\mathbf{e}^{\mu}_{(2)}),$$

which means $a_1 = -(\tau\frac{\omega_1}{\omega_3} + \kappa\frac{\omega_5}{\omega_3}),\ \omega_3 \neq 0$ or equivalently $a_1 = -(\tau\frac{\omega_2}{\omega_4} + \kappa\frac{\omega_6}{\omega_4}),\ \omega_4 \neq 0.$

Corollary: Let $\omega_i,\ 1 \le i \le 6$ be arbitrary smooth functions given in Proposition. Then, we have the following relation.

$$\frac{\tau}{\kappa}=\frac{\omega_6\omega_3-\omega_5\omega_4}{\omega_1\omega_4-\omega_2\omega_3},\ \omega_1\omega_4-\omega_2\omega_3\neq 0 \tag{16}$$

Proof: By definition we get $\phi(\mathcal{B})=\mathcal{B}\times\mathcal{B}=0$. Thus, we have

$$0=\phi(\mathcal{B})=\tau\phi(\mathbf{e}^{\mu}_{(0)})-(\tau\frac{\omega_1}{\omega_3}+\kappa\frac{\omega_5}{\omega_3})\phi(\mathbf{e}^{\mu}_{(1)})+\kappa\phi(\mathbf{e}^{\mu}_{(2)}). \tag{17}$$

If we also use (13), (14), and (15), then the proof is completed.

Corollary: Let ω_i, $1\leq i\leq 6$ be arbitrary smooth functions along the curve ζ such that the ratio given in Eq. (16) is constant. Then $\mathbf{N_f}-$magnetic curve is a general helix in the $\mathcal{B}$ magnetic field.

Proof: If we use Lancret theorem, then it gives the result immediately [18].

4. Energy on the $\mathbf{N_f}-$Magnetic curves

Definition: Let (M,ρ) and (N,h) be two Riemannian manifolds. Then, the energy of a differentiable map $r:(M,\rho)\to(N,h)$ can be defined as

$$\varepsilon nergy(r)=\frac{1}{2}\int_M\sum_{a=1}^{n}h(df(e_a),df(e_a))v, \tag{18}$$

where $\{e_a\}$ is a local basis of the tangent space and v is the canonical volume form in M [19].

Proposition: Let $Q:T(T^1M)\to T^1M$ be the connection map. Then following two conditions hold:

i) $\omega\circ Q=\omega\circ d\omega$ and $\omega\circ Q=\omega\circ\tilde{\omega}$, where $\tilde{\omega}:T(T^1M)\to T^1M$ is the tangent bundle projection;

ii) for $\rho\in T_xM$ and a section $\varsigma:M\to T^1M$; we have

$$Q(d\varsigma(\rho))=\nabla_{\rho}\varsigma, \tag{19}$$

where ∇ is the Levi-Civita covariant derivative [19,20].

Definition: Let $\sigma_1,\sigma_2\in T_{\varsigma}(T^1M)$, then we define

$$\rho_S(\sigma_1,\sigma_2)=\rho(d\omega(\sigma_1),d\omega(\sigma_2))+\rho(Q(\sigma_1),Q(\sigma_2)). \tag{20}$$

This yields a Riemannian metric on TM. As known ρ_S is called the Sasaki metric that also makes the projection $\omega: T^1M \to M$ a Riemannian submersion.

Main Theorem: Let Γ be a moving charged particle such that it corresponds to a unit speed $\mathbf{N_f}-$ magnetic curve ζ on the 3D Riemannian manifold (M^3,ρ) in the magnetic field $\mathcal{B}$. Then, energy on the particle in the magnetic vector field $\mathcal{B}$ is stated by

$$\varepsilon nergy(\mathbf{N_f}) = \frac{1}{2}\int_0^s (1+(\tau' + \kappa\tau\frac{\omega_1}{\omega_3} + \kappa^2\frac{\omega_5}{\omega_3})^2 + (\tau'\frac{\omega_1}{\omega_3} + \tau(\frac{\omega_1'\omega_3 - \omega_1\omega_3'}{\omega_3^2}) + \kappa'\frac{\omega_5}{\omega_3} + \tau(\frac{\omega_5'\omega_3 - \omega_5\omega_3'}{\omega_3^2}))^2 + (\kappa' - \kappa\tau\frac{\omega_5}{\omega_3} - \tau^2\frac{\omega_1}{\omega_3})^2)ds,$$

or equivalently

$$\varepsilon nergy(\mathbf{N_f}) = \frac{1}{2}\int_0^s (1+(\tau' + \kappa\tau\frac{\omega_2}{\omega_4} + \kappa^2\frac{\omega_6}{\omega_4})^2 + (\tau'\frac{\omega_2}{\omega_4} + \tau(\frac{\omega_2'\omega_4 - \omega_2\omega_4'}{\omega_4^2}) + \kappa'\frac{\omega_6}{\omega_4} + \tau(\frac{\omega_6'\omega_4 - \omega_6\omega_4'}{\omega_4^2}))^2 + (\kappa' - \kappa\tau\frac{\omega_6}{\omega_4} - \tau^2\frac{\omega_2}{\omega_4})^2)ds.$$

Proof: From (18) and (19), we get

$$\varepsilon nergy(\mathbf{N_f}) = \frac{1}{2}\int_0^s \rho_S\left(d\mathcal{B}(\mathbf{e}^{\mu}_{(0)}), d\mathcal{B}(\mathbf{e}^{\mu}_{(0)})\right)ds.$$

By using Eq. (20), we have

$$\rho_S(d\mathcal{B}(\mathbf{e}^{\mu}_{(0)}), d\mathcal{B}(\mathbf{e}^{\mu}_{(0)})) = \rho(d\omega(\mathcal{B}(\mathbf{e}^{\mu}_{(0)})), d\omega(\mathcal{B}(\mathbf{e}^{\mu}_{(0)}))) + \rho(Q(\mathcal{B}(\mathbf{e}^{\mu}_{(0)})), Q(\mathcal{B}(\mathbf{e}^{\mu}_{(0)}))).$$

Since $\mathbf{e}^{\mu}_{(0)}$ is a section, we also get

$$d(\omega)\circ d(\mathcal{B}) = d(\omega\circ\mathcal{B}) = d(id_C) = id_{TC}.$$

Moreover, it is clear that

$$Q(\mathcal{B}(\mathbf{e}^{\mu}_{(0)})) = \nabla_{\zeta'}\mathcal{B} = (\tau' + \kappa\tau\frac{\omega_1}{\omega_3} + \kappa^2\frac{\omega_5}{\omega_3}) - (\tau'\frac{\omega_1}{\omega_3} + \tau(\frac{\omega_1'\omega_3 - \omega_1\omega_3'}{\omega_3^2}) + \kappa'\frac{\omega_5}{\omega_3} + \tau(\frac{\omega_5'\omega_3 - \omega_5\omega_3'}{\omega_3^2})) + (\kappa' - \kappa\tau\frac{\omega_5}{\omega_3} - \tau^2\frac{\omega_1}{\omega_3}).$$

Thus, we find from (6) and (14)

$$\begin{aligned}\rho_S(d\mathcal{B}(\mathbf{e}^{\mu}_{(0)}),d\mathcal{B}(\mathbf{e}^{\mu}_{(0)})) &= \rho(\mathbf{e}^{\mu}_{(0)},\mathbf{e}^{\mu}_{(0)})+\rho(\nabla_{\zeta'}\mathcal{B},\nabla_{\zeta'}\mathcal{B}) \\ &= 1+(\tau'+\kappa\tau\frac{\omega_1}{\omega_3}+\kappa^2\frac{\omega_5}{\omega_3})^2+(\tau'\frac{\omega_1}{\omega_3}+\tau(\frac{\omega_1'\omega_3-\omega_1\omega_3'}{\omega_3^2}) \\ &\quad +\kappa'\frac{\omega_5}{\omega_3}+\tau(\frac{\omega_5'\omega_3-\omega_5\omega_3'}{\omega_3^2}))^2+(\kappa'-\kappa\tau\frac{\omega_5}{\omega_3}-\tau^2\frac{\omega_1}{\omega_3})^2).\end{aligned}$$

Finally, this gives the desired result. The equivalent relation can be proved by a similar method.

Corollary: Let ζ be a unit speed $\mathbf{N_f}-$magnetic curve on 3D Riemannian manifold (M^3,ρ) in the magnetic field $\mathcal{B}$. Then, energy on the particle in each Lorentz force field of the magnetic field $\mathcal{B}$ given in (13) is stated by using Sasaki metric as the following.

$$\begin{aligned}\varepsilon nergy\phi(\mathbf{e}^{\mu}_{(0)}) &= \frac{1}{2}\int_0^s(1+(\omega_1'-\kappa^2)^2+(\kappa\omega_1+\kappa'-\omega_2\tau)^2+(\kappa\tau+\omega_2')^2)ds, \\ \varepsilon nergy\phi(\mathbf{e}^{\mu}_{(1)}) &= \frac{1}{2}\int_0^s(1+(\omega_3')^2+(\omega_3\kappa-\omega_4\tau)^2+(\omega_4')^2)ds, \\ \varepsilon nergy\phi(\mathbf{e}^{\mu}_{(2)}) &= \frac{1}{2}\int_0^s(1+(\omega_5'-\kappa\tau)^2+(\omega_5\kappa-\tau'-\omega_6\tau)^2+(-\tau^2+\omega_6')^2)ds.\end{aligned}$$

Proof: It is obvious from Eqs. (13), (18), (19), and (20).

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Application of Fractional Calculus Operators to the functions in the certain subclasses of analytic functions

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Abstract

In this study, we investigate the growth and distortion properties of functions in the a certain subclasses of analytic functions which involves the operator Fractional Calculus.

Keywords: Analytic functions, Univalent functions, starlike and convex functions, Fractional calculus operators, Differential operator.

1. Introduction and Definitions

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open disc $\mathcal{U} = \{z : z \in \mathcal{C} : \; |z| < 1\}$. Suppose that $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of functions that are the univalent in $\mathcal{U}$. Also denote by $\mathcal{T}$ the subclass of $\mathcal{A}$ consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n. \tag{2}$$

A function $f \in \mathcal{A}$ is said to be in the class of uniformly convex functions of order α and type β, denoted by $\beta - \mathcal{UCV}(\alpha)$ (see [14]) if

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)} - \alpha\right\} > \beta \left|\frac{zf''(z)}{f'(z)}\right|, \quad z \in \mathcal{U}, \tag{3}$$

where $-1 \le \alpha < 1$ and $\beta \ge 0$.

A function $f \in \mathcal{A}$ is said to be in the class of uniformly starlike functions of order α and type β, denoted by $\beta - \mathcal{SP}(\alpha)$ (see [2]) if

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)} - \alpha\right\} > \beta \left|\frac{zf'(z)}{f(z)} - 1\right|, \quad z \in \mathcal{U}, \tag{4}$$

where and $-1 \le \alpha < 1$ and $\beta \ge 0$.

These classes generalize various other classes which are worth mentioning here. The class $\beta - \mathcal{UCV}(0) = \beta - \mathcal{UCV}$ is the class of $\beta-$uniformly convex functions [9].

Indeed it follows from (3) and (4) that

$$D_\lambda^m f(z) \in \beta - \mathcal{UCV}(\alpha) \Leftrightarrow z\left(D_\lambda^m f(z)\right)' \in \beta - \mathcal{SP}(\alpha).$$

Especially the classes $1-\mathcal{UCV}(0)=\mathcal{UCV}$ and $1-\mathcal{SP}(0)=\mathcal{SP}$, defined by Goodman [8] and Ronning [14], respectivelly.

For a functions f in $\mathcal{A}$, Deniz and Özkan [5] (see also [6]) introduced the following differential operatör $\mathcal{D}_\lambda^m$ as follows:

Definition 1. Let $f \in \mathcal{A}$. For the parametres $\lambda \geq 0$ and $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ the differential operatör $\mathcal{D}_\lambda^m$ on $\mathcal{A}$ defined by

$$\mathcal{D}_\lambda^0 f(z) = f(z)$$

$$\mathcal{D}_\lambda^1 f(z) = \lambda z^3 f'''(z) + (2\lambda+1) z^2 f''(z) + z f'(z)$$

$$\mathcal{D}_\lambda^m f(z) = \mathcal{D}(\mathcal{D}_\lambda^{m-1} f(z)$$

for $z \in \mathcal{U}$.

For a function f in $\mathcal{A}$ from the definition of the differential operatör $\mathcal{D}_\lambda^m$, we can easily see that

$$\mathcal{D}_\lambda^m f(z) = z + \sum_{n=2}^{\infty} n^{2m} (\lambda(n-1)+1)^m a_n z^n. \tag{5}$$

Also, $\mathcal{D}_\lambda^m f(z) \in \mathcal{A}$.

For $f \in \mathcal{A}$ given by (1) and $g(z) \in \mathcal{A}$ given by $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, the Hadamard product (or Convolution) of f and g defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z), \quad z \in \mathcal{U}.$$

Special cases of this operator include the Salagean derivative operator $\mathcal{S}^m$ (see [15]) as follows:

$$\mathcal{D}_0^m f(z) = \mathcal{S}^m f(z) * \mathcal{S}^m f(z) = \mathcal{S}^{2m} f(z)$$

and

$$\mathcal{D}_1^m f(z) = \mathcal{S}^m f(z) * \mathcal{S}^m f(z) * \mathcal{S}^m f(z) = \mathcal{S}^{3m} f(z).$$

For $-1 \leq \alpha < 1$, $\lambda \geq 0$, $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\beta \geq 0$, let $\beta - \mathcal{SP}_\lambda^m(\alpha)$ be the subclass of $\mathcal{A}$ consising of functions of the form (1) and satisfying the analytic criterion

$$\operatorname{Re}\left\{ \frac{z\left(\mathcal{D}_\lambda^m f(z)\right)'}{\mathcal{D}_\lambda^m f(z)} - \alpha \right\} > \beta \left| \frac{z\left(\mathcal{D}_\lambda^m f(z)\right)'}{\mathcal{D}_\lambda^m f(z)} - 1 \right|,$$

where $\mathcal{D}_\lambda^m f(z)$ is given by (5). We also let $\beta - \mathcal{TSP}_\lambda^m(\alpha) = \beta - \mathcal{SP}_\lambda^m(\alpha) \cap \mathcal{T}$.

Not that $\mathcal{D}_\lambda^m f(z) \in \beta - \mathcal{SP}_\lambda^m(\alpha)$ if and only if $\mathcal{D}_\lambda^m f(z) \in \beta - \mathcal{SP}(\alpha)$. Using the Alexander type relation, we define the class $\beta - \mathcal{UCV}_\lambda^m(\alpha)$ as follows

$$\mathcal{D}_\lambda^m f(z) \in \beta - \mathcal{UCV}_\lambda^m(\alpha) \Leftrightarrow z\left(\mathcal{D}_\lambda^m f(z)\right)' \in \beta - \mathcal{SP}_\lambda^m(\alpha).$$

We also let $\beta - \mathcal{TUCV}_\lambda^m(\alpha) = \beta - \mathcal{UCV}_\lambda^m(\alpha) \cap \mathcal{T}$.

We note that by specializing the parameters α, β, λ and m, the subclasses $\beta - \mathcal{SP}_\lambda^m(\alpha)$ and $\beta - \mathcal{UCV}_\lambda^m(\alpha)$ reduces to several well-known subclasses of analytic functions. This subclasses are:

i. $\beta - \mathcal{SP}_\lambda^0(\alpha) = \beta - \mathcal{SP}(\alpha),\ \beta - \mathcal{UCV}_\lambda^0(\alpha) = \beta - \mathcal{UCV}(\alpha)$

ii. $\beta - \mathcal{SP}_\lambda^0(0) = \beta - \mathcal{SP},\ \beta - \mathcal{UCV}_\lambda^0(0) = \beta - \mathcal{UCV}$

iii. $1 - \mathcal{SP}_\lambda^0(0) = \mathcal{SP},\ 1 - \mathcal{UCV}_\lambda^0(0) = \mathcal{UCV}$.

In [7], authors obtained the following results:

Theorem 1. A function $f(z)$ of the form (1) is in $\beta - \mathcal{SP}_\lambda^m(\alpha)$ if

$$\sum_{n=2}^{\infty}\left[n(1+\beta)-(\alpha+\beta)\right]n^{2m}\left(\lambda(n-1)+1\right)^m |a_n| \le 1-\alpha \tag{6}$$

where $-1 \le \alpha < 1$, $\lambda \ge 0$, $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\beta \ge 0$.

Theorem 2. A necessary and sufficient condition for $f(z)$ of the form (2) to be in the class $\beta - \mathcal{TSP}_\lambda^m(\alpha)$ for $-1 \le \alpha < 1$, $\lambda \ge 0$, $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\beta \ge 0$ is that

$$\sum_{n=2}^{\infty}\left[n(1+\beta)-(\alpha+\beta)\right]n^{2m}\left(\lambda(n-1)+1\right)^m a_n \le 1-\alpha. \tag{7}$$

Theorem 3. A function $f(z)$ of the form (1) is in $\beta - \mathcal{UCV}_\lambda^m(\alpha)$ if

$$\sum_{n=2}^{\infty}\left[n(1+\beta)-(\alpha+\beta)\right]n^{2m+1}\left(\lambda(n-1)+1\right)^m |a_n| \le 1-\alpha \tag{8}$$

where $-1 \le \alpha < 1$, $\lambda \ge 0$, $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\beta \ge 0$.

Theorem 4. A necessary and sufficient condition for $f(z)$ of the form (2) to be in the class $\beta - \mathcal{TUCV}_\lambda^m(\alpha)$ for $-1 \le \alpha < 1$, $\lambda \ge 0$, $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\beta \ge 0$ is that

$$\sum_{n=2}^{\infty}\left[n(1+\beta)-(\alpha+\beta)\right]n^{2m+1}\left(\lambda(n-1)+1\right)^m a_n \le 1-\alpha. \tag{9}$$

2. Main Results

Applications of Fractional Calculus Operators

Various operators of fractional calculus (that is, fractional derivatives) have been studied in the literatüre extensivelly (cf., e.g., [13,16,19]; see also [3,4,12,17,18] and the various references cited therein). In our present investigation, we recall the following definitions.

Definition 2. Let $f(z)$ be analytic in a simply connected region of the $z-$plane containing the origin. The fractional integral of f order v is defined by

$$D_z^{-v} f(z) = \frac{1}{\Gamma(v)} \int_0^z \frac{f(\zeta)}{\left(z-\zeta\right)^{1-v}} d\zeta \quad (v > 0)$$

where the multiplicity of $\left(z-\zeta\right)^{1-v}$ is removed by requiring that $\log\left(z-\zeta\right)$ is real for $z-\zeta > 0.$

Definition 3. Let $f(z)$ be analytic in a simply connected region of the $z-$plane containing the origin. The fractional integral of f order v is defined by

$$\mathcal{D}_z^{v} f(z) = \frac{1}{\Gamma(1-v)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{\left(z-\zeta\right)^{v}} d\zeta \quad (0 \le v < 1).$$

Where the multiplicity of $\left(z-\zeta\right)^{-v}$ is removed by requiring that $\log\left(z-\zeta\right)$ is real for $z-\zeta > 0.$

Definition 4. From the hypothesis of Definition 3, the fractional derivative of order $n+v$ is defined, for a function $f(z)$, by

$$\mathcal{D}_z^{n+v} f(z) = \frac{d^n}{dz^n}\left\{\mathcal{D}_z^{v} f(z)\right\} \; (0 \le v < 1;\; n \in \mathbb{N}_0).$$

In the present paper, we make use of the familiar integral operator L_ϑ defined by (see, for details, [1,10,11]; see also [20])

$$\left(L_\vartheta\right)(z) = \frac{\vartheta+1}{z} \int_0^z t^{\vartheta-1} f(t)dt \quad f \in A;\; \vartheta > -1$$

as well as the fractional calculus operator $\mathcal{D}_z^{v}$ for which it is well known that (see, for details, [13,16,19])

$$\mathcal{D}_z^{v}\left\{z\right\} = \frac{1}{\Gamma(2-v)} z^{1-v} \quad (v \in \mathbb{R})$$

in terms of the gamma function.

In this section, we investigate the growth and distortion properties of functions in the class $\beta-\mathcal{TSP}_\lambda^m(\alpha)$ and $\beta-\mathcal{TUCV}_\lambda^m(\alpha)$ which involves the operators L_ϑ and $\mathcal{D}_z^{v}$. To derive our results, we need following lemma given by Chen et al. [3].

Lemma (see [3]) Let the function $f(z)$ be given defined by (2). Then

$$\mathcal{D}_z^{v}\left\{\left(L_{\vartheta} f\right)(z)\right\}=\frac{z^{1-v}}{\Gamma(2-v)}-\sum_{n=2}^{\infty}\frac{(\vartheta+1)\Gamma(n+1)}{(\vartheta+n)\Gamma(n+1-v)}a_n z^{n-v} \quad (v\in\mathbb{R};\ \vartheta>-1;\ n\in\mathbb{N}) \tag{10}$$

and

$$L_{\vartheta}\left\{\left(\mathcal{D}_z^{v} f\right)(z)\right\}=\frac{(\vartheta+1)}{(\vartheta+1-v)\Gamma(2-v)}z^{1-v}-\sum_{n=2}^{\infty}\frac{(\vartheta+1)\Gamma(n+1)}{(\vartheta+n-v)\Gamma(n+1-v)}a_n z^{n-v} \tag{11}$$
$$(v\in\mathbb{R};\ \vartheta>-1;\ n\in\mathbb{N})$$

provided that no zeros appear in the denominators (10) and (11).

Theorem 5. Let the functions $f(z)$ defined by (2) be in the $\beta-\mathcal{TSP}_{\lambda}^{m}(\alpha)$. Then

$$\begin{aligned}&\left|\mathcal{D}_z^{-v}\left\{\left(L_{\vartheta} f\right)(z)\right\}\right|\\ &\geq\left\{\frac{1}{\Gamma(2+v)}-\frac{(1+\vartheta)(1-\alpha)}{\left(2+\vartheta\right)\Gamma\left(3+v\right)2^{2m-1}(\beta-\alpha+2)(\lambda+1)^m}|z|\right\}|z|^{1+v}\\ &(z\in\mathcal{U};\ v>0,\vartheta>-1,\beta>\alpha;\ n\in\mathbb{N}).\end{aligned} \tag{12}$$

and

$$\begin{aligned}&\left|\mathcal{D}_z^{-v}\left\{\left(L_{\vartheta} f\right)(z)\right\}\right|\\ &\leq\left\{\frac{1}{\Gamma(v+2)}+\frac{(1+\vartheta)(1-\alpha)}{\left(\vartheta+2\right)\Gamma\left(3+v\right)2^{2m-1}(\beta-\alpha+2)(\lambda+1)^m}|z|\right\}|z|^{1+v}\\ &(z\in\mathcal{U};\ v>0,\vartheta>-1,\beta>\alpha;\ n\in\mathbb{N}).\end{aligned} \tag{13}$$

Each of the assertion (12) and (13) is sharp.

Proof: In view of Theorem 2, we have

$$\frac{\left(\left(\beta-\alpha+2\right)2^{2m}\left(\lambda+1\right)^m\right)}{1-\alpha}\sum_{n=2}^{\infty}|a_n|\leq\sum_{n=2}^{\infty}\frac{\left[n(1+\beta)-\left(\alpha+\beta\right)\right]n^{2m}\left(\lambda\left(n-1\right)+1\right)^m}{1-\alpha}|a_n|\leq 1, \tag{14}$$

which readily yields

$$\sum_{n=2}^{\infty}|a_n|\leq\frac{1-\alpha}{(\beta-\alpha+2)2^{2m}\left(\lambda+1\right)^m}. \tag{15}$$

Consider the function $F(z)$ defined in $\mathcal{U}$ by

$$\begin{aligned}F(z)&=\Gamma(2-v)z^{-v}\mathcal{D}_z^{-v}\left\{\left(L_{\vartheta} f\right)(z)\right\}\\ &=z-\sum_{n=2}^{\infty}\frac{(\vartheta+1)\Gamma(n+1)\Gamma(v+2)}{(\vartheta+n)\Gamma(n+v+1)}|a_n|z^n\\ &=z-\sum_{n=2}^{\infty}\Theta(n)|a_n|z^n \quad (z\in\mathcal{U}),\end{aligned}$$

where

$$\Theta(n)=\frac{(1+\vartheta)\Gamma(n+1)\Gamma(v+2)}{(n+\vartheta)\Gamma(n+v+1)},\quad (n\geq 2,\ v>0). \tag{16}$$

Since $\Theta(n)$ is a decreasing function of n when $v>0,$ we get

$$0<\Theta(n)\leq\Theta(2)=\frac{2(1+\vartheta)}{(2+\vartheta)(2+v)},\quad (\vartheta>-1,\ v>0). \tag{17}$$

Thus, from (15) ve (17) for all $z\in\mathcal{U},$ we deduce that

$$\begin{aligned}\left|F(z)\right|&\geq|z|-\Theta(2)|z|^2\sum_{n=2}^{\infty}|a_n|\\ &\geq|z|-\frac{(1+\vartheta)(1-\alpha)}{(2+\vartheta)(2+v)2^{2m-1}(\beta-\alpha+2)(\lambda+1)^m}|z|^2\end{aligned}$$

and

$$\begin{aligned}\left|F(z)\right|&\leq|z|+\Theta(2)|z|^2\sum_{n=2}^{\infty}|a_n|\\ &\leq|z|+\frac{(1+\vartheta)(1-\alpha)}{(2+\vartheta)(2+v)2^{2m-1}(\beta-\alpha+2)(\lambda+1)^m}|z|^2,\end{aligned}$$

which yields the inequalities (12) and (13) of Theorem 5. Equalities in (12) and (13) are attained for the function $f(z)$ given by

$$\mathcal{D}_z^{-v}\left\{(L_\vartheta f)(z)\right\}=\left\{\frac{1}{\Gamma(2+v)}-\frac{(\vartheta+1)(1-\alpha)}{(2+\vartheta)\Gamma(3+v)2^{2m-1}(\beta-\alpha+2)(\lambda+1)^m}z\right\}z^{1+v}$$

$$(z\in\mathcal{U};\ v>0,\vartheta>-1,\beta>\alpha;n\in\mathbb{N}).$$

or, equivalently, by

$$(L_\vartheta f)(z)=z-\frac{(1+\vartheta)(1-\alpha)}{(2+\vartheta)2^{2m}(\beta-\alpha+2)(\lambda+1)^m}z^2.$$

Thus we complete the proof of Theorem 5.

Theorem 6. Let the functions $f(z)$ defined by (2) be in the $\beta-\mathcal{TSP}_\lambda^m(\alpha)$. Then

$$\begin{aligned}&\left|\mathcal{D}_z^{v}\left\{(L_\vartheta f)(z)\right\}\right|\\ &\geq\left\{\frac{z^{1-v}}{\Gamma(2-v)}-\frac{(\vartheta+1)(1-\alpha)}{(\vartheta+2)\Gamma(3-v)2^{2m-1}(\beta-\alpha+2)(\lambda+1)^m}|z|\right\}|z|^{1-v}\end{aligned} \tag{18}$$

$$(z\in\mathcal{U};\ v>0,\vartheta>-1,\beta>\alpha;n\in\mathbb{N}).$$

and

$$\begin{aligned}&\left|\mathcal{D}_z^{v}\left\{\left(L_{\vartheta} f\right)(z)\right\}\right| \\ &\leq \left\{\frac{z^{1-v}}{\Gamma(2-v)} + \frac{(1+\vartheta)(1-\alpha)}{\left(2+\vartheta\right)\Gamma\left(3-v\right)2^{2m-1}(\beta-\alpha+2)(\lambda+1)^m}|z|\right\}|z|^{1-v} \\ &(z\in\mathcal{U};\ v>0, \vartheta>-1, \beta>\alpha;\ n\in\mathbb{N}).\end{aligned} \tag{19}$$

Each of the assertions (18) and (19) is sharp.
Proof: It follows from Theorem 2 that

$$\sum_{n=2}^{\infty} n|a_n| \leq \frac{1-\alpha}{2^{2m-1}(\beta-\alpha+2)(\lambda+1)^m}. \tag{20}$$

Consider the function $\varphi(z)$ defined in $\mathcal{U}$ by

$$\begin{aligned}\varphi(z) &= \Gamma(2-v)z^{v}\mathcal{D}_z^{v}\left\{\left(L_{\vartheta} f\right)(z)\right\} \\ &= z - \sum_{n=2}^{\infty}\frac{(1+\vartheta)\Gamma(n)\Gamma(2-v)}{(n+\vartheta)\Gamma(n+1-v)} n|a_n| z^n \\ &= z - \sum_{n=2}^{\infty}\varphi(n) n|a_n| z^n \quad (z\in\mathcal{U}),\end{aligned}$$

where, for convenience

$$\varphi(n) = \frac{(1+\vartheta)\Gamma(n)\Gamma(2-v)}{(n+\vartheta)\Gamma(n+1-v)} \quad (n\geq 2,\ 0\leq v<1). \tag{21}$$

Since $\varphi(n)$ is a decreasing function of n when $0\leq v<1$, we find that

$$0<\varphi(n)\leq\varphi(2)=\frac{(1+\vartheta)}{(2+\vartheta)(2-v)}. \tag{22}$$

Hence, with the aid of (20) and (22), for all $z\in\mathcal{U}$, we have

$$\begin{aligned}|\varphi(z)| &\geq |z| - \varphi(2)|z|^2\sum_{n=2}^{\infty} n|a_n| \\ &\geq |z| - \frac{(1+\vartheta)(1-\alpha)}{(2+\vartheta)(2-v)2^{2m-1}(\beta-\alpha+2)(\lambda+1)^m}|z|^2\end{aligned}$$

and

$$\begin{aligned}|\varphi(z)| &\leq |z| + \varphi(2)|z|^2\sum_{n=2}^{\infty} n|a_n| \\ &\leq |z| + \frac{(1+\vartheta)(1-\alpha)}{(2+\vartheta)(2-v)2^{2m-1}(\beta-\alpha+2)(\lambda+1)^m}|z|^2\end{aligned}$$

which yields the inequalities (18) and (19) of Theorem 6. Equalities in (18) and (19) are attained for the functions $f(z)$ given by

$$\mathcal{D}_z^{v}\left\{\left(L_{\vartheta} f\right)(z)\right\}$$
$$=\left\{\frac{1}{\Gamma(2-v)}-\frac{(1+\vartheta)(1-\alpha)}{\left(2+\vartheta\right)\Gamma(3-v)2^{2m-1}(\beta-\alpha+2)(\lambda+1)^{m}}z\right\}z^{1-v}$$

or, equivalently, by

$$\left(L_{\vartheta} f\right)(z)=z-\frac{(\vartheta+1)(1-\alpha)}{\left(\vartheta+2\right)2^{2m}(\beta-\alpha+2)(\lambda+1)^{m}}z^{2}.$$

Consequently, we complete the proof of Theorem 6.

Corollary 1. Let the functions $f(z)$ defined by (2) be in the $\beta-\mathcal{TUCV}_{\lambda}^{m}(\alpha)$. Then

$$\left|\mathcal{D}_z^{-v}\left\{\left(L_{\vartheta} f\right)(z)\right\}\right|$$
$$\geq\left\{\frac{1}{\Gamma(2+v)}-\frac{(1+\vartheta)(1-\alpha)}{\left(2+\vartheta\right)\Gamma\left(3+v\right)2^{2m}(\beta-\alpha+2)(\lambda+1)^{m}}|z|\right\}|z|^{1+v}$$
$$(z\in\mathcal{U};\ v>0,\vartheta>-1,\beta>\alpha;\ n\in\mathbb{N}).$$

and

$$\left|\mathcal{D}_z^{-v}\left\{\left(L_{\vartheta} f\right)(z)\right\}\right|$$
$$\leq\left\{\frac{1}{\Gamma(2+v)}+\frac{(1+\vartheta)(1-\alpha)}{\left(2+\vartheta\right)\Gamma\left(3+v\right)2^{2m}(\beta-\alpha+2)(\lambda+1)^{m}}|z|\right\}|z|^{1+v}$$
$$(z\in\mathcal{U};\ v>0,\vartheta>-1,\beta>\alpha;\ n\in\mathbb{N}).$$

Corollary 2. Let the functions $f(z)$ defined by (2) be in the $\beta-\mathcal{TUCV}_{\lambda}^{m}(\alpha)$. Then

$$\left|\mathcal{D}_z^{v}\left\{\left(L_{\vartheta} f\right)(z)\right\}\right|$$
$$\geq\left\{\frac{z^{1-v}}{\Gamma(2-v)}-\frac{(1+\vartheta)(1-\alpha)}{\left(2+\vartheta\right)\Gamma\left(3-v\right)2^{2m}(\beta-\alpha+2)(\lambda+1)^{m}}|z|\right\}|z|^{1-v}$$
$$(z\in\mathcal{U};\ v>0,\vartheta>-1,\beta>\alpha;\ n\in\mathbb{N}).$$

and

$$\left|\mathcal{D}_z^{v}\left\{\left(L_{\vartheta,1} f\right)(z)\right\}\right|$$
$$\leq\left\{\frac{z^{1-v}}{\Gamma(2-v)}+\frac{(1+\vartheta)(1-\alpha)}{\left(2+\vartheta\right)\Gamma\left(3-v\right)2^{2m}(\beta-\alpha+2)(\lambda+1)^{m}}|z|\right\}|z|^{1-v}.$$
$$(z\in\mathcal{U};\ v>0,\vartheta>-1,\beta>\alpha;\ n\in\mathbb{N}).$$

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Neighborhoods of Certain Classes of Analytic Functions Defined By Rabotnov Function

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Abstract

We introduce a new subclass of analytic functions in the open unit disk $\mathcal{U}$ with negative coefficients defined by normalized of the Rabotnov function. The object of the present paper is to determine coefficient inequalities, inclusion relations and neighborhoods properties for Rabotnov function belonging to this subclass.

Keywords: Analytic function, starlike and convex functions, Rabotnov function, neighborhoods, coefficient inequality.

1. Introduction

Let $\mathcal{A}$ be a class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

that are analytic in the open disk $\mathcal{U} = \{z : |z| < 1\}$. Denote by $\mathcal{A}(n)$ the class of functions consisting of functions f of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad (a_n \geq 0) \tag{2}$$

which are analytic in $\mathcal{U}$.

We recall that the convolution (or Hadamard product) of two functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n$$

is given by

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n =: (g * f)(z), \quad (z \in \mathcal{U}).$$

Note that $f * g \in \mathcal{A}$.

Next, following the earlier investigations by Goodman [7], Ruscheweyh [12], Silverman [13], Altıntaş et al. [2,3] and Srivastava and Bulut [14] (see also [1], [4]-[6], [8]-[10]), we define the $(n,\delta)-$ neighborhood of a function $f \in \mathcal{A}(n)$ by

$$\mathcal{N}_{n,\delta}(f)=\left\{g\in\mathcal{A}(n):g(z)=z-\sum_{n=2}^{\infty}b_nz^n \text{ and } \sum_{n=2}^{\infty}n\left|a_n-b_n\right|\leq\delta\right\}. \tag{3}$$

For $e(z)=z,$ we have

$$\mathcal{N}_{n,\delta}(e)=\left\{g\in\mathcal{A}(n):g(z)=z-\sum_{n=2}^{\infty}b_nz^n \text{ and } \sum_{n=2}^{\infty}n\left|b_n\right|\leq\delta\right\}. \tag{4}$$

A function $f\in\mathcal{A}(n)$ is $\alpha-$starlike of complex order γ, denoted by $f\in\mathcal{S}_n^*(\alpha,\gamma)$ if it satisfies the following condition

$$\Re\left\{1+\frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)}-1\right)\right\}>\alpha,\quad \left(\gamma\in\mathbb{C}\setminus\{0\},\ 0\leq\alpha<1,\ z\in\mathcal{U}\right)$$

and a function $f\in\mathcal{A}(n)$ is $\alpha-$convex of complex order γ, denoted by $f\in\mathcal{C}_n^*(\alpha,\gamma)$ if it satisfies the following condition

$$\Re\left\{1+\frac{1}{\gamma}\frac{zf''(z)}{f'(z)}\right\}>\alpha,\quad \left(\gamma\in\mathbb{C}\setminus\{0\},\ 0\leq\alpha<1,\ z\in\mathcal{U}\right).$$

The Rabotnov [11] function $R_{\alpha,\beta}(z),$ defined by

$$R_{\alpha,\beta}(z)=z^{\alpha}\sum_{n=0}^{\infty}\frac{\beta^n}{\Gamma\left((n+1)(1+\alpha)\right)}z^{n(1+\alpha)},\quad \left(\alpha>-1,\ \beta\geq0,\ z\in\mathcal{U}\right). \tag{5}$$

The Rabotnov function $R_{\alpha,\beta}(z)$ does not belong to the class $\mathcal{A}$. Therefore, we consider the following normalization for the function $R_{\alpha,\beta}(z)$:

$$\mathbb{R}_{\alpha,\beta}(z)=\Gamma(1+\alpha)z^{1/(1+\alpha)}R_{\alpha,\beta}\left(z^{1/(1+\alpha)}\right)=\sum_{n=0}^{\infty}\frac{\beta^n\Gamma(1+\alpha)}{\Gamma\left((n+1)(1+\alpha)\right)}z^{n+1},\quad (z\in\mathcal{U}). \tag{6}$$

In terms of Hadamard product and $\mathbb{R}_{\alpha,\beta}(z)$ given by (6), a new operator $\Theta_{\alpha,\beta}:\mathcal{A}\to\mathcal{A}$ can be defined as follows:

$$\Theta_{\alpha,\beta}f(z)=\left(\Theta_{\alpha,\beta}*f\right)(z)=z+\sum_{n=1}^{\infty}\frac{\beta^n\Gamma(1+\alpha)a_{n+1}}{\Gamma\left((n+1)(1+\alpha)\right)}z^{n+1},\quad (z\in\mathcal{U}). \tag{7}$$

If $f\in\mathcal{A}(n)$ is given by (2), then we have

$$\Theta_{\alpha,\beta}f(z)=z-\sum_{n=1}^{\infty}\frac{\beta^n\Gamma(1+\alpha)a_{n+1}}{\Gamma\left((n+1)(1+\alpha)\right)}z^{n+1},\quad (z\in\mathcal{U}). \tag{8}$$

Finally, by using the differential operator defined by (8), we investigate the subclasses $\mathcal{M}_{\alpha,\beta}^n(\kappa,\gamma)$ and $\mathcal{R}_{\alpha,\beta}^n(\kappa,\gamma;\vartheta)$ of $\mathcal{A}(n)$ consisting of functions f as the followings:
However, throughout this paper, we restrict our attention to the case real-valued α,β with $\alpha>-1$ and $\beta\geq0$.

Definition 1. The subclass $\mathcal{M}_{\alpha,\beta}^{n}(\kappa,\gamma)$ of $\mathcal{A}(n)$ is defined as the class of functions f such that

$$\left|\frac{1}{\gamma}\left(\frac{z\left[\Theta_{\alpha,\beta}f(z)\right]'}{\Theta_{\alpha,\beta}f(z)}-1\right)\right|<\kappa, \quad (z\in\mathcal{U}), \tag{9}$$

where $\gamma\in\mathbb{C}\setminus\{0\}$ and $0\le\kappa<1$.

Definition 2. Let $\mathcal{R}_{\alpha,\beta}^{n}(\kappa,\gamma;\vartheta)$ denote the subclass of $\mathcal{A}(n)$ consisting of f which satisfy the inequality

$$\left|\frac{1}{\gamma}\left[(1-\vartheta)\right]\frac{\Theta_{\alpha,\beta}f(z)}{z}+\vartheta\left(\Theta_{\alpha,\beta}f(z)\right)'-1\right|<\kappa, \tag{10}$$

where $\gamma\in\mathbb{C}\setminus\{0\}$, $0\le\kappa<1$ and $0\le\vartheta\le1$.
In this paper, we obtain the coefficient inequalities, inclusion relations and neighborhood properties of the subclasses $\mathcal{M}_{\alpha,\beta}^{n}(\kappa,\gamma)$ and $\mathcal{R}_{\alpha,\beta}^{n}(\kappa,\gamma;\vartheta)$.

2. Coefficient inequalities for $\mathcal{M}_{\alpha,\beta}^{n}(\kappa,\gamma)$ and $\mathcal{R}_{\alpha,\beta}^{n}(\kappa,\gamma;\vartheta)$

Theorem 1. Let $f\in\mathcal{A}(n)$. Then $f\in\mathcal{M}_{\alpha,\beta}^{n}(\kappa,\gamma)$ if and only if

$$\sum_{n=2}^{\infty}\frac{\beta\Gamma(1+\alpha)}{\Gamma(n(1+\alpha))}\left[n-1+\kappa|\gamma|\right]a_n\le\kappa|\gamma|, \quad (z\in\mathcal{U}) \tag{11}$$

for $\gamma\in\mathbb{C}\setminus\{0\}$ and $0\le\kappa<1$.

Proof. Let $f\in\mathcal{A}(n)$. Then, by (9) we can write

$$\Re\left\{\frac{z\left[\Theta_{\alpha,\beta}f(z)\right]'}{\Theta_{\alpha,\beta}f(z)}-1\right\}>-\kappa|\gamma|, \quad (z\in\mathcal{U}). \tag{12}$$

Using (2) and (8), we have,

$$\Re\left\{\frac{-\sum_{n=2}^{\infty}\frac{\beta^{n-1}\Gamma(1+\alpha)}{\Gamma(n(1+\alpha))}[n-1]a_nz^n}{z-\sum_{n=2}^{\infty}\frac{\beta^{n-1}\Gamma(1+\alpha)}{\Gamma(n(1+\alpha))}a_nz^n}\right\}>-\kappa|\gamma|, \quad (z\in\mathcal{U}). \tag{13}$$

Since (13) is true for all $z\in\mathcal{U}$, choose values of z on the real axis. Letting $z\to1$, through the real values, the inequality (13) yields the desired inequality

$$\sum_{n=2}^{\infty}\frac{\beta^{n-1}\Gamma(1+\alpha)}{\Gamma(n(1+\alpha))}\left[n-1+\kappa|\gamma|\right]a_n\le\kappa|\gamma|.$$

Conversely, supposed that the inequality (11) holds true and $|z|=1,$ then we obtain

$$\begin{aligned}\left|\frac{z\left[\Theta_{\alpha,\beta}f(z)\right]'}{\Theta_{\alpha,\beta}f(z)}-1\right| &\leq \left|\frac{\sum_{n=2}^{\infty}\frac{\beta^{n-1}\Gamma(1+\alpha)}{\Gamma(n(1+\alpha))}[n-1]a_n z^n}{z-\sum_{n=2}^{\infty}\frac{\beta^{n-1}\Gamma(1+\alpha)}{\Gamma(n(1+\alpha))}a_n z^n}\right| \\ &\leq \frac{\sum_{n=2}^{\infty}\frac{\beta^{n-1}\Gamma(1+\alpha)}{\Gamma(n(1+\alpha))}[n-1]a_n}{1-\sum_{n=2}^{\infty}\frac{\beta^{n-1}\Gamma(1+\alpha)}{\Gamma(n(1+\alpha))}a_n} \\ &\leq \kappa|\gamma|.\end{aligned}$$

Hence, by the maximum modulus theorem, we have $f(z)\in\mathcal{M}_{\alpha,\beta}^{n}(\kappa,\gamma),$ which establishes the required result.

Theorem 2. Let $f\in\mathcal{A}(n).$ Then $f\in\mathcal{R}_{\alpha,\beta}^{n}(\kappa,\gamma;\vartheta)$ if and only if

$$\sum_{n=2}^{\infty}\frac{\beta^{n-1}\Gamma(1+\alpha)}{\Gamma(n(1+\alpha))}\left[1+\vartheta(n-1)\right]a_n\leq\kappa|\gamma| \tag{14}$$

for $\gamma\in\mathbb{C}\setminus\{0\}$, $0\leq\kappa<1$ and $0\leq\vartheta\leq1.$

Proof. We omit the proofs since it is similar to Theorem 1.

3. Inclusion relations involving $\mathcal{N}_{n,\delta}(e)$ of $\mathcal{M}_{\alpha,\beta}^{n}(\kappa,\gamma)$ and $\mathcal{R}_{\alpha,\beta}^{n}(\kappa,\gamma;\vartheta)$

Theorem 3. If

$$\delta=\frac{2\kappa|\gamma|\Gamma(2+2\alpha)}{\beta(1+\kappa|\gamma|)\Gamma(1+\alpha)},\quad (|\gamma|<1), \tag{15}$$

then $\mathcal{M}_{\alpha,\beta}^{n}(\kappa,\gamma)\subset\mathcal{N}_{n,\delta}(e).$

Proof. Let $f(z)\in\mathcal{M}_{\alpha,\beta}^{n}(\kappa,\gamma).$ By Theorem 1, we have

$$\frac{\beta\Gamma(1+\alpha)}{\Gamma(2+2\alpha)}(1+\kappa|\gamma|)\sum_{n=2}^{\infty}a_n\leq\kappa|\gamma|,$$

which implies

$$\sum_{n=2}^{\infty} a_n \leq \frac{\kappa|\gamma|}{\frac{\beta\Gamma(1+\alpha)}{\Gamma(2+2\alpha)}(1+\kappa|\gamma|)}. \tag{16}$$

Using (11) and (16), we get

$$\begin{aligned} \frac{\beta\Gamma(1+\alpha)}{\Gamma(2+2\alpha)}\sum_{n=2}^{\infty} na_n &\leq \kappa|\gamma| + \frac{\beta\Gamma(1+\alpha)}{\Gamma(2+2\alpha)}(1-\kappa|\gamma|)\sum_{n=2}^{\infty} a_n \\ &\leq \frac{2\kappa|\gamma|}{(1+\kappa|\gamma|)} = \delta, \end{aligned}$$

that is,

$$\sum_{n=2}^{\infty} na_n \leq \frac{2\kappa|\gamma|}{\frac{\beta\Gamma(1+\alpha)}{\Gamma(2+2\alpha)}(1+\kappa|\gamma|)} = \delta.$$

Thus, by the definition given by (4), $f(z) \in \mathcal{N}_{n,\delta}(e)$, which completes the proof.

Theorem 4. If

$$\delta = \frac{2\kappa|\gamma|\Gamma(2+2\alpha)}{\beta(1+\vartheta)\Gamma(1+\alpha)}, \quad (|\gamma|<1), \tag{17}$$

then $\mathcal{R}_{\alpha,\beta}^{n}(\kappa,\gamma;\vartheta) \subset \mathcal{N}_{n,\delta}(e)$.

Proof. For $f(z) \in \mathcal{R}_{\alpha,\beta}^{n}(\kappa,\gamma;\vartheta)$ and making use of the condition (14), we obtain

$$\frac{\beta\Gamma(1+\alpha)}{\Gamma(2+2\alpha)}(1+\vartheta)\sum_{n=2}^{\infty} a_n \leq \kappa|\gamma|$$

so that

$$\sum_{n=2}^{\infty} a_n \leq \frac{\kappa|\gamma|}{\frac{\beta\Gamma(1+\alpha)}{\Gamma(2+2\alpha)}(1+\vartheta)}. \tag{18}$$

Thus, using (14) along with (18), we also get

$$\begin{aligned}\vartheta\frac{\beta\Gamma(1+\alpha)}{\Gamma(2+2\alpha)}\sum_{n=2}^{\infty}na_n &\leq \kappa|\gamma|+(\vartheta-1)\frac{\beta\Gamma(1+\alpha)}{\Gamma(2+2\alpha)}\sum_{n=2}^{\infty}a_n\\ &\leq \kappa|\gamma|+\frac{\beta(\vartheta-1)\Gamma(1+\alpha)}{\Gamma(2+2\alpha)}\frac{\kappa|\gamma|\Gamma(2+2\alpha)}{\beta(1+\vartheta)\Gamma(1+\alpha)}\\ &\leq \frac{2\vartheta\kappa|\gamma|}{(1+\vartheta)}=\delta.\end{aligned}$$

Hence,

$$\sum_{n=2}^{\infty}na_n\leq\frac{2\kappa|\gamma|}{\frac{\beta\Gamma(1+\alpha)}{\Gamma(2+2\alpha)}(1+\vartheta)}=\delta$$

which in view of (4), completes the proof of theorem.

4. Neighborhood properties for the classes $\mathcal{M}_{\alpha,\beta}^{n}(\kappa,\gamma,\eta)$ and $\mathcal{R}_{\alpha,\beta}^{n}(\kappa,\gamma,\eta;\vartheta)$

Definition: For $0\leq\eta<1$ and $z\in\mathcal{U}$, A function $f(z)\in\mathcal{A}(n)$ is said to be in the class $\mathcal{M}_{\alpha,\beta}^{n}(\kappa,\gamma,\eta)$ if there exists a function $g(z)\in\mathcal{M}_{\alpha,\beta}^{n}(\kappa,\gamma)$ such that

$$\left|\frac{f(z)}{g(z)}-1\right|<1-\eta. \tag{19}$$

Analogously, for $0\leq\eta<1$ and $z\in\mathcal{U}$, $f(z)\in\mathcal{A}(n)$ is said to be in the class $\mathcal{R}_{\alpha,\beta}^{n}(\kappa,\gamma,\eta;\vartheta)$ if there exists a function $g(z)\in\mathcal{R}_{\alpha,\beta}^{n}(\kappa,\gamma;\vartheta)$ such that the inequality (19) holds true.

Theorem 5. If $g(z)\in\mathcal{M}_{\alpha,\beta}^{n}(\kappa,\gamma)$ and

$$\eta=1-\frac{\delta\beta\Gamma(1+\alpha)(1+\kappa|\gamma|)}{2\left[\beta(1+\kappa|\gamma|)\Gamma(1+\alpha)-\kappa|\gamma|\Gamma(2+2\alpha)\right]} \tag{20}$$

then $\mathcal{N}_{n,\delta}(g)\subset\mathcal{M}_{\alpha,\beta}^{n}(\kappa,\gamma,\eta)$.

Proof. Let $f(z)\in\mathcal{N}_{n,\delta}(g)$. Then,

$$\sum_{n=2}^{\infty}n|a_n-b_n|\leq\delta \tag{21}$$

which yields the coefficient inequality,

$$\sum_{n=2}^{\infty}|a_n - b_n| \leq \frac{\delta}{2}, \quad (n \in \mathbb{N}).$$

Since $g(z) \in \mathcal{M}_{\alpha,\beta}^{n}(\kappa,\gamma)$ by (16), we have

$$\sum_{n=2}^{\infty} b_n \leq \frac{\kappa|\gamma|}{\frac{\beta\Gamma(1+\alpha)}{\Gamma(2+2\alpha)}(1+\kappa|\gamma|)}, \tag{22}$$

and so

$$\begin{aligned}\left|\frac{f(z)}{g(z)}-1\right| &< \frac{\sum_{n=2}^{\infty}|a_n - b_n|}{1-\sum_{n=2}^{\infty} b_n} \\ &\leq \frac{\delta}{2}\frac{\frac{\beta\Gamma(1+\alpha)}{\Gamma(2+2\alpha)}(1+\kappa|\gamma|)}{\frac{\beta\Gamma(1+\alpha)}{\Gamma(2+2\alpha)}(1+\kappa|\gamma|)-\kappa|\gamma|} \\ &= 1-\eta.\end{aligned}$$

Thus, by definition, $f(z) \in \mathcal{M}_{\alpha,\beta}^{n}(\kappa,\gamma,\eta)$ for η given by (20), which establishes the desired result.

Theorem 6. If $g(z) \in \mathcal{R}_{\alpha,\beta}^{n}(\kappa,\gamma;\vartheta)$ and

$$\eta = 1 - \frac{\delta\beta(1+\vartheta)\Gamma(1+\alpha)}{2\left[\beta(1+\vartheta)\Gamma(1+\alpha)-\kappa|\gamma|\Gamma(2+2\alpha)\right]}, \tag{23}$$

then $\mathcal{N}_{n,\delta}(g) \subset \mathcal{R}_{\alpha,\beta}^{n}(\kappa,\gamma,\eta;\vartheta)$.

Proof. We omit the proofs since it is similar to Theorem 5.

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From The First Remarkable Limit to a Nonlinear Differential Equation

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Abstract

In the paper a nonlinear differential equation arising from an elementary geometry problem was discussed. This geometry problem was inspired by one of the proofs of the first remarkable limit known from the 1st semester undergraduate Calculus course. It is known that the involved differential equation can be reduced to Abel's differential equation of the first kind. In the paper the problem was solved using an approximate geometric method which constructs broken line approximation for the curve. Compass tool of GeoGebra was extensively used for these constructions. At the end of the paper, some generalizations were discussed. A new transformation of curves, named as Interception, was introduced and its approximate construction by GeoGebra was described.

Keywords: Length of a curve, nonlinear differential equations, Abel's equation, The First Remarkable Limit.

1. Introduction

One of the first theorems that an undergraduate student learns from Calculus 1 course is the First Remarkable Limit:

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1.$$

Most of the Calculus textbooks provide the following proof based on the inequality $\sin\theta<\theta<\tan\theta$ for $\theta\in\left[0,\frac{\pi}{2}\right]$. Dividing all the sides of this double inequality by positive number θ and then tending $\theta\to 0$ we obtain via, by then already covered Sandwich Theorem, the required limit. To prove the double inequality $\sin\theta<\theta<\tan\theta$ for $\theta\in\left[0,\frac{\pi}{2}\right]$ the following standard diagram is used.

In the following diagram, a circle of unit radius with the centre at the point O is drawn, OA and OB are its radii, and the tangent of the circle at the point B intersects the extension of the radius OA at the point C. It is easy to see that the area of the triangle AOB is lesser than the area of the sector AOB, which in turn is lesser than the area of triangle BOC. Therefore,

$$\frac{\sin\theta}{2}<\frac{\theta}{2}<\frac{\tan\theta}{2},$$

where $\theta=\angle AOB$.

The inequality $\theta < \tan\theta$ for $\theta \in \left[0, \frac{\pi}{2}\right]$ also means that the length of the arc AB is lesser than the length of the tangent BC, as the line OA rotates counterclockwise around the point O. So, it is natural to ask, whether it is possible to replace the unit circle with another smooth curve passing through the point B such that now the length of the curve AB is equal to the length of the tangent BC as the line OA rotates counterclockwise around the point O.

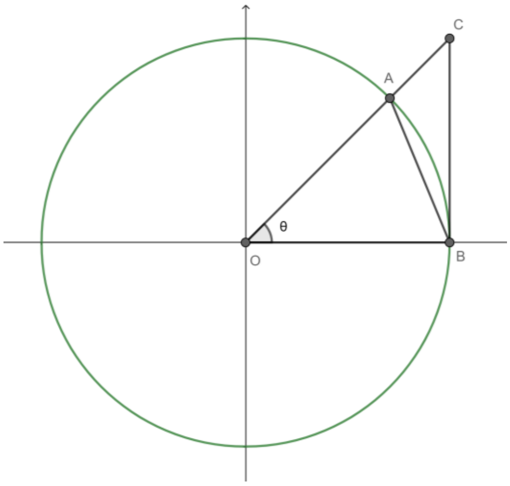

Figure 1. The proof of The First Remarkable Limit

It will be shown in the Section II that this problem is equivalent to a nonlinear differential equation of the first order and overview of the literature about the equation will be given here. In Section III we will use an approximate method to determine the shape of this curve. In Section IV we generalize the problem and in Section V we give some elementary examples. In Section VI we discuss one possible application of this theory. For diagrams and numerical experiments, we used GeoGebra Calculator. This paper also intends to become a motivation for professors and students interested in undergraduate research projects. The current paper can be a motivating example to show that it is possible to jump from a familiar textbook topic directly to advanced research problems.

2. The Differential Equation

We want to find a curve whose polar equation $r = r(\theta)$ satisfies $r(0) = 1$ and the length of its arc in the interval $[0, \theta]$ is equal to the length of the tangent line at $\theta = 0$ in the same interval $[0, \theta]$. In the following diagram the length of the line segment BC which is perpendicular to OB, is equal to the length of the arc AB of the required curve as OC rotates around O.

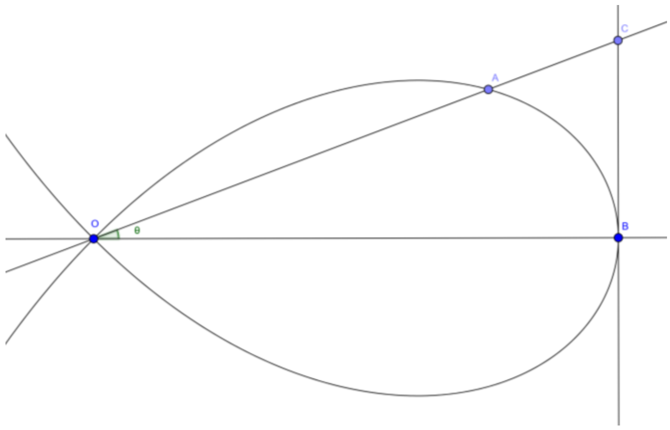

Figure 2. The length of BA is equal to BC.

Using the well-known formula for the length of a curve given in polar form $r = r(\theta)$, we obtain the equation

$$\int_0^\theta \sqrt{r^2 + (r')^2}\, d\theta = \tan\theta,$$

where $\angle COB = \theta$ and $OA = r(\theta)$. By taking the derivative of both sides with respect to θ we obtain the differential equation

$$\sqrt{r^2 + (r')^2} = \frac{1}{cos^2\theta}, \tag{1}$$

with initial condition $r(0) = 1$. One of the solutions of the differential equation is its known solution $r = \frac{1}{\cos\theta}$, which is the equation of tangent line itself. We are interested in the existence of the other solution. Using standard methods of series solutions of ODE, we can find the first terms of the Maclaurin's series of the two solutions:

$$r_1 = 1 + \frac{1}{2}\theta^2 + \frac{5}{24}\theta^4 + \frac{61}{720}\theta^6 + \frac{277}{8064}\theta^8 + O(\theta^{10}),$$

$$r_2 = 1 - \theta^2 - \frac{2}{21}\theta^4 - \frac{1933}{24255}\theta^6 - \frac{6004}{169785}\theta^8 + O(\theta^{10}).$$

We used Maple 2021 to obtain these series.

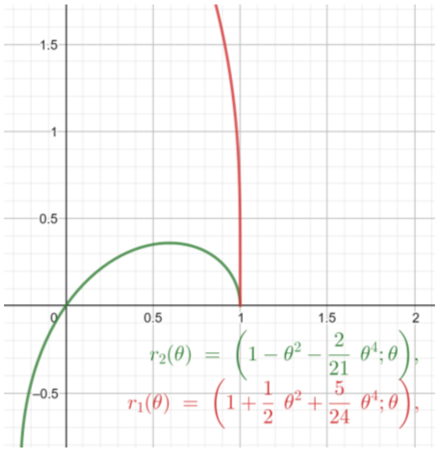

Figure 3. 4th order series approximations of solutions r_1, r_2.

It is obvious that $r_1 = \frac{1}{\cos\theta}$. But it is not clear what r_2 is. Let us try to solve (1) explicitly. It was noted in [1, Part C (Part 3 in Russian Translation), Sect. 1.370] that a differential equation of the form

$$r^2 + (r')^2 = f^2(x)$$

can always be transformed into the form

$$fu' + f' \tan u = \pm f,$$

using the substitution $r = f \sin u(x)$ (see [1, Sect. 1.370]), which in turn can be transformed into the equation

$$fu' + gu^3 + hu^2 + gu + h = 0,$$

using the substitution $u(x) = \tan y$ (see [1, Sect. 1.202]). The last equation is I type Abel equation (see [1, Sect. 4.10], [2, Sect. 4-1]). The special case

$$y^2 + (y')^2 = \frac{a^2}{\cos^4 x},$$

was discussed in [3] in relation to one kinematics problem which is dilational version of our problem (See also [1, Sect. 1.460]). Solution in quadratures for the last equation was given in [4] (See also [1, Sect. 1.460]). One can follow the following steps:

1) Use substitution $y' = y \cdot \cot u$ to obtain $y \cdot \cos^2 x = \pm a \cdot \sin u$. Then differentiating and excluding y and y' we obtain $u' + 2\tan u \tan x = 1$. [1, Sect. 1.460]

2) Use substitutions $\eta(\xi) = \tan u$, $\xi = \tan x$ to obtain Abel's equation $(\xi^2 + 1)\eta' = (\eta^2 + 1)(1 - 2\xi\eta)$. [1, Sect. 1.81]

3) Use substitution $\xi^4\eta(\xi) = (\xi^2 + 1)z + \xi^3$ to obtain again Abel's equation $\xi^7 z' + 2(\xi^2 + 1)z^3 + 5\xi^3 z^2 = 0$. [1, Sect. 1.151]

4) Use substitution $v = \frac{1}{z}$ to obtain $\xi^7 v v' = 2(\xi^2 + 1) + 5\xi^3 v$. [1, Sect. 1.185]

5) Use substitution $\xi w = \xi^3 v + 1$ to obtain linear equation $\frac{d\xi}{dw} - \frac{\xi w}{2(w^2+1)} + \frac{1}{2(w^2+1)} = 0$, which can be solved using quadratures [1, Sect. 1.185].

As one can understand from these steps, the solution of (1) as a closed formula is not going to be simple. Therefore, the approximate solutions are very helpful to find the shape of the curve.

It is also worthwhile to note that similar problems were also discussed in the context of aeronautics [5]. These real-life applications are not the main objective of the current paper, and they will be shortly touched in the conclusion part of the current paper.

I thank my former colleague E. Hasanalizade (currently PhD student at The University of Lethbridge) for the discussion of the references in [1].

3. Approximate geometric solution.

Let us draw the rays $\theta_1 = \frac{\pi}{n}$, $\theta_2 = \frac{2\pi}{n}$, $\theta_3 = \frac{3\pi}{n}$, ..., $\theta_n = \frac{n\pi}{n} = \pi$ for the given n. Denote the intersections of these rays with the vertical line $r = \frac{1}{\cos\theta}$ by $C_1, C_2, C_3, \ldots$. Let us construct the points $A_1, A_2, A_3, \ldots$ on the rays $OC_1, OC_2, OC_3, \ldots$, respectively, so that $BA_1 = BC_1$, $A_1A_2 = C_1C_2$, $A_2A_3 = C_2C_3$,... We can use the compass tool of GeoGebra for this purpose. To construct the point A_1 we draw circle with radius BC_1 at the centre B and denote its second intersection with the line OC_1 by A_1. Similarly, to find the point A_2, we draw circle with radius C_1C_2 at the centre A_1, and denote its second inersection with the line OC_2 by A_2. The other points $A_3, A_4, \ldots$ are constructed in the same way.

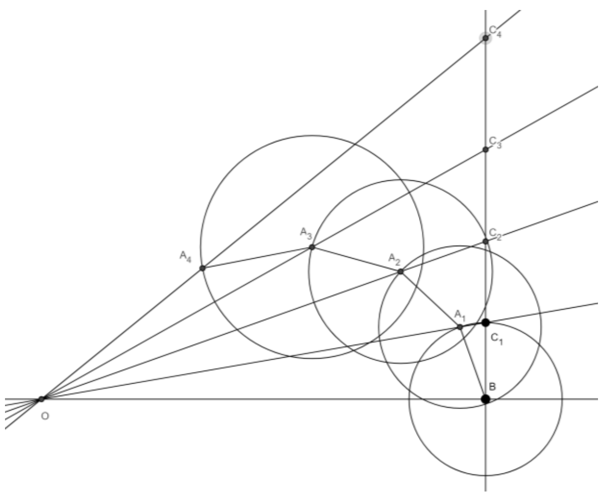

Figure 4. The construction of the broken line.

It is now obvious that the length of the line segment $BC_1C_2C_3 \dots C_n$ is equal to the length of the broken line $BA_1A_2A_3 \dots A_n$ and as $n \to \infty$ this broken line approaches the required curve. It is noteworthy that all these constructions can be done using only ruler and compass. One can also observe that the locus of each of the points $A_1, A_2, A_3, \dots, A_n$ as the point C_1 moves, can be interpreted as approximations of the required curve. We used one of these approximations to draw the curve at the beginning of Section II. Each circle in the next picture is used as a compass to find the centre of the next circle. The centre of each next circle is taken at the point obtained in the previous construction.

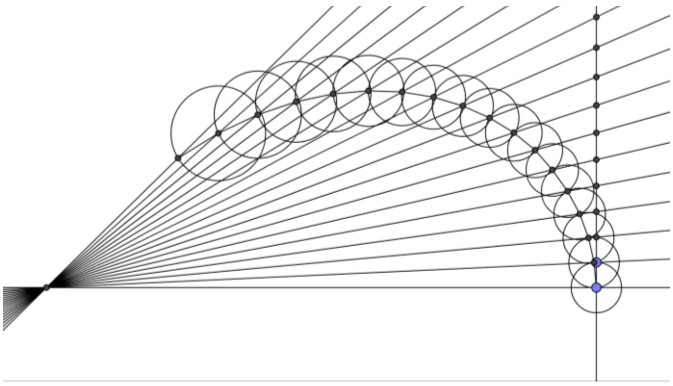

Figure 5. The broken line approximates the curve.

Note that at around $\theta = \theta_0 \approx 0.9235 \approx 52.9°$ the curve $r = r(\theta)$ passes through the point O. It would be interesting to find out how the constant θ_0 is related to the other constants in mathematics such as e or π. We could only prove that

$$\frac{1}{\theta_0(\tan^2\theta_0 + 1)} = 1 - \theta_0 \tan\theta_0 + \theta_0^2\left(\tan^2\theta_0 + \frac{1}{6}\right) + \cdots$$

where arbitrary number of terms of the series on the right-hand side can be calculated.

4. Generalizations and "Interception".

Let us now replace the vertical line BC by an arbitrary differentiable curve $r = \Phi(\theta)$ passing through the point $B(1,0)$. Then the differential equation becomes

$$\sqrt{r^2 + (r')^2} = \varphi(\theta),$$

where $\varphi(\theta) = \sqrt{\Phi(\theta)^2 + \Phi'(\theta)^2}$. One of the solutions is obviously $r_1 = \Phi(\theta)$. It is interesting to find the other solution $r_2 = r_2(\theta)$. As the case $\Phi(\theta) = \frac{1}{\cos\theta}$ suggests, it is not always possible to do this analytically. So, it is reasonable to have an approximate method for the solution. The approximate method described in Section 3 can be applied here again with obvious modifications. We just need to take the points $C_1, C_2, C_3, \dots, C_n$ on the curve $r = \Phi(\theta)$ and measure the distances BC_1, C_1C_2, C_2C_3,... using the broken line approximation $BC_1C_2C_3 \dots C_n \dots$ of the curve $r = \Phi(\theta)$. As in the previous case we use the compass to construct the points $A_1, A_2, A_3, \dots$ on the rays $OC_1, OC_2, OC_3, \dots$, respectively, so that

$$BA_1 = BC_1,\ A_1A_2 = C_1C_2,\ A_2A_3 = C_2C_3, \dots$$

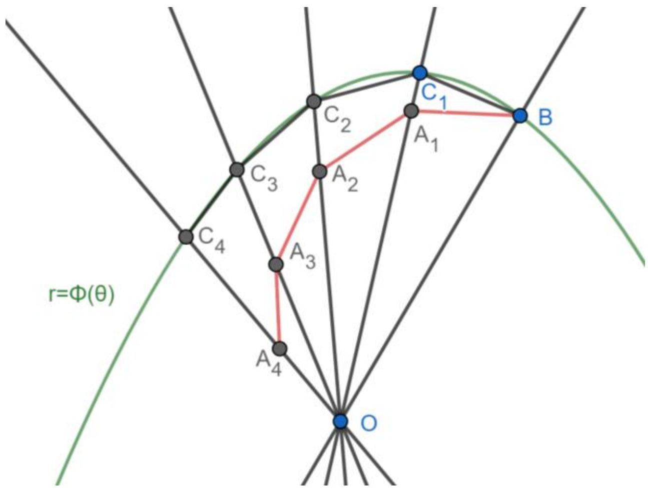

Figure 6: Construction of "The Interception Curve".

The broken line $BA_1A_2A_3 \dots A_n$ approximates $r_2 = r_2(\theta)$ and as $n \to \infty$ this broken line approaches the curve $r_2 = r_2(\theta)$. We will call the process of obtaining the curve $r_2 = r_2(\theta)$ from the given curve $r_1 = \Phi(\theta)$ as *Interception*. The reasons for the choice of this name will be clear later in the last section of the paper. This transformation preserves the distances on the curves. It would be interesting to find an analogue of this transformation for surfaces in space.

5. Examples.

It would be satisfying to see some examples of elementary functions $r_1 = \Phi(\theta)$ for which r_2 is again an elementary function. First, note that if

$$r_1^2 + (r_1')^2 = r_2^2 + (r_2')^2$$

then

$$(r_2 - r_1)(r_2 + r_1) = -(r_2' - r_1')(r_2' + r_1').$$

Let us denote $r_2 - r_1 = 2x$, $r_2 + r_1 = 2y$. Then we obtain $yx = -y'x'$. The last equality can also be rewritten as $\frac{x}{x'} = -\frac{y'}{y}$ or $(\ln y)' = -\frac{1}{(\ln x)'}$. From geometrical point of view this means that the solutions $r_1,\ r_2$ of the differential equation $r^2 + (r')^2 = \varphi(\theta)^2$ can be represented as $r_1 = y - x,\ r_2 = y + x$, where the functions $x = x(\theta), y = y(\theta)$ have the nice property that the tangent lines of the functions $\ln x(\theta)$ and $\ln y(\theta)$ at an arbitrary point θ are perpendicular to each other. The equality $(\ln y)' = -\frac{1}{(\ln x)'}$ can be the starting point to find infinitely many elementary examples of such $r_1,\ r_2$. Using the easily verifiable fact that

$$(\ln \sin \theta)' = -\frac{1}{(\ln \cos \theta)'},$$

we obtain the pair of circles

$$r_1 = \cos \theta - \sin \theta\,,\ r_2 = \cos \theta + \sin \theta.$$

Note that the equality of the arcs BA and BC follows easily from the elementary properties of inscribed angles and the fact that these circles have the same radius.

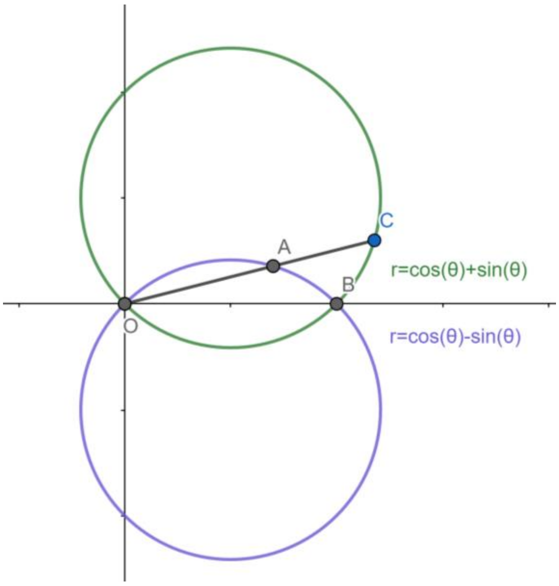

Figure 7. Circles as an example of Interception.

Similarly, if we take $x = \theta$, then we get $y = e^{-\frac{\theta^2}{2}}$.

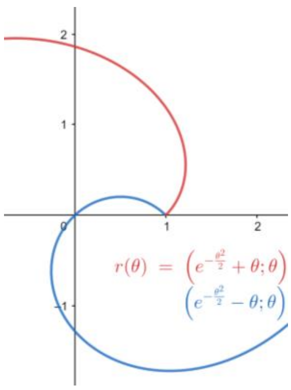

Figure 8. Another example of Interception.

Finally, if we take $x = \sqrt{\theta}$, then we obtain $y = e^{-\theta^2}$.

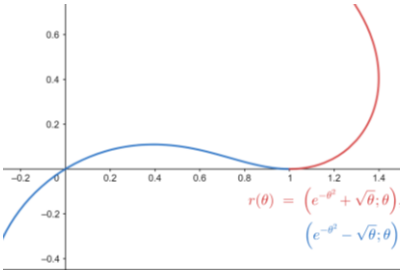

Figure 9. One more example of Interception.

Note that the curves r_1, r_2 do not need to pass through the same point $B(0,1)$ and can still have the distance preserving property.

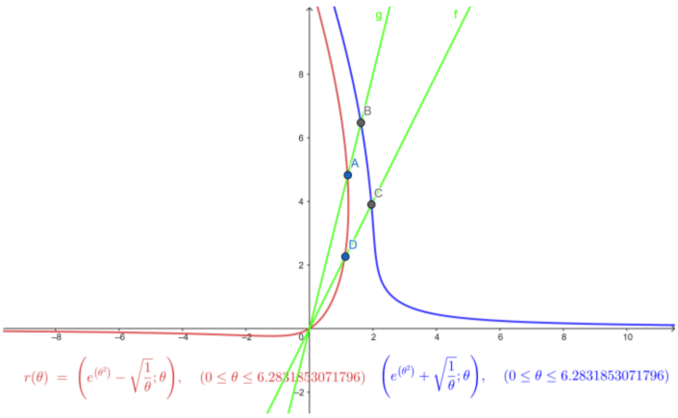

Figure 10. Non-intersecting example of Interception.

For example, if we take $x = \frac{1}{\sqrt{\theta}}$ we obtain $y = e^{\theta^2}$. The obtained curves $r_1 = e^{\theta^2} + \frac{1}{\sqrt{\theta}}$, $r_2 = e^{\theta^2} - \frac{1}{\sqrt{\theta}}$ never intersect but they have the property that their arcs between two arbitrary lines $f: \theta = \theta_1$ and $g: \theta = \theta_2$, have the same length (length of arc $AD =$ length of arc BC). The transformation *Interception* that was described in the previous section should be modified for such cases. In this case instead of the point B we should take two points: B on $r_1 = \Phi(\theta)$ and B' on r_2, and then construct the broken line $B'A_1A_2A_3 \dots A_n$ so that

$$B'A_1 = BC_1,\ A_1A_2 = C_1C_2,\ A_2A_3 = C_2C_3, \dots$$

Note that two such broken line approximations $B'A_1A_2A_3 \dots A_n$ are possible here and only one was drawn below.

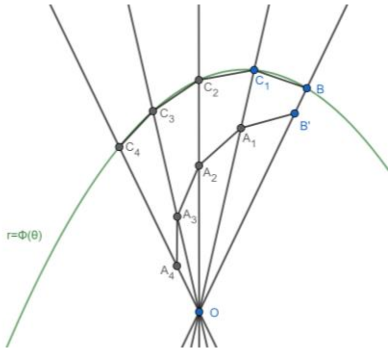

Figure 11. More general Interception.

6. Conclusion

The study of curves and their properties has a long history dating back to the time of the ancient Greeks. Modern mathematics supplied the theory of curves with analytical tools and abstract viewpoint. Although it is not a mainstream research topic today, for undergraduate research projects and expository papers the theory of curves can be a source of inspiration and motivation. In the current paper one interesting curve was studied in detail. Its approximate shape was drawn as a broken line. This construction was done with the help of the compass tool of GeoGebra. After this, a generalization was discussed and a new transformation (named as *Interception*) preserving the distances on the curves was introduced. Elementary function examples for which *Interception* gives again an elementary function were given at the end of the paper.

Figure 12. Real life Interception REUTERS/Amir Cohen.

The discussed topics also have some connections with problems related to laser-beam riding interception (hence the name) of high-speed missiles in defence technology [5]. The discussed method in the current paper can have some applications outside of mathematics.

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On Strongly ⊕-g-Rad-Supplemented Modules

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Abstract

In this work, all rings have unity and all modules are unitary left modules. It is investigated some new properties of strongly ⊕-g-Rad-supplemented modules, in this work. Let *M* be a strongly ⊕-g-Rad-supplemented *R*-module. If *M* is supplemented, then *M* is strongly ⊕-supplemented.

Keywords: Essential Submodules, g-Small Submodules, Supplemented Modules, g-Supplemented Modules.

2020 Mathematics Subject Classification: 16D10, 16D80.

1. INTRODUCTION

Throughout this paper all rings will be associative with identity and all modules will be unital left modules.

Let R be a ring and M be an R-module. We will denote a submodule N of M by $N\leq M$. Let M be an R-module and $N\leq M$. If $L=M$ for every submodule L of M such that $M=N+L$, then N is called a *small* submodule of M and denoted by $N\ll M$. Let M be an R-module and $N\leq M$. If there exists a submodule K of M such that $M=N+K$ and $N\cap K=0$, then N is called a *direct summand* of M and it is denoted by $M=N\oplus K$. For any R-module M, we have $M=M\oplus 0$. The intersection of all maximal submodules of M is called the *radical* of M and denoted by $RadM$. If M have no maximal submodules, then it is defined $RadM=M$. M is said to be *semilocal* if $M/RadM$ is semisimple. A submodule N of an R-module M is called an *essential* submodule of M and denoted by $N\trianglelefteq M$ in case $K\cap N\neq 0$ for every submodule $K\neq 0$, or equivalently, $K=0$ for every $K\leq M$ with $N\cap K=0$. Let M be an R-module and K be a submodule of M. K is called a *generalized small* (or briefly, *g-small*) submodule of M if for every essential submodule T of M with the property $M=K+T$ implies that $T=M$, then we write $K\ll_g M$ (in [16], it is called an *e-small* submodule of M and denoted by $K\ll_e M$). It is clear that every small submodule is a generalized small submodule but the converse is not true in general. Let M be an R-module. M is called a *hollow* module if every proper submodule of M is small in M. M is called a *local* module if M has the largest submodule, i.e. a proper submodule which contains all other proper submodules. Let U and V be submodules of M. If $M=U+V$ and V is minimal with respect to this property, or equivalently, $M=U+V$ and $U\cap V\ll V$, then V is called a *supplement* of U in M. M is said to be *supplemented* if every submodule of M has a supplement in M. If every submodule of M has a supplement that is a direct summand in M, then M is called a *$\oplus$-supplemented* module. Let M be a supplemented R-module. If every supplement submodule of M is a direct summand of M, then M is called a *strongly $\oplus$-supplemented* module. Let M be an R-module and $U;V\leq M$. If $M=U+V$ and $M=U+T$ with $T\trianglelefteq V$ implies that $T=V$, or equivalently, $M=U+V$ and $U\cap V\ll_g V$, then V is called a *g-supplement* of U in M. M is said to be *g-supplemented* if every submodule of M has a g-supplement in M. M is said to be *$\oplus$-g-supplemented* if every submodule of M has a g-supplement that is a direct summand in M (see [9]). Let M be an R-module and $U,V\leq M$. If $M=U+V$ and $U\cap V\leq RadV$, then V is called a *generalized* (*radical*) *supplement* (briefly, *Rad-supplement*) of U in M. M is said to be *generalized* (*radical*) *supplemented* (briefly, *Rad-supplemented*) if every submodule of M has a Rad-supplement in M. M is said to be *generalized* (*radical*) *$\oplus$-supplemented* (briefly, *Rad-$\oplus$-supplemented*) if every submodule of M has a Rad-supplement that is a direct summand in M. The intersection of all essential maximal submodules of an R-module M is called the *generalized radical* of M and denoted by Rad_gM (in [16], it is denoted by Rad_eM). If M have no essential maximal submodules, then we denote $Rad_gM=M$. An R-module M is said to be *g-semilocal* if M/Rad_gM is semisimple (see [8]). Let M be an R-module and $U,V\leq M$. If $M=U+V$ and $U\cap V\leq Rad_gV$, then V is called a *generalized radical supplement* (or briefly, *g-radical supplement*) of U in M. M is said to be *generalized radical supplemented* (briefly, *g-radical supplemented*) if every submodule of M has a g-radical supplement in M. M is said to be *$\oplus$-g-Rad-supplemented* if every submodule of M has a g-radical supplement that is a direct summand in M (see [11]). Let M be an R-module and $X\leq Y\leq M$. If $Y/X\ll M/X$, then we say Y lies above X in M. If every submodule of M lies above a direct summand in M, then we say M satisfies ($D1$) property. Let M be an R-module. M is said to be *π-projective* if for every $U,V\leq M$ such that $M=U+V$ there exists an R-module homomorphism $f:M\to M$ such that $Imf\leq U$ and $Im(1-f)\leq V$.

More informations about supplemented modules are in [1] and [15]. More results about $\oplus$-supplemented modules are in [4] and [7]. More details about strongly $\oplus$-supplemented modules are in [10]. More details about generalized (radical) supplemented modules are in [14]. More details about generalized (radical) $\oplus$-supplemented modules are in [2], [3] and [13]. More informations about g-supplemented modules are in [5]. More informations about g-radical supplemented modules are in [6].

Lemma 1.1. Let M be an R-module.

(1) If $K\leq L\leq M$, then $K\trianglelefteq M$ if and only if $K\trianglelefteq L\trianglelefteq M$.

(2) Let N be an R-module and $f:M\to N$ be an R-module homomorphism. If $K\trianglelefteq N$, then $f^{-1}(K)\trianglelefteq M$.

(3) For $N\leq K\leq M$, if $K/N\trianglelefteq M/N$, then $K\trianglelefteq M$.

(4) If $K_1\trianglelefteq L_1\leq M$ and $K_2\trianglelefteq L_2\leq M$, then $K_1\cap K_2\trianglelefteq L_1\cap L_2$.

(5) If $K_1 \trianglelefteq M$ and $K_2 \trianglelefteq M$, then $K_1 \cap K_2 \trianglelefteq M$.

Proof. See [15, 17.3].

Lemma 1.2. Let M be an R-module. The following conditions hold.
(1) $Rad_g M$ is equal to the sum of g-small submodules of M.
(2) $Rm \ll_g M$ for every $m \in Rad_g M$.
(3) If $N \leq M$, then $Rad_g N \leq Rad_g M$.
(4) If $K,L \leq M$, then $Rad_g K + Rad_g L \leq Rad_g (K+L)$.
(5) Let N be an R-module and $f: M \rightarrow N$ be an R-module homomorphism. Then $f(Rad_g M) \leq Rad_g N$.
(6) If $K,L \leq M$, then $(Rad_g K+L)/L \leq Rad_g[(K+L)/L]$. If $L \leq Rad_g K$, then $(Rad_g K)/L \leq Rad_g(K/L)$.
(7) If $M = \oplus_{i \in I} M_i$, then $Rad_g M = \oplus_{i \in I} Rad_g M_i$.
(8) $RadM \leq Rad_g M$.
Proof. See [6, Lemma 2, Lemma 3 and Lemma 4].

2. STRONGLY ⊕-g-RAD-SUPPLEMENTED MODULES

Definition 2.1. Let M be a g-supplemented R-module. If every g-supplement submodule is a direct summand in M, then M is called a *strongly ⊕-g-supplemented* module.

Definition 2.2. Let M be a g-radical supplemented R-module. If every g-radical supplement submodule is a direct summand in M, then M is called a *strongly ⊕-g-Rad-supplemented* module. (See also [12])

Proposition 2.3. Every strongly ⊕-g-supplemented module is ⊕-g-supplemented.
Proof. Clear, from definitions.

Proposition 2.4. Every strongly ⊕-g-Rad-supplemented module is ⊕-g-Rad-supplemented.
Proof. Clear from definitions.

Proposition 2.5. Let M be a strongly ⊕-g-Rad-supplemented R-module.
Then M is g-semilocal.
Proof. Since M is strongly ⊕-g-Rad-supplemented, M is g-radical supplemented. Then by [6, Theorem 1], M is g-semilocal.

Proposition 2.6. Every strongly ⊕-g-supplemented module is ⊕-g-Rad-supplemented.
Proof. Let M be a strongly ⊕-g-supplemented module. Then by Proposition 2.3, M is ⊕-g-supplemented. It is clear that every ⊕-g-supplemented module is ⊕-g-Rad-supplemented. Hence M is ⊕-g-Rad-supplemented.

Proposition 2.7. Let *M* be a strongly ⊕-g-Rad-supplemented module. If *M* is g-supplemented, then *M* is strongly ⊕-g-supplemented.
Proof. Since every g-supplement submodule is a g-radical supplement submodule and every g-radical supplement submodule is a direct summand in *M*, every g-supplement submodule is a direct summand in *M*. Hence *M* is strongly ⊕-g-supplemented.

Proposition 2.8. Let *M* be a strongly ⊕-g-Rad-supplemented module. If *M* is supplemented, then *M* is strongly ⊕-supplemented.
Proof. Clear, since every supplement submodule in *M* is a g-radical supplement submodule in *M*.

Proposition 2.9. Let *M* be a strongly ⊕-g-Rad-supplemented module. If *M* is supplemented, then *M* is ⊕-supplemented.
Proof. By Proposition 2.8, *M* is strongly ⊕-supplemented. Hence *M* is ⊕-supplemented.

Proposition 2.10. Let *M* be a g-supplemented module. If every g-radical supplement submodule is a direct summand in *M*, then *M* is strongly ⊕-g-Rad-supplemented.
Proof. Since *M* is g-supplemented, *M* is g-radical supplemented. Then by definition, *M* is strongly ⊕-g-Rad-supplemented.

Proposition 2.11. Let *M* be a supplemented module. If every g-radical supplement submodule is a direct summand in *M*, then *M* is strongly ⊕-g-Rad-supplemented.
Proof. Since *M* is supplemented, *M* is g-supplemented. Since *M* is g-supplemented, *M* is g-radical supplemented. Then by definition, *M* is strongly ⊕-g-Rad-supplemented.

Proposition 2.12. Let *M* be a hollow module. If every g-radical supplement submodule is a direct summand in *M*, then *M* is strongly ⊕-g-Rad-supplemented.
Proof. Since *M* is hollow, *M* is supplemented and hence *M* is g-supplemented. Since *M* is g-supplemented, *M* is g-radical supplemented. Then by definition *M* is strongly ⊕-g-Rad-supplemented.

Proposition 2.13. Let *M* be a local module. If every g-radical supplement submodule is a direct summand in *M*, then *M* is strongly ⊕-g-Rad-supplemented.
Proof. Clear from Proposition 2.12, since every local module is hollow.

Proposition 2.14. Let *M* be a g-radical supplemented module and $K \leq M$. If every g-radical supplement submodule is a direct summand in M/K, then M/K is strongly ⊕-g-Rad-supplemented.
Proof. Since *M* is g-radical supplemented M/K is g-radical supplemented. Then by definition M/K is strongly ⊕-g-Rad-supplemented.

Proposition 2.15. Let M be a g-supplemented module and $K \leq M$. If every g-radical supplement submodule is a direct summand in M/K, then M/K is strongly ⊕-g-Rad-supplemented.
Proof. Since M is g-supplemented, M is g-radical supplemented. Then by Proposition 2.14, M/K is strongly ⊕-g-Rad-supplemented.

Proposition 2.16. Let M be a supplemented module and $K \leq M$. If every g-radical supplement submodule is a direct summand in M/K, then M/K is strongly ⊕-g-Rad-supplemented.
Proof. Since M is supplemented, M is g-radical supplemented. Then by Proposition 2.14, M/K is strongly ⊕-g-Rad-supplemented.

3. CONCLUSION

Strongly ⊕-g-Rad-supplemented modules are special parts of ⊕-g-Rad-supplemented modules.

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Focal Curves of Adjoint Curves

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Abstract

In this study, we firstly characterize focal curves of adjoint curves by considering quasi frame in the ordinary space. Then, we obtain the relation of each quasi curvatures of curve in terms of focal curvatures. Finally, we give some new conditions with constant quasi curvatures in the ordinary space.

Keywords: Quasi frame, focal curve, adjoint curve.

1. Introduction

By way of design and style, this is model to kind of a moving frame with regards to a particle. In the quick stages of regular differential geometry, the Frenet-Serret frame was applied to create a curve in location. After that, Frenet-Serret frame is established by way of subsequent equations for a presented framework [1-14],

$$\begin{bmatrix} \nabla_{\mathbf{T}}\mathbf{T} \\ \nabla_{\mathbf{T}}\mathbf{N} \\ \nabla_{\mathbf{T}}\mathbf{B} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix},$$

where $\kappa = \|\mathbf{t}\|$ and τ are the curvature and torsion of γ, respectively.
The quasi frame(q -frame) of a regular curve γ is given by

$$\mathbf{T}_q^{\gamma} = \mathbf{T}, \mathbf{N}_q^{\gamma} = \frac{\mathbf{T} \wedge \mathbf{k}}{\|\mathbf{T} \wedge \mathbf{k}\|}, \mathbf{B}_q^{\gamma} = \mathbf{T}_q^{\gamma} \wedge \mathbf{N}_q^{\gamma},$$

where $\mathbf{k}$ is the projection vector [4].

For simplicity, we have chosen the projection vector $\mathbf{k} = (0,0,1)$ in this paper. However, the q-frame is singular in all cases where $\mathbf{t}$ and $\mathbf{k}$ are parallel. Thus, in those cases where $\mathbf{t}$ and $\mathbf{k}$ are parallel the projection vector $\mathbf{k}$ can be chosen as $\mathbf{k} = (0,1,0)$ or $\mathbf{k} = (1,0,0)$.

If the angle between the quasi normal vecctor $\mathbf{N}_q^{\gamma}$ and the normal vector $\mathbf{N}$ is choosen as ψ, then following relation is obtained between the quasi and FS frame.

$$
\begin{aligned}
\mathbf{T}_q^\gamma &= \mathbf{T}, \\
\mathbf{N}_q^\gamma &= \cos\psi\mathbf{N} + \sin\psi\mathbf{B}, \\
\mathbf{B}_q^\gamma &= -\sin\psi\mathbf{N} + \cos\psi\mathbf{B},
\end{aligned}
$$

such that short computation by using Eqs. $(1-3)$ yields that the variation of parallel adapted q-frame is given by

$$
\begin{aligned}
\nabla_{\mathbf{T}_q^\gamma}\mathbf{T}_q^\gamma &= \grave{u}_{\gamma_1}\mathbf{N}_q^\gamma + \grave{u}_{\gamma_2}\mathbf{B}_q^\gamma, \\
\nabla_{\mathbf{T}_q^\gamma}\mathbf{N}_q^\gamma &= -\grave{u}_{\gamma_1}\mathbf{T}_q^\gamma + \grave{u}_{\gamma_3}\mathbf{B}_q^\gamma, \\
\nabla_{\mathbf{T}_q^\gamma}\mathbf{B}_q^\gamma &= -\grave{u}_{\gamma_2}\mathbf{T}_q^\gamma - \grave{u}_{\gamma_3}\mathbf{N}_q^\gamma,
\end{aligned}
$$

where

$$
\grave{u}_{\gamma_1} = \kappa\cos\psi, \quad \grave{u}_{\gamma_2} = -\kappa\sin\psi, \quad \grave{u}_{\gamma_3} = \psi' + \tau,
$$

curvatures of γ and

$$
\mathbf{T}_q^\gamma \times \mathbf{N}_q^\gamma = \mathbf{B}_q^\gamma, \mathbf{N}_q^\gamma \times \mathbf{B}_q^\gamma = \mathbf{T}_q^\gamma, \mathbf{B}_q^\gamma \times \mathbf{T}_q^\gamma = \mathbf{N}_q^\gamma.
$$

Definition 1: 1 Let γ be a regular curve arc-length parametrized, $\{\mathbf{T}_q^\gamma, \mathbf{N}_q^\gamma, \mathbf{B}_q^\gamma\}$ be q-frame of γ. Then, the adjoint curve of γ according to q-frame is given as

$$
\beta(s) = \int_{s_0}^{s} \mathbf{B}_q^\gamma(s)ds.
$$

Theorem 2: 2 Let γ be a regular curve arc-length parametrized, $\{\mathbf{T}_q^\gamma, \mathbf{N}_q^\gamma, \mathbf{B}_q^\gamma\}$ be q-frame of γ and β be adjoint curve of γ according to q-frame. Denote by $\{\mathbf{T}_q^\beta, \mathbf{N}_q^\beta, \mathbf{B}_q^\beta\}$ q-frame of β. Then, q-frame elements of β can given by

$$
\begin{aligned}
\mathbf{T}_q^\beta &= \mathbf{B}_q^\gamma, \\
\mathbf{N}_q^\beta &= -\frac{\grave{u}_{\gamma_2}\mathbf{T}_q^\gamma + \grave{u}_{\gamma_3}\mathbf{N}_q^\gamma}{\sqrt{\grave{u}_{\gamma_2}^2 + \grave{u}_{\gamma_3}^2}}, \\
\mathbf{B}_q^\beta &= \frac{\grave{u}_{\gamma_3}\mathbf{T}_q^\gamma - \grave{u}_{\gamma_2}\mathbf{N}_q^\gamma}{\sqrt{\grave{u}_{\gamma_2}^2 + \grave{u}_{\gamma_3}^2}}.
\end{aligned}
$$

2. Quasi Focal Curve of Adjoint Curves with q-Frame in E^3

The focal curve of β is given by

$$\beta_f = \beta + \phi_1 \mathbf{N}_q^\beta + \phi_2 \mathbf{B}_q^\beta, \tag{2.1}$$

where the coefficients ϕ_1, ϕ_2 are smooth functions of the parameter of the curve β, called the first and second focal curvatures of β, respectively.

Theorem 3: 3 Let $\beta : I \to \mathsf{E}^3$ be adjoint curve of γ, β_f its focal curve on E^3 and ù_{β_1}, ù_{β_2}, ù_{β_3} be curvatures of β according to q-frame. Then,

$$\begin{aligned}\beta_f = \beta + \Bigg[&\frac{(\text{ù}_{\gamma_3} - \text{ù}_{\gamma_3}\text{ù}_{\beta_1})\left(\int\left(\frac{\text{ù}_{\beta_3}}{\text{ù}_{\beta_2}} e^{\int \frac{\text{ù}_{\beta_1}\text{ù}_{\beta_3}}{\text{ù}_{\beta_2}} ds}\right) ds + C\right)}{\text{ù}_{\beta_2}\sqrt{\text{ù}_{\gamma_2}^2 + \text{ù}_{\gamma_3}^2}\, e^{\int \frac{\text{ù}_{\beta_1}\text{ù}_{\beta_3}}{\text{ù}_{\beta_2}} ds}} \\ &- \frac{\text{ù}_{\gamma_2}}{\sqrt{\text{ù}_{\gamma_2}^2 + \text{ù}_{\gamma_3}^2}} e^{\int \frac{\text{ù}_{\beta_1}\text{ù}_{\beta_3}}{\text{ù}_{\beta_2}} ds} \left(\int\left(\frac{\text{ù}_{\beta_3}}{\text{ù}_{\beta_2}} e^{\int \frac{\text{ù}_{\beta_1}\text{ù}_{\beta_3}}{\text{ù}_{\beta_2}} ds}\right) ds + C\right)\Bigg]\mathbf{T}_q^\gamma \\ +\Bigg[&\frac{\text{ù}_{\gamma_3}}{\sqrt{\text{ù}_{\gamma_2}^2 + \text{ù}_{\gamma_3}^2}} e^{-\int \frac{\text{ù}_{\beta_1}\text{ù}_{\beta_3}}{\text{ù}_{\beta_2}} ds} \left(\int\left(\frac{\text{ù}_{\beta_3}}{\text{ù}_{\beta_2}} e^{\int \frac{\text{ù}_{\beta_1}\text{ù}_{\beta_3}}{\text{ù}_{\beta_2}} ds}\right) ds + C\right) \\ &- \frac{(\text{ù}_{\gamma_2} - \text{ù}_{\gamma_2}\text{ù}_{\beta_1})\left(\int\left(\frac{\text{ù}_{\beta_3}}{\text{ù}_{\beta_2}} e^{\int \frac{\text{ù}_{\beta_1}\text{ù}_{\beta_3}}{\text{ù}_{\beta_2}} ds}\right) ds + C\right)}{\text{ù}_{\beta_2}\sqrt{\text{ù}_{\gamma_2}^2 + \text{ù}_{\gamma_3}^2}\, e^{\int \frac{\text{ù}_{\beta_1}\text{ù}_{\beta_3}}{\text{ù}_{\beta_2}} ds}}\Bigg]\mathbf{N}_q^\gamma,\end{aligned} \tag{2.2}$$

where C is a constant of integration.

Proof: Assume that β is a unit speed curve and β_f its focal curve in E^3. So, by differentiating of the formula (2.1), we get

$$\beta_f' = (1 - \text{ù}_{\beta_1}\phi_1 - \text{ù}_{\beta_2}\phi_2)\mathbf{T}_q^\beta + (\phi_1' - \text{ù}_{\beta_3}\phi_2)\mathbf{N}_q^\beta + (\phi_2' + \text{ù}_{\beta_3}\phi_1)\mathbf{B}_q^\beta$$

From above equation, the first 2 components vanish, we get

$$
\begin{aligned}
1-\grave{u}_{\beta_1}\phi_1-\grave{u}_{\beta_2}\phi_2 &= 0,\\
\phi_1'-\grave{u}_{\beta_3}\phi_2 &= 0.
\end{aligned}
$$

Using the above equations, we obtain

$$\phi_1'-\frac{\grave{u}_{\beta_3}}{\grave{u}_{\beta_2}}(1-\grave{u}_{\beta_1}\phi_1)=0,$$

$$\phi_1'+\frac{\grave{u}_{\beta_1}\grave{u}_{\beta_3}}{\grave{u}_{\beta_2}}\phi_1=\frac{\grave{u}_{\beta_3}}{\grave{u}_{\beta_2}}.$$

By integrating this equation, we find

$$\phi_1=e^{-\int\frac{\grave{u}_{\beta_1}\grave{u}_{\beta_3}}{\grave{u}_{\beta_2}}ds}\left(\int e^{\int\frac{\grave{u}_{\beta_1}\grave{u}_{\beta_3}}{\grave{u}_{\beta_2}}ds}\frac{\grave{u}_{\beta_3}}{\grave{u}_{\beta_2}}ds+C\right),$$

$$\phi_2=\frac{1}{\grave{u}_{\beta_2}}-\frac{\grave{u}_{\beta_1}}{\grave{u}_{\beta_2}}e^{-\int\frac{\grave{u}_{\beta_1}\grave{u}_{\beta_3}}{\grave{u}_{\beta_2}}ds}\left(\int e^{\int\frac{\grave{u}_{\beta_1}\grave{u}_{\beta_3}}{\grave{u}_{\beta_2}}ds}\frac{\grave{u}_{\beta_3}}{\grave{u}_{\beta_2}}ds+C\right).$$

By means of obtained equations and using Theroem 2, we express (2.2). This completes the proof of the theorem.

As an immediate consequence of the above theorem, we have:

Corollary 4: 4 Let $\beta: I\to \mathsf{E}^3$ be a unit speed curve and β_f its focal curve on E^3. Then, the focal curvatures of β are

$$\phi_1=e^{-\int\frac{\grave{u}_{\beta_1}\grave{u}_{\beta_3}}{\grave{u}_{\beta_2}}ds}\left(\int e^{\int\frac{\grave{u}_{\beta_1}\grave{u}_{\beta_3}}{\grave{u}_{\beta_2}}ds}\frac{\grave{u}_{\beta_3}}{\grave{u}_{\beta_2}}ds+C\right),$$

$$\phi_2=\frac{1}{\grave{u}_{\beta_2}}-\frac{\grave{u}_{\beta_1}}{\grave{u}_{\beta_2}}e^{-\int\frac{\grave{u}_{\beta_1}\grave{u}_{\beta_3}}{\grave{u}_{\beta_2}}ds}\left(\int e^{\int\frac{\grave{u}_{\beta_1}\grave{u}_{\beta_3}}{\grave{u}_{\beta_2}}ds}\frac{\grave{u}_{\beta_3}}{\grave{u}_{\beta_2}}ds+C\right).$$

Proof: From above theorem, we have above system, which completes the proof.

In the light of Theorem 3 and Corollary 4, we express the following corollary without proof:

Corollary 5: 5 Let $\beta : I \to \mathbb{E}^3$ be a unit speed curve and β_f its focal curve on $\mathbb{E}^3$. If the curvatures of β are constant, then the focal curve of β are

$$\beta_f = \beta - [\frac{\grave{u}_{\gamma_3}\grave{u}_{\beta_1} e^{-\frac{\grave{u}_{\beta_1}\grave{u}_{\beta_3}}{\grave{u}_{\beta_2}}s} C}{\grave{u}_{\beta_2}\sqrt{\grave{u}_{\gamma_2}^2 + \grave{u}_{\gamma_3}^2}} + \frac{\grave{u}_{\gamma_2} + \grave{u}_{\gamma_2}\grave{u}_{\beta_1} e^{-\frac{\grave{u}_{\beta_1}\grave{u}_{\beta_3}}{\grave{u}_{\beta_2}}s} C}{\grave{u}_{\beta_1}\sqrt{\grave{u}_{\gamma_2}^2 + \grave{u}_{\gamma_3}^2}}]\mathbf{T}_q^{\gamma}$$

$$+ [\frac{\grave{u}_{\gamma_3} + \grave{u}_{\gamma_3}\grave{u}_{\beta_1} e^{-\frac{\grave{u}_{\beta_1}\grave{u}_{\beta_3}}{\grave{u}_{\beta_2}}s} C}{\grave{u}_{\beta_1}\sqrt{\grave{u}_{\gamma_2}^2 + \grave{u}_{\gamma_3}^2}} + \frac{\grave{u}_{\gamma_2}\grave{u}_{\beta_1} e^{-\frac{\grave{u}_{\beta_1}\grave{u}_{\beta_3}}{\grave{u}_{\beta_2}}s} C}{\grave{u}_{\beta_2}\sqrt{\grave{u}_{\gamma_2}^2 + \grave{u}_{\gamma_3}^2}}]\mathbf{N}_q^{\gamma}.$$

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Approximation by Nonlinear Multivariate Convolution Operators in Differentiation Sense

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Abstract

In this work, we will prove some theorems for nonlinear multivariate convolution operators in order to approximate one-sided partial derivatives of functions of multivariables by using extended definition of the notion of one-sided derivative in univariate case. Our proofs will be guided by previous studies in the literature.

Keywords: nonlinear multivariate convolution operators, Lipschitz condition, unified approach, approximation of partial derivatives.

1. Introduction

Let Ω be a non-empty index set and $[a,b)$ be a bounded interval in $\mathbb{R}$. Musielak [1] investigated some problems of nonlinear approximation using integral operators of the form:

$$\mathrm{T}_\sigma(f;x) = \int_a^b K_\sigma\big(t-x, f(t)\big)dt$$

where $\sigma \in \Omega$, $K_\sigma\colon [a,b) \times \mathbb{R} \to \mathbb{R}$, and proved a theorem for the functions which belong to generalized Orlicz space. In order to overcome the nonlinearity barrier occurring in the proof, the Lipschitz condition imposed on the kernel function K_σ in [1]. Bardaro, Musielak and Vinti [2] gave extensive information about unified approach and approximation by nonlinear integral operators.

The linear convolution-type integral operators are the building blocks for their nonlinear counterparts. In this context, detailed information can be found in [3]. Multivariate versions of convolution-type operators were widely studied in [4] and [5].

The convergence method given by Fatou [6] is studied in several works presenting some results concerning convergence of linear integral operators. Some results concerning linear convolution-type operators' Fatou-type (pointwise) convergences can be found in [7-12]. For the nonlinear case, one may see [13] and so on. When it comes to the convergence of derivatives of operators, Taberski [7] gave a complicated result for higher order derivatives of linear convolution-type integral operators depending on two parameters. He put some conditions conditions on the kernel function in order to obtain (Fatou-type) convergence. In the same year, Gadjiev, Džafarov and Labsker [14] obtained the asymptotic approximation of derivatives of general summation-integral type operators to the functions whose higher order one-sided derivatives exist. In [15], Gadjiev dealt with approximation of functions whose one-sided derivatives exist at a point by means of Fatou-type convergence of linear integral operators. For further reading, we refer the reader to [11, 16-20].

In [20] and [21], the authors respectively approximated first and higher order derivatives of functions using univariate nonlinear convolution-type integral operators depending on two parameters. They also obtained some convergence theorems concerning right and left derivatives of the functions. Some related works can be given as [22] and [23]. Recently, Uysal and Yılmaz [24] obtained some results which are analogues to that of proved in [7, 11, 12, 15] for multivariate singular integral operators of convolution-type using similar approach. Nonlinear counterparts of convolution-type integral operators concerning multidimensional case were studied, for example, in [25-27].

Let E be a non-empty index set consisting of non-negative parameters σ and $\mathbb{R}^n$ denote usual finite $n-$dimensional Euclidean space. The accumulation point of E is denoted by σ_0 by allowing it to be also infinity. As a continuation of the works [7, 15, 20, 21, 24], in this study, we consider the nonlinear multivariate integral operators in the following form:

$$T_\sigma(g;x)=\int_A K_\sigma\big(t-x,g(t)\big)dt,\ t,x\in A,$$

where $A:=(a_1,b_1)\times\cdots\times(a_n,b_n)$ is arbitrary bounded $n-$dimensional open interval in $\mathbb{R}^n$ with finite end points and $\bar{A}:=[a_1,b_1]\times\cdots\times[a_n,b_n]$. Here, for each fixed $\sigma\in E$, K_σ is a real-valued function defined on $\mathbb{R}^n\times\mathbb{R}$ and satisfies some additional conditions. We denote the space of all measurable functions g which are integrable in the sense of Lebesgue on $\bar{A}$ such that the norm $\|g\|_{L(\bar{A})}:=\int_{\bar{A}}|g(t)|dt<\infty$ by $L(\bar{A})$. Using notations $t:=(t_1,\dots,t_n)$ and $x:=(x_1,\dots,x_n)$, we restate the operators T_σ as

$$T_\sigma(g;x)=\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}K_\sigma\big(t_1-x_1,\dots,t_n-x_n,g(t_1,\dots,t_n)\big)dt_n\cdots dt_1.$$

Let $\sigma\in E$ and **K** be the family of all functions K_σ, $K_\sigma:\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}$, such that the following conditions hold there:

(I) $K_\sigma(t,0)=0$ for all $t\in\mathbb{R}^n$ and $K_\sigma(t,u)$ is Lebesgue integrable over $\mathbb{R}^n$ for each fixed $\sigma\in E$ for all values of u.

(II) Partial derivative(s) of $K_\sigma(t-x,\cdot)$ up to the order $m+1$ with respect to x_j (resp. t_j), where j is a fixed integer with $1\le j\le n$, finitely exist(s) for each fixed $\sigma\in E$. There exists a function $\mathcal{L}_\sigma:\mathbb{R}^n\to\mathbb{R}_0^+$ which is Lebesgue integrable over $\mathbb{R}^n$ and partial derivative(s) of $\mathcal{L}_\sigma(t-x,\cdot)$ up to the order $m+1$ with respect to x_j (resp. t_j), where j is a fixed integer with $1\le j\le n$, finitely exist(s) for each fixed $\sigma\in E$ such that the condition

$$\frac{\partial^{m+1}}{\partial x_j^{m+1}}K_\sigma(t-x,u)-\frac{\partial^{m+1}}{\partial x_j^{m+1}}K_\sigma(t-x,v)=\frac{\partial^{m+1}}{\partial x_j^{m+1}}\mathcal{L}_\sigma(t-x)[u-v], \tag{1.1}$$

where $t, x \in \mathbb{R}^n$ and $u, v \in \mathbb{R}$, holds for each fixed $\sigma \in E$ and $m \in \mathbb{N}_0$.

(III) $\lim\limits_{\sigma\to\sigma_0}\left[\sup\limits_{|t|\geq\xi}\mathcal{L}_\sigma(t)\right] = 0,\ \forall\xi > 0.$

(IV) $\lim\limits_{\sigma\to\sigma_0}\int_{|t|\geq\xi}\mathcal{L}_\sigma(t)dt = 0, \forall\xi > 0.$

(V) $\lim\limits_{\sigma\to\sigma_0}\int_{\mathbb{R}^n}\mathcal{L}_\sigma(t)dt = 1$ and $\|\mathcal{L}_\sigma\|_{L(\mathbb{R}^n)} \leq M$, where $M > 0$ is independent of σ.

Remark 1. Condition (II) covering equation (1.1) is the condition whose univariate analogue is used in the works [20-23]. The remaining conditions were used in several works including [1, 2, 20-23, 27]. Especially, conditions (III-V) are well-known approximate identity properties (see, e.g., [3]).

2. Main Result

Now, we prove the following theorem based on the results and corresponding proofs of [7, 12, 15, 20, 21]. More specifically, we generalize, mutatis mutantis, Theorem 3 in [21] and its proof steps.

Theorem 1. Let $K_\sigma \in$ **K**, $m \in \mathbb{N}_0$ and j be a fixed integer such that $1 \leq j \leq n$. Assume that $\mathcal{L}_\sigma(t)$ and $\frac{\partial^v}{\partial t_j^v}\mathcal{L}_\sigma(t)$ are continuous functions with respect to t on $\mathbb{R}^n$ for each fixed $\sigma \in E$ and $v = 1, 2, \dots, m+1$. Suppose that the following conditions:

$$\lim_{\sigma\to\sigma_0}\sup_{|t|\geq\xi}\left|\frac{\partial^v}{\partial t_j^v}\mathcal{L}_\sigma(t)\right| = 0,\ \forall\xi > 0 \tag{2.1}$$

hold for each fixed $v = 1, 2, \dots, m+1$. If $g \in L(\bar{A})$ is continuous at a point $c := (c_1, \dots, c_n) \in A$ and for each fixed $v = 1, 2, \dots, m+1$, its partial derivatives with respect to the $j-th$ variable $g_j^{(v)}$ and right-hand $(m+1)-th$ order partial derivative with respect to the $j-th$ variable $g_{+,j}^{(m+1)}$ and left-hand $(m+1)-th$ order partial derivative with respect to the $j-th$ variable $g_{-,j}^{(m+1)}$ finitely exist and are continuous at c, then we have

$$\lim_{(x,\sigma)\to(c,\sigma_0)}\frac{\partial^{m+1}}{\partial x_j^{m+1}}T_\sigma(g;x) = \beta g_{+,j}^{(m+1)}(c) + (1-\beta)g_{-,j}^{(m+1)}(c),$$

where

$$\lim_{(x,\sigma)\to(c,\sigma_0)}\int_{-\infty}^{\infty}\cdots\int_{c_j}^{\infty}\cdots\int_{-\infty}^{\infty}\mathcal{L}_\sigma\big(t_1 - x_1, \dots, t_j - x_j, \dots, t_n - x_n\big)dt_n\cdots dt_j\cdots dt_1 = \beta,\ \beta \in [0,1],$$

on any set S consisting of $(x,\sigma) \in A \times E$ on which the following functions:

$$\sup_{\sigma\in E} \int_{c_1-x_1-\zeta}^{c_1-x_1+\zeta} \cdots \int_{c_n-x_n-\zeta}^{c_n-x_n+\zeta} |t_j|^{m+1} \left| \frac{\partial^{m+1}}{\partial t_j^{m+1}} \mathcal{L}_\sigma(t_1,\dots,t_n) \right| dt_n \cdots dt_1, \tag{2.2}$$

$$\sup_{\sigma\in E} \int_{c_1-x_1-\zeta}^{c_1-x_1+\zeta} \cdots \int_{c_n-x_n-\zeta}^{c_n-x_n+\zeta} \left| \frac{\partial^{m+1}}{\partial t_j^{m+1}} \mathcal{L}_\sigma(t_1,\dots,t_n) \right| dt_n \cdots dt_1 \tag{2.3}$$

and

$$|x_j - c_j|^v \int_{c_1-x_1-\zeta}^{c_1-x_1+\zeta} \cdots \int_{c_n-x_n-\zeta}^{c_n-x_n+\zeta} |t_j|^{m+1-v} \left| \frac{\partial^{m+1}}{\partial t_j^{m+1}} \mathcal{L}_\sigma(t_1,\dots,t_n) \right| dt_n \cdots dt_1 \tag{2.4}$$

are bounded for sufficiently large real number(s) $\zeta > 0$ as $(x,\sigma) \to (c,\sigma_0)$ for each fixed $v = 1,2,\dots,m+1$.

Proof. Let $c \in A$ be fixed. Further, there exists a number $\delta > 0$ satisfying $0 < |c_j - x_j| < \frac{\delta}{2}$ with $j = 1,2,\dots,n$ such that the hypotheses on the function g hold. We consider the function h defined as

$$h(t) := h_{t_j^-} \text{ and } h_{t_j^-} := g(c) + \cdots + \frac{(t_j-c_j)^m}{m!} g_j^{(m)}(c) + \frac{(t_j-c_j)^{m+1}}{(m+1)!} g_{-,j}^{(m+1)}(c)$$

for all $t \in (a_1,b_1) \times \cdots \times (a_j,c_j) \times \cdots \times (a_n,b_n)$ and

$$h(t) := h_{t_j^+} \text{ and } h_{t_j^+} := g(c) + \cdots + \frac{(t_j-c_j)^m}{m!} g_j^{(m)}(c) + \frac{(t_j-c_j)^{m+1}}{(m+1)!} g_{+,j}^{(m+1)}(c)$$

for all $t \in (a_1,b_1) \times \cdots \times [c_j,b_j) \times \cdots \times (a_n,b_n)$.

We also take into account the following function: $f(t) := h(t)$, for all $t \in A$ and $f(t) := 0$ for all $t \in \mathbb{R}^n \setminus A$. In view of this, we have

$$\begin{aligned} T_\sigma(g;x) &= \int_A K_\sigma(t-x, g(t))dt \\ &= \int_A K_\sigma(t-x, g(t))dt - \int_{\mathbb{R}^n} K_\sigma(t-x, f(t))dt + \int_{\mathbb{R}^n} K_\sigma(t-x, f(t))dt \\ &=: I_1(\sigma,x) + I_2(\sigma,x), \end{aligned}$$

where

$$I_1(\sigma,x) = \int_A K_\sigma\big(t-x, g(t)\big)dt - \int_{\mathbb{R}^n} K_\sigma\big(t-x, f(t)\big)dt$$

and

$$I_2(\sigma,x) = \int_{\mathbb{R}^n} K_\sigma\big(t-x, f(t)\big)dt.$$

Differentiating both sides of the integral $I_2(\sigma,x)$ up to the order $m+1$ with respect to x_j, we have

$$\frac{\partial^{m+1}}{\partial x_j^{m+1}} I_2(\sigma,x) = (-1)^{m+1}\int_{\mathbb{R}^n} f(t)\frac{\partial^{m+1}}{\partial t_j^{m+1}}\mathcal{L}_\sigma(t-x)dt.$$

Applying integration by parts $m+1$ times with respect to t_j, we get

$$\begin{aligned}\frac{\partial^{m+1}}{\partial x_j^{m+1}} I_2(\sigma,x) = {}& g_{+,j}^{(m+1)}(c)\int_{-\infty}^{\infty}\cdots\int_{c_j}^{\infty}\cdots\int_{-\infty}^{\infty}\mathcal{L}_\sigma\big(t_1-x_1,\dots,t_j-x_j,\dots,t_n-x_n\big)dt_n\cdots dt_j\cdots dt_1\\ &+g_{-,j}^{(m+1)}(c)\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{c_j}\cdots\int_{-\infty}^{\infty}\mathcal{L}_\sigma\big(t_1-x_1,\dots,t_j-x_j,\dots,t_n-x_n\big)dt_n\cdots dt_j\cdots dt_1.\end{aligned}$$

Evaluating the limits of both sides of resulting equality as $(x,\sigma)\to(c,\sigma_0)$, we get

$$\lim_{(x,\sigma)\to(c,\sigma_0)}\frac{\partial^{m+1}}{\partial x_j^{m+1}} I_2(\sigma,x) = \beta g_{+,j}^{(m+1)}(c) + (1-\beta)g_{-,j}^{(m+1)}(c), \qquad \beta\in[0,1].$$

In view of conditions (2.1)-(2.4), the remaining part of the proof is quite similar to the proof of Theorem 1 in [24]. We skip that part. The proof is completed.

3. Conclusion

As is mentioned in [21], main theorem of this paper is valid for periodic kernel functions too. Our ongoing research is on finding an appropriate nonlinear kernel function which is compatible with computer applications. It is thought that nonlinear periodic kernel functions may help to achive this goal.

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The Spectral Expansion Formula for a Discontinuous Equation of Second Order

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Abstract

In this study, an inverse scattering problem for a discontinuous Sturm-Liouville equation on the half-line $[0,\infty)$ is considered. The special solutions and scattering datas are defined for the equation. Green function is obtained and the resolvent operator is constructed. Applying the contour integration , the spectral expansion formulas in terms of scattering datas are obtained.

Keywords: Expansion formula., scattering data, singular differential operator, spectral parameter.

1. Introduction

In this paper, on the half line $[0,\infty)$, we are interested in obtaining the expansion formula of the problem (1.1)-(1.2) in terms of scattering data. These types of problems with spectral parameter in the boundary condition are investigated in [1-3]. Unlike the classical works, in present work, the boundary value problem for the differential equation with discontinuous coefficient

$$\ell \qquad \frac{d^2\psi}{dx^2}+q(x)\psi=\rho^2 r(x)\psi,\quad 0<x<\infty, \tag{1.1}$$

with the boundary condition

$$U(\psi)\equiv(a_0+ia_1\mu)\psi(0)+(b_0+ib_1\mu)\psi'(0)=0, \tag{1.2}$$

is considered. Here μ spectral parameter, $r(x)$ is a positive function with a finite number of discontinuity points, $q(x)$ is real valued function satisfying the condition

$$\int_0^\infty (1+x)|q(x)|dx<\infty,$$

where $(1+x)|q(x)|\in L_1(0,\infty)$ $(1+x)|q(x)|\in L_1(0,\infty)$, a_0,a_1,b_0,b_1 are real numbers such that $\delta=a_0b_1-a_1b_0\geq 0$, $\rho\neq 1$.

It is known that, the asymptotic behavior of wave functions at infinity is shown by scattering data (see [4]). Scattering datas are defined and expansion formula is expressed by these datas.

Firstly, let's define the special solutions for equation (1.1). Assume that, the function $r(x)$ in the following form

$$r(x)=\begin{cases}\alpha^2, & 0\leq x\leq d,\\ 1, & x>d.\end{cases}$$

It is known that [6], the Jost solution $\varphi(x,\rho)$ can be represented in the form

$$\varphi(x,\rho)=\varphi_0(x,\rho)+\int_{\eta^+(x)}^{\infty}A(x,t)e^{i\rho t}dt,$$

where the function $A(x,t)\in L_1(\eta^+(x),\infty)$. For real $\rho\neq 0$, the functions $\varphi(x,\rho)$ and $\overline{\varphi(x,\rho)}$ form a fundamental system of solutions for (1.1) and their Wronskian is

$$W\left\{\varphi(x,\rho),\overline{\varphi(x,\rho)}\right\}=2i\rho,$$

where $W\{y_1,y_2\}\equiv y_1'y_2-y_1y_2'$.

Let $\omega(x,\rho)$ be solution of equation (1.1) under initial condition

$$\omega(0,\rho)=a_0+ia_1\rho,\ \ \omega'(0,\rho)=-b_0-ib_1\rho.$$

Denote

$$\phi(\rho)=(a_0+ia_1\rho)\varphi(0,\rho)+(b_0+ib_1\rho)\varphi'(0,\rho),$$
$$\phi_1(\rho)=(a_0+ia_1\rho)\overline{\varphi(0,\rho)}+(b_0+ib_1\rho)\overline{\varphi'(0,\rho)},$$

and

$$S(\rho)=\frac{\phi_1(\rho)}{\phi(\rho)}.$$

The function $S(\rho)$ is called the scattering function of the boundary value problem (1.1)-(1.2).

Similarly to the Lemma 2.1. in [7], it is show that the identity

$$\frac{2i\rho\omega(x,\rho)}{\phi(\rho)} = \overline{\varphi(x,\rho)} - S(\rho)\varphi(x,\rho),$$

holds for real $\rho \neq 0$ and $S(\rho) = \overline{S(-\rho)} = \left[S(-\rho)\right]^{-1}$. The function is meromorphic in half plane $\operatorname{Im}\rho > 0$ with poles at the function $\phi(\rho)$. The function $\phi(\rho)$ may have only a finite number of zeros in the half plane $\operatorname{Im}\rho > 0$.

They are all simple and lie on the imaginary axis. For $\rho = i\rho_\kappa \left(\rho_\kappa > 0,\ \kappa = 1,2,...,n\right)$, we get

$$m_\kappa^{-2} \equiv \int_0^\infty \left|\varphi(x, i\rho_\kappa)^2\right| dx - \frac{\delta}{2\rho_\kappa\left(b_0 - b_1\rho_\kappa\right)^2} = -\frac{i\phi(i\rho_\kappa)}{2\rho_\kappa\left(b_0 - b_1\rho_\kappa\right)^2}.$$

The numbers are called the normalizing numbers of the boundary value problem (1.1)-(1.2) has finite numbers eigenvalue. The function is a characteristic of continuous spectrum of the boundary value problem (1.1)-(1.2)

$$F_s(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(S_0 - S(\rho)\right)\exp(i\rho x)\,d\rho,$$

where

$$S_0 = \frac{1 + \gamma\exp(-2i\rho\alpha d)}{\gamma + \exp(-2i\rho\alpha d)},$$

where

$$\gamma = \frac{\alpha - 1}{\alpha + 1}.$$

The collection $\left\{S(\lambda)(-\infty < \lambda < \infty);\ \lambda_\kappa, m_\kappa(\kappa = 1,2,...,n)\right\}$ provides a complete description of the behaviour at infinity of all eigenfunctions of the boundary value problem (1.1)-(1.2). The set of values $\left\{S(\lambda),\ \lambda_\kappa, m_\kappa\right\}(\kappa = 1,2,...,n)$ is called the scattering data of the boundary value problem (1.1)-(1.2).

One can introduce the operator $L_\rho y = \ell$ with the domain of definition

$$D\left(L_\rho\right)=\left\{y \mid y\in L_2\left(R_+\right)\cap AC_\ell\left(R_+\right),\ y'\in AC_\ell\left(R_+\right),\ \ell \quad \left(R_+\right),\ U\left(y\right)=0\right\},$$

where $R_+ =\left[0,\infty\right)$. If ρ runs through the set of all points of the $\rho-$plane, then we obtain a family of singular operators L_ρ depending on the parameter ρ (see [5]).]

2. The Construction of Resolvent Operator

Assume that ρ is not a spectrum point of operator L_ρ. Let us find the expression of the resolvent of operator L_ρ. For this reason, let's find the solution of the following problem

$$-\psi''+q\left(x\right)\psi=\mu+f\left(x\right), \tag{2.1}$$

$$\left(a_0+ia_1\mu\right)\psi\left(0\right)+\left(b_0+ib_1\mu\right)\psi'\left(0\right)=0, \tag{2.2}$$

where $f\left(x\right)\in D\left(L_\rho\right)$ is a finite function and equal to zero on outside of finite interval $\left[0,a\right]$.

By applying the Lagrange method, we obtain solution of (2.1)-(2.2)

$$\psi\left(x,\rho\right)=\int_0^\infty G\left(x,t,\rho\right)f\left(t\right)dt, \tag{2.3}$$

where

$$G\left(x,t,\rho\right)=-\frac{1}{\phi\left(\rho\right)}\begin{cases}\varphi\left(x,\rho\right)\omega\left(t,\rho\right), & x\le t<\infty,\\ \omega\left(x,\rho\right)\varphi\left(t,\rho\right), & 0\le t\le x.\end{cases}$$

Therefore, for $\phi\left(\rho\right)\neq 0,\ \operatorname{Im}\rho\ge 0,$ all these numbers ρ belong to resolvent set of operator L_ρ and the resolvent operator R_ρ is the integral operator

$$R_\rho f=\int_0^\infty G\left(x,t,\rho\right)f\left(t\right)r\left(t\right)dt. \tag{2.4}$$

Moreover, for the kernel function $G\left(x,t,\rho\right)$, by using the properties of the solutions of $\varphi\left(x,\rho\right)$ and $\omega\left(x,\rho\right)$, the inequality

$$G\left(x,t,\rho\right)\le c\left(x\right)\frac{\exp\left(-\operatorname{Im}\rho\left|x-t\right|\right)}{\phi\left(\rho\right)}, \tag{2.5}$$

is obtained.

Theorem 2.1. Assume that, let the function $g(x)$ be twice continuously differentiable finite function. Then, the following equality is valid:

$$\int_0^\infty G(x,t,\rho)g(t)r(t)dt = -\frac{g(x)}{\rho^2} + \frac{1}{\rho^2}\int_0^\infty G(x,t,\rho)g(t)r(t)dt, \tag{2.6}$$

where

$$g(t) = g''(t) + q(t)g(t)dt.$$

Proof: For the kernel $G(x,t,\rho)$, the following equality is valid

$$-G''(x,t,\rho) + q(x)G(x,t,\rho) - \rho^2 G(x,t,\rho) = \delta(x-t).$$

Here $\delta(x)$ is Dirac-delta function. By integrating by parts and considering the boundary condition, the formula (2.6) is obtained.

3. The Expansion Formula

Let $f(x) \in D(L_\rho)$ and assume that it is a finite function at finite interval $[0,a] \subset [0,\infty)$. Let us C_R denotes the circle of radius R and center at the origin which contour is positive oriented. Let C_R that doesn't include points z satisfying the conditions $\operatorname{Im} z > \varepsilon$ and let $C_{R,\varepsilon}^{(2)}$ be a half arc that does not include $\operatorname{Im} z < -\varepsilon$ points of C_R and $C_{R,\varepsilon} = C_{R,\varepsilon}^{(1)} \cup C_{R,\varepsilon}^{(2)}$ is positive oriented. Let $C_{R,\varepsilon}^{(3)}$ be negative oriented curve formed with $\operatorname{Im} z = \pm\varepsilon$ lines and be arcs including points z satisfying the conditions $|\operatorname{Im} z| < \varepsilon$. Then, we can use the property of the integration

$$\int_{C_{R,\varepsilon}} = \int_{C_R} + \int_{C_{R,\varepsilon}^{(3)}} . \tag{3.1}$$

Let's call $\phi(x,\rho) = \int_0^\infty G(x,t,\rho)g(t)r(t)dt.$ According to (2.6), it is clear that

$$\phi(x,\rho) = -\frac{g(x)}{\rho^2} + \frac{1}{\rho^2}\int_0^\infty G(x,t,\rho)g(t)dt.$$

Now multiplying both sides of the equality by $\frac{1}{2\pi i}\rho$ and and integrating over the contour $C_{R,\varepsilon}$

$$\frac{1}{2\pi i}\int_{C_{R,\varepsilon}}\rho\phi(x,\rho)d\rho=-\frac{1}{2\pi i}\int_{C_{R,\varepsilon}}\frac{g(x)}{\rho}d\rho+\frac{1}{2\pi i}\int_{C_{R,\varepsilon}}\frac{1}{\rho}\left\{\int_0^\infty G(x,t,\rho)\right\}.$$

According to (3.1), we obtain

$$\frac{1}{2\pi i}\int_{c_{R,\varepsilon}}\rho F(x,\rho)d\rho=\frac{1}{2\pi i}\int_{C_R}\rho F(x,\rho)d\rho+\frac{1}{2\pi i}\int_{C_{R,\varepsilon}^{(3)}}\rho\phi(x,\rho)d\rho.$$

From the relations $(R\to\infty,\in\to 0)$, we get

$$\begin{aligned} g(x)=&-\sum_{K=1}^{n}\underset{\rho=i\rho_K}{Res}[\rho\phi(x,\rho)]-\sum_{K=1}^{n}\underset{\rho=i\rho_K}{Res}[\rho\phi(x,\rho)]\\ &+\int_{-\infty}^{\infty}\rho\left[\phi(x,\rho+i0)-\phi(x,\rho-i0)\right]d\rho. \end{aligned} \tag{3.2}$$

and with calculations

$$\sum_{\kappa=1}^{n}\underset{\rho=i\rho_\kappa}{Res}[\rho\phi(x,\rho)]=\frac{i\rho_\kappa}{\dot{F}(i\rho_\kappa)}.\frac{b_1\varphi'(0,i\rho_\kappa)+a_1\varphi(0,i\rho_\kappa)}{\delta}\omega(x,i\rho_\kappa)\times$$

$$\times\int_0^\infty\omega(t,i\rho_\kappa)g(t)r(t)dt+u(x,i\rho_\kappa)\int_0^\infty u(t,i\rho_\kappa)g(t)r(t)dt,$$

where

$$\begin{aligned} u(x,i\rho_K)&=\frac{1}{2}m_K\varphi(x,i\rho_K),\\ \varphi(x,i\rho_K)&=-\frac{b_1\varphi'(0,i\rho_K)+a_1\varphi(0,i\rho_K)}{\delta}\omega(x,i\rho_K), \end{aligned}$$

and

$$m_\kappa^2=-\frac{2i\rho_\kappa}{\dot{F}(i\rho_\kappa)}.\frac{\delta}{a_1\varphi(0,i\rho_\kappa)+b_1\varphi'(0,i\rho_\kappa)}.$$

Now taking

$$\sum_{\kappa=1}^{n} \underset{\rho=i\rho_\kappa}{Res}[\rho\phi(x,\rho)] + \sum_{\kappa=1}^{n} \underset{\rho=-i\rho_\kappa}{Res}[\rho\phi(x,\rho)] = \sum_{\kappa=1}^{n} u(x,i\rho_\kappa)\int_0^\infty u(t,i\rho_\kappa)g(t)r(t)\,dt, \tag{3.3}$$

since

$$\phi(x,\rho-i0) = \overline{\phi(x,\rho+i0)},$$

we have

$$\begin{aligned} &\frac{1}{2\pi i}\int_{-\infty}^{+\infty} \rho\left[\phi(x,\rho+i0)-\phi(x,\rho-i0)\right]d\rho = -\int_0^\infty \frac{\left|a_1\varphi(0,\rho)+b_1\varphi'(0,\rho)\right|^2}{\delta}\times \\ &\times u(x,\rho)\int_0^\infty u(t,\rho)g(t)r(t)\,dtd\rho + \int_0^\infty u(x,\rho)\int_0^\infty u(t,\rho)g(t)r(t)\,dtd\rho. \end{aligned} \tag{3.4}$$

Substituting (3.2), (3.3) in (3.4), we obtain expansion formula in the following form:

$$\begin{aligned} g(x) &= \sum_{\kappa=1}^{n} u(x,i\rho_\kappa)\int_0^\infty u(t,i\rho_\kappa)g(t)r(t)\,dt - \\ &-\int_0^\infty \frac{\left|a_1\varphi(0,\rho)+b_1\varphi'(0,\rho)\right|^2}{\delta} u(x,\rho)\int_0^\infty u(t,\rho)g(t)r(t)\,dtd\rho + \\ &+\int_0^\infty u(x,\rho)\int_0^\infty u(t,\rho)g(t)r(t)\,dtd\rho. \end{aligned}$$

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Prime Ideals of Gamma Nearness Near Rings

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Abstract

In this article, we define the notion of prime ideals of Γ-near-rings on weak nearness approximation spaces and explain some of the concepts and definitions. Then, we study some basic properties of prime ideals of Γ-nearness near-rings. Γ-nearness near-rings is different from Γ-nearness rings and Γ-nearness semirings since Γ does not have to be group in Γ-nearness near-rings. Because of this, some properties defined in Γ-nearness rings and Γ-nearness semirings show some changes in Γ-nearness near-rings.

Keywords: Near set, Near ring, Nearness approximation space, Weak nearness approximation space, Near-ring, Gamma-near-ring, Nearness near-ring, Gamma nearness near-ring.

1. Introduction

The concept Γ-rings, a generalization of a ring was introduced by Nobusawa in 1964 [1] and generalized by Barnes in 1966 [2]. Pilz defined near-rings (also near ring or nearring) that is an algebraic structure similar to a ring but satisfying some axioms [3]. A generalization of both the concepts near-ring and the ring, namely Γ-near-ring was introduced by Satyanarayana in 1984 and later studied by the authors like Satyanarayana [4], [5], Booth [6], Booth and Groenewald [7], [8], Jun, Sapancı and Öztürk [9].

In 2002, Peters introduced near set theory, which is a generalization of rough set theory [10]. In this theory, Peters defined an indiscernibility relation by using the features of the objects to determine the nearness of the objects [11], [12], [13]. The concept of nearness has a different approach for algebraic structures. Because, in the concept of ordinary algebraic structures, such a structure that consists of a nonempty set of abstract points with one or more binary operations, which are required to satisfy certain axioms. Also, the sets are composed of abstract points. Perceptual objects (non-abstract points) can be used on weak nearness approximation space to define nearness algebraic structures. This is more useful than working with abstract points for many areas such as engineering applications, image analysis and so on. In 2012, İnan and Öztürk investigated the concept of nearness groups [14] and other algebraic approaches of near sets in [15], [16], [17], [18], [19], [20], [21], [22], [23]. In 2021, Uçkun and Genç defined near-rings on nearness approximation spaces [24].

The aim of this paper is to define the concept of prime ideals of Γ-nearness near-rings and to study some properties. Γ-nearness near-rings is different from Γ-nearness rings [20] and Γ-nearness semirings [21] because Γ does not have to be group in Γ-nearness near-rings. Because of this, some properties defined in Γ-nearness rings and Γ-nearness semirings show some changes in Γ-nearness near-rings.

2. Preliminaries

An object description is specified by means of a tuple of function values $\Phi(x)$ deal with an object $x \in X$. $B \subseteq \mathcal{F}$ is a set of probe functions and these functions stand for features of sample objects $X \subseteq \mathcal{O}$. Let $\varphi_i \in B$, that is $\varphi_i : \mathcal{O} \to \mathbb{R}$. The functions showing object features supply a basis for $\Phi: \mathcal{O} \to \mathbb{R}^L$, $\Phi(x) = (\varphi_1(x), \varphi_2(x), \ldots, \varphi_L(x))$ a vector consisting of measurements deal with each functional value $\varphi_i(x)$, where the description length $|\Phi| = L$ ([2]).

The selection of functions $\varphi_i \in B$ is very fundamental by using to determine sample objects. $X \subseteq \mathcal{O}$ are near each other if and only if the sample objects have similar characterization. Each φ shows a descriptive pattern of an object. Hence, $\triangle_{\varphi_i}$ means $\triangle_{\varphi_i} = |\varphi_i(x)' - \varphi_i(x)|$, where $x, 'x \in \mathcal{O}$. The difference φ means to a description of the indiscernibility relation " $\sim_B$ " defined by Peters in [2]. B_r is probe functions in B for $r \leq |B|$.

Definition 1: [11]

$$[c]l \sim_B = \{(x, x)' \in \mathcal{O} \times \mathcal{O} \mid \triangle_{\varphi_i} = 0 \forall \varphi_i \in B \; B \subseteq \mathcal{F}\}$$

means indiscernibility relation on $\mathcal{O}$, where description length $i \leq |\Phi|$. $\sim_{B_r}$ is also indiscernibility relation determined by utilizing B_r.

Near equivalence class is stated as $[x]_{B_r} = \{x \in '\mathcal{O} \mid x \sim_{B_r} x\}'$. After getting near equivalence classes, quotient set $\mathcal{O} / \sim_{B_r} = \{[x]_{B_r} \mid x \in \mathcal{O}\} = \xi_{\mathcal{O},B_r}$ and set of partitions $N_r(B) = \{\xi_{\mathcal{O},B_r} \mid B_r \subseteq B\}$ can be found. By using near equivalence classes, $N_r(B)^* X = \bigcup_{[x]_{B_r} \cap X \neq \varnothing} [x]_{B_r}$ upper approximation set can be attained.

Definition 2: [18]
Let $\mathcal{O}$ be a set of sample objects, $\mathcal{F}$ a set of the probe functions, $\sim_{\mathrm{B_r}}$ an indiscernibility relation,and $\mathrm{N_r}$ a collection of partitions. Then, $(\mathcal{O}, \mathcal{F}, \sim_{\mathrm{B_r}}, \mathrm{N_r}(\mathrm{B}))$ is called a weak nearness approximation space.

Theorem 1:[18]
Let $(\mathcal{O}, \mathcal{F}, \sim_{\mathrm{B_r}}, \mathrm{N_r})$ be a weak nearness approximation space and $\mathrm{X}, \mathrm{Y} \subset \mathcal{O}$. Then the following statements hold:

i) $X \subseteq N_r(B)^*X$,
ii)$N_r(B)^*(X \cup Y) = N_r(B)^*X \cup N_r(B)^*Y$,
iii) $X \subseteq Y$ implies $N_r(B)^*X \subseteq N_r(B)^*Y$,
iv) $N_r(B)^*(X \cap Y) \subseteq N_r(B)^*X \cap N_r(B)^*Y$.

Definition 3: [25]
Let $\left(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r(B)\right)$ be a weak nearness approximation space, $G \subseteq \mathcal{O}$ and " $\cdot$ "a operation by $\cdot : G \times G \to N_r(B)^*G$. G is called a group on $\mathcal{O}$ or shortly nearness group if the following properties are satisfied:

i) $x \cdot y \in N_r(B)^*G$ for all $x, y \in G$,

ii) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ property holds in $N_r(B)^*G$ for all $x, y, z \in G$,

iii) There exists $e \in N_r(B)^*G$ such that $x \cdot e = x = e \cdot x$ for all $x \in G$,

iv) There exists $y \in G$ such that $x \cdot y = e = y \cdot x$ for all $x \in G$.

Lemma 1: [21]
Let S be a Γ-nearness semiring. If $\sim_{B_r}$ is a congruence indiscernibility relation on S, then $[x]_{B_r} + [y]_{B_r} \subseteq x + y]_{B_r}$, $[\beta]_{B_r} + [\gamma]_{B_r} \subseteq \beta + \gamma]_{B_r}$, $[x]_{B_r}\alpha y]_{B_r} \subseteq x\alpha y]_{B_r}$ for all $x, y \in S$, and $\alpha, \beta, \gamma \in \Gamma$.

Lemma 2: [21] Let S be a Γ-nearness semiring. The following properties hold:

i) If $X, Y \subseteq S$, then $(N_r(B)^*X) + (N_r(B)^*Y) \subseteq N_r(B)^*(X + Y)$,

ii) If $X, Y \subseteq S$, then $(N_r(B)^*X)\Gamma(N_r(B)^*Y) \subseteq N_r(B)^*(X\Gamma Y)$.

Definition 4: [2] Let M and Γ be additive Abelian groups. M is said to be a Γ-ring if there exists a mapping $M \times \Gamma \times M \to M$ (the image of (a, α, b) is denoted by $a\alpha b$). M is called a Γ-near-ring (in the sense of Barnes*)* on $\mathcal{O} - \mathcal{O}'$ or shortly a Γ-nearness near-ring and denoted by $(M, +, \cdot)$ satisfying the following conditions:
i)
$$(a + b)\alpha c = a\alpha c + b\alpha c,$$
$$a(\alpha + \beta)b = a\alpha b + a\beta b,$$
$$a\alpha(b + c) = a\alpha b + a\alpha c,$$

ii) $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$.

Definition 5: [24] Let $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r(B), \nu_{N_r})$ be a nearness approximation spaces, "$+$" and "$\cdot$" be binary operations defined on $\mathcal{O}$. $M \subseteq \mathcal{O}$ is called a near-ring on nearness approximation spaces or shortly nearness near-ring if the following properties are satisfied:

NN_1) $(M, +)$ is a nearness group (it does not need to be commutative),
NN_2) $(M, \cdot)$ is a nearness semigroup,
NN_3) for all $x, y, z \in M$, $(x + y) \cdot z = (x \cdot z) + (y \cdot z)$ property holds in $N_r(B)^* M$.

Definition 6: [26] Let $M = \{a, b, c, ...\} \subseteq \mathcal{O}$, and $\Gamma = \{\alpha, \beta, ...\} \subseteq \mathcal{O}'$ where $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r(B))$ and $(\mathcal{O}', \mathcal{F}, \sim_{B_r}, N_r(B))$ are two different weak near approximation spaces. " $\cdot$ "a operation by $\cdot: M \times M \to N_r(B)^* M$. M is called a Γ-near-ring (in the sense of Barnes) on $\mathcal{O} - \mathcal{O}'$ or shortly a Γ-nearness near-ring and denoted by $(M, +, \cdot)$ if the following conditions are satisfied:

GNR_1) $(M, +)$ is a nearness group on $\mathcal{O}$ with identity element 0_M (not necessarily abelian),

GNR_2) for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$ such that $(a\alpha b)\beta c = a\alpha(b\beta c)$ hold in $N_r(B)^* M$,

GNR_3) for all $a, b, c \in M$ and $\alpha \in \Gamma$ such that $(a + b)\alpha c = a\alpha c + b\alpha c$ hold in $N_r(B)^* M$.

Theorem 2: [26] Let M be a Γ-nearness near-ring and $\{H_i | i \in I\}$ be a nonempty family of Γ-ideal of M, where an arbitrary index set I.

i) If $N_r(B)^*(\bigcap_{i\in I} H_i) = \bigcap_{i\in I} N_r(B)^* H_i$, then $\bigcap_{i\in I} H_i$ is a Γ-ideal of M.

ii) $\bigcup_{i\in I} H_i$ is a Γ-ideal of M.

3. Prime Ideals of Γ-nearness near-ring

Definition 7: Let P be an ideal of Γ-nearness near-ring M. P is called

i) a Γ-prime ideal of M if for all ideals I and J of M, $I\Gamma J \subseteq N_r(B)^* P$ implies $I \subseteq P$ or $J \subseteq P$.

ii) a Γ-semiprime ideal of M if for all ideals I and J of M, $I^2 = I\Gamma J \subseteq N_r(B)^* P$ implies $I \subseteq P$ or $J \subseteq P$.

Definition 8: Let X be a nonempty subset of a Γ-nearness near-ring M. Let $\{A_i : i \in I\}$ be a family of all ideals in M that contain X. If

$$\bigcap_{i\in\Delta} (N_r(B)^* A_i) = N_r(B)^*(\bigcap_{i\in\Delta} A_i),$$

then $\bigcap_{i\in I} A_i$ is called the ideal generated by the set X and it is denoted by (X).

The elements of X is called the generators of ideal (X). If $X = \{x_1, x_2, \cdots, x_n\}$, then $(X) = (x_1, x_2, \cdots, x_n)$. Thus, we call (X) is finitely generated.

If $X = \{a\}$, then $(X) = (a)$ is called the principal ideal generated by a.

Theorem 3: Let P be a Γ-prime ideal of M. Then, the following conditions are equivalent.

$i)$ P is prime,

$ii)$ For every two ideals I, J of M, it implies that $I \nsubseteq P$ and $J \nsubseteq P \Rightarrow I\Gamma J \nsubseteq N_r(B)^*P$,

$iii)$ For every two elements $a, b \in M$, $a \notin P$ and $b \notin P \Rightarrow (a)\Gamma(b) \nsubseteq N_r(B)^*P$.

Proof.

$i) \Rightarrow ii)$ Assume that P is a Γ-prime ideal of M, $I \nsubseteq P$ and $J \nsubseteq P$. If possible, suppose that $I\Gamma J \subseteq N_r(B)^*$, then $I \subseteq P$ or $J \subseteq P$ since P is a Γ-prime ideal of M. Thus, we received a contradiction. From here, we have $I\Gamma J \nsubseteq N_r(B)^*$.

$ii) \Rightarrow iii)$ Let $a \notin P$ and $b \notin P$ be elements of M. In this case, we get $(a) \nsubseteq P$ and $(b) \nsubseteq P$. Therefore, by hypothesis, $(a)\Gamma(b) \nsubseteq N_r(B)^*P$.

$iii) \Rightarrow i)$ For elements $a, b \in M$, $a \notin P$ and $b \notin P$, and so $(a) \nsubseteq P$ and $(b) \nsubseteq P$. Suppose that $(a) \nsubseteq P$ and $(b) \nsubseteq P$ such that $(a)\Gamma(b) \subseteq N_r(B)^*P$. Since $a \notin P$ and $b \notin P$, then by hypothesis $(a)\Gamma(b) \nsubseteq N_r(B)^*P$, which is contradiction. In this case, $(a)\Gamma(b) \subseteq N_r(B)^*P \Rightarrow (a) \subseteq P$ and $(b) \subseteq P$. Therefore, P is a prime ideal of M.

Definition 9: Let M be a Γ-nearness near-ring. Then, M is called Γ-prime near ring if 0 is a Γ-prime ideal of M.

Theorem 4: Let M be a Γ-nearness near-ring and $\{A_i | i \in I\}$ be a nonempty family of Γ-prime ideal of M, where an arbitrary index set I.

$i)$ If $N_r(B)^*(\bigcap_{i\in I} A_i) = \bigcap_{i\in I} N_r(B)^* A_i$, then $\bigcap_{i\in I} A_i$ is a Γ-prime ideal of M.

$ii)$ If $A_1 \subseteq A_2 \subseteq A_3$ ..., then $\bigcup_{i\in I} A_i$ is a Γ-prime ideal of M.

Proof.

$i)$ $\bigcap_{i\in I} A_i$ is a Γ-ideal of M by Theorem 2. Suppose that $P_1\Gamma P_2 \subseteq N_r(B)^*(\bigcap_{i\in I} A_i)$ for any Γ-ideals P_1 and P_2 of M. In this case, $P_1\Gamma P_2 \subseteq N_r(B)^*(\bigcap_{i\in I} A_i) = \bigcap_{i\in I} N_r(B)^* A_i$ from hypothesis. Thus, $P_1\Gamma P_2 \subseteq \bigcap_{i\in I} N_r(B)^* A_i$, and so $P_1\Gamma P_2 \subseteq N_r(B)^* A_i$ for all $i \in I$. Because A_i's are Γ-prime ideals of M for all $i \in I$, then $P_1 \subseteq A_i$ or $P_2 \subseteq A_i$ for all $i \in I$. From here, we attain that $P_1 \subseteq \bigcap_{i\in I} A_i$ or $P_2 \subseteq \bigcap_{i\in I} A_i$.

$ii)$ From Theorem 2.(ii), $\bigcup_{i\in I} A_i$ is a Γ-ideal of M. Assume that $P_1\Gamma P_2 \subseteq N_r(B)^*(\bigcup_{i\in I} A_i)$ for any ideals P_1 and P_2 of M. In this case, we get $P_1\Gamma P_2 \subseteq \bigcup_{i\in I} N_r(B)^* A_i$ by Theorem 1.(ii). There is at least one $i_n \in I$ such that $P_1\Gamma P_2 \subseteq N_r(B)^* A_{i_n}$. As A_{i_n} is prime ideal of M for $i_n \in I$, $P_1 \subseteq A_{i_n}$ or $P_2 \subseteq A_{i_n}$ for $i_n \in I$.

Therefore, $P_1 \subseteq \bigcup_{i \in I} A_i$ or $P_2 \subseteq \bigcup_{i \in I} A_i$.

Definition 10: A Γ-nearness near-ring M is called simple if M has no proper ideal.

Theorem 5: If Γ-nearness near-ring M is simple, then either M is Γ-prime or $\mathrm{M\Gamma M} = \{0\} \in \mathrm{N_r(B)^*M}$.

Proof.

Suppose that I and J are ideals of M. Since M is simple, we have $I = M$ or $I = 0$ and $J = M$ or $I = 0$. Therefore, for the ideals I and J of M, we have the equation $I\Gamma J = 0$, then $I = 0$ or $J = 0$, or $I = J = M$. If $I = 0$ or $J = 0$, then M is Γ-prime near ring. Otherwise, $M\Gamma M = \{0\} \in N_r(B)^*M$ if $I = J = M$.

4. Conclusion

As a recent study of Γ-nearness near-ring, it is defined that the notion of prime ideals in Γ-nearness near-ring. Afterward, it is explained that some of the concepts and definitions. We believe that these properties will be more useful theoretical development for Γ-nearness near-ring theory.

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Comparison of the order-type integrals in Riesz Space

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Abstract

In this paper we begin with investigating the order-type Pettis, Bochner, Dunfort and McShane integrals in Banach lattice and give some comparison results. One interesting difference between these kinds of integration is the fact that they possess the properties represented by Hake and Henstock lemmas. We observe that on the case of L-spaces the order integral of Pettis is stronger as Bochner one (by norm).

Keyword(s) Banach lattice, (o)-Pettis integral, (o)- McShane integrals.

1. Introduction and preliminaries

It is known that the McShane integral and the Henstock-Kurzweil integral are two kinds of the Riemann-type integral. Relations of different generalizations of Riemann-type integral was done in the last decades and afterwards the notions of order-type integrals were introduced and studied for functions taking their values in ordered vector spaces, and in Banach lattices. In particular we can see [11], [7], [4], [8], [9], [13], [3], [5], [10], [12], [14]. We are inspired from the works of Candeloro and Sambucini [1], [2] as well as Boccuto et al. [3]-[6] about order –type integrals. In this paper we begin with investigating the order-type Pettis, Bochner, Dunfort and McShane integrals in Banach lattice and give some comparison results. One interesting difference between these kinds of integration is the fact that they possess the properties represented by Hake and Henstock lemmas. We observe that on the case of L-spaces that are separable the order integral of Pettis is stronger as Bochner one.

From now on, T will denote a compact metric space, and $\mu: \mathfrak{B} \to \mathbb{R}_0^+$ any regular, nonatomic σ-additive measure on the σ-algebra $\mathfrak{B}$ of Borel subsets of T.

A sequence $(r_n)_n$ is said to be order-convergent (or (o)-convergent) to r, if there exists a sequence $(p_n)_n \in R$, such that $p_n \downarrow 0$ and $|r_n - r| \leq p_n$, $\forall\, n \in \mathbb{N}$.

(see also [12], [14]), and we will write$(o) \lim{}_n r_n = r$.

A *gage* is any map $\gamma: T \to \mathbb{R}^+$. A *partition* Π of T is a finite family $\Pi = \{(E_i, t_i): i = 1, \dots, k\}$ of pairs such that the sets E_i are pairwise disjoint sets whose union is T and the points t_i are called *tags*. If all tags satisfy the condition $t_i \in E_i$ then the partition is said to be of *Henstock* type, or a *Henstock partition*. Otherwise, if t_i is not necessary to be in E_i, we say that it is a *free* or *McShane* partition.

Given a gage γ, we say that Π is γ-*fine* if $d(w, t_i) < \gamma(t_i)$ for every $w \in E_i$ and $i = 1, \dots, k$. Clearly, a gage γ can also be defined as a mapping associating with each point $t_i \in T$ an open ball centered at $t_i$and cover E_i.

Let us assume now that X is any Banach lattice with an order-continuous norm. For the sake of completeness we recall the main notions of integral we are interested in.

Definition 1.1.

A function $f: T \to X$ is called (o)- McShane integrable ((oH)-integrable) and $J \in X$ is its (o)-McShane integral ((oH)-integral) if for every (o)- sequence $(b_n)_n$ in X, there is a corresponding sequence $(\gamma_n)_n$of gauges $(\gamma_n(t): T \to]0, +\infty[$ such that for every n and (γ_n) -fine M-partition (H-partition) $\{(E_i, t_i), i = 1, \dots, p\}$ of T holds the inequality

$$|\sigma(f, \Pi) - J| \leq b_n. \qquad (1)$$

Where $\sigma(f, \Pi) = \sum_{i=1}^{p} f(t_i)\, \mu(E_i)$. We denote

$J = (oM) \int_T f.$

respectively

$$J = (oH) \int_T f.$$

a

Definition1.2.

A function$f: T \to$ X, is called (o)- Bochner integrable if there is an(o)- Cauchy sequence $\{f_n\}_n$ of simple functions which converges to f almost everywhere in T, i.e.

$$(o)\text{-}\lim \int_T |f_n(t) - f_m(t)| = 0 \text{ for almost all } t \in T \qquad (2)$$

$(o)\text{-}\lim\limits_{n\to\infty} \int_T f_n(t)$ is called the (o)- Bochner integral of the function f: (3)

$$(oB) \int_T f(t).$$

Where f_n is an arbitrary sequence of simple functions which determines f.

Definition 1.3.

If $f: T \to X$ is weakly measurable and that for each $x^* \in X^*$, the function $x^*(f): T \to \mathbb{R}$ is (oM) integrable then f is called (o)- Dunford integrable $(oD)\int_E f$, of f over a measurable $E \subset T$,is defined by the element $x_E^{**} \in X^{**}$

$$(oD)\int_E f = x_E^{**} \in X^{**}, (4)$$

for all $x^* \in X^*$. $x_E^{**}(x^*) = \int_E x^*(f)$

The Dunford integral $(oD)\int_E f$ is an element of the second dual X^{**} of the Banach lattice X. This situation is not very pleasant,one would expect that the values of an integral of an X-valued function belong to the same space X

The space X itself is in a natural way embedded intoX^{**}, if $(oD)\int_E f \in X \subset X^{**}$the following definition can be presented.

Definition1.4.

If $f: T \to X$ is (oD)- integrable where $(oD)\int_E f \in e(X) \subset X^{**}$, (e is the canonical embedding of $X \subset X^{**}$) for every measurable $E \subset T$, then f is called (o)- Pettis integrable and

$$(oP)\int_e f = (oD)\int_E f \ (5)$$

is called the (o) -Pettis integral of $f: T \to X$ over the set E.

The (oP)- integrability of can be defined equivalently as follows

Definition1.5.

A weakly measurable $f: T \to X$, with $x^*(f)$ (o)-Bochner integrable for every $x^* \in X^*$ is (o) -Pettis i integrable definition can be presented.

$x^*(x_E) = \int_E x^*(f) \quad (6)$

For every $x^* \in X^*$.

If X is a reflexive space $(X^{**} = X)$, then the (oD) and (oP) integrals coincide.

Definition.1.6

If $f: [a, b] \to X$ is such that the function $x^*(f): [a, b] \to \mathbb{R}$ is (o)-Denjoy integrable for each $x^* \in X^*$and if for every interval $E \subset [a, b]$ there is an element . $x_E^{**} \in X^{**}$ such that

$x_E^{**}(x^*) = \int_E \; x^*(f)$ for all $x^* \in X^*$ then f is called (o)-Denjoy-Dunford integrable on $[a, b]$

We write

$$(oDD)\int_E \; f{=}x_E^{**}$$

Lemma 1.7. (Saks-Henstock).

Assume that $f\colon T \to X$ is (o)– McShane integrable. Given (o) - sequence $(b_n)_n$ assume that a corresponding sequence $(\gamma_n)_n$ of gauges $(\gamma_n(t)\colon T \to]0, +\infty[$ on T such that for every n and for every γ_n-fine M- partition $\Pi = \{(E_i, t_i)\colon i = 1, \dots, k\}$, of T holds the inequality

$$\left|\sum_{i=1}^{k} f(t_i)\,\mu(E_i) - (oH)\int_T \; f\right| \le b_n \ (7)$$

Then if $\{(F_j, \tau_j)\colon j = 1, \dots, m\}$ is an arbitrary γ_n-fine M-system we have

$$\left|\sum_{j=1}^{m} (f(\tau_j)\,\mu(F_j) - (oH)\int_{F_j} f)\right| \le b_n \ \ (8)$$

The same holds if (oM) is replaced by (oH) and H- partitions are used instead of M- partitions.

Theorem 1.8.(Hake)

Let $[\mathrm{a}, \mathrm{b}] \subset \mathbb{R}$, $f\colon [\mathrm{a}, \mathrm{b}] \to X$. If the integral $(\mathrm{oH})\int_c^b f d\mu$ and (o)-$\lim\limits_{c\to a^+}(oH)\int_c^b f = L \in X$ exists for every $a < c \le b$ then the integral $(oH)\int_a^b f$ exists and holds the equality:

$(oH)\int_a^b f = L.$ (9)

Theorem 1.9. [1].

Let $f\colon T \to X$ be any mapping. Then f is (o)– Henstock integrable ((o)- McShane integrable) if and only if there exist an (o) –sequence $(b_n)_n$ and a corresponding sequence $(\gamma_n)_n$ of gages, such that for every n, as soon as Π'',Π' are two -γ_n fine Henstock (McShane) partitions, the following holds true:

$$|\ \sigma(f, \Pi'') - \sigma(f, \Pi')\ | \le b_n \ (10)$$

Theorem 1.10.[1].

Let $f\colon T \to X$ be (oM)-integrable and assume that X is an L- space. Then f is Bochner integrable.

2.Comparison of the order-type integrals

Proposition 2.1.

If $f: T \to X$ is (oB)- integrable then f is (oP)-integrable :

$$(oP) \int_E f = (oB) \int_E f \ (11)$$

For every measurable $E \subset T$.

Proof. Since $f \in (oB)$**,** let (f_q) be an (o) -Cauchy sequence of simple functions determining f . Then

$$(oB) \int_E f = {}^{O-lim}_{q\to\infty}(oB) \int_E f_q$$

And for $x^* \in X^*$ we have:

$$x^*\left((oB) \int_E f\right) = x^*\left({}^{O-lim}_{q\to\infty}(oB) \int_E f_q\right) = {}^{O-lim}_{q\to\infty} x^*\left((oB) \int_E f_q\right)$$

$$={}^{O-lim}_{q\to\infty}(oB) \int_E x^*(f_q) = \int_E x^* f,$$

Because

$$\left|(oB) \int_E x^*(f_q - f)\right| \leq (oB) \int_E |x^*(f_q - f)|$$

$$\leq (oB) \int_E sup\,|x^*(f_q - f)| \leq \|x^*\|(oB) \int_E |f_q - f|$$

And

$$={}^{O-lim}_{q\to\infty}(oB) \int_E |f_q - f| = 0.$$

Hence $f \in (oP)$.

Proposition 2.2.

If $f: T \to X$ is (oM) integrable with $(oM) \int_T f \in X$, then for every $x^* \in X^*$ the real function $x^*(f): T \to \mathbb{R}$ is order McShane integrable and

$$(oM) \int_T x^*(f) = x^*\left((oM) \int_T f\right).(12)$$

Proof. By Definition 1.1 for every (o)- sequence $(b_n)_n$ in X, there is a corresponding sequence $(\gamma_n)_n$of gauges $(\gamma_n(t): T \to]0, +\infty[$ such that for every n and (γ_n) -fine M-partition $\{(I_i, t_i), i = 1, \ldots, p\}$ of T holds the inequality

$$\left|\sum_{i=1}^{p} f(t_i)\,\mu(I_i) - (oM) \int_T f\right| \leq b_n,$$

If $x^* \in X^*$, then by previous inequality we have

$$\left|\sum_{i=1}^{p} x^*\left(f(t_i)\mu(I_i) - x^*(oM)\int_T f\right)\right|$$

$$=\left| x^* \sum_{i=1}^{p}\left(f(t_i)\mu(I_i) - (oM)\int_T f\right)\right|$$

$$\leq \|x^*\| \left|\sum_{i=1}^{p} f(t_i)\,\mu(I_i) - (oM)\int_T f\right| \leq \|x^*\| \,.\, b_n,$$

for every n and (γ_n) -fine M-partition $\{(I_i, t_i), i = 1, \dots, p\}$ of T

The same holds if (oM) is replaced by (oH) and H- partitions are used instead of M- partitions.

■

Remark. The (o)-McShane, (o)- Bochner (Lebesgue) integrals of $x^*(f): T \to \mathbb{R}$, coincide $(X = \mathbb{R})$ and therefore we can replace in proposition 2.2 the (o)-McShane integrability of $x^*(f): T \to \mathbb{R}$, by its (o)-Bochner (Lebesgue) integrability. Consequently, we also have that the function $f: T \to X$ is weakly measurable.

Theorem 2.3.

If $f: T \to X$ is (o)-McShane integrabile with $(oM)\int_T f \in X$, then f is also (oP)-integrable and

$$(oP)\int_E f = (oM)\int_T f\,\chi_E = (oM)\int_E f \qquad (13)$$

for every measurable $E \subset T$. Hence we have $(oM) \subset (oP)$.

Proof. According to the previous Remark the function $f: T \to X$ is weakly measurable. For every measurable set E⊂T the function $f.\chi_E$ is McShane integrable and :

$$(oM)\int_T f.\chi_E = (oM)\int_E f \in X.$$

Hence by Proposition 2.2 for every $x^* \in X^*$ the real function $x^*(f.\chi_E)$ is (o)-McShane integrable and

$$(oM)\int_T x^*(f.\chi_E) = (oM)\int_E x^* f = x^*\left(oM\int_E f\right).$$

By Definition 1.5 this implies that f is(o)-Pettis integrable

Theorem.2.4.

If $f_u: T \to X, u \in N$ are (o)- McSshane integrable functions such that

1. $f_u(t) \to f(t)$ for $t \in T$,
2. the set $\{f_u; u \in N\}$ forms an (oM)-equi-integrable sequence.

Then $f_u \,.\, \chi_E, u \in N$ is an (oM)-equi-integrable sequence for every measurable set $E \subset T$.

Proof. Then for every (o)- sequence $(b_n)_n$ there exists an $\eta > 0$ and assume that $E \subset T$ measurable.. Then there exist $F \subset T$ closed and $G \subset T$ open such $F \subset E \subset G$ where $\mu(G\backslash F) < \eta$. Assume that the sequence $(\gamma_n)_n$ corresponding gauge $\gamma_n: T \to]0, \infty[$ such that

$$B(t, (\gamma_n(t) \subset G \text{ for } t \in G,$$

$$B(t, (\gamma_n(t) \cap T \subset T\backslash F \text{ for } t \in T\backslash F$$

and that $\{(E_i, t_i)\}$, $\{(K_j, r_j)\}$, are γ_n-fine M-partitions of T.

We have

$$\text{if } t_i \in E \text{ then } E_i \subset G,\ F \subset \text{int}\bigcup_{t_i \in F} E_i$$

$$\text{if } r_j \in E \text{ then } K_j \subset G,\ F \subset \text{int}\bigcup_{r_j \in F} K_j$$

$$\left|\sum_{i, t_i \in E} f_u(t_i)\mu(E_i) - \sum_{j, r_j \in E} f_u(r_j)\mu(K_j)\right| \leq\ b_n$$

And therefore also

$$\left|\sum_i\ f_u(t_i)\chi_E(t_i)\mu(E_i) - \sum_j\ f_u(r_j)\chi_E(r_j)\mu(K_j)\right| \leq\ b_n$$

This is the Bolzano -Cauchy condition for (oM) equi-integrability of the sequence $f_u\ .\ \chi_E, u \in N$ and the proof is complete.

Theorem.2.5.

Let $f: T \to X$ be measurable. If f is (o)- Pettis integrable on T, then f is (o)- McShane integrable on T.

Proof. The measurability of f implies that there is a bounded measurable function $g: T \to X$ and a measurable function : $h: T \to X$

$$\text{h}(t) = \sum_{n=1}^{\infty} x_n\ \chi_{E_n}(t), \text{t} \in T$$

with $x_n \in X, E_n \subset T, n \in \mathbb{N}$, E_n being pairwise disjoint measurable sets such that:

$$f(t) = g(t) + h(t), \text{t} \in T.$$

The function g is bounded and measurable, therefore it is (oB)-integrable because T is assumed to be a compact interval. Therefore g is (o)- McShane integrable by Theorem 2.4. and by Theorem 2.3 it is also (o)-Pettis integrable. Since f i assumed to be (o)-Pettis integrable, the function $h = f - g$ must be (o)-Pettis integrable and therefore the series $\sum_{n=1}^{\infty} x_n\ \mu(E_n)$ converges unconditionally in X. The sequence $h_n(t) = \sum_{j=1}^{n} x_j \chi_{E_j}(t)$, t $\in T$ $n \in \mathbb{N}$ is (oM)-equi-integrable and it is easy to see that

$$\text{(o)-}\lim_{n\to\infty}\ h_n(t) = h(t) \text{ por t} \in T.$$

Then function h is (o)-McShane integrable

$(oM)\int_T h = \lim_{n\to\infty}(oM)\int_T h_n = \lim_{n\to\infty}\sum_{j=1}^{n} x_j\mu(E_j) = \sum_{j=1}^{\infty} x_j\mu(E_j)$

and therefore f is also (o)-McShane integrable.

Theorem 2.6.

Assume that the Banach lattice X is separable and that $f: T \to X$ is (o)-Pettis integrable. Then f is (o)-McShane integrable.

Proof . The (o) -Pettis integrability of f assumes that f is weakly measurable and the function f is measurable. Theorem 2.5. gives the (o) - McShane integrability of f .

Using Theorem 2.3 and Theorem 2.6 we obtain immediately the following result.

Corollary 2.7.

Assume that the Banach lattice X is separable. Then $f: T \to X$ is (o) - Pettis integrable if and only if it is (o) - McShane integrable,i.e. (oM) =(oP) holds in this case.

This fact has interesting consequences where (o) -Pettis integrability is stronger than Bochner (norme) one.

Theorem 2.8.(Hake)

Assume that $f: [a, b] \to X$ is (o)-Denjoy-Dunford integrable on $[a, t]$ for all $t \in [a, b[$ and for each $x^* \in X^*$ the limit (o) $- \lim_{t\to b}\int_a^t x^*(f)$ exists. Then f is (o)-Denjoy-Dunford integrable on $[a, b]$, and

$$x^*((oDD)\int_a^b f) = \text{(o)}\lim_{t\to b} x^*((oDD)\int_a^b f)$$

For each $x^* \in X^*$.

Proof. Since f is (o)- Denjoy-Dunford integrable on $[a, t]$ for all $t \in [a, b[$ and for each $x^* \in X^*$

the limit (o) $- \lim_{t\to b}\int_a^t x^*(f)$ exists, $x^*(f)$ is (o)- Denjoy integrable on $[a, b]$ for all $x^* \in X^*$.On the other hand ,take any (o)-sequence (t_n) in $[a, b[$ convergent to b.

Define $L(x^*) = (o)\lim_n \int_a^{t_n} x^*(f) = \text{(o)}\lim_n x^*((oDD)\int_a^{t_n} f)$.

The uniform boundedness principle guarantees that the linear functional L is continuous on X^*.Then it is immeditiate that f is (o)Denjoy-Dunford integrable on$[a, b]$.

Conclusion

İn this paper we have investigated the notions of Order gauge integrals for functions taking values in a Banach lattice with an order -continuous norm. We compare the norm -and order-type integral, which become more interesting in L-spaces. Though in case of Banach space-valued functions the stronger type of integral is the Bochner one which is stronger both of McShane and Pettis integrals, we have seen that in L-spaces the orde-type Pettis integrals is indeed a Bochner one.

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Invariance of Special Class of Rational Numbers

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Abstract

The 3x+1 Problem is one of the longstanding problems in mathematics [2]. It is about the following simple algorithm. Take a positive integer. If it is even then divide it by 2, otherwise i.e., if it is odd then multiply the number by 3 and add 1. Continue to do the same with the resulting number. This can be interpreted as a discrete dynamical system. For example, if you start with 7, then the next numbers are 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, …. As one can observe from this example the algorithm in the long run reaches 1. It is still unproven that for any starting number, eventually, there is a step where the algorithm touches 1. The other observation is that after 1 we obtain a periodic sequence 4, 2, 1, 4, 2, 1, … Again, it is still unproven that it is the only such possible cycle for this algorithm. In the current paper, a new approach for the 5x+1 incarnation of the second problem will be discussed. Instead of choosing a starting number and letting it dictate what operations are applied to the following numbers, a fixed sequence of Conway [1] type operations are chosen and the starting number, which returns to itself when the given operations are applied to it in the given order, is determined using these operations. Because of this change in the perspective, the starting number and the numbers which follow it are, in general, not always positive, and not always integer anymore. The obtained rational numbers enjoy some interesting divisibility properties, discussion, and proof of which is the main focus of the current paper. In some simpler cases these properties can also be expressed in the context of 5-adic representation of numbers where patterns of repeating digits appear and visualize those divisibility results.

Keywords: 3x+1 problem, Collatz problem, p-adic numbers, 5-adic numbers, periods, rational numbers, algorithm.

1. Introduction

For arbitrarily chosen initial positive integer x_0, the recursive sequence defined by

$$x_{n+1} = \begin{cases} \dfrac{3x_n+1}{2}, \text{if } x_n \text{ is an odd number} \\ \dfrac{x_n}{2}, \quad \text{if } x_n \text{ is an even number} \end{cases}$$

returns $x_N = 1$ for some positive integer N. Another observation is that after $x_N = 1$ it is periodic $1, 2, 1, 2, \ldots$. There are two non-trivial problems associated with this recursive sequence.

Problem 1. Prove that for any positive integer x_0 there exists a positive integer N such that $x_N = 1$.

Problem 2. Prove that the only periodic sequence that this recursive formula can produce is $1, 2, 1, 2, \ldots$..

Both problems remain unsolved. For more information about the history of the problem and the various attempts to solve it, we refer the reader to the encyclopedic book [2]. If x_0 and therefore the subsequent numbers are allowed to be negative or zero, then there are other possible cycles, but the essence of the above problems doesn't change much. Problem 1 can be stated to prove that $\lim_{n\to+\infty} |x_n| \neq +\infty$. Problem 2 can be stated to prove that there are only finitely many different periodic sequences that this recursive formula can generate.

We are currently interested only with Problem 2. Note that it is not clear at all, how the solution of Problem 2 will be helpful for the solution of Problem 1. Because, even if it is known that there is only one periodic sequence, it remains to prove that $\lim_{n\to+\infty} x_n \neq +\infty$.

When we choose x_0, it determines all the numbers x_n coming after x_0, and all the operations $\frac{3x_n+1}{2}$ and $\frac{x_n}{2}$ that are applied to them. Since these numbers and the corresponding operations appear chaotically, it is reasonable to fix finite number of operations first and then determine the number x_0, which returns to itself as a periodic sequence (cycle), using these operations. When we do this, we also determine the other numbers $x_1, \ldots, x_{n+m-1}$ that together with the number x_0 form a periodic sequence (cycle) $x_0, x_1, \ldots, x_{n+m-1}$. This problem in a more general setting was discussed in the recent paper of the author [3] in detail. In the current paper, we applied this idea to more general problem, involving Conway type operations $S_k(x) = \frac{5x+k}{2}$ and $T(x) = \frac{x}{2}$. John Conway used this type of operations in his paper [4] to prove some results about undecidability (see [1], [5] for new results in this direction). The results obtained in the current paper can be extended for any operation of the type $S_k(x) = \frac{px+k}{2}$ with arbitrary odd number $p > 2$ as a coefficient of x. The special case when $p = 3$ and $k = 0, 1$, or 2, was discussed in [3]. It is even possible to take several operations $S_k(x) = \frac{px+k}{2}$ with different p. But then the proof becomes more complicated and the visualizations, that we are going to present at the end of the current paper (see also [3]), are not so obvious. In that case one needs to switch from one base to another in p-

adic representations. So, to be able to explain the main idea of the proof, to keep the visuals simple and the notations short, we decided to constrain ourselves only to the case of the operations $S_k(x) = \frac{5x+k}{2}$.

We will see that the order of operations doesn't affect an invariant which is dependent only on the total number of S_k operations (m) and the number of $T(x) = \frac{x}{2}$ operations (n). The invariant that we are going to investigate is a pair of nonnegative integers (a, b), for which $5^a U_0 - U_b$ is an integer, where $U_i = \frac{2^i}{2^{n+m}-5^m}$. This is equivalent to say that $2^{n+m} - 5^m$ divides $5^a - 2^b$. Proof of the existence of such pairs (a, b), which is a pure number theoretic question, will not be discussed in the current paper. We will constrain ourselves by stating the obvious fact that one can always choose, for example, trivial pairs like $(a, b) = (\varphi(2^{n+m} - 5^m), 0)$, $(a, b) = (0,\ \varphi(2^{n+m} - 5^m))$ or $(a, b) = (m, n + m)$. As numerical experiments suggest, some of which are given in the current paper, there are infinitely many such pairs for any $m > 0$ and $n \geq 0$. The same remarks can be made for the existence of a pair of nonnegative integers (a, b), for which $5^a U_0 + U_b$ is an integer. Again, it is equivalent to say that $2^{n+m} - 5^m$ divides $5^a + 2^b$. Finally, we will also need a pair of nonnegative integers (a', b'), for which $5^{a'} U_{b'} \pm U_0$ is an integer. It is equivalent to say that $2^{n+m} - 5^m$ divides $5^a \pm 2^b$. For some reasons, that will be explained at the end of the paper, we are more interested with pairs (a, b), for which a is minimal and $b \leq n + m$. Further discussion of these divisibility problems should be a topic of another paper.

2. Operations and Cycles.

2.1. Operations.

Consider the set of operations $O = \left\{T(x) = \frac{x}{2}\right\} \cup \left\{S_k \mid S_k(x) = \frac{5x+k}{2},\ k \in Z\right\}$. Suppose that a sequence of operations $P = B_0 B_1 \dots B_{n+m-1}$ is given, where $B_i \in O$ for $i = 0,1,2, \dots, n + m - 1$. Suppose that n is the number of T operations and m is the total number of S_k operations for all k. We assume that $m > 0$ and $n \geq 0$. We will need also infinite extension of P defined by $B_i = B_j$ if $i \equiv j \pmod{n + m}$. In other words, we interpret P as a sequence infinite in both directions:

$$P = \cdots B_0 B_1 \dots B_{n+m-1} B_0 B_1 \dots B_{n+m-1} B_0 B_1 \dots B_{n+m-1} \cdots.$$

2.2. Cycles.

Consider the equation $B_0 B_1 \dots B_{n+m-2} B_{n+m-1}(x) = x$. Note that this is a linear equation and therefore its solution x_0 is a rational number. We are not going to write a formula for x_0, although it is possible. We just mention the fact that x_0 is completely defined by the given sequence of operations. In a similar way, let x_1 be the solution of $B_1 \dots B_{n+m-2} B_{n+m-1} B_0(x) = x$ $\left(x_0 = B_0(x_1)\right)$. The other numbers x_i are defined similar to these. The last number x_{n+m-1} is the solution of $B_{n+m-1} B_0 B_1 \dots B_{n+m-2}(x) = x$ $\left(x_{n+m-1} = B_{n+m-1}(x_0)\right)$. Again, we can extend the numbers x_i $(i = 0, 1, \dots, n + m - 1)$ using the equalities $x_i = x_j$ if $i \equiv j \pmod{n + m}$. This means that the numbers x_i are also interpreted as a sequence infinite in both directions:

$\dots, x_0, x_1, \dots, x_{n+m-1}, x_0, x_1, \dots, x_{n+m-1}, x_0, x_1, \dots, x_{n+m-1}, \dots$

3. Special numbers and Main Results.

3.1. Special numbers.

Let $U_i = \frac{2^i}{2^{n+m}-5^m}$, where $i = 0, 1, 2, \ldots, n+m$. There are infinitely many pairs of nonnegative integers (a, b), for which $5^a U_0 - U_b$ is an integer, or equivalently, $5^a U_i - U_{i+b}$ is an integer $(0 \le i \le i+b \le n+m)$. Similarly, there are infinitely many pairs of non-negative integers (a', b'), for which $5^{a'} U_{b'} - U_0$ is an integer, or equivalently, $5^{a'} U_{i+b'} - U_i$ is an integer $(0 \le i \le i+b' \le n+m)$. Let $\sigma(s, r)$, where $s \le r$, be the number of all S_k operations in the fragment $B_s B_{s+1} \ldots B_{r-1}$ of the infinitely extended sequence P. In particular $\sigma(s, s) = 0$, because $\sigma(s, s)$ corresponds to an empty fragment of the sequence P.

3.2. Main Results.

Theorem 1.

If the pair of nonnegative integers $(a,\ b)$ satisfy $5^a U_0 - U_b \in Z$ then for the numbers x_i defined above, the difference $5^a x_i - 5^{\sigma(i,i+b)} x_{i+b}$ is also an integer $(0 \le i \le i+b \le n+m)$.

Proof. The proof is based on the method of mathematical induction. We can first show that the claim is true for one particular sequence of operations consisted of only S_0 and T operations. Then we can prove that the correctness of the claim does not change when we change one S_0 operation in the sequence to S_1, or to S_{-1}. Similarly, the claim does not change when we change those S_1 and S_{-1} operations in the sequence to S_2 and S_{-2}, respectively. In this way we can obtain any sequence from a special sequence of operations consisted of only S_0 and T operations. Suppose now that all S_k operations in P are replaced by S_0. This means that the given sequence of operations is consisted of only S_0 and T operations. For example, $P = S_0 T S_0 S_0 S_0 T$. It is obvious that in this case $x_i = 0$ for all i, where $0 \le i \le n+m$. So, $5^a x_i - 5^{\sigma(i,i+b)} x_{i+b} = 0$ and therefore the claim is trivially true. Now, we will replace some of the S_0 operations by S_{-1} and prove that the claim remains true. Suppose that the claim is already true for a sequence of operations P. First, make a cyclic permutation of P so that the operation S_0 which will be changed to S_{-1}, comes at the beginning of P. For example, suppose we are going to change the operation S_0 with a bar above it in the sequence $P = S_0 T\ S_0\ \overline{S_0}\ S_0 T$. We do a cycling permutation to write it as $P_1 = \overline{S_0}\ S_0 T\ S_0 T\ S_0$. The numbers x_i $(0 \le i \le n+m)$ also make the same cyclic permutation. Let us rename them so that they are labelled again as x_i $(0 \le i \le n+m)$ with respect to P_1. Now consider the sequence of operations P_2 which is obtained from P_1 by replacing the S_0 operation in the beginning by S_{-1}. In the example, it will look like $P_2 = S_{-1}\ S_0 T\ S_0 T\ S_0$. Let us denote by x_i' $(0 \le i \le n+m)$ the rational numbers corresponding to the sequence of operations P_2.

Let us denote by $R_i = B_{n+m-i} \ldots B_{n+m-2} B_{n+m-1}$ the sequence formed by the last i operations in P_1. Denote by $R_i[b]$ the number obtained from application of the operations of the sequence R_i to some number b in the given order (from right to left). In particular, $R_0[b] = b$. As we said before, the sequence P_1 starts with S_0 and the sequence P_2 starts with S_{-1}, all the other operations are the same, that is $P_1 = S_0 R_{n+m-1}$ and $P_2 = S_{-1} R_{n+m-1}$. Since $P_1[x_0] = x_0$ and $P_2[x_0'] = x_0'$, we can write

$$x_0 = \frac{5R_{n+m-1}[x_0]}{2}, x_0' = \frac{5R_{n+m-1}[x_0'] - 1}{2},$$

and find the difference

$$x_0' - x_0 = \frac{5R_{n+m-1}[x_0' - x_0] - 1}{2} = \frac{5^m(x_0' - x_0)}{2^{n+m}} - \frac{1}{2}.$$

By solving this equation for $x_0' - x_0$, we obtain

$$x_0' = x_0 - U_{n+m-1}.$$

Using this we prove step by step that

$$x_{n+m-1}' = B_{n+m-1}(x_{n+m}') = B_{n+m-1}(x_0') =$$
$$= B_{n+m-1}(x_0 - U_{n+m-1}) =$$
$$= x_{n+m-1} - 5^{\sigma(n+m-1,n+m)} \cdot U_{n+m-2}.$$

Similarly,

$$x_{n+m-2}' = B_{n+m-2}(x_{n+m-1}') =$$
$$= B_{n+m-2}\left(x_{n+m-1} - 5^{\sigma(n+m-1,n+m)} \cdot U_{n+m-2}\right) =$$
$$= x_{n+m-2} - 5^{\sigma(n+m-2,n+m)} \cdot U_{n+m-3}.$$

By continuing in this manner, we obtain that

$$x_{n+m-k}' = x_{n+m-k} - 5^{\sigma(n+m-k,n+m)} \cdot U_{n+m-k-1},$$

for all $k = 0,1,2,\dots,n+m-1$. In particular, for $k = n+m-1$ we obtain $x_1' = x_1 - 5^{\sigma(1,n+m)} \cdot U_0$. Noting the fact that $\sigma(1, n+m) = m-1$, we can write $x_1' = x_1 - 5^{m-1} \cdot U_0$. It is also possible to write the above equalities with simpler indices: $x_i' = x_i - 5^{\sigma(i,n+m)} \cdot U_{i-1}$, where $i = 1,2,\dots,n+m$. Going one more step and using the fact that B_0 is not a T operation, we obtain again

$$x_0' = B_0(x_1') = B_0(x_1 - 5^{m-1} \cdot U_0) = x_0 - \frac{1}{2}(5^m \cdot U_0 + 1) = x_0 - U_{n+m-1}.$$

This can also be interpreted as

$$x_{n+m}' = x_{n+m} - U_{n+m-1}.$$

Suppose now that the pair of nonnegative integers (k, j) satisfy $5^a U_0 - U_b \in Z$ and for the numbers x_i $(0 \leq i \leq n+m)$ the difference $5^a x_i - 5^{\sigma(i,i+b)} x_{i+b}$ is known to be an integer. We want to prove that the last property holds true for the numbers x_i' $(0 \leq i \leq n+m)$, too. Assume first that $i > 0$. Using the above results, we can write

$$5^a x_i' - 5^{\sigma(i,i+b)} x_{i+b}' = 5^a \left(x_i - 5^{\sigma(i,n+m)} \cdot U_{i-1}\right) -$$

$$-5^{\sigma(i,i+b)} \cdot \left(x_{i+b} - 5^{\sigma(i+b,n+m)} \cdot U_{i+b-1}\right) =$$

$$= 5^a x_i - 5^{\sigma(i,i+b)} x_{i+b} - 5^{\sigma(i,n+m)} \cdot (5^a \cdot U_{i-1} - U_{i+b-1}),$$

which is an integer by assumption. For the case $i = 0$ we obtain

$$5^a x_0' - 5^{\sigma(0,b)} x_b' = 5^k (x_0 - U_{n+m-1}) - 5^{\sigma(0,b)} \cdot \left(x_b - 5^{\sigma(b,n+m)} \cdot U_{b-1}\right) =$$

$$= \left(5^a x_0 - 5^{\sigma(0,b)} x_b\right) - \left(5^a \cdot U_{n+m-1} - 5^{\sigma(0,n+m)} \cdot U_{b-1}\right),$$

where the summand $5^a x_0 - 5^{\sigma(0,b)} x_b$ is again an integer by assumption. For the second summand we can write

$$5^a \cdot U_{n+m-1} - 5^{\sigma(0,n+m)} \cdot U_{b-1} = 5^a \cdot U_{n+m-1} - 5^m \cdot U_{b-1} =$$

$$= \frac{5^a}{2}(5^m \cdot U_0 + 1) - 5^m \cdot U_{b-1} = \frac{5^m (5^a \cdot U_0 - U_b) + 5^a}{2},$$

which is again an integer, because $5^a \cdot U_0 - U_b$ is odd and therefore $5^m (5^a \cdot U_0 - U_b) + 5^a$ is an even number.

The case when S_0 operation is replaced by S_1 is considered in a similar way. Now, it is possible to prove that when S_1 operation is replaced by S_2, similarly, S_2 operation is replaced by S_3, etc. in general, S_k operation is replaced by S_{k+1}, then the property of $5^a x_i - 5^{\sigma(i,i+b)} x_{i+b}$ being an integer remains true. The same can be stated if S_{-1} operation is replaced by S_{-2}, S_{-2} operation is replaced by S_{-3}, etc. in general, S_k operation is replaced by S_{k-1}. These replacements of operations allow us to obtain any sequence of operations $P = B_0 B_1 \dots B_{n+m-1}$ from the sequence of operations consisted of only S_0 and T operations. After each replacement, the claim of the current theorem, which says that $5^a x_i - 5^{\sigma(i,i+b)} x_{i+b}$ is an integer, remains true. So, when we will reach the desired sequence of operations $P = B_0 B_1 \dots B_{n+m-1}$, the claim will be true for this sequence, too. Proof is complete.

The following three theorems are proved analogously.

Theorem 2.

If the pair of nonnegative integers $(a,\ b)$ satisfy $5^a U_0 + U_b \in Z$ then for the numbers x_i defined above, the sum $5^a x_i + 5^{\sigma(i,i+b)} x_{i+b}$ is also an integer.

Theorem 3.

If the pair of nonnegative integers (a', b'), for which $5^{a'}U_{b'} - U_0 \in Z$ then for the numbers x_i defined above, the difference $5^{a'+\sigma(i,i+b')}x_{i+b'} - x_i$ is also an integer $(0 \leq i \leq i + b' \leq n + m)$.

Theorem 4.

If the pair of nonnegative integers (a', b'), for which $5^{a'}U_{b'} + U_0 \in Z$ then for the numbers x_i defined above, the sum $5^{a'+\sigma(i,i+b')}x_{i+b'} + x_i$ is also an integer $(0 \leq i \leq i + b' \leq n + m)$.

4. Examples.

Consider the sequence of operations $P = B_0B_1B_2 = S_2TS_1$. Here $m = 2$ and $n = 1$. The solution of the linear equation $S_2TS_1(x) = x$ is the number $x_0 = -13/17$. Note that $x_0 = x_3$. We can also find the other numbers $x_1 = -12/17$, $x_2 = -24/17$. We also find the numbers $U_i = -2^i/17$ $(i = 0,1,2,3)$. Note that

I. $5^6 \cdot U_0 - U_1 = -919 \in Z$, ($a = 6$, $b = 1$ in Theorem 1)

II. $5^4 \cdot U_0 + U_2 = -37 \in Z$, ($a = 4$, $b = 2$ in Theorem 2)

III. $5^4 \cdot U_2 - U_0 = -147 \in Z$, ($a' = 4$, $b' = 2$ in Theorem 3)

IV. $5^2 \cdot U_1 + U_0 = -3 \in Z$. ($a' = 2$, $b' = 1$ in Theorem 4)

We observe that

I. $5^6 \cdot x_0 - 5^{\sigma(0,1)} \cdot x_1 = 5^6 \cdot \left(-\frac{13}{17}\right) - 5^1 \cdot \left(-\frac{12}{17}\right) = -11945 \in Z$. Note that $\sigma(0,1) = 1$ is the number of non-T operations in the fragment $B_0 = S_2$ of P, (cf. Theorem 1)

II. $5^4 \cdot x_0 + 3^{\sigma(0,2)} \cdot x_2 = 5^4 \cdot \left(-\frac{13}{17}\right) + 5^1 \cdot \left(-\frac{24}{17}\right) = -485 \in Z$, (cf. Theorem 2)

III. $5^{4+\sigma(0,2)} \cdot x_2 - x_0 = 5^{4+1} \cdot \left(-\frac{24}{17}\right) - \left(-\frac{13}{17}\right) = -4411 \in Z$, (cf. Theorem 3)

IV. $5^{2+\sigma(1,2)} \cdot x_2 + x_1 = 5^{2+0} \cdot \left(-\frac{24}{17}\right) + \left(-\frac{12}{17}\right) = -36 \in Z$. (cf. Theorem 4)

These observations are in perfect agreement with the main results of the current paper.

5. Visualization: 5-adic representations.

In some simple cases it is possible to give a visual representation for the above results. Let us return to the example $P = B_0B_1B_2 = S_2TS_1$. Let us try to visualise the fact that $5^6 \cdot x_0 - 5^{\sigma(0,1)} \cdot x_1 \in Z$ which was given as an example for Theorem 1. We will write the obtained 5-adic representations from right to left and omit powers of 5. Also, if a non-T operation of P is applied to that number, then we shift that number and all the numbers above it, to the left by one digit. The non-T operation applied to that number is also indicated at the end of the number, but it is separated from the number by a line.

$$\begin{array}{llllll}
x_3 = & \ldots & 0\ 2\ 4\ 3\ 2\ 3\ 1\ 0\ 4\ 2\ 0\ 1\ 2\ 1 & |1 & & S_1 = B_2 \\
x_2 = & \ldots & 0\ 1\ 2\ 1\ 3\ 4\ 0\ 2\ 4\ 3\ 2\ 3\ 1\ 0\ 3 & & & T = B_1 \\
x_1 = & \ldots & \boxed{2\ 3\ 1\ 0}\ 4\ 2\ 0\ 1\ 2\ 1\ 3\ 4\ 0\ 2\ 4 & |2 & & S_2 = B_0 \\
x_0 = & \ldots & 3\ 4\ 0\ 2\ 4\ 3\ \boxed{2\ 3\ 1\ 0}\ 4\ 2\ 0\ 1\ 2\ 1 & & &
\end{array}$$

Each of the rows should be periodic because they are 5-adic representations of rational numbers. But there are also repeated digits in the different rows. The fact that $5^6 \cdot x_0 - 5^{\sigma(0,1)} \cdot x_1 \in Z$ can be explained by the appearance of the same digits in 5-adic representations of x_0 and x_1. The similarities are indicated using rectangular boxes. Note that the size of the boxes in this and the following constructions is not important. The same pattern is true for all the other rows of the above construction. In all the rows the boxes have the same horizontal shift ($a = 6$), and the same vertical shift ($b = 1$).

$$\begin{array}{llllll}
x_3 = & \ldots & 0\ 2\ 4\ 3\ 2\ 3\ 1\ 0\ 4\ 2\ 0\ 1\ 2\ 1 & |1 & & S_1 = B_2 \\
x_2 = & \ldots & 0\ \boxed{1\ 2\ 1\ 3}\ 4\ 0\ 2\ 4\ 3\ 2\ 3\ 1\ 0\ 3 & & & T = B_1 \\
x_1 = & \ldots & 2\ 3\ 1\ 0\ 4\ 2\ 0\ \boxed{1\ 2\ 1\ 3}\ 4\ 0\ 2\ 4 & |2 & & S_2 = B_0 \\
x_0 = & \ldots & 3\ 4\ 0\ 2\ 4\ 3\ 2\ 3\ 1\ 0\ 4\ 2\ 0\ 1\ 2\ 1 & & &
\end{array}$$

For Theorem 2, it is possible to give a similar visualization. The fact that $5^4 \cdot x_0 + 5^{\sigma(0,2)} \cdot x_2 \in Z$ can be explained by appearance of opposite digits in the rows of the construction. By opposite digits we mean the digits which complete each other to 4.

$$
\begin{array}{lll}
x_3 = & \dots\ 0\ 2\ 4\ 3\ 2\ 3\ 1\ 0\ 4\ 2\ 0\ 1\ 2\ 1\ |1 & S_1 = B_2 \\
x_2 = & \dots\ 0\ 1\ \boxed{2\ 1\ 3\ 4}\ 0\ 2\ 4\ 3\ 2\ 3\ 1\ 0\ 3 & T = B_1 \\
x_1 = & \dots\ 2\ 3\ 1\ 0\ 4\ 2\ 0\ 1\ 2\ 1\ 3\ 4\ 0\ 2\ 4\ |2 & S_2 = B_0 \\
x_0 = & \dots\ 3\ 4\ 0\ 2\ 4\ 3\ \boxed{2\ 3\ 1\ 0}\ 4\ 2\ 0\ 1\ 2\ 1 &
\end{array}
$$

Again, it is not important where the boxes are put, how large the boxes are, the only restriction is that they have the same horizontal shift ($a = 4$), and the same vertical shift ($b = 2$). Also, the boxes should not be too close to the right side of the construction where this and the above patterns can fail.

$$
\begin{array}{lll}
x_3 = & \dots\ 0\ 2\ 4\ 3\ \boxed{2\ 3\ 1\ 0\ 4}\ 2\ 0\ 1\ 2\ 1\ |1 & S_1 = B_2 \\
x_2 = & \dots\ 0\ 1\ 2\ 1\ 3\ 4\ 0\ 2\ 4\ 3\ 2\ 3\ 1\ 0\ 3 & T = B_1 \\
x_1 = & \dots\ 2\ 3\ 1\ 0\ 4\ 2\ 0\ 1\ \boxed{2\ 1\ 3\ 4\ 0}\ 2\ 4\ |2 & S_2 = B_0 \\
x_0 = & \dots\ 3\ 4\ 0\ 2\ 4\ 3\ 2\ 3\ 1\ 0\ 4\ 2\ 0\ 1\ 2\ 1 &
\end{array}
$$

In the previous visualizations the boxes were traversed in the "northwest" direction. For Theorem 3 and Theorem 4 these boxes are traversed in the "northeast" direction. For Theorem 3 we gave the example $5^{4+\sigma(0,2)} \cdot x_2 - x_0 \in Z$. It can be interpreted as the appearance of the same digits in the following boxes.

$$
\begin{array}{lll}
x_3 = & \dots\ 0\ 2\ 4\ 3\ 2\ 3\ 1\ 0\ 4\ 2\ 0\ 1\ 2\ 1\ |1 & S_1 = B_2 \\
x_2 = & \dots\ 0\ 1\ 2\ 1\ 3\ \boxed{4\ 0\ 2\ 4\ 3}\ 2\ 3\ 1\ 0\ 3 & T = B_1 \\
x_1 = & \dots\ 2\ 3\ 1\ 0\ 4\ 2\ 0\ 1\ 2\ 1\ 3\ 4\ 0\ 2\ 4\ |2 & S_2 = B_0 \\
x_0 = & \dots\ 3\ \boxed{4\ 0\ 2\ 4\ 3}\ 2\ 3\ 1\ 0\ 4\ 2\ 0\ 1\ 2\ 1 &
\end{array}
$$

In the above construction the horizontal shift is $a' = 4$, and the vertical shift b' =2. In the following constraction, which visualises Theorem 4 with the example $5^{2+\sigma(1,2)} \cdot x_2 + x_1 \in Z$, the horizontal shift is $a' = 2$, and the vertical shift $b' = 1$.

$$
\begin{array}{ll}
x_3 = \; \ldots\; 0\;2\;4\;3\;2\;3\;1\;0\;4\;2\;0\;1\;2\;1\;|1 & S_1 = B_2 \\
x_2 = \; \ldots\; 0\;1\;2\;\boxed{1\;3\;4\;0\;2}\;4\;3\;2\;3\;1\;0\;3 & T = B_1 \\
x_1 = \; \ldots\; 2\;\boxed{3\;1\;0\;4\;2}\;0\;1\;2\;1\;3\;4\;0\;2\;4\;|2 & S_2 = B_0 \\
x_0 = \; \ldots\; 3\;4\;0\;2\;4\;3\;2\;3\;1\;0\;4\;2\;0\;1\;2\;1 &
\end{array}
$$

Note that in all the above examples we chose (a, b), so that a is minimal and $b \leq n + m = 3$. Minimality of a makes it possible to avoid further extension to the left side of these already long constructions. Another advantage is the decrease of stopping time if the algorithm is designed to determine if the numbers x_i are all integers or not. More about these integer cases in the Conclusion below. The limitation $b \leq n + m$ is useful because one doesn't need to extend the above constructions vertically to see the arising patterns. Otherwise, it is necessary to put several copies of the above constructions to see the pattern of repeating or opposing digits as in the following construction.

$$
\begin{array}{ll}
x_3 = \; \ldots\; 0\;2\;4\;3\;2\;3\;1\;\boxed{0\;4\;2\;0\;1}\;2\;1\;|1 & S_1 = B_2 \\
x_2 = \; \ldots\; 0\;1\;2\;1\;3\;4\;0\;2\;4\;3\;2\;3\;1\;0\;3 & T = B_1 \\
x_1 = \; \ldots\; 2\;3\;1\;0\;4\;2\;0\;1\;2\;1\;3\;4\;0\;2\;4\;|2 & S_2 = B_0 \\
x_0 = \; \ldots\; 3\;4\;0\;2\;4\;3\;2\;3\;1\;0\;4\;2\;0\;1\;2\;1\;|1 & S_1 = B_2 \\
\quad x_2 = \; \ldots\; 0\;1\;2\;1\;3\;4\;0\;2\;4\;3\;2\;3\;1\;0\;3 & T = B_1 \\
\quad x_1 = \; \ldots\; 2\;3\;1\;\boxed{0\;4\;2\;0\;1}\;2\;1\;3\;4\;0\;2\;4\;|2 & S_2 = B_0 \\
\quad x_0 = \; \ldots\; 3\;4\;0\;2\;4\;3\;2\;3\;1\;0\;4\;2\;0\;1\;2\;1 &
\end{array}
$$

The above construction visualises the fact that $5^{2+\sigma(1,6)} \cdot x_6 - x_1 = 5^{2+3} \cdot x_0 - x_1 = 5^5 \cdot \left(-\frac{13}{17}\right) - \left(-\frac{24}{17}\right) = -2389$ is an integer.

6. Conclusion

We proved that for the numbers x_i, there are invariants determined by the numbers U_i, which are not dependent, to some extent, on the order and nature of operations in P. The main results of the paper can be nicely visualized if the numbers x_i and U_i are represented as 5-adic numbers. These visualizations show that there are patterns in the p-adic representations which can be studied further and in more detail. Special cases of these visualizations about 3-adic representations can be found in [3]. These constructions consisted of rows of 5-adic numbers can be interpreted as new mathematical objects. These constructions deserve to be investigated more systematically in the future. The main results of the paper can be further generalized if the number 5 is replaced by any prime $p > 2$.

There are sequences of operation P for which the numbers x_i are all integers, e.g., $P = S_3S_3TTT$.

$x_5 =$	$3 =$	$\cdots$	0	0	0	0	0	0	0	0	0	0	0	0	0	3	$\vert 3$		$S_3 = B_4$
$x_4 =$	$9 =$	$\cdots$	0	0	0	0	0	0	0	0	0	0	0	0	0	1	4	$\vert 3$	$S_3 = B_3$
$x_3 =$	$24 =$	$\cdots$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4	4	$T = B_2$
$x_2 =$	$12 =$	$\cdots$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	2	$T = B_1$
$x_1 =$	$6 =$	$\cdots$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	$T = B_0$
$x_0 =$	$3 =$	$\cdots$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3	

I leave as an exercise to write the above construction for the well-known cycle {13, 33, 83, 208, 104, 52, 26} corresponding to the sequence of operations $P = S_1S_1S_1TTTT$. This cycle is usually given as a counterexample for "$5x + 1$" incarnation of $3x + 1$ problem. Another such counterexample is {17, 43, 108, 54, 27, 68, 34} which corresponds to $P = S_1S_1TTS_1TT$.

The main motivation behind the current study was the determination of all such sequences P for which all the numbers x_i are integers. For now, this task is incomplete, and it should be a topic for the future papers. The fact that the numbers x_i in the above construction are all integers, is evident from the presence of zeros in their 5-adic representations. But note that if the integers x_i are negative then the zeros will be replaced by 4s. The above observations about the presence of patterns of repeating digits (indicated by boxes) in the examples of Theorem 1 and Theorem 3, are trivially true here, because all the digits are the same. So, there is a hope that better understanding of the nature of the patterns in these constructions will be helpful for the determination and classification of all such integer cases.

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On Finitely e-Supplemented Modules

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Abstract

In this work, some new properties of finitely e-supplemented (briefly, fe-supplemented) modules are investigated. All rings have unity and all modules are unitary left modules, in this work. Let M be an fe-supplemented R-module and N be a finitely generated submodule of M. Then M/N is fe-supplemented.

Keywords: Small Submodules, Essential Submodules, Supplemented Modules, f-Supplemented Modules.

2020 Mathematics Subject Classification: 16D10, 16D70.

1. INTRODUCTION

Throughout this paper all rings will be associative with identity and all modules will be unital left modules.

Let R be a ring and M be an R-module. We will denote a submodule N of M by $N \leq M$. Let M be an R-module and $N \leq M$. If there exists a submodule K of M such that $M=N+K$ and $N \cap K=0$, then N is called a *direct summand* of M and it is denoted by $M=N \oplus K$. For any R-module M, we have $M=M \oplus 0$. Let M be an R-module and $N \leq M$. If $L=M$ for every submodule L of M such that $M=N+L$, then N is called a *small* (or *superfluous*) submodule of M and denoted by $N \ll M$. Let M be an R-module. M is called a *hollow* module if every proper submodule of M is small in M. M is called a *local* module if M has the largest submodule, i.e. a proper submodule which contains all other proper submodules. A submodule N of an R-module M is called an *essential* submodule and denoted by $N \trianglelefteq M$ in case $K \cap N \neq 0$ for every submodule $K \neq 0$, or equvalently, $N \cap L=0$ for $L \leq M$ implies that $L=0$. Let M be an R-module and $U,V \leq M$. If $M=U+V$ and V is minimal with respect to this property, or equivalently, $M=U+V$ and $U \cap V \ll V$, then V is called a *supplement* of U in M. M is called a *supplemented* module if every submodule of M has a supplement in M. M is said to be *essential supplemented* (briefly, *e-supplemented*) if every essential submodule of M has a supplement in M. M is said to be *finitely supplemented* (briefly, *f-supplemented*) if every finitely generated submodule of M has a supplement in M. Let M be an R-module and $U \leq M$. If for every $V \leq M$ such that $M=U+V$, U has a supplement X with $X \leq V$, we say U has *ample supplements* in M. If every submodule of M has ample supplements in M, then M is called an *amply supplemented* module. If every essential submodule of M has ample supplements in M, then M is called an *amply essential supplemented* module. The intersection of maximal submodules of an R-module M is called the *radical* of M and denoted by $RadM$. If M have no maximal submodules, then we denote $RadM=M$. A module M is said to be *noetherian*, if every submodule of M is finitely generated.

More informations about (amply) supplemented modules are in [1], [5] and [6]. More details about (amply) essential supplemented modules are in [3] and [4].

Lemma 1.1. Let M be an R-module.

(1) If $K\leq L\leq M$, then $K\trianglelefteq M$ if and only if $K\trianglelefteq L\trianglelefteq M$.

(2) Let N be an R-module and $f : M\rightarrow N$ be an R-module homomorphism. If $K\trianglelefteq N$, then $f^{-1}(K)\trianglelefteq M$.

(3) For $N\leq K\leq M$, if $K/N\trianglelefteq M/N$, then $K\trianglelefteq M$.

(4) If $K_1\trianglelefteq L_1\leq M$ and $K_2\trianglelefteq L_2\leq M$, then $K_1\cap K_2\trianglelefteq L_1\cap L_2$.

(5) If $K_1\trianglelefteq M$ and $K_2\trianglelefteq M$, then $K_1\cap K_2\trianglelefteq M$.

Proof. See [5, 17.3].

Lemma 1.2. Let M be an R-module. The following assertions are hold.

(1) If $K\leq L\leq M$, then $L\ll M$ if and only if $K\ll M$ and $L/K\ll M/K$.

(2) Let N be an R-module and $f : M\rightarrow N$ be an R-module homomorphism. If $K\ll M$, then $f(K)\ll N$. The converse is true if f is an epimorphism and $Kef\ll M$.

(3) If $K\ll M$, then $(K+L)/L\ll M/L$ for every $L\leq M$.

(4) If $L\leq M$ and $K\ll L$, then $K\ll M$.

(5) If $K_1,K_2,...,K_n\ll M$, then $K_1+K_2+...+K_n\ll M$.

(6) Let $K_1,K_2,...,K_n,L_1,L_2,...,L_n\leq M$. If $K_i\ll L_i$ for every $i=1,2,...,n$, then $K_1+K_2+...+K_n\ll L_1+L_2+...+L_n$.

Proof. See [1, 2.2] and [5, 19.3].

Lemma 1.3. Let M be an R-module. The following assertions are hold.

(1) $RadM$ is the sum of all small submodules of M.

(2) Let N be an R-module and $f : M\rightarrow N$ be an R-module homomorphism. Then $f(RadM)\leq RadN$. If $Kef\leq RadM$, then $f(RadM)=Radf(M)$.

(3) If $N\leq M$, then $RadN\leq RadM$.

(4) For $K,L\leq M$, $RadK+RadL\leq Rad(K+L)$.

(5) $Rx\ll M$ for every $x\in RadM$.

Proof. See [5, 21.5 and 21.6].

2. FINITELY e-SUPPLEMENTED MODULES

Definition 2.1. Let M be an R-module. If every finitely generated essential submodule of M has a supplement in M or M have no finitely generated essential submodules, then M is called a *finitely e-supplemented* (or briefly, *fe-supplemented*) module. (See also [2])

Lemma 2.2. Every f-supplemented module is fe-supplemented.
Proof. Let M be a f-supplemented module. Then every finitely generated submodule of M has a supplement in M. By this, every finitely generated essential submodule of M has a supplement in M. Hence M is fe-supplemented, as required.

Corollary 2.3. Let M be an R-module and $L\ll M$. If M is f-supplemented, then M/L is fe-supplemented.

Proof. Since M is f-supplemented and $L\ll M$, by [5, 41.3 (2) (i)], M/L is f-supplemented. Then by Lemma 2.2, M/L is fe-supplemented.

Corollary 2.4. Let M be an R-module and L be a finitely generated submodule of M. If M is f-supplemented, then M/L is fe-supplemented.
Proof. Since M is f-supplemented and L is a finitely generated submodule of M, by [5, 41.3 (2) (i)], M/L is f-supplemented. Then by Lemma 2.2, M/L is fe-supplemented.

Proposition 2.5. Let M be a fe-supplemented R-module. If every nonzero finitely generated submodule of M is essential in M, then M is f-supplemented.
Proof. Let U be a finitely generated submodule of M. If U=0, then M is a supplement of U in M. If $U\neq 0$, then by hypothesis U is a finitely generated essential submodule of M and since M is fe-supplemented, U has a supplement in M. Hence M is f-supplemented.

Lemma 2.6. Let M be a fe-supplemented R-module and N be a finitely generated submodule of M. Then M/N is fe-supplemented.

Proof. Let K/N be a finitely generated essential submodule of M/N. Since $K/N\trianglelefteq M/N$, by Lemma 1.1, $K\trianglelefteq M$. Since K/N and N are finitely generated, K is also finitely generated. Since M is fe-supplemented, K has a supplement T in M. Then by [5, 41.1 (7)], $(T+N)/N$ is a supplement of K/N in M/N. Hence M/N is fe-supplemented.

Corollary 2.7. Let M be a fe-supplemented R-module and N be a cyclic submodule of M. Then M/N is fe-supplemented.
Proof. Since N is cyclic, then N is finitely generated. Then by Lemma 2.6, M/N is fe-supplemented, as desired.

Corollary 2.8. Let $f : M \to N$ be an R-module epimomorphism and Kef be finitely generated. If M is fe-supplemented, then N is also fe-supplemented.
Proof. Clear from Lemma 2.6.

Corollary 2.9. Let $f : M \to N$ be an R-module epimomorphism with cyclic kernel. If M is fe-supplemented, then N is also fe-supplemented.
Proof. Clear from Corollary 2.8.

Proposition 2.10. Every supplemented module is fe-supplemented.
Proof. Clear, since every supplemented module is f-supplemented.

Proposition 2.11. Every amply supplemented module is fe-supplemented.
Proof. Clear from Proposition 2.10, since every amply supplemented module is supplemented.

Proposition 2.12. Hollow and local modules are fe-supplemented.
Proof. Clear from Proposition 2.10, since hollow and local modules are supplemented.

Proposition 2.13. Every essential supplemented module is fe-supplemented.
Proof. Clear from definitions.

Proposition 2.14. Every amply essential supplemented module is fe-supplemented.
Proof. Clear from Proposition 2.13, since every amply essential supplemented module is essential supplemented.

3. CONCLUSION

fe-supplemented modules are more general than f-supplemented modules.

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On eg-Supplemented Modules

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Abstract

In this work, some new properties of eg-supplemented modules are investigated. All rings have unity and all modules are unitary left modules. It is clear that every essential supplemented module is eg-supplemented. Hence eg-supplemented modules are more general than essential supplemented modules.

Keywords: g-Small Submodules, Essential Submodules, Supplemented Modules, g-Supplemented Modules.

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1. INTRODUCTION

Throughout this paper all rings are associative with identity and all modules are unital left modules.

Let R be a ring and M be an R-module. We denote a submodule N of M by $N \leq M$. Let M be an R-module and $N \leq M$. If there exists a submodule L of M such that $M=N+L$ and $N \cap L=0$, then N is called a *direct summand* of M and denoted by $M=N \oplus L$. Let M be an R-module and $N \leq M$. If $L=M$ for every submodule L of M such that $M=N+L$, then N is called a *small* (or *superfluous*) submodule of M and denoted by $N \ll M$. A module M is said to be *hollow* if every proper submodule of M is small in M. M is said to be *local* if M has a proper submodule which contains all proper submodules. A submodule N of an R-module M is called an *essential* submodule, denoted by $N \trianglelefteq M$, in case $K \cap N \neq 0$ for every submodule $K \neq 0$, or equvalently, $N \cap L=0$ for $L \leq M$ implies that $L=0$. Let M be an R-module and K be a submodule of M. K is called a *generalized small* (briefly, *g-small*) submodule of M if for every essential submodule T of M with the property $M=K+T$ implies that $T=M$, we denote this by $K \ll_g M$. A module M is said to be *generalized hollow* (briefly, *g-hollow*) if every proper submodule of M is g-small in M. It is clear that every small submodule is a generalized small but the converse is not true generally. Let M be an R-module and $U,V \leq M$. If $M=U+V$ and V is minimal with respect to this property, or equivalently, $M=U+V$ and $U \cap V \ll V$, then V is called a *supplement* of U in M. M is said to be *supplemented* if every submodule of M has a supplement in M. M is said to be *essential supplemented* (briefly, *e-supplemented*) if every essential submodule of M has a supplement in M. Let M be an R-module and $U\leq M$. If for every $V\leq M$ such that $M=U+V$, U has a supplement X with $X\leq V$, we say U has *ample supplements* in M. M is said to be *amply supplemented* if every submodule of M has ample supplements in M. M is said to be *amply essential supplemented* (briefly, *amply e-supplemented*) if every essential submodule of M has ample supplements in M. Let M be an R-module and $U,V\leq M$. If $M=U+V$ and $M=U+T$ with $T\trianglelefteq V$ implies that $T=V$, or equivalently, $M=U+V$ and $U\cap V\ll_g V$, then V is called a *g-supplement* of U in M. M is said to be *g-supplemented* if every submodule of M has a g-supplement in M. Let M be an R-module and $U\leq M$. If for every $V\leq M$ such that $M=U+V$, U has a g-supplement X with $X\leq V$, we say U has *ample g-supplements* in M. M is said to be *amply g-supplemented* if every submodule of M has ample g-supplements in M. The intersection of maximal submodules of an R-module M is called the *radical* of M and denoted by $RadM$. If M have no maximal submodules, then we denote $RadM=M$. The intersection of essential maximal submodules of an R-module M is called the *generalized radical* (briefly, *g-radical*) of M and denoted by Rad_gM. If M have no essential maximal submodules, then we denote $Rad_gM=M$.

More details about (amply) supplemented modules are in [1], [8] and [9]. More details about (amply) essential supplemented modules are in [6] and [7]. More informations about g-small submodules and g-supplemented modules are in [2], [3] and [4].

Lemma 1.1. Let M be an R-module.

(1) If $K\leq L\leq M$, then $K\trianglelefteq M$ if and only if $K\trianglelefteq L\trianglelefteq M$.

(2) Let N be an R-module and $f: M\rightarrow N$ be an R-module homomorphism. If $K\trianglelefteq N$, then $f^{-1}(K)\trianglelefteq M$.

(3) For $N\leq K\leq M$, if $K/N\trianglelefteq M/N$, then $K\trianglelefteq M$.

(4) If $K_1\trianglelefteq L_1\leq M$ and $K_2\trianglelefteq L_2\leq M$, then $K_1\cap K_2\trianglelefteq L_1\cap L_2$.

(5) If $K_1\trianglelefteq M$ and $K_2\trianglelefteq M$, then $K_1\cap K_2\trianglelefteq M$.

Proof. See [8, 17.3].

Lemma 1.2. Let M be an R-module. The following assertions are hold.

(1) If $K\leq L\leq M$, then $L\ll M$ if and only if $K\ll M$ and $L/K\ll M/K$.

(2) Let N be an R-module and $f: M\rightarrow N$ be an R-module homomorphism. If $K\ll M$, then $f(K)\ll N$. The converse is true if f is an epimorphism and $Kef\ll M$.

(3) If $K\ll M$, then $(K+L)/L\ll M/L$ for every $L\leq M$.

(4) If $L\leq M$ and $K\ll L$, then $K\ll M$.

(5) If $K_1,K_2,...,K_n \ll M$, then $K_1+K_2+...+K_n \ll M$.

(6) Let $K_1,K_2,...,K_n,L_1,L_2,...,L_n \leq M$. If $K_i \ll L_i$ for every $i=1,2,...,n$, then $K_1+K_2+...+K_n \ll L_1+L_2+...+L_n$.

Proof. See [1, 2.2] and [8, 19.3].

Lemma 1.3. Let M be an R-module. The following assertions are hold.

(1) Every small submodule in M is g-small in M.

(2) If $K \leq L \leq M$ and $L \ll_g M$, then $K \ll_g M$ and $L/K \ll_g M/K$.

(3) Let N be an R-module and $f : M \to N$ be an R-module homomorphism. If $K \ll_g M$, then $f(K) \ll_g N$.

(4) If $K \ll_g M$, then $(K+L)/L \ll_g M/L$ for every $L \leq M$.

(5) If $L \leq M$ and $K \ll_g L$, then $K \ll_g M$.

(6) If $K_1,K_2,...,K_n \ll_g M$, then $K_1+K_2+...+K_n \ll_g M$.

(7) Let $K_1,K_2,...,K_n,L_1,L_2,...,L_n \leq M$. If $K_i \ll_g L_i$ for every $i=1,2,...,n$, then $K_1+K_2+...+K_n \ll_g L_1+L_2+...+L_n$.

Proof. See [2] and [3].

2. ESSENTIAL g-SUPPLEMENTED MODULES

Definition 2.1. Let M be an R-module. If every essential submodule of M has a g-supplement in M, then M is called an *essential g-supplemented* (or briefly, *eg-supplemented*) module. (See [5])

Proposition 2.2. Every essential supplemented module is eg-supplemented.

Proof. Clear from definitions.

Proposition 2.3. Every factor module of an essential supplemented module is eg-supplemented.

Proof. Clear from Proposition 2.2, since every factor module of an essential supplemented module is essential supplemented.

Proposition 2.4. Every homomorphic image of an essential supplemented module is eg-supplemented.

Proof. Clear from Proposition 2.2, since every homomorphic image of an essential supplemented module is essential supplemented.

Proposition 2.5. The finite sum of essential supplemented modules is eg-supplemented.

Proof. Clear from Proposition 2.2, since the finite sum of essential supplemented modules is essential supplemented.

Proposition 2.6. The finite direct sum of essential supplemented modules is eg-supplemented.

Proof. Clear from Proposition 2.5.

Proposition 2.7. Let M be an essential supplemented module. Then every finitely M-generated module is eg-supplemented.
Proof. Clear from Proposition 2.2, since every finitely M-generated module is essential supplemented.

Proposition 2.8. Let R be any ring. If $_RR$ is essential supplemented, then every finitely generated R-module is eg-supplemented.
Proof. Clear from Proposition 2.7.

Proposition 2.9. Hollow and local modules are eg-supplemented.
Proof. Clear from definitions.

Proposition 2.10. Every g-hollow module is eg-supplemented.
Proof. Clear from definitions.

Proposition 2.11. Every supplemented module is eg-supplemented.
Proof. Clear from definitions.

Proposition 2.12. Every factor module of a supplemented module is eg-supplemented.
Proof. Clear from Proposition 2.11, since every factor module of a supplemented module is supplemented.

Proposition 2.13. Every homomorphic image of a supplemented module is eg-supplemented.
Proof. Clear from Proposition 2.11, since every homomorphic image of a supplemented module is supplemented.

Proposition 2.14. The finite sum of supplemented modules is eg-supplemented.
Proof. Clear from Proposition 2.11, since the finite sum of supplemented modules is supplemented.

Proposition 2.15. The finite direct sum of supplemented modules is eg-supplemented.
Proof. Clear from Proposition 2.14.

Lemma 2.16. Every g-supplemented module is eg-supplemented.
Proof. Clear from definitions.

Corollary 2.17. Every factor module of a g-supplemented module is eg-supplemented.

Proof. Let M be a g-supplemented module and M/N be a factor module of M. By [2, Theorem 2], M/N is g-supplemented. Then by Lemma 2.16, M/N is eg-supplemented.

Corollary 2.18. The homomorphic image of a g-supplemented module is eg-supplemented.
Proof. Clear from Corollary 2.17.

3. CONCLUSION

eg-supplemented modules are more general than essential supplemented modules.

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Feedback Control for Attractor Dynamics in the Epileptor Model

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Abstract

The variety of dynamic mechanisms leading to epileptic behavior in the human brain demands the development of different theoretical approaches to the modeling of this disease. The Epileptor is a generic phenomenological model recently developed to describe fast-slow limit cycles in the dynamics of seizures. To make the control over the co-existing attractors in the Epileptor model we propose here the algorithm based on Kolesnikov's "synergetic" target attractor feedback. We investigate the pros and cons of our approach to compare with other methods and discuss the perspectives of the further development of the control algorithm for modeling the real epileptic processes *in vivo*.

Keywords: Epileptor, nonlinear feedback, dynamical attractors.

1. Introduction

The mathematical modeling of epilepsy involves the set of fundamental concepts from the non-linear theory of dynamical systems and control theory: stability analysis, multistability, bifurcations, attractors, feedback driving of the system dynamics, and others [1]. Epileptiform seizes are modeled at all scales of the brain dynamics: micro-, meso- and macro-. The detailed investigation of different dynamical regimes in a given model is important to establish a taxonomy of the seizes [2] while the development of new control algorithms serves for detecting pre-ictal and ictal phases and their subsequent suppressing [3].

Here we investigate the application of non-linear control feedback to one of the mesoscopic models of epilepsy. The Epileptor is a generic phenomenological model recently developed to describe fast-slow limit cycles in the dynamics of seizures [4]. To make the control over the co-existing attractors in the Epileptor model we propose here the algorithm based on Kolesnikov's "synergetic" target attractor feedback [5]. We discuss the pros and cons of our approach to compare with other methods and the perspectives of the further development of the control algorithm for modeling the real epileptic processes *in vivo*.

2. Neural Mass (Mesoscopic) Epileptor Model

Differential Equations for the Epileptor Model

The Epileptor system is a neural mass (mesoscopic) model of partial seizures [4]. Its five state variables $\{x_1, y_1, x_2, y_2, z\}$ comprise the set of ordinary differential equations [6], and includes Subsystem 1 for the fast discharges:

$$\begin{aligned} \frac{dx_1}{dt} &= y_1 - f_1(x_1, x_2) - z + I_{\text{ext}1} \; ; \\ \frac{dy_1}{dt} &= c_1 - d_1 x_1^2 - y_1 \; , \end{aligned} \tag{1}$$

Subsystem 2 for the spike-wave events (SWEs) [4]:

$$\begin{aligned} \frac{dx_2}{dt} &= -y_2 + x_2 - x_2^3 + I_{\text{ext}2} + 0.002 g(x_1) - 0.3(z - 3.5) \; ; \\ \frac{dy_2}{dt} &= \frac{1}{\tau_2}\left(-y_2 + f_2(x_2)\right) , \end{aligned} \tag{2}$$

and the variable z for the slow time-scale evolution:

$$\frac{dz}{dt} = \begin{cases} r(s(x_1 - x_0) - z - 0.1z^7) \; , & \text{if } \; z < 0; \\ r(s(x_1 - x_0) - z) \; , & \text{if } \; z \ge 0 \; . \end{cases} \tag{3}$$

Here in (1)-(2) we used notations:

$$\begin{aligned} f_1(x_1, x_2) &= \begin{cases} a x_1^3 - b x_1^2 \; , & \text{if } \; x_1 < 0; \\ -(m - x_2 + 0.6(z-4)^2) x_1 , & \text{if } \; x_1 \ge 0 \; , \end{cases} \\ f_2(x_2) &= \begin{cases} 0 \; , & \text{if } \; x_2 < -0.25; \\ a_2(x_2 + 0.25), & \text{if } \; x_2 \ge -0.25 \; , \end{cases} \\ g(x_1) &= \int_{t_0}^{t} e^{-\gamma(t-\tau)} x_1(\tau) d\tau \; . \end{aligned} \tag{4}$$

Thus, the Epileptor set (1)-(4) is a generic phenomenological model describing fast-slow limit cycles in the dynamics of seizes.

The quasi-periodical dynamics of the Epileptor model are presented in Fig. 1: **A**. Time series of the Epileptor model (its enlarged view is shown on the right), the first (middle), and second (bottom) subsystem are plotted to show the principal components of a seizure-like event, that is an inter-ictal period with no spikes, the emergence of pre-ictal spikes, ictal onset, seizure evolution, and the emergence of sharp-wave events toward ictal offset; **B.** The trajectory of the whole system is sketched in the (y_1, c, z) phase space. Seizure offset and ictal onset emerge through the z evolution.

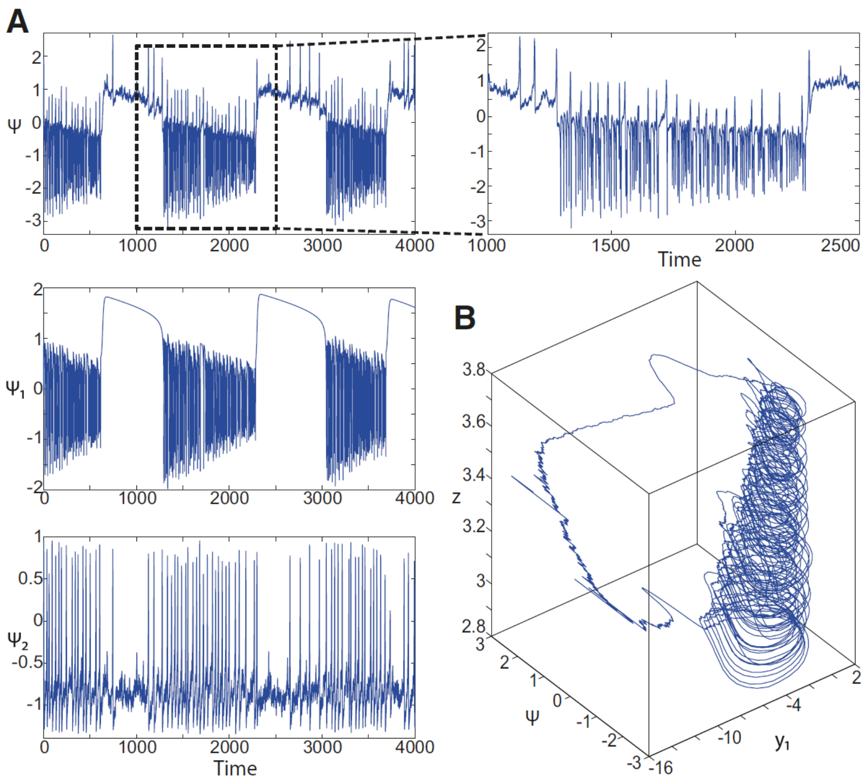

Figure 1. Quasi-periodical dynamics in the Epileptor model [6].

The set of numerical parameters in Fig. 1 is: $a = 1$, $b = 3$, $c = 1$, $d = 5$, $m = 0$, and $I_{ext1} = 3.1$ for Subsystem 1; $a_2 = 6$, $\tau_2 = 10$, $\gamma = 0.01$, and $I_{ext2} = 0.45$ for Subsystem 2; $r = 0.035$, $s = 4$, and $x_0 = -1.6$ for the slow z dynamics.

For further details on the Epileptor dynamical regimes, one should turn to [6].

Subsystem 2 Generating Sharp-Wave Events

As an example of our approach, we investigate here Subsystem 2, which is responsible for generating sharp-wave events. A typical shape of SWEs in the recorded human EEG signals is presented in Fig. 2.

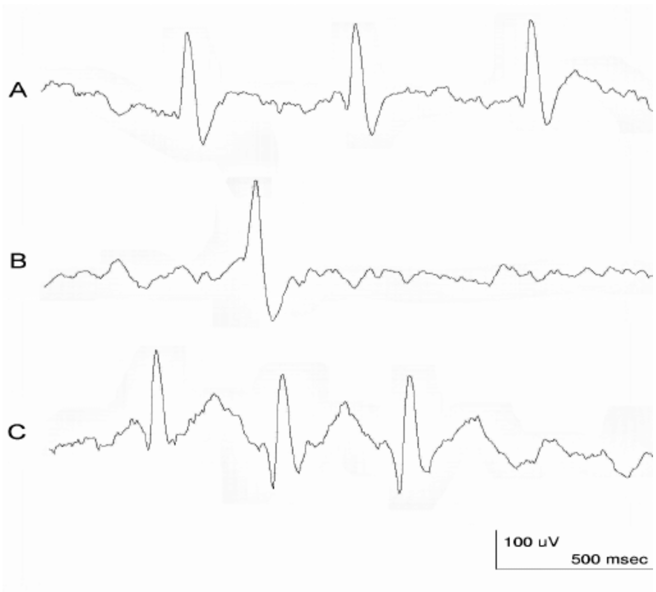

Figure 2. The epileptiform activity in human EEG signals [7].
B: Sharp-wave event signals.

SWEs often occur during sleep and quiet restfulness states, and they are believed to play a critical role in the consolidation of episodic memory.

3. Target Attractor Feedback for Elipeptor Dynamics Control

To make the control over the dynamics in the Epileptor model we propose here the algorithm based on Kolesnikov's "synergetic" target attractor feedback [5]. To do it, one should define the attracting manifolds ψ_s as functions of the state variables. This subset refers to the goal function of the control.

As an example, suppose that we study the stabilization of the state variable y_2 in Subsystem 2 by the application of the external control current I_{ext2}. Then, the subsystem (2) can be re-written as:

$$\frac{dx_2}{dt} = -y_2 + x_2 - x_2^3 + I_{\text{ext}\,2} \;; \qquad \frac{dy_2}{dt} = \frac{1}{\tau_2}\left(-y_2 + f_2(x_2)\right). \tag{5}$$

The f_2 is defined in (4)

The goal of the stabilization is:

$$\frac{dy_2}{dt} = 0 \quad . \tag{6}$$

To achieve it, let’s define the target attractor variable by:

$$\psi = f_2(x_2) - y_2 \quad . \tag{7}$$

It must satisfy Kolesnikov’s dynamical equation [5]:

$$\frac{d\psi}{dt} = -\frac{\psi}{T} \;;\; T = \text{const} > 0. \tag{8}$$

Then by the substitution (7) to (8), we get:

$$f_2'(x_2)\frac{dx_2}{dt} - \frac{dy_2}{dt} = -\frac{1}{T}\left[f_2(x) - y_2\right] \;, \tag{9}$$

which implies by (5):

$$f_2'(x_2)\left[-y_2 + x_2 - x_2^3 + I_{\text{ext}\,2}\right] - \frac{1}{\tau_2}\left[f_2(x_2) - y_2\right] = -\frac{1}{T}\left[f_2(x_2) - y_2\right] . \tag{10}$$

For the case $T << \tau_2$, when Kolesnikov’s attractor is formed much faster than the subsystem (5) makes its evolution, we obtain:

$$f_2'(x_2)\left[-y_2 + x_2 - x_2^3 + I_{\text{ext}\,2}\right] = -\frac{1}{T}\left[f_2(x_2) - y_2\right] . \tag{11}$$

Finally, by (11) we express the control current:

$$I_{\text{ext}2} = -\frac{f_2(x_2) - y_2}{T \cdot f_2'(x_2)} + y_2 - x_2 + x_2^3 \ ;$$
$$, \ f_2'(x_2) = \begin{cases} 0, \ x_2 < -0.25, \\ a_2, \ x_2 \geq -0.25. \end{cases} \tag{12}$$

The external control current (12) drives the subsystem (5) towards the stabilization of the state coordinate y_2 exponentially fast. The x_2-derivative of the function f_2 is equal to 0 if $x_2 < -0.25$. It means that our control algorithm (12) does not work for the stabilization in this domain.

4. Conclusions

The usage of Kolesnikov's "synergetic" feedback provides by (8) exponentially fast stabilization or tracking. Such a type of control works more efficiently than gradient optimal and sub-optimal methods, and it seems to be very natural for control over different attractor regimes in the Epileptor model.

Forming different target attractors or repellers opens a gate for complex control over particular subsystems in the Epileptor independently. It allows driving separately the fast and slow dynamics of the epileptiform behavior in the different subsystems.

The algorithm proposed here for the forming control attractors in the Epileptor model can be used in perspectives for the further development of the control algorithms for *in silico* or *in vivo* detecting and suppressing the real epileptic processes in the human brain.

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Commutativity and Wangerin Differential Equation

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Abstract

The commutativity conditions of the Wangerin differential equation are studied. It is shown that the only class of commutative pairs of Wangerin differential equation is its constant feedback and constant-forward feedback conjugate pairs, and the possibility of commutative pairs other than feedback conjugates is proven to be strictly negative. The commutativity with its feedback conjugate pairs is also studied under nonzero initial conditions.

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Keywords: Commutativity, Wangerin dfiferential equation, time-varying system, linear system

1. Introduction

The commutativty of linear time-varying differential systems has gained attraction in the last few decades. In addition to the exhaustive study [1], which sets the basic theory of commutativty, there have been many other publications in the literature concerning higher-order systems [2], studying commutativty of most of the second-order linear time-varying systems [3, 4], dealing with commutativty in cryptology [5], about the effects of commutativity on system performances such as robustness, stability, noise, sensitivity [6, 7] and subjecting many other aspects.

The commutativity is not only restricted for analogue systems; the subject is also impressive for discrete-time systems modelled by linear difference equations with time-varying coefficients [8, 9].

Although most of the famous second-order linear time-varying differential were subjected from the commutativity point of view in [3, 4], Wangerin differential equation [10] is not among them. So the purpose of this study is to study the commutativity conditions of Wangerin differential equation; namely

$$\ddot{y}+\frac{1}{2}\left(\frac{1}{t-a}+\frac{1}{t-b}+\frac{1}{t-c}\right)\dot{y}+\frac{A+Bt+Ct^2}{4(t-a)(t-b)(t-c)}y=0;\ t\geq t_0, \qquad (1)$$

where a,b,c,A,B,C are constants; and $y(t_0), y'(t_0)$ are the initial conditions at the initial time $t=t_0$. Note that the independent variable x is changed to the continuous-time variable t in order to be coherent with the theory of linear time-varying systems and the theory of commutativity. So, it is kept in mind that t can be any variable, space for example, other than time.

2. Summary of the Relevant Theory

Let A be linear time-varying second-order differential system described by

$$A:\ a_2(t)\ddot{y}_A(t) + a_1(t)\dot{y}_A(t) + a_0(t)y_A(t) = x_A(t);\ t \geq 0 \tag{2}$$

where $x_A(t)$ is the independent excitation and $y_A(t)$ is the resulting response. We assume time variable $t \geq t_0$ (initial time) and $y_A(t_0),\ \dot{y}_A(t_0\,)$ are the initial conditions. For the unique solution of Eq. (2) for $t \geq t_0$, it is sufficient that the excitation and the time-varying coefficients $a_2(t),\ a_1(t),\ a_0(t)$ be piece-wise continuous functions of time with $a_2(t) \neq 0$ [11]. Note that $\ddot{y}_A(t) = y_A''(t) = \frac{d^2}{dt^2}y_A(t),\ \dot{y}_A(t) = y_A'(t) = \frac{d}{dt}y_A(t)$, and similar notations will be followed throughout the paper.

It is shown in [12] that all the commutative pairs of (2) are obtained by

$$\begin{bmatrix} b_2 \\ b_1 \\ b_0 \end{bmatrix} = \begin{bmatrix} a_2 & 0 & 0 \\ a_1 & a_2^{0.5} & 0 \\ a_0 & a_2^{-0.5}(2a_1 - \dot{a}_2)/4 & 1 \end{bmatrix} \begin{bmatrix} k_2 \\ k_1 \\ k_0 \end{bmatrix} \tag{3}$$

if the coefficients of A satisfy

$$-a_2^{0.5}\frac{d}{dt}\left[a_0 - \frac{1}{16a_2}(4a_1^2 + 3\dot{a}_2^2 - 8a_1\dot{a}_2 + 8\dot{a}_1a_2 - 4a_2\ddot{a}_2)\right]k_1 = 0. \tag{4}$$

In Eqs. (3) and (4), $k_2,\ k_1,\ k_0$ are some constants. Then, any commutative pair B of A is described by

$$b_2(t)\ddot{y}_B(t) + b_1(t)\dot{y}_B(t) + b_0(t)y_B(t) = x_B(t) \tag{5}$$

with the paper initial conditions $y_B(t_0)$ and $\dot{y}_B(t_0)$. If $k_1 = 0$, Eq. (4) is automatically satisfied for any second-order linear time-varying differential system A and its commutative pairs are obtainable from A by constant feed-forward gain $\alpha_A = \frac{1}{k_2}$ and constant feedback gain $\sigma_A = k_0$ [2]. Since Eq. (4) is automatically satisfied for $k_1 = 0$, every second-order linear time-varying system has a commutative pair which is its constant feedback conjugate. In this work, we look for the commutativity not of this type, so $k_1 \neq 0$, and the coefficients of A must satisfy

$$a_0 - \frac{1}{16a_2}(4a_1^2 + 3\dot{a}_2^2 - 8a_1\dot{a}_2 + 8\dot{a}_1a_2 - 4a_2\ddot{a}_2) = K \tag{6}$$

for all $t \geq t_0$ where K is some constant.

When the case of nonzero initial conditions is considered, A and B must satisfy (see Eqs. (18a) and (18b) in [13] for $k = 0,1,\cdots,n+m-1$; see also (3) in [14])

$$y(t_0) = y_A(t_0) = y_B(t_0) \tag{7}$$

$$\dot{y}(t_0) = \dot{y}_A(t_0) = \dot{y}_B(t_0) \tag{8}$$

$$\ddot{y}(t_0) = \frac{(1-a_0)ty_A - a_1\dot{y}_A}{a_2} = \frac{(1-b_0)ty_B - b_1\dot{y}_B}{b_2} \tag{9}$$

$$\dddot{y}(t_0) = -\left[\frac{\dot{a}_0}{a_2} + \frac{(1-a_0)(\dot{a}_2+a_1)}{a_2^2}\right] y_A + \left[\frac{1-a_0\dot{a}_1}{a_2} + \frac{a_1(\dot{a}_2+a_1)}{a_2^2}\right] \dot{y}_A$$

$$= -\left[\frac{\dot{b}_0}{b_2} + \frac{(1-b_0)(\dot{b}_2+b_1)}{b_2^2}\right] y_B + \left[\frac{1-b_0\dot{b}_1}{b_2} + \frac{b_1(\dot{b}_2+b_1)}{b_2^2}\right] \dot{y}_B. \tag{10}$$

Eqs. (7) and (8) clearly indicate the equivalence of initial conditions of A and B. With this knowledge Eqs. (9) and (10) yield

$$\begin{bmatrix} \frac{1-a_0}{a_2} - \frac{1-b_0}{b_2} & -\frac{a_1}{a_2} + \frac{b_1}{b_2} \end{bmatrix} \begin{bmatrix} y_A \\ \dot{y}_A \end{bmatrix} = 0 \tag{11a}$$

$$\begin{bmatrix} \frac{\dot{b}_0}{b_2} - \frac{\dot{a}_0}{a_2} + \frac{(1-b_0)(\dot{b}_2+b_1)}{b_2^2} - \frac{(1-a_0)(\dot{a}_2+a_1)}{a_2^2} \\ \frac{1-a_0\dot{a}_1}{a_2} - \frac{1-b_0\dot{b}_1}{b_2} + \frac{a_1(\dot{a}_2+a_1)}{a_2^2} - \frac{b_1(\dot{b}_2+b_1)}{b_2^2} \end{bmatrix}^T \begin{bmatrix} y_A \\ \dot{y}_A \end{bmatrix} = 0 \tag{11b}$$

Eqs. (7-11) are derived for the requirement that the differential systems A and B be commutative under nonzero initial conditions as well and their cascade connections AB and BA are described by a fourth-order linear time-varying differential system C described by

$$c_4(t)y^{(4)}(t) + c_3(t)\dddot{y}(t) + c_2(t)\ddot{y}(t) + c_1(t)\dot{y}(t) + c_0(t)y(t) = x(t) \tag{12}$$

with the initial conditions $y(t_0), \dot{y}(t_0), \ddot{y}(t_0), \dddot{y}(t_0)$.

3. Commutativity of Wangerin Differential Equation

Under the light of the above given theoretical bases, we investigate the commutativity conditions for Wangerin differential equations given in (1).

Regarding the comparison of Wagnerian differential equation in (1) as the system A and comparing it with the general form of the second-order linear time-varying differential system of Eq. (2), the coefficients of Wangerin differential equation are

$$a_2(t) = 1, \tag{13a}$$

$$a_1(t) = \frac{1}{2}\left(\frac{1}{t-a} + \frac{1}{t-b} + \frac{1}{t-c}\right), \tag{13b}$$

$$a_0(t) = \frac{A + Bt + Ct^2}{4(t-a)(t-b)(t-c)}. \tag{13c}$$

The commutativity condition in Eq. (6) yields

$$4a_0a_1^2 - 2\dot{a}_1 = -\frac{A+Bt+Ct^2}{(t-a)(t-b)(t-c)} + \frac{1}{4}\left[\frac{1}{(t-a)^2} + \frac{1}{(t-b)^2} + \frac{1}{(t-c)^2}\right]$$
$$-\frac{1}{4}\left[\frac{1}{(t-a)^2} + \frac{1}{(t-b)^2} + \frac{1}{(t-c)^2}\right.$$
$$\left. + \frac{2}{(t-a)(t-b)} + \frac{2}{(t-a)(t-c)} + \frac{2}{(t-b)(t-c)}\right] = constant \tag{14a}$$

Rearranging the above expression, we obtain

$$\frac{4(A+Bt+Ct^2)}{(t-a)^2(t-b)^2(t-c)^2} + 3\left[\frac{1}{(t-a)^2} + \frac{1}{(t-b)^2} + \frac{1}{(t-c)^2}\right]$$
$$-2\left[\frac{1}{(t-a)(t-b)} + \frac{1}{(t-a)(t-c)} + \frac{1}{(t-b)(t-c)}\right] = K. \tag{14b}$$

As the limit $t \to \infty$, the left side of Eq. (14) approaches $0 - \frac{6}{t^2}$; hence for large t it must hold that

$$-\frac{6}{t^2} = K \tag{15}$$

which is not possible for K being a constant.

Similar result can be obtained as follows: Arranging Eq. (14b), we obtain

$$K(t-a)^2(t-b)^2(t-c)^2 = 4(A+Bt+Ct^2)$$
$$+3[(t-b)^2(t-c)^2 + (t-a)^2(t-c)^2 + (t-a)^2(t-b)^2]$$
$$-2[(t-a)(t-b)(t-c)^2 + (t-a)(t-b)^2(t-c) + (t-a)^2(t-b)(t-c)]. \tag{16}$$

Since the right-hand side is a fourth-degree polynomial, the left-hand side should have "0" coefficient for t^6 and t^5, which is possible only if $K = 0$. In this case, the left-hand side is zero. So to satisfy the equality for the general values of t, the coefficients of t^4, t^3, t^2, t^1, t^0
On the right-hand side must be equal to "0". And for t^4 this yields $0 = 3$ which is a contradiction. Hence, we arrive the result that Eq. (6) can never be satisfied for the Wangerin differential equation. Therefore, the only second-order commutative pairs of Wangerin differential equation are its feedback conjugates obtained by constant gain feed-forward and constant gain feedback structures [15].

4. Commutativity of Constant Feedback Conjugates

For this case $k_1 = 0$ and the commutative pairs (B's) are obtained by Eq. (3):

$$b_2(t) = k_2a_2(t) = k_2, \tag{17a}$$
$$b_1(t) = k_2a_1(t) = \frac{k_2}{2}\left(\frac{1}{t-a} + \frac{1}{t-b} + \frac{1}{t-c}\right), \tag{17b}$$
$$b_0(t) = k_2a_0(t) + k_0 = \frac{k_2(A+Bt+Ct^2)}{4(t-a)(t-b)(t-c)} + k_0. \tag{17c}$$

It is true that with zero initial conditions, Wagnerian differential equation A and B defined by Eq. (17) are commutative. For the nonzero initial conditions, insertion of these in Eq. (11) yields the matrix equation

$$\frac{k_1 + k_0 - 1}{k_2 a}\begin{bmatrix} 1 & 0 \\ -\frac{\dot{a}_2 + a_1}{a_2} & 1 \end{bmatrix}\begin{bmatrix} y_A \\ \dot{y}_A \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{18}$$

It is obvious from Eq. (18) that for

$$k_2 + k_0 = 1 \tag{19}$$

It is obvious from Eq. (18) that for $y_A(t_0)$ and $\dot{y}_A(t_0)$ can be chosen arbitrarily. Hence $\ddot{y}(t_0)$ and $\dddot{y}(t_0)$ can be computed from Eqs. (9) and (10) respectively. On the other hand, if Eq. (19) is not satisfied, then Eq. (18) yields that $y_A(t_0) = 0, \dot{y}_A(t_0) = 0$; and then Eqs. (9) and (10) yield $\ddot{y}(t_0) = 0,\ \dddot{y}(t_0) = 0$. This means that all initial conditions of A, B and of the combined system $C = AB$ or $C = BA$ must be zero for the commutativity holds.

Since $a_2(\mathrm{t}) = 1$ for the investigated differential system, the initial conditions formula can be summarized for $k_1 = 0$ and $k_2 + k_0 = 1$ as

$$y = y_A = y_B, \tag{20a}$$
$$\dot{y} = \dot{y}_A = \dot{y}_B, \tag{20b}$$
$$\ddot{y} = (1 - a_0)y_A - a_1\dot{y}_A, \tag{20c}$$
$$\dddot{y} = -[a_1(1 - a_0) + \dot{a}_0]y_A + (1 - a_0 + a_1^2 - \dot{a}_1)\dot{y}_A, \tag{20d}$$

where all time-dependent parameters are evaluated at the initial time t_0.

5. Simulations

Assume $a = b = c = 1,\ A = 4,\ B = 12,\ C = -12$. Then from Eq. (13), the differential equation for A has the coefficients

$$a_2 = 1, \tag{21a}$$
$$a_1 = \frac{1.5}{\mathrm{t} - 1}, \tag{21b}$$
$$a_0 = \frac{-3\mathrm{t}^2 + 3\mathrm{t} - 1}{(\mathrm{t} - 1)^3}. \tag{21c}$$

The initial condition in Eq. (20) for $y_A(t_0) = 0.1,\ \dot{y}_A(t_0) = 0.4$

$$y = y_A = y_B = 0.1, \tag{22a}$$
$$\dot{y} = \dot{y}_A = \dot{y}_B = 0.4, \tag{22b}$$
$$\ddot{y} = 0.2, \tag{22c}$$
$$\dddot{y} = 2.3. \tag{22d}$$

Within remind $k_1 = 0$, choosing $k_2 = 2,\ k_0 = -1$ for which Eq. (19) is satisfied, Eq. (3) together with Eq. (24) yields the differential system B as obtained by the coefficients

$$b_2 = 2, \tag{23a}$$

$$b_1 = \frac{3}{t-1}, \tag{23b}$$

$$b_0 = \frac{-t^3 - 3t^2 + 3t - 1}{(t-1)^3}. \tag{23c}$$

Then the resulting connection $C = AB$ or BA has the differential representation as in Eq. (24) (See Eq. (3) for $n = m = 2$ in [1]) with the initial conditions in Eq. (22).

$$2y^{(4)} + \frac{6}{t-1}y^{(3)} - \frac{t^3 + 9t^2 - 7.5t + 1.5}{(t-1)^3}y^{(2)} + 1.5\frac{t^3 + t^2 - 10t + 4}{(t-1)^4}y^{(1)} + \frac{3t^5 + 6t^4 - 20t^3 + 6t^2 + 6t + 1}{(t-1)^6}. \tag{24}$$

Within $k_2 = 2\ , k_0 = -2$ where Eq. (19) is not satisfied, we obtain B'

$$b_2 = 2, \tag{25a}$$

$$b_1 = \frac{3}{t-1}, \tag{25b}$$

$$b_0 = \frac{-2t^3}{(t-1)^3}. \tag{25c}$$

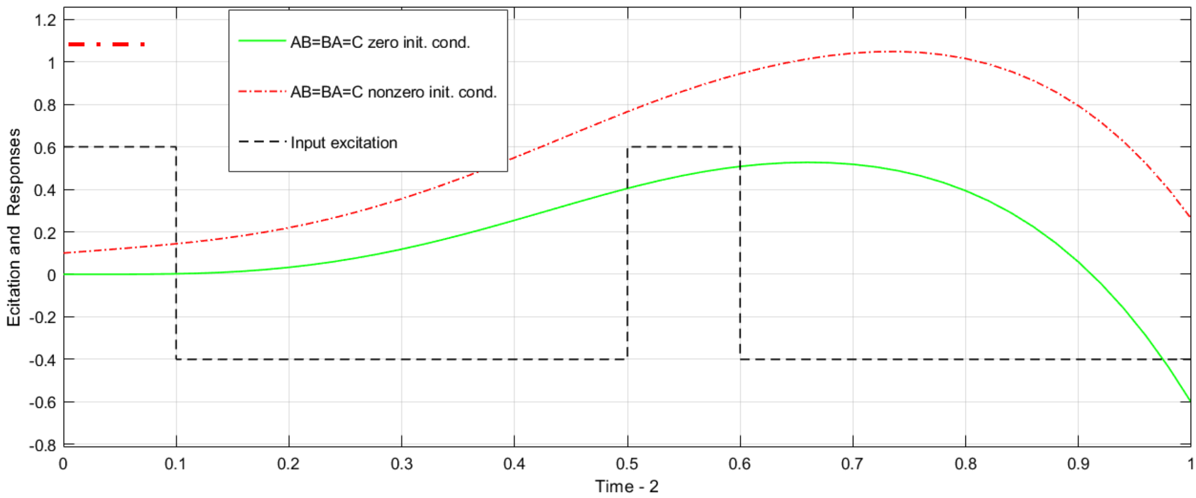
Then the resulting connection $C' = AB'$ or $C' = B'A$ has the differential representation as in Eq. (26); note that the commutativity of A and B' gets valid only for zero initial conditions.

$$2y^{(4)} + \frac{6}{t-1}y^{(3)} + \frac{-2t^3 - 6t^2 + 4.5t + 0.5}{(t-1)^3}y^{(2)} + 1.5\frac{-3t^3 + 3t^2 + 10.5t + 4.5}{(t-1)^4}y^{(1)} + \frac{6t^5 - 6t^4 - t^3 - 9t^2 + 12t}{(t-1)^6}y = x. \tag{26}$$

Wangerin differential system A defied by the coefficients in Eq. (21) together with its constant feedback conjugate pair B defined by Eq. (23) are connected in cascade in the form AB and BA .The connections and their mathematical fourth-order equivalence system C defined by Eq. (24) are excited by a pulse train of a period 0.5, duty cycle of 20%, amplitude of 2000, and bias of -800. The starting time is assumed $t_0 = 2$.

The solutions for the output is obtained by ode45 (Dormend-Prince) of Simulink toolbox in Matlab 2019a. It is observed that the output of $AB,\ BA,\ C$ are indentical for zero intial conditions; see $AB = BA = C$ zero initial condition in Fig. 1. On the same figure, the input signal is shown by input excitation. When the proper initial conditions in Eq. (22) are chosen, the commutativity still holds and all the differential systems $AB,\ BA,\ C$ give the same solution as shown by $AB = BA = C$ nonzero initial conditions.

In Fig. 2, the results relevant to the case where the necessary condition for commutativity with nonzero initial conditions are not satisfied. The commutativity of relaxed systems A' and B' are obvious from the plot $AB' = BA' = C'$ zero initial condition. When the condition in Eq. (19) is not satisfied, as the case for systems A, and B' as defined by coefficients in Eq. (25), the commutativity with nonzero initial condition is spoiled and the connections AB' and $B'A$ yield the different response (see AB' nonzero initial condition $B'A = C'$ nonzero initial condition). Here, the responses of $B'A$ and C' coincide since initial conditions are satisfied for the connection of $B'A$ only [13].



Figigure 1. Responses for the case $k_2 + k_0 = 1$; all the commutativity conditions are satisfied.

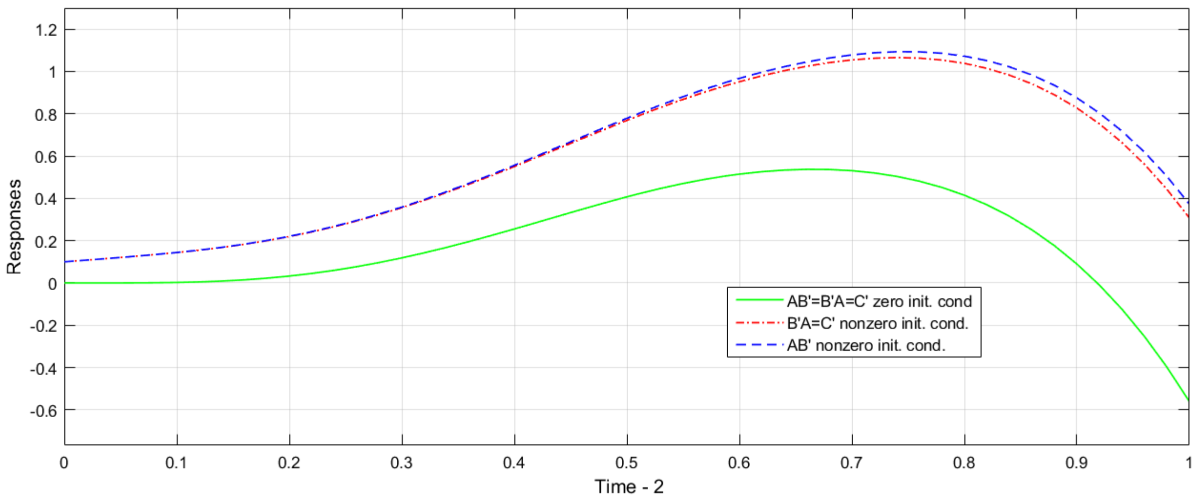


Figure 2. Responses for the case $k_2 + k_0 \neq 1;$ commutativity conditions relevant to nonzero initial conditions are not satisfied.

6. Conclusion

Wangerin differential equation is studied in this work from the point of view of commutativity. It is shown that Wangerian differential system is commutative with a second-order linear time-varying differential system only if this system is one of constant forward and constant feedback conjugate pair of the given Wangerin system. The case of nonzero initial conditions is also examined in this study. The results are well-validated by the MATLAB simulation software Simulink.

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Coincidence and Common Fixed Soft Point Theorems ın Parametric Soft Metric Spaces

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Abstract

Parameters play the main role to model real life problems involving uncertainly. Researchers working in the area of metric spaces have been inspired by this idea, and the soft metric spaces gave birth. By the similar reason, we defined parametric soft metric spaces as the parametric extension of the soft metric spaes [1]. The existence and the uniqueness of the fixed points in the metric (-like) saces take important place. Since the theory of fixed points is the backbone of the several applied sciences.In this article, we investigate some common and coincidence fixed soft point results in the parametric soft metric spaces.

Keywords: Parametric soft metric, soft set, coincidence and common fixed soft point.

1. Introduction

Investigations on indicating the existence and also uniqueness of fixed points of self mappings have extensive applications in different branches of mathematics, economics, engineering, and statistics. In mathematical aspect, fixed point theory is worthwhile to investigate by its applicability in various problems which consist differential and integral equations, approximations, games, and so on. For these reasons, to determine the existence and uniqueness of fixed points, common fixed points, coincident fixed points in different types of metric spaces, take attentions of the researchers working in the different branches of mathematics.

The notion of a parametric metric space defined by Hussain et al. [7] in 2014. Later, Rao et al. [9] presented parametric S-metric spaces and prove common fixed point theorems in this spaces. Çetkin [12] introduced the concept of parametric 2-metric spaces and investigated some fixed point results in such spaces. Different versions of parametric metric spaces and investigations on fixed points of the proposed spaces have been considered by several authors [10,1314,15]. Besides, the notion of soft metric spaces [4] is one of the generalizations of metric spaces, which based on the parameterization tool. Researches on soft metric spaces and their fixed point theorems are very popular, nowadays. By expanding the role of the parameterization tool in the parametric metric spaces, Tunçay and Çetkin [8] defined the concept of a parametric soft metric space and observed some basic features of these spaces.

The goal of this study is to show the existence and the uniqueness of coincidence fixed soft points and common fixed soft points of self soft mappings described on a (complete) parametric soft metric space. So this study is arranged in the following manner. In section 2, we recall some basic notions and notations which are necessary for the main sections. In section 3, we observe some common and coincidence fixed soft point results for self soft mappings in the described spaces. Also we have some useful results on the existence of fixed soft points in such spaces.

2. Preliminaries

Definition 2.1. [1] Let X denote the universal set, E denote the set of parameters and $\emptyset \neq A \subseteq E$.Then a mapping $F: A \to \mathcal{P}(X)$ is called a soft set over X, and it is denoted by the pair (F, A). In other words, a soft set over X can be thought as a parametrized family of crisp subsets of the universe X.

For any soft set (F, A), we can extend the soft set (F, A) to the soft set $(\bar{F}, E)$ where

$$\bar{F} : E \to \mathcal{P}(X)\,, \bar{F}(e) = \begin{cases} F(e) & , \ if\ e \in A \\ \emptyset & , \ if\ e \notin A \end{cases}$$

Example 2.2. [6] Let X be the set of houses under consideration. E is the set of parameters. Each parameter is a word or a sentence. E = {expensive; beautiful; wooden; cheap; in the green surroundings; modern; in good repair; in bad repair}. In this case, to define a soft set means to point out expensive houses, beautiful houses, and so on. The soft set *(F, E)* describes the "attractiveness of the houses" which Mr. X is going to buy. Suppose that there are six houses in the universe X given by

$X = \{h_1, h_2, h_3, h_4, h_5, h_6\}$ and $E = \{e_1, e_2, e_3, e_4, e_5\}$:

where e_1 stands for the parameter 'expensive', e_2 stands for the parameter 'beautiful', e_3 stands for the parameter 'wooden', e_4 stands for the parameter 'cheap', e_5 stands for the parameter 'in the green surroundings'.

Suppose that

$F(e_1) = \{h_2, h_4\}$, $F(e_2) = \{h_1, h_3\}$, $F(e_3) = \{h_3, h_4, h_5\}$, $F(e_4) = \{h_1, h_3, h_5\}$, $F(e_5) = \{h_1\}$

The soft set

$(F, E) = \{(e_1, \{h_2, h_4\}),\ (e_2, \{h_1, h_3\}), (e_3, \{h_3, h_4, h_5\}),\ (e_4, \{h_1, h_3, h_5\}),\ (e_5, \{h_1\})\}$ is a parametrized family $\{F(e_i),\ i = 1,2,3,4, \ldots, 8\}$ of subsets of the set X and gives us a collection of approximate descriptions of an object.

Definition 2.3. [1] (1) The union of two soft sets (F, A) and (G, B) over X is a soft set (H, C), where $C = A \cup B$ and for all $e \in C$ $H(e) = F(e) \cup G(e)$. We express it as $(F, A)\ \tilde{\cup}\ (G, B) = (H, C)$.

(2) The intersection of two soft sets (F, A) and (G, B) over X is the soft set (H, D), where $D = A \cap B$ and for all $e \in D$, $H(e) = F(e) \cap G(e)$. We express it as $(F, A)\ \tilde{\cap}\ (G, B) = (H, D)$.

Definition 2.4. [2] A soft set (F, E) over X is said to be an absolute soft set denoted by $\tilde{X}$ if $F(e) = X$,for all $e \in E$.

A soft set (F, E) over X is said to be a null soft set denoted by $\widetilde{\emptyset}$ if $F(e) = \emptyset$, for all $e \in E$.

Definition 2.5. [3] Let $\mathbb{R}$ be the set of real numbers and $\mathcal{B}(\mathbb{R})$ be the collection of all non-empty bounded subsets of $\mathbb{R}$. Then the mapping described by $F: A \to \mathcal{B}(\mathbb{R})$ is called a soft real set. In this case F is called a soft real number.

We use notations $\tilde{r}, \tilde{s}, \tilde{t}$ to denote soft real numbers whereas $\bar{r}, \bar{s}, \bar{t}$ will denote a particular type of soft real numbers such that $\bar{r}(\lambda) = r$, for all $\lambda \in A$.

For example 0 is the soft real number where $\bar{0}(\lambda) = 0$, for all $\lambda \in A$.

Definition 2.6. [4] For two soft real numbers $\tilde{r}, \tilde{s}$ the orderings are defined as follows:

i. $\tilde{r}(\alpha) \tilde{\leq} \tilde{s}(\alpha) \Rightarrow \tilde{r} \leq \tilde{s}$

ii. $\tilde{r}(\alpha) \tilde{\geq} \tilde{s}(\alpha) \Rightarrow \tilde{r} \geq \tilde{s}$

iii. $\tilde{r}(\alpha) \tilde{<} \tilde{s}(\alpha) \Rightarrow \tilde{r} < \tilde{s}$

iv. $\tilde{r}(\alpha) \tilde{>} \tilde{s}(\alpha) \Rightarrow \tilde{r} > \tilde{s}$

Definition 2.7. [4] If $\mathcal{B}$ is a collection of soft points then the soft set generated by taking all the soft points of $\mathcal{B}$ is denoted by the symbol $SS(\mathcal{B})$. In addition, the collection of all soft points of a soft set (F, A) is described by $SP(F, A)$.

Notation 2.8. [4] $\mathbb{R}(A)^*$ denotes the set of all non-negative soft real numbers.

Definition 2.9. [4] A soft set (F, A) over X is said to be a soft point and it is denoted by P_λ^x, where $\lambda \in A$ and $x \in X$. A soft point P_λ^x is defined by follows:

$$P_\lambda^x(\alpha) = \begin{cases} \{x\} & \alpha = \lambda \\ \emptyset & otherwise \end{cases} \quad \text{, for all } \alpha \in A.$$

Definition 2.10. [6] Let (F, A) and (G, B) be two soft sets over X and Y, respectively and let φ_ψ be a soft mapping from $(\mathcal{P}(X))^A$ into $(\mathcal{P}(Y))^B$.

(1) The image of (F, A) under the soft mapping φ_ψ is the soft set over Y, defined by

$$\varphi_\psi\big((F, A)\big)(k) = \bigcup_{e \in \psi^{-1}(k)} \varphi((F, A)(e)) \quad for\ all\ k \in B$$

(2) The pre-image of (G, B) under the soft mapping φ_ψ is the soft set over X, defined by

$$\varphi_\psi^{-1}\big((G, B)\big)(e) = \varphi^{-1}\Big((G, B)\big(\psi(e)\big)\Big) \ for\ all\ e \in A.$$

For further information on notions and notations for the soft sets, we refer the papers [1,2,4].

Definition 2.11. [8] Let $\tilde{X}$ be the universal soft set and

$d: SP(\tilde{X}) \times SP(\tilde{X}) \times \mathbb{R}(A)^* \rightarrow \mathbb{R}(A)^*$ be a mapping. Then the mapping d is called a parametric soft metric on $\tilde{X}$ if the following axioms are valid:

(P1) d(P_λ^x, P_μ^y, $\bar{t}$) = $\bar{0}$ for all $\bar{t} \gtrsim \bar{0}$if and only if P_λ^x = P_μ^y,

(P2) d(P_λ^x, P_μ^y, $\bar{t}$) = d(P_μ^y, P_λ^x, $\bar{t}$) for all $\bar{t} \gtrsim \bar{0}$,

(P3) d(P_λ^x, P_μ^y, $\bar{t}$)$\tilde{\leq}$d(P_λ^x, P_γ^z, $\bar{t}$) + d(P_γ^z, P_μ^y, $\bar{t}$) for all

P_λ^x, P_μ^y, P_γ^z $\tilde{\in}$ SP($\tilde{X}$) and all $\bar{t} \gtrsim \bar{0}$.

The pair ($\tilde{X}$, d) is said to be a parametric soft metric space. In case when the parameter set is one-pointed, then we get the definition of a parametric metric.

Example 2.12. [8] Let X be any nonempty set and describe a mapping

$d: SP(\tilde{X}) \times SP(\tilde{X}) \times \mathbb{R}(A)^* \rightarrow \mathbb{R}(A)^*$ by

d(P_λ^x, P_μ^y, $\bar{t}$) $= \begin{cases} \bar{1}, & P_\lambda^x = P_\mu^y \\ \bar{0} & P_\lambda^x \neq P_\mu^y \end{cases}$ for all $\bar{t} \gtrsim \bar{0}$.

Then the mapping d identified above is a parametric soft metric on $\tilde{X}$. This parametric soft metric is called the discrete parametric soft metric and ($\tilde{X}$, d) is called the discrete parametric soft metric space.

Example 2.13. [8] Let $X = \mathbb{R}$ be the set of all reals and define a mapping

$d: SP(\tilde{X}) \times SP(\tilde{X}) \times \mathbb{R}(A)^* \rightarrow \mathbb{R}(A)^*$ by d(P_λ^x, P_μ^y, $\bar{t}$) $= \bar{t}\left[|\bar{x} - \bar{y}| + |\bar{\lambda} - \bar{\mu}|\right] \, f \, or \, all \, \bar{t} > \bar{0}$.

Then d is a parametric soft metric.

Definition 2.14. [8] Let $\left\{P_{\lambda_n}^{x_n}\right\}_{n \in N}$ be a sequence of soft points in a parametric soft metric space ($\tilde{X}$, d).

1. $\left\{P_{\lambda_n}^{x_n}\right\}_{n \in N}$ is said to be convergent to P_λ^x $\tilde{\in}$ SP($\tilde{X}$), written as $\lim_{n\to\infty} P_{\lambda_n}^{x_n}$ = P_λ^x, for all $\bar{t} \gtrsim \bar{0}$, if $\lim\limits_{n\to\infty}$d($P_{\lambda_n}^{x_n}$, P_λ^x, $\bar{t}$) = $\bar{0}$.

2. $\left\{P_{\lambda_n}^{x_n}\right\}_{n \in N}$ is said to be a Cauchy sequence in $\tilde{X}$ if for all $\bar{t} \gtrsim \bar{0}$, if $\lim_{n,m\to\infty}$ d($P_{\lambda_n}^{x_n}$, $P_{\lambda_m}^{x_m}$, $\bar{t}$) = $\bar{0}$.

3. $(\tilde{X}, d)$ is said to be complete if every Cauchy sequence is a convergent sequence.

Definition 2.15. [8] Let ($\tilde{X}$, d_1) and $(\tilde{Y}, d_2)$ be two parametric soft metric spaces, and

$\varphi_\psi : (\tilde{X}, d_1) \rightarrow (\tilde{Y}, d_2)$ be a soft mapping. We say φ_ψ is a continuous mapping at P_λ^x in $\tilde{X}$, if for any sequence $\left\{P_{\lambda_n}^{x_n}\right\}_{n \in N}$ in X such that $\lim_{n\to\infty} P_{\lambda_n}^{x_n}$= P_λ^x, then $\lim\limits_{n\to\infty} \varphi_\psi$ ($P_{\lambda_n}^{x_n}$) = $\varphi_\psi(P_\lambda^x)$.

Lemma 2.22. [8] Let $\left\{P_{\lambda_n}^{x_n}\right\}_{n\in N}$ be a sequence of soft points in a parametric soft metric space $(\tilde{X}, d)$ such that

$d\left(P_{\lambda_n}^{x_n}, P_{\lambda_{n+1}}^{x_{n+1}}, \bar{t}\right) = \bar{\lambda} d\left(P_{\lambda_{n-1}}^{x_{n-1}}, P_{\lambda_n}^{x_n}, \bar{t}\right)$ ... (1)

where $\bar{\lambda} \tilde{\in} [\bar{0}, \bar{1})$ and $n \in \mathbb{N}$. Then $\left\{P_{\lambda_n}^{x_n}\right\}_{n\in N}$ is a Cauchy sequence in $(\tilde{X}, d)$.

Proof. Let $m >$ n ≥ 1 be chosen. Then it implies that the follwoing

$$d\left(P_{\lambda_n}^{x_n}, P_{\lambda_m}^{x_m}, \bar{t}\right) \tilde{\leqq} d\left(P_{\lambda_n}^{x_n}, P_{\lambda_{n+1}}^{x_{n+1}}, \bar{t}\right) + d\left(P_{\lambda_{n+1}}^{x_{n+1}}, P_{\lambda_{n+2}}^{x_{n+2}}, \bar{t}\right) + \cdots + d(P_{\lambda_{m-1}}^{x_{m-1}}, P_{\lambda_m}^{x_m}, \bar{t})$$

$$\tilde{\leqq} (\bar{\lambda}^n + \bar{\lambda}^{n+1} + \cdots + \bar{\lambda}^{m-1}) d(P_{\lambda_0}^{x_0}, P_{\lambda_1}^{x_1}, \bar{t})$$

$$\tilde{\leqq} \frac{\bar{\lambda}^n}{\bar{1}-\bar{\lambda}} d(P_{\lambda_0}^{x_0}, P_{\lambda_1}^{x_1}, \bar{t}), \text{ for all } \bar{t} \tilde{>} \bar{0}. \;.....(2)$$

Since $\bar{\lambda} \tilde{<} \bar{1}$. Assume that $d(P_{\lambda_0}^{x_0}, P_{\lambda_1}^{x_1}, \bar{t}) \tilde{>} \bar{0}$. If we take limit as $m, n \to +\infty$ in the above inequality, then we gain the following

$\lim_{n,m\to\infty} d(P_{\lambda_n}^{x_n}, P_{\lambda_m}^{x_m}, \bar{t}) = \bar{0}$....(3)

Therefore, $\left\{P_{\lambda_n}^{x_n}\right\}_{n\in N}$ is a Cauchy sequence in $\tilde{X}$. Also, if $d(P_{\lambda_0}^{x_0}, P_{\lambda_1}^{x_1}, \bar{t}) = \bar{0}$, then $d\left(P_{\lambda_n}^{x_n}, P_{\lambda_m}^{x_m}, \bar{t}\right) = \bar{0}$, for all $m > n$ and hence $\left\{P_{\lambda_n}^{x_n}\right\}_{n\in N}$ is a Cauchy sequence in $\tilde{X}$.

3. Coincidence and Common Fixed Soft Point Theorems ın Parametric Soft Metric Spaces

Definition 3.1: Let δ_ρ and φ_ψ be two self soft mappings of the soft universe $\tilde{X}$. Then δ_ρ and φ_ψ are said to be weakly compatible if they commute at all of their coincidence soft points; that is, $\delta_\rho(P_\lambda^x) = \varphi_\psi(P_\lambda^x)$, for some $P_\lambda^x \tilde{\in} SP(\tilde{X})$ and then $\delta_\rho(\varphi_\psi(P_\lambda^x)) = \varphi_\psi(\delta_\rho(P_\lambda^x))$.

Theorem 3.2: Let $(\tilde{X}, d)$ be a complete parametric soft metric space. Let δ_ρ and φ_ψ be a weakly compatible self soft mappings of $\tilde{X}$ and $\varphi_\psi(\tilde{X}) \subseteq \delta_\rho(\tilde{X})$. Suppose that there exists $\bar{k} \tilde{\geqq} \bar{1}$ such that

$d(\delta_\rho(P_\lambda^x), \delta_\rho(P_\mu^y), \bar{t}) \tilde{\geqq} \bar{k}\, d(\varphi_\psi(P_\lambda^x), (P_\mu^y), \bar{t}), \forall P_\lambda^x, P_\mu^y \tilde{\in} \tilde{X}$ ... (4)

If one of the subspaces $\varphi_\psi(\tilde{X})$ or $\delta_\rho(\tilde{X})$ is complete, then δ_ρ and φ_ψ have a unique common fixed point in $\tilde{X}$.

Proof. Let $P_{\lambda_0}^{x_0} \tilde{\in} SP(\tilde{X})$ be taken arbitary. Since $\varphi_\psi(\tilde{X}) \subseteq \delta_\rho(\tilde{X})$, choose $P_{\lambda_1}^{x_1}$ such that $P_{\mu_1}^{y_1} = \delta_\rho(P_{\lambda_1}^{x_1}) = \varphi_\psi(P_{\lambda_0}^{x_0})$. In general choose $P_{\lambda_{n+1}}^{x_{n+1}}$ such that $P_{\mu_{n+1}}^{y_{n+1}} = \delta_\rho(P_{\lambda_{n+1}}^{x_{n+1}}) = \varphi_\psi(P_{\lambda_n}^{x_n})$, then from condition (4), we gain the following

$$d\left(P_{\mu_{n+1}}^{y_{n+1}}, P_{\mu_{n+2}}^{y_{n+2}}, \bar{t}\right) = \mathcal{P}\left(\varphi_\psi\left(P_{\lambda_n}^{x_n}\right), \varphi_\psi\left(P_{\lambda_{n+1}}^{x_{n+1}}\right), \bar{t}\right) \ldots (5)$$

By repeating (5) $(n+1)$ -times, we obtain the following inequality

$$d\left(P_{\mu_{n+1}}^{y_{n+1}}, P_{\mu_{n+2}}^{y_{n+2}}, , \bar{t}\right) \lesseqqgtr \bar{\lambda}^{n+1} d\left(P_{\mu_0}^{y_0}, P_{\mu_1}^{y_1}, \bar{t}\right) \ldots (6)$$

Hence for $n > m$, we have

$$d\left(P_{\mu_n}^{y_n}, P_{\mu_m}^{y_m}, \bar{t}\right) \lesseqqgtr d\left(P_{\mu_n}^{y_n}, P_{\mu_{n+1}}^{y_{n+1}}, \bar{t}\right) + d\left(P_{\mu_n}^{y_n}, P_{\mu_{n+1}}^{y_{n+1}}, \bar{t}\right) + \ \ldots \ \ldots + d\left(P_{\mu_{m-1}}^{y_{m-1}}, P_{\mu_m}^{y_m}, \mathrm{t}\right) \ldots (7)$$

$$\lesseqqgtr \left(\bar{\lambda}^n + \bar{\lambda}^{n+1} + \cdots + \bar{\lambda}^{m-1}\right)\mathcal{P}\left(P_{\mu_0}^{y_0}, P_{\mu_1}^{y_1}, \bar{t}\right), \text{for all } \bar{t} \gtrsim \bar{0}.$$

By taking $\mathrm{n}, \mathrm{m} \to \infty$ in the above inequality, we obtain that $\lim_{n\to\infty} d(P_{\lambda_n}^{x_n}, P_{\mu_m}^{y_m}, \bar{t}) = \bar{0}$.

Therefore, $\left\{P_{\lambda_n}^{x_n}\right\}_{n\in N}$ is a Cauchy Sequence. Since $(\tilde{X}, d)$ is a complete parametric soft metric space, there exists $P_\rho^{\varpi} \tilde{\in} SP(\tilde{X})$ such that $P_{\mu_n}^{y_n} \to P_\rho^{\varpi}$ as $\mathrm{n} \to +\infty$. Hence we get

$$\lim_{n\to\infty} P_{\mu_n}^{y_n} = \lim_{n\to\infty} \varphi_\psi\left(P_{\lambda_n}^{x_n}\right) = \lim_{n\to\infty} \delta_\rho\left(P_{\lambda_n}^{x_n}\right) = P_\rho^{\varpi} \ldots (8)$$

Since $\varphi_\psi(\tilde{X})$ or $\delta_\rho(\tilde{X})$ is complete and $\varphi_\psi(\tilde{X}) \subseteq \delta_\rho(\tilde{X})$, there exists a soft point $P_u^v \tilde{\in} \tilde{X}$ such that $\delta_\rho(P_u^v) = P_\rho^{\varpi}$. Now from (4), for all $\bar{t} \gtrsim \bar{0}$,

$$d\left(\varphi_\psi(P_u^v), \varphi_\psi\left(P_{\lambda_n}^{x_n}\right), \bar{t}\right) \lesseqqgtr \frac{\bar{1}}{\bar{k}} d\left(\delta_\rho(P_u^v), \delta_\rho\left(P_{\mu_n}^{y_n}\right), \bar{t}\right) \ldots . (9)$$

Proceeding to the limit as $\mathrm{n} \to +\infty$ in (9), we have

$$d\left(\varphi_\psi(P_u^v), P_\rho^{\varpi}, \bar{t}\right) \lesseqqgtr \frac{\bar{1}}{\bar{k}} d\left(\delta_\rho(P_u^v), P_\rho^{\varpi}, \bar{t}\right) \ldots (10)$$

for all $\bar{t} \gtrsim \bar{0}$, which implies that $\varphi_\psi(P_u^v) = P_\rho^{\varpi}$. Therefore $\varphi_\psi(P_u^v) = \delta_\rho(P_u^v) = P_\rho^{\varpi}$. Since φ_ψ and δ_ρ are weakly compatible self soft mappings, we have $\delta_\rho(\varphi_\psi(P_u^v)) = \varphi_\psi(\delta_\rho(P_u^v))$, that is $\delta_\rho(P_\rho^{\varpi}) = \varphi_\psi(P_\rho^{\varpi})$.

Now we show that P_ρ^{ϖ} is a fixed point of δ_ρ and φ_ψ . From (4), we have

$d(\delta_\rho(P_\rho^\varpi), \delta_\rho(P_{\lambda_n}^{x_n}), \bar{t})) \cong\geq \bar{k}\, d(\varphi_\psi\,(P_\rho^\varpi\,), \varphi_\psi\,(P_{\lambda_n}^{x_n}), \bar{t})$ ... (11)

Proceeding to the limit as n → ∞ in (11), we have

$d(\delta_\rho(P_\rho^\varpi), P_\rho^\varpi, \bar{t})) \cong\geq \bar{k} d(\varphi_\psi\,(P_\rho^\varpi), P_\rho^\varpi, \bar{t}))$ ... (12)

which implies that $\delta_\rho(P_\rho^\varpi) = P_\rho^\varpi$. Hence $\delta_\rho(P_\rho^\varpi) = \varphi_\psi\,(P_\rho^\varpi) = P_\rho^\varpi$.

Uniquenes: Suppose that $P_\rho^\varpi \neq P_\gamma^z$ is also another common fixed point of δ_ρ and φ_ψ. Then

$d(\delta_\rho(P_\rho^\varpi), \delta_\rho(P_\gamma^z), \bar{t})) \cong\geq \bar{k} d(\varphi_\psi\,(P_\rho^\varpi), \varphi_\psi\,(P_\gamma^z), \bar{t}))$, for all $\bar{t} \,\tilde{>}\, \bar{0}$, which witnesses the fact that $P_\rho^\varpi = P_\gamma^z$. This completes the proof.

Theorem 3.3: Let $(\tilde{X}, d)$ be a complete parametric soft metric space and $\varphi_\psi, \delta_\rho : \tilde{X} \to \tilde{X}$ be two surjective self soft mappings which satisfy the following conditions.

$d(\varphi_\psi\ \delta_\rho(P_\lambda^x),), \delta_\rho(P_\lambda^x), \bar{t}) + \bar{k} d(\varphi_\psi\, \delta_\rho(P_\lambda^x), P_\lambda^x, \bar{t}) \cong\geq \bar{a}\, d(\delta_\rho(P_\lambda^x), P_\lambda^x, \bar{t})$ ... (13)

and

$d(\delta_\rho\ (\varphi_\psi\,(P_\lambda^x), \varphi_\psi\,(P_\lambda^x), \bar{t}) + \bar{k} d(\delta_\rho\left(\varphi_\psi\,(P_\lambda^x), P_\lambda^x, \bar{t}\right) \cong\geq \bar{b} d\left(\varphi_\psi\,(P_\lambda^x), P_\lambda^x, \bar{t}\right)$ ... (14)

for all $P_\lambda^x \,\tilde{\in}\, SP(\tilde{X})$, all $\bar{t} \,\tilde{>}\, \bar{0}$, and some nonnegative soft real numbers $\bar{a}, \bar{b}$ and $\bar{k}$ with $\bar{a} > \bar{1} + 2\bar{k}$ and $\bar{b} > \bar{1} + 2\bar{k}$. If φ_ψ or δ_ρ is continuous, then φ_ψ and δ_ρ have a common fixed point.

Proof. Let $P_{\lambda_0}^{x_0} \,\tilde{\in}\, SP(\tilde{X})$ be an arbitrary soft point in $\tilde{X}$. Since φ_ψ is surjective, there exists $P_{\lambda_1}^{x_1} \,\tilde{\in}\, SP(\tilde{X})$ such that $P_{\lambda_0}^{x_0} = \varphi_\psi\,(P_{\lambda_1}^{x_1})$. Since δ_ρ is also surjective, there exists $P_{\lambda_2}^{x_2} \,\tilde{\in}\, SP(\tilde{X})$ such that $P_{\lambda_2}^{x_2} = \delta_\rho\ (P_{\lambda_1}^{x_1})$. Continuing this process, we construct a sequence of soft points $\left\{P_{\lambda_n}^{x_n}\right\}_{n\in N}$ in $\tilde{X}$ such that

$P_{\lambda_{2n}}^{x_{2n}} = \varphi_\psi\left(P_{\lambda_{2n+1}}^{x_{2n+1}}\right)$ and $P_{\lambda_{2n+1}}^{x_{2n+1}} = \delta_\rho\left(P_{\lambda_{2n+2}}^{x_{2n+2}}\right), \forall\, n \in \mathbb{N} \cup \{0\}$ ... (15)

Now for $n \in \mathbb{N} \cup \{0\}$, we have

$d(\varphi_\psi\ (\delta_\rho(P_{\lambda_{2n+2}}^{x_{2n+2}}), \delta_\rho(P_{\lambda_{2n+2}}^{x_{2n+2}}), \bar{t}) + \bar{k}\ d(\varphi_\psi\ (\delta_\rho(P_{\lambda_{2n+2}}^{x_{2n+2}}), P_{\lambda_{2n+2}}^{x_{2n+2}}, \bar{t}) \cong\geq \bar{a} d\left(\delta_\rho\left(P_{\lambda_{2n+2}}^{x_{2n+2}}\right), P_{\lambda_{2n+2}}^{x_{2n+2}}, \bar{t}\right)$ ... (16)

Thus we have

$d\left(P_{\lambda_{2n}}^{x_{2n}}, P_{\lambda_{2n+1}}^{x_{2n+1}}, \bar{t}\right) + \bar{k}\ d\left(P_{\lambda,2n}^{x_{2n}}, P_{\lambda,2n+2}^{x_{2n+2}}, \bar{t}\right) \cong\geq \bar{a} d\left(P_{\lambda_{2n+1}}^{x_{2n+1}}, P_{\lambda_{2n+2}}^{x_{2n+2}}, \bar{t}\right)$ ... (17)

Since $d(P_{\lambda_{2n}}^{x_{2n}}, P_{\lambda_{2n+2}}^{x_{2n+2}}, \mathrm{t}) \cong\leq d(P_{\lambda_{2n}}^{x_{2n}}, P_{\lambda_{2n+1}}^{x_{2n+1}}, \bar{t}) + d(P_{\lambda_{2n+1}}^{x_{2n+1}}, P_{\lambda_{2n+2}}^{x_{2n+2}}, \bar{t})$

Hence from (17),

$$d\left(P_{\lambda_{2n+1}}^{x_{2n+1}}, P_{\lambda_{2n+2}}^{x_{2n+2}}, \bar{t}\right) \widetilde{\leq} \frac{\bar{1}+\bar{k}}{\bar{a}-\bar{k}} d\left(P_{\lambda_{2n}}^{x_{2n}}, P_{\lambda_{2n+2}}^{x_{2n+2}}, \bar{t}\right) \ldots (18)$$

On other hand, we have

$$d(\delta_\rho\, (\varphi_\psi\, (P_{\lambda_{2n+1}}^{x_{2n+1}}), \varphi_\psi\, (P_{\lambda_{2n+1}}^{x_{2n+1}}), \bar{t}) + \bar{k}\ d(\delta_\rho\, (\varphi_\psi\, (P_{\lambda_{2n+1}}^{x_{2n+1}})), P_{\lambda_{2n+1}}^{x_{2n+1}}, \bar{t}) \widetilde{\geq} \bar{b} d\varphi_\psi\, (P_{\lambda_{2n+1}}^{x_{2n+1}}), P_{\lambda_{2n+1}}^{x_{2n+1}}, \bar{t}) \ldots (19)$$

Thus, we have

$$d\left(P_{\lambda_{2n-1}}^{x_{2n-1}}, P_{\lambda_{2n}}^{x_{2n}}, \bar{t}\right) + \bar{k}\, d\left(P_{\lambda_{2n-1}}^{x_{2n-1}}, P_{\lambda_{2n+1}}^{x_{2n+1}}, \bar{t}\right) \widetilde{\geq} \bar{b} d\left(P_{\lambda_{2n}}^{x_{2n}}, P_{\lambda_{2n+1}}^{x_{2n+1}}, \bar{t}\right) \ldots (20)$$

Since $d(P_{\lambda_{2n-1}}^{x_{2n-1}}, P_{\lambda_{2n+1}}^{x_{2n+1}}, \bar{t}) \widetilde{\leq} d(P_{\lambda_{2n-1}}^{x_{2n-1}}, P_{\lambda_{2n}}^{x_{2n}}, \bar{t}) + d(P_{\lambda_{2n}}^{x_{2n}}, P_{\lambda_{2n+1}}^{x_{2n+1}}, \bar{t})$

Hence from (20), we have

$$d\left(P_{\lambda_{2n}}^{x_{2n}}, P_{\lambda_{2n+1}}^{x_{2n+1}}, \bar{t}\right) \widetilde{\leq} \frac{\bar{1}+\bar{k}}{\bar{b}-\bar{k}} d\left(P_{\lambda_{2n-1}}^{x_{2n-1}}, P_{\lambda_{2n}}^{x_{2n}}, \bar{t}\right) \ldots (21)$$

Let $\bar{\lambda} = \max\{\frac{\bar{1}+\bar{k}}{\bar{b}-\bar{k}}, \frac{\bar{1}+\bar{k}}{\bar{a}-\bar{k}}\}$. Then by combining (18) and (21), we have

$$d\left(P_{\lambda_n}^{x_n}, P_{\lambda_{n+1}}^{x_{n+1}}, \bar{t}\right) \widetilde{\leq} \bar{\lambda}\, d\left(P_{\lambda_{n-1}}^{x_{n-1}}, P_{\lambda_n}^{x_n}, \bar{t}\right), \forall\, n \in \mathrm{N} \cup \{0\} \text{and for all } \bar{t} \widetilde{>} \bar{0} \ldots (22)$$

By repeating (21) ntimes, we get

$$d\left(P_{\lambda_n}^{x_n}, P_{\lambda_{n+1}}^{x_{n+1}}, \bar{t}\right) \widetilde{\leq} \bar{\lambda} d\left(P_{\lambda_0}^{x_0}, P_{\lambda_1}^{x_1}, \bar{t}\right) \ldots (23)$$

for all $n \in \mathrm{N} \cup \{0\}$ and for all $\bar{t} \widetilde{>} \bar{0}$.By Lemma 2.21, $\left\{P_{\lambda_n}^{x_n}\right\}_{n\in N}$ is a Cauchy sequence in the complete parametric soft metric space $(\tilde{X}, d)$. Then there exists $P_\rho^\varpi \widetilde{\in} \tilde{X}$ such that

$P_{\lambda_n}^{x_n} \to P_\rho^\varpi$ as $n \to +\infty$. Therefore $P_{\lambda_{2n+1}}^{x_{2n+1}} \to P_\rho^\varpi$ and $P_{\lambda_{2n+2}}^{x_{2n+2}} \to P_\rho^\varpi$ as $n \to +\infty$. Without loss of generality, we may assume that φ_ψ is continuous, then $\varphi_\psi\, (P_{\lambda_{2n+1}}^{x_{2n+1}}) \to \varphi_\psi\, (P_\rho^\varpi)$ as $n \to +\infty$. But $\varphi_\psi\, (P_{\lambda_{2n+1}}^{x_{2n+1}}) = P_{\lambda_{2n}}^{x_{2n}} \to P_\rho^\varpi$ as $n \to +\infty$. Thus, we have $\varphi_\psi\, (P_\rho^\varpi) = P_\rho^\varpi$. Since δ_ρ is surjective, there exists $P_\sigma^\vartheta \widetilde{\in} \tilde{X}$ such that $\delta_\rho(P_\lambda^{x^*}) = P_\rho^\varpi$. Now

$$d(\varphi_\psi\ (\delta_\rho(P_\sigma^\vartheta), \delta_\rho P_\sigma^\vartheta), \bar{t}) + \bar{k} d(\varphi_\psi\ \left(\delta_\rho\left(P_\sigma^\vartheta\right), P_\sigma^\vartheta, \bar{t}\right) \widetilde{\geq}\ \bar{a} d\left(\delta_\rho\left(P_\sigma^\vartheta\right), P_\sigma^\vartheta, \bar{t}\right) \ldots (24)$$

implies that $\bar{k} d(P_\rho^\varpi, P_\sigma^\vartheta, \bar{t}) \widetilde{\geq} \bar{a} d(P_\rho^\varpi, P_\sigma^\vartheta, \bar{t})$. Thus we gain $d\left(P_\rho^\varpi, P_\sigma^\vartheta, \mathrm{t}\right) \widetilde{\leq} \frac{\bar{k}}{\bar{a}} d\left(P_\rho^\varpi, P_\sigma^\vartheta, \bar{t}\right) \ldots (25)$

Since $\bar{a} \gtrsim \bar{k}$, we conclude that $d(P_\rho^\varpi, P_\sigma^\vartheta, \bar{t}) = \bar{0}$. So $P_\rho^\varpi = P_\sigma^\vartheta$. Hence, $\varphi_\psi(P_\rho^\varpi) = \delta_\rho(P_\rho^\varpi) = P_\rho^\varpi$. Therefore P_ρ^ϖ is a common fixed soft point of φ_ψ and δ_ρ. This completes the proof.

Theorem 3.4. Let $(\tilde{X}, d)$be a parametric soft metric space. Let $\varphi_\psi, \phi_\alpha : \tilde{X} \to \tilde{X}$ be mappings satisfying

$d(\varphi_\psi(P_\lambda^x), \varphi_\psi(P_\mu^y), \bar{t}) \lesseqqgtr \bar{a}\, d(\phi_\alpha(P_\lambda^x), \phi_\alpha(P_\mu^y), \bar{t}) + \bar{b} d(\phi_\alpha(P_\lambda^x), \varphi_\psi(P_\lambda^x), \bar{t}) + \bar{c} d(\phi_\alpha(P_\mu^y), \varphi_\psi(P_\mu^y), \bar{t}) \ldots .(26)$

for all $P_\lambda^x, P_\mu^y \tilde{\in} SP(\tilde{X})$ and all $\bar{t} \gtrsim \bar{0}$ where $\bar{a}, \bar{b}, \bar{c} \gtreqqless \bar{0}$ with

$\bar{a} + \bar{b} + \bar{c} \gtrsim \bar{1}$. Suppose the following hypotheses:

1) $\bar{b} \gtrsim \bar{1}$ or $\bar{c} \lesssim \bar{1}$;

2) $\phi_\alpha(\tilde{X}) \subseteq \varphi_\psi(\tilde{X})$;

3) $\varphi_\psi(\tilde{X})$ is a complete subspace of $\tilde{X}$.

Then φ_ψ and ϕ_α have a coincidence fixed soft point.

Proof. Let $P_{\lambda_0}^{x_0} \tilde{\in} SP(\tilde{X})$. Since $\phi_\alpha(\tilde{X}) \subseteq \varphi_\psi(\tilde{X})$, we choose $P_{\lambda_1}^{x_1} \tilde{\in} \tilde{X}$ such that $\varphi_\psi(P_{\lambda_1}^{x_1}) = \phi_\alpha(P_{\lambda_0}^{x_0})$. Again we can choose $P_{\lambda_2}^{x_2} \tilde{\in} \tilde{X}$such that $\varphi_\psi(P_{\lambda_2}^{x_2}) = \phi_\alpha(P_{\lambda_1}^{x_1})$. Continuing in the same way, we construct a sequence $\left\{P_{\lambda_n}^{x_n}\right\}_{n\in N}$ in $\tilde{X}$ such that $\varphi_\psi(P_{\lambda_{n+1}}^{x_{n+1}}) = \phi_\alpha(P_{\lambda_n}^{x_n})\ \forall\ n \in \mathbb{N} \cup \{0\}$.

If $\phi_\alpha(P_{\lambda_{m-1}}^{m-1}) = \phi_\alpha(P_{\lambda_m}^{x_m})$ for m ∈ N, then $\varphi_\psi(P_{\lambda_m}^{x_m}) = \phi_\alpha(P_{\lambda_m}^{x_m})$. Thus $P_{\lambda_m}^{x_m}$ is a coincidence point of φ_ψ and ϕ_α. Now assume that $P_{\lambda_{n-1}}^{x_{n-1}} \neq P_{\lambda_n}^{x_n}$ for all n ∈ ℕ. We have the following two cases

Case (1) Suppose $\bar{b} \lesssim \bar{1}$. By (26), we have

$d(\phi_\alpha(P_{\lambda_{n-1}}^{x_{n-1}}), \phi_\alpha(P_{\lambda_n}^{x_n}), \bar{t}) = d(\varphi_\psi(P_{\lambda_n}^{x_n}), \varphi_\psi(P_{\lambda_{n+1}}^{x_{n+1}}), \bar{t})$

$\gtreqqless \bar{a} d(\phi_\alpha(P_{\lambda_n}^{x_n}), \phi_\alpha(P_{\lambda_{n+1}}^{x_{n+1}}), \bar{t}) + \bar{b}\, d(\phi_\alpha(P_{\lambda_n}^{x_n}), \varphi_\psi(P_{\lambda_n}^{x_n}), \bar{t}) + \bar{c} d(\phi_\alpha(P_{\lambda_{n+1}}^{x_{n+1}}), \varphi_\psi(P_{\lambda_{n+1}}^{x_{n+1}}), \bar{t})$

$= \bar{a} d(\phi_\alpha\left(P_{\lambda_n}^{x_n}\right), \phi_\alpha(P_{\lambda_{n+1}}^{x_{n+1}}), \bar{t}) + \bar{b} d(\phi_\alpha(P_{\lambda_n}^{x_n}), \phi_\alpha(P_{\lambda_{n-1}}^{x_{n-1}}), \bar{t}) + \bar{c} d(\phi_\alpha(P_{\lambda_{n+1}}^{x_{n+1}}), \phi_\alpha\left(P_{\lambda_n}^{x_n}\right), \bar{t})$

Thus, we have

$(\bar{1} - \bar{b}) d(\phi_\alpha(P_{\lambda_{n-1}}^{x_{n-1}}), \phi_\alpha(P_{\lambda_n}^{x_n}), \bar{t}) \gtreqqless (\bar{a} + \bar{c}) d(\phi_\alpha(P_{\lambda_{n+1}}^{x_{n+1}}), \phi_\alpha(P_{\lambda_n}^{x_n}), \bar{t})$

Hence $d(\phi_\alpha(P_{\lambda_{n+1}}^{x_{n+1}}), \phi_\alpha(P_{\lambda_n}^{x_n}), \bar{t}) \lesseqqgtr \frac{\bar{1}-\bar{b}}{\bar{a}+\bar{c}} d(\phi_\alpha(P_{\lambda_{n-1}}^{x_{n-1}}), \phi_\alpha(P_{\lambda_n}^{x_n}), \bar{t}) \ldots$ (27)

Case (2) Suppose $\bar{c} \lesssim \bar{1}$. Also from (26), we have

$d(\phi_\alpha(P_{\lambda_n}^{x_n}), \left(P_{\lambda_{n-1}}^{x_{n-1}}\right), \bar{t}) = d(\varphi_\psi\left(P_{\lambda_{n+1}}^{x_{n+1}}\right), \varphi_\psi(P_{\lambda_n}^{x_n}), \bar{t}) \ldots$(28)

$\gtreqqless \bar{a} d(\phi_\alpha(P_{\lambda_{n+1}}^{x_{n+1}}), \phi_\alpha(P_{\lambda_n}^{x_n}), \bar{t}) + \bar{b} d(\phi_\alpha(P_{\lambda_{n+1}}^{x_{n+1}}), \varphi_\psi\left(P_{\lambda_{n+1}}^{x_{n+1}}\right), \bar{t}) + \bar{c}\, d(\phi_\alpha(P_{\lambda_n}^{x_n}), \varphi_\psi(P_{\lambda_n}^{x_n}), \bar{t})$

$= \bar{a} d(\phi_\alpha(P_{\lambda_n}^{x_n}), \phi_\alpha(P_{\lambda_{n+1}}^{x_{n+1}}), \bar{t}) + \bar{b} d(\phi_\alpha(P_{\lambda_{n+1}}^{x_{n+1}}), \phi_\alpha(P_{\lambda_n}^{x_n}), \bar{t}) + \bar{c}\, d(\phi_\alpha(P_{\lambda_n}^{x_n}), \phi_\alpha(P_{\lambda_{n-1}}^{x_{n-1}}), \bar{t})$

Thus, we have $(\bar{1} - \bar{c})d(\phi_\alpha(P_{\lambda_{n-1}}^{x_{n-1}}), \phi_\alpha(P_{\lambda_n}^{x_n}), \bar{t}) \geqq (\bar{a} + \bar{b})d(\phi_\alpha(P_{\lambda_{n+1}}^{x_{n+1}}), \phi_\alpha(P_{\lambda_n}^{x_n}), \bar{t})$

Hence

$d(\phi_\alpha (P_{\lambda_{n-1}}^{x_{n-1}}), \phi_\alpha(P_{\lambda_n}^{x_n}), \bar{t}) \leqq \frac{\bar{1}-\bar{c}}{\bar{a}+\bar{b}} d(\phi_\alpha (P_{\lambda_{n+1}}^{x_{n+1}}), \phi_\alpha(P_{\lambda_n}^{x_n}), \bar{t})$… (29)

In case (1), we let

$$\bar{\lambda}_1 = \frac{\bar{1} - \bar{b}}{\bar{a} + \bar{c}}$$

and in case (2), we let

$$\bar{\lambda}_2 = \frac{\bar{1} - \bar{c}}{\bar{a} + \bar{b}}$$

Define $\bar{\lambda} = \max\{\bar{\lambda}_1, \bar{\lambda}_2\}$. Thus in both cases, we have $\bar{\lambda} \lessapprox \bar{1}$. Hence

$d(\phi_\alpha (P_{\lambda_{n+1}}^{x_{n+1}}), \phi_\alpha(P_{\lambda_n}^{x_n}), \bar{t}) \leqq \bar{\lambda} d(\phi_\alpha(P_{\lambda_{n-1}}^{x_{n-1}}), \phi_\alpha(P_{\lambda_n}^{x_n}), \bar{t})$… (30)

Repeating (30), n-times, we obtain

$d(\phi_\alpha(P_{\lambda_{n+1}}^{x_{n+1}}), \phi_\alpha(P_{\lambda_n}^{x_n}), \bar{t}) \leqq \bar{\lambda}^n d(\phi_\alpha(P_{\lambda_0}^{x_0}), \phi_\alpha(P_{\lambda_1}^{x_1}), \bar{t})$…(31)

So for m > n, we have

$d\left(\phi_\alpha\left(P_{\lambda_n}^{x_n}\right), \phi_\alpha\left(P_{\lambda_m}^{x_m}\right), \bar{t}\right) \leqq d\left(\phi_\alpha\left(P_{\lambda_n}^{x_n}\right), \phi_\alpha\left(P_{\lambda_{n+1}}^{x_{n+1}}\right), \bar{t}\right) + d\left(\phi_\alpha\left(P_{\lambda_{n+1}}^{x_{n+1}}\right) \phi_\alpha\left(P_{\lambda_{n+1}}^{x_{n+1}}\right), \bar{t}\right) + \cdots +$
$d(\phi_\alpha\left(P_{\lambda_{m-1}}^{x_{m-1}}\right), \phi_\alpha\left(P_{\lambda_m}^{x_m}\right), \bar{t}) \leqq \left(\bar{\lambda}^n + \bar{\lambda}^{n+1} + \cdots + \bar{\lambda}^{m-1}\right) d(P_{\lambda_0}^{x_0}, P_{\lambda_1}^{x_1}, \bar{t})$

$$\leqq \frac{\bar{\lambda}^n}{\bar{1}-\bar{\lambda}} d\left(\phi_\alpha(P_{\lambda_0}^{x_0}\right), \phi_\alpha\left(P_{\lambda_1}^{x_1}\right), \bar{t}) \quad \ldots(32)$$

for all $\bar{t} \gtrapprox \bar{0}$. Since $\bar{\lambda} \lessapprox \bar{1}$.

By taking limit as n,m→+∞ in above inequality (32), we get $\lim_{n,m\to\infty} d(\phi_\alpha (P_{\lambda_n}^{x_n}), \phi_\alpha (P_{\lambda_m}^{x_m}), \bar{t}) = \bar{0}$ for all $\bar{t} \gtrapprox \bar{0}$. Therefore $\left\{\varphi_\psi (P_{\lambda_n}^{x_n})\right\}_{n\in N}$ is a Cauchy sequence in $\varphi_\psi \tilde{X}$. Since $\varphi_\psi \tilde{X}$ is a complete subspace of $\tilde{X}$, there is $P_\rho^\varpi \tilde{\in} SP(\tilde{X})$ such that $\left\{\varphi_\psi (P_{\lambda_n}^{x_n})\right\}_{n\in N}$ converges $\varphi_\psi (P_\rho^\varpi)$ as n → +∞. Hence $\phi_\alpha(P_{\lambda_n}^{x_n})$ converges to $\varphi_\psi (P_\rho^\varpi)$ as n → +∞ . Since $\bar{a} + \bar{b} + \bar{c} > \bar{1}$. We have $\bar{a}, \bar{b}$ *and* $\bar{c}$ are not all $\bar{0}$. So we have the following cases.

Step 1: If $\bar{a} \neq \bar{0}$, then

$d(\varphi_\psi (P_{\lambda_n}^{x_n}), \varphi_\psi (P_\rho^\varpi), \bar{t}) \geqq \bar{a} d(\phi_\alpha (P_{\lambda_n}^{x_n}), \phi_\alpha(P_\rho^\varpi), \bar{t}) + \bar{b}\, d(\phi_\alpha (P_{\lambda_n}^{x_n}), \varphi_\psi (P_{\lambda_n}^{x_n}), \bar{t}) + \bar{c}\, d(\phi_\alpha(P_\rho^\varpi), \varphi_\psi (P_\rho^\varpi), \bar{t})$

$\geqq \bar{a} d(\phi_\alpha (P_{\lambda_n}^{x_n}), \phi_\alpha (P_\lambda^x)$

Hence $d(\phi_\alpha (P_{\lambda_n}^{x_n}), (\phi_\alpha(P_\rho^\varpi), \bar{t}) \leqq \frac{1}{\bar{a}} d(\varphi_\psi (P_{\lambda_n}^{x_n}), \varphi_\psi (P_\rho^\varpi), \bar{t})$… (33)

Since $\frac{1}{\bar{a}}d(\varphi_\psi(P_{\lambda_n}^{x_n}), \varphi_\psi(P_\rho^{\varpi}), \bar{t}) \to 0$ as $n \to +\infty$. Thus $\phi_\alpha(P_{\lambda_n}^{x_n}) \to (\phi_\alpha(P_\rho^{\varpi})$ as $n \to +\infty$. By uniqueness of limit, we have $\varphi_\psi(P_\rho^{\varpi}) = \phi_\alpha(P_\rho^{\varpi})$. Therefore φ_ψ and ϕ_α have a coincidence point.

Step 2: If $b \neq 0$, then

$$d(\varphi_\psi(P_\rho^{\varpi}), \varphi_\psi(P_{\lambda_n}^{x_n}), \bar{t}) \gtrsim \bar{a}d(\phi_\alpha\left(P_{\lambda_n}^{x_n}\right), \phi_\alpha(P_\rho^{\varpi}), \bar{t}) + \bar{b}d(\phi_\alpha(P_\rho^{\varpi}), \varphi_\psi(P_\rho^{\varpi}), \bar{t}) + \bar{c}\, d(\phi_\alpha\left(P_{\lambda_n}^{x_n}\right), \varphi_\psi\left(P_{\lambda_n}^{x_n}\right), \bar{t})$$

$$\gtrsim \bar{b}d(\phi_\alpha\left(P_\mu^{y^*}\right), \phi_\alpha\left(P_\mu^{y^*}\right), \bar{t})$$

Hence $\quad d(\phi_\alpha\left(P_\rho^{\varpi}\right), \varphi_\psi(P_\rho^{\varpi}), \bar{t}) \lesssim \frac{1}{\bar{b}}d(\varphi_\psi(P_\rho^{\varpi}), \varphi_\psi(P_{\lambda_n}^{x_n}), \bar{t})$… (34)

As similar proof of case (1), we can show that $\phi_\alpha\left(P_\rho^{\varpi}\right) = \varphi_\psi\left(P_\rho^{\varpi}\right)$. thus ϕ_α and φ_ψ have a coincidence point.

Step 3: If $\bar{c} \neq \bar{0}$, then

$$d(\varphi_\psi\left(P_{\lambda_n}^{x_n}\right), \varphi_\psi(P_\rho^{\varpi}), \bar{t}) \gtrsim \bar{a}d(\phi_\alpha\left(P_{\lambda_n}^{x_n}\right), \phi_\alpha(P_\rho^{\varpi}), \bar{t}) + \bar{b}d(\phi_\alpha\left(P_{\lambda_n}^{x_n}\right), \varphi_\psi\left(P_{\lambda_n}^{x_n}\right), \bar{t} + \bar{c}d(\varphi_\psi(P_\rho^{\varpi}), \phi_\alpha(P_\rho^{\varpi}), \bar{t})$$

$$\gtrsim \bar{c}d(\phi_\alpha\left(P_\rho^{\varpi}\right), \varphi_\psi\left(P_\rho^{\varpi}\right), \bar{t})$$

Hence $\quad d(\phi_\alpha\left(P_\rho^{\varpi}\right), \varphi_\psi\left(P_\rho^{\varpi}\right), \bar{t}) \lesssim \frac{1}{\bar{c}}d(\varphi_\psi(P_{\lambda_n}^{x_n}), \varphi_\psi(P_\rho^{\varpi}), \bar{t})$…(35)

for all $\bar{t} \gtrsim \bar{0}$. As similar proof of case (1), we can Show that $\phi_\alpha\left(P_\rho^{\varpi}\right) = \varphi_\psi(P_\rho^{\varpi})$. thus ϕ_α and φ_ψ have a coincidence fixed soft point.

Corollary 3.5. Let $(\tilde{X}, d)$ be a parametric soft metric space. Let $\varphi_\psi, \phi_\alpha: \tilde{X} \to \tilde{X}$ mappings satisfying

$d(\varphi_\psi(P_\lambda^x), \varphi_\psi(P_\mu^y), \bar{t}) \gtrsim \bar{a}d(\phi_\alpha(P_\lambda^x), \phi_\alpha(P_\mu^y), \bar{t}) + \bar{b}d(\phi_\alpha(P_\lambda^x), \varphi_\psi(P_\lambda^x), \bar{t})$…(36)

For all $P_\lambda^x, P_\mu^y \tilde{\in} SP(\tilde{X})$, $P_\lambda^x \neq P_\mu^y$ and all $\bar{t} \gtrsim \bar{0}$, where $\bar{a}, \bar{b} \gtrsim \bar{0}$ with $\bar{a} + \bar{b} \gtrsim \bar{1}$ and $\bar{b} \gtrsim \bar{1}$. Suppose the following hypotheses:

1) $\phi_\alpha(\tilde{X}) \subseteq \varphi_\psi(\tilde{X})$;

2) $\varphi_\psi(\tilde{X})$, is a complete subspace of $\tilde{X}$.

Then φ_ψ and ϕ_α have a coincidence point.

Corollary 3.6. Let $(\tilde{X}, d)$ be a parametric soft metric space. Let $\varphi_\psi, \phi_\alpha: \tilde{X} \to \tilde{X}$ mappings satisfying

$d\left(\varphi_\psi(P_\lambda^x), \varphi_\psi\left(P_\mu^y\right), \bar{t}\right) \gtrsim \bar{a}d(\phi_\alpha(P_\lambda^x), \phi_\alpha(P_\mu^y), \bar{t})$…(37)

for all $P_\lambda^x, P_\mu^y \tilde{\in} SP(\tilde{X})$, and all $\bar{t} \gtrsim \bar{0}$, where $\bar{a} \gtrsim \bar{1}$. Suppose the following hypotheses:

1) $\phi_\alpha(\tilde{X}) \subseteq \varphi_\psi(\tilde{X})$;

2) $\varphi_\psi(\tilde{X})$ is a complete subspace of $\tilde{X}$.

Then φ_ψ and ϕ_α have a coincidence point.

Corollary 3.7. Let $(\tilde{X}, d)$ be a parametric soft metric space. Let $\varphi_\psi: \tilde{X} \to \tilde{X}$ mappings satisfying

$d\big(\varphi_{\psi}(P_{\lambda}^{x}),\varphi_{\psi}(P_{\mu}^{y}),\bar{t}\big) \cong \bar{a}d\big(P_{\lambda}^{x},P_{\mu}^{y},\bar{t}\big)+\bar{b}d(P_{\lambda}^{x},\varphi_{\psi}(P_{\lambda}^{x}),\bar{t})+\bar{c}dP_{\mu}^{y},\varphi_{\psi}(P_{\mu}^{y}),\bar{t})$…(38)

for all $P_{\lambda}^{x},P_{\mu}^{y} \tilde{\in} SP(\tilde{X})$ and all $\bar{t} \tilde{>} \bar{0}$, where $\bar{a},\bar{b},\bar{c} \tilde{\geq} \bar{0}$ with $\bar{a}+\bar{b}+\bar{c} \tilde{>} \bar{1}$.

Suppose $\bar{b} \tilde{>} \bar{1}$ or $\bar{c} \tilde{<} \bar{1}$. Then φ_{ψ} has a fixed point.

Corollary 3.8. Let $(\tilde{X},d)$ be a parametric soft metric space. Let $\varphi_{\psi}:\tilde{X} \to \tilde{X}$ mappings satisfying

$d(\varphi_{\psi}(P_{\lambda}^{x}),\varphi_{\psi}(P_{\mu}^{y}),\bar{t}) \tilde{\geq} \bar{a}d(P_{\lambda}^{x},P_{\mu}^{y},\bar{t})$….(39)

for all $P_{\lambda}^{x},P_{\mu}^{y} \tilde{\in} SP(\tilde{X})$ and all $\bar{t} \tilde{>} \bar{0}$, where $\bar{a} \tilde{>} \bar{1}$. Then φ_{ψ} has a fixed point.

Corollary 3.9. Let $(\tilde{X},d)$ be a parametric soft metric space. Let $\varphi_{\psi}:\tilde{X} \to \tilde{X}$ mappings satisfying

$d(\varphi_{\psi}(P_{\lambda}^{x}),\varphi_{\psi}(P_{\mu}^{y}),\bar{t}) \tilde{\geq} \bar{a}d(P_{\lambda}^{x},P_{\mu}^{y},\bar{t})+\bar{b}d(P_{\lambda}^{x},\varphi_{\psi}(P_{\lambda}^{x}),\bar{t})$….(40)

for all $P_{\lambda}^{x},P_{\mu}^{y} \tilde{\in} SP(\tilde{X})$ and all $\bar{t} \tilde{>} \bar{0}$, where $\bar{a},\bar{b} \tilde{\geq} \bar{0}$ with $\bar{a}+\bar{b} \tilde{>} \bar{1}$.

Suppose $\bar{b} \tilde{>} \bar{1}$. Then φ_{ψ} has a fixed point.

Corollary 3.10. Let $(\tilde{X},d)$ be a parametric soft metric space. Let $\varphi_{\psi}:\tilde{X} \to \tilde{X}$ mappings satisfying

$d(\varphi_{\psi}(P_{\lambda}^{x}),\varphi_{\psi}(P_{\mu}^{y}),\bar{t}) \tilde{\geq} \bar{a}d(P_{\lambda}^{x},P_{\mu}^{y},\bar{t})+\bar{b}d(P_{\mu}^{y},\varphi_{\psi}(P_{\mu}^{y}),\bar{t})$…(41)

for all $P_{\lambda}^{x},P_{\mu}^{y} \tilde{\in} SP(\tilde{X})$ and all $\bar{t} \tilde{>} \bar{0}$, where $\bar{a},\bar{b} \tilde{\geq} \bar{0}$ with $\bar{a}+\bar{b} \tilde{>} \bar{1}$. Suppose $\bar{b} \tilde{>} \bar{1}$. Then φ_{ψ} has a fixed point.

4. Conclusion

Fixed point theory has fascinated many researchers since 1922 with the celebrated Banach's fixed point theorem. There exists a vast literature on the topic and is a very active field of research at present. Many researchers have been applied/embedded this theory in different types of metric spaces as well as applicable sciences. In this merit, we decide to investigate the existence and the uniqueness of fixed soft points of self soft mappings in parametric soft metric spaces, which spaces the parameterization tool plays the key role. For further research, we aim to investigate some different kinds of fixed soft point theorems, some fixed circle theorems and also we plan to give some applications.

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Artificial Neural Networks for Non-parametric Regression with Biological Data

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Abstract

In this paper, we consider the deep learning network model in the construction of biological networks. This approach has been commonly used in recent years for modeling complex nonlinear regression model and for the classification problem. Here, we have adapted this approach as a regression model for the representation of the protein-protein interaction networks. Thus, we have compared its performance with the Gaussian graphical model (GGM) which is one of the well known graphical models to describe the biological systems. In the calculation of GGM, we have also implemented the bootstrap procedure to increase the number of observations and the consistent AIC as well as information and complexity approaches as the model selection criteria within GGM to improve the accuracy of estimates. In the analyses, we have used two real bench-mark datasets and compared the accuracy of the deep learning method with the underling GGM.

Keywords: Deep neural networks, nonparametric regression, biological data

1. Introduction

Deep neural networks have huge success in different areas such as image progressing, analyses of biomedical signals and financial time series in recent years. In this paper, we have suggested this neural network model as generalized additive models(GAMs). Because, GAMs have great flexibility to explain explanatory variables while capturing non-linearities in the regression model. In order to solve the problem of nonlinearity, Hastie and Tibshirani (1990) have suggested the multilayer feed forward neural network approach. Then, the non- parametric regressions are proposed by Scmidt-Hieber (2020) , in particular, when the number of network's parameter is greater than the number of samples which is called the problem of over-parametrization. Moreover, we have also compared the proposed generalized additive neural networks (GANNs) and Gaussian Graphical model (GGM) with bootstrap scheme and two model selection criteria inserted in GGM. Bootstrap scheme is known as a computationally efficient variance reduction technique. Furthermore, the consistent AIC (CAICF) and ICOMP selection criteria are suggested by Bozdogan (1987,2010) to obtain more consistent selection criterion derived from the Fisher information matrix. For this reason, Bülbül et al. (2019) and Kaygusuz and Purutçuoğlu (2019) have suggested CAICF and ICOMP selection criteria for Gaussian graphical model. We have examined the performance of the proposed generalized additive model and GGM for two biological datasets.

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2 Theory and Methods

2.1. Feed-forward Neural Networks

Feed-forward neural network (FNN), also known as deep learning network, is one of the fundamental artificial neural networks in the literature. In this method, the information goes always in one direction from input layer to output layer. In other words, the network does not consist of any cycles or loops. The multilayer feed-forward neural network is developed by the Bauer and Kohler (2019) since it has been considered the curse of dimensionality problem for single-index models. FNN uses the sigmoid activation function σ : R → [0,1] with the following equation. In the above expression, we have assumed that there are L- hidden layers and $N_1, N_2, \ldots, N_L$ neurons for the L-th layer.

$$\mathrm{f(x)}=\sum_{i=1}^{N_L} c_i^{(L)} f_i^{(L)}(x) + c_{\,0}^{(L)} \tag{1}$$

where $c_{\,0}^{(L)}, \ldots, c_{N_L}^{(0)}$ and $f_i^{(L)}$ may be defined via:

$$f_i^{(s)}(x) = \sigma(\sum_{L=1}^{N_{s-1}} c_{i,j}\, f_j^{(s-1)}(x) + c_{\,j,0}) + c_{i,0}^{(s-1)} \tag{2}$$

$w$hile $c_{i,0}^{(s-1)}, \ldots, c_{i,N_{s-1}}^{(s-1)}$ and s=2,…,L

$$f_i^{(1)}(x) = \sigma(\sum_{j=1}^{d} c_{i,j}^{(0)}\, x^{(i)} + c_{\,i,0}^{(0)}) \tag{3}$$

Neural networks apply activation function to estimate parameters with σ : R → [0,1] which is non-decreasing and must satisfy

$$\lim_{z\to -\infty} \sigma(\mathrm{z}) = 0 \text{ and } \lim_{z\to \infty} \sigma(\mathrm{z}) = 1 \tag{4}$$

this expression might be called the sigmoidal function or logistic squasher

$$\sigma(\mathrm{z}) = \frac{1}{1+\exp(-z)} \tag{5}$$

while ($z \in R$). In the literatüre of the deep learning, besides the sigmodial activation function, there are other alternative activation functions too. One of the well known choices is the Re-Lu activation function due to it is flexibility in our analyses, we prefer this choice. On the other hand, we can define neural networks with distinct number of hidden layers and in here, we use one hidden layer by the following form:

$$f(x) = \sum_{j=1}^{N} c_j\, \sigma(\sum_{L=1}^{d} c_{j,n}\, x^k + c_{\,j,0}) + c_{\,0} \tag{6}$$

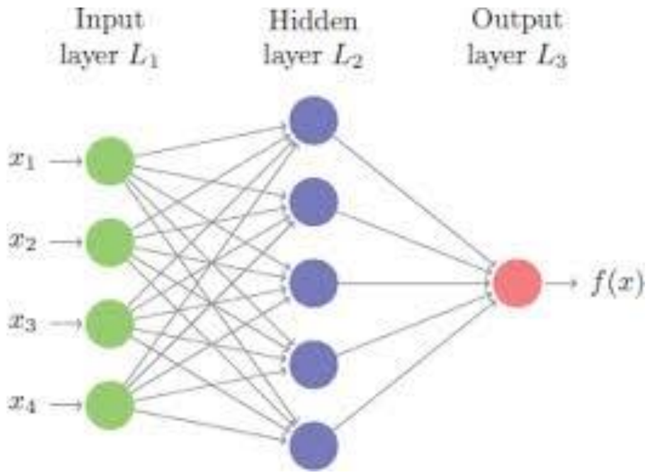

Figure 1 : An example of Feed-forward neural networks

In general deep neural network is used for the classification purpose. But in this study, we apply it as regression model by keeping the weight coefficients as the regression coefficient and conducting this model for each gene in the system iteratively so that each gene can be modelled in terms of other genes in the system. Then, the regression coefficients obtained from each gene model are converted as a binary value via a threshold value. In this analysis, we set the coefficient to zero if the number in absolute value is closer to zero; otherwise, we set it to one.

Once all the weights, i.e., regression co-efficients of gene specific model, are converted to binary values, we generate two rules in the construction of the adjacency matrix:

1- OR rule: If one of the genes indicates the value 1 in binary conversion between the selected gene in response side and other gene is predictor side and vice versa, we accept as 1 between these two genes if they have both 1 binary coefficients or 0-1 as well as 1-0 binary coefficients.

2- AND rule: We accept link between two genes if they both show 1 in binary values when they are response and predictor of each other. Therefore we consider that they are connected if both regression coefficients produce 1 in binary entries.

Once we decide on the rule, we can generate a symmetric adjacency matrix which indicates an undirected graphical structure. In the decision of final rule, currently we check the accuracy of both results with the true networks. The rule which produces the highest accuracy is selected as the ultimate rule of the network construction. Whereas, for a realistic application when the true network is unknown, we consider to evaluate this question in a comprehensive simulation study.

2.2. Generalized Additive Models

Linear model can be too restrictive for some scientific applications to explain dependent variables. For this reason, Hastie and Tibshirani (1990) proposed the generalized additive models (GAMs). In this modelling, let X_i and Y_i (i = 1, . . . , n) be independent samples of random pairs (X, Y) with following regression model:

$$Y = \alpha + f(X) + \varepsilon \tag{7}$$

where ε is distributed with N(mu,sigma). Another assumption for GAMs is

$$f(x_1, \ldots, x_d) = f_1(x_1) + \ldots + f_d(x_d) \tag{8}$$

where the component functions f_j's are obtained non-parametrically by the method of Hastie and Tibshirani (1990). Now, we can present GAMs by the following equation:

$$Y = \alpha + \sum_{j=1}^{p} f_j(x_j) + \varepsilon \tag{9}$$

in which the error term ε has zero mean.

3. Bootstrap Scheme for Graphical Lasso Algorithm

This section includes the definition of graphical lasso algorithm and bootstrap scheme inserted in this algorithm to improve the accuracy in high dimensional settings. We also suggest to use CAICF and ICOMP selection criteria in the lasso method so that we can compare the performance of both the underlying graphical lasso approach with the deep learning regression model in two real biological datasets.

3.1. Graphical Lasso Algorithm

Graphical lasso algorithm is suggested by Bühlmann and Meinhausen (2006) to estimate the inverse covariance matrix when the number of variables (p) is much more than the number of observations (n). Because the ordinary maximum likelihood approach have some limitations in high dimensional settings. For this reason, Yuan and Lin (2007) suggested a penalized likelihood method to solve this problem.

Let us assume that independent identically distributed (i.i.d.) sample ($X = X_{(1)}, \ldots, X_{(n)}$) comes from normal distribution with mean μ and unknown variance Σ,i.e., $X \sim N_d(\mu, \Sigma)$. Thereby, the aim is to infer $(\Sigma)^{-1} = (\Theta)^*$ when the number of observations (n) is more than the number of random variables (p). Accordingly, Friedman et al. (2007) gave the following definition for the graphical lasso procedure.

$$L(\Theta) = \log\det\Theta - tr(S\Theta) - \lambda\|\Theta\|_1 \tag{10}$$

where λ is a non-negative tuning parameter controlling the amount of l_1-shrinkage, S is the empirical covariance matrix and $\Theta = \Sigma^{-1}$ refers to a non-negative precision matrix while an element θ_{ij} implies that corresponding variables X_j and X_k are conditionally independent. Moreover, tr(.) denotes the trace and det(.) shows the determinant of the given matrix. In the above expression, if S is singular and λ is enough big, the estimate of $\Theta^{\hat{}}$ can be sparse.

3.2. Bootstrap Scheme

Bootstrap scheme is very beneficial, in particular, when we work with limited observations' datasets. Hence, this regime can be applicable in the estimation of the sampling distribution from almost any statistics by using splitting methods. Furthermore, it can be used when the approximate distribution is available. We also assume that observations come from i.i.d. population F.

Thus, we can summarize the scheme for the non-parametric bootstrap algorithm as follows:

. (i) Build the sample probability distribution $\hat{F}$ which divides n at each point $x_1, x_2, x_3 \dots, x_n$
(ii) When $\hat{F}$ is constant, draw a random sample of size n from $\hat{F}$ via $X_i^* = x_i^*, X_i^* = \hat{F}$ when i = 1,2,...,n. It is called bootstrap sample, $X* = (X_1^*, \dots, X_n^*)$,

. while $x* = (x_1^*, \dots, x_n^*)$.

. (iii) Compute the bootstrapped estimators $\hat{\theta}_n^*$= f($X_1^*, \dots, X_n^*$) X_1 , Y_1 based on the bootstrap sample.

. (iv) Repeat step 1, 2, 3,..., B times to obtain $\hat{\theta}^{*,1}_{.}, \dots, \hat{\theta}^{*,B}_{.}$ for B repetitions.

(v) Bootstrapped estimators are approximately the bootstrap expectation and the bootstrap

variance via E($\hat{\theta}_n^*$)= 1/B $\sum_{i=1}^{B} \theta_n^{*,i}$ and V ar($\hat{\theta}_n^*$)= 1/(B − 1) $*$ $\sum_{i=1}^{B} \theta_n^{*,i}$- $\sum_{j=1}^{B} \theta_n^{*,j2}$

respectively.

3.3. CAICF and ICOMP Selection Criteria

There are two classical model selection criteria, namely, Akaike's information criterion (AIC) and Schwartz's Bayesian information criterion that are applicabe in different fields. In order to improve the performance of accuracy, Bozdoğan (1987,2010) proposed two alternative approaches. The first method is called the consistent AIC selections criterion (CAIC) which makes the distance between the true model and the real value as small as possible. In the study of Bozdoğan (1987), the smallest distance is computed by the Kullback-Leibler divergence and this criterion has the followinf form:

$$CAIC(k) = -2\log L(\hat{\theta}_k) + k[logn + 1] \tag{11}$$

in which the likelihood of θ is shown by logL($\hat{\theta}_k$) and k denotes the degrees of freedom of the distribution. There is no need to say the similarity between the CAIC(k) and the BIC of k log n and k[log n + 1] terms that have a stronger penalty term.

The second Bozdoğan's model selection criterion which is based on the consistent AIC with Fisher information matrix, also shown by CAICF(k)), has an increasing penalty term for the over-parametrization whose expression is as follows:

$$\text{CAICF(k)} = -2\log L(\hat{\theta}_k) + k[\log n + 2] + \log|I(\hat{\theta}_k)|. \tag{12}$$

Here, logL($\hat{\theta}_k$) indicates the likelihood estimation of θ, as used beforehand, k presents the degrees of freedom of the distribution and I($\hat{\theta}_k$) $\hat{F}^{-1}$ represents the Fisher information matrix.

As an extension of this method, Bozdoğan (2010) also proposed the Information and COMPlexity (ICOMP) measure. Basically, ICOMP can penalize the free parameters and the covariance matrix directly with a third term. This third term in the loss function has a capability to calculate the distance when the parameter estimates are correlated in the model fitting stage. Hence, the expression for ICOMP can be presented as below.

$$ICOMP = -2\log L(\hat{\theta}_k) + 2C(\hat{\Sigma}\ ) \tag{13}$$

where logL($\hat{\theta}_k$) is the log-likelihood of E, $\hat{\theta}_k$ shows the maximum likelihood estimate of the parameter vector of θ_k, C expresses a real-valued complexity measure and finally, $\hat{\Sigma} = \widehat{cov}(\hat{\theta}_k)$ refers to the estimated covariance matrix of the parameter vector of the candidate model. This covariance matrix can be obtained in different ways. Bozdoğan (2010)` s choice is the computation of the inverse of the Cramer-Rao lower bound matrix that is obtained from the estimated inverse Fisher information matrix with the following equation.

$$\hat{F}^{-1} = \left\{-E\left(\frac{\partial^2 logL(\theta)}{\partial\theta\partial\theta'}\right)\right\}^{-1} \tag{14}$$

In this expression, the (s × s)-dimensional second-order partial derivatives of the log-likelihood function of the estimated model is denoted by $\hat{F}^{-1}$. As a result, a more general form of ICOMP can be expressed via

$$ICOMP = -2\log L(\hat{\theta}_k) + 2C(\hat{F}^{-1}) \quad (15)$$

when

$C(\hat{F}^{-1})$=$\frac{s}{2}\log\left[\frac{\widehat{tr F}^{-1}}{s}\right] - \frac{1}{2}\log\left|\hat{F}^{-1}\right|$.

In this expression, the second term shows the information complexity of the estimated inverse Fisher information matrix of the model and $s = \dim\,(\hat{F}^{-1}) = rank(\hat{F}^{-1})$ while dim(.) shows the dimension of the given matrix.

4. Application

We implement the proposed deep neural networks and Gaussian graphical model in two biological datasets, namely, cell signalling data and human genome expression data. In the calculation of Gaussian graphical model, we infer the model parameters via the lasso regression approach and apply the non-parametric bootstrap scheme to increase the sample size. Finally, we compute CAICF and ICOMP model selection criteria for the decision of the optimal model. On the other hand, we calculate distinct accuracy measures from the estimated values of neural networks and its adjacency matrix. We compute the accuracy, precision, F-score and recall values in the comparison of models. In the representation of the adjacency matrix via the deep learning model, since we need to construct a symmetrix matrix, we use both AND and OR rule. Shortly, the AND rule implies that if both X_i and X_j indicate significant regression coefficients for the regression model while the response is X_i and X_j , respectively, then we can denote the entries of the X_{ij} and X_{ji} by one. Whereas in the OR rule, if we find either X_i and X_j 's regression coefficient as significant, then we can present the entries of the adjacency matrix for X_{ij} and X_{ji} as one. Otherwise, we denote the zero value for the associated entries. In the estimation part, we presented the results from AND rule since its accuracy is higher than the OR rule in both datasets.

Below, we presented the description of the selected real datasets:

. **Cell signaling dataset:** This dataset contains the flow of cytometry results of 11 phosphorylated proteins and phospholipids measured on 11.672 red blood cells. These components are the part of the cellular protein-signaling network of human immune system cells. This dataset is studied by

Sacks et al.(2005) to investigate the reactions of the native state tissue signaling biology and drug actions. Thus, the aim of the construction of this network is to understand the native-state tissue signaling biology, complex drug actions and the dysfunctional signals in diseased cells.

. **Human genome expression dataset:** This dataset involves the gene expression of B- lymphocyte cells from the Utah residents with Northern and Western European ancestry sample. The genes of 60 unrelated individuals are examined for 100 different transcripts. From the 55 biologically validated links, 45 have the names of the transcription factor and target genes in the network of gene expression data [0]). For this reason, for the inference of both models we use these 45 links for the calculation of the accuracy measures.

From the tabulated values it is seen that the accuracy and recall values are best under the deep learning method from both datasets. On the other side, F-score of the cell signalling data is high under the deep learning model. For the remaining entries the inference conducted by the graphical lasso approach with bootstrap is better. Furthermore, we do not observe difference in the selected two model selection criteria under the graphical lasso approach.

Table.1 The results of graphical lasso algorithm with non-parametric bootstrap scheme with ICOMP and CAICF model selection criteria.

Data	Measures	ICOMP	CAICF
Cell signal n = 11672 p = 11	Accuracy	0.719	0.719
	Precision	1.000	1.000
	F-score	0.393	0.393
	Recall	0.244	0.244
Gene expression n = 60 p = 100	Accuracy	0.991	0.991
	Precision	1.000	1.000
	F-score	0.690	0.690
	Recall	0.526	0.526

Table.2 The results of deep learning model whose accuracy measures are computed based on the AND rule

Data	Measures	Deep Learning
Cell signal n = 11672 p = 11	Accuracy Precision F-score Recall	0.661 0.577 0.436 0.344
Gene expression n = 60 p = 100	Accuracy Precision F-score Recall	0.954 0.383 0.433 0.926

5. Conclusion

We have proposed an alternative model selection algorithm for Gaussian graphical model whose inference is done via the L_1-norm penalized lasso approach with bootstrap scheme and CAICF as well as ICOMP model selection criteria. On the other hand, we have used the Re-Lu function for one layer deep learning method whose adjacency matrix is generated by the AND rule. The results have indicated that the deep learning is best under accuracy and recall measures and it is comparable under the F-score regarding the estimated network by the Gaussian graphical model. Therefore, we believe that the deep learning approach can be a promising alternative method for the construction of biological networks specifically under large model parameters. However, in order to make more precise decision, we consider to perform this approach in more datasets too.

In future work, we also think that CAICF and ICOMP selection procedures may be good candidates for multilayer feedforward neural networks, i.e. the deep learning approach, as well. Moreover, the deep neural network can be extended by inserting the bootstrap scheme while the data have limited observations. Furthermore, we consider to apply other bootstrap schemes such as wild bootstrap and

Bayesian bootstrap in both Gaussian graphical model and deep learning models so that we can evaluate whether the accuracy can be improved for model specific techniques.

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Some Results on Rough Weighted Ideal Statistical Convergence of Sequences

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Abstract

Two classes of sets are examined: rough weighted I-statistical limit set and weighted I-statistical cluster points set which are generalizations of rough I-limit set and I-cluster points set respectively. Our main aim is to give the different behaviors of the new convergences and characterize both the sets with topological approach like closedness, boundedness, compactness etc.

Keywords: Rough weighted I-statistical limit set, rough weighted I-statistical cluster points set, rough weighted ideal statistical convergence.

1. Introduction

In order to extend convergence of sequences, the notion of statistical convergence was given by Fast [1] for the real sequences. Afterward, it was further researched from sequence point of view and connected with the summability theory (see [2-7]). The concept of ideal convergence which is a generalization of statistical convergence was presented by Kostyrko et al. [8]. Kostyrko et al. [9] examined some features of I-convergence and extremal I-limit points. Ideal convergence became a noteworthy topic in summability theory after the studies of [10-11].

The idea of rough convergence was first defined by Phu [12] in finite-dimensional normed spaces. In another paper [13] related to this subject, Phu worked the rough continuity of linear operators and denoted that every linear operator $f: X \to Y$ r-continuous at every point $x \in X$ under the assumption $dim Y < \infty$ and $r > 0$, where X and Y are normed spaces. In [14], Phu extended the results given in [12] to infinite-dimensional normed spaces. Aytar [15] investigated the rough statistical convergence. Also, Aytar [16] studied that the rough limit set and the core of a real sequence. Pal et al. [17] and Dündar et al. [18] independently extended the result given in [15] to rough I-convergence. Savaş et al. [19] examined rough I-statistical convergence as an extension of rough convergence and defined the set of rough I-statistical limit points of a sequence and analyzed the results with proofs.

Weighted statistical convergence introduced by Karakaya and Chisti in [20]. Also Küçükaslan studied this concept in [21]. Then the modified definition is given by Mursaleen et al. in [22] as follow:
Let p_k be a positive sequence of nonnegative numbers such that $p_0 = 0$ and $P_n = \sum_{k=0}^{n} p_k \to \infty$ as $n \to \infty$. A sequence x_k is weighted statistically convergent to x if for $\varepsilon > 0$, the set $\{k \in \mathbb{N}: p_k|x_k - x| \geq \varepsilon\}$ has weighted density zero, i.e.

$$\lim_{n\to\infty}\frac{1}{P_n}|\{k\leq P_n: p_k|x_k-x|\geq\varepsilon\}|=0.$$

It is indicated by $S_{\bar{N}}-\lim_{k\to\infty}x_k=x$. $S_{\bar{N}}$ denote the set of these sequences.
Let $K\subseteq\mathbb{N}$, the set of positive integers, the weighted density of K is defined by

$$\delta_{\bar{N}}=\lim_{n\to\infty}\frac{1}{P_n}|\{k\leq P_n: k\in K\}|.$$

In particular, if we choose $p_k=1$, then it reduces to natural density. After that we use (s_n) instead of (p_k) in our results to avoid confusing and $S_n=\sum_{k=0}^{n}s_n$.

In the year 2017 weighted lacunary statistical convergence is a generalization of lacunary statistical convergence was introduced by Ghosal et al. [23]. In the study [24] rough weighted *I*-lacunary statistical limit set and weighted *I*-lacunary statistical cluster points set which are natural generalizations of rough I-limit set and *I*-cluster points set.

On further progress, we combine the approaches of *I*-statistical convergence [10], rough *I*-convergence [17, 18], statistical cluster point and *I*-cluster point [7, 24, 25] and weighted statistical convergence [20] and investigate new and more advance summability methods namely, rough weighted *I*-statistical limit set and weighted *I*-statistical cluster points set of a sequence in a metric space.

2. Main Results

Definition 2.1. Let *r* be a non-negative real number and (s_n) be a weighted sequence. Then, the sequence of $w=\{w_n\}$ is known as rough weighted *I*-statistically convergent to w_* w.r.t the roughness of degree r (or briefly: r-weighted *I*-statistically convergent to w_* if for every $\varepsilon,\delta>0$,

$$\left\{n\in\mathbb{N}:\frac{1}{S_n}|\{k\leq S_n: s_k\rho(w_k,w_*)\geq r+\varepsilon\}|\geq\delta\right\}\in I.$$

We indicate $w_n\xrightarrow{r-WI-st}w_*$. The set

$$\left\{WI-stLIM^r w=\left\{w_*\in\mathbb{R}: w_n\xrightarrow{r-WI-st}w_*\right\}\right\}$$

is called the r-weighted *I*-statistical limit set of the sequence $w=\{w_n\}$ with degree of the roughness r. The sequence $w=\{w_n\}$ is said to be r-weighted *I*-statistically convergent as long as $WI-stLIM^r w\neq\emptyset$.

Definition 2.2. A sequence of $w = \{w_n\}$ is called to be weighted *I*-statistically bounded if there is an element ζ in *X* and a positive real number *M* such that for each $\delta > 0$,

$$\left\{n \in \mathbb{N}: \frac{1}{S_n} |\{k \leq S_n: s_k \rho(w_k, \zeta) \geq M\}| \geq \delta\right\} \in I.$$

From the above definition, a weighted sequence $w = \{w_n\}$ of real numbers is called to be self-weighted *I*-statistically bounded if there exists $M > 0$ such that for each $\delta > 0$,

$$\left\{n \in \mathbb{N}: \frac{1}{S_n} |\{k \leq S_n: s_k \geq M\}| \geq \delta\right\} \in I.$$

In general, for a weighted sequence $\{s_n\}$; the *r*-weighted statistical limit of a sequence $w = \{w_n\}$ may not be unique actually it can be infinite for some roughness of degree $r > 0$.

Theorem 2.1. The rough weighted *I*-statistical limit set $WI - stLIM^r w$ includes at most one element in *X* if the weighted sequence (s_n) is not self-weighted *I*-statistically bounded.

Proof. Presume that that there are two points $y_* \neq z_*$ such that $y_*, z_* \in WI - stLIM^r w$. Select $2\varepsilon = \rho(y_*, z_*)$.

Case 1: Let the weighted sequence $\{s_n\}$ is properly divergent to $+\infty$. Then,

$$\begin{aligned} \mathbb{N}/\{\text{a finite subset of } \mathbb{N}\} &= \left\{n \in \mathbb{N}: \frac{1}{S_n} \left|\left\{k \leq S_n: s_k \geq \frac{2r + 2\varepsilon}{\rho(y_*, z_*)}\right\}\right| \geq 1\right\} \\ &\subseteq \left\{n \in \mathbb{N}: \frac{1}{S_n} |\{k \leq S_n: s_k \rho(w_k, y_*) \geq r + \varepsilon\}| \geq \frac{1}{2}\right\} \\ &+ \left\{n \in \mathbb{N}: \frac{1}{S_n} |\{k \leq S_n: s_k \rho(w_k, z_*) \geq r + \varepsilon\}| \geq \frac{1}{2}\right\} \in I, \end{aligned}$$

which is a contradiction.

Case 2: Let the weighted sequence $\{s_n\}$ be unbounded but not properly divergent to $+\infty$. Then, there are two infinite subsets P and R of $\mathbb{N}$ such that $P \cup R = \mathbb{N}, P \cap R = \emptyset$ and $\{s_n\}_{n \in P}$ is a unbounded subsequence and $\{s_n\}_{n \in R}$ is a bounded subsequence of $\{s_n\}_{n \in \mathbb{N}}$.

Subcase 2(i): Let

$$\left\{n \in \mathbb{N}: \frac{1}{S_n} |\{k \leq S_n: k \in P\}| \geq \delta\right\} \in I.$$

Since $\{s_n\}_{n\in R}$ is a bounded subsequence of $\{s_n\}_{n\in\mathbb{N}}$, so there is a $M > 0$ such that $s_n < M, \forall n \in R$. Then

$$\left\{n \in \mathbb{N}: \frac{1}{S_n} |\{k \leq S_n: s_k \geq M\}| \geq \delta\right\} \subseteq \left\{n \in \mathbb{N}: \frac{1}{S_n} |\{k \leq S_n: k \in P\}| \geq \delta\right\} \in I,$$

which contradicts that $\{s_n\}_{n\in\mathbb{N}}$ is not self weighted *I*-statistically bounded.

Subcase 2(ii): Let

$$\left\{n \in \mathbb{N}: \frac{1}{S_n} |\{k \leq S_n: k \in P\}| \geq \delta\right\} \notin I.$$

Then,

$$\left\{n \in \mathbb{N}: \frac{1}{S_n} \left|\{k \leq S_n: k \in P/\{\text{a finite subset of } \mathbb{N}\}\}\right| \geq \delta\right\} \subseteq \left\{n \in \mathbb{N}: \frac{1}{S_n} \left|\left\{k \leq S_n: s_k \geq \frac{2r + 2\varepsilon}{\rho(y_*, z_*)}\right\}\right| \geq \delta\right\}$$
$$\subseteq \left\{n \in \mathbb{N}: \frac{1}{S_n} \left|\{k \leq S_n: s_k\rho(w_k, y_*) \geq r + \varepsilon\}\right| \geq \frac{\delta}{2}\right\}$$
$$\cup \left\{n \in \mathbb{N}: \frac{1}{S_n} |\{k \leq S_n: s_k\rho(w_k, z_*) \geq r + \varepsilon\}| \geq \frac{\delta}{2}\right\} \in I,$$

which is a contradiction. Hence the proof is completed.

If the weighted sequence $\{s_n\}_{n\in\mathbb{N}}$ is self weighted *I*-statistically bounded then there is a $M > 0$ such that

$$\left\{n \in \mathbb{N}: \frac{1}{S_n} |\{k \leq S_n: s_k \geq M\}| \geq \delta\right\} \in I$$

$$\Rightarrow \left\{n \in \mathbb{N}: 1 - \frac{1}{S_n} |\{k \leq S_n: k \in T\}| < \delta\right\} \in F(I),$$

where $T = \{k \in \mathbb{N}: s_k < M\}$. Then, the subsequence $\{s_n\}_{n\in T}$ of the sequence $\{s_n\}_{n\in\mathbb{N}}$ is bounded and so the limit inferior exists. The notation $\liminf_{n\in T} s_n$ indicates the limit inferior of the sequence $\{s_n\}_{n\in T}$ when the weighted sequence $\{s_n\}_{n\in\mathbb{N}}$ is self weighted *I*-statistically bounded.

Theorem 2.2. For a sequence $w = \{w_n\}_{n\in\mathbb{N}}$, we have

$$0 \le \operatorname{diam}(WI - stLIM^r w) \le \begin{cases} \dfrac{2r}{\operatorname{liminf}_{n\in T} s_n}, & \text{if } \{s_n\}_{n\in\mathbb{N}} \text{ is self weighted } I-\text{statistically bounded,} \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Let $\{s_n\}_{n\in\mathbb{N}}$ be self weighted I-statistically bounded sequence. By contradiction we presume that $\operatorname{diam}(WI - stLIM^r w) \ge \frac{2r}{\operatorname{liminf}_{n\in T} s_n}$. Then, there is a positive real number $\sigma \in (0, \operatorname{liminf}_{n\in T} s_n)$ such that $\operatorname{diam}(WI - stLIM^r w) \ge \frac{2r}{\sigma} \ge \frac{2r}{\operatorname{liminf}_{n\in T} s_n}$.
So, there are $y_*, z_* \in WI - stLIM^r w$ such that $\rho(y_*, z_*) > \frac{2r}{\sigma}$. Since $\sigma < \operatorname{liminf}_{n\in T} s_n$ then, there exists a natural number m such that $\sigma < s_n$ for all $n > m$ and $n \in T$.
Let $\varepsilon \in \left(0, \frac{\sigma\rho(y_*,z_*)}{2} - r\right)$, $\delta \in (0,1)$ and $T_m = \{1,2,\dots,m\}$. Then,

$$B = \left\{n \in \mathbb{N}: \frac{1}{S_n}\left|\{k \le S_n: s_k\rho(w_k, y_*) \ge r + \varepsilon\}\right| < \frac{\delta}{3}\right\} \in F(I),$$

$$C = \left\{n \in \mathbb{N}: \frac{1}{S_n}\left|\{k \le S_n: s_k\rho(w_k, z_*) \ge r + \varepsilon\}\right| < \frac{\delta}{3}\right\} \in F(I),$$

$$D = \left\{n \in \mathbb{N}: \frac{1}{S_n}\left|\{\{k \le S_n: s_k \ge M\} \cup T_m\}\right| < \frac{\delta}{3}\right\} \in F(I).$$

and

$$\left\{n \in \mathbb{N}: \frac{1}{S_n}\left|\{k \le S_n: s_k\rho(w_k, z_*) \ge r + \varepsilon\}\right| \ge \delta\right\} \in I.$$

Since $B \cap C \cap D \in F(I)$ and $\emptyset \notin F(I)$, we can select $n \in B \cap C \cap D$ such that

$$\frac{1}{S_n}\left|\{k \le S_n: s_k\rho(w_k, y_*) \ge r + \varepsilon\}\right| < \frac{\delta}{3} < \frac{1}{3},$$

$$\frac{1}{S_n}\left|\{k \le S_n: s_k\rho(w_k, z_*) \ge r + \varepsilon\}\right| < \frac{\delta}{3} < \frac{1}{3},$$

and

$$\frac{1}{S_n}\left|\left\{\{k \le S_n : s_k \ge M\} \cup T_m\right\}\right| < \frac{\delta}{3} < \frac{1}{3}.$$

This gives

$$\frac{1}{S_n}|\{k \le (S_n - T_m): s_k\rho(w_k, y_*) \ge r + \varepsilon \vee s_k\rho(w_k, z_*) \ge r + \varepsilon \vee s_k \ge M\}| < 1.$$

Since $(B \cup C \cup D)^c = B^c \cap C^c \cap D^c$ so there is a $k_0 \le (S_n - T_m)$ such that $s_{k_0}\rho\left(w_{k_0}, y_*\right) < r + \varepsilon, s_{k_0}\rho\left(w_{k_0}, z_*\right) < r + \varepsilon, s_{k_0} < M$.

$$\sigma\rho(y_*, z_*) \le \sigma\rho\left(w_{k_0}, y_*\right) + \sigma\rho\left(w_{k_0}, z_*\right) \le s_{k_0}\rho\left(w_{k_0}, y_*\right) + \sigma s_{k_0}\rho\left(w_{k_0}, z_*\right) < 2(r + \varepsilon) < \sigma\rho(y_*, z_*),$$

which is a contradiction. Hence the proof of theorem is completed.

Theorem 2.3. The set $WI - stLIM^r w$ of a sequence $w = \{w_n\}$ is closed.

Proof. Case 1: Let $\{s_n\}_{n\in\mathbb{N}}$ be self-weighted I-statistically bounded and $WI - stLIM^r w \ne \emptyset$. Then, there is a sequence $t = \{t_n\}_{n\in\mathbb{N}}$ in $WI - stLIM^r w$ such that $t_n \to t_*$ as $n \to \infty$. We have to denote that $t_* \in WI - stLIM^r w$.
Since $t_n \to t_*$ as $n \to \infty$, then for any $\varepsilon > 0$, then is a $n_0 \in \mathbb{N}$ such that $\rho(t_n, t_*) < \frac{\varepsilon}{2M}, \forall n \ge n_0$.
Then, from the triangle inequality we get

$$\left\{k \le S_n : s_k\rho\left(w_k, t_{k_0}\right) < r + \frac{\varepsilon}{2}\right\} \cap \{k \le S_n : s_k < M\} \subseteq \{k \le S_n : s_k\rho(w_k, t_*) < r + \varepsilon\}.$$

So $t_* \in WI - stLIM^r w$. Therefore, the set $WI - stLIM^r w$ of a sequence $w = \{w_k\}$ is closed.

Case 2: Let $\{s_n\}_{n\in\mathbb{N}}$ be self weighted I-statistically bounded. Then, by the Theorem 2.1, the set $WI - stLIM^r w$ is closed.

Definition 2.3. Let $\{s_n\}_{n\in\mathbb{N}}$ be a weighted sequence. A point $c \in X$ is named a weighted I-statistical cluster point of a sequence $w = \{w_n\}_{n\in\mathbb{N}}$ if for every $\varepsilon, \beta > 0$,

$$\left\{n \in \mathbb{N}: \frac{1}{S_n}|\{k \le S_n : s_k\rho(w_k, c) < \varepsilon\}| \ge \delta\right\} \notin I.$$

We indicate the set of all a weighted I-statistical cluster point of a sequence $w = \{w_n\}_{n\in\mathbb{N}}$ by $W\Gamma_w(I)$.

Theorem 2.4. For an arbitrary $c \in W\Gamma_w(I)$ of a sequence $w = \{w_n\}_{n\in\mathbb{N}}$ we get

$$\rho(w_*, c) = \begin{cases} \dfrac{r}{\liminf_{n\in T} s_n}, \text{ if } \{s_n\}_{n\in\mathbb{N}} \text{ is self weighted } I - \text{statistically bounded,} \\ \dfrac{r}{\inf_{n\in T} s_n}, \quad \text{otherwise.} \end{cases}$$

for all $w_* \in WI - stLIM^r w$.

Proof. Case 1: Let $\{s_n\}_{n\in\mathbb{N}}$ be self weighted I-statistically bounded sequence. By contradiction we suppose that there is a point $c \in W\Gamma_w(I)$ and $w_* \in WI - stLIM^r w$ such that $\rho(w_*, c) = \frac{r}{\liminf_{n\in T} s_n} > 0$.
This gives $\frac{(\liminf_{n\in T} s_n)\rho(w_*,c) - r}{3} > 0$. Then, there is a positive real number $\sigma \in (0, \liminf_{n\in T} s_n)$ such that

$$\frac{(\liminf_{n\in T} s_n)\rho(w_*, c) - r}{3} > \frac{\sigma\rho(w_*, c) - r}{3} > 0.$$

Define $\varepsilon = \frac{\sigma\rho(w_*,c) - r}{3} > 0$. Since $\sigma < iminf_{n\in T} s_n$, so there is a $k_0 \in \mathbb{N}$ such that $s_n > \sigma\ \forall n > k_0$ and $n \in T$ where $T = \{k \in \mathbb{N}: s_k < M\}$.
Let $T_0 = T/\{1,2, \dots, k_0 - 1\}$ and $Q = \{k \in \mathbb{N}: s_k\rho(w_k, c) < \varepsilon\}$. Then four subcases may arise.
Subcase 1(i): If $T_0 \cap Q = \emptyset$, then $Q \subseteq \mathbb{N}/T_0$, which is a contradiction since $c \in W\Gamma_w(I)$. So this case can never happen.
Subcase 1(ii): If $Q \subseteq T_0$, then $T_0 \cap Q = Q$.
Subcase 1(iii): If $T_0 \subseteq Q$, then $T_0 \cap Q = T_0$.
Subcase 1(iv): If $T_0 \cap Q \neq \emptyset$, $T_0/Q \neq \emptyset$ and $Q/T_0 \neq \emptyset$ then $Q/(T_0 \cap Q) \subseteq \mathbb{N}/T_0$.

$$\left\{n \in \mathbb{N}: \frac{1}{S_n}|\{k \le S_n: k \in Q/(T_0 \cap Q)\}| \ge \frac{\delta}{2}\right\} \subseteq \left\{n \in \mathbb{N}: \frac{1}{S_n}|\{k \le S_n: k \in \mathbb{N}/T_0\}| \ge \frac{\delta}{2}\right\} \in I,$$

So, we obtain,

$$\left\{n \in \mathbb{N}: \frac{1}{S_n}|\{k \le S_n: k \in Q/(T_0 \cap Q)\}| \ge \frac{\delta}{2}\right\} \in I.$$

Since $c \in W\Gamma_w(I)$, then we acquire

$$\left\{n \in \mathbb{N}: \frac{1}{S_n}|\{k \leq S_n: k \in Q\}| \geq \delta\right\} \notin I.$$

Again $Q = \left[Q/\left((T_0 \cap Q)\right)\right] \cup (T_0 \cap Q)$. Then, we get

$$\left\{n \in \mathbb{N}: \frac{1}{S_n}|\{k \leq S_n: k \in Q\}| \geq \delta\right\}$$
$$\subseteq \left\{n \in \mathbb{N}: \frac{1}{S_n}|\{k \leq S_n: k \in Q/(T_0 \cap Q)\}| \geq \frac{\delta}{2}\right\} \cup \left\{n \in \mathbb{N}: \frac{1}{S_n}|\{k \leq S_n: k \in T_0 \cap Q\}| \geq \frac{\delta}{2}\right\}.$$

So we know that if $A, B, C \subseteq \mathbb{N}$, $A \notin I$, $B \in I$ and $A \subseteq B \cup C$ then $C \notin I$, this gives

$$\left\{n \in \mathbb{N}: \frac{1}{S_n}|\{k \leq S_n: k \in T_0 \cap Q\}| \geq \frac{\delta}{2}\right\} \notin I.$$

This shows that for all existing cases (i.e., 1(ii); 1(iii) and 1(iv)) we obtain

$$\left\{n \in \mathbb{N}: \frac{1}{S_n}|\{k \leq S_n: k \in T_0 \cap Q\}| \geq \frac{\delta}{2}\right\} \notin I.$$

So there is a natural number $k \in T_0 \cap Q$ such that

$$s_k\rho(w_k, w_*) \geq s_k\rho(w_*, c) - s_k\rho(w_k, c) > 3\varepsilon + r - \varepsilon = r + 2\varepsilon > r + \varepsilon$$

(since $s_k > \sigma$ and $3\varepsilon = \sigma\rho(w_*, c) - r$).

$$\Rightarrow T_0 \cap Q \subseteq \{k \in \mathbb{N}: s_k\rho(w_k, w_*) \geq r + \varepsilon\}.$$

Then,

$$\left\{n \in \mathbb{N}: \frac{1}{S_n}|\{k \leq S_n: s_k\rho(w_k, w_*) \geq r + \varepsilon\}| \geq \frac{\delta}{2}\right\} \notin I.$$

This contradicts the fact that $w_* \in WI - stLIM^r w$.

Case 2: Let the weighted sequence $\{s_n\}_{n\in\mathbb{N}}$ be not self weighted I-statistically bounded.
By contradiction we presume that there is a point $c \in W\Gamma_w(I)$ and $w_* \in WI - stLIM^r w$ such that $\rho(w_*, c) = \frac{r}{inf_{n\in\mathbb{N}} s_n}$. Select $\varepsilon = \frac{\xi\rho(w_*,c)-r}{2}$, where $\xi = inf_{n\in\mathbb{N}} s_n$. We know that

$$s_k\rho(w_*, w_k) \geq s_k\rho(w_*, c) - s_k\rho(w_k, c) \geq \xi\rho(w_*, c) - s_k\rho(w_k, c) = r + 2\varepsilon - s_k\rho(w_k, c), \forall k \in \mathbb{N}.$$

Then, we obtain

$$\left\{n \in \mathbb{N}: \frac{1}{S_n}|\{k \leq S_n: s_k\rho(w_k, c) < \varepsilon\}| \geq \delta\right\} \subseteq \left\{n \in \mathbb{N}: \frac{1}{S_n}|\{k \leq S_n: s_k\rho(w_k, w_*) \geq r + \varepsilon\}| \geq \delta\right\}.$$

$w_* \in WI - stLIM^r w$ contradicts the fact that $c \in W\Gamma_w(I)$.

Theorem 2.5.

i. For an arbitrary $c \in W\Gamma_w(I)$ of a sequence $w = \{w_n\}_{n\in\mathbb{N}}$, we have

$$WI - stLIM^r w \subseteq \begin{cases} \bar{B}_{\frac{r}{p}}(c), & \text{if } \{s_n\}_{n\in\mathbb{N}} \text{ is self weighted } I - \text{statistically bounded}, \\ \bar{B}_{\frac{r}{q}}(c), & \text{otherwise where } p = \liminf_{n\in T} s_n,\ q = \liminf_{n\in\mathbb{N}} s_n \text{ and } \bar{B}_\varepsilon(c) = \{y \in X: \rho(y,c) \leq \varepsilon\} \end{cases},$$

ii.

$$WI - stLIM^r w = \begin{cases} \bigcap_{c\in W\Gamma_w(I)} \bar{B}_{\frac{r}{p}}(c) \subseteq \left\{w_* \in \mathbb{R}: W\Gamma_w(I) \subseteq \bar{B}_{\frac{r}{p}}(w_*)\right\}, \\ \quad \text{if } \{s_n\}_{n\in\mathbb{N}} \text{ is self weighted } I - \text{statistically bounded}, \\ \bigcap_{c\in W\Gamma_w(I)} \bar{B}_{\frac{r}{q}}(c) \subseteq \left\{w_* \in \mathbb{R}: W\Gamma_w(I) \subseteq \bar{B}_{\frac{r}{q}}(w_*)\right\}, \text{otherwise.} \end{cases}$$

Proof. The results are obvious so omitted.

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On The Ruled Surface According to Dual Bézier Curves

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Abstract

The set of dual numbers is defined by $D = \{a + \varepsilon a^*: a, a^* \in R, \varepsilon \neq 0; \varepsilon^2 = 0\}$. Every unit dual spherical point corresponds to a directed line in real 3- space R^3 by E. Study's theorem. So a dual unit spherical curve corresponds to a ruled surface. In this paper we consider the dual Bézier Curve $\hat{B}(t) = B(t) + \varepsilon B^*(t)$ where $B,\ B^*$ are Bézier curve of degree n with cotrol points $P_i, P_i^* \in R^3$ for $i = 0,1, \dots, n$. It is investigated the ruled surface according to Study's theorem in this paper.

Keywords: Bézier curves, ruled surface, dual spherical projection.

1.Introduction:

Dual numbers denoted by *D* were introduced in 1873 by William Clifford [1], and developed by Eduard Study [2]. It is necessary to make the following distinction: the word "dual vector" is defined in the literature as an element of dual vector space which is the duality of a vector space. However, the "dual vector" used in this study is an element of a vector space defined as the cartesian set of dual numbers ($D^3 = D \times D \times D$) given by Clifford in 1843. After E.Study, with a dual spherical point corresponding to a directed line in R^3 to study of a ruled surface is reduced to study of a spherical curve, many scientists studied in this area. Especially Hoschek [10] investigated integral invariants for characterization of the closed ruled surfaces. Gürsoy, Gürsoy and Küçük [3-7], Hacisalihoğlu [17] were studied the ruled surfaces with integral invariants which are stated as dual quantities.

1. Materials and method

1.1. Dual Numbers and D-module

Let two dual vectors $\hat{U}$ and $\hat{V}$ be given as $\hat{U} = U + \varepsilon U^*$ and $\hat{V} = V + \varepsilon V^*$. Then the inner product of two dual vectors $\hat{U}$ and $\hat{V}$ is $\langle \hat{U}, \hat{V} \rangle = \langle U, V \rangle + \varepsilon(\langle U^*, V \rangle + \langle U, V^* \rangle)$. The norm of a dual vector $\hat{U} = U + \varepsilon U^*$ is a dual number such that

$$\|\hat{U}\| = \sqrt{\langle \hat{U}, \hat{U} \rangle} = \sqrt{\langle U, U \rangle + 2\varepsilon \langle U, U^* \rangle} = \|U\| + \varepsilon \frac{\langle U, U^* \rangle}{\|U\|} = u + \varepsilon u^* \in D \qquad (1)$$

if the real part of the dual vector is different from zero. i.e. $U \neq 0$. If the norm of a dual vector $\hat{U}$ is $1 + \varepsilon 0 = 1$ then the vector $\hat{U}$ is called dual unit vector [16].

Proposition 2.1 [16]:Let a dual vector $\hat{U} = U + \varepsilon U^*$ be given. If $\|\hat{U}\| = 1$ then $\|U\| = 1$ and $\langle U, U^* \rangle = 0$.

Proposition 2.2 [16]:Let a dual vector $\hat{U} = U + \varepsilon U^*$ be given. If $\|\hat{U}\| \neq 1$ and $U \neq 0$ then

$$\breve{U} = \frac{\hat{U}}{\|\hat{U}\|} = \frac{U}{\|U\|} + \varepsilon \frac{U^* - \frac{\langle U, U^* \rangle}{\|U\|^2} U}{\|U\|} = \frac{U}{\|U\|} + \varepsilon \left(\frac{U^*}{\|U\|} - \frac{\langle U, U^* \rangle U}{\|U\|^3} \right) = \dot{U} + \varepsilon \dot{U}^* \tag{2}$$

is a dual unit vector with direction of $\hat{U}$.

Definition 2.1: The set of dual unit vectors in *D*-module D^3 is called unit dual sphere. i.e. unit dual sphere is defined as

$$\left\{ \breve{U} = \dot{U} + \varepsilon \dot{U}^* \in D^3 : \|\breve{U}\| = 1 \right\} \tag{3}$$

Theorem 2.1. (E.Study)[3]: Every unit vector on the dual unit sphere $\breve{U} = \dot{U} + \varepsilon \dot{U}^*$ $(U \neq 0)$ corresponds to oriented line in real space R^3 by one to one.

According to this theorem, a unit dual vector $\breve{U} = \dot{U} + \varepsilon \dot{U}^*$ $(U \neq 0)$ corresponds only one oriented line where the real vector $\dot{U}$ shows the direction of this line and the real vector $\dot{U}^*$ shows the vectorial moment of the unit vector $\dot{U}$ with respect to the origin point 0. The vectorial moment of $\dot{U}$ is given as $\dot{U}^* = 0M \wedge \dot{U}$ where M is a point on $\dot{U}$-oriented line and $\wedge$ denotes the cross product in R^3 [4].

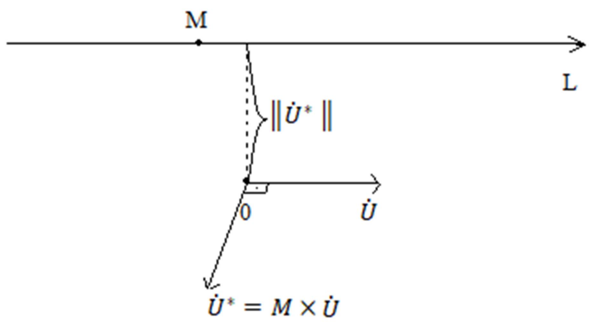

Figure 1: The directed line L corresponding to dual unit vector $\breve{U} = \dot{U} + \varepsilon \dot{U}^*$

1.2. Ruled Surfaces

A ruled surface can be described by a parametric representation of the form

$$X(u, v) = \alpha(u) + v\beta(u), \quad v \in R \tag{4}$$

where $\alpha(u)$ and $\beta(u)$ are curves in R^3 with $\alpha'(u) \neq 0$ for every u. The curve $\alpha(u)$ is called the *directrix* or base curve of the ruled surface and $\beta(u)$ is **called** the director curve. Any curve $X(u_0, v)$ with fixed parameter $u = u_0$ is a generator line [11].

Lemma 2.1: The Gaussian curvature of a ruled surface $X(u, v) \subset R^3$ is everywhere nonpositive [11].

If the Gaussian curvature of a ruled surface $X(u,v) \subset R^3$ is zero then the ruled surface is called developable ruled surface or flat ruled surface [11].

Definition 2.2: A ruled surface parametrized by (4) is said to be noncylindrical provided $\beta(u) \times \beta'(u)$ never vanishes [11].

Lemma 2.2: Any noncylindrical ruled surface $\bar{X}(u,v)$ has a reparametrization of the form

$$X(u,v) = \sigma(u) + v\delta(u), \quad v \in R \tag{5}$$

where $\|\delta(u)\| = 1$ and $\langle \sigma'(u), \delta'(u) \rangle = 0$. The curve σ is called the striction curve of $\bar{X}(u,v)$ [11].

Lemma 2.3: The striction curve of a noncylindrical ruled surface does not depend on the choice of base curve [11].

Definition 2.3 [11]: Let $X(u,v)$ be a noncylindrical ruled surface given by (5) then the distribution parameter of $X(u,v)$ is the function p defined by

$$p(u) = \frac{(\sigma\prime(u), \delta(u), \delta\prime(u))}{\langle \delta\prime(u), \delta\prime(u) \rangle}, \tag{6}$$

Theorem 2.2 [11]: Let $X(u,v) = \beta(u) + v\delta(u)$ with $\|\delta(u)\| = 1$ parametrize a flat ruled surface. Then

i) If $\beta'(u) = 0$, then X is a cone,
ii) If $\delta'(u) = 0$, then X is a cylinder
iii) If both $\beta'(u)$ and $\delta'(u)$ never vanish, then X is tangent developable of its striction curve.

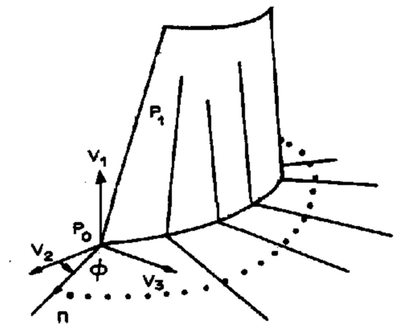

Figure 2: A ruled surface

3. Results

Let $\hat{B}(t)$ be a dual Bézier curve with control points $\hat{P}_0, \hat{P}_1, \ldots, \hat{P}_n \in D^3$ where $\hat{P}_i = P_i + \varepsilon P_i{}^*$, $P_i, P_i{}^* \in R^3$ for $i = 0,1,\ldots,n$. $\hat{P}_i = P_i + \varepsilon P_i{}^* \in D^3$ Then for $t \in [0,1]$, the dual Bézier curve can be defined as

$$\hat{B}(t) = \sum_{i=0}^{n} B_i^n(t)\hat{P}_i \tag{7}$$

Since each control point $\hat{P}_i = P_i + \varepsilon P_i{}^*$ then for $t \in [0,1]$ the dual Bézier curve can be written as

$$\begin{aligned}\hat{B}(t) &= \sum_{i=0}^{n} B_i^n(t) P_i + \varepsilon \sum_{i=0}^{n} B_i^n(t) P_i{}^* \\ &= B(t) + \varepsilon B^*(t)\end{aligned} \tag{8}$$

where $B(t)$ and $B^*(t)$ are real Bézier curves of degree n with control points $P_0, P_1, \dots, P_n$ and $P_0{}^*, P_1{}^*, \dots, P_n{}^*$ respectively.

Let the coordinate frame in R^3 be denoted as $\{e_1, e_2, e_3\}$. Then the j.th coordinat element of any control point $P_i = (P_{i_1}, P_{i_2}, P_{i_3})$ in R^3 for $j = 1,2,3$ is the inner product $P_{ij} = \langle P_i, e_j \rangle$. So any control point P_i is stated as

$$P_i = \sum_{j=1}^{3} \langle P_i, e_j \rangle\, e_j. \tag{9}$$

Similarly since any control point of $\hat{P}_i$ in D^3 is stated as $\hat{P}_i = P_i + \varepsilon P_i{}^* = (P_{i_1}, P_{i_2}, P_{i_3}) + \varepsilon(P_{i_1}{}^*, P_{i_2}{}^*, P_{i_3}{}^*)$ where $P_i, P_i{}^* \in R^3$ then

$$\hat{P}_i = \sum_{j=1}^{3} \langle P_i, e_j \rangle\, e_j + \varepsilon \sum_{j=1}^{3} \langle P_i{}^*, e_j \rangle\, e_j \tag{10}$$

can be stated. The norm of the curve $B(t)$ at any time t is

$$\begin{aligned}\|B(t)\| &= \left\| \sum_{i=0}^{n} B_i^n(t) P_i \right\| = \sqrt{\sum_{j=1}^{3} \left(\sum_{i=0}^{n} B_i^n(t) P_{ij} \right)^2} \\ &= \sqrt{\sum_{j=1}^{3} \left(\sum_{i=0}^{n} B_i^n(t) \langle P_i, e_j \rangle \right)^2} = \sqrt{\sum_{j=1}^{3} \langle \sum_{i=0}^{n} B_i^n(t) P_i\,, e_j \rangle^2} = \sqrt{\sum_{j=1}^{3} \langle B(t), e_j \rangle^2}\end{aligned} \tag{11}$$

Now from (26) for $t \in [0,1]$ the dual Bézier curve $\hat{B}(t)$ can be expressed as

$$\begin{aligned}\hat{B}(t) &= \sum_{i=0}^{n} B_i^n(t) P_i + \varepsilon \sum_{i=0}^{n} B_i^n(t) P_i{}^* \\ &= \sum_{i=0}^{n} B_i^n(t) \left(\sum_{j=1}^{3} \langle P_i, e_j \rangle\, e_j \right) + \varepsilon \sum_{i=0}^{n} B_i^n(t) \left(\sum_{j=1}^{3} \langle P_i{}^*, e_j \rangle\, e_j \right) \\ &= \sum_{i=0}^{n} \sum_{j=1}^{3} B_i^n(t) \langle P_i, e_j \rangle\, e_j + \varepsilon \sum_{i=0}^{n} \sum_{j=1}^{3} B_i^n(t) \langle P_i{}^*, e_j \rangle\, e_j\end{aligned} \tag{12}$$

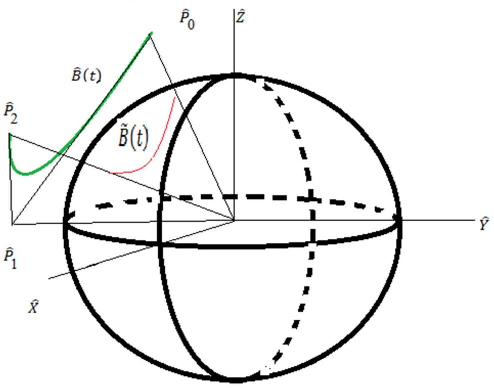

Figure 3: Unit dual Sphere and Projection curve $\tilde{B}(t)$ of the Bézier curve $\hat{B}(t)$ to unit dual sphere

The projection of the dual Bézier curve $\hat{B}(t)$ to unit dual sphere D-module is a curve in Fig. 3 denoted by $\tilde{B}(t)$ and defined by

$$\tilde{B}(t)=\frac{\hat{B}(t)}{\|\hat{B}(t)\|}=\frac{\sum_{i=0}^{n}B_i^n(t)\hat{P}_i}{\left\|\sum_{i=0}^{n}B_i^n(t)\hat{P}_i\right\|}=\frac{\sum_{i=0}^{n}B_i^n(t)(P_i+\varepsilon P_i^*)}{\left\|\sum_{i=0}^{n}B_i^n(t)\hat{P}_i\right\|}=\bar{B}(t)+\varepsilon\bar{B}^*(t) \tag{13}$$

Since from (3) the norm of the curve $\hat{B}(t)$ is

$$\left\|\hat{B}(t)\right\|=\|B(t)+\varepsilon B^*(t)\|=\|B(t)\|+\varepsilon\frac{\langle B(t),B^*(t)\rangle}{\|B(t)\|} \tag{14}$$

The projection curve $\tilde{B}(t)$ can be stated as

$$\tilde{B}(t)=\frac{\hat{B}(t)}{\|\hat{B}(t)\|}=\frac{\hat{B}(t)}{\|B(t)\|+\varepsilon\frac{\langle B(t),B^*(t)\rangle}{\|B(t)\|}}=\left(\frac{1}{\|B(t)\|}-\varepsilon\frac{\langle B(t),B^*(t)\rangle}{\|B(t)\|^3}\right)\hat{B}(t) \tag{15}$$

When (8) and (10) is also replaced by (15) the projection curve $\tilde{B}(t)$ of the dual Bézier curve $\hat{B}(t)$ to unit dual sphere

$$\begin{aligned}\tilde{B}(t)&=\left(\frac{1}{\|B(t)\|}-\varepsilon\frac{\langle B(t),B^*(t)\rangle}{\|B(t)\|^3}\right)\left(\sum_{i=0}^{n}B_i^n(t)\left(\sum_{j=1}^{3}\langle P_i,e_j\rangle e_j\right)+\varepsilon\sum_{i=0}^{n}B_i^n(t)\left(\sum_{j=1}^{3}\langle P_i^*,e_j\rangle e_j\right)\right) \qquad (16)\\
&=\left(\frac{1}{\sqrt{\sum_{j=1}^{3}\left(\sum_{i=0}^{n}B_i^n(t)\langle P_i,e_j\rangle\right)^2}}-\varepsilon\frac{\sum_{i=0}^{n}\sum_{j=0}^{n}B_i^n(t)B_j^n(t)\langle P_i,P_j^*\rangle}{\left(\sum_{j=1}^{3}\left(\sum_{i=0}^{n}B_i^n(t)\langle P_i,e_j\rangle\right)^2\right)^{3/2}}\right)\left(\sum_{i=0}^{n}\sum_{j=1}^{3}\langle P_i,e_j\rangle e_jB_i^n(t)+\varepsilon\sum_{i=0}^{n}\sum_{j=1}^{3}\langle P_i^*,e_j\rangle e_jB_i^n(t)\right)\\
&=\frac{\sum_{i=0}^{n}\sum_{j=1}^{3}\langle P_i,e_j\rangle e_jB_i^n(t)}{\sqrt{\sum_{j=1}^{3}\left(\sum_{i=0}^{n}B_i^n(t)\langle P_i,e_j\rangle\right)^2}}\\
&\quad+\varepsilon\left(\frac{\sum_{i=0}^{n}\sum_{j=1}^{3}\langle P_i^*,e_j\rangle e_jB_i^n(t)}{\sqrt{\sum_{j=1}^{3}\left(\sum_{i=0}^{n}B_i^n(t)\langle P_i,e_j\rangle\right)^2}}-\frac{\sum_{i=0}^{n}\sum_{j=0}^{n}B_i^n(t)B_j^n(t)\langle P_i,P_j^*\rangle}{\left(\sum_{j=1}^{3}\left(\sum_{i=0}^{n}B_i^n(t)\langle P_i,e_j\rangle\right)^2\right)^{3/2}}\sum_{i=0}^{n}\sum_{j=1}^{3}\langle P_i,e_j\rangle e_jB_i^n(t)\right)\\
&=\bar{B}(t)+\varepsilon\bar{B}^*(t)\end{aligned}$$

can be written. Therefore this theorem can be stated as follows

Theorem 3.1: Let $\hat{B}(t)=B(t)+\varepsilon B^*(t)$ be a dual Bézier curve with control points $\hat{P}_0,\hat{P}_1,\dots,\hat{P}_n\in D^3$ where $\hat{P}_i=P_i+\varepsilon P_i^*$, $P_i,P_i^*\in R^3$ for $i=0,1,\dots,n$. Then the projection curve $\tilde{B}(t)$ of the dual Bézier curve $\hat{B}(t)$ to unit dual sphere is

$$\tilde{B}(t)=\bar{B}(t)+\varepsilon\bar{B}^*(t)$$

where

$$\bar{B}(t)=\frac{\sum_{i=0}^{n}\sum_{j=1}^{3}\langle P_i,e_j\rangle e_jB_i^n(t)}{\sqrt{\sum_{j=1}^{3}\left(\sum_{i=0}^{n}B_i^n(t)\langle P_i,e_j\rangle\right)^2}} \tag{17}$$

and

$$\bar{B}^*(t)=\frac{\sum_{i=0}^{n}\sum_{j=1}^{3}\langle P_i^*,e_j\rangle e_jB_i^n(t)}{\sqrt{\sum_{j=1}^{3}\left(\sum_{i=0}^{n}B_i^n(t)\langle P_i,e_j\rangle\right)^2}}-\frac{\sum_{i=0}^{n}\sum_{j=0}^{n}B_i^n(t)B_j^n(t)\langle P_i,P_j^*\rangle}{\left(\sum_{j=1}^{3}\left(\sum_{i=0}^{n}B_i^n(t)\langle P_i,e_j\rangle\right)^2\right)^{3/2}}\sum_{i=0}^{n}\sum_{j=1}^{3}\langle P_i,e_j\rangle e_jB_i^n(t) \tag{18}$$

It can be written more simply as

$$\bar{B}(t)=\frac{B(t)}{\|B(t)\|}\quad\text{and}\quad\bar{B}^*(t)=\frac{B^*(t)}{\|B(t)\|}-\frac{\langle B(t),B^*(t)\rangle}{\|B(t)\|^3}B(t) \tag{19}$$

From Proposition 2.1 the inner product these vectors $\langle\bar{B}(t),\bar{B}^*(t)\rangle=0$ satisfies.

According to E.Study's theorem any dual unit vector corresponds to a oriented line in R^3. Since for every $t \in [0,1]$ the projection curve $\tilde{B}(t)$ of the dual Bézier curve $\hat{B}(t)$ to unit dual sphere is a dual unit vector, for any $t_0 \in [0,1]$, the projection curve $\tilde{B}(t_0)$ also corresponds to a oriented line in R^3. So the projection curve $\tilde{B}(t)$ corresponds to a ruled surface in R^3. The oriented line corresponding to $\tilde{B}(t_0)$ is a line with direction of the vector $\bar{B}(t_0)$ and its distance from origine is $\|\bar{B}^*(t_0)\|$.

If $\bar{B}^*(t)$ is denoted from (19) as $\bar{B}^* = \frac{B^*}{\|B\|} - \frac{\langle B,B^*\rangle}{\|B\|^3}B$ for shortness, the magnitude $\|\bar{B}^*\|$ of the dual part of the projection curve $\tilde{B}(t)$ of the dual Bézier curve $\hat{B}(t)$ is obtained as follows:

$$\begin{aligned}
\|\bar{B}^*\| &= \left\|\frac{B^*}{\|B\|} - \frac{\langle B,B^*\rangle}{\|B\|^3}B\right\| \\
&= \frac{1}{\|B\|^3}\|(\|B\|^2B^* - \langle B,B^*\rangle B)\| \\
&= \frac{1}{\|B\|^3}\sqrt{\langle(\|B\|^2B^* - \langle B,B^*\rangle B),(\|B\|^2B^* - \langle B,B^*\rangle B)\rangle} \\
&= \frac{1}{\|B\|^3}\sqrt{\|B\|^4\langle B^*,B^*\rangle - 2\|B\|^2\langle B,B^*\rangle^2 + \langle B,B^*\rangle^2\langle B,B\rangle} \\
&= \frac{1}{\|B\|^3}\sqrt{\|B\|^4\langle B^*,B^*\rangle - 2\|B\|^2\langle B,B^*\rangle^2 + \langle B,B^*\rangle^2\|B\|^2} \\
&= \frac{1}{\|B\|^2}\sqrt{\|B\|^2\langle B^*,B^*\rangle - \langle B,B^*\rangle^2} \\
&= \frac{1}{\|B\|^2}\sqrt{\langle B,B\rangle\langle B^*,B^*\rangle - \langle B,B^*\rangle^2} \\
&= \frac{1}{\|B\|^2}\sqrt{\langle B\times B^*,B\times B^*\rangle} = \frac{\|B\times B^*\|}{\|B\|^2} = \frac{\|B^*\|sin\theta}{\|B\|}
\end{aligned} \tag{20}$$

where θ is an angle between the vectors B and B^*.

Now the vector $\bar{B}(t)$ and the magnitude of the vector $\bar{B}^*(t)$ for $t = 0$ and $t = 1$ can be easily stated by end point interpolation property of Bézier curves. In case for $t_0 \neq 0$ or $t_0 \neq 1$ they can be calculated by the de Casteljau algorithm as follows:

Theorem 3.2: From (20) the projection curve $\tilde{B}(t)$ of the dual Bézier curve $\hat{B}(t)$ for $t_0 = 0$ and $t_0 = 1$ are

$$\tilde{B}(t)\big|_{t=t_0=0} = \bar{B}(t)|_{t=t_0=0} + \varepsilon\bar{B}^*(t)|_{t=t_0=0} = \frac{P_0}{\|P_0\|} + \varepsilon\left(\frac{P_0{}^*}{\|P_0\|} - \frac{\langle P_0,P_0{}^*\rangle}{\|P_0\|^3}P_0\right) \tag{21}$$

$$\tilde{B}(t)\big|_{t=t_0=1} = \bar{B}(t)|_{t=t_0=1} + \varepsilon\bar{B}^*(t)|_{t=t_0=1} = \frac{P_n}{\|P_n\|} + \varepsilon\left(\frac{P_n{}^*}{\|P_n\|} - \frac{\langle P_n,P_n{}^*\rangle}{\|P_n\|^3}P_n\right) \tag{22}$$

Theorem 3.3: The projection curve $\tilde{B}(t)$ of the dual Bézier curve $\hat{B}(t)$ for any $t_0 \in (0,1)$ is

$$\tilde{B}(t)\big|_{t=t_0} = \bar{B}(t)|_{t=t_0} + \varepsilon\bar{B}^*(t)|_{t=t_0} = \frac{P_0^n}{\|P_0^n\|} + \varepsilon\left(\frac{P_0^{n^*}}{\|P_0^n\|} - \frac{\langle P_0^n,P_0^{n^*}\rangle}{\|P_0^n\|^3}P_0^n\right) \tag{23}$$

where P_0^n and $P_0^{n^*}$ are the points computed in the de Casteljau algorithm.

Since the projection curve $\tilde{B}(t)$ corresponds to a ruled surface in R^3, we can obtain this ruled surface as follows:

For every $t \in [0,1]$ the dual unit vector $\tilde{B}(t)$ corresponds to a directed line in R^3 according to Theorem 2.1 mentioned as above. If the positions of the points on the oriented lines with direction $\bar{B}(t)$ for every $t \in [0,1]$ is denoted by $\gamma(t)$, then the vectorial moment of the unit vector $\bar{B}(t)$ with respect to the origin is written as

$$\bar{B}^*(t) = \gamma(t) \times \bar{B}(t) \tag{24}$$

[18]. Here since the position vector $\gamma(t)$ for every $t \in [0,1]$ is orthogonal to the moment vector $\bar{B}^*(t)$, the position vector $\gamma(t)$ lies on the plane of two vectors $\bar{B}(t)$ and $\bar{B}(t) \times \bar{B}^*(t)$. So for every $t \in [0,1]$ the position vector $\gamma(t)$ is written as

$$\gamma(t) = \bar{B}(t)cos\theta + \left(\bar{B}(t) \times \bar{B}^*(t)\right)sin\theta \tag{25}$$

If (25) is substituted to (24) , then

$$\begin{aligned} \bar{B}^*(t) &= \left[\, \bar{B}(t)cos\theta + \left(\bar{B}(t) \times \bar{B}^*(t)\right)sin\theta\right] \times \bar{B}(t) \\ &= sin\theta\left[\left(\bar{B}(t) \times \bar{B}^*(t)\right) \times \bar{B}(t)\right] \\ &= sin\theta[\langle \bar{B}(t), \bar{B}(t)\rangle \bar{B}^*(t) - \langle \bar{B}(t), \bar{B}^*(t)\rangle \bar{B}(t)] \\ &= \|\bar{B}(t)\|^2 \bar{B}^*(t) sin\theta \end{aligned}$$

can be written. Since $\|\bar{B}(t)\| = 1$ then

$$sin\theta = 1 \quad \text{and so} \quad cos\theta = 0 \tag{26}$$

is founded. Consequently the position vectors $\gamma(t)$ for every $t \in [0,1]$

$$\gamma(t) = \bar{B}(t) \times \bar{B}^*(t) \tag{27}$$

is obtained. From (19)

$$\gamma(t) = \frac{B(t) \times B^*(t)}{\|B(t)\|^2} \tag{28}$$

This curve is the directrix curve of corresponding ruled surface of the projection curve $\tilde{B}(t)$. Since the director curve of ruled surface is $\beta(t) = \bar{B}(t)$ [17] , the ruled surface corresponding to the projection curve $\tilde{B}(t)$ of dual Bézier curve $\hat{B}(t)$ is

$$\begin{aligned} X(t, v) &= \gamma(t) + v\beta(t) \\ &= \frac{B(t) \times B^*(t)}{\|B(t)\|^2} + v\frac{B(t)}{\|B(t)\|} \end{aligned} \tag{29}$$

is obtained. So this theorem can be given as

Theorem 3.4 : Let $\hat{B}(t) = B(t) + \varepsilon B^*(t)$ be a dual Bézier curve with control points $\hat{P}_0, \hat{P}_1, \ldots, \hat{P}_n \in D^3$ where $\hat{P}_i = P_i + \varepsilon P_i{}^*$, $P_i, P_i{}^* \in R^3$ for $i = 0,1,\ldots,n$ and let $\tilde{B}(t)$ be the projection curve of $\hat{B}(t)$ to unit dual sphere in D^3. Then the ruled surface $X(t,v)$ corresponding to $\tilde{B}(t)$ by E.Study's theorem is

$$X(t,v) = \frac{B(t)\times B^*(t)}{\|B(t)\|^2} + v\frac{B(t)}{\|B(t)\|} \tag{30}$$

where $t \in [0,1]$ and $v \in R$.

In case $t = 0$ or $t = 1$ the parameter curves of this ruled surface $X(0,v)$ and $X(1,v)$ can be easily stated as

Theorem 3.5 : Let $\hat{B}(t) = B(t) + \varepsilon B^*(t)$ be a dual Bézier curve with control points $\hat{P}_0, \hat{P}_1, \ldots, \hat{P}_n \in D^3$ where $\hat{P}_i = P_i + \varepsilon P_i{}^*$, $P_i, P_i{}^* \in R^3$ for $i = 0,1,\ldots,n$. Then the parameter curves $X(0,v)$ and $X(1,v)$ of the ruled surface $X(t,v)$ corresponding to dual unit curve $\tilde{B}(t)$ obtained by projection of the curve $\hat{B}(t)$ to the dual unit sphere under E.Study's theorem are

$$i)\ X(0,v) = \frac{P_0\times P_0{}^*}{\|P_0\|^2} + v\frac{P_0}{\|P_0\|} \text{ for } t = 0; \tag{31}$$

$$ii)\ X(1,v) = \frac{P_n\times P_n{}^*}{\|P_n\|^2} + v\frac{P_n}{\|P_n\|} \text{ for } t = 1; \tag{32}$$

where $v \in R$.

In case for $t_0 \neq 0$ and $t_0 \neq 1$ the parameter curve $X(t_0,v)$ can be calculated by the de Casteljau algorithm in theorem 2.1 as follows

Theorem 3.6 : Let $\hat{B}(t) = B(t) + \varepsilon B^*(t)$ be a dual Bézier curve with control points $\hat{P}_0, \hat{P}_1, \ldots, \hat{P}_n \in D^3$ where $\hat{P}_i = P_i + \varepsilon P_i{}^*$, $P_i, P_i{}^* \in R^3$ for $i = 0,1,\ldots,n$. Then the parameter curve $X(t_0,v)$ of the ruled surface $X(t,v)$ corresponding to dual Bézier curve $\hat{B}(t)$ for $t_0 \in (0,1)$ is

$$X(t_0,v) = \frac{P_0^n\times P_0^{n*}}{\|P_0^n\|^2} + v\frac{P_0^n}{\|P_0^n\|}\ ; \tag{33}$$

where P_0^n and P_0^{n*} are the points computed in the de Casteljau algorithm in Theorem 2.1.

Now from (46) the directrix and director curve of the ruled surface $X(t,v)$ corresponding to the dual Bézier curve $\hat{B}(t) = B(t) + \varepsilon B^*(t)$ are the curves as follows

$$\gamma(t) = \frac{B(t)\times B^*(t)}{\|B(t)\|^2} \text{ and } \delta(t) = \frac{B(t)}{\|B(t)\|} \tag{34}$$

respectively.

Proposition 3.1: Let $\hat{B}(t) = B(t) + \varepsilon B^*(t)$ be a dual Bézier curve with control points $\hat{P}_0, \hat{P}_1, \ldots, \hat{P}_n \in D^3$ where $\hat{P}_i = P_i + \varepsilon P_i{}^*$, $P_i, P_i{}^* \in R^3$ for $i = 0,1,\ldots,n$. Then

$$\|B(t)\|' = \frac{\langle B(t), B'(t)\rangle}{\|B(t)\|} \tag{35}$$

satisfies.

Proof: $\|B(t)\|' = \left(\sqrt{\sum_{j=1}^{3}\langle B(t), e_j\rangle^2}\right)' = \frac{\left(\sum_{j=1}^{3}\langle B(t),e_j\rangle^2\right)'}{2.\sqrt{\sum_{j=1}^{3}\langle B(t),e_j\rangle^2}} = \frac{2\sum_{j=1}^{3}\left(\langle B(t),e_j\rangle\langle B'(t),e_j\rangle\right)}{2\|B(t)\|} = \frac{\langle B(t),B'(t)\rangle}{\|B(t)\|}$

İs obtained.

Proposition 3.2: Let $\hat{B}(t) = B(t) + \varepsilon B^*(t)$ be a dual Bézier curve with control points $\hat{P}_0, \hat{P}_1, \dots, \hat{P}_n \in D^3$ where $\hat{P}_i = P_i + \varepsilon P_i^*$, $P_i, P_i^* \in R^3$ for $i = 0,1,\dots,n.$ and $\delta(t)$ be the director curve of the ruled surface $X(t,v)$ corresponding to the dual Bézier curve $\hat{B}(t)$. Then

$$\delta'(t) = \frac{B(t)\times\left(B'(t)\times B(t)\right)}{\|B(t)\|^3} \tag{36}$$

satisfies.

Proof: From (34), $\delta'(t) = \left(\frac{B(t)}{\|B(t)\|}\right)' = \frac{B'(t)\|B(t)\| - B(t)(\|B(t)\|)'}{\|B(t)\|^2}$ can be written. So from (51),

$$\delta'(t) = \left(\frac{B(t)}{\|B(t)\|}\right)' = \frac{B'(t)\|B(t)\| - B(t)\frac{\langle B(t),B'(t)\rangle}{\|B(t)\|}}{\|B(t)\|^2} = \frac{B'(t)\|B(t)\|^2 - B(t)\langle B(t),B'(t)\rangle}{\|B(t)\|^3} = \frac{B'(t)\langle B(t),B(t)\rangle - B(t)\langle B(t),B'(t)\rangle}{\|B(t)\|^3}$$
$$= \frac{B(t)\times\left(B\prime(t)\times B(t)\right)}{\|B(t)\|^3}$$

or

$$\delta'(t) = \frac{B'(t)}{\|B(t)\|} - \frac{\langle B(t),B'(t)\rangle B(t)}{\|B(t)\|^3}$$

is obtained.

Proposition 3.3: Let $\hat{B}(t) = B(t) + \varepsilon B^*(t)$ be a dual Bézier curve with control points $\hat{P}_0, \hat{P}_1, \dots, \hat{P}_n \in D^3$ where $\hat{P}_i = P_i + \varepsilon P_i^*$, $P_i, P_i^* \in R^3$ for $i = 0,1,\dots,n.$ and $\gamma(t)$ be the directrix curve of the ruled surface $X(t,v)$ corresponding to the dual Bézier curve $\hat{B}(t)$. Then

$$\gamma'(t) = \frac{B'(t)\times B^*(t)}{\|B(t)\|^2} + \frac{B(t)\times (B^*)'(t)}{\|B(t)\|^2} - 2\frac{\langle B(t),B'(t)\rangle\left(B(t)\times B^*(t)\right)}{\|B(t)\|^4} \tag{37}$$

satisfies.

Proof: $\gamma'(t) = \left(\frac{B(t)\times B^*(t)}{\|B(t)\|^2}\right)' = \frac{\left(B(t)\times B^*(t)\right)'\|B(t)\|^2 - \left(B(t)\times B^*(t)\right)\left(\|B(t)\|^2\right)'}{\|B(t)\|^4}$

$$= \frac{\left[\left(B'(t)\times B^*(t)\right) + \left(B(t)\times (B^*)'(t)\right)\right]\|B(t)\|^2 - 2\left(B(t)\times B^*(t)\right)\|B(t)\|\frac{\langle B(t),B'(t)\rangle}{\|B(t)\|}}{\|B(t)\|^4}$$
$$= \frac{B'(t)\times B^*(t)}{\|B(t)\|^2} + \frac{B(t)\times (B^*)'(t)}{\|B(t)\|^2} - 2\frac{\langle B(t),B'(t)\rangle\left(B(t)\times B^*(t)\right)}{\|B(t)\|^4}$$

is obtained.

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Some Properties of Cofinitely eg-Supplemented Modules

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Abstract

In this work, some new properties of cofinitely eg-supplemented modules are studied. Every ring has unity and every module is unitary left module, in this work. It is clear that every cofinitely essential supplemented module is cofinitely eg-supplemented. Because of this, cofinitely eg-supplemented modules are more general than cofinitely essential supplemented modules.

Keywords: Cofinite Submodules, Essential Submodules, Cofinitely Supplemented Modules, g-Supplemented Modules.

2020 Mathematics Subject Classification: 16D10, 16D70.

1. INTRODUCTION

Throughout this paper all rings are associative with identity and all modules are unital left modules.

Let R be a ring and M be an R-module. We denote a submodule N of M by $N \leq M$. A submodule U of an R-module M is called a *cofinite* submodule of M if M/U is finitely generated. Let M be an R-module and $N \leq M$. If $L=M$ for every submodule L of M such that $M=N+L$, then N is called a *small* (or *superfluous*) submodule of M and denoted by $N \ll M$. A submodule N of an R-module M is called an *essential* submodule, denoted by $N \trianglelefteq M$, in case $K \cap N \neq 0$ for every submodule $K \neq 0$, or equvalently, $N \cap L=0$ for $L \leq M$ implies that $L=0$. Let M be an R-module and K be a submodule of M. K is called a *generalized small* (briefly, *g-small*) submodule of M if for every essential submodule T of M with the property $M=K+T$ implies that $T=M$, we denote this by $K \ll_g M$ (in [15] , it is called an *e-small* submodule of M and denoted by $K \ll_e M$). Let M be an R-module and $U,V \leq M$. If $M=U+V$ and V is minimal with respect to this property, or equivalently, $M=U+V$ and $U \cap V \ll V$, then V is called a *supplement* of U in M. M is said to be *supplemented* if every submodule of M has a supplement in M. M is said to be *cofinitely supplemented* if every cofinite submodule of M has a supplement in M. M is said to be *essential supplemented* (briefly, *e-supplemented*) if every essential submodule of M has a supplement in M. M is said to be *cofinitely essential supplemented* (briefly, *cofinitely e-supplemented*) if every cofinite essential submodule of M has a supplement in M. Let M be an R-module and $U,V \leq M$. If $M=U+V$ and $M=U+T$ with $T \trianglelefteq V$ implies that

$T=V$, or equivalently, $M=U+V$ and $U\cap V\ll_g V$, then V is called a *g-supplement* of U in M. M is said to be *g-supplemented* if every submodule of M has a g-supplement in M. M is said to be *essential g-supplemented* if every essential submodule of M has a g-supplement in M. M is said to be *cofinitely g-supplemented* if every cofinite submodule of M has a g-supplement in M. The intersection of maximal submodules of an R-module M is called the *radical* of M and denoted by $RadM$. If M have no maximal submodules, then we denote $RadM=M$. The intersection of essential maximal submodules of an R-module M is called the *generalized radical* (briefly, *g-radical*) of M and denoted by Rad_gM (in [15] , it is denoted by Rad_eM). If M have no essential maximal submodules, then we denote $Rad_gM=M$. Let M be an R-module and $K\leq V\leq M$. We say V *lies above* K in M if $V/K\ll M/K$.

More details about supplemented modules are in [3] and [14]. More informaitons about cofinitely supplemented modules are in [1]. More details about essential supplemented modules are in [11] and [12]. More details about cofinitely essential supplemented modules are in [5] and [6]. More informations about g-small submodules and g-supplemented modules are in [7] and [8]. The definition of cofinitely g-supplemented modules and more informations about these modules are in [4]. The definition of essential g-supplemented modules and some properties of them are in [9].

Lemma 1.1. Let M be an R-module. The following assertions hold.
(1) Every small submodule in M is g-small in M.
(2) If $K\leq L\leq M$ and $L\ll_g M$, then $K\ll_g M$ and $L/K\ll_g M/K$.
(3) Let N be an R-module and $f: M\to N$ be an R-module homomorphism. If $K\ll_g M$, then $f(K)\ll_g N$.
(4) If $K\ll_g M$, then $(K+L)/L\ll_g M/L$ for every $L\leq M$.
(5) If $L\leq M$ and $K\ll_g L$, then $K\ll_g M$.
(6) If $K_1,K_2,...,K_n\ll_g M$, then $K_1+K_2+...+K_n\ll_g M$.
(7) Let $K_1,K_2,...,K_n,L_1,L_2,...,L_n\leq M$. If $K_i\ll_g L_i$ for every $i=1,2,...,n$, then $K_1+K_2+...+K_n\ll_g L_1+L_2+...+L_n$.
Proof. See [7] and [8].

Lemma 1.2. Let M be an R-module. The following conditions hold.
(1) Rad_gM is equal to the sum of g-small submodules of M.
(2) $Rm\ll_g M$ for every $m\in Rad_gM$.
(3) If $N\leq M$, then $Rad_gN\leq Rad_gM$.
(4) If $K,L\leq M$, then $Rad_gK+Rad_gL\leq Rad_g(K+L)$.
(5) Let N be an R-module and $f: M\to N$ be an R-module homomorphism. Then $f(Rad_gM)\leq Rad_gN$.
(6) If $K,L\leq M$, then $(Rad_gK+L)/L\leq Rad_g[(K+L)/L]$. If $L\leq Rad_gK$, then $(Rad_gK)/L\leq Rad_g(K/L)$.
(7) If $M=\oplus_{i\in I}M_i$, then $Rad_gM=\oplus_{i\in I}Rad_gM_i$.

(8) $RadM \leq Rad_g M$.
Proof. See [8, Lemma 2, Lemma 3 and Lemma 4].

2. COFINITELY ESSENTIAL g-SUPPLEMENTED MODULES

Definition 2.1. Let M be an R-module. If every cofinite essential submodule of M has a g-supplement in M, then M is called a *cofinitely essential g-supplemented* (or briefly, *cofinitely eg-supplemented*) module. (See also [10])

Lemma 2.2. Every cofinitely supplemented module is cofinitely eg-supplemented.
Proof. Let M be a cofinitely supplemented module and U be a cofinite essential submodule of M. Since U is a cofinite submodule of M and M is cofinitely supplemented, U has a supplement V in M. Here $M=U+V$ and $U\cap V\ll V$. Since $U\cap V\ll V$, $U\cap V\ll_g V$. Hence V is a g-supplement of U in M. Therefore, M is cofinitely eg-supplemented.

Corollary 2.3. Let $M=\Sigma_{i\in I}M_i$. If M_i is cofinitely supplemented for every $i\in I$, then M is cofinitely eg-supplemented.
Proof. Since M_i is cofinitely supplemented for every $i\in I$, by [1, Lemma 2.3], $M=\Sigma_{i\in I}M_i$ cofinitely supplemented. Then by Lemma 2.2, M is cofinitely eg-supplemented.

Corollary 2.4. Let M be a cofinitely supplemented module. Then every M-generated module is cofinitely eg-supplemented.
Proof. Clear from Corollary 2.3.

Corollary 2.5. Let R be a ring. If ${}_RR$ is supplemented, then every R-module is cofinitely eg-supplemented.
Proof. Since ${}_RR$ is supplemented, ${}_RR$ is cofinitely supplemented. Then by Corollary 2.4, every R-module is cofinitely eg-supplemented.

Lemma 2.6. Every essential supplemented module is cofinitely eg-supplemented.
Proof. Let M be an essential supplemented module and U be a cofinite essential submodule of M. Since M is essential supplemented and $U\trianglelefteq M$, U has a supplement V in M. Here $M=U+V$ and $U\cap V\ll V$. Since $U\cap V\ll V$, $U\cap V\ll_g V$. Hence V is a g-supplement of U in M. Thus every cofinite essential submodule of M has a g-supplement in M and M is cofinitely eg-supplemented.

Corollary 2.7. Every supplemented module is cofinitely eg-supplemented.
Proof. Since every supplemented module is essential supplemented, by Lemma 2.6, every supplemented module is cofinitely eg-supplemented.

Lemma 2.8. Every cofinitely g-supplemented module is cofinitely eg-supplemented. (See also [10])
Proof. Let M be a cofinitely g-supplemented module and U be a cofinite essential submodule of M. Since M is cofinitely g-supplemented and U is a cofinite submodule of M, U has a g-supplement in M. Hence every cofinite essential submodule of M has a g-supplement in M and M is cofinitely eg-supplemented.

Corollary 2.9. Let $M=\Sigma_{i\in I}M_i$. If M_i is cofinitely g-supplemented for every $i\in I$, then M is cofinitely eg-supplemented.
Proof. Since M_i is cofinitely g-supplemented for every $i\in I$, by [4, Lemma 2.6], $M=\Sigma_{i\in I}M_i$ cofinitely g-supplemented. Then by Lemma 2.8, M is cofinitely eg-supplemented.

Corollary 2.10. Let M be a cofinitely g-supplemented module. Then every M-generated module is cofinitely eg-supplemented.
Proof. Clear from Corollary 2.9.

Corollary 2.11. Let R be a ring. If ${}_RR$ is g-supplemented, then every R-module is cofinitely eg-supplemented.
Proof. Since ${}_RR$ is g-supplemented, ${}_RR$ is cofinitely g-supplemented. Then by Corollary 2.10, every R-module is cofinitely eg-supplemented.

Proposition 2.12. Let M be a cofinitely eg-supplemented module. If every nonzero submodule of M is essential in M, then M is cofinitely supplemented.
Proof. Let U be a cofinite submodule of M. If $U=0$, then M is a supplement of U in M. Let $U\neq 0$. Then $U\trianglelefteq M$ and since M is cofinitely eg-supplemented, U has a g-supplement V in M. Here $M=U+V$ and $U\cap V\ll_g V$. Since every nonzero submodule of M is essential in M, $U\cap V\ll V$. Then V is a supplement of U in M. Hence M is cofinitely supplemented.

3. CONCLUSION

Supplemented and g-supplemented modules are actual subjects in Module Theory and can be studied on these modules.

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Commutativity of Third-Order Discrete-Time Linear Time-Varying Systems

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Abstract

In this study, third-order cascade-connected discrete-time linear time-varying system is considered. This system is described by a third-order linear difference equation. Commutativity conditions (CCs) of third-order discrete-time linear time-varying systems are presented.

Keywords: Commutativity, Discrete-time, Third-order, Linear system

1. Introduction

Cascade connection of several simple subsystems to form a complex system is a common method for the realization of many engineering designs; this is important for the synthesis of especially electronic and electrical and systems. The order of connection of subsystems may be arbitrary or might depend on the special design methods and traditional techniques. However, when the system performance parameters such as sensitivity, linearity, stability, noise quality, robustness are important, it may cause drastic changes. Hence, the best order should be chosen so that the main function of the total system remains the same (commutativity). This is why commutativity appears to be significant in view of engineering applications.

As shown in Fig. 1, by changing the connection order of two cascade-connected time-varying linear systems A and B, we say that A and B are commutative systems and (A, B) constitutes a commutative pair if input-output relations of the assembled systems AB and BA are identical.

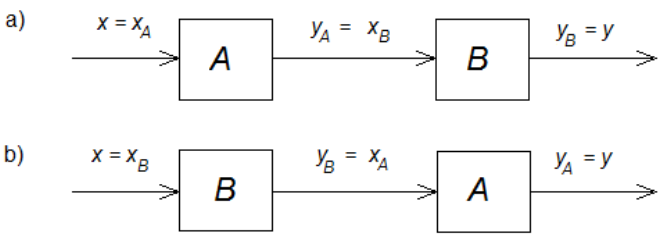

Figure 1: Cascade connections of differential systems

The commutativity was studied for the first time by Marshal [1] in 1977. He developed CCs of first-order continuous time-varying linear systems and proved that a linear time-varying system could be commutative only with another time-varying linear system. In 1982, CCs of second-order continuous time-varying linear systems were obtained by M. Koksal [2]. Then, in 1985, CCs of third- and fourth-order continuous time-varying linear systems were presented by the same author [3]. After a long period of time, in 2011, commutativity of Euler differential systems was investigated, and explicit CCs of fifth-order continuous time-varying linear systems were studied by M. Koksal and M. E. Koksal [4]. For the CCs of continuous time-varying linear systems, the last literature containing CCs of sixth-order systems was studied by S. Ibrahim and M. E. Koksal in 2021 [5].

Even though there are many papers on the commutativity of time-varying linear analog systems, there are only a few literature on the commutativity of time-varying linear digital systems. The trend in the new technology is moving to the digital world from the analog world. The advantages of digital systems over analog systems in many cases are well known. Some of these advantages are; digital systems do not necessitate any hardware changes for achieving different transfer characteristics, and they can be tuned by only software. From this point of view, the investigation of CCs of discrete time-varying linear systems is very important.

In 2015, explicit CCs of second-order discrete-time linear time-varying systems were studied by M. Koksal and M. E. Koksal [6]. The results were illustrated by numerical experiments. After that, in 2019, CCs of first-order discrete-time linear time-varying systems were obtained by M. E. Koksal [7] with illustrative examples. In this study, third-order discrete-time linear time-varying systems are considered. Some CCs are given for these systems.

2. Third-order Systems

Let systems A and B be discrete-time linear time-varying systems defined as in the following equations:

$$A: a_3(k)y_A(k+3) + a_2(k)y_A(k+2) + a_1(k)y_A(k+1) + a_0(k)y_A(k) = x_A(k), \tag{1}$$

$$B: b_3(k)y_B(k+3) + b_2(k)y_B(k+2) + b_1(k)y_B(k+1) + b_0(k)y_B(k) = x_B(k), \tag{2}$$

From the cascade connection in Fig. 1a, we know that

$$x(k) = x_A(k), \qquad y_A(k) = x_B(k), \qquad y_B(k) = y(k).$$

So, Eq. (2) can be written as

$$b_3(k)y_B(k+3) + b_2(k)y_B(k+2) + b_1(k)y_B(k+1) + b_0(k)y_B(k) = x_B(k) = y_A(k).$$

By writing $k+1,\ k+2,\ k+3$ instead of k in the above equation, we obtain the following three equations:

$$b_3(k+1)y_B(k+4) + b_2(k+1)y_B(k+3) + b_1(k+1)y_B(k+2) + b_0(k+1)y_B(k+1)$$

$$= x_B(k+1) = y_A(k+1),$$

$$b_3(k+2)y_B(k+5) + b_2(k+2)y_B(k+4) + b_1(k+2)y_B(k+3) + b_0(k+2)y_B(k+2) \\ = x_B(k+2) = y_A(k+2),$$

$$b_3(k+3)y_B(k+6) + b_2(k+3)y_B(k+5) + b_1(k+3)y_B(k+4) + b_0(k+3)y_B(k+3) \\ = x_B(k+3) = y_A(k+3).$$

Substituting the above three formulas (of $y_A(k+1), y_A(k+2), y_A(k+3)$) in Eq. (1) and considering $x_A(k) = x(k), x_B(k) = y_A(k)$ for the cascade-connected system AB, we obtain the following sixth-order difference equation:

$$\begin{aligned} AB\colon & a_3(k)b_3(k+3)y(k+6) + [a_3(k)b_2(k+3) + a_2(k)b_3(k+2)]y(k+5) \\ & +[a_3(k)b_1(k+3) + a_2(k)b_2(k+2) + a_1(k)b_3(k+1)]y(k+4) \\ & +[a_3(k)b_0(k+3) + a_2(k)b_1(k+2) + a_1(k)b_2(k+1) + a_0(k)b_3(k)]y(k+3) \\ & +[a_2(k)\mathrm{b}_0(k+2) + a_1(k)\mathrm{b}_1(k+1) + a_0(k)\mathrm{b}_2(k)]\mathrm{y}(k+2) \\ & +[a_1(k)b_0(k+1) + a_0(k)b_1(k)]y(k+1) + a_0(k)b_0(k)y(k) = x(k), \end{aligned} \tag{3}$$

In a similar manner in obtaining the equation of cascade-connected system AB, we can obtain the equation of BA. From the cascade connection in Fig. 1b, we know that

$$x(k) = x_B(k), \qquad y_B(k) = x_A(k), \qquad y_A(k) = y(k).$$

Then, Eq. (1) can be written as

$$a_3(k)y_A(k+3) + a_2(k)y_A(k+2) + a_1(k)y_A(k+1) + a_0(k)y_A(k) = x_A(k) = y_B(k).$$

By writing $k+1$, $k+2$, $k+3$ instead of k in the above equation, we obtain the following three equations:

$$a_3(k+1)y_A(k+4) + a_2(k+1)y_A(k+3) + a_1(k+1)y_A(k+2) + a_0(k+1)y_A(k+1) \\ = x_A(k+1) = y_B(k+1),$$

$$a_3(k+2)y_A(k+5) + a_2(k+2)y_A(k+4) + a_1(k+2)y_A(k+3) + a_0(k+2)y_A(k+2) \\ = x_A(k+2) = y_B(k+2),$$

$$a_3(k+3)y_A(k+6) + a_2(k+3)y_A(k+5) + a_1(k+3)y_A(k+4) + a_0(k+3)y_A(k+3) \\ = x_A(k+3) = y_B(k+3),$$

If we substitute the above formulas of $y_B(k+1), y_B(k+2), y_B(k+3)$ in Eq. (2) and use $x_B(k) = x(k), x_A(k) = y_B(k)$ for the cascade-connected system BA, we obtain the following sixth-order difference equation:

$$\begin{aligned} BA\colon & b_3(k)a_3(k+3)y(k+6) + [b_3(k)a_2(k+3) + b_2(k)a_3(k+2)]y(k+5) \\ & +[b_3(k)a_1(k+3) + b_2(k)a_2(k+2) + b_1(k)a_3(k+1)]y(k+4) \\ & +[b_3(k)a_0(k+3) + b_2(k)a_1(k+2) + b_1(k)a_2(k+1) + b_0(k)a_3(k)]y(k+3) \\ & +[b_2(k)a_0(k+2) + b_1(k)a_1(k+1) + b_0(k)a_2(k)]y(k+2) \end{aligned}$$

$$+[b_1(k)\,a_0(k+1)+b_0(k)a_1(k)]y(k+1)+b_0(k)a_0(k)y(k)=x(k) \tag{4}$$

3. Commutativity

If Eqs. (3) and (4) are compared, since systems AB and BA systems are identical (A and B are commutative conjugates), the coefficients of the same $y(i)$ values are equalized. By equalizing, we obtain the following six equations:

$$a_3(k)b_3(k+3)=b_3(k)a_3(k+3),\quad k\geq 0, \tag{5}$$

$$a_3(k)b_2(k+3)+a_2(k)b_3(k+2)=b_3(k)a_2(k+3)+b_2(k)a_3(k+2),\quad k\geq 1 \tag{6}$$

$$\begin{aligned}&a_3(k)b_1(k+3)+a_2(k)b_2(k+2)+a_1(k)b_3(k+1)\\&=b_3(k)a_1(k+3)+b_2(k)a_2(k+2)+b_1(k)a_3(k+1),\quad k\geq 2,\end{aligned} \tag{7}$$

$$\begin{aligned}&a_3(k)b_0(k+3)+a_2(k)b_1(k+2)+a_1(k)b_2(k+1)+a_0(k)b_3(k)\\&=b_3(k)a_0(k+3)+b_2(k)a_1(k+2)+b_1(k)a_2(k+1)+b_0(k)a_3(k),\quad k\geq 3\end{aligned} \tag{8}$$

$$\begin{aligned}&a_2(k)b_0(k+2)+a_1(k)b_1(k+1)+a_0(k)b_2(k)\\&=b_2(k)a_0(k+2)+b_1(k)a_1(k+1)+b_0(k)a_2(k),\quad k\geq 4,\end{aligned} \tag{9}$$

$$a_1(k)b_0(k+1)+a_0(k)b_1(k)=b_1(k)\,a_0(k+1)+b_0(k)a_1(k),\qquad k\geq 5. \tag{10}$$

In the above, we have four unknown coefficients $b_3(n), b_2(n), b_1(n), b_0(n)$ and six equations. Here, we can find the unknown coefficients by using the iterative method. The general solution of $b_3(n)$ is found by giving the values $0,1,2,\cdots$ to k, respectively in Eq. (5), and using the previous result each time as follows:

$$b_3(n)=\begin{cases}\dfrac{a_3(n)}{a_3(0)}b_3(0), & n=3,6,9,\cdots\\[2mm] \dfrac{a_3(n)}{a_3(1)}b_3(1), & n=4,7,10,\cdots\\[2mm] \dfrac{a_3(n)}{a_3(2)}b_3(2), & n=5,8,11,\cdots\end{cases}$$

Following the same procedure in Eq. (6) and using the values of $b_3(n)$, We can find the formula of $b_2(n)$;

$$b_2(n)=b_2(0)\prod_{i=1}^{\frac{n}{3}}\frac{a_3(3i-1)}{a_3(3i-3)}+\frac{b_3(0)}{a_3(0)}\sum_{i=1}^{\frac{n}{3}}\left[\prod_{k=0}^{i-2}\frac{a_3(n-1-3k)}{a_3(n-3-3k)}\right]a_2(n+3-3i)$$

$$-\frac{b_3(2)}{a_3(2)}\sum_{i=1}^{\frac{n}{3}}\left[\prod_{k=0}^{i-1}\frac{a_3(n-1-3k)}{a_3(n-3-3k)}\right]a_2(n-3i), \quad for\ n=0,3,6,\cdots$$

$$b_2(n)=b_2(1)\prod_{i=1}^{\frac{n-1}{3}}\frac{a_3(3i)}{a_3(3i-2)}-\frac{b_3(0)}{a_3(0)}\sum_{i=1}^{\frac{n-1}{3}}\left[\prod_{k=0}^{i-1}\frac{a_3(n-1-3k}{a_3(n-3-3k)}\right]a_2(n-3i)$$
$$+\frac{b_3(1)}{a_3(1)}\sum_{i=1}^{\frac{n-1}{3}}\left[\prod_{k=0}^{i-2}\frac{a_3(n-1-3k}{a_3(n-3-3k)}\right]a_2(n+3-3i), \quad for\ n=1,4,7,\cdots$$

$$b_2(n)=b_2(2)\prod_{i=1}^{\frac{n-2}{3}}\frac{a_3(3i+1)}{a_3(3i-1)}-\frac{b_3(1)}{a_3(1)}\sum_{i=1}^{\frac{n-2}{3}}\left[\prod_{k=0}^{i-1}\frac{a_3(n-1-3k}{a_3(n-3-3k)}\right]a_2(n-3i)$$
$$+\frac{b_3(2)}{a_3(2)}\sum_{i=1}^{\frac{n-2}{3}}\left[\prod_{k=0}^{i-2}\frac{a_3(n-1-3k)}{a_3(n-3-3k)}\right]a_2(n+3-3i), \quad for\ n=2,5,8,\cdots$$

In Eq. (7), by giving the values $0,1,2,\cdots$ to k, respectively, and using the previous value of $b_1(n)$, we obtain the formula of $b_1(n)$ as follows:

$$b_1(n)=\sum_{i=1}^{\frac{n}{3}}\left(\prod_{k=i}^{\frac{n-3}{3}}\frac{a_3(3k+1)}{a_3(3k)}\right)\left[\frac{a_1(3i)}{a_3(3i-3)}b_3(3i-3)-\frac{a_1(3i-3)}{a_3(3i-3)}b_3(3i-2)\right]$$
$$+\sum_{i=1}^{\frac{n}{3}}\left(\prod_{k=i}^{\frac{n-3}{3}}\frac{a_3(3k+1)}{a_3(3k)}\right)\left[\frac{a_2(3i-1)}{a_3(3i-3)}b_2(3i-3)-\frac{a_2(3i-3)}{a_3(3i-3)}b_2(3i-1)\right]$$
$$+\prod_{i=1}^{\frac{n}{3}}\frac{a_3(3i-2)}{a_3(3i-3)}b_1(0), \quad for\ n=3,6,9,\cdots$$

$$b_1(n)=\sum_{i=1}^{\frac{n-1}{3}}\left[\prod_{k=i}^{\frac{n-4}{3}}\frac{a_3(3k+2)}{a_3(3k+1)}\right]\left[\frac{a_1(3i+1)}{a_3(3i-2)}b_3(3i-2)-\frac{a_1(3i-2)}{a_3(3i-2)}b_3(3i-1)\right]$$

$$+\sum_{i=1}^{\frac{n-1}{3}}\left[\prod_{k=i}^{\frac{n-4}{3}}\frac{a_3(3k+2)}{a_3(3k+1)}\right]\left[\frac{a_2(3i)}{a_3(3i-2)}b_2(3i-2)-\frac{a_2(3i-2)}{a_3(3i-2)}b_2(3i)\right]$$

$$+\prod_{i=1}^{\frac{n-1}{3}}\frac{a_3(3i-1)}{a_3(3i-2)}b_1(1)\,,\quad for\; n=4,7,10,\cdots$$

$$b_1(n)=\sum_{i=1}^{\frac{n-2}{3}}\left[\prod_{k=i}^{\frac{n-5}{3}}\frac{a_3(3k+3)}{a_3(3k+2)}\right]\left[\frac{a_1(3i+2)}{a_3(3i-1)}b_3(3i-1)-\frac{a_1(3i-1)}{a_3(3i-1)}b_3(3i)\right]$$

$$+\sum_{i=1}^{\frac{n-2}{3}}\left[\prod_{k=i}^{\frac{n-5}{3}}\frac{a_3(3k+3)}{a_3(3k+2)}\right]\left[\frac{a_2(3i+1)}{a_3(3i-1)}b_2(3i-1)-\frac{a_2(3i-1)}{a_3(3i-1)}b_2(3i+1)\right]$$

$$+\prod_{i=1}^{\frac{n-2}{3}}\frac{a_3(3i)}{a_3(3i-1)}b_1(2)\,,\quad for\; n=5,8,11,\cdots$$

Finally, using the same way in Eq. (10), we find the formula of $b_0(n)$;

$$b_0(n)=\sum_{i=1}^{n}\frac{a_0(i)-a_0(i-1)}{a_1(i-1)}b_1(i-1)+b_0(0)\,,\qquad n=1,2,3,\cdots$$

So, we obtain the formulas of $b_3(n), b_2(n), b_1(n), b_0(n)$ and these are required for finding the commutative pair of system A, but additional conditions among the coefficients $b_i(n)$ for $i=0,1,2,3$ are required because of the fact that we have not used Eq. (8) and (9) for finding these coefficients.

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Cut locus of L^1 sub-Finlser problems in R^3: two case studies.

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Abstract

We study the local geometry of the sub-Finsler structure associated with two bracket-generating vector fields in R^3 and the L^1 norm of the controls. We provide a normal form of the two vector fields and we study two particular non-generic cases, describing the upper part of the cut locus for short geodesics.

Keywords: Optimal Control, sub-Finsler geometry, cut locus

1. Introduction

Let f and g be two smooth vector fields on R^3 such that f,g and their Lie brackets $[f,g]$ are linearly independent at every point. We endow R^3 with a *L^1 sub-Finsler distance*, in the following way: for every pair of points x_0, x_1 in R^3, we define the distance between them as the infimum of the functional

$$J(\boldsymbol{u}) = \int_0^1 |u_1(t)| + |u_2(t)| dt, \quad \boldsymbol{u} = (u_1, u_2), \tag{1}$$

taken over all real-valued L^1 functions u_1 and u_2 such that the solution of the Cauchy problem

$$\begin{cases} \dot{\xi}(t) = u_1(t) f\big(\xi(t)\big) + u_2(t) g\big(\xi(t)\big) \\ \quad \xi(0) = x_0 \end{cases} \tag{2}$$

satisfies $\xi(1) = x_1$.

Stated as above, the problem of computing the distance between the two points is an optimal control problem, that can be treated with classical tools, such as Pontryagin Maximum Principle (PMP). Yet, in most cases, an admissible trajectory (that is, a solution of (2)) that satisfies PMP is not optimal on its whole length, but at some point, called the *cut point*, it loses its global optimality (that is, it ceases to be length minimizing). The set of all cut points of the trajectories with origin in x_0 is called the *cut locus to x_0*. Determining the cut locus (to some point x_0) is thus important in order to compute the distances from x_0 and to describe the properties of the metric, such as, for instance, the shape of the spheres centered at x_0.

The simplest case of L^1 sub-Finsler distance on R^3 is the Heisenberg group, which has been studied in [4]. The cut locus and the form of the spheres of any radius have been characterized. A similar analysis in the generic case is however vary hard, as many issues due to the nonlinearity of f and g may arise;

on the other hand, it is very interesting to see how *small perturbations* of the Heisenberg system affect the shapes of the cut locus and of the sphere; this study can be done by concentrating on *small* (i.e., with small radius) *spheres*: indeed, for short distances, the linear parts of the vector fields f and g (the ones corresponding to the Heisenberg system) dominate.

In the article [3], the authors study the small spheres in the most generic case; more precisely, they provide a normal form for the vector fields f and g and single out two invariants of the metric (in the following, C_1 and C_2), whose signs allow to localize where a trajectory loses its local optimality (*conjugate point*), and, as byproduct, to give a bound on the cut points. Under the hypothesis that these invariants are non zero, the authors provide a complete description of the cut locus for small spheres; the classification of the different cases is given in terms of the values of C_1 and C_2. In particular, in [3] it is shown that the cut locus is always symmetric with respect to the vertical axis and, depending on the values of two invariants, it can be constituted either by five smooth branches, or by only one smooth branch.

In this paper, we are going on with the analysis started in [3]. In particular, we assume that one (and only one) between C_1 and C_2 is zero. New invariants (called E_1 and D_1 in the following) appear in the determination of conjugate points and in the classification of the possible shapes of the cut locus. Some of these cases are quite different from the ones found in [3, 4] (for instance, the symmetry with respect to the vertical axis is broken): in particular, in this paper we are showing two of them, one in which the cut locus is made by three smooth branches, one other in which the cut locus is not connected.

2. Preliminaries

Normal form of the vector fields f and g

Before tackling the resolution of the problem, it is worth to find a *normal form* of the vector fields f and g. In this paper, we are restricting our attention to a specific sub-class of sub-Finsler L^1 structures, that is, those that are related to a specific sub-Riemannian structure. Indeed, let h_1, h_2 be two vector fields on $\mathbb{R}^3$, such that h_1, h_2, and $[h_1, h_2]$ are linearly independent at every point and that h_1 and h_1 are orthonormal with respect to some scalar product on $\mathbb{R}^3$. Then, it has been proved in [1, 5] that there exist local coordinates (q_1, q_2, q_3) in which the vector fields h_1, h_2 have the form

$$h_1 = \begin{pmatrix} 1 + q_2^2\beta(q_1, q_2, q_3) \\ -q_1 q_2 \beta(q_1, q_2, q_3) \\ \frac{-q_2}{2}\left(1 + \gamma(q_1, q_2, q_3)\right) \end{pmatrix} \qquad h_2 = \begin{pmatrix} -q_1 q_2 \beta(q_1, q_2, q_3) \\ 1 + q_1^2\beta(q_1, q_2, q_3) \\ \frac{q_1}{2}\left(1 + \gamma(q_1, q_2, q_3)\right) \end{pmatrix},$$

where β,γ are two functions that vanish on $(0,0,q_3)$, together with the (first order) partial derivatives of γ with respect to q_1 and q_2.

We assume that the vector fields f, g are related to h_1, h_2 in the following way: there exist a non-singular matrix M such that $\begin{pmatrix} f \\ g \end{pmatrix} = M \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$. We call such sub-Finsler L^1 structures *compatible with a sub-Riemannian structure*.

We can apply another change of variables and write, in the new variables (x,y,z),

$$f = \begin{pmatrix} 1 + xyL_{11}(x,y,z) + y^2L_{12}(x,y,z) \\ xyL_{21}(x,y,z) + y^2L_{22}(x,y,z) \\ -\frac{y}{2} + xyL_{31}(x,y,z) + y^2L_{32}(x,y,z) \end{pmatrix} \qquad g = \begin{pmatrix} -x^2L_{11}(x,y,z) - xyL_{12}(x,y,z) \\ 1 - x^2L_{21}(x,y,z) - xyL_{22}(x,y,z) \\ \frac{x}{2} - x^2L_{31}(x,y,z) - xyL_{32}(x,y,z) \end{pmatrix}$$

where L_{ij}, for $i=1,2,3$, $j=1,2$, are smooth functions that can be developed in power series as

$$L_{ij}(x,y,z) = ax_{ij}x + ay_{ij}y + \frac{1}{2}(xyz)\begin{pmatrix} \omega_{xxij} & \omega_{xyij} & \omega_{xzij} \\ \omega_{yxij} & \omega_{yyij} & \omega_{yzij} \\ \omega_{zxij} & \omega_{zyij} & \omega_{zzij} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} + h.o.t$$

The terms ax_{ij}, ay_{ij} and ω_{abij}, $i=1,2,3$, $j=1,2$, $a,b=x,y,z$, are called the *invariants* of the metric. Indeed, the Heisenberg system corresponds to the case where all invariants are null. If only a finite (possibly zero) number of invariants are assumed to be zero, then we say that the case is *generic*.

The minimum time problem

Let us consider the optimal control problem (1)-(2) with $x_0=0$. Performing the time reparametrization $s(t) = \int_0^t |u_1(r)| + |u_2(r)| dr$, the problem (1)-(2) can be rewritten as the following minimum-time problem

Problem (T) : *minimize the final time T, over all trajectories of the control system (2) such that* $\xi(T)=x_1$ *and the controls* u_1,u_2 *are measurable functions satisfying the constraint*

$$|u_1(t)|+|u_2(t)|\leq 1 .$$

Pontryagin Maximum Principle (PMP) is a celebrated first-order necessary optimality condition. Here below, we recall it in a formulation adapted to the minimum time problem under study; for more references, see for instance [2]. First of all, let us define the control-dependent Hamiltonian $h(P,X,u)=u_1F(P,X)+u_1G(P,X)$, where $P=(p_x,p_y,p_z)\in\mathbb{R}^3$ and $F(P,X)=\langle P,f(X)\rangle$, $G(P,X)=\langle P,g(X)\rangle$. PMP states that, if $\underline{\xi}$: $[0,T]\rightarrow\mathbb{R}^3$ is an optimal solution for the minimum-time problem and $\underline{\boldsymbol{u}}(.)$ is its associated control function, then there exist a Lipschitz curve $\underline{P}(.)$ and a constant $\nu\in\{0, 1\}$ such that $\underline{P}(t)\neq 0$ and $h(\underline{P}(t),\underline{\xi}(t),\underline{\boldsymbol{u}}(t))= \nu$ for all $t\in[0,T]$, and

$$\frac{d\underline{P}(t)}{dt} = \frac{-\partial h}{\partial X}\left(\underline{P}(t),\underline{\xi}(t),\underline{u}(t)\right) \qquad \frac{d\underline{X}(t)}{dt} = \frac{\partial h}{\partial P}\left(\underline{P}(t),\underline{\xi}(t),\underline{u}(t)\right) \quad (3)$$

$$h\left(\underline{P}(t), \underline{\xi}(t), \underline{u}(t)\right) = \max_{|w_1|+|w_2|\leq 1} h\left(\underline{P}(t), \underline{\xi}(t), w\right). \quad (4)$$

The curves $(\underline{P},\underline{\xi})$: $[0,T]\to R^3\times R^3$ satisfying the conditions here above (for some admissible control $\boldsymbol{u}$) are called *extremals* of the optimal control problem; their projections $\underline{\xi}(.)$ are called *geodesics*. If the constant ν is equal to 1, we say that the extremal is *normal*. Otherwise, it is said to be *abnormal*.

Remark 1: Under the assumption that $\{f,g,[f,g]\}$ is a basis for R^3 at every point, it is easy to prove that there is no abnormal extremals and that the control associated with an extremal takes values *on the boundary* of the set $Q=\{(u_1,u_2):|u_1|+|u_2|\leq 1\}$. Moreover, as along any extremal the Hamiltonian h is equal to 1, and since $\underline{\xi}(0)=0$, at least one between $|p_x(0)|$ or $|p_y(0)|$ is 1; in particular, both $|p_x(0)|$ are $|p_y(0)|$ less than or equal to 1.

Thanks to (4), the value of the control is determined by the relative values of F and G along the extremal: take some interval $I \subset [0, T]$, and let $(P,\xi) : [0, T]\to R^3\times R^3$ be some extremal; then

• if $|F(P(t),\xi(t))|\neq|G(P(t),\xi(t))|$ $\forall t \in I$, then, on I, the control takes value on one of the *vertices* of Q. In this case, $(P(t),\xi(t))|_I$ is said to be a *regular bang arc*.

• if $|F(P(t),\xi(t))|=|G(P(t),\xi(t))|$ $\forall t \in I$, then the control takes values on one of the *sides* of Q, and, in particular, it is not uniquely determined. Indeed, if for instance $F(P(t),\xi(t))= G(P(t),\xi(t)) > 0$ $\forall t \in I$, then any control of the form $(\alpha,1-\alpha)$, $\alpha \in [0, 1]$, realizes the maximum in equation (4). In this case, $(P(t),\xi(t))|_I$ is said to be a *singular arc*.

When an extremal crosses transversely one of the subsets $\{F = G\}$ or $\{F = -G\}$, the control *switches* from one vertex of Q to another one; in particular, by continuity of the extremals and from the fact that F and G cannot be both zero along an extremal, a control satisfying PMP can switch only from one vertex of Q to a neighboring one; an extremal such that the associate control switches from vertex to vertex is called *bang-bang*. F and G are called the *switching functions*, and the subsets $\{F = G\}$ and $\{F = -G\}$ are called switching surfaces. The derivatives of the switching functions along an extremal are respectively given by $\dot{F} = -u_2$ and $\dot{G} = u_1\Theta$, where $\Theta(P,X)=\langle P,[f,g](X)\rangle$.

Local and global optimality of geodesics

Clearly, if a geodesic is time-minimizing between its two endpoints, then it is time-minimizing also between any two intermediate points of its. However, in general a geodesic is not time-minimizing on its whole length, but at some point it ceases to be optimal; the point where a geodesics loses its optimality is called a *cut point,* defined as it follows.

Definition 1. Let ξ be an admissible curve of the control system (2). We define $t_{cut}(\xi) = \{\sup t > 0 : \xi|_{[0,t]}$ is time-minimizing$\}$, and we call it *cut time* of the geodesic ξ. If $t_{cut}(\xi) < +\infty$, we say that $\xi(t_{cut})$ is the *cut point* to $\xi(0)$ along ξ. We call the *cut locus* (to the origin) the set of all cut points of geodesics starting from x_0.

In optimal control, it is worth investigating also the *local* optimality of geodesics; in particular, in this paper we are interested in optimality in the strong topology, accordingly to the following definition: we say that an admissible trajectory ξ is *locally optimal* if there exists a neighborhood U of its graph in $R \times R^3$ such that ξ is time minimizing among all admissible trajectories with graph contained in U.

In optimal control, a usual method to detect the loss of local optimality is to look for the points of non-invertibility of the exponential map (see for instance [3, 4]); these points are called *conjugate points*. For the problem under concern, the exponential map is defined as

$$exp: (P^0, s) \mapsto \xi(s),$$

where $(P(s),\xi(s))$ is the value at the time s of the extremal with initial condition equal to (P^0,x_0). We remark that, as we always have $|p_x(0)| = 1$ or $|p_y(0)| = 1$ (see Remark 1), then the exponential mapping is a function of three variables.

This map is well defined only if $(P(s),\xi(s))$ is a regular bang-bang extremal (see [7]), therefore we can use it only for studying the local optimality of regular bang-bang extremals. Moreover, this map is smooth, except at the pairs (P^0,s) such that $(P(s), \xi(s))$ belongs to a switching surface: at this points, the exponential map has well-defined, but different, "left" and "right" derivatives. On the other hand, it is well known (see again [7]) that bang-bang extremals cannot lose their local optimality along bang arcs, so that we must concentrate on what happens at the switching points. This justifies the following definition.

Definition 2. Let (P,ξ) be a regular bang-bang extremal for Problem **(T)**. The first *conjugate time* along (P,ξ) is defined as

$$t_{conj}(P,\xi) = \inf \{t > 0 : \exists\, t_1 < t < t_2 \text{ such that } JExp(P^0, t_1) JExp(P^0, t_2) < 0\},$$

where $JExp(P^0, t)$ denotes the Jacobian of $Exp(P^0, t)$. We also call $\xi(t_{conj})$ the first *conjugate point* along (P,ξ). The set of all conjugate points associated to extremals starting from the origin is called the first *conjugate locus* (from the origin).

The Heisenberg system

As already mentioned in the introduction, the simplest pair of vector fields such that f, g and their Lie brackets $[f,g]$ are linearly independent at every point corresponds to the Heisenberg group (also known as Brockett integrator), and it has already been studied in [4]. Since it constitutes a starting point for the study of the generic case, it is worth recalling the main properties of its time-optimal synthesis; we refer to

[4] for all details. First, we recall that the vector fields are respectively given by $f=(1,0,-y/2)$ and $g=(0,1,x/2)$.

Let us apply PMP to problem **(P)**; since the control system does not depend on z, the adjoint vector p_z is constant along every extremal; moreover, $\Theta(P,X)\equiv p_z$. This fact has two consequences:

- Singular arcs are characterized by $p_z\equiv 0$; therefore, if an extremal contains a singular arc, then the whole extremal is singular.
- Given any bang-bang concatenation, the sequence of its associated controls is uniquely determined by the initial one and the sign of p_z : indeed, if $p_z > 0$ (respectively, $p_z < 0$), the controls follow the vertices of Q in the counterclockwise (respectively, clockwise) sense.

In particular, the only possible extremals are either singular or bang-bang.

We notice moreover that the problem exhibits a discrete symmetry: it is invariant by rotations of multiples of $\pi/2$ around the z axis. For these reasons, it suffices to compute only the extremals whose initial adjoint

vector $P(0)=(p^0_x,p^0_y,p^0_z)$ satisfies $p^0_x=1$, $|p^0_y|\leq 1$ and $p^0_z \geq 0$; indeed, all other extremals can be recovered from these ones by applying a suitable transformation.

First of all, we consider the case in which $p^0_y \in [-1, 1)$ and $p^0_z > 0$. Since $F(P(0),\xi(0)) = 1 > |p^0_y| = |G(P(0),\xi(0))|$, then the control associated with the extremal at $t=0$ is (1,0) ; actually, the control is (1,0) on the whole interval $[0,T_1)$, where T_1 is the smallest (positive) time satisfying $F(P(T_1),\xi(T_1)) = G(P(T_1),\xi(T_1))$; by computations, $T_1 =(1-p^0_y)/ p^0_z$. After this time, the control equals (0,1), until the time T_2 satisfying $F(P(T_2),\xi(T_2)) = -G(P(T_2),\xi(T_2))$, given by $T_2 = T_1 + \Delta T$, with $\Delta T =2/p^0_z$. After T_2, the control switches every ΔT.

In the special case $p^0_x= p^0_y = 1$, explicit integrating the Hamiltonian system and applying equation (4), it is easy to see that the first control is (0,1); repeating the same analysis as above, we see that the control switches every ΔT , following the same sequence as above.

Let ξ be a geodesic corresponding to an extremal with $p(0) = (1, p^0_y,p^0_z)$, $|p^0_y| < 1$; integrating the system, we see that, for $t\in[T_4 , 8/p^0_z]$, the expression of the geodesic is given by

$$\begin{cases} x(t) = t - 4\Delta T \\ y(t) = 0 \\ z(t) = 4\Delta T^2 \end{cases}$$

that is, its value depends only on t and on p^0_z, but not on p^0_y. In particular, at $t = T_4 = (7-p^0_y)/ p^0_z$, ξ meets all other geodesics with initial adjoint vector $(1,\hat{p}^0_y,p^0_z)$, $\hat{p}^0_y > p^0_y$, and coincides with them up to the time $8/ p^0_z$. Moreover, for every $\varepsilon>0$, we can always find some $q > p^0_y$ such that the graph of the geodesic with initial adjoint covector equal to $(1,q,p^0_z)$ is ε-close (in the uniform norm) to the graph of ξ. In other words, at $t = T_4$ the trajectory ξ loses its local optimality; the fourth switching time is then its conjugate time.

The cut time is obviously less than or equal to the conjugate time. To verify if these two times are different, we must look at intersections of the trajectory under study with other trajectories whose graph does not belong to a neighborhood of its graph; in particular, these trajectories are those whose initial control is not (1,0) (that is, the first component of their initial adjoint vector is not 1). By computations, it is possible to prove that such intersections occur either at the conjugate time, either at $8/p^0_z$. Therefore, for every trajectory, the fourth switching time T_4 is *both the conjugate time and the cut time*.

To complete the analysis, we now consider a singular extremal with $p^0_x=p^0_y= 1$ and $p_z = 0$. As already said, every control of the form (α(t), 1 − α(t)), with α(t) ∈ [0, 1] ∀t, realizes the maximum in (4). First of all, by direct integration we notice that $x(T) + y(T) = \int_0^T |u_1(t)| + |u_2(t)| dt = T$, so that a singular extremal is necessarily optimal (as no point (x_1,y_1,z_1) can be reached in a time less than $|x_1|+|y_1|$). Moreover, it is possible to prove that every point reached by a singular trajectory satisfies $|z(T)| \leq |x(T)y(T)|/2$, and, conversely, that every point (x_1,y_1,z_1) satisfying this bound can be reached by a singular trajectory. In particular, the minimum time for reaching such points is $|x_1|+|y_1|$.

3. Local expansion of geodesics

Jets of short geodesics

As already anticipated in the Introduction, we are interested in the local problem only, that is, we study the optimal synthesis from the origin for *small times*. In this paper, we focus on final points (x_1,y_1,z_1) such that $|z_1| > |x_1 y_1|/2$, that is, they are reached by means of bang-bang extremals.

Since, in the nilpotent case, the switching times are inversely proportional to the third component of the adjoint vector (at time zero) p_z^0, small times correspond to *large values* for p_z^0. Let $(P(t),\xi(t))$ be an extremal of **(T)** with $p_z^0>>1$, and let $\boldsymbol{u}(t)$ be its associated control. Following the same techniques of [1, 3, 5], we perform the time reparametrization

$$\tau(t) = \int_0^t p_z(s) ds$$

and we define the new variables $\underline{p}_x=p_x/p_z$, $\underline{p}_y=p_y/p_z$, $\rho=1/p_z$ and $\overline{\Theta}=\Theta/p_z$. We set $\rho_0=\rho(0)$.

In order to compute the jets of the extremal $(P(t),\xi(t))$, we expand it in power series of ρ_0:

$$\begin{cases} x(\tau) = \rho_0 x_1(\tau) + \rho_0^2 x_2(\tau) + \rho_0^3 x_3(\tau) + \rho_0^4 x_4(\tau) + O(\rho_0^5) \\ y(\tau) = \rho_0 y_1(\tau) + \rho_0^2 y_2(\tau) + \rho_0^3 y_3(\tau) + \rho_0^4 y_4(\tau) + O(\rho_0^5) \\ z(\tau) = \rho_0 z_1(\tau) + \rho_0^2 z_2(\tau) + \rho_0^3 z_3(\tau) + \rho_0^4 z_4(\tau) + \rho_0^5 z_5(\tau) + O(\rho_0^6) \\ \bar{p_x}(\tau) = \rho_0 \underline{p}_{x1}(\tau) + \rho_0^2 \underline{p}_{x2}(\tau) + \rho_0^3 \underline{p}_{x3}(\tau) + \rho_0^4 \underline{p}_{x4}(\tau) + O(\rho_0^5) \\ \bar{p_y}(\tau) = \rho_0 \underline{p}_{y1}(\tau) + \rho_0^2 \underline{p}_{y2}(\tau) + \rho_0^3 \underline{p}_{y3}(\tau) + \rho_0^4 \underline{p}_{y4}(\tau) + O(\rho_0^5) \\ \rho(\tau) = \rho_0 + \rho_0^2 \rho_2(\tau) + \rho_0^3 \rho_3(\tau) + \rho_0^4 \rho_4(\tau) + O(\rho_0^5) \end{cases} \quad (5)$$

where x_i, y_i, z_i ,$\underline{p}_{xi}$, $\underline{p}_{yi}$ and ρ_i are function of the reparametrized time to be determined. Plugging (5) into (3), we can see that $\frac{d\rho}{d\tau} = O(\rho_0^5)$,so that $\rho(\tau)=\rho_0 + O(\rho_0^5)$. The control system (2) can be thus written as

$$\begin{cases} \frac{dx}{d\tau}(\tau) = u_1\rho_0 + (u_2x_1 - u_1y_2)(ax_{11}x_1^2 + (ax_{12} + ay_{11})x_1y_1 + ay_{12}y_1^2)\rho_0^4 + O(\rho_0^5) \\ \frac{dy}{d\tau}(\tau) = u_2\rho_0 + (u_2x_1 - u_1y_2)(ax_{21}x_1^2 + (ax_{22} + ay_{21})x_1y_1 + ay_{22}y_1^2)\rho_0^4 + O(\rho_0^5) \\ \frac{dz}{d\tau}(\tau) = (u_2x_1 - u_1y_2)\left(\frac{1}{2}\rho_0^2 + (ax_{31}x_1^2 + (ax_{32} + ay_{31})x_1y_1 + ay_{32}y_1^2)\rho_0^4\right) \\ \frac{+1}{2}\begin{pmatrix} (u_2x_4 - u_1y_4) + (u_2x_1 - u_1y_2)\omega_{xx31}x_1^3 + \left(\omega_{xx32} + \omega_{xy31} + \omega_{yx31}\right)x_1^2y_1 \\ +\left(\omega_{xy32} + \omega_{yx32} + \omega_{yy31}\right)x_1y_1^2 + \omega_{yy32}y_1^3 \end{pmatrix}\rho_0^5 + O(\rho_0^6). \end{cases}$$

Analogously, plugging (5) into (3), we obtain the equations for the jets of the (rescaled) adjoint vectors $\underline{p}_x$ and $\underline{p}_y$. These equations can be integrated at each order in ρ_0 , in order to obtain the expression of the functions x_i, y_i, z_i ,$\underline{p}_{xi}$, $\underline{p}_{yi}$. For the analysis carried out in this paper, it is sufficient to stop at the fourth order for x and y, at the third one for $\underline{p}_x$ and $\underline{p}_y$, and at the fifth one for z (that is, we need three order after the first nonzero one).

The power series for the switching times are computed analogously. Consider, for instance, an extremal with $p_x(0) = 1$ and $|p_y(0)| < 1$ which implies that, along the first bang-arc, the associated control is $\boldsymbol{u}=(1,0)$. The switching (reparametrized) time $\boldsymbol{\tau_1}$ is determined by the condition $F(t(\boldsymbol{\tau_1}))=G(t(\boldsymbol{\tau_1}))=1$ (we recall indeed that $F(t(0))=1$ and that F is constant for $u_2=0$). Plugging the expansions for $(P(t(\tau)),\xi(t(\tau)))$ into $\overline{\Theta}$ and factorizing in powers of ρ_0, we obtain the coefficients $\bar{\Theta}_k$ in the expansion $\overline{\Theta} = 1 + \overline{\Theta_1}\rho_0 + \overline{\Theta_2}\rho_0^2 + \overline{\Theta_3}\rho_0^3$ Writing $\boldsymbol{\tau_1}$ as the power series $\boldsymbol{\tau_1}=\tau_1^0+\tau_1^1\rho_0^2+\tau_1^2\rho_0^2+...$, and imposing, at each order in ρ_0, the equality

$$1 = G(t(\tau_1)) = p_y(0) + \int_0^{\tau_1} \bar{\Theta}\left(\mu(t(\tau)), \xi(t(\tau))\right) d\tau$$

$$p_y(0) + \int_0^{\tau^0{}_1} (1 + \rho_0\bar{\Theta}_1 + \ldots)\, d\tau + \int_{\tau^0{}_1}^{\tau^0{}_1 + \rho_0\tau^1{}_1} (1 + \rho_0\bar{\Theta}_1 + \ldots)\, d\tau + \ldots,$$

we can identify all the coefficients τ_1^k, $k \geq 0$. We proceed in the same way for the other switching times. In particular, we obtain $\tau_1^0 = p_x(0) - p_y(0)$ and $(\tau_{k+1}^0 - \tau_k^0) = 2\ \forall k \geq 1$, that is, at the zeroth order in ρ_0 the switching times for the generic system coincide with those of the nilpotent case (see [4]).

Notation: In the following, we are classifying the bang-bang geodesics according to their initial adjoint vector:

- γ_f and γ_{-f} denote the set of the geodesics ξ associated with extremals satisfying $F(P(0),\xi(0)) > |G(P(0),\xi(0))|$ and $F(P(0),\xi(0))<-|G(P(0),\xi(0))|$, respectively; such trajectories start with control equal to $\boldsymbol{u} = (1, 0)$ and $\boldsymbol{u} = (-1, 0)$, respectively, and are characterized by $p_x(0)=\pm1$.

- γ_g and γ_{-g} denote a geodesics whose extremal satisfies $G(P(0),\xi(0)) > |F(P(0),\xi(0))|$ and $G(P(0),\xi(0)) <- |F(P(0),\xi(0))|$, respectively; in particular, these trajectories start with control equal to $\boldsymbol{u} = (0, \pm1)$ and are characterized by $p_y(0)=\pm1$.

Conjugate times and loss of local optimality

We are now ready to compute the (power expansion of the) Jacobian of the exponential map.
For extremals such that $p^0_x=1$ (respectively, $p^0_x=-1$), and $|p^0_y|\leq 1$ we obtain

$$JExp(p_y^0,p_z^0,\tau)=\begin{cases} 0 & if \quad 0\leqslant\tau<\tau_2(p_y^0,p_z^0) \\ 4\rho_0^3+O(\rho_0^4) & if \quad \tau_2(p_y^0,p_z^0)<\tau<\tau_3(p_y^0,p_z^0) \\ 8\rho_0^3+O(\rho_0^4) & if \quad \tau_3(p_y^0,p_z^0)<\tau<\tau_4(p_y^0,p_z^0) \\ 32C_1\rho_0^5+32(D_1p_{y^0}\pm E_1)\rho_0^6+O(\rho_0^7) & if \quad \tau_4(p_y^0,p_z^0)<\tau<\tau_5(p_y^0,p_z^0) \\ -8\rho_0^3+O(\rho_0^4) & if \quad \tau_5(p_y^0,p_z^0)<\tau<\tau_6(p_y^0,p_z^0) \end{cases} \quad (6)$$

where $C_1=ax_{31}$, $D_1 = 9ax_{21} -15\omega_{xx31}$ and $E_1 = 3ax_{11} -3ax_{22} -3ay_{21} +5\omega_{xx32} +5\omega_{xy31} +5\omega_{yx31}$. In particular, we notice that the Jacobian is be constant (with respect to time) along each bang arc, at least up to the order we computed.
Analogously, for extremals such that $p^0_y=1$ (respectively, $p^0_y=-1$), and $|p^0_x|\leq 1$ we obtain

$$JExp(p_x^0,p_z^0,\tau)=\begin{cases} 0 & if \quad 0\leqslant\tau<\tau_2(p_x^0,p_z^0) \\ 4\rho_0^3+O(\rho_0^4) & if \quad \tau_2(p_x^0,p_z^0)<\tau<\tau_3(p_y^0,p_z^0) \\ 8\rho_0^3+O(\rho_0^4) & if \quad \tau_3(p_x^0,p_z^0)<\tau<\tau_4(p_x^0,p_z^0) \\ 32C_2\rho_0^5-32(D_2p_{x^0}\mp E_2)\rho_0^6+O(\rho_0^7) & if \quad \tau_4(p_x^0,p_z^0)<\tau<\tau_5(p_x^0,p_z^0) \\ -8\rho_0^3+O(\rho_0^4) & if \quad \tau_5(p_x^0,p_z^0)<\tau<\tau_6(p_x^0,p_z^0) \end{cases} \quad (7)$$

where $C_2=ay_{32}$, $D_2 =-9ay_{12} -15\omega yy_{32}$ and $E_2 = -3ax_{12} -3ay_{11} -3ay_{22} -5\omega_{xy32} +5\omega_{yx32} -5\omega_{yy31}$.
Based on Definition 2 and on formulas (6) and (7), we can state the following results.

Proposition 1. *If* $ax_{31} < 0$, *then the geodesics with initial adjoint vector* (p^0_x,p^0_y,p^0_z) *satisfying* $p^0_x=1$, $p^0_y\in[-1,1)$ *and* p^0_z *large enough lose local optimality at the fourth switching time* ($\boldsymbol{\tau_4}$)*; if* $ax_{31} > 0$, *they lose local optimality at the fifth switching time* ($\boldsymbol{\tau_5}$)*. The same result holds for geodesics with initial adjoint vector satisfying* $p^0_x=-1$ and $p^0_y\in(-1,1]$ *(and* p^0_z *large enough).*
Analogously, the conjugate time for all extremals satisfying $p^0_y=1$, $p^0_x\in[-1,1)$ *and* p^0_z *large enough coincides with* $\boldsymbol{\tau_4}$ *if* $ay_{32} < 0$ *and with* $\boldsymbol{\tau_5}$ *if* $ay_{32} > 0$.

Proposition 2. *Consider the geodesics with initial adjoint vector* (p^0_x,p^0_y,p^0_z) *satisfying* $|p^0_x|=1,|p^0_y|\leq 1$ *and* p^0_z *large enough. Assume that* $ax_{31} = 0$ *and* $|E_1|\neq|D_1|$. *If* $E_1 <-|D_1|$ *(respectively,* $E_1>|D_1|$*), then the fourth* *(respectively, fifth) switching time is the conjugate time for all considered geodesics. If $|E_1| < |D_1|$, then the value of p^0_y determines whether the conjugate time coincides with the fourth or the fifth switching time. In particular, the conjugate time coincides with the fourth (respectively, fifth) switching time for geodesics satisfying $D_1 p^0_y + E_1 < 0$ (respectively, $D_1 p^0_y + E_1 > 0$).*

Propositions 1 and 2 show that the conjugate locus (and, as we will see later, the cut locus) is determined by the value of the invariants C_1 and C_2, and, if these are zero, also by D_1, E_1, D_2, and E_2. In the following, we will see that another invariant intervene in the classification of the cut locus: for the purpose of future computations, we set $A=4ax_{32}+4ay_{31}$.

4. Local structure of the cut locus for small spheres

Computation of the intersection of fronts

The first conjugate time of an extremal tells when the extremal ceases to be locally optimal and, as we already saw, is determined by the change in sign of the Jacobian of the exponential map; it corresponds to the first time when a geodesic meets other geodesics of the same kind (for instance, a trajectory with $p^0_x = 1$ meets other trajectories with $p^0_x = 1$, but different p^0_y). On the other hand, a geodesic may lose its optimality before the first conjugate time, if it intersects some ''far'' geodesic, that is, a geodesic with a different initial control (that is, belonging to another family $\gamma_\cdot$).
This phenomenon is shown in Figure 1; in the picture on the left, we see that the geodesics of the set γ_f are crossing those of the sets γ_g and γ_{-g}, giving rise to the intersections plotted in dotted line; as we will see below, these intersections will contribute to the cut locus (this configuration in particular occurs when $A>0$, $C_2<0$ and $C_1<0$ or, if $C_1<0$, $E_1<-|D_1|$). On the right, we see another case, in which the geodesics of the set γ_g are intersecting those of the sets γ_g and γ_{-f} (this configuration occurs for instance when $A<0$, $C_2<0$ and $C_1<0$ or, if $C_1<0$, $E_1<-|D_1|$).

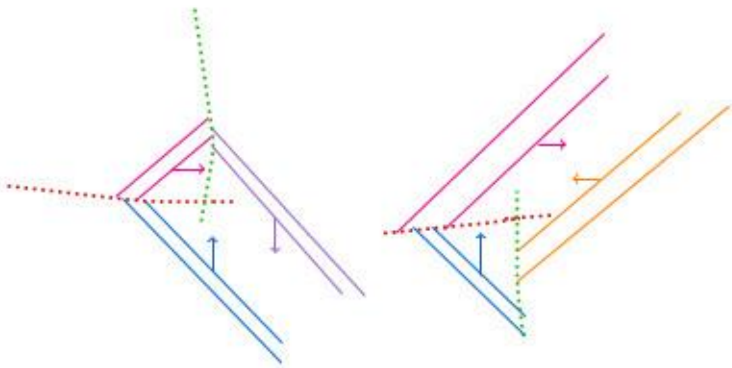

Figure 1: In magenta (respectively orange, blue, purple), the intersection with a plane of constant coordinate z of the fronts of the geodesics of the sets γ_f (respectively γ_{-f}, γ_g and γ_{-g}), *before* the fourth switching time. The intersection between the fronts are plotted in dotted line. The arrows show the direction in which the fronts are moving as time increases.

The strategy we use to detect and describe the intersections among "far" geodesics is sketched here below, for the particular case of the geodesics of the kind γ_f and those of the kind γ_g. All other cases are computed similarly.
We fix $\rho_0 > 0$ small enough, $q\in[-1,1)$ and a time $T>0$. Let ξ_f be the geodesics associated with the extremal (P_f,ξ_f) with initial adjoint vector $P_f(0)=(1,q,1/\rho_0)$, and let ξ_g be the geodesic associated with some extremal with initial covector $(\beta,1,1/\tilde{\rho}_0)$ (that is, ξ_f is of the kind γ_f and ξ_g of the king γ_g). As we are looking for intersections between the two geodesics, $\tilde{\rho}_0$ must be close to ρ_0; for this, we set $\tilde{\rho}_0 = \rho_0 + a_2\rho_0^2 + a_3\rho_0^3 + \ldots$, where the coefficient a_k will be determined later. As, for ρ_0 small enough, the geodesics are close to those of the Heisenberg system (the difference being $O(\rho_0^4)$), we are also expecting that the two geodesics intersect only if β is close to 1; for this, we set $\beta = 1 + \beta_2\rho_0^2 + \beta_3\rho_0^3 + \ldots$.
Finally, since the jets of the geodesics are computed with respect to the reparametrized time, and each geodesics possesses its own reparametrization, we must make sure that we are comparing the two geodesics *at the same (real) time T.* This correspond to imposing

$$T = \int_0^{\tau} \rho(s)ds = \int_0^{\tilde{\tau}} \tilde{\rho}(s)ds$$

where $\tau = \boldsymbol{\tau}_3 + \delta_0 + \delta_1\rho_0 + \delta_2\rho_0^2 + \cdots$ and $\tilde{\tau} = \tilde{\tau}_3 + \tilde{\delta}_0 + \tilde{\delta}_1\rho_0 + \tilde{\delta}_2\rho_0^2 + \ldots$, $\boldsymbol{\tau}_3$ and $\tilde{\tau}_3$ denoting respectively the third switching times of ξ_f and ξ_g. Remark that the analytical expression of the functions $x_i, y_i, z_i, p_{xi}, p_{yi}$ in (5) change at each switching time. Then we must to choose which arc bang of the geodesic ξ_f we want to consider (same for the geodesic ξ_g). Inspired from the Heisenberg system, in which they met along the fourth bang arc, we impose that both $\delta_0 + \delta_1\rho_0 + \delta_2\rho_0^2 + \ldots$ and $\tilde{\delta}_0 + \tilde{\delta}_1\rho_0 + \tilde{\delta}_2\rho_0^2 + \ldots$ are non-positive.

We now compute the jets of $\xi_f(t(\tau))$ and $\xi_g(t(\tilde{\tau}))$, and we impose the equality $\xi_f(t(\tau))=\xi_g(t(\tilde{\tau}))$, up to the fourth order in ρ_0 for the coordinates x, y and to the fifth one for the coordinate z. Thanks to this, we are able to recover the values for the coefficients a_k, β_k, δ_k, $k \leq 3$, as functions of the invariants and of q. In particular, we obtain

$$\beta = 1 + 2(1-q)C_1\rho_0^2 + 2(1-q)\left(E_1 + \frac{q+2}{3}D_1\right)\rho_0^4 + O(\rho_0^4). \quad (8)$$

As β must be contained in the interval $[-1, 1]$, this intersection occurs only if $C_1 \leq 0$ and, for $C_1 = 0$ (the case we are studying), if $3E_1+(q+2)D_1\leq 0$. We obtain also

$$\tau_4 - \tau = 2(q-1)C_1\rho_0^2 + 2(q-1)\left(E_1 + \frac{2q+1}{3}D_1\right)\rho_0^4. \quad (9)$$

Then, this intersection occurs before the conjugate time only if $C_1 \leq 0$ and, if $C_1 = 0$, if $3E_1+(2q+1)D_1\leq 0$. These two conditions are indeed *existence conditions* for the intersection.

In order to picture the shape of this intersection, we compute its suspension, that is, the intersection of its graph with the plane $\{z = 4\zeta^2\}$, where ζ is a positive constant close to ρ_0 (that is $\zeta=\rho_0+O(\rho_0^{\,2})$); to do so, it suffices to set $\rho_0 = \zeta + r_2\zeta^2 + r_3\zeta^3 + \ldots$, and to find those r_k that guarantee $z_f(t(\tau))=4\zeta^2 + O(\zeta^6)$. Then, we plug this expansion for ρ_0 into the expressions of $x_f(t(\tau))$ and $y_f(t(\tau))$. If $C_1 = 0$, we obtain the curve

$$\begin{cases} x = -(1+q)\zeta + \left((1+12q-5q^2)\frac{A}{4} + (q-3)C_2\right)\zeta^3 + O(\zeta^4) \\ y = 4A\zeta^3 - \frac{2}{3}(D_1q^2 + (3E_1 + D_1)(1+q) - c_1)\zeta^4 + O(\zeta^5). \end{cases} \quad (10)$$

where c_1 is a constant depending on the invariants of the metric. We remark that, up to fourth order terms in ζ, (10) is a horizontal segment of length $\sim 2\zeta$.

All other kinds of intersections may be computed in an analogous way; for instance, consider a geodesic, that we call ξ_{-f}, associated with the extremal (P_{-f},ξ_{-f}) with initial adjoint vector $P_{-f}(0)=(-1,q,1/\rho_0)$, and a geodesic ξ_{-g} associated with some extremal with initial covector $(\beta,-1, 1/\rho_0)$ (in particular, ξ_f is a geodesic of the kind γ_f and ξ_{-g} is of the kind γ_{-g}). Proceeding as above, we see that this intersection may occur only if $C_1\leq 0$ and, if $C_1=0$ if $3E_1+(2-q)D_1\geq 0$ and $3E_1+(1-2q)D_1\geq 0$. If these existence conditions are met, the suspension at the plane $\{z = 4\zeta^2\}$ is given by

$$\begin{cases} x = (1-q)\zeta - \left((1-12q-5q^2)\frac{A}{4} - (q+3)C_2\right)\zeta^3 + O(\zeta^4) \\ y = -4A\zeta^3 - \frac{2}{3}(D_1q^2 + (3E_1 + D_1)(1-q) - c_1)\zeta^4 + O(\zeta^5). \end{cases} \quad (11)$$

Remark 2 : in the nilpotent case, because of the discrete symmetry around the *z*-axis, the suspension of cut locus has a “cross” shape and is symmetric for rotations of kπ/2 around the *z*-axis. In the generic case $C_1C_2\neq 0$, studied in [3], the symmetry among the geodesics of the sets γ_f and γ_{-f} (as well as the one between the sets γ_g and γ_{-g}) remains, and the cut locus is invariant for rotations of kπ around the z-axis. As we will see below, in the cases presented in this paper, this symmetry does not survive.

Case study 1

In the first case we are presenting, we assume that the main invariants of the metric satisfy the following inequalities:

$$A>0 \quad C_1=0 \quad C_2<0 \quad E_1<-|D_1|<0.$$

As stated in Proposition 1 the conjugate times are completely determined by the values of the invariants C_1, C_2, E_1 and D_1. More precisely:

- the geodesics of the kind γ_f, γ_g and γ_{-g} lose optimality before their fourth switching time;
- the geodesics of the kind γ_{-f} may be optimal up to their fifth switching time, if they do not cross other geodesics before.

To detect the lost of local optimality of the geodesics of the sets γ_f, γ_g and γ_{-g}, whose conjugate time is the fourth switching time, as in the Heisenberg case, we at what happens in the Heisenberg case itself. Therefore, we search for their intersection with the geodesics of the type γ_g, γ_{-f} and γ_f, respectively.

For what concerns the geodesics of the kind γ_f, we notice that, as $E_1 < -|D_1|$, then both conditions (8) and (9) hold true for every $|q|\leq 1$, so that the intersection is always possible.

We then study the intersections among the geodesics of the kind γ_g, (respectively, γ_{-g}) with those of the kind γ_{-f} (γ_f), following the same procedure explained above; we obtain that these intersection always occur when $C_2<0$ (as in the case we are studying) and that their suspension at the plane $\{z = 4\zeta^2\}$ are respectively described by the curves

$$\begin{cases} x = (4A + 2(1-\beta)C_2)\zeta^3 + O(\zeta^4) \\ y = (\beta - 1)\zeta + \left((1 - 12\beta - 5\beta^2)\frac{A}{4} - \frac{1}{24}(5 + 9\beta - 9\beta^2 - 5\beta^3)C_2\right)\zeta^3 + O(\zeta^4), \end{cases}$$

$$\begin{cases} x = -(4A + 2(1+\eta)C_2)\zeta^3 + O(\zeta^4) \\ y = (\eta + 1)\zeta - \left((1 + 12\eta - 5\eta^2)\frac{A}{4} - \frac{1}{24}(5 - 9\eta - 9\eta^2 + 5\eta^3)C_2\right)\zeta^3 + O(\zeta^4), \end{cases} \tag{12}$$

where $\beta,\eta \in [-1, 1)$ respectively denote the second component of the initial covector of the geodesic γ_g and γ_{-g}.

Taking the limit of (10) and (12) for $q, \eta \to -1$, we see that the two curves must intersect. In particular (as shown in Figure 1), they "bound" the wavefront made by the geodesics of the set γ_f, so that we conclude that these geodesics lose their optimality by intersecting these curves.

On the contrary, it is easy to see that the curves (10) and (11) do not intersect. Moreover, the existence condition $3E_1+(1-2q)D_1\geq 0$ for the intersection between the geodesics of the set γ_{-f} and those of the set γ_{-g} is never satisfied. Then, we cannot repeat the same reasonment as above to "bound" the wavefront made by the geodesics of the kinds γ_g and γ_{-g} ; on the other hand, looking for instance at Figure 1, we see that the two fronts will likely meet. Studying this intersection as usual, we can find that indeed it is allowed when $C_2\leq 0$, and that its suspension is a segment with slope equal to -1. This sets the cut locus for the geodesics of the kinds γ_g and γ_{-g}.

It is left to prove how the geodesics of the set γ_{-f} lose optimality; some of them, with initial covector $(-1,p_y^0,p_z^0)$ with $p_y^0 \sim 1$, lose optimality before the fourth switching time, intersecting the geodesics of the set γ_g, as seen above. On the other hand, by studying all cases we can prove that all other intersection of the geodesics of the kind γ_{-f} *before* their fourth switching time do not occur: therefore, we must consider these geodesics *after* their fourth switching time. By computations, we can prove that they lose optimality when intersecting the front of the geodesics of the set γ_g; the suspension of the intersection is a small arc of length $O(\zeta^4)$, described by

$$\begin{cases} x = 4A\zeta^3 + \left(\frac{-2}{3}D_1q^3 + (E_1 + D_1)q^2 - 2E_1q - \frac{1}{3}(D_1 + 5E_1 + 4d_1)\right)\zeta^4 + O(\zeta^5) \\ y = -4A\zeta^3 - \left(2D_1q^2 - 4E_1q - \frac{2}{3}c_1\right)\zeta^4 + O(\zeta^5). \end{cases} \tag{13}$$

Summing up, the suspension at the plane $\{z = 4\zeta^2\}$ of the cut locus is the three-branches C^1-smooth curve plotted in Figure 2.

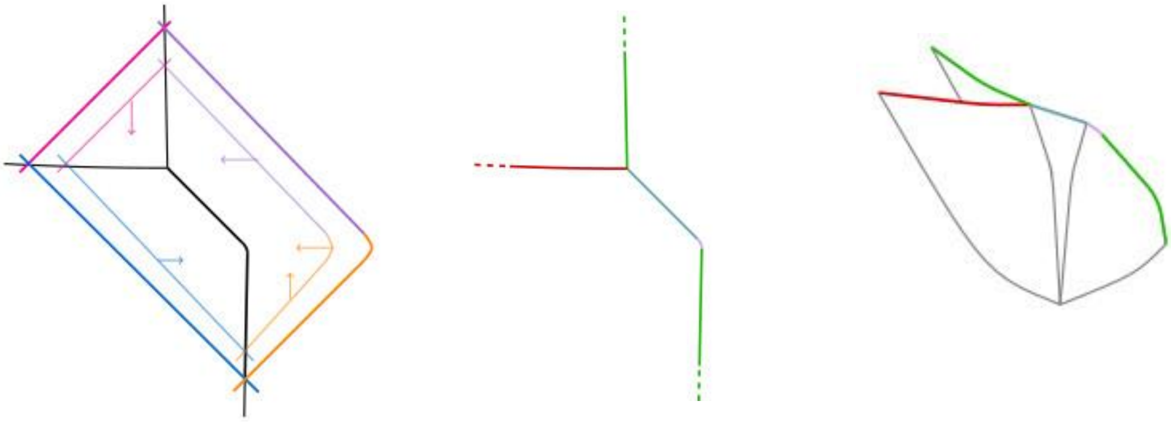

Figure 2. The cut locus in the case 1. On the left, in magenta (respectively blue, orange, purple) the wavefronts of the geodesics of the set γ_f (respectively, γ_g, γ_{-f} and γ_{-g}). The suspension of the cut locus is shown in the central figure: in red the intersections between the geodesics of the set γ_f and those of the set γ_g, in green the intersections between γ_{-g} and γ_f (respectively, γ_g and γ_{-f}), in purple the intersection of the geodesics of the set γ_{-f} *after* the fourth switching time with those of the set γ_g. On the right, the cut locus.

Case study 2

We now analyze the case in which the main invariants of the metric satisfy the following inequalities:

$$A>0 \qquad C_1 = 0 \qquad C_2 < 0 \qquad 0 < E_1 < D_1.$$

As stated in Proposition 1, the conjugate times are completely determined by the values of the invariants C_1, C_2, E_1 and D_1. In particular:

- the geodesics of the sets γ_g and γ_{-g} lose optimality before their fourth switching time;
- the geodesics of the set the γ_f with initial adjoint vector $(1,p_y^0,p_z^0)$ satisfying $p^0_y \leq -E_1/D_1$ lose optimality before the fourth switching time, the other ones lose local optimality at the fifth switching time (see Proposition 2);
- analogously, the geodesics of the kind γ_{-f} with initial adjoint vector $(1,p_y^0,p_z^0)$ satisfying $p^0_y \leq E_1/D_1$ lose optimality before the fourth switching time, the other ones lose local optimality at the fifth switching time.

We moreover spot these main differences with respect to the case we studied above:

- the condition (8) is never satisfied, so that the intersection between the geodesics of the set γ_f (before their fourth switching time) with those of the set γ_g does not occur;

- on the other hand, the condition $3E_1+(1-2p_y^0)D_1\geq 0$ for the intersections of the geodesics of the set γ_{-f} (before their fourth switching time) with those of the kind γ_{-g} is verified only if $p_y^0\leq(1+3E_1/D_1)/2$. In particular, if $3E_1/D_1<1$, then $(1+3E_1/D_1)/2<1$, so that the geodesics of the kind γ_{-f} with $p_y>(1+3E_1/D_1)/2$ do not intersect those of the set γ_{-g}.

To understand how the geodesics lose their optimality, we plot the suspension of the wavefront (that is, we fix some time T and some constant $\zeta>0$, and we compute $\mathrm{Exp}(P^0,T)\cap\{z=4\zeta^2\}$). As can be seen in Figure(fronte 2), the wavefront of geodesics of the family γ_f is *self-intersecting:* in particular, the cut locus for the geodesics of the set γ_f with $p_y^0\leq E_1/D_1$ corresponds with their intersection with (a part of) the geodesics of the same family that have already passed their fifth switching time. The geodesics with $p_y^0\geq E_1/D_1$ lose optimality after the fourth switching time, either for the self-intersection of the wavefront just described, or because they meet the geodesics of the kind γ_g.

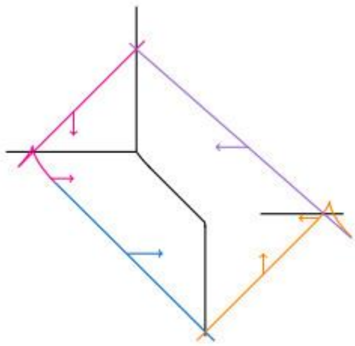

Figure 3 in magenta (respectively blue, orange, purple) the wavefronts of the geodesics of the set γ_f (respectively, γ_g, γ_{-f} and γ_{-g}).

As for the geodesics of the kind γ_{-f}, we can distinguish two cases: if $3E_1/D_1\geq 1$, then their cut point is determined by the intersection with the geodesics of the kind γ_{-g}, before the fourth switching time (see Figure 3, on the left). If instead $3E_1/D_1<1$, then only the geodesics with $p_y\leq(1+3E_1/D_1)/2$ intersects those of the set γ_{-g}; for the other ones, the cut point is determined by the intersection with the geodesics of the kind γ_g, after the fourth switching time (see Figure 4, on the right). This gives rise to a disconnected cut locus.

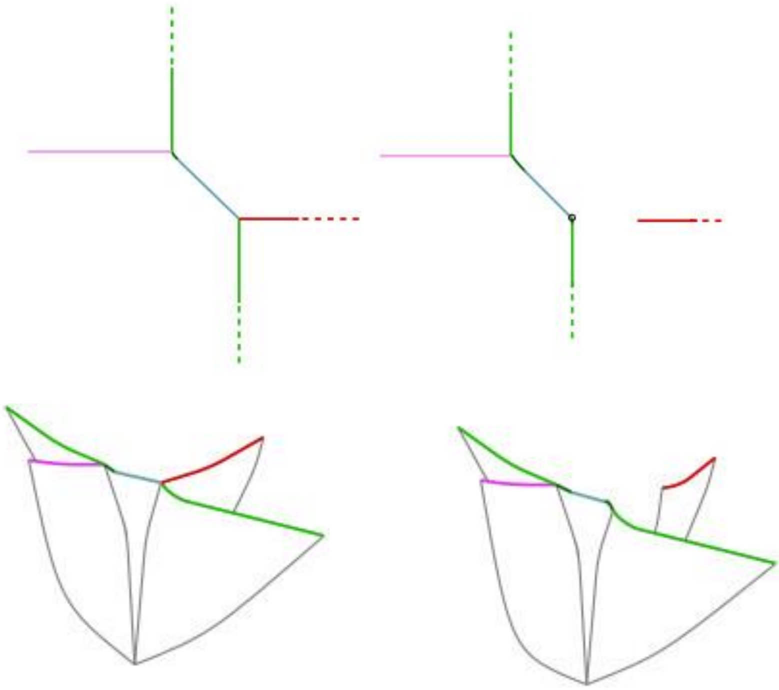

Figure 4 : The cut locus (below) and its suspension (above) in the case 2. On the left, the case $3E_1/D_1 \geq 1$, on the right, $3E_1/D_1 < 1$.

6. Conclusion

In this paper, we study the upper part of the cut locus for the sub-Finsler L^1 problem, in some generic cases. In particular, we present two cases quite different from those founded in [3], where the most generic cases are studied.

Further researches will include the classification of all cases, and the study of the other parts of the cut locus.

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Tangent Surfaces of Adjoint Curves

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Abstract

In this study, we define new surfaces of adjoint curves as tangent surface under Frenet-Serret frame in 3D Euclidean space. Also, we construct new characterizations for tangent surfaces of adjoint curves.

Keywords: Tangent surface, Adjoint curve, Frenet-Serret frame.

1. Backround on Serret-Frenet Frame and Adjoint Curve

By way of design and style, this model is kind of a moving frame with regards to a particle. In the quick stages of regular differential geometry, the Frenet-Serret(F-S) frame was applied to create a curve in location. After that, F-S frame is established by way of subsequent equations for a presented framework [1],

$$\begin{bmatrix} \nabla_{\mathbf{T}}\mathbf{T} \\ \nabla_{\mathbf{T}}\mathbf{N} \\ \nabla_{\mathbf{T}}\mathbf{B} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}, \tag{1.1}$$

where $\kappa = \|\mathbf{T}\|$ and τ are the curvature and torsion of γ, respectively. Let s be arc-length parameter. Then, these formulas are written as

$$\mathbf{T} = \lambda'(s), \quad \mathbf{N} = \frac{\lambda''(s)}{\|\lambda''(s)\|}, \quad \mathbf{B} = \mathbf{T}\times\mathbf{N}.$$

Definition 11: Let γ be a regular curve s arc-length parametrized, $\{\mathbf{T}_\gamma, \mathbf{N}_\gamma, \mathbf{B}_\gamma\}$ be F-S frame of γ. Then, the adjoint curve of γ according to F-S frame is given as [2]

$$\beta(s) = \int_{s_0}^{s} \mathbf{B}(s)ds.$$

Theorem 2: 2 Let γ be a regular curve arc-length parametrized, $\{\mathbf{T}_\gamma, \mathbf{N}_\gamma, \mathbf{B}_\gamma\}$ be F-S frame of γ, β be adjoint curve of γ according to F-S frame and κ_γ and τ_γ be curvature and torsion of γ. Denote by $\{\mathbf{T}_\beta, \mathbf{N}_\beta, \mathbf{B}_\beta\}$ F-S frame of β and by κ_β and τ_β be curvature and torsion of β. Then, the following equations hold [2]:

$$\mathbf{T}_\beta = \mathbf{B}_\gamma, \quad \mathbf{N}_\beta = -\mathbf{N}_\gamma, \quad \mathbf{B}_\beta = \mathbf{T}_\gamma.$$

$$\kappa_\beta = \tau_\gamma \quad \tau_\beta = \kappa_\gamma.$$

2. Tangent Surfaces of Adjoint Curves with F-S Frame in $\mathbb{E}^3$

In this topic, after making some reminders about the characterization of surfaces, we get results and characterizations about tangent surfaces of adjoint curves.

Let n be the standard unit normal vector field on a surface ϕ defined by

$$n = \frac{\phi_s \wedge \phi_t}{\|\phi_s \wedge \phi_t\|},$$

where s is arc-lenght, t is time parameter and $\phi_s = \partial\phi/\partial s, \phi_t = \partial\phi/\partial t$. Then, the first fundamental form $\mathbf{I}$ and the second fundamental form $\mathbf{II}$ of a surface ϕ are defined by

$$\begin{aligned} \mathbf{I} &= Eds^2 + 2Fdsdt + Gdt^2, \\ \mathbf{II} &= eds^2 + 2fdsdt + gdt^2, \end{aligned}$$

where

$$\begin{aligned} E &= \langle \phi_s, \phi_s \rangle, F = \langle \phi_s, \phi_t \rangle, G = \langle \phi_t, \phi_t \rangle, \\ e &= \langle \phi_{ss}, n \rangle, \ f = \langle \phi_{st}, n \rangle, \ g = \langle \phi_{tt}, n \rangle. \end{aligned}$$

On the other hand, the Gaussian curvature K and the mean curvature H are

$$K = \frac{eg - f^2}{EG - F^2}, H = \frac{Eg - 2Ff + Ge}{2(EG - F^2)}$$

and the principal curvatures k_1 and k_2 are respectively, written as [3-7]

$$k_1 = H + \sqrt{H^2 - K},\ k_2 = H - \sqrt{H^2 - K}.$$

Theorem 3: 3 The surface is minimal if and only if it has vanishing mean curvature [1,4].

Definition 4: 4 The tangent surface of a regular space curve γ is given as [5]

$$\phi^{\mathbf{T}}(s,t) = \gamma + t\mathbf{T}.$$

Theorem 5: 5 Let γ be a regular curve s arc-length parametrized, β be adjoint curve of γ. Denote by $\mathbf{I}_\beta$, $\mathbf{I}_\gamma$ and $\mathbf{II}_\beta$, $\mathbf{II}_\gamma$ be the first and the second fundamental forms of the tangent surfaces of β and γ. Then the following states hold:

$$\begin{aligned}
\mathbf{I}_\beta &= (1+t^2\tau_\gamma^2)ds^2 + 2dsdt + dt^2,\\
\mathbf{I}_\gamma &= (1+t^2\kappa_\gamma^2)ds^2 + 2dsdt + dt^2,\\
\mathbf{II}_\beta &= \mathbf{II}_\gamma = -t\kappa_\gamma\tau_\gamma ds^2.
\end{aligned}$$

Proof: From the definition of tangent surface, the tangent surface of β is written as

$$\phi^{\mathbf{T}_\beta}(s,t) = \gamma + t\mathbf{T}_\beta.$$

Therefore, the following equalities are obtained:

$$\begin{aligned}
\phi_s^{\mathbf{T}_\beta} &= \mathbf{T}_\beta + t\kappa_\beta\mathbf{N}_\beta,\\
\phi_{ss}^{\mathbf{T}_\beta} &= t\kappa_\beta^2\mathbf{T}_\beta + (\kappa_\beta + t\kappa_\beta^{'})\mathbf{N}_\beta + t\kappa_\beta\tau_\beta\mathbf{B}_\beta,\\
\phi_t^{\mathbf{T}_\beta} &= \mathbf{T}_\beta,\quad \phi_{tt}^{\mathbf{T}_\beta} = 0,\quad \phi_{st}^{\mathbf{T}_\beta} = \kappa_\beta\mathbf{N}_\beta,
\end{aligned}$$

and, from the equalities, the unit normal vector field of $\phi^{\mathbf{T}_\beta}$ surface is found as

$$n_\beta = \frac{\phi_s^{\mathbf{T}_\beta} \times \phi_t^{\mathbf{T}_\beta}}{\left\|\phi_s^{\mathbf{T}_\beta} \times \phi_t^{\mathbf{T}_\beta}\right\|} = \frac{(\mathbf{T}_\beta + t\kappa_\beta\mathbf{N}_\beta)\times\mathbf{T}_\beta}{t\kappa_\beta} = -\mathbf{B}_\beta.$$

These equalities are obtained similarly for the tangent surface of γ curve. Then, with the help of Theorem 2 and of the equations we gave at the beginning of this section, we obtain

$$
\begin{aligned}
E_\beta &= 1+t^2\tau_\lambda^2, E_\lambda = 1+t^2\kappa_\gamma^2 \\
F_\beta &= F_\gamma = G_\beta = G_\lambda = 1, \\
e_\beta &= e_\gamma = -t\kappa_\gamma\tau_\gamma, \\
f_\beta &= f_\gamma = g_\beta = g_\lambda = 0.
\end{aligned}
$$

Hence, the first and the second fundamental forms of the tangent surfaces of β and γ are obtained as

$$
\begin{aligned}
\mathbf{I}_\beta &= (1+t^2\tau_\gamma^2)ds^2 + 2dsdt + dt^2, \\
\mathbf{I}_\gamma &= (1+t^2\kappa_\gamma^2)ds^2 + 2dsdt + dt^2, \\
\mathbf{II}_\beta &= \mathbf{II}_\gamma = -t\kappa_\gamma\tau_\gamma ds^2.
\end{aligned}
$$

Corollary 6: 6 Let γ be a regular curve s arc-length parametrized, β be adjoint curve of γ. Denote by K_β, K_γ and H_β, H_γ be the Gaussian curvatures and the mean curvatures of the tangent surfaces of β and γ, respectively. Then the following states hold:

$$
\begin{aligned}
K_\beta &= K_\gamma = 0, \\
H_\beta &= \frac{1}{4t^2 H_\gamma}.
\end{aligned}
$$

Proof: With the proof of Theorem 5, we get

$$
\begin{aligned}
K_\beta &= \frac{e_\beta g_\beta - f_\beta^2}{E_\beta G_\beta - F_\beta^2} = 0, \\
H_\beta &= \frac{E_\beta g_\beta - 2F_\beta f_\beta + G_\beta e_\beta}{2\left(E_\beta G_\beta - F_\beta^2\right)} = -\frac{\tau_\beta}{2t\kappa_\beta} = -\frac{\kappa_\gamma}{2t\tau_\gamma},
\end{aligned}
$$

and, similarly,

$$
\begin{aligned}
K_\gamma &= \frac{e_\gamma g_\gamma - f_\gamma^2}{E_\gamma G_\gamma - F_\gamma^2} = 0, \\
H_\gamma &= \frac{E_\gamma g_\gamma - 2F_\gamma f_\gamma + G_\gamma e_\gamma}{2\left(E_\gamma G_\gamma - F_\gamma^2\right)} = -\frac{\tau_\gamma}{2t\kappa_\gamma}.
\end{aligned}
$$

Hence, we obtain

$$K_\beta = K_\gamma = 0,$$
$$H_\beta = \frac{1}{4t^2 H_\gamma}.$$

Corollary 7: 7 Let γ be a regular curve s arc-length parametrized, β be adjoint curve of γ. Then, the tangent surfaces of β and γ can never be minimal.

Proof: From the Theorem 3 and the proof of Corollary 6, since $\tau_\beta = \kappa_\gamma \neq 0$ and $\tau_\gamma = \kappa_\beta \neq 0$, the proof is clear.

Corollary 8: 8 Let γ be a regular curve s arc-length parametrized, β be adjoint curve of γ. Denote by k_{β_1}, k_{γ_1} and k_{β_2}, k_{γ_2} be the first and the second principal curvatures of the tangent surfaces of β and γ, respectively. Then the following states hold:

$$k_{\beta_1} = k_{\gamma_1} = 0,$$
$$k_{\beta_2} = \frac{1}{t^2 k_{\gamma_2}}.$$

Proof: With the proof of Theorem 5, we get

$$\begin{aligned} k_{\beta_1} &= H_\beta + \sqrt{H_\beta^2 - K_\beta} = 0, \\ k_{\beta_2} &= H_\beta - \sqrt{H_\beta^2 - K_\beta} = -\frac{\tau_\beta}{t\kappa_\beta} = -\frac{\kappa_\gamma}{t\tau_\gamma}, \end{aligned}$$

and, similarly,

$$\begin{aligned} k_{\gamma_1} &= H_\gamma + \sqrt{H_\gamma^2 - K_\gamma} = 0, \\ k_{\gamma_2} &= H_\gamma - \sqrt{H_\gamma^2 - K_\gamma} = -\frac{\tau_\gamma}{t\kappa_\gamma}. \end{aligned}$$

Hence, we obtain

$$k_{\beta_1} = k_{\gamma_1} = 0,$$
$$k_{\beta_2} = \frac{1}{t^2 k_{\gamma_2}}.$$

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Several New Bounds of Hermite-Hadamard Type Integral Inequalities Pertaining to s-Convex Functions And Their Applications

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Abstract

In this paper, authors found a new result regarding Hermite-Hadamard type integral inequalities using generalized fractional integral operators. Furthermore, a new interesting integral identity about Hermite-Hadamard type integral is derived. By using this identity as an auxiliary result, some new bounds with respect to Hermite-Hadamard type integral inequalities pertaining to s-convex functions are established. It is pointed out that several special cases are deduced from the main results for suitable choices of function inside the generalized fractional integral operators. In order to show the efficient of our main results, some applications to special means for different positive real numbers and error bound estimates for trapezoidal quadrature formula are obtain as well.

Keywords: Hermite-Hadamard type integral inequality, s-convex functions, generalized fractional integral operators, special means, error estimates.

1. Introduction and Preliminaries

Definition 1. A function $f: I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be convex on I, if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \tag{1}$$

holds for all $x, y \in I$ and $t \in [0,1]$.

In their paper [1], Hudzik and Maligranda considered, among others, the class of functions which are s-convex in the second sense. This class is defined in the following way:

Definition 2. A function $f: [0, +\infty[\longrightarrow \mathbb{R}$ is said to be s-convex in the second sense, if

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y) \tag{2}$$

holds for all $x, y \in [0, +\infty[$, $t \in [0,1]$ and for some fixed $s \in]0,1]$.

Authors of recent decades have studied convex and s-convex in the second sense function, see [2] - [8].

The following inequality, named Hermite – Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

Theorem 1. Let $f: I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function on I and $a_1, a_2 \in I$, with $a_1 < a_2$. Then the following inequality holds:

$$f\left(\frac{a_1 + a_2}{2}\right) \leq \frac{1}{a_2 - a_1}\int_{a_1}^{a_2} f(x)dx \leq \frac{f(a_1 + a_2)}{2}. \tag{3}$$

The trapezium type inequality has remained an area of great interest due to its wide applications in the field of mathematical analysis. For other recent results which generalize, improve and extend the inequality (1) through various classes of convex functions interested readers are referred to [2]-[10].

Let us recall some special functions and evoke some basic definitions as follows:

Definition 3. For $\mathrm{a}_1, \mathrm{a}_2 > 0$ the incomplete beta function is defined by

$$\beta_x(\mathrm{a}_1, \mathrm{a}_2) = \int_0^x \mathrm{t}^{\mathrm{a}_1 - 1}(1 - \mathrm{t})^{\mathrm{a}_2 - 1}\mathrm{dt}, \qquad 0 < \mathrm{x} \leq 1. \tag{4}$$

Definition 4. For $\alpha > 0$, the integral representation of gamma function is given as

$$\Gamma(\alpha) = \int_0^{+\infty} \mathrm{t}^{\alpha - 1}\exp(-\mathrm{t})\mathrm{dt}. \tag{5}$$

One can note that

$$\Gamma(\alpha + 1) = \alpha\Gamma(\alpha) \tag{6}$$

Definition 5. [11] Let $\mathrm{f} \in \mathrm{L}[\mathrm{a}_1, \mathrm{a}_2]$. Then Riemann-Liouville fractional integrals of order $\alpha > 0$ with $\mathrm{a}_1 \geq 0$ are defined as

$$\mathcal{J}_{\mathrm{a}_1^+}^{\alpha}\mathrm{f}(\mathrm{x}) = \frac{1}{\Gamma(\alpha)}\int_{\mathrm{a}_1}^{\mathrm{x}} (\mathrm{x} - \mathrm{t})^{\alpha - 1}\mathrm{f}(\mathrm{t})\mathrm{dt}, \qquad (\mathrm{x} > \mathrm{a}_1) \tag{7}$$

and

$$\mathcal{J}_{\mathrm{a}_2^-}^{\alpha}\mathrm{f}(\mathrm{x}) = \frac{1}{\Gamma(\alpha)}\int_{\mathrm{x}}^{\mathrm{a}_2} (\mathrm{t} - \mathrm{x})^{\alpha - 1}\mathrm{f}(\mathrm{t})\mathrm{dt}, \qquad (\mathrm{x} < \mathrm{a}_2). \tag{8}$$

Definition 6. [12] Let $f \in L[a_1, a_2]$ (the set of all integrable functions on $[a_1, a_2]$). Left- and right-hand sided fractional integral operators of order $\alpha \in (0,1)$ with exponential kernel are given as

$$\mathfrak{I}_{a_1^+}^{\alpha} f(x) = \frac{1}{\alpha}\int_{a_1}^{x} \exp\left[-\frac{1-\alpha}{\alpha}(x-t)\right] f(t)dt, \quad (x > a_1) \tag{9}$$

and

$$\mathfrak{I}_{a_2^-}^{\alpha} f(x) = \frac{1}{\alpha}\int_{x}^{a_2} \exp\left[-\frac{1-\alpha}{\alpha}(t-x)\right] f(t)dt, \quad (x < a_2). \tag{10}$$

Remark 1. The function $\varphi\colon [0,+\infty[\longrightarrow [0,+\infty[$, which is constructed from the work of Sarikaya *et al.* [13], has the following four conditions:

$$\int_0^1 \frac{\varphi(\tau)}{\tau} d\tau < +\infty,$$

$$\frac{1}{A_1} \le \frac{\varphi(\tau_1)}{\varphi(\tau_2)} \le A_1 \text{ for } \frac{1}{2} \le \frac{\tau_1}{\tau_2} \le 2, \tag{11}$$

$$\frac{\varphi(\tau_2)}{\tau_2^2} \le A_2 \frac{\varphi(\tau_1)}{\tau_1^2} \text{ for } \tau_1 \le \tau_2$$

and

$$\left|\frac{\varphi(\tau_2)}{\tau_2^2} - \frac{\varphi(\tau_1)}{\tau_1^2}\right| \le A_3 |\tau_2 - \tau_1| \frac{\varphi(\tau_2)}{\tau_2^2} \text{ for } \frac{1}{2} \le \frac{\tau_1}{\tau_2} \le 2,$$

where A_1, A_2 and $A_3 > 0$ are independent of $\tau_1, \tau_2 > 0$.

Moreover, Sarikaya *et al.* [13] used the above function ς in order to define the following fractional integral operators.

Definition 7. The generalized left-side and right-side fractional integrals are given as follows:

$$_{a_1^+}I_{\varphi} f(x) = \int_{a_1}^{x} \frac{\varphi(x-t)}{x-t} f(t)dt \qquad (x > a_1) \tag{12}$$

and

$$_{a_2^-}I_{\varphi} f(x) = \int_{x}^{a_2} \frac{\varphi(t-x)}{t-x} f(t)dt \qquad (x < a_2), \tag{13}$$

respectively.

Furthermore, Sarikaya *et al.* [13] noticed that the generalized fractional integrals given by Definition 7 may contain some types of fractional integrals such as the Riemann-Liouville and other fractional integrals for some special choices of function φ.

Motivated by the above results and literature, the aim of this paper is to establish in the next section, a new interesting result regarding Hermite-Hadamard type integral inequalities using generalized fractional integral operators. Furthermore, a new integral identity about Hermite-Hadamard type integral will be derived. By using this identity as an auxiliary result, some new bounds with respect to Hermite-Hadamard type integral inequalities pertaining to s-convex functions will be established. It is pointed out that several special cases will be deduced from the main results for suitable choices of function inside the generalized fractional integral operators. In order to show the efficient of our main results, some applications to special means for different positive real numbers and error bound estimates for trapezoidal quadrature formula will be obtain as well.

2. Main Results

Before we give our main results, let us denote by

$$\mathcal{A}(t) := \int_0^t \frac{\varphi\big((a_2 - a_1)s\big)}{s} f(s)ds.$$

Theorem 2. Let $f\colon [a_1, a_2] \to \mathbb{R}$ be a s-convex function on $[a_1, a_2]$, then the following inequalities for generalized fractional integral hold:

$$f\left(\frac{a_1 + a_2}{2}\right) \le \frac{1}{2^s \mathcal{A}(1)} \left[{}_{a_2^-}I_\varphi f(a_1) + {}_{a_1^+}I_\varphi f(a_2)\right] \le \frac{\Phi^*(s)}{2^s \mathcal{A}(1)} [f(a_1) + f(a_2)], \qquad (14)$$

where

$$\Phi^*(s) := \int_0^1 \frac{\varphi((a_2 - a_1)t)}{t} [t^s + (1-t)^s]\, dt.$$

Proof. For $t \in [0,1]$, let $x = a_1 + t(a_2 - a_1)$ and $y = a_1 + (1-t)(a_2 - a_1)$. From s-convexity we have

$$f\left(a_1 + \frac{a_2 - a_1}{2}\right) = f\left(\frac{a_1 + a_2}{2}\right) = f\left(\frac{x+y}{2}\right) \le \frac{f(x) + f(y)}{2^s},$$

i.e.

$$2^s f\left(\frac{a_1 + a_2}{2}\right) \le f\big((1-t)a_1 + ta_2\big) + f(ta_1 + (1-t)a_2).$$

Multiplying both sides by $\frac{\varphi((a_2-a_1)t)}{t}$ and integrating the resulting inequality with respect to t over $(0,1]$, we obtain

$$2^s f\left(\frac{a_1+a_2}{2}\right)\int_0^1 \frac{\varphi((a_2-a_1)t)}{t}dt$$
$$\leq \int_0^1 \frac{\varphi\big((a_2-a_1)t\big)}{t} f\big((1-t)a_1+ta_2\big)dt + \int_0^1 \frac{\varphi\big((a_2-a_1)t\big)}{t} f(ta_1+(1-t)a_2)dt.$$

Hence,

$$2^s f\left(\frac{a_1+a_2}{2}\right)\mathcal{A}(1) \leq \left[{}_{a_2^-}I_\varphi f(a_1) + {}_{a_1^+}I_\varphi f(a_2)\right]. \tag{15}$$

So, the first inequality is proved.

To prove the other half of the inequality, since f is s-convex, we have

$$f\big((1-t)a_1+ta_2\big) + f(ta_1+(1-t)a_2) \leq (1-t)^s f(a_1) + t^s f(a_2) + t^s f(a_1) + (1-t)^s f(a_2)$$
$$= [f(a_1)+f(a_2)][(1-t)^s + t^s]$$

so, $f\big((1-t)a_1+ta_2\big) + f(ta_1+(1-t)a_2) \leq [f(a_1)+f(a_2)][(1-t)^s + t^s]$.

Multiplying both sides by $\frac{\varphi((a_2-a_1)t)}{t}$ and integrating by t over $(0,\ 1]$, we get

$$\int_0^1 \frac{\varphi\big((a_2-a_1)t\big)}{t} f\big((1-t)a_1+ta_2\big)dt + \int_0^1 \frac{\varphi\big((a_2-a_1)t\big)}{t} f(ta_1+(1-t)a_2)dt$$
$$\leq [f(a_1)+f(a_2)]\int_0^1 \frac{\varphi\big((a_2-a_1)t\big)}{t}[(1-t)^s+t^s]dt.$$

So,

$${}_{a_2^-}I_\varphi f(a_1) + {}_{a_1^+}I_\varphi f(a_2) \leq [f(a_1)+f(a_2)]\Phi^*(s). \tag{16}$$

Using (15) and (16), then the inequality (14) holds.■

Corollary 1. In Theorem 2, if we take $\varphi(t)=t$, then we have the following inequality

$$f\left(\frac{a_1+a_2}{2}\right) \leq \frac{1}{2^{s-1}(a_2-a_1)}\int_{a_1}^{a_2} f(t)dt \leq \frac{1}{2^{s-1}(s+1)}[f(a_1)+f(a_2)].$$

Corollary 2. In Theorem 2, if we choose $\varphi(t)=\frac{t^\alpha}{\Gamma(\alpha)}$, then the inequalities (14) become the inequalities

$$f\left(\frac{a_1+a_2}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2^s(a_2-a_1)^\alpha}\left[\mathcal{J}^\alpha_{a_2^-}f(a_1)+\mathcal{J}^\alpha_{a_1^+}f(a_2)\right] \leq \left[\frac{1}{\alpha+s}+\beta(\alpha,s+1)\right][f(a_1)+f(a_2)],$$

where $\mathcal{J}^\alpha_{a_2^-}f$ and $\mathcal{J}^\alpha_{a_1^+}f$ are the fractional Riemann integrals.

Corollary 3. If we take $\varphi(t)=\frac{t}{\alpha}\exp\left[\left(-\frac{1-\alpha}{\alpha}\right)t\right]$ in Theorem 2, then the following inequality is obtained

$$\begin{aligned} f\left(\frac{a_1+a_2}{2}\right) &\leq \frac{1-\alpha}{2^s\left[\exp\left(\left(-\frac{1-\alpha}{\alpha}\right)(a_2-a_1)\right)-1\right]}\left[\mathfrak{I}^\alpha_{a_2^-}f(a_1)+\mathfrak{I}^\alpha_{a_1^+}f(a_2)\right] \\ &\leq \frac{(1-\alpha)\Delta(s)}{2^s\left[\exp\left(\left(-\frac{1-\alpha}{\alpha}\right)(a_2-a_1)\right)-1\right]}[f(a_1)+f(a_2)], \end{aligned}$$

where $\mathfrak{I}^\alpha_{a_2^-}$ and $\mathfrak{I}^\alpha_{a_1^+}$ are the left- and right-hand sided fractional integral operators with exponential kernel and

$$\Delta(s):=\int_0^1\frac{(a_2-a_1)}{\alpha}\exp\left[\left(-\frac{1-\alpha}{\alpha}\right)(a_2-a_1)t\right][t^s+(1-t)^s]dt.$$

For establishing some new results regarding general fractional integrals we need to prove the following basic lemma.

Lemma 1. Let $f:[a_1,a_2]\to\mathbb{R}$ be a differentiable function on (a_1,a_2). If $f'\in L(P)$, then the following identity for generalized fractional integrals holds:

$$\begin{aligned} &\frac{f(a_1)+f(a_2)}{2}-\frac{1}{2\mathcal{A}(1)}\left[{}_{a_1^+}I_\varphi f(a_2)+{}_{a_2^-}I_\varphi f(a_1)\right] \\ &\qquad=\frac{(a_2-a_1)}{2\mathcal{A}(1)}\int_0^1[\mathcal{A}(1-t)-\mathcal{A}(t)]f'(ta_1+(1-t)a_2)\,dt. \end{aligned}$$

We denote

$$\mathcal{H}_{f,\mathcal{A}}(a_1,a_2):=\frac{(a_2-a_1)}{2\mathcal{A}(1)}\int_0^1[\mathcal{A}(1-t)-\mathcal{A}(t)]f'(ta_1+(1-t)a_2)\,dt. \tag{17}$$

Proof. Integrating by parts (2.4) and changing the variable of integration, we have

$$\mathcal{H}_{f,\mathcal{A}}(a_1,a_2)=\frac{(a_2-a_1)}{2\mathcal{A}(1)}\cdot\left\{\int_0^1\mathcal{A}(1-t)f'(ta_1+(1-t)a_2)dt-\int_0^1\mathcal{A}(t)f'(ta_1+(1-t)a_2)dt\right\}$$
$$=\frac{(a_2-a_1)}{2\mathcal{A}(1)}$$
$$\cdot\left\{-\frac{\mathcal{A}(1-t)f(ta_1+(1-t)a_2)}{(a_2-a_1)}\Big|_0^1-\frac{1}{a_2-a_1}\int_0^1\frac{\varphi\big((a_2-a_1)(1-t)\big)}{(1-t)}f(ta_1+(1-t)a_2)\right.$$
$$\left.+\ \frac{\mathcal{A}(t)f(ta_1+(1-t)a_2)}{a_2-a_1}\Big|_0^1-\frac{1}{a_2-a_1}\int_0^1\frac{\varphi\big((a_2-a_1)t\big)}{t}f(ta_1+(1-t)a_2)dt\right\}$$
$$=\frac{(a_2-a_1)}{2\mathcal{A}(1)}\cdot\left\{\frac{\mathcal{A}(1)f(a_2)}{(a_2-a_1)}-\frac{1}{(a_2-a_1)}\,{}_{a_2^-}I_\varphi f(a_1)+\frac{\mathcal{A}(1)f(a_1)}{(a_2-a_1)}-\frac{1}{(a_2-a_1)}\,{}_{a_1^+}I_\varphi f(a_2)\right\}$$
$$=\frac{f(a_1)+f(a_2)}{2}-\frac{1}{2\mathcal{A}(1)}\left[{}_{a_1^+}I_\varphi f(a_2)+{}_{a_2^-}I_\varphi f(a_1)\right].$$

Lemma 1 is proved. ■

Theorem 3. Let $f:[a_1,a_2]\to\mathbb{R}$ be a differentiable function on (a_1,a_2). If $|f'|^q$ is s-convex function on $[a_1,a_2]$ for $q>1$ with $p^{-1}+q^{-1}=1$, then the folowing inequality for generalized fractional integrals holds:

$$\left|\mathcal{H}_{f,\mathcal{A}}(a_1,a_2)\right|\le\frac{(a_2-a_1)}{2\mathcal{A}(1)}\sqrt[p]{K_{\mathcal{A}}(p)}\cdot\left[\frac{|f'(a_1)|^q+|f'(a_2)|^q}{s+1}\right]^{\frac{1}{q}},\tag{18}$$

where

$$K_{\mathcal{A}}(p):=\int_0^1|\mathcal{A}(1-t)-\mathcal{A}(t)|^p dt.$$

Proof. From Lemma 1, s-convexity of $|f'|^q$, Hölder's inequality and properties of the modulus, we have

$$\left|\mathcal{H}_{f,\mathcal{A}}(a_1,a_2)\right|\le\frac{(a_2-a_1)}{2\mathcal{A}(1)}\int_0^1|\mathcal{A}(1-t)-\mathcal{A}(t)|\left|f'(ta_1+(1-t)a_2)\right|dt$$
$$\le\frac{(a_2-a_1)}{2\mathcal{A}(1)}\left(\int_0^1|\mathcal{A}(1-t)-\mathcal{A}(t)|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'(ta_1+(1-t)a_2)\right|^q dt\right)^{\frac{1}{q}}$$
$$\le\frac{(a_2-a_1)}{2\mathcal{A}(1)}\sqrt[p]{K_{\mathcal{A}}(p)}\left(\int_0^1\left((1-t)^s|f'(a_2)|^q+t^s|f'(a_1)|^q\right)dt\right)^{\frac{1}{q}}$$
$$=\frac{(a_2-a_1)}{2\mathcal{A}(1)}\sqrt[p]{K_{\mathcal{A}}(p)}\cdot\left[\frac{|f'(a_1)|^q+|f'(a_2)|^q}{s+1}\right]^{\frac{1}{q}}$$

Theorem 3 is proved. ■

Corollary 4. Taking $p = q = 2$ in Theorem 3, we get

$$\left|\mathcal{H}_{f,\mathcal{A}}(a_1,a_2)\right| \leq \frac{(a_2-a_1)}{2\mathcal{A}(1)}\sqrt{K_{\mathcal{A}}(2)}\sqrt{\frac{|f'(a_1)|^q+|f'(a_2)|^q}{s+1}}$$

Corollary 5. Choosing $\varphi(t) = t$, and $s = 1$ in Theorem 3, we obtain Theorem 2.3 of [3].

Corollary 6. Taking $\varphi(t) = \frac{t^{\alpha}}{\Gamma(\alpha)}$ and $s = 1$ in Theorem 3, we have Theorem 8 of [14].

Theorem 4. Let $f: [a_1, a_2] \to \mathbb{R}$ b*e a* differentiable function on (a_1, a_2). If $\left|f'\right|^q$ is s-convex function on $[a_1, a_2]$ for $q \geq 1$, then the following inequality for generalized fractional integrals holds:

$$\left|\mathcal{H}_{f,\mathcal{A}}(a_1,a_2)\right| \leq \frac{(a_2-a_1)}{2\mathcal{A}(1)}[K_{\mathcal{A}}(1)]^{1-\frac{1}{q}}[D_{\mathcal{A}}(s)]^{\frac{1}{q}}\left[\left|f'(a_1)\right|^q+\left|f'(a_2)\right|^q\right]^{\frac{1}{q}}, \tag{19}$$

where

$$D_{\mathcal{A}}(s) := \int_0^1 t^s|\mathcal{A}(1-t)-\mathcal{A}(t)|dt.$$

Proof. From Lemma 1, s -convexity of $|f'|^q$, the well–known power mean inequality and properties of the modulus, we have

$$\begin{aligned}
\left|\mathcal{H}_{f,\mathcal{A}}(a_1,a_2)\right| &\leq \frac{(a_2-a_1)}{2\mathcal{A}(1)}\int_0^1|\mathcal{A}(1-t)-\mathcal{A}(t)|\left|f'(ta_1+(1-t)a_2)\right|dt \\
&\leq \frac{(a_2-a_1)}{2\mathcal{A}(1)}\cdot\left(\int_0^1|\mathcal{A}(1-t)-\mathcal{A}(t)|dt\right)^{1-\frac{1}{q}} \\
&\cdot\left(\int_0^1|\mathcal{A}(1-t)-\mathcal{A}(t)|\left|f'(ta_1+(1-t)a_2)\right|^q dt\right)^{\frac{1}{q}} \\
&\leq \frac{(a_2-a_1)}{2\mathcal{A}(1)}[K_{\mathcal{A}}(1)]^{1-\frac{1}{q}}\left(\int_0^1|\mathcal{A}(1-t)-\mathcal{A}(t)|\left(t^s\left|f'(a_1)\right|^q+(1-t)^s\left|f'(a_2)\right|^q\right)dt\right)^{\frac{1}{q}} \\
&= \frac{(a_2-a_1)}{2\mathcal{A}(1)}[K_{\mathcal{A}}(1)]^{1-\frac{1}{q}}[D_{\mathcal{A}}(s)]^{\frac{1}{q}}\left[\left|f'(a_1)\right|^q+\left|f'(a_2)\right|^q\right]^{\frac{1}{q}}.
\end{aligned}$$

Theorem 4 is proved. ■

We point out some special cases of Theorem 4.

Corollary 7. Taking q = 1 in Theorem 4, we get

$$\left|\mathcal{H}_{f,\mathcal{A}}(a_1,a_2)\right| \leq \frac{(a_2-a_1)}{2\mathcal{A}(1)}D_{\mathcal{A}}(s)\left[\left|f'(a_1)\right|+\left|f'(a_2)\right|\right].$$

Corollary 8. Choosing $\varphi(t) = t$ in Theorem 4, we have

$$\left|\mathcal{H}_{f,\mathcal{A}}(a_1, a_2)\right| \leq \frac{a_2 - a_1}{4 \cdot 2^{\frac{s-1}{q}}} \cdot \left(\frac{1}{(s+1)(s+2)}\right)^{q} \cdot \left[\left|f'(a_1)\right|^q + \left|f'(a_2)\right|^q\right]^{\frac{1}{q}}.$$

Corollary 9. Taking $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$ in Theorem 4, we obtain

$$\left|\mathcal{H}_{f,\mathcal{A}}(a_1, a_2)\right| \leq \frac{a_2 - a_1}{2}\left[\frac{1}{\alpha+1}\left(1 - \frac{1}{2^\alpha}\right)\right]^{1-\frac{1}{q}}\left[\beta_{\frac{1}{2}}(s+1, \alpha+1) - \beta_{\frac{1}{2}}(\alpha+1, s+1) + \frac{1}{\alpha+s+1}\right] \cdot \left[\left|f'(a_1)\right|^q + \left|f'(a_2)\right|^q\right]^{\frac{1}{q}}.$$

3. Applications

Consider the following special means for different positive real numbers a_1, a_2, where $a_1 < a_2$:

- The arithmetic mean

$$A(a_1, a_2) = \frac{a_1 + a_2}{2};$$

- The generalized log–mean

$$L_n(a_1, a_2) = \left[\frac{a_2^{\,n+1} - a_1^{\,n+1}}{(n+1)(a_2 - a_1)}\right]^{\frac{1}{n}}; \;\; n \in \mathbb{Z}\backslash\{-1, 0\}.$$

Now, by using the main results in Section 2, we give the following interesting applications.

Proposition 1 Let $0 < a_1 < a_2$. Then for some fixed $s \in (0, 1]$, where $q > 1$ with $p^{-1} + q^{-1} = 1$, the following inequality holds:

$$\left|A\left(a_1^{\frac{s}{q}+1}, a_2^{\frac{s}{q}+1}\right) - L_{\frac{s}{q}+1}^{\frac{s}{q}+1}(a_1, a_2)\right| \leq \frac{(s+q)}{2q} \cdot \frac{(a_2 - a_1)}{\sqrt[p]{p+1}} \cdot \left(\frac{2}{s+1}\right)^{\frac{1}{q}} \cdot A^{\frac{1}{q}}(a_1^s, a_2^s). \tag{20}$$

Proof. Applying $f(x) = \frac{x^{\frac{s}{q}+1}}{\frac{s}{q}+1}$ and $\varphi(t) = t$ in Theorem 3, we can obtain the result immediately. ■

Proposition 2. Let $0 < a_1 < a_2$. Then for some fixed $s \in (0, 1]$ with $q > 1$, the following inequality holds:

$$\left|A\left(a_1^{\frac{s}{q}+1}, a_2^{\frac{s}{q}+1}\right) - L_{\frac{s}{q}+1}^{\frac{s}{q}+1}(a_1, a_2)\right| \leq \frac{(s+q)(a_2 - a_1)}{q \cdot 2^{2+\frac{s-2}{q}}} \cdot \left[\frac{1}{(s+1)(s+2)}\right]^{\frac{1}{q}} \cdot A^{\frac{1}{q}}(a_1^s, a_2^s). \tag{21}$$

Proof. Taking $f(x) = \frac{x^{\frac{s}{q}+1}}{\frac{s}{q}+1}$ and $\varphi(t) = t$ in Theorem 4, we can derive the result directly. ■

Next, we provide some new error estimates for the trapezoidal formula.

Let Q be the partition of the points $a_1 = x_0 < x_1 < \cdots < x_n = a_2$ of the closed interval $[a_1, a_2]$.
Let consider the following quadrature formula

$$\int_a^b f(x)dx := T(f, Q) + E(f, Q),$$

where

$$T(f, Q) := \sum_{i=0}^{n-1} \frac{f(x_i) + f(x_{i+1})}{2} (x_{i+1} - x_i)$$

is the trapezoidal version and E(f, Q) is denote their associated approximation error.

Proposition 3. Let $f: [a_1, a_2] \longrightarrow \mathbb{R}$ be a differentiable function on (a_1, a_2), where $a_1 < a_2$. If $|f'|^q$ is s-convex function on $[a_1, a_2]$ for $q > 1$ with $p^{-1} + q^{-1} = 1$, then the following inequality holds:

$$|E(f, Q)| \leq \frac{1}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \cdot \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2 \cdot \left[\frac{|f'(x_i)|^q + |f'(x_{i+1})|^q}{s+1}\right]^{\frac{1}{q}}. \tag{22}$$

Proof. Applying Theorem 3 for $\varphi(t) = t$ on the subintervals $[x_i, x_{i+1}]$ $(i = 0, 1, \ldots, n-1)$ of partition Q, we have

$$\left|\frac{f(x_i) + f(x_{i+1})}{2} - \frac{1}{x_{i+1} - x_i}\int_{x_i}^{x_{i+1}} f(x)dx\right| \leq \frac{(x_{i+1} - x_i)^2}{2(x_{i+1} - x_i)} \cdot \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \cdot \left[\frac{|f'(x_i)|^q + |f'(x_{i+1})|^q}{s+1}\right]^{\frac{1}{q}}.$$

$$\left|\frac{f(x_i) + f(x_{i+1})}{2}(x_{i+1} - x_i) - \int_{x_i}^{x_{i+1}} f(x)dx\right| \leq \frac{(x_{i+1} - x_i)^2}{2} \cdot \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \cdot \left[\frac{|f'(x_i)|^q + |f'(x_{i+1})|^q}{s+1}\right]^{\frac{1}{q}}.$$

Summing up with respect to i for $i = 0, 1, \ldots, n-1$, we get

$$|E(f, Q)| \leq \frac{1}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \cdot \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2 \cdot \left[\frac{|f'(x_i)|^q + |f'(x_{i+1})|^q}{s+1}\right]^{\frac{1}{q}}.$$

Proposition 3 is proved. ■

Proposition 4. Let $f: [a_1, a_2] \longrightarrow \mathbb{R}$ be a differentiable function on (a_1, a_2), where $a_1 < a_2$. If $|f'|^q$ is s-convex function on $[a_1, a_2]$ for $q \geq 1$, then the following inequality holds:

$$|E(f, Q)| \leq \frac{1}{2^{2+\frac{s-2}{q}}} \cdot \left(\frac{1}{(s+1)(s+2)}\right)^{\frac{1}{q}} \cdot \sum_{i=0}^{n-1} (x_{i+1} - x_i)^2 \cdot \left[\frac{|f'(x_i)|^q + |f'(x_{i+1})|^q}{s+1}\right]^{\frac{1}{q}}. \tag{23}$$

Proof. The proof is analogous as in Proposition 3 but using Theorem 4 and $\varphi(t) = t$. ■

4. Conclusion

In this paper, we found a new interesting integral identity regarding Hermite-Hadamard type integral inequalities using generalized fractional integral operators. Furthermore, a new interesting integral identity about Hermite-Hadamard type integral is derived. By using this identity as an auxiliary result, some new bounds with respect to Hermite-Hadamard type integral inequalities pertaining to s-convex functions are established. It is pointed out that several special cases are deduced from the main results for suitable choices of function inside the generalized fractional integral operators. In order to show the efficient of our main results, some applications to special means for different positive real numbers and error bound estimates for trapezoidal quadrature formula are obtain as well. These ideas and techniques of this paper may stimulate further research in these directions for different class of convex functions for interested readers.

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Prime Ideals of Gamma Nearness Near Rings

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Abstract

In this article, we define the notion of prime ideals of Γ-near-rings on weak nearness approximation spaces and explain some of the concepts and definitions. Then, we study some basic properties of prime ideals of Γ-nearness near-rings. Γ-nearness near-rings is different from Γ-nearness rings and Γ-nearness semirings since Γ does not have to be group in Γ-nearness near-rings. Because of this, some properties defined in Γ-nearness rings and Γ-nearness semirings show some changes in Γ-nearness near-rings.

Keywords: Near set, Near ring, Nearness approximation space, Weak nearness approximation space, Near-ring, Gamma-near-ring, Nearness near-ring, Gamma nearness near-ring.

1. Introduction

The concept Γ-rings, a generalization of a ring was introduced by Nobusawa in 1964 [1] and generalized by Barnes in 1966 [2]. Pilz defined near-rings (also near ring or nearring) that is an algebraic structure similar to a ring but satisfying some axioms [3]. A generalization of both the concepts near-ring and the ring, namely Γ-near-ring was introduced by Satyanarayana in 1984 and later studied by the authors like Satyanarayana [4], [5], Booth [6], Booth and Groenewald [7], [8], Jun, Sapancı and Öztürk [9].

In 2002, Peters introduced near set theory, which is a generalization of rough set theory [10]. In this theory, Peters defined an indiscernibility relation by using the features of the objects to determine the nearness of the objects [11], [12], [13]. The concept of nearness has a different approach for algebraic structures. Because, in the concept of ordinary algebraic structures, such a structure that consists of a nonempty set of abstract points with one or more binary operations, which are required to satisfy certain axioms. Also, the sets are composed of abstract points. Perceptual objects (non-abstract points) can be used on weak nearness approximation space to define nearness algebraic structures. This is more useful than working with abstract points for many areas such as engineering applications, image analysis and so on. In 2012, İnan and Öztürk investigated the concept of nearness groups [14] and other algebraic approaches of near sets in [15], [16], [17], [18], [19], [20], [21], [22], [23]. In 2021, Uçkun and Genç defined near-rings on nearness approximation spaces [24].

The aim of this paper is to define the concept of prime ideals of Γ-nearness near-rings and to study some properties. Γ-nearness near-rings is different from Γ-nearness rings [20] and Γ-nearness semirings [21] because Γ does not have to be group in Γ-nearness near-rings. Because of this, some properties defined in Γ-nearness rings and Γ-nearness semirings show some changes in Γ-nearness near-rings.

2. Preliminaries

An object description is specified by means of a tuple of function values $\Phi(x)$ deal with an object $x \in X$. $B \subseteq \mathcal{F}$ is a set of probe functions and these functions stand for features of sample objects $X \subseteq \mathcal{O}$. Let $\varphi_i \in B$, that is $\varphi_i : \mathcal{O} \to \mathbb{R}$. The functions showing object features supply a basis for $\Phi: \mathcal{O} \to \mathbb{R}^L, \Phi(x) = (\varphi_1(x), \varphi_2(x), \ldots, \varphi_L(x))$ a vector consisting of measurements deal with each functional value $\varphi_i(x)$, where the description length $|\Phi| = L$ (**Hata! Başvuru kaynağı bulunamadı.**).

The selection of functions $\varphi_i \in B$ is very fundamental by using to determine sample objects. $X \subseteq \mathcal{O}$ are near each other if and only if the sample objects have similar characterization. Each φ shows a descriptive pattern of an object. Hence, $\triangle_{\varphi_i}$ means $\triangle_{\varphi_i} = |\varphi_i(x)' - \varphi_i(x)|$, where $x, 'x \in \mathcal{O}$. The difference φ means to a description of the indiscernibility relation " $\sim_B$ " defined by Peters in **Hata! Başvuru kaynağı bulunamadı.**. B_r is probe functions in B for $r \leq |B|$.

Definition 1 [11]:

$$[c]l \sim_B = \{(x, x)' \in \mathcal{O} \times \mathcal{O} | \triangle_{\varphi_i} = 0 \forall \varphi_i \in B \ B \subseteq \mathcal{F}\}$$

means indiscernibility relation on $\mathcal{O}$, where description length $i \leq |\Phi|$. $\sim_{B_r}$ is also indiscernibility relation determined by utilizing B_r.

Near equivalence class is stated as $[x]_{B_r} = \{x \in '\mathcal{O} | x \sim_{B_r} x\}'$. After getting near equivalence classes, quotient set $\mathcal{O} / \sim_{B_r} = \{[x]_{B_r} | x \in \mathcal{O}\} = \xi_{\mathcal{O},B_r}$ and set of partitions $N_r(B) = \{\xi_{\mathcal{O},B_r} | B_r \subseteq B\}$ can be found. By using near equivalence classes, $N_r(B)^* X = \bigcup_{[x]_{B_r} \cap X \neq \emptyset} [x]_{B_r}$ upper approximation set can be attained.

Definition 2 [18]:
Let $\mathcal{O}$ be a set of sample objects, $\mathcal{F}$ a set of the probe functions, $\sim_{\mathrm{B_r}}$ an indiscernibility relation,and $\mathrm{N_r}$ a collection of partitions. Then, $(\mathcal{O}, \mathcal{F}, \sim_{\mathrm{B_r}}, \mathrm{N_r}(\mathrm{B}))$ is called a weak nearness approximation space.

Theorem 1[18]:
Let $(\mathcal{O}, \mathcal{F}, \sim_{\mathrm{B_r}}, \mathrm{N_r})$ be a weak nearness approximation space and $\mathrm{X}, \mathrm{Y} \subset \mathcal{O}$. Then the following statements hold:

i) $X \subseteq N_r(B)^*X$,
ii)$N_r(B)^*(X \cup Y) = N_r(B)^*X \cup N_r(B)^*Y$,
iii) $X \subseteq Y$ implies $N_r(B)^*X \subseteq N_r(B)^*Y$,
iv) $N_r(B)^*(X \cap Y) \subseteq N_r(B)^*X \cap N_r(B)^*Y$.

Definition 3 Hata! Başvuru kaynağı bulunamadı.**:**
Let $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r(B))$ be a weak nearness approximation space, $G \subseteq \mathcal{O}$ and " $\cdot$ "a operation by $\cdot: G \times G \rightarrow N_r(B)^*G$. G is called a group on $\mathcal{O}$ or shortly nearness group if the following properties are satisfied:

i) $x \cdot y \in N_r(B)^*G$ for all $x, y \in G$,

ii) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ property holds in $N_r(B)^*G$ for all $x, y, z \in G$,

iii) There exists $e \in N_r(B)^*G$ such that $x \cdot e = x = e \cdot x$ for all $x \in G$,

iv) There exists $y \in G$ such that $x \cdot y = e = y \cdot x$ for all $x \in G$.

Lemma 1 Hata! Başvuru kaynağı bulunamadı.**:**
Let S be a Γ-nearness semiring. If $\sim_{B_r}$ is a congruence indiscernibility relation on S, then $[x]_{B_r} + [y]_{B_r} \subseteq x + y]_{B_r}$, $[\beta]_{B_r} + [\gamma]_{B_r} \subseteq \beta + \gamma]_{B_r}$, $[x]_{B_r}\alpha y]_{B_r} \subseteq x\alpha y]_{B_r}$ for all $x, y \in S$, and $\alpha, \beta, \gamma \in \Gamma$.

Lemma 2 Hata! Başvuru kaynağı bulunamadı.**:** Let S be a Γ-nearness semiring. The following properties hold:

i) If $X, Y \subseteq S$, then $(N_r(B)^*X) + (N_r(B)^*Y) \subseteq N_r(B)^*(X + Y)$,

ii) If $X, Y \subseteq S$, then $(N_r(B)^*X)\Gamma(N_r(B)^*Y) \subseteq N_r(B)^*(X\Gamma Y)$.

Definition 4 Hata! Başvuru kaynağı bulunamadı.**:** Let M and Γ be additive Abelian groups. M is said to be a Γ-ring if there exists a mapping $M \times \Gamma \times M \rightarrow M$ (the image of (a, α, b) is denoted by $a\alpha b$). M is called a Γ-near-ring (in the sense of Barnes*)* on $\mathcal{O} - \mathcal{O}'$ or shortly a Γ-nearness near-ring and denoted by $(M, +, \cdot)$ satisfying the following conditions:

i)
$$(a + b)\alpha c = a\alpha c + b\alpha c,$$
$$a(\alpha + \beta)b = a\alpha b + a\beta b,$$
$$a\alpha(b + c) = a\alpha b + a\alpha c,$$

ii) $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$.

Definition 5 Hata! Başvuru kaynağı bulunamadı.**:** Let $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r(B), \nu_{N_r})$ be a nearness approximation spaces, "$+$" and "$\cdot$" be binary operations defined on $\mathcal{O}$. $M \subseteq \mathcal{O}$ is called a near-ring on nearness approximation spaces or shortly nearness near-ring if the following properties are satisfied:

NN_1) $(M, +)$ is a nearness group (it does not need to be commutative),
NN_2) $(M, \cdot)$ is a nearness semigroup,
NN_3) for all $x, y, z \in M$, $(x + y) \cdot z = (x \cdot z) + (y \cdot z)$ property holds in $N_r(B)^*M$.

Definition 6 Hata! Başvuru kaynağı bulunamadı.**:** Let $M = \{a, b, c, ...\} \subseteq \mathcal{O}$, and $\Gamma = \{\alpha, \beta, ...\} \subseteq \mathcal{O}'$ where $(\mathcal{O}, \mathcal{F}, \sim_{B_r}, N_r(B))$ and $(\mathcal{O}', \mathcal{F}, \sim_{B_r}, N_r(B))$ are two different weak near approximation spaces. " $\cdot$ "a operation by $\cdot: M \times M \to N_r(B)^*M$. M is called a Γ-near-ring (in the sense of Barnes) on $\mathcal{O} - \mathcal{O}'$ or shortly a Γ-nearness near-ring and denoted by $(M, +, \cdot)$ if the following conditions are satisfied:

GNR_1) $(M, +)$ is a nearness group on $\mathcal{O}$ with identity element 0_M (not necessarily abelian),

GNR_2) for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$ such that $(a\alpha b)\beta c = a\alpha(b\beta c)$ hold in $N_r(B)^*M$,

GNR_3) for all $a, b, c \in M$ and $\alpha \in \Gamma$ such that $(a + b)\alpha c = a\alpha c + b\alpha c$ hold in $N_r(B)^*M$.

Theorem 2 Hata! Başvuru kaynağı bulunamadı.**:** Let M be a Γ-nearness near-ring and $\{H_i | i \in I\}$ be a nonempty family of Γ-ideal of M, where an arbitrary index set I.

i) If $N_r(B)^*(\bigcap_{i\in I} H_i) = \bigcap_{i\in I} N_r(B)^* H_i$, then $\bigcap_{i\in I} H_i$ is a Γ-ideal of M.

ii) $\bigcup_{i\in I} H_i$ is a Γ-ideal of M.

3. Prime Ideals of Γ-nearness near-ring

Definition 7: Let P be an ideal of Γ-nearness near-ring M. P is called

i) a Γ-prime ideal of M if for all ideals I and J of M, $I\Gamma J \subseteq N_r(B)^*P$ implies $I \subseteq P$ or $J \subseteq P$.

ii) a Γ-semiprime ideal of M if for all ideals I and J of M, $I^2 = I\Gamma J \subseteq N_r(B)^*P$ implies $I \subseteq P$ or $J \subseteq P$.

Definition 8: Let X be a nonempty subset of a Γ-nearness near-ring M. Let $\{A_i : i \in I\}$ be a family of all ideals in M that contain X. If

$$\bigcap_{i\in\Delta} (N_r(B)^* A_i) = N_r(B)^* (\bigcap_{i\in\Delta} A_i),$$

then $\bigcap_{i\in I} A_i$ is called the ideal generated by the set X and it is denoted by (X).

The elements of X is called the generators of ideal (X). If $X = \{x_1, x_2, \cdots, x_n\}$, then $(X) = (x_1, x_2, \cdots, x_n)$. Thus, we call (X) is finitely generated.

If $X = \{a\}$, then $(X) = (a)$ is called the principal ideal generated by a.

Theorem 3: Let P be a Γ-prime ideal of M. Then, the following conditions are equivalent.

$i)$ P is prime,

$ii)$ For every two ideals I, J of M, it implies that $I \nsubseteq P$ and $J \nsubseteq P \Rightarrow I\Gamma J \nsubseteq N_r(B)^* P$,

$iii)$ For every two elements $a, b \in M$, $a \notin P$ and $b \notin P \Rightarrow (a)\Gamma(b) \nsubseteq N_r(B)^* P$.

Proof.

$i) \Rightarrow ii)$ Assume that P is a Γ-prime ideal of M, $I \nsubseteq P$ and $J \nsubseteq P$. If possible, suppose that $I\Gamma J \subseteq N_r(B)^*$, then $I \subseteq P$ or $J \subseteq P$ since P is a Γ-prime ideal of M. Thus, we received a contradiction. From here, we have $I\Gamma J \nsubseteq N_r(B)^*$.

$ii) \Rightarrow iii)$ Let $a \notin P$ and $b \notin P$ be elements of M. In this case, we get $(a) \nsubseteq P$ and $(b) \nsubseteq P$. Therefore, by hypothesis, $(a)\Gamma(b) \nsubseteq N_r(B)^* P$.

$iii) \Rightarrow i)$ For elements $a, b \in M$, $a \notin P$ and $b \notin P$, and so $(a) \nsubseteq P$ and $(b) \nsubseteq P$. Suppose that $(a) \nsubseteq P$ and $(b) \nsubseteq P$ such that $(a)\Gamma(b) \subseteq N_r(B)^* P$. Since $a \notin P$ and $b \notin P$, then by hypothesis $(a)\Gamma(b) \nsubseteq N_r(B)^* P$, which is contradiction. In this case, $(a)\Gamma(b) \subseteq N_r(B)^* P \Rightarrow (a) \subseteq P$ and $(b) \subseteq P$. Therefore, P is a prime ideal of M.

Definition 9: Let M be a Γ-nearness near-ring. Then, M is called Γ-prime near ring if 0 is a Γ-prime ideal of M.

Theorem 4: Let M be a Γ-nearness near-ring and $\{A_i | i \in I\}$ be a nonempty family of Γ-prime ideal of M, where an arbitrary index set I.

$i)$ If $N_r(B)^*(\bigcap_{i\in I} A_i) = \bigcap_{i\in I} N_r(B)^* A_i$, then $\bigcap_{i\in I} A_i$ is a Γ-prime ideal of M.

$ii)$ If $A_1 \subseteq A_2 \subseteq A_3$ ..., then $\bigcup_{i\in I} A_i$ is a Γ-prime ideal of M.

Proof.

$i)$ $\bigcap_{i\in I} A_i$ is a Γ-ideal of M by Theorem 2. Suppose that $P_1\Gamma P_2 \subseteq N_r(B)^*(\bigcap_{i\in I} A_i)$ for any Γ-ideals P_1 and P_2 of M. In this case, $P_1\Gamma P_2 \subseteq N_r(B)^*(\bigcap_{i\in I} A_i) = \bigcap_{i\in I} N_r(B)^* A_i$ from hypothesis. Thus, $P_1\Gamma P_2 \subseteq \bigcap_{i\in I} N_r(B)^* A_i$, and so $P_1\Gamma P_2 \subseteq N_r(B)^* A_i$ for all $i \in I$. Because A_i's are Γ-prime ideals of M for all $i \in I$, then $P_1 \subseteq A_i$ or $P_2 \subseteq A_i$ for all $i \in I$. From here, we attain that $P_1 \subseteq \bigcap_{i\in I} A_i$ or $P_2 \subseteq \bigcap_{i\in I} A_i$.

$ii)$ From Theorem 2.(ii), $\bigcup_{i\in I} A_i$ is a Γ-ideal of M. Assume that $P_1\Gamma P_2 \subseteq N_r(B)^*(\bigcup_{i\in I} A_i)$ for any ideals P_1 and P_2 of M. In this case, we get $P_1\Gamma P_2 \subseteq \bigcup_{i\in I} N_r(B)^* A_i$ by Theorem 1.(ii). There is at least one $i_n \in I$

such that $P_1 \Gamma P_2 \subseteq N_r(B)^* A_{i_n}$. As A_{i_n} is prime ideal of M for $i_n \in I$, $P_1 \subseteq A_{i_n}$ or $P_2 \subseteq A_{i_n}$ for $i_n \in I$. Therefore, $P_1 \subseteq \bigcup_{i \in I} A_i$ or $P_2 \subseteq \bigcup_{i \in I} A_i$.

Definition 10: A Γ-nearness near-ring M is called simple if M has no proper ideal.

Theorem 5: If Γ-nearness near-ring M is simple, then either M is Γ-prime or $\mathrm{M\Gamma M} = \{0\} \in \mathrm{N_r(B)^*M}$.

Proof.

Suppose that I and J are ideals of M. Since M is simple, we have $I = M$ or $I = 0$ and $J = M$ or $I = 0$. Therefore, for the ideals I and J of M, we have the equation $I\Gamma J = 0$, then $I = 0$ or $J = 0$, or $I = J = M$. If $I = 0$ or $J = 0$, then M is Γ-prime near ring. Otherwise, $M\Gamma M = \{0\} \in N_r(B)^* M$ if $I = J = M$.

4. Conclusion

As a recent study of Γ-nearness near-ring, it is defined that the notion of prime ideals in Γ-nearness near-ring. Afterward, it is explained that some of the concepts and definitions. We believe that these properties will be more useful theoretical development for Γ-nearness near-ring theory.

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A Note on the Comparison Theorems for Second Order Neutral Dynamic Equations on Time Scales

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Abstract

In [1], Agarwal et al. presented some comparison theorems on the oscillation of second order functional dynamic equations with a neutral term. They studied a class of neutral dynamic equations under assumptions that allow applications to equations with both delayed and advanced arguments. In this work, by extending the ideas exploited in [1] and [2], we attempt to establish several new comparison theorems for oscillation of second order mixed neutral noncanonical dynamic equations, based on comparisons with associated second order linear non-neutral dynamic equations on time scales.

1. INTRODUCTION

The study of analysis on time scales was introduced by Hilger in his Ph. D. dissertation [3] in 1988 in order to unify continuous and discrete analysis. This new and exciting type of mathematics is more general and versatile than the traditional theories of differential and difference equations as it can, under one framework, mathematically describe continuous-discrete hybrid processes. In fact, the progressive field of dynamic equations on time scales contains links and extends the classical theories of differential and difference equations. For instance, if $\mathrm{T}=\mathbb{Z}$, we have a result for difference equations, if $\mathrm{T}=\mathbb{R}$, we obtain a result for differential equations [4]. The usual notation and concepts from the time scale calculus as can be found in Bohner and Peterson [5] will be used throughout the study without further mention.

In [1], Agarwal et al. considered the second order dynamic equation of neutral type of the form

$$\left[r(t)\left[x(t)+p(t)x(\eta(t))\right]^{\Delta}\right]^{\Delta}+q(t)x(g(t))=0 \tag{1}$$

on an arbitrary time scale T, where r, p and q are real-valued positive right dense continuous functions on T, the deviating arguments $\eta, g:\mathrm{T}\to\mathrm{T}$ are $rd-$continuous, and $\lim_{t\to\infty}\eta(t)=\lim_{t\to\infty}g(t)=\infty$. On the basis of conditions $0\le p(t)<1$ or $p(t)>1$ and under assumption of

$$\int_{t_0}^{\infty} \frac{\Delta s}{r(s)} < \infty, \tag{2}$$

they studied both neutral delay $(i.e., \eta(t) \le t)$ and neutral advanced $(i.e., \eta(t) \ge t)$ cases separately, and derived several comparison theorems that guarantee the oscillation of all solutions of equation (1). The main objective of this study is to generalize the results given in [1], to the second order dynamic equations with delayed and advanced arguments in the neutral term of the form

$$\left[r(t)\left[x(t) + p(t)x(\eta(t)) + m(t)x(\vartheta(t)) \right]^{\Delta} \right]^{\Delta} + q(t)x(g(t)) = 0 \tag{E}$$

on an arbitrary time scale $\mathbb{T}$, where r, p, m and q are real-valued positive $rd-$continuous functions on $\mathbb{T}$, the deviating arguments $\eta, \vartheta, g : \mathbb{T} \to \mathbb{T}$ are $rd-$continuous functions such that, η and ϑ are strictly increasing, $\eta(t) \le t \le \vartheta(t)$, and $\lim_{t\to\infty} \eta(t) = \lim_{t\to\infty} g(t) = \infty$. For these formulation, we consider equation (E) in the case when (2) holds, that is, in noncanonical form. For the simplicity and without further mention, we use the notations:

$$y(t) = x(t) + p(t)x(\eta(t)) + m(t)x(\vartheta(t)),$$

$$A(t) := \int_t^{\infty} \frac{\Delta s}{r(s)} \quad and \quad B(t) := \int_{t_1}^{t} \frac{\Delta s}{r(s)}.$$

As usual, all occurring functional inequalities considered in this work are assumed to hold eventually, that is, they are satisfied for all sufficiently large t.

2. MAIN RESULTS

Set

$$\psi(t) = \frac{1}{m(t)}\left[1 - \frac{1}{m(\vartheta^{-1}(t))} - \frac{p(t)}{m(\vartheta^{-1}(\eta(t)))} \right] > 0$$

and

$$\phi(t) = \frac{1}{m(t)}\left[1 - \frac{1}{m(\vartheta^{-1}(t))} \frac{F(\vartheta^{-1}(t))}{F(t)} - \frac{p(t)}{m(\vartheta^{-1}(\eta(t)))} \frac{F(\vartheta^{-1}(\eta(t)))}{F(t)} \right] > 0$$

where ϑ^{-1} denotes the inverse function of ϑ, and F is a $\Delta-$differentiable function that will be specified later. Our first result is the following theorem.

Theorem 1. Suppose that (2) holds, $\vartheta(\sigma(t)) \geq g(t)$, $p(t) > 0$, $m(t) > 1$, $r(t)A(t) - \mu(t) > 0$ for all $t \in [t_0, \infty)_{\mathrm{T}}$, where $\sigma(t)$ is the forward jump operator and $\mu(t)$ is the graininess function defined on the time scale T. Assume further that there exist positive real valued $\Delta-$ differentiable functions f, F such that

$$\frac{f(t)}{r(t)B(t)} - f^{\Delta}(t) \leq 0 \quad and \quad \frac{F(t)}{r(t)A(t)} + F^{\Delta}(t) \leq 0 \tag{3}$$

for all sufficiently large $t_1 \in [t_0, \infty)_{\mathrm{T}}$. If the second order dynamic equations

$$\left[r(t) z^{\Delta}(t)\right]^{\Delta} + q(t)\psi\left(\vartheta^{-1}(g(t))\right)\frac{f\left(\vartheta^{-1}(g(t))\right)}{f(\sigma(t))} z(\sigma(t)) = 0 \tag{4}$$

and

$$\left[r(t) z^{\Delta}(t)\right]^{\Delta} + q(t)\phi\left(\vartheta^{-1}(g(t))\right) z(\sigma(t)) = 0 \tag{5}$$

are oscillatory, then Eq. (E) is oscillatory.

Proof. Suppose to the contrary that $x(t)$ is a nonoscillatory (say positive) solution of Eq. (E). Then we may assume that $x(t) > 0$, $x(\eta(t)) > 0$, $x(\vartheta(t)) > 0$, and $x(g(t)) > 0$ for $t \in [t_0, \infty)_{\mathrm{T}}$. In view of (E), we obtain

$$\left[r(t) y^{\Delta}(t)\right]^{\Delta} = -q(t) x(g(t)) < 0 \quad for \quad t \in [t_0, \infty)_{\mathrm{T}}. \tag{6}$$

So, $r(t) y^{\Delta}(t)$ is strictly decreasing, and $y^{\Delta}(t)$ is eventually of one sign. Assume first that $y^{\Delta}(t) > 0$ for $t \in [t_1, \infty)_{\mathrm{T}} \subseteq [t_0, \infty)_{\mathrm{T}}$. From the definition of the neutral term, we see that the inequality

$$\begin{aligned} x(t) &= \frac{1}{m(t)}\left[y(t) - x(t) - p(t) x(\eta(t))\right] \\ &\geq \frac{1}{m(t)}\left[y(t) - \frac{y\left(\vartheta^{-1}(t)\right)}{m\left(\vartheta^{-1}(t)\right)} - p(t)\frac{y\left(\vartheta^{-1}(\eta(t))\right)}{m\left(\vartheta^{-1}(\eta(t))\right)}\right] \end{aligned} \tag{7}$$

holds for $t \in [t_1, \infty)_{\mathrm{T}}$. Since y, η and ϑ are increasing functions, the inequalities $y\left(\vartheta^{-1}(t)\right) \leq y(t)$ and $y\left(\vartheta^{-1}(\eta(t))\right) \leq y(t)$ are fulfilled. So, we obtain from (7) that

$$x\left(\vartheta(t)\right) \geq \frac{1}{m(t)}\left[1-\frac{1}{m\left(\vartheta^{-1}(t)\right)}-\frac{p(t)}{m\left(\vartheta^{-1}\left(\eta(t)\right)\right)}\right]y(t)=\psi(t)y(t),$$

and it follows from (E) that

$$\left[r(t)y^{\Delta}(t)\right]^{\Delta}+q(t)\psi\left(\vartheta^{-1}\left(g(t)\right)\right)y\left(\vartheta^{-1}\left(g(t)\right)\right)\leq 0.$$

On the other hand, in view of (3) and the fact that

$$y(t)=y(t_1)+\int_{t_1}^{t}\frac{r(s)y^{\Delta}(s)}{r(s)}\Delta s \geq r(t)\left(\int_{t_1}^{t}\frac{\Delta s}{r(s)}\right)y^{\Delta}(t),$$

we arrive to the conclusion

$$\left(\frac{y(t)}{f(t)}\right)^{\Delta}=\frac{y^{\Delta}(t)f(t)-y(t)f^{\Delta}(t)}{f(t)f\left(\sigma(t)\right)}$$

$$\leq \frac{y(t)}{f(t)f\left(\sigma(t)\right)}\left(\frac{f(t)}{r(t)B(t)}-f^{\Delta}(t)\right)\leq 0,$$

that is, y/f is nonincreasing on $[t_1,\infty)_{\mathbb{T}}$. If we set

$$u(t)=\frac{r(t)y^{\Delta}(t)}{y(t)}, \tag{8}$$

then we obtain

$$u^{\Delta}(t)\leq -q(t)\psi\left(\vartheta^{-1}\left(g(t)\right)\right)\frac{y\left(\vartheta^{-1}\left(g(t)\right)\right)}{y\left(\sigma(t)\right)}-\frac{u^2(t)}{r(t)}\frac{y(t)}{y\left(\sigma(t)\right)} \tag{9}$$

$$\leq -q(t)\psi\left(\vartheta^{-1}\left(g(t)\right)\right)\frac{f\left(\vartheta^{-1}\left(g(t)\right)\right)}{f\left(\sigma(t)\right)}-\frac{u^2(t)}{r(t)}\frac{y(t)}{y(t)+\mu(t)y^{\Delta}(t)}.$$

Thus,

$$u^{\Delta}(t)+q(t)\psi\left(\vartheta^{-1}\left(g(t)\right)\right)\frac{f\left(\vartheta^{-1}\left(g(t)\right)\right)}{f\left(\sigma(t)\right)}+\frac{u^2(t)}{r(t)+\mu(t)u(t)}\leq 0$$

for large t. Therefore, we get by results of [6] that, Equation (4) is nonoscillatory, which is a contradiction.

Next, assume that $y^{\Delta}(t)<0$ for $t\in[t_1,\infty)_{\mathrm{T}}\subseteq[t_0,\infty)_{\mathrm{T}}$. Then, the inequality (7) holds again, and if we define the function u by (8), we see that $u(t)<0$ for $t\in[t_1,\infty)_{\mathrm{T}}$. Since $r(t)y^{\Delta}(t)$ is strictly decreasing, we get $r(s)y^{\Delta}(s)\le r(t)y^{\Delta}(t)$ for $s\in[t,\infty)_{\mathrm{T}}$. Integrating this inequality from t to ℓ, we have

$$y(\ell)\le y(t)+r(t)y^{\Delta}(t)\int_t^{\ell}\frac{\Delta s}{r(s)},\quad \ell\in[t,\infty)_{\mathrm{T}}.$$

Letting $\ell\to\infty$ in the latter inequality yields

$$0\le y(t)+r(t)y^{\Delta}(t)A(t),\quad t\in[t_1,\infty)_{\mathrm{T}}.$$

Thus, we obtain $u(t)\ge -1/A(t)$ and $y^{\Delta}(t)/y(t)\ge -1/r(t)A(t)$. In view of (3) and these inequalities, we have

$$\left(\frac{y(t)}{F(t)}\right)^{\Delta}\ge\frac{y(t)}{F(t)F(\sigma(t))}\left(\frac{F(t)}{r(t)A(t)}+F^{\Delta}(t)\right)\ge 0, \tag{10}$$

that is, y/F is nondecreasing. So, we obtain from (7) and (10) that

$$x(\vartheta(t))\ge\frac{1}{m(t)}\left[1-\frac{1}{m(\vartheta^{-1}(t))}\frac{F(\vartheta^{-1}(t))}{F(t)}-\frac{p(t)}{m(\vartheta^{-1}(\eta(t)))}\frac{F(\vartheta^{-1}(\eta(t)))}{F(t)}\right]y(t)=\phi(t)y(t),$$

Differentiating $u(t)$ and using this last inequality, we obtain

$$u^{\Delta}(t)\le -q(t)\phi(\vartheta^{-1}(g(t)))\frac{y(\vartheta^{-1}(g(t)))}{y(\sigma(t))}-\frac{u^2(t)}{r(t)}\frac{y(t)}{y(t)+\mu(t)y^{\Delta}(t)} \tag{11}$$

$$\le -q(t)\phi(\vartheta^{-1}(g(t)))-\frac{u^2(t)}{r(t)}\frac{y(t)}{y(t)+\mu(t)y^{\Delta}(t)}$$

$$\le -q(t)\phi(\vartheta^{-1}(g(t)))-\frac{u^2(t)}{r(t)+\mu(t)u(t)}.$$

Due to

$$r(t)+\mu(t)u(t)\geq\frac{r(t)B(t)-\mu(t)}{B(t)}>0,$$

the function $u(t)$ satisfies

$$u^{\Delta}(t)+q(t)\phi\left(\vartheta^{-1}(g(t))\right)+\frac{u^{2}(t)}{r(t)+\mu(t)u(t)}\leq 0$$

for large t. Therefore, we get by results of [6] that, Equation (5) is nonoscillatory, which is a contradiction that proves the theorem.

Theorem 2. Suppose that (2) holds, $\vartheta(\sigma(t))\leq g(t)$, $p(t)>0$, $m(t)>1$, $r(t)A(t)-\mu(t)>0$ for all $t\in[t_0,\infty)_{\mathrm{T}}$, where $\sigma(t)$ is the forward jump operator and $\mu(t)$ is the graininess function defined on the time scale T. Assume further that there exist positive real valued $\Delta-$ differentiable functions f,F such that (3) holds for all sufficiently large $t_1\in[t_0,\infty)_{\mathrm{T}}$. If the second order dynamic equations

$$\left[r(t)z^{\Delta}(t)\right]^{\Delta}+q(t)\psi\left(\vartheta^{-1}(g(t))\right)z(\sigma(t))=0$$

and

$$\left[r(t)z^{\Delta}(t)\right]^{\Delta}+q(t)\phi\left(\vartheta^{-1}(g(t))\right)\frac{F\left(\vartheta^{-1}(g(t))\right)}{F(\sigma(t))}z(\sigma(t))=0$$

are oscillatory, then Eq. (E) is oscillatory.

Proof. Suppose to the contrary that $x(t)$ is a nonoscillatory (say positive) solution of Eq. (E). Then we may assume that $x(t)>0$, $x(\eta(t))>0$, $x(\vartheta(t))>0$, and $x(g(t))>0$ for $t\in[t_0,\infty)_{\mathrm{T}}$. In view of (E), one can see that (6) holds again, $r(t)y^{\Delta}(t)$ is strictly decreasing, and $y^{\Delta}(t)$ is eventually of one sign. Assume first that $y^{\Delta}(t)>0$ for $t\in[t_1,\infty)_{\mathrm{T}}\subseteq[t_0,\infty)_{\mathrm{T}}$. Following a similar procedure to the proof of first case in Theorem 1, we see that inequality (9) holds. Due to $\vartheta(\sigma(t))\leq g(t)$, and y is strictly increasing, we have from (9) that

$$u^{\Delta}(t)\leq -q(t)\psi\left(\vartheta^{-1}(g(t))\right)-\frac{u^{2}(t)}{r(t)}\frac{y(t)}{y(t)+\mu(t)y^{\Delta}(t)}.$$

The remainder of the proof is similar to that of first case in Theorem 1. Next, assume that $y^{\Delta}(t)<0$ for $t\in[t_1,\infty)_{\mathrm{T}}\subseteq[t_0,\infty)_{\mathrm{T}}$. From the proof of Theorem 1, we see that y/F is nondecreasing and inequality (11) holds. Due to $\vartheta(\sigma(t))\le g(t)$, we derive from (11) that

$$u^{\Delta}(t)\le -q(t)\phi\left(\vartheta^{-1}(g(t))\right)\frac{F\left(\vartheta^{-1}(g(t))\right)}{F(\sigma(t))}-\frac{u^2(t)}{r(t)}\frac{y(t)}{y(t)+\mu(t)y^{\Delta}(t)}.$$

The rest of the proof is similar to that of second case in Theorem 1. We omit the details.

Remark 1. One can obtain similar comparison theorems for the cases where

$$\eta(\sigma(t))\ge g(t),\quad p(t)>1,\quad m(t)>0,\quad r(t)A(t)-\mu(t)>0$$

and

$$\eta(\sigma(t))\le g(t),\quad p(t)>1,\quad m(t)>0,\quad r(t)A(t)-\mu(t)>0\,.$$

In order to obtain these results, it is necessary to define some appropriate functions that to replace the functions ψ and ϕ, and will provide the inequalities arising in the proofs, see for example [2, Lemma 2.5]. The details are left to the reader.

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Comparison of Static Path Planning Models by Time Requirement

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Abstract

In the last decade, much research has been done on localization for wireless sensor networks. The location information of a sensor node is the main problem in processing the sensed data in wireless sensor networks (WSNs). A promising solution for static distributed sensors is localization using mobile beacons. The main challenge in WSN is to design and develop an optimal path planning scheme for a mobile beacon to reduce the time required for location determination. In this paper, we compare six path plans described in the literature with different numbers of nodes in Uniform, Beta, Weibull and Gamma networks. Accuracy-Priority Trilateration is used as the method for position determination. The performance of network localization is evaluated and compared using MATLAB simulations.

Keywords: Path planning, sensor localization, wireless sensor networks

1. Introduction

Wireless sensor network technology has advanced in many different applications, such as environmental monitoring, military battlefield information gathering, remote medical care, disaster relief, and so on. In the field of wireless sensor network research, there are several important issues such as localization, deployment, high bandwidth demand, energy consumption, coverage, etc. Localization is one of the most important topics because location information is usually useful for coverage, deployment, routing, localization service, target tracking and rescue. Most existing localization techniques for WSNs can be classified into two main groups based on a key classification: range-based or range-free. Range-free techniques use only connectivity information between sensors and beacons [1]. Range-based techniques use distance/angle information to locate a node, while range-free techniques use connectivity information to locate it. Range-free techniques are inexpensive, less complex, and do not require additional hardware, but range-based techniques provide better results in terms of localization accuracy [2]. Some common range-based techniques are Received Signal Strength Indicator (RSSI), Time of Arrival (ToA), Time Difference of Arrival (TDoA), and Angle of Arrival (AoA) [3].

Many outdoor positioning applications require the configuration of a GPS receiver to support sensor node positioning. The device GPS enables mobile anchors to determine their position and transmit their current position. Based on the position reports, the surrounding nodes are able to calculate their own position [4]. However, the cost is too high and each sensor node has a significant GPS positioning error, resulting in a large gap in sensor node location information. Localization algorithms can overcome this problem if they are able to estimate the location of sensors based on knowledge of the absolute positions of a few sensors. Generally, these small portions of sensor nodes with known location information are called beacons. Ordinary sensors that need to be located urgently are called unknown nodes.

Localization methods fall into four groups:

(1) static beacons and static nodes such as the methods proposed in [5],
(2) static beacons and mobile nodes such as the systems proposed in [6],
(3) mobile beacons and static nodes such as the methods proposed in [7],
(4) mobile beacons and mobile nodes such as the methods in [8].

The rest of the paper is divided into four sections: Section 2, provides the overview of simulation environment. Localization technique is discussed in Section 3. Finally, the paper is concluded with simulation results in Section 4.

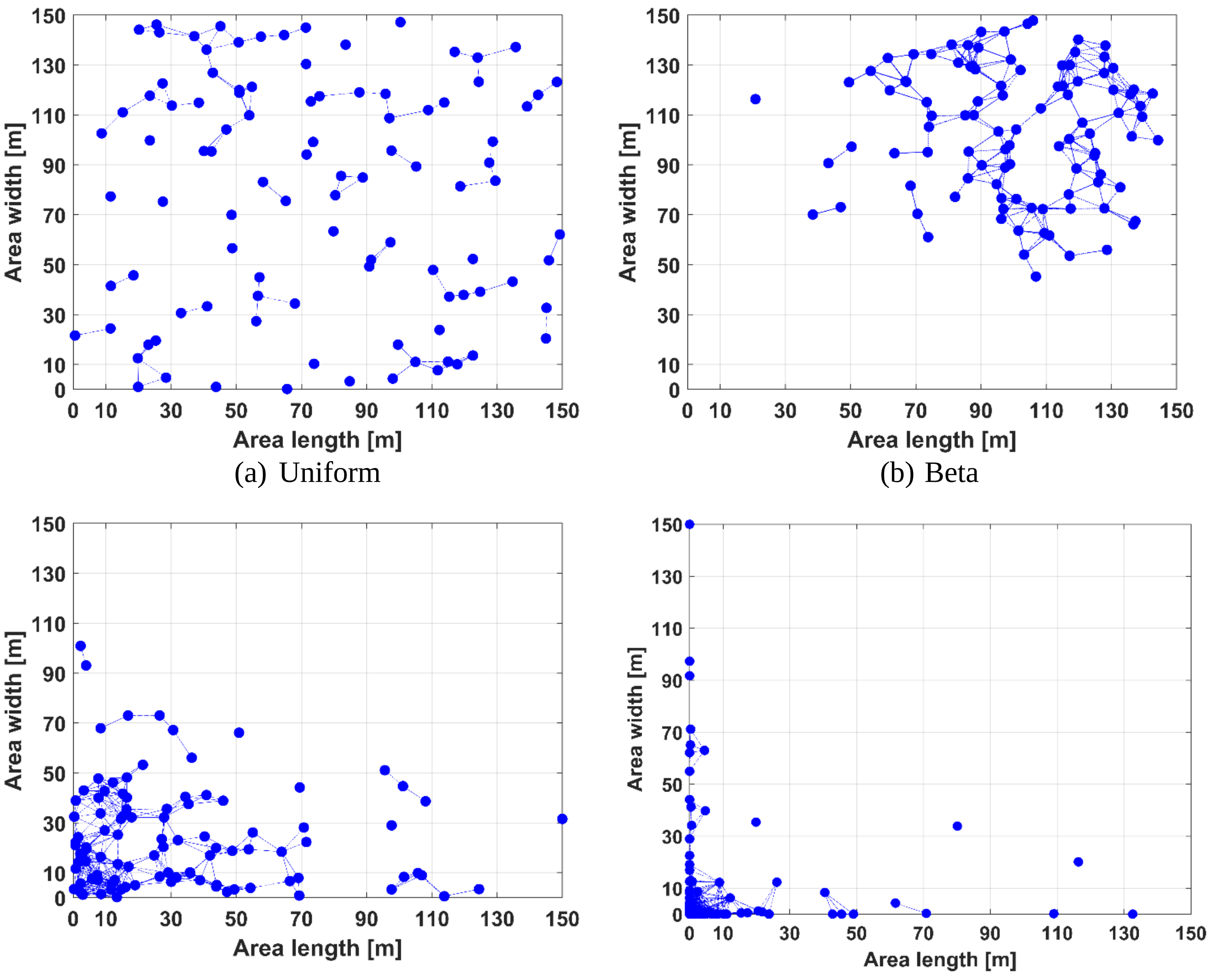
2. Design of Simulation Environment

Node placement is either deterministic or nondeterministic depending on the region in which it is applied. Due to the different detection probabilities, the same sensor placement approach is not suitable for all applications. Some of the applications are well suited and achieve better performance when the sensors are uniformly distributed, but a few require intensive distribution of nodes in particularly sensitive locations, especially the intrusion detection applications. Equations 1 to 4 apply respectively to the Uniform, Beta, Weibull and Gamma probability distributions [9-12].

Table 1. Distributions and density functions

Distributions	Density Functions	Eqs.
Uniform Distribution	$f(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b \\ 0, & \text{otherwise} \end{cases}$	(1)
Beta Distribution	$f(x) = \begin{cases} \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$ $B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx$	(2)
Weibull Distribution	$f(x) = \frac{\alpha x^{\alpha-1}}{\lambda^{\alpha}} e^{\left(\frac{x}{\lambda}\right)^{\alpha}}$	(3)
Gamma Distribution	$f(x) = \begin{cases} \frac{\beta^{\alpha} x^{\alpha-1} x^{-\beta x}}{\Gamma(\alpha)}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$	(4)

Figure 1 shows the Uniform, Beta, Weibull and Gamma distributions of 100 nodes in a 150x150 area. Sensors within the communication range are shown with blue dashed lines. For the distributions other than the uniform distribution, it can be seen that the sensors are congregated in a certain area.

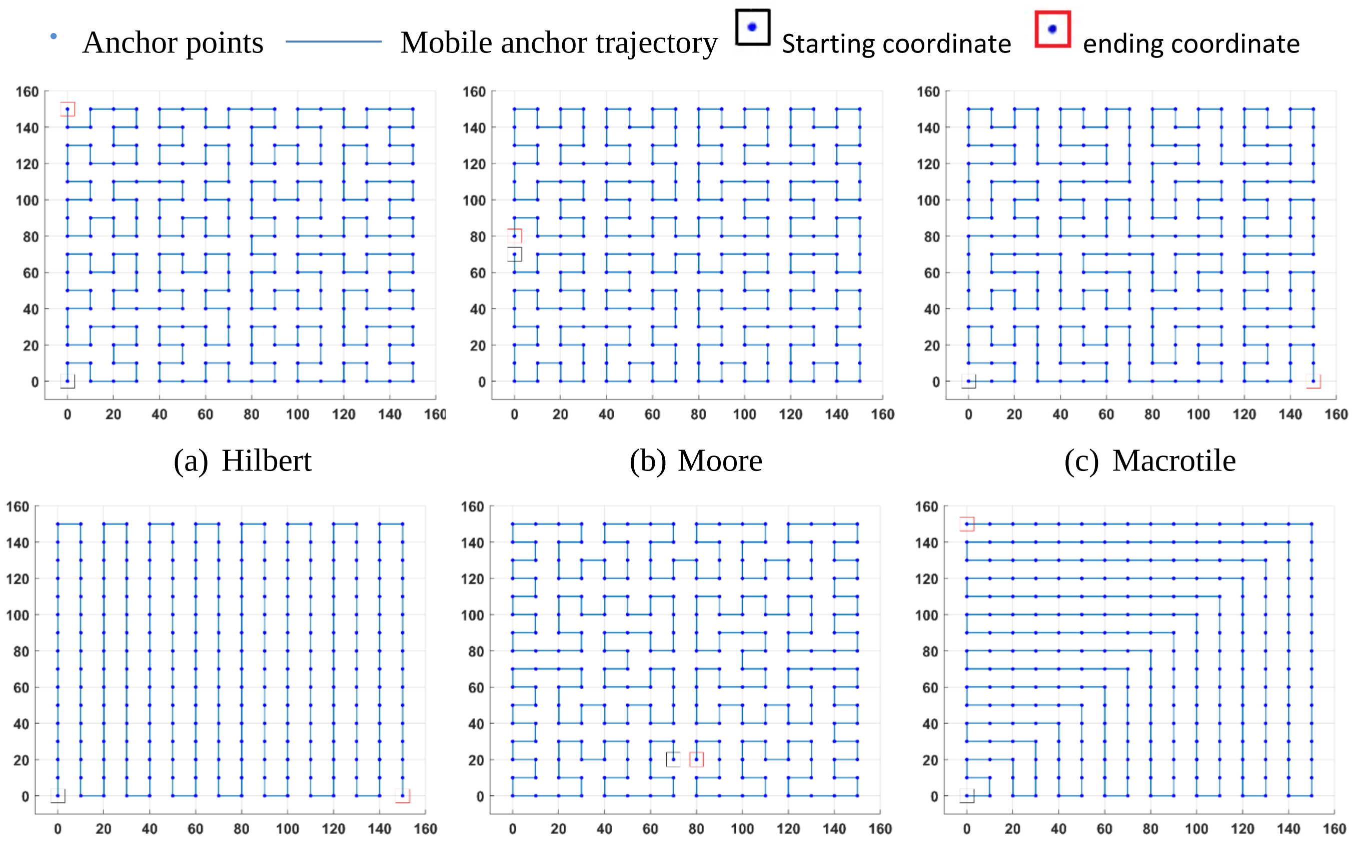


(a) Uniform (b) Beta

(c) Weibull (d) Gamma

Figure 1: Distribution of 100 nodes (a) Uniform, (b) Beta, (c) Weibull, and (d) Gamma

One of the biggest challenges in applications that use a mobile anchor and unknown static nodes is designing the path that the mobile anchor will follow. In this paper, we consider six models for planning mobile paths based on a grid layout (Figure 2). The most obvious difference between the path models is that the start and end coordinates are different and they follow different paths in the grid layout.



(a) Hilbert (b) Moore (c) Macrotile

(d) Scan (e) Beta (f) Gamma

Figure 2: Paths, (a) Hilbert, (b) Moore, (c) Macrotile, (d) Scan, (d) Beta and (f) Gamma

3. Localization technique

Trilateration is the most commonly used technique to estimate the position of unknown nodes in WSN. Each unknown node receives beacon messages from an anchor node. Once an unknown node receives adequate number of beacon messages, it starts the process of localization. Each unknown node calculates its respective distances using the received beacon messages. The position of the node is calculated using trilateration.

Considering the 2-D environment for the unknown sensor node S at position (x, y) and the anchor nodes at positions (x_0, y_0), (x_1, y_1), (x_2, y_2) and respective distances d_0, d_1 and d_2 the position of S (in Figure 3) can be estimated by solving the following equations.

$$d_0 = \sqrt{(x_0 - x)^2 + (y_0 - y)^2} \tag{5}$$

$$d_1 = \sqrt{(x_1 - x)^2 + (y_1 - y)^2} \tag{6}$$

$$d_2 = \sqrt{(x_2 - x)^2 + (y_2 - y)^2} \tag{7}$$

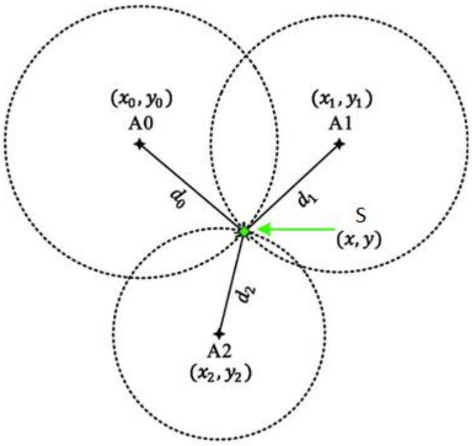


Figure 3: Accuracy-Priority Trilateration (APT)

Accuracy-Priority Trilateration (APT) is a technique that estimates the location of sensors by considering the three closest beacon messages among the received messages. This technique provides high localization accuracy compared to other trilateration techniques.

To simplify the complexity of the environment, the following assumptions were made and a simulation environment was used in this paper;

- The unknown nodes and the mobile anchor are two types of sensor nodes in the network
- 100 unknown nodes are randomly introduced into the network with distributions of Uniform, Beta, Weibull and Gamma
- All sensors in WSN are identical and homogeneous
- The distance between two consecutive anchor points in the trajectory is set to 10 m
- A mobile anchor is ready to traverse the whole network in straight lines, depending on each path pattern
- There are no obstacles in the environment
- The mobile anchor has sufficient energy to move and broadcast location information during the localization process
- The communication range (CR) of each sensor node in the network is 12.5m, 15.625m, 18.75m and 25m
- Each mobile node and the anchor node communicate with each other when both nodes are in their transmission range
- Each Monte Carlo simulation is created with a specific set of input parameters and run 100 times with different fields. The results are then averaged.

4. Simulation Results and Conclusions

The 150m x150m region of the simulation field was created using uniform, Beta, Weibull, and Gamma distributions. We tested six path plans available in the literature using 100 nodes and one mobile node. Table 2 shows the maximum time required to find the sensor nodes. Each Monte Carlo simulation is created with a specific set of input parameters and run 100 times with different fields. The results are then averaged. The simulations were repeated for the 12.5m (R12), 15.625m (R15), 18.75m (R18) and 25m (R25) coverage areas for each distribution. From the table, it can be seen that the Beta, Gamma and Weibull distributions provide the best time for Hilbert, and Scan statics path planning respectively. The environment with obstacles is not considered in this paper. They will be addressed in our future work.

Table 2: The maximum time required to locate the sensor nodes

	Coverage	Hilbert	Moore	Macrotile	Scan	Beta	Gamma
Uniform	R12	254.31	254.95	254.27	254.29	255.36	**253.68**
	R15	254.11	255.07	254.08	**253.99**	255.77	254.11
	R18	254.15	254.78	254.08	254.03	255.59	**253.69**
	R25	254.43	255.09	254.05	**253.99**	255.51	254.04
Weibull	R12	247.60	252.63	229.12	**223.56**	254.54	245.11
	R15	248.95	254.92	234.04	**228.04**	252.64	244.14
	R18	247.58	254.53	232.80	**230.55**	254.26	243.88
	R25	246.35	252.35	233.09	**228.51**	254.08	243.97
Beta	R12	**237.62**	241.61	240.54	248.35	240.18	248.41
	R15	**240.10**	243.66	240.36	248.60	241.29	248.75
	R18	**239.13**	242.61	241.67	248.46	243.92	248.06
	R25	**237.19**	241.22	241.44	248.21	244.41	248.54
Gamma	R12	248.79	249.04	225.51	**225.29**	243.40	245.19
	R15	244.43	247.06	231.79	**230.70**	247.45	244.62
	R18	250.42	251.80	226.15	**225.71**	246.65	245.54
	R25	247.49	252.22	228.12	**228.03**	248.47	242.43

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A new approach for solving distributed order fractional partial differential equations

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Abstract

In this study, a numerical scheme based on fractional Bernoulli wavelets is introduced for solving distributed –order fractional partial differential equations, where the fractional derivative is considered in the Caputo definition. For this purpose, the spectral collocation method via Gauss quadrature is employed to reduce the main problem to an algebraic linear system.

Keywords: distributed-order fractional partial differential equation, Bernoulli wavelets, collocation method, Gauss quadrature.

1. Introduction

Distributed-order fractional differential equations (FDEs) can be seen as a generalization of constant-order FDEs. Distributed-order FDEs might contain fractional derivatives which are integrated over the order of the differentiation within a given interval. These differential equations are widely used in different disciplines such as control systems, distributed order system identifcation, diffusion- wave phenomena, dielectric material, viscoelasticity model, and electronic oscillator. Analytic solutions for distributed-order FDEs have been deeply studied. The existence and uniqueness properties of solutions of distributed-order FDEs can be found, for instance, in [1]-[2]. In [3], the authors study the well-posedness of the solutions. In general, it is not easy to determine an exact solution for a distributed-order FDE.
Therefore several numerical methods for solving distributed-order FDEs have been proposed, such as Legendre spectral element method, block-pulse wavelet method, hybrid of block-pulse functions and Taylor polynomials, finite volume method, finite difference schemes, hybrid functions of block-pulse functions and Bernoulli polynomials, the Petrov-Galerkin spectral method, Chebyshev collocation method, a method using the trapezoidal quadrature rule, and Laguerre Petrov-Galerkin spectral method [4]-[5]. Different from the distributed-order fractional ordinary differential equations, there are only a few numerical methods available for solving distributed-order fractional partial differential equations (DOFPDEs), such as meshless method [6], finite element method [7], fractional pseudo-spectral method [8], and Legendre operational matrix methods [9]. In recent decades, wavelets, and their operational matrices of fractional and non-fractional derivetives and integration, played important role in numerically solving the problems in fractional calculus. The

solutions of fractional differential equations (FDEs) can contain some fractional power terms which cannot be approximated by using classical integer order bases. To increase the efficiency of the numerical methods, fractional-order polynomials and wavelets were introduced by applying the transformation $x \to t^{\alpha},\ \ \alpha > 0$ of variables to the integer-order polynomials and wavelets.

Main problem

In the current research, we consider the following two-dimensional DOFPDE:

$$\int_0^1 \rho(\alpha)\, {}_cD_t^{\alpha} u(x,t)\, d\alpha = u_{xx}(x,t) + H(x,t), \qquad (x,t) \in [0,1] \times [0,1], \tag{1}$$

subject to the following initial and boundary conditions

$$u(x,0) = f(x), \qquad u(0,t) = q_0(t), \qquad u(1,t) = q1(t), \tag{2}$$

where H, f, q_0 and q_1 are continuous functions, and $\rho(\alpha)$ is a continuous non-negative weight function satisfying [2]:

$$\rho(\alpha) \geq 0, \qquad \int_0^1 \rho(\alpha)\, \mathrm{d}\alpha = \mu > 0.$$

The proofs for the existence, continuity, smoothness and uniqueness of the solution were given in [2]. The aim of this study is introducing a new approach based on fractional order Bernoulli wavelets. The operational matrices of Bernoulli wavelets and the Gauss-Legendre quadrature are applied to acquire the approximated solution of problem (1)-(2).

2. Preliminaries on fractional calculus

In this section, we present some basic definitions and concepts on fractional calculus that are essential for subsequent discussion. There are various definitions for fractional integration and derivative operators. However, the fractional Riemann-Liouville integration and fractional Caputo derivative operators have been used in this study.

Definition 2.1. The Riemann-Liouville fractional integral operator of nonnegative order α is defined as [10]:

$$J^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, dt, \qquad x > 0, \tag{3}$$

where $J^0 f(x) = f(x)$.

2.1. Riemann-Liouville fractional integral for polynomials

The Riemann-Liouville fractional integrals for the polynomials are defined as

$$J^{\alpha}x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}x^{\beta+\alpha}, \qquad \beta > -1. \tag{4}$$

Also the mentioned operator is linear, that is for real constant λ we have

$$J^{\alpha}\{\lambda f(x) + g(x)\} = \lambda J^{\alpha}\{f(x)\} + J^{\alpha}\{g(x)\}. \tag{5}$$

Definition 2.2. The Caputo fractional derivative operator of nonnegative order α is defined as [10]

$$_{c}D^{\alpha}f(x) = \frac{1}{\Gamma(n-\alpha)}\int_0^x \frac{f^{(n)}(t)}{(x-t)^{\alpha+1-n}}\,dt, \qquad n-1<\alpha\le n, \qquad n\in\mathbb{N}. \tag{6}$$

2.2. Caputo fractional derivative for polynomials

For the Caputo derivative we have [10]:

$$_{c}D^{\alpha}x^{\beta} = 0, \qquad \beta\in\mathbb{N}_0, \qquad \beta<\lceil\alpha\rceil,$$

and

$$_{c}D^{\alpha}x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}x^{\beta-\alpha}, \qquad \beta\in\mathbb{N}_0, \qquad \beta\ge\lceil\alpha\rceil \quad or \quad \beta\in\mathbb{R}-\mathbb{N}_0, \qquad \beta>\lfloor\alpha\rfloor.$$

Similar to the Riemann-Liouville fractional integral operator, the Caputo fractional derivative operator is linear, that is, for real constant λ, we have

$$_{c}D^{\alpha}\{\lambda f(x)+g(x)\} = \lambda \;\; _{c}D^{\alpha}\{f(x)\} + \;\; _{c}D^{\alpha}\{g(x)\}.$$

2.3. Relations between Reimann-Liouville fractional integral and Caputo fractional derivative

The relations between Reimann-Liouville fractional integral and Caputo fractional derivative operators can be addressed by the following identities [11]:

$$_{c}D^{\alpha}J^{\alpha}f(x) = f(x), \tag{7}$$

and

$$J^{\alpha}D_c^{\alpha}f(x) = f(x) - \sum_{j=0}^{n-1}\frac{f^{(j)}(0)}{j!}x^j. \tag{8}$$

3. Review on Bernoulli wavelets

In this section definitions of Fractional Bernoulli Wavelets (FBWs) and their operational matrix of Caputo fractional derivative are described.

3.1. BWs and FBWs

Definition 3.1. BWs of order m, which are denoted by $\psi_{nm}(t) = \psi(k, \hat{n}, m, t)$, consist of four arguments, k: a positive integer, $n = 1,2,\dots,2^{k-1}, \hat{n} = n-1$ and t is the normalized time. These wavelets are defined on the interval $[0, 1)$ ([12]) as:

$$\psi_{nm}(t) = 2^{\frac{k-1}{2}} \tilde{B}_m(2^{k-1}t - \hat{n})\chi_{\left[\frac{\hat{n}}{2^{k-1}}, \frac{\hat{n}+1}{2^{k-1}}\right)}, \tag{9}$$

where $\tilde{B}_0(t) = 1$ and

$$\tilde{B}_0(t) = \frac{B_m(t)}{\Lambda_m}, \qquad m > 0, \tag{10}$$

and $\Lambda_m = \sqrt{\frac{(-1)^{m-1}(m!)^2}{(2m)!}\vartheta_{2m}}$ is the normality coefficient. The functions B_m, $m = 0,1,\dots,M-1$ are known Bernoulli polynomials, defined as

$$B_m(t) = \sum_{j=0}^{m} \binom{m}{j} \vartheta_{m-j} t^j \tag{11}$$

where $\vartheta_j \coloneqq B_j(0)$ are the Bernoulli numbers. Therefore Bernoulli wavelets for $m > 0$ can be rewritten as

$$\psi_{n,m}(t) = \Theta_m \sum_{j=0}^{m} \binom{m}{j} \vartheta_{m-j} 2^{j(k-1)} \left(t - \frac{\hat{n}}{2^{k-1}}\right)^j \chi_{\left[\frac{\hat{n}}{2^{k-1}}, \frac{\hat{n}+1}{2^{k-1}}\right)}, \tag{12}$$

where$\Theta_m = \sqrt{\frac{2^{k-1}\,(2m!)}{(-1)^{m-1}(m!)^2\vartheta_{2m}}}$ and $\psi_{n,0}(t) = 2^{\frac{k-1}{2}}\chi_{\left[\frac{\hat{n}}{2^{k-1}}, \frac{\hat{n}+1}{2^{k-1}}\right)}$.

Definition 3.2. Fractional Bernoulli Wavelets are denoted by $\psi_{n,m}^{\alpha}$ and constructed by changing the variable t to x^{α}, $(\alpha > 0)$ on the BWs [12], that is

$$\psi_{n,m}^{\alpha} \coloneqq \psi_{n,m}(x^{\alpha}) = \Theta_m \sum_{j=0}^{m} \binom{m}{j} \vartheta_{m-j} 2^{j(k-1)} \left(x^{\alpha} - \frac{\hat{n}}{2^{k-1}}\right)^j \chi_{\left[\left(\frac{\hat{n}}{2^{k-1}}\right)^{1/\alpha}, \left(\frac{\hat{n}+1}{2^{k-1}}\right)^{1/\alpha}\right)}. \tag{13}$$

Remak. The Bernoulli polynomials satisfies the following relation [12]:

$$\int_0^1 B_m(x)B_n(x)dx = \frac{(-1)^{n-1}m!\,n!}{(m+n)!}\vartheta_{m+n}, \qquad m,n \geq 1. \tag{14}$$

Thus these polynomials are not orthogonal, consequently the FBWs, which are constructed by Bernoulli polynomials, are not orthogonal, too.

3.2. Function approximation by FBWs

A function $f \in L^2[0,1]$ could be approximated by FBWs, as

$$f(x) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_{n,m}\psi_{n,m}^{\alpha}(x), \tag{15}$$

by truncating the infinite series (15) in some suitable k and M, we get

$$f(x) \simeq \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}^{\alpha}(x) = C_M^T\Psi_{k,M}^{\alpha}(x), \tag{16}$$

where C_M and $\Psi_{k,M}^{\alpha}$ are $2^{k-1} \times M$-dimensional column vectors and defined as

$$C_M = \left(c_{1,0}, \dots, c_{1,M-1}, \dots, c_{2^{k-1},0}, \dots, c_{2^{k-1},M-1}\right)^T, \tag{17}$$

$$\Psi_{k,M}^{\alpha} = \left(\psi_{1,0}^{\alpha}, \dots, \psi_{1,M-1}^{\alpha}, \dots, \psi_{2^{k-1},0}^{\alpha}, \dots, \psi_{2^{k-1},M-1}^{\alpha}\right)^T. \tag{18}$$

In order to determine the coefficients in (16), we put

$$\eta_{ij} \coloneqq \int_{\sqrt[\alpha]{\frac{\hat{n}}{2^{k-1}}}}^{\sqrt[\alpha]{\frac{\hat{n}+1}{2^{k-1}}}} f(x)\psi_{i,j}^{\alpha}(x)x^{\alpha-1}\,dx, \tag{19}$$

and

$$\lambda_{n,m}^{i,j} \coloneqq \int_{\sqrt[\alpha]{\frac{\hat{n}}{2^{k-1}}}}^{\sqrt[\alpha]{\frac{\hat{n}+1}{2^{k-1}}}} \psi_{n,m}^{\alpha}(x)\psi_{i,j}^{\alpha}(x)x^{\alpha-1}\,dx. \tag{20}$$

Now substituting (16) in (19), we get

$$\eta_{ij} \simeq \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \int_{\sqrt[\alpha]{\frac{\hat{n}}{2^{k-1}}}}^{\sqrt[\alpha]{\frac{\hat{n}+1}{2^{k-1}}}} \psi_{n,m}^{\alpha}(x)\psi_{i,j}^{\alpha}(x)x^{\alpha-1}dx = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}\ \lambda_{n,m}^{i,j} = C_M^T \Lambda_M^{i,j},$$

where

$$\Lambda_M^{i,j} = \left(\lambda_{1,0}^{i,j}, \dots, \lambda_{1,M-1}^{i,j}, \dots, \lambda_{2^{k-1},0}^{i,j}, \dots, \lambda_{2^{k-1},M-1}^{i,j}\right)^T,$$

so by putting

$$T_M = \left(\eta_{1,0}, \dots, \eta_{1,M-1}, \dots, \eta_{2^{k-1},0}, \dots, \eta_{2^{k-1},M-1}\right)^T,$$

and

$$\Lambda_M = \left(\Lambda_M^{1,0}, \dots, \Lambda_M^{1,M-1}, \dots, \Lambda_M^{2^{k-1},0}, \dots, \Lambda_M^{2^{k-1},M-1}\right)_{(2^{k-1}\times M)\times(2^{k-1}\times M)},$$

the vector C_M is evaluated by

$$C_M^T = T_M \Lambda_M^{-1}\ . \tag{21}$$

The two variable function $v(x,t)$ could be approximated by two dimensional FBWs as

$$v(x,t) = \sum_{n=1}^{2^{k_2-1}} \sum_{m=0}^{M_2-1} \sum_{r=1}^{2^{k_1-1}} \sum_{s=0}^{M_1-1} v_{r,s,n,m}\, \psi_{r,s}^{\alpha}(x)\psi_{n,m}^{\alpha}(t) = \Psi_{k_1,M_1}^{\alpha}(x) V \left(\Psi_{k_1,M_1}^{\alpha}(x)\right)^T, \tag{22}$$

where V is $(2^{k_1-1} \times M_1) \times (2^{k_2-1} \times M_2)$ dimensional coefficient matrix, where

$$v_{r,s,n,m} = \alpha^2 \langle\langle v(x,t), \psi_{r,s}^{\alpha}(t)\, \rangle_{t^{\alpha-1}}, \psi_{n,m}^{\alpha}(x)\rangle_{x^{\alpha-1}}\ , \tag{23}$$

$$n = 1,2,\dots,2^{k_1-1}, \qquad r = 1,2,\dots,2^{k_2-1}, \qquad m = 0,1,\dots,M_1-1, \qquad s = 0,1,\dots,M_2-1.$$

It is clear that for $k_1 = k_2 = k$ and $M_1 = M_2 = M, V$ is $(2^{k-1} \times M)$ dimensional square coefficient matrix.

Theorem 3.1. ([12]) Let $u(x,t) \in C^{M_1,M_2}(D)$ be approximated by two dimensional FBWs as $u(x,t) \simeq u_{k_1,M_1,k_2,M_2}(x,t) = \left(\Psi_{k_2,M_2}^{\alpha}\right)^T(t) V \Psi_{k_1,M_1}^{\alpha}(x)$, there exist constants $C_i \in \mathbb{R}^+$, $i = 1,2,3$ such that

$$\left\| u(x,t) - u_{k_1,M_1,k_2,M_2}(x,t) \right\|_2 \le \frac{C_1}{A_1} + \frac{C_2}{A_2} + \frac{C_3}{A_1 A_2}, \tag{24}$$

where $A_i = M_i!\, 2^{M_i(k_i+1)-1}$, $i = 1,2$.

3.3. Operational matrix of Riemann-Liouville fractional integration

The Riemann-Liouville fractional integration of Ψ^{α} can be obtained as

$$J^{\xi}\Psi^{\alpha}(x) = \mathcal{F}^{\xi,\alpha}\Psi^{\alpha}(x), \tag{25}$$

where $\mathcal{F}^{\xi,\alpha}$ is relative operational square matrix of dimension $2^{k-1} \times M$ and could be evaluated by using equation (4) and (13) as follows

$$J^{\xi}\psi_{n,m}^{\alpha}(x) = \Theta_m \sum_{j=0}^{m} \binom{m}{j} \vartheta_{m-j} 2^{j(k-1)} J^{\xi}\left(x^{\alpha} - \frac{\hat{n}}{2^{k-1}}\right)^{j}, \tag{26}$$

$$\left(\frac{\hat{n}}{2^{k-1}}\right)^{1/\alpha} \leq x \leq \left(\frac{\hat{n}+1}{2^{k-1}}\right)^{1/\alpha}.$$

On the other hand

$$J^{\xi}\left(x^{\alpha} - \frac{\hat{n}}{2^{k-1}}\right)^{j} = \sum_{i=0}^{j} \binom{j}{i} \left(-\frac{\hat{n}}{2^{k-1}}\right)^{j-i} \frac{\Gamma(\alpha i + 1)}{\Gamma(\alpha i + \xi + 1)} x^{\alpha i + \xi}. \tag{27}$$

Thus, using equation (26)-(27), we can write

$$J^{\xi}\psi_{n,m}^{\alpha}(x) = \Theta_m \sum_{j=0}^{m} \sum_{i=0}^{j} A_{j,i} x^{\alpha i + \xi}, \tag{28}$$

where

$$A_{j,i} = \binom{m}{j} \vartheta_{m-j} 2^{j(k-1)} \binom{j}{i} \left(-\frac{\hat{n}}{2^{k-1}}\right)^{j-i} \frac{\Gamma(\alpha i + 1)}{\Gamma(\alpha i + \xi + 1)}.$$

Now we expand $x^{\alpha i + \xi}$ in terms of FBWs:

$$x^{\alpha i + \xi} \simeq \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} d_{n,m}^{\xi,\alpha} \psi_{n,m}^{\alpha}(x), \tag{29}$$

By (28)-(29), we get

$$J^{\xi}\psi_{n,m}^{\alpha}(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} \eta_{n,m,j,i}^{\xi,\alpha} \psi_{n,m}^{\alpha}(x),$$

where $\eta_{n,m,j,i}^{\xi,\alpha} = \sum_{j=0}^{m} \sum_{i=0}^{j} A_{j,i} d_{n,m}^{\xi,\alpha}$. Thus we have

$$\mathcal{F}^{\xi,\alpha} = \begin{pmatrix} \mathcal{F}_1^{\xi,\alpha} & 0 & \cdots & 0 \\ 0 & \mathcal{F}_2^{\xi,\alpha} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{F}_{2^{k-1}}^{\xi,\alpha} \end{pmatrix}_{(2^{k-1}\times M)\times(2^{k-1}\times M)}, \tag{30}$$

where

$$\mathcal{F}_l^{\mathcal{F}^{\xi,\alpha}} = \left(\eta_{l,0,0,0}^{\xi,\alpha}, \eta_{l,1,0,0}^{\xi,\alpha}, \dots, \eta_{l,M-1,i,j}^{\xi,\alpha}\right)_{1\times M}$$

and $\mathbf{0}$ is a $1 \times M$ -dimensional row matrix which its all entries are zero.

4. Numerical implementation

By using properties of FBWs, their operational matrices of derivative and Riemann-Liouville fractional integration, collocation method and Gauss-Legendre quadrature, a new approach is introduced in this section for solving DOFPDEs. For this purpose, we first expand $\frac{\partial^3 u(x,t)}{\partial x^2 \partial t}$ by FBWs of order as

$$\frac{\partial^3 u(x,t)}{\partial x^2 \partial t} \simeq \Psi_{k_1,M_1}^{\alpha}(x)^T U \Psi_{k_2,M_2}^{\alpha}(t), \tag{37}$$

where U is $2^{k_1-1}M_1 \times 2^{k_2-1}M_2$-dimensional unknown coefficients matrix. Integrating (37) with respect to t and using the initial condition, we have

$$\frac{\partial^2 u(x,t)}{\partial x^2} \simeq \Psi_{k_1,M_1}^{\alpha}(x)^T U I^1 \Psi_{k_2,M_2}^{\alpha}(t) + f''(x), \tag{38}$$

and by integrating (38) twice with respect to x and using the Dirichlet boundary conditions, we get

$$u(x,t) \simeq I^2 \Psi_{k_1,M_1}^{\alpha}(x)^T U I^1 \Psi_{k_2,M_2}^{\alpha}(t) - x I^2 \Psi_{k_1,M_1}^{\alpha}(1)^T U I^1 \Psi_{k_2,M_2}^{\alpha}(t)$$

$$+(1-x)\big(q_0(t) - q_0(0)\big) + x\big(q_1(t) - q_1(0)\big) + f(t), \tag{39}$$

Now for obtaining ${}_cD_t^{\alpha}\, u(x,t)$, we can write

$${}_cD_t^{\alpha}\, u(x,t) \simeq I^2 \Psi_{k_1,M_1}^{\gamma}(x)^T U I^{1-\alpha} \Psi_{k_2,M_2}^{\gamma}(t) - x I^2 \Psi_{k_1,M_1}^{\gamma}(1)^T U I^{1-\alpha} \Psi_{k_2,M_2}^{\gamma}(t)$$

$$+(1-x)\,{}_cD_t^{\alpha}\{q_0(t)\} + x\,{}_cD_t^{\alpha}\{q_1(t)\}. \tag{40}$$

Now we apply the Gauss-Legendre quadrature to determine the left hand side integral of equation (1):

$$\int_0^1 \rho(\alpha)\ {}_cD_t^{\alpha}\{u(x,t)\}\,dt \simeq \frac{1}{2}\sum_{j=1}^{n}\omega_j\rho\left(\frac{\eta_j+1}{2}\right)\ {}_cD_t^{\frac{\eta_j+1}{2}}\{u(x,t)\}, \tag{41}$$

where ω_j and η_j are the weights and the nodes of Gauss-Legendre quadratre, respectively. Also ${}_cD_t^{\frac{\eta_j+1}{2}}\{u(x,t)\}$ is achivied by (41). So by substituting equations (38)-(41) in the main problem, we obtain an algebraic equation. For solving it, we employ collocation method, that is; the algebraic equation is discritize in the collocation nodes (x_i, t_j), where

$$\begin{cases} x_i = \left(\dfrac{2i-1}{2^{k_1}M_1}\right)^{1/\gamma}, & i = 1, \dots, 2^{k_1}M_1, \\ t_j = \left(\dfrac{2j-1}{2^{k_2}M_2}\right)^{1/\gamma}, & j = 1, \dots, 2^{k_1}M_1. \end{cases}$$

Therefore an algebraic system with $2^{k_1+k_1-2}M_1M_2$ linear equations with unknown coefficients, which can be solved by some iterative technique.

5. Illustrative example

In this section for showing the accuracy and efficiency of the proposed method, we solve an example.

Example 1. Consider the following DOFPDE

$$\int_0^1 \Gamma\left(\frac{5}{2}-\alpha\right)\ {}_cD_t^{\alpha}u(x,t)dt = u_{xx}(x,t) + H(x,t),$$

where

$$H(x,t) = \frac{\sqrt{t}\,(x-1)^2\left[3\sqrt{\pi}\,(t-1)(x-1)^2x^2 - 8t(15x^2-10x+1)\mathrm{Ln}\,(t)\right]}{4Ln(t)},$$

subject to the initial and boundary conditions

$$u(x,0) = 0, \qquad u(0,t) = 0, \qquad u(1,t) = 0, \qquad (x,t) \in [0,1] \times [0,1],$$

where the exact solution of this problem is $u(x,t) = t^{3/2}\,x^2\,(x-1)^4$.

We put $k = k_1 = k_2$ and $M = M_1 = m_2$ and solved this problem for $\gamma = 0.5, k = 2$ and $M = 2,3,4$. The numerical results are tabulated in tables $1-2$.

Table 1. Numerical solutions of example 1 for $\gamma = 0.5, t = 1, k = 2$ and $M = 2,3,4$.

x	$M=2$	$M=3$	$M=4$	$Exact$
0	0.000000	0.000000	0.000000	0.000000
0.1	0.006513	0.006558	0.006561	0.006561
0.2	0.016402	0.016370	0.016386	0.016384
0.3	0.021584	0.021615	0.021609	0.021609
0.4	0.020705	0.020729	0.020736	0.020736
0.5	0.015596	0.015621	0.015625	0.015625
0.6	0.009244	0.009224	0.009216	0.009216
0.7	0.003881	0.003977	0.003969	0.003969
0.8	0.001109	0.001019	0.001024	0.001024
0.9	0.000007	0.000084	0.000081	0.000081
1	0.000000	0.000000	0.000000	0.000000

Table 2. ϵ_{L^2} and ϵ_{L^∞} of example 1 for $\gamma = 0.5$, $t = 1$, $k = 2$ and $M = 2,3,4$.

	$M=2$	$M=3$	$M=4$
ϵ_{L^2}	1.689×10^{-5}	7.240×10^{-6}	1.349×10^{-7}
ϵ_{L^∞}	7.772×10^{-5}	1.086×10^{-6}	5.149×10^{-7}

6. Conclusion

In this paper, a numerical method for solving distributed –order Caputo fractional partial differential equations is presented. First fractional Bernoulli wavelets and their operational matrices of integration are defined and then applied to transform the main problem to an algebraic system. The spectral collocation method and Gauss quadrature are employed for solving the obtained system. The introduced method is simple and applicable and can be extended for solving other classes of distributed-order fractional equations.

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On Summability of Infinite Series and Fourier Series

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Abstract
In this paper, two known theorems on absolute Riesz summability factors of infinite and Fourier series are generalized to $|A, p_n, \beta;\gamma|_k$ summability method by using δ-quasi-monotone sequences.

Keywords: δ-quasi-monotone sequences, infinite series, Fourier series.

1. Introduction

A sequence (B_n) is said to be δ-quasi-monotone if $B_n \to 0$, $B_n > 0$ ultimately and $\Delta B_n \geq -\delta_n$, where $\delta = (\delta_n)$ is a sequence of positive numbers (see [1]). Let $\sum a_n$ be a given infinite series with the partial sums (s_n). Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \ as \ \ n \to \infty, \ (P_{-m} = p_{-m} = 0, \ m \geq 1).$$

The sequence-to-sequence transformation

$$\sigma_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v$$

defines the sequence (σ_n) of the $(\bar{N}, p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [2]). The series $\sum a_n$ is said to be summable $\left|\bar{N}, p_n\right|_k$, $k \geq 1$, if (see [3])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} \left|\sigma_n - \sigma_{n-1}\right|^k < \infty,$$

and it is said to be summable $\left|\bar{N}, p_n, \beta;\gamma\right|_k$, $k \geq 1, \gamma \geq 0$ and β is real number, if (see [4])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\beta(\gamma k + k - 1)} \left|\sigma_n - \sigma_{n-1}\right|^k < \infty.$$

Let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of non-zero diagonal entries. The series $\sum a_n$ is said to be summable $|A, p_n, \beta;\gamma|_k$, $k \geq 1$, $\gamma \geq 0$ and β is real number, if (see [5])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\beta(\gamma k + k - 1)} | A_n(s) - A_{n-1}(s) |^k < \infty,$$

where

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v , \qquad n = 0,1,...$$

If we take $\beta = 1$, then $|A, p_n, \beta; \gamma|_k$ summability reduces to $|A, p_n; \gamma|_k$ summability method (see [6]). If we take $\beta = 1$ and $\gamma = 0$, then $|A, p_n, \beta; \gamma|_k$ summability reduces $\left|A, p_n\right|_k$ summability method (see [7]). Also, if we take $\beta = 1$, then $\left|\bar{N}, p_n, \beta; \gamma\right|_k$ summability reduces to $\left|\bar{N}, p_n; \gamma\right|_k$ summability method (see [8]). Finally, if we take $\beta = 1$ and $\gamma = 0$, then $\left|\bar{N}, p_n, \beta; \gamma\right|_k$ summability reduces to $\left|\bar{N}, p_n\right|_k$ summability method.

If we write $X_n = \sum_{v=0}^{n} p_v / P_v$, then (X_n) is a positive increasing sequence tending to infinity with n.

2. Known Results

In [9], Bor has proved the following theorem and lemmas.

Theorem 2.1. Let $\lambda_n \to 0$ as $n \to \infty$ and (p_n) be a sequence of positive numbers such that $P_n = O(np_n)$ as $n \to \infty$. Suppose that there exists a sequence of numbers (B_n) which is δ-quasi-monotone with $\sum nX_n\delta_n < \infty$, $\sum B_n X_n$ is convergent and $\left|\Delta\lambda_n\right| \le \left|B_n\right|$ for all $n.$ If the condition

$$\sum_{n=1}^{m} \frac{p_n}{P_n} \left|t_n\right|^k = O(X_m) \quad as \quad m \to \infty \tag{1}$$

is satisfied, where (t_n) denotes the n-th $(C,1)$ mean of the sequence (na_n), then the series $\sum a_n \lambda_n$ is summable $\left|\bar{N}, p_n\right|_k$, $k \ge 1$.

Lemma 2.1. Under the conditions of Theorem 2.1, we have

$$|\lambda_n| X_n = O(1) \quad as \quad n \to \infty. \tag{2}$$

Lemma 2.2. Let (B_n) is δ-quasi-monotone with $\sum nX_n\delta_n < \infty$, $\sum B_n X_n$ is convergent, then

$$mX_m B_m = O(1) \quad as \quad \mathrm{m} \to \infty, \tag{3}$$

$$\sum_{n=1}^{\infty} nX_n \left|\Delta B_n\right| < \infty. \tag{4}$$

3. Main Result

There are some different works on absolute matrix summability of infinite series (see [10-19]). Now, let us mention some notations. Let $A=\left(a_{nv}\right)$ be a normal matrix, two lower semimatrices $\overline{A}=(\overline{a}_{nv})$ and $\hat{A}=(\hat{a}_{nv})$ are given as follows:

$$\overline{a}_{nv}=\sum_{i=v}^{n} a_{ni}\ ,\ \ n,v=0,1,... \tag{5}$$

$$\hat{a}_{00}=\overline{a}_{00}=a_{00}\ ,\ \ \hat{a}_{nv}=\overline{a}_{nv}-\overline{a}_{n-1,v}\ ,\ \ n=1,2,... \tag{6}$$

$$A_n\left(s\right)=\sum_{v=0}^{n} a_{nv}s_v=\sum_{v=0}^{n}\overline{a}_{nv}a_v \tag{7}$$

$$\overline{\Delta}A_n\left(s\right)=\sum_{v=0}^{n}\hat{a}_{nv}a_v\ . \tag{8}$$

Theorem 3.1. Let $A=\left(a_{nv}\right)$ be a positive normal matrix such that

$$\overline{a}_{n0}=1\ ,\quad n=0,1,..., \tag{9}$$

$$a_{n-1,v}\geq a_{nv}\ \text{ for }\ n\geq v+1, \tag{10}$$

$$a_{nn}=O\left(\frac{p_n}{P_n}\right), \tag{11}$$

$$\left|\hat{a}_{n,v+1}\right|=O\left(v\left|\Delta_v(\hat{a}_{nv})\right|\right), \tag{12}$$

$$\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\gamma k+k-1)-k+1}\left|\Delta_v(\hat{a}_{nv})\right|=O\left(\left(\frac{P_v}{p_v}\right)^{\beta(\gamma k+k-1)-k}\right)\quad as\quad m\to\infty, \tag{13}$$

where $\Delta_v(\hat{a}_{nv})=\hat{a}_{nv}-\hat{a}_{n,v+1}$. If all conditions of Theorem 2.1 are satisfied with the condition (1) replaced by

$$\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\beta(\gamma k+k-1)-k}\left|t_n\right|^k=O\left(X_m\right)\quad as\quad m\to\infty, \tag{14}$$

then the series $\sum a_n\lambda_n$ is summable $|A,p_n,\beta;\gamma|_k$, $k\geq 1$, $\gamma\geq 0$ and $-\beta(\gamma k+k-1)+k>0$.

4. Proof of Theorem 3.1

Let (Θ_n) denotes A-transform of the series $\sum a_n\lambda_n$. Then, by (7) and (8), we have $\bar{\Delta}\Theta_n = \sum_{v=1}^{n} \hat{a}_{nv}\lambda_v a_v$.

By Abel's transformation, we get

$$\bar{\Delta}\Theta_n = \sum_{v=1}^{n-1} \Delta_v\left(\frac{\hat{a}_{nv}\lambda_v}{v}\right)\sum_{r=1}^{v} ra_r + \frac{\hat{a}_{nn}\lambda_n}{n}\sum_{r=1}^{n} ra_r$$

$$= \sum_{v=1}^{n-1} \Delta_v\left(\frac{\hat{a}_{nv}\lambda_v}{v}\right)(v+1)t_v + \frac{\hat{a}_{nn}\lambda_n}{n}(n+1)t_n$$

$$= \sum_{v=1}^{n-1} \frac{v+1}{v}\Delta_v(\hat{a}_{nv})\lambda_v t_v + \sum_{v=1}^{n-1}\frac{v+1}{v}\hat{a}_{n,v+1}\Delta\lambda_v t_v + \sum_{v=1}^{n-1}\hat{a}_{n,v+1}\lambda_{v+1}\frac{t_v}{v} + \frac{n+1}{n}a_{nn}\lambda_n t_n$$

$$= \Theta_{n,1} + \Theta_{n,2} + \Theta_{n,3} + \Theta_{n,4}.$$

To prove Theorem 3.1, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{\beta(\gamma k+k-1)}\left|\Theta_{n,r}\right|^k < \infty \quad for \quad r = 1,2,3,4.$$

First, using Hölder's inequality, we have

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\gamma k+k-1)}\left|\Theta_{n,1}\right|^k = O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\gamma k+k-1)}\left(\sum_{v=1}^{n-1}\left|\Delta_v(\hat{a}_{nv})\right|\left|\lambda_v\right|\left|t_v\right|\right)^k$$

$$= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\gamma k+k-1)}\left(\sum_{v=1}^{n-1}\left|\Delta_v(\hat{a}_{nv})\right|\left|\lambda_v\right|^k\left|t_v\right|^k\right)\left(\sum_{v=1}^{n-1}\left|\Delta_v(\hat{a}_{nv})\right|\right)^{k-1}$$

$$= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\gamma k+k-1)-k+1}\left(\sum_{v=1}^{n-1}\left|\Delta_v(\hat{a}_{nv})\right|\left|\lambda_v\right|^k\left|t_v\right|^k\right)$$

$$= O(1)\sum_{v=1}^{m}\left|\lambda_v\right|^{k-1}\left|\lambda_v\right|\left|t_v\right|^k\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\gamma k+k-1)-k+1}\left|\Delta_v(\hat{a}_{nv})\right|$$

$$= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\beta(\gamma k+k-1)-k}\left|\lambda_v\right|\left|t_v\right|^k$$

$$= O(1)\sum_{v=1}^{m-1}\Delta\left|\lambda_v\right|\sum_{r=1}^{n}\left(\frac{P_r}{p_r}\right)^{\beta(\gamma k+k-1)-k}\left|t_r\right|^k + O(1)\left|\lambda_m\right|\sum_{r=1}^{m}\left(\frac{P_r}{p_r}\right)^{\beta(\gamma k+k-1)-k}\left|t_r\right|^k$$

$$= O(1)\sum_{v=1}^{m-1}B_v X_v + O(1)\left|\lambda_m\right|X_m = O(1) \quad as \quad m\to\infty,$$

by virtue of the hypotheses of Theorem 3.1 and Lemma 2.1.

Again, using Hölder's inequality, we get

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\gamma k+k-1)}\left|\Theta_{n,2}\right|^k &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\gamma k+k-1)}\left(\sum_{v=1}^{n-1} v\left|\Delta_v(\hat{a}_{nv})\right|\left|\Delta\lambda_v\right|\left|t_v\right|\right)^k \\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\gamma k+k-1)}\left(\sum_{v=1}^{n-1} v\left|\Delta_v(\hat{a}_{nv})\right|\left|\Delta\lambda_v\right|\left|t_v\right|^k\right)\left(\sum_{v=1}^{n-1} v\left|\Delta_v(\hat{a}_{nv})\right|B_v\right)^{k-1} \\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\gamma k+k-1)}\left(\sum_{v=1}^{n-1} v\left|\Delta_v(\hat{a}_{nv})\right|\left|\Delta\lambda_v\right|\left|t_v\right|^k\right)\left(\sum_{v=1}^{n-1}\left|\Delta_v(\hat{a}_{nv})\right|\right)^{k-1} \\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\gamma k+k-1)-k+1}\left(\sum_{v=1}^{n-1} v\left|\Delta_v(\hat{a}_{nv})\right|\left|\Delta\lambda_v\right|\left|t_v\right|^k\right) \\
&= O(1)\sum_{v=1}^{m} vB_v\left|t_v\right|^k\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\gamma k+k-1)-k+1}\left|\Delta_v(\hat{a}_{nv})\right| \\
&= O(1)\sum_{v=1}^{m} v\,B_v\left|t_v\right|^k\left(\frac{P_v}{p_v}\right)^{\beta(\gamma k+k-1)-k} \\
&= O(1)\sum_{v=1}^{m-1}\Delta\left(v\,B_v\right)\sum_{r=1}^{v}\left(\frac{P_r}{p_r}\right)^{\beta(\gamma k+k-1)-k}\left|t_r\right|^k + O(1)\,m\,B_m\sum_{r=1}^{m}\left(\frac{P_r}{p_r}\right)^{\beta(\gamma k+k-1)-k}\left|t_r\right|^k \\
&= O(1)\sum_{v=1}^{m-1} v\left|\Delta B_v\right|X_v + O(1)\sum_{v=1}^{m-1} B_v X_v + O(1)\,m\,B_m X_m = O(1) \quad as \quad m\to\infty,
\end{aligned}$$

by virtue of the hypotheses of Theorem 3.1 and Lemma 2.2.

Now, for $r=3$, we get

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\gamma k+k-1)}\left|\Theta_{n,3}\right|^k &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\gamma k+k-1)}\left(\sum_{v=1}^{n-1}\left|\Delta_v(\hat{a}_{nv})\right|\left|\lambda_{v+1}\right|\left|t_v\right|\right)^k \\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\gamma k+k-1)}\left(\sum_{v=1}^{n-1}\left|\Delta_v(\hat{a}_{nv})\right|\left|\lambda_{v+1}\right|^k\left|t_v\right|^k\right)\left(\sum_{v=1}^{n-1}\left|\Delta_v(\hat{a}_{nv})\right|\right)^{k-1} \\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\gamma k+k-1)-k+1}\left(\sum_{v=1}^{n-1}\left|\Delta_v(\hat{a}_{nv})\right|\left|\lambda_{v+1}\right|\left|t_v\right|^k\right) \\
&= O(1)\sum_{v=1}^{m}\left|\lambda_{v+1}\right|\left|t_v\right|^k\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\gamma k+k-1)-k+1}\left|\Delta_v(\hat{a}_{nv})\right| \\
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\beta(\gamma k+k-1)-k}\left|\lambda_{v+1}\right|\left|t_v\right|^k = O(1) \quad as \quad m\to\infty,
\end{aligned}$$

as in $\Theta_{n,1}$.

Finally, we get

$$\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\beta(\gamma k+k-1)} \left|\Theta_{n,4}\right|^k = O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\beta(\gamma k+k-1)} a_{nn}^k \left|\lambda_n\right|^k \left|t_n\right|^k$$

$$= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\beta(\gamma k+k-1)-k} \left|\lambda_n\right|\left|t_n\right|^k = O(1) \quad as \quad m \to \infty,$$

as in $\Theta_{n,1}$.

5. An Application to Fourier Series

Recently, some works on absolute summability of Fourier series have been done (see [20]-[23]). Let f be a periodic function with period 2π and Lebesgue integrable over $(-\pi,\pi)$. The trigonometric Fourier series of f is defined as

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right) = \sum_{n=0}^{\infty} C_n(x)$$

where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx, \; a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx, \; b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx.$$

Write $\phi(t) = \frac{1}{2}\{f(x+t)+f(x-t)\}$ and $\phi_1(t) = \frac{1}{t}\int_0^t \phi(u)du$. If $\phi_1(t) \in BV(0,\pi)$, then $t_n(x) = O(1)$, where $t_n(x)$ is the n-th $(C,1)$ mean of the sequence $(nC_n(x))$ (see [24]). By using this fact, in [9], Bor has obtained the following theorem.

Theorem 5.1. If $\phi_1(t) \in BV(0,\pi)$ and the sequences (p_n), (λ_n) and (X_n) satisfy the conditions of Theorem 2.1, then the series $\sum C_n(x)\lambda_n$ is summable $\left|\bar{N}, p_n\right|_k$, $k \geq 1$.

Theorem 5.1 is generalized as in the following form.

Theorem 5.2. If $\phi_1(t) \in BV(0,\pi)$ and the sequences (p_n), (λ_n), (B_n) and (X_n) satisfy the conditions of Theorem 3.1, then the series $\sum C_n(x)\lambda_n$ is summable $\left|A, p_n, \beta; \gamma\right|_k$, $k \geq 1$, $\gamma \geq 0$ and $-\beta(\gamma k+k-1)+k > 0$.

6. Conclusions

If we take $\beta = 1$, $\gamma = 0$ and $a_{nv} = p_v / P_n$ in Theorem 3.1, then we get Theorem 2.1. Similarly, if we take $\beta = 1$, $\gamma = 0$ and $a_{nv} = p_v / P_n$ in Theorem 5.2, then we get Theorem 5.1.

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α-Admissible multi-valued mappings and related common fixed point theorems

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Abstract

In this paper, new common fixed point theorems are presented. Indeed, we will propose new theorems related to the fixed points of some operators. We discuss the admissibility of two multi-valued mappings in the category of complete b-metric spaces to obtain the existence of a common fixed point.

Keywords: Common fixed point, α-admissible multi valued mapping.

1. Introduction

Fixed point theory is a branch of pure mathematics, it is showed that this theory is one of the main tools to use in order to tackle the qualitative properties of differential and integral equations in general and existence and uniqueness of solutions to these equations in particular.
In 1973, Geraghty [1], studied a generalization of Banach contraction principle. Popescu [6], defined the concept of triangular α -orbital admissible mappings and proved the unique fixed point theorems for the mentioned mappings. On the other hand, Karapinar [2], proved the existence of a unique fixed point for a triangular α -admissible mapping which is a generalized $\alpha - \psi$ -Geraghty contraction type mapping.
A significant number of mathematicians utilized the classical results in the fixed point theory to discuss solutions of initial and boundary value problems ([8,4,5]). While others establish new fixed point theorems and applied them to prove the existence and oneness of solutions to variety of differential equations [3,7,9].
In this work, we study the admissibility of two multi-valued mappings in the category of complete b-metric spaces to obtain the existence of a common fixed point.

2. Preliminaries

Let Φ be a set of all increasing and continuous functions $\varphi: [0, +\infty) \to [0, +\infty)$ with the property $\varphi(y) = 0$ and only if $y = 0$ and $\varphi(cy) \leq c\varphi(y)$ for $c > 1$.
Let Ω be the family of all functions $\theta: [0, +\infty) \to [0, \frac{1}{s^2})$ such that for any bounded sequence $\{y_n\}$ of positive real numbers, $\theta(y_n) \to 1$ implies $y_n \to 0$.

Let (X, d) be a b-metric space. Take $CB(X)$ the set of bounded and closed sets in X. For $x \in X$ and $A, B \in CB(X)$, we define

$$D(x, A) = \inf_{a \in A} d(x, a),$$

$$D(A, B) = \sup_{a \in A} D(a, B).$$

Define a mapping $H: CB(X) \times CB(X) \to [0, \infty)$ such that

$$H(A, B) = max\left\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, B)\right\},$$

for every $A, B \in CB(X)$. Then the mapping H forms a b-metric.
Throughout the article I denotes I.

Definition 2.1. Let $T_1, T_2: X \to CB(X)$ be two multi-valued mappings and $\rho: X \times X \to [0, +\infty)$ be a function. Then the pair (T_1, T_2) is said to be triangurlar ρ_*-admissible if the following conditions hold:

(i) (T_1, T_2) is ρ_*-admissible; that is, $\rho(y, \eta) \geq 1$ implies $\rho_*(T_1 y, T_2 \eta) \geq 1$ and $\rho_*(T_2 y, T_1 \eta) \geq 1$, where

$$\rho_*(A, B) = inf\{\rho(y, \eta):\ y \in A, \eta \in B\},$$

(ii) $\rho(y, u) \geq 1$ and $\rho(u, \eta) \geq 1$ imply $\rho(y, \eta) \geq 1$.

3. Main results

Now, we are ready to state and prove our main results.
The following key lemma is essential to proceed in proving the main results. It states that the admissibility of a pair of multi-valued functions will guarantee the existence of a sequence of points with diameter greater than 1.

Lemma 3.1. Let $T_1, T_2: X \to CB(X)$ be two multi-valued mappings such that the pair (T_1, T_2) is triangular ρ_*-admissible. Assume that there exists $z_0 \in X$ with $\rho_*(z_0, T_1 z_0) \geq 1$. Define a sequence $\{z_n\}$ in $X$by $z_{2i+1} \in T_1 z_{2i}$ and $z_{2i+2} \in T_2 z_{2i+1}$, where $i = 0,1,2, \ldots$. Then for $m, n \in \mathbb{N} \cup \{0\}$ with $m > n$, we have $\rho(z_n, z_m) \geq 1$.

Proof. Form $\rho_*(z_0, T_1 z_0) \geq 1$ we get $\rho(z_0, z_1) \geq 1$. Since (T_1, T_2) is ρ_*-admissible, we obtain $\rho_*(T_1 z_0, T_2 z_1) \geq 1$, hence $\rho(z_1, z_2) \geq 1$ and so $\rho_*(T_2 z_1, T_1 z_2) \geq 1$, then $\rho(z_2, z_3) \geq 1$, with continuing this process we obtain, $\rho(z_m, z_{m+1}) \geq 1$.

By (ii) from definition of triangular ρ_*-admissible and regarding as; $\rho(z_n, z_{n+2}) \geq 1$ and , deduce. Again with continuing this process and from $m > n$, we find out $\rho(z_n, z_m) \geq 1$.

The following theorem gives the existence of a common fixed point for two mappings T_1 and T_2 under less hypotheses than the results existing in literature.

Theorem 3.2. Let (X, d) be an ρ-complete b-metric space (with $s \geq 1$), and $\rho: X \times X \to [0, +\infty)$ be a function. Suppose that $T_1, T_2: X \to CB(X)$ are mappings such that

$$\rho(y,\eta)\varphi\big(s^3 H(T_1 y, T_2\eta)\big) \leq \theta\Big(\varphi\big(M(y,\eta)\big)\Big)\varphi\big(M(y,\eta)\big) + L\phi\big(N(y,\eta)\big), \tag{1}$$

where

$$M(y,\eta) = \max\left\{d(y,\eta), D(y,T_1y), D(\eta,T_2\eta), \frac{D(\eta,+T_1y) + D(y,T_2\eta)}{2s}\right\},$$

and

$$N(y,\eta) = \min\{D(\eta,+T_1y), D(\eta,T_2\eta)\}. \tag{2}$$

for $\theta \in \Omega$ and $\varphi, \phi \in \Phi$ .Moreover suppose

(i) (T_1, T_2) is triangular ρ_*-admissible;

(ii) there exists $z_0 \in X$ with $\rho_*(z_0, T_1 z_0) \geq 1$;

(iii) if for every sequence $\{z_n\}$ in X with $\rho(z_n, z_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $z_n \to z \in X$, then there exists a subsequence $\{z_{n(k)}\}$ of $\{z_n\}$ with $\rho(z_n(k), z) \geq 1$.

Then T_1 and T_1 have a common fixed point $z \in X$.

Corollar 3.3. Let (X, d) be an ρ -complete b -metric space, and $\rho: X \times X \to [0, +\infty)$ be a function. Suppose that $T_1, T_2: X \to X$ are mappings such that

$$\rho(y,\eta)\varphi\big(s^3 H(T_1 y, T_2\eta)\big) \leq \theta\Big(\varphi\big(d(y,\eta)\big)\Big)\varphi\big(d(y,\eta)\big), \tag{3}$$

for$z \in \Omega$ and $\varphi \in \Phi$. Moreover suppose

(i) (T_1, T_2) is triangular ρ_*-admissible;

(ii) there exists $z_0 \in X$ with $\rho(z_0, T_1 z_0) \geq 1$;

(iii) if for every sequence $\{z_n\}$ in X with $\rho(z_n, z_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $z_n \to z \in X$, then there exists a subsequence $\{z_{n(k)}\}$ of $\{z_n\}$ with $\rho\big(z_{n(k)}, z\big) \geq 1$ for all k.

Then T_1 and T_2 have a common fixed point $z \in X$.

Moreover, if the following condition hold:

H_1: Either $\rho(u, v) \geq 1$ or $\rho(v, u) \geq 1$ when ever $T_1 u = T_2 u = u$ and $T_1 v = T_2 v = v$.

Then T_1 and T_2 have a unique common fixed point.

Proof. The proof of the existence of a common fixed point of T_1 and T_2 is based on Theorem 3.2.
We claim that if $T_1u = T_2u = u$ and $T_1v = T_2v = v$, then $u = v$. By hypotheses, if $u \neq v$, then either $\rho(u, v) \geq 1$ or $\rho(u, v) \geq 1$ .Suppose that $\rho(u, v) \geq 1$, then

$$\psi\big(d(u,v)\big) = \psi\big(d(Su,Tv)\big) \leq \psi\big(s^3 d(Su,Tv)\big) \leq \rho(u,v)\psi\big(s^3 d(Su,Tv)\big)$$

$$\leq \theta\left(\psi\big(d(u,v)\big)\right)\psi\big(d(u,v)\big) < \psi\big(d(u,v)\big),$$

which is contradiction. So $u = v$. Similarly, if $\rho(v, u) \geq 1$, we can prove $u = v$.

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Pseudo Concircular Ricci Symmetric Spacetimes Admitting Special Conditions

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Abstract

This paper deals with pseudo concircular Ricci Symmetric spacetimes with some special conditions. In the first section, we give the definition of pseudo concircular Ricci symmetric manifold $(PCRS)_n$. In the second section, some properties of the Z-symmetric tensor are mentioned. In the third section, we consider this tensor on $(PCRS)_n$ manifold and we give some theorems. In the last section, considering some special conditions, we discuss the properties of these spacetimes.

Keywords: Pseudo concircular Ricci symmetric manifold, Z-symmetric tensor, divergence-free Z-symmetric tensor, perfect fluid.

1. Introduction

Because of its property of free mobility, the manifold of constant sectional curvature which is the most simple non-flat manifold as the model space for the universe was considered by Riemann and Helmholtz. In 1926, Cartan [1] generalized this manifold to locally symmetric manifold and he obtained a classification of such a manifold. Cartan [1] introduced locally symmetric manifold if its curvature tensor R satisfies the relation $\nabla R=0$ where ∇ denotes the operator of the covariant differentation with respect to the metric tensor g.

During the last eight decades, the generalization of these manifolds have been carried out by many authors around the world by several ways. Some of them are conformally symmetric manifold by Chaki and Gupta [2], recurrent manifold by Walker [3], conformally recurrent space by Adati and Miyazawa [4], pseudosymmetric manifold by Chaki [5], pseudo-symmetric manifold by Deszcz [6], semi-symmetric manifold by Szabo [7], weakly symmetric manifold by Binh and Tamassy, weakly symmetric manifold by Selberg [8], pseudo Ricci symmetric manifolds by Chaki [9], weakly Ricci symmetric manifold by Tamassy and Binh [10], generalized pseudo Ricci symmetric manifold by Chaki and Koley [11], pseudo symmetric and pseudo Ricci symmetric manifolds by De and Gazi [12], generalized pseudo Ricci symmetric manifolds by Altay Demirbag [13], weakly and pseudo symmetric spaces by Özen and Altay [14], etc.

However, the notion of pseudo symmetry by Chaki and Deszcz are different and that of weakly symmetry by Selberg and Tamassy and Binh are also different.

A non-flat Riemannian manifold (M^n,g) $(n>2)$ is said to be pseudo symmetric [5] if its curvature tensor R satisfies the condition

$$(\nabla_X R)(Y,Z)W=2A(X)R(Y,Z)W+A(Y)R(X,Z)W+A(Z)R(Y,X)W$$
$$A(W)R(Y,Z)X+g(R(Y,Z)W,X)\rho,$$

where A is a non-zero associated 1-form, ρ is a vector field defined by $g(X,\rho)=A(X)$, for every vector field X and ∇ denotes the operator of the covariant differentiation with respect to the metric tensor g. If A=0 then this manifold reduces to a symmetric manifold in the sense of Cartan.

Again, a Riemannian manifold is said to be Ricci symmetric if the condition $\nabla S=0$ holds, where S is the Ricci tensor of type (0,2).

A Riemannian manifold is called Ricci recurrent [15] if the Ricci tensor S satisfies the relation $\nabla S=A\otimes S$ where A is a non-zero 1-form.

Every locally symmetric manifold is Ricci symmetric but not conversely and every recurrent manifold is Ricci recurrent but the converse does not hold, in general. Every Ricci symmetric manifold is Ricci recurrent but not conversely.

A non-flat Riemannian manifold which is called generalized Ricci recurrent [16] if the Ricci tensor of this manifold satisfies the relation

$$(\nabla_X S)(Y,Z)=A(X)S(Y,Z)+B(X)g(Y,Z),$$

where A and B are two non-zero 1-forms.

If the 1-form B=0 then the generalized Ricci recurrent manifold reduces to a Ricci recurrent manifold.

In 1988, Chaki [9] introduced pseudo Ricci symmetric manifold and it is defined as a non-flat Riemannian manifold $(M^n,g)(n\geq 3)$ whose Ricci tensor S of type (0,2) satisfies the condition

$$(\nabla_X S)(Y,Z) = 2A(X)S(Y,Z) + A(Y)S(X,Z) + A(Z)S(Y,X), \tag{1}$$

where A and ∇ are stated as in the definition of pseudosymmetric manifold. Such an n-dimensional manifold is denoted by $(PRS)_n$.

A transformation of an n-dimensional Riemannian manifold M, which transforms every geodesic circle of M into a geodesic circle, is called a concircular transformation, [17, 18].

A concircular transformation is always a conformal transformation [17]. A geodesic circle means a curve in M whose first curvature is constant and whose second curvature is identically zero. Hence, the geometry of concircular transformation is named by the concircular geometry, is a generalization of inverse geometry in the sense that the change of metric is more general than that induced by a circle preserving diffeomorphism.

The concircular curvature tensor, an interesting invariant under this transformation is defined by [18, 19]

$$\bar{C}(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y], \tag{2}$$

where R is the Riemannian curvature tensor and r is the scalar curvature.

The notion of pseudo concircularly symmetric manifold was introduced by De and Tarafdar [20]. A non-flat semi-Riemannian manifold (M^n, g) is called pseudo concircularly symmetric manifold if its curvature tensor satisfies the condition

$$(\nabla_X \bar{C})(Y,Z)W = 2A(X)\bar{C}(Y,Z)W + A(Y)\bar{C}(X,Z)W$$
$$+A(Z)\bar{C}(Y,X)W + A(W)\bar{C}(Y,Z)X + g(\bar{C})(Y,Z)W,X)\rho$$

where A is a non-zero 1-form, $g(X,\rho) = A(X)$, for every vector field X and ∇ denotes the operator of the covariant differentiation with respect to the metric tensor g.

Let $\{e_i, \quad i = 1,2,\dots,n\}$ be an orthonormal basis of the tangent space at each point of the manifold and let

$$P(X,Y) = \sum_{i=1}^{n} \bar{C}(X, e_i, e_i, Y), \tag{3}$$

then from (1.2), we get

$$P(X,Y) = S(X,Y) - \frac{r}{n} g(X,Y). \tag{4}$$

The tensor P is called the concircular Ricci tensor [21], which is a symmetric tensor of type (0,2). The present paper deals with a type of non-flat Riemannian manifold $(M^n, g)(n \geq 2)$ whose concircular Ricci tensor P is not identically zero and satisfies the condition (1)

$$(\nabla_X P)(Y,Z) = 2A(X)P(Y,Z) + A(Y)P(X,Z) + A(Z)P(Y,X),$$

(5)

where A and ∇ has the meaning as before. Such a manifold is called pseudo concircular Ricci symmetric manifold and it is denoted by $(PCRS)_n$.

A vector field ξ in a Riemannian manifold M is called torse-forming if it satisfies the condition $\nabla_X \xi = \alpha X + \lambda(X)\xi$, where $X \in TM$, $\lambda(X)$ is a linear form and α is a scalar function, [22, 23, 24].

In the local transcription, this reads

$$\xi^h_{,i} = \alpha\delta^h_i + \xi^h\lambda_i \tag{6}$$

where ξ^h and λ_i are the components of ξ and λ respectively, and δ^h_i is the Kronecker symbol. A torse-forming vector field ξ is called, [23, 24],

i. recurrent if $\alpha = 0$, i.e,

$$\xi^h_{,i} = \xi^h\lambda_i \tag{7}$$

ii. concircular if the 1-form λ_i is gradient covector (i.e., $\lambda_i = \lambda_{,i}$), i.e,

$$\xi^h_{,i} = \alpha\delta^h_i \tag{8}$$

iii. convergent if it is concircular and $\alpha = const.exp(\lambda)$.

A $\varphi(Ric)$-vector field is a vector field on an n dimensional Riemannian manifold (M, g) with the metric g and Levi-Civita connection ∇, which satisfies the condition, [25]

$$\nabla\varphi = \mu Ric, \tag{9}$$

where μ is some constant and Ric is the Ricci tensor. Obviously, when (M, g) is an Einstein space, the vector field φ is concircular. Moreover, when $\mu = 0$, the vector field φ is covariantly constant. In the following we suppose that $\mu \neq 0$ and (M, g) is neither an Einstein space nor a vacuum solution of the Einstein equations. In a locally coordinate neighbourhood U(x), the equation (9) is written as

$$\varphi^h_{,i} = \mu S^h_i, \tag{10}$$

where φ^i and S^h_i are components of φ and Ric, respectively. After lowering indices, (10) has the form

$$\varphi_{i,j} = \mu S_{ij}, \tag{11}$$

where $\varphi_i = \varphi^\alpha g_{i\alpha}$ and $S_{ij} = g_{i\alpha}S^\alpha_j$.

2. Z-Tensor

A (0,2) symmetric tensor is generalized Z tensor if it satisfies

$$Z_{kl} = S_{kl} + \phi g_{kl}, \tag{12}$$

where ϕ is an arbitrary scalar function. The scalar $\bar{Z}$ is the trace of Z-tensor of from (12)

$$\bar{Z} = g^{kl} Z_{kl} = r + n\phi. \tag{13}$$

The classical Z tensor is obtained with the choice $\phi = -\frac{1}{n} r$. Shortly, the generalized Z-tensor is called as the Z-tensor. From the Z-tensor, we can find several well known structures on Riemannian manifolds:

1) If $Z_{kl} = 0$ then this manifold (Z-flat) reduces to an Einstein manifold, $S_{ij} = (\frac{r}{n}) g_{ij}$,[26].
2) If $\nabla_j Z_{kl} = \lambda_j Z_{kl}$ then this manifold (Z-recurrent) reduces to a generalized Ricci recurrent manifold [16, 27]. The condition is equivalent to $\nabla_j S_{kl} = \lambda_j S_{kl} + (n-1)\mu_j g_{kl}$ where $(n-1)\mu_j = (\lambda_j - \nabla_j)\phi$. Moreover, if $(\lambda_j - \nabla_j)\phi = 0$ then our manifold reduces to a Ricci recurrent manifold.
3) If
$$\nabla_j Z_{kl} = \nabla_k Z_{jl}, \tag{14}$$
i.e, Z is a Codazzi tensor [28], then $\nabla_j S_{kl} - \nabla_k S_{jl} = (g_{jl}\nabla_k - g_{kl}\nabla_j)\phi$. Multiplying the last equation by g^{kl}, we find $\nabla_j[r + 2(n-1)\phi] = 0$. Then, we obtain
$$\nabla_j S_{kl} - \nabla_k S_{jl} = \frac{1}{2(n-1)} (g_{kl}\nabla_j - g_{jl}\nabla_k) r.$$
This condition defines a nearly conformal symmetric manifold, $(NCS)_n$. This condition was introduced and studied by Roter [29]. Conversely, a $(NCS)_n$ manifold has a Z-tensor of Codazzi type if the condition $\nabla_j[r + 2(n-1)\phi] = 0$ is satisfied.
4) Einstein' s equations [30] with cosmological constant Λ and energy-stress tensor T_{kl} may be written as $Z_{jl} = kT_{jl}$ where $\phi = -\frac{1}{2} r + \Lambda$ and k is the gravitational constant.
The Z-tensor may be considered as a generalized Einstein gravitational tensor with arbitrary scalar function ϕ.

Conditions on the enery-momentum tensor determine constraints on the tensor Z: the vacuum solution Z=0 determines an Einstein space $\Lambda = (\frac{n-2}{2n})r$; conservation of total energy-momentum ($\nabla^l T_{kl} = 0$) gives $\nabla^l Z_{kl} = 0$ and $\nabla_k(\frac{1}{2}r + \phi) = 0$; the condition $\nabla_j Z_{kl} = 0$ describes a spacetime with conserved enery-momentum density.

This manifold has received a great deal of attention and is studied in considerable detail by many authors [31, 32, 33, 34, 35]. Motivated by the above studies, in the present, we examine the properties of Z-tensor on $(PCRS)_4$ spacetime.

3. Z-tensor on $(PCRS)_n$

In this section, we consider a manifold $(PCRS)_n$ with the Z-tensor.

In local coordinates, from (4) and (12), the relation between the Z-tensor and the concircular Ricci symmetric tensor is found as

$$Z_{ij} = P_{ij} + (\frac{r}{n} + \phi)g_{ij}. \tag{15}$$

The covariant derivative of (15) can be obtained as follows

$$Z_{ij,k} = P_{ij,k} + (\frac{r_{,k}}{n} + \phi_k)g_{ij}. \tag{16}$$

Let us consider an n-dimensional pseudo concircular Ricci symmetric manifold. Hence, we have from (5)

$$P_{ij,k} = 2A_k P_{ij} + A_i P_{kj} + A_j P_{ik}. \tag{17}$$

Then using (17), the expression (16) can be written as

$$Z_{ij,k} = 2A_k P_{ij} + A_i P_{kj} + A_j P_{ik} + (\frac{r_{,k}}{n} + \phi_k)g_{ij}. \tag{18}$$

By the aid of (4), (18) reduces to

$$Z_{ij,k} = 2A_k S_{ij} + A_i S_{kj} + A_j S_{ik} - \frac{r}{n}(2A_k g_{ij} + A_i g_{kj} + A_j g_{ik}) + (\frac{r_{,k}}{n} + \phi_k) g_{ij} \tag{19}$$

Differentiating the equation (12) then we get

$$Z_{ij,k} = S_{ij,k} + \phi_k g_{ij}. \tag{20}$$

Multiplying the equation (20) by g^{ik}, (20) transforms into

$$Z^k_{j,k} = S^k_{j,k} + \phi_j. \tag{21}$$

Now, let us assume that the Z-tensor is divergence-free. Thus, from (21), we find

$$S^k_{j,k} = -\phi_j. \tag{22}$$

Considering the Ricci identity $S^k_{j,k} = \frac{1}{2} r_{,j}$, we get from (22)

$$r_{,j} = -2\phi_j. \tag{23}$$

On the other hand, if we multiply the equation (19) by g^{ik} and if we assume that the Z-tensor is divergence-free then we obtain

$$A^i S_{ij} = \frac{r}{n} A_j - \frac{1}{3}(\frac{r_{,j}}{n} + \phi_j). \tag{24}$$

Putting (23) in (24), the equation (24) reduces to

$$A^i S_{ij} = \frac{r}{n} A_j + \frac{(2-n)}{3n} \phi_j. \tag{25}$$

In the next section, we consider spacetimes with the property (25).

4. Perfect Fluid $(PCRS)_4$ Spacetime with Z-Tensor

Now, we consider a perfect fluid $(PCRS)_4$ spacetime with the Z-tensor. To find a model of universe, Einstein obtained the field equations of general relativity. The universe on a large scale shows isotropy and homogeneity and the matter contents of the universe(stars, galaxies, nebulas, etc.) can be assumed to be that of a perfect fluid. Let us consider the Einstein's field equation without the cosmological constant given by

$$S(X,Y) - \frac{r}{2} g(X,Y) = kT(X,Y), \tag{26}$$

where S, r, T, k denote the Ricci tensor, the scalar curvature, the energy momentum tensor and gravitational constant, respectively.

The energy momentum tensor T of a perfect fluid is given by [36]

$$T(X,Y) = (\sigma + p)A(X)A(Y) + pg(X,Y), \tag{27}$$

where σ is the energy density, p is the isotropic pressure, $g(X,\rho) = A(X)$ and ρ is a unit timelike vector field.

Now, if we compare the equations (26) and (27) then we find

$$S_{ij} = k(\sigma + p)A_i A_j + (kp + \frac{r}{2})g_{ij}. \tag{28}$$

From (28), we get

$$r = k(\sigma - 3p). \tag{29}$$

Thus, putting (29) in (28), it can be found that

$$S_{ij} = k(\sigma + p)A_i A_j + \frac{k}{2}(\sigma - p)g_{ij}. \tag{30}$$

For a perfect fluid spacetime $(PCRS)_4$ admitting divergence-free Z-tensor, we get from (25)

$$A^i S_{ij} = \frac{r}{4} A_j - \frac{1}{6}\phi_j. \tag{31}$$

On the other hand, multiplying (30) by A^i, it can be obtained that

$$A^i S_{ij} = -\frac{k}{2}(\sigma + 3p)A_j. \tag{32}$$

Thus, comparing the equations (29), (31) and (32), we find

$$\phi_j = \frac{9k}{2}(\sigma + p)A_j. \tag{33}$$

Hence, we have the following theorem:

Theorem 1. *In a perfect fluid $(PCRS)_4$ spacetime with divergence-free Z-tensor, the associated vector fields A_j and ϕ_j are parallel and the relation between them is*

$$\phi_j = \frac{9k}{2}(\sigma + p)A_j.$$

Now, differentiating the equation (26), it can be found

$$kT_{ij,l} = S_{ij,l} - \frac{r_{,l}}{2} g_{ij}. \tag{34}$$

Hence, multiplying (34) by g^{ij}, we get

$$T_{,l} = -\frac{1}{k} r_{,l}. \tag{35}$$

Comparing (23) and (35), we find

$$T_{,l} = \frac{2}{k}\phi_l. \tag{36}$$

If we differentiate (36) then

$$T_{,lm} = \frac{2}{k}\phi_{l,m}. \tag{37}$$

Now, we assume that ϕ_l is a torse-forming vector field. Then from (6) and (37), we can find

$$T_{,lm} = \frac{2}{k}(\alpha g_{lm} + \gamma_m \phi_l). \tag{38}$$

ϕ_l is a vector field generated by ϕ with the condition $\phi^l \phi_l = -1$ in this spacetime, α is a scalar function and γ_m is a vector field. Thus, by the aid of (6),

$$\phi^l \phi_{l,m} = 0 = \alpha \phi_m - \gamma_m.$$

Then

$$\gamma_m = \alpha \phi_m. \tag{39}$$

If we put (39) in (38), we get

$$T_{,lm} = \frac{2\alpha}{k}(g_{lm} + \phi_m \phi_l). \tag{40}$$

Multiplying (40) by g^{lm},

$$\Delta T = g^{lm} T_{,lm} = \frac{6\alpha}{k} \tag{41}$$

Thus, we have the following theorem:

Theorem 2. *In a perfect fluid $(PCRS)_4$ spacetime admitting divergence-free Z-tensor if the associated vector field ϕ_l is torse-forming then the Laplacian of the trace function of the energy momentum tensor is found as*

$$\Delta T = \frac{6\alpha}{k}$$

where α is a scalar function, k is the gravitational constant.

Corollary 2.1. *In a perfect fluid $(PCRS)_4$ spacetime admitting divergence-free Z-tensor if the associated vector field ϕ_l is torse-forming then the trace function of the energy momentum tensor cannot be harmonic.*

Proof. If we assume that T is harmonic then from (41) we find $\alpha = 0$. Thus, we cannot define a torse-forming vector field such as (6). This completes the proof.

Now, assume that ϕ_l is $\phi(Ric)$ vector field. Then, from (11), we have

$$\phi_{l,m} = \mu S_{lm} \tag{42}$$

where μ is a scalar function.

Multiplying (42) by ϕ^l, and remembering that $\phi^l \phi_{l,m} = 0$, we get

$$\mu S_{lm} \phi^l = 0. \tag{43}$$

Since $\mu \neq 0$ then one can obtain

$$S_{lm} \phi^l = 0. \tag{44}$$

On the other hand, multiplying (30) by ϕ^l, we find

$$S_{lm} \phi^l = k(\sigma + p) A_l A_m \phi^l + \frac{k}{2}(\sigma - p)\phi_m. \tag{45}$$

If we put (44) in (45) then one can get

$$k(\sigma + p) A_l A_m \phi^l + \frac{k}{2}(\sigma - p)\phi_m = 0. \tag{46}$$

Now, multiplying (46) by A^m, we obtain

$$A_l \phi^l (\sigma + 3p) = 0. \tag{47}$$

We know from (33) that A_l is parallel to ϕ_l. Thus, $A_l \phi^l \neq 0$. Finally, we get

$$\sigma = -3p. \tag{48}$$

In this case, by putting the condition (48) in (30), the Ricci tensor reduces to

$$S_{ij} = -2kp(A_iA_j + g_{ij}). \tag{49}$$

Hence, we have the following theorem:

Theorem 3. *In a perfect fluid $(PCRS)_4$ spacetime admitting divergence-free Z-tensor if the associated vector field ϕ_l is $\phi(Ric)$ then the Ricci tensor of this spacetime is in the following form*

$$S_{ij} = -2kp(A_iA_j + g_{ij}).$$

Now, by using (48) in (27), we get

$$T_{ij} = p(-2A_iA_j + g_{ij}). \tag{50}$$

Multiplying (50) by A^i, we find

$$A^iT_{ij} = 3pA_j. \tag{51}$$

Hence, we have the following theorem:

Theorem 4. *In a perfect fluid $(PCRS)_4$ spacetime admitting divergence-free Z-tensor if the vector field ϕ_l is $\phi(Ric)$ then $3p$ is an eigenvalue of the energy-momentum tensor T_{ij} corresponding to the eigenvector ρ where $g(X,\rho) = A(X)$.*

Now, if we use (42) in (37), we obtain

$$T_{,lm} = \frac{2\mu}{k}S_{lm}. \tag{52}$$

Multiplying (52) by g^{lm}, the Laplacian of the trace function of the energy-momentum tensor is found as from (29) and (48)

$$g^{lm}T_{,lm} = \Delta T = -12\mu p, \quad (\mu \neq 0). \tag{53}$$

Thus, we have the following theorem:

Theorem 5. *In a perfect fluid $(PCRS)_4$ spacetime admitting divergence-free Z-tensor if the vector field ϕ_l is $\phi(Ric)$ then the trace function of the energy-momentum tensor is proportional with the isotropic pressure and the relation between them is*

$$\Delta T = -12\mu p, \quad (\mu \neq 0)$$

where p is the isotropic pressure and μ is constant.

Corollary 5.1. *In a perfect fluid $(PCRS)_4$ spacetime admitting divergence-free Z-tensor if the vector field ϕ_l is $\phi(Ric)$ then the trace function of the energy-momentum tensor cannot be harmonic.*

Proof. From (53), if $\Delta T = 0$ then p must be zero since $(\mu \neq 0)$. If $p = 0$ then from (48), $\sigma = 0$. In this case, $S_{ij} = 0$. So, the trace function of the energy-momentum tensor cannot be harmonic. Thus, the proof is completed.

5. References

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The strong versions of the order-McShane and Henstock integrals in Rıesz space

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Abstract

In this article we consider a strong versions of the order-McShane (Henstock) integral on Banach lattice. We define the property $o(S^*\mathrm{M})$, $o(S^*\mathrm{H})$ and we compare the order type integrals, showing that order - strongly type integrals respect almost everywhere equality for order- bounded functions. Another interesting difference is that the order- strong McShane integrability of a function imply that it has the property $o(S^*\mathrm{M})$,but this condition cannot be used for the order-strong Henstock integrability and the property $o(S^*\mathrm{H})$.

Keyword(s) Riesz space, (o)-strongy Henstock integral, (o)- strongly Mcshane integral.

1. Introduction and preliminaries

Recently, there are many papers paying attention to the integration in Riesz space. There are introduced and studied the notions of order-type integrals, for functions taking their values in ordered vector spaces, and in Banach lattices. In particular we can see [3], [7], [11], [10], [9], [5], [4], [8] [12]. We are affected from the works of Candeloro and Sambucini [6] as well as Boccuto et al.[1-2] about order –type integrals. In this article we consider a strong versions of the order-McShane (Henstock) integral on Banach lattice. We define the property o(S^*M), $o(S^*\mathrm{H})$ and we compare the order type integrals, showing that order - strongly type integrals respect almost everywhere equality for order- bounded functions. Another interesting difference is that the order-strong McShane integrability of a function imply that it has the property o(S^*M) but this condition cannot be used for the order- strong Henstock integrability and the property o(S^*H).

From now on, T will denote a compact metric space, and $\mu: \mathfrak{B} \rightarrow \mathbb{R}_0^+$ any regular, nonatomic σ-additive measure on the σ-algebra $\mathfrak{B}$ of Borel subsets of T.

A sequence $(r_n)_n$ is said to be order-convergent (or (o)-convergent) to r, if there exists a sequence $(p_n)_n \in R$, such that $p_n \downarrow 0$ and $|r_n - r| \leq p_n$, $\forall\, n \in \mathbb{N}$.

(see also [4], [10]), and we will write$(o)\ \lim{}_n r_n = r$.

A *gage* is any map $\gamma: T \to \mathbb{R}^+$. A *partition* Π of T is a finite family $\Pi = \{(E_i, t_i): i = 1, \ldots, k\}$ of pairs such that the sets E_i are pairwise disjoint sets whose union is T and the points t_i are called *tags*. If all tags satisfy the condition $t_i \in E_i$ then the partition is said to be of *Henstock* type, or a *Henstock partition*. Otherwise, if t_i is not necessary to be in E_i, we say that it is a *free* or *McShane* partition.

Given a gage γ, we say that Π is γ*-fine* if $\mathrm{d}(w, t_i) < \gamma(t_i)$ for every $w \in E_i$ and $i = 1, \ldots, k$. Clearly, a gage γ can also be defined as a mapping associating with each point $t_i \in T$ an open ball centered at $t_i$and cover E_i.

Let us assume now that *X* is any Banach lattice with an order-continuous norm. For the sake of completeness we recall the main notions of integral we are interested in.

Definition 1.1.

A function $f: T \to X$ is called (o)- McShane integrable ((oH)-integrable) and $J \in X$ is its (o)-McShane integral ((oH)-integral) if for every (o)- sequence $(b_n)_n$ in X, there is a corresponding sequence $(\gamma_n)_n$of gauges $(\gamma_n(t): T \to]0, +\infty[$ such that for every n and (γ_n) -fine M-partition (H-partition) $\{(E_i, t_i), i = 1, \ldots, p\}$ of T holds the inequality

$$|\sigma(f, \Pi) - J| \leq b_n. \quad (1)$$

Where $\sigma(f, \Pi) = \sum_{i=1}^{p} f(t_i)\, \mu(E_i)$. We denote

$$J = (oM) \int_T f.$$

respectively

$$J = (oH) \int_T f.$$

Proposition1.2.

Let $f, g: T \to X$ be two bounded maps, such that $f = g$ μ −almost everywhere in T. Then f is (oM)-integrable if and only if g is (oM)- integrable, and the integral is the same.

Proof. Let use denote by M any majorant for $|f|$ and $|g|$. Assume that f is (oM)-i integrable.

Let $N = \{\tau \in T: \varphi(\tau) \neq 0\}$

and

$$N_i = \{\tau \in N, \frac{2M}{n} i < \varphi \leq \frac{2M}{n}(i+1)\}.$$

Since $\mu(N) = 0$, we have also $\mu(N_i) \leq \frac{b}{n}$, and pick any open set $G_i$and $\mu(G_i) < \frac{1}{n}$.

Define a gauge

$$\delta_n(\tau) = \begin{cases} \gamma_n(\tau) & p\ddot{e}r\ \tau \in T \setminus N \\ \overline{\gamma_n}\,(\tau)\ i\ till\ddot{e} & p\ddot{e}r\ \tau\ \in N_i :]\tau - \overline{\gamma_n}\,(\tau), \tau + \overline{\gamma_n}\,(\tau)[\ \subset G_i \end{cases}$$

Let as fix any tagged δ_n-fine partition Π: $(E_i, \tau_i)_{\mathrm{i}}$ and we remark the following facts :

$$|\ \sigma(f, \Pi) - J\ | \leq b_n$$

$$\sigma(f, \Pi) = \sum_{\tau_i \in N} f(\tau_i)\,\mu\,(E_i) + \sum_{\tau_i \notin N} f(\tau_i)\,\mu\,(E_i),$$

$$\sigma(g, \Pi) = \sum_{\tau_i \in N} g(\tau_i)\,\mu\,(E_i) + \sum_{\tau_i \notin N} g(\tau_i)\,\mu\,(E_i),$$

And

$$\text{Sup}\,\{\sum_{\tau_i \in N} |\ f(\tau_i)\ |\,\mu\,(E_i), \sum_{\tau_i \in N} |\ g(\tau_i)\ |\,\mu\,(E_i)\} \leq \frac{M}{n},$$

While

$$\sum_{\tau_i \notin N} f(\tau_i)\,\mu\,(E_i) = \sum_{\tau_i \notin N} g(\tau_i)\,\mu\,(E_i).$$

So,we deduce :

$$|\ \sigma(g, \Pi) - J\ | \leq \left|\sum_{\tau_i \notin N} f(\tau_i)\,\mu\,(E_i) - J\ \right| + \sum_{\tau_i \in N} |\ g(\tau_i)\ |\,\mu\,(E_i) \leq$$

$$\leq |\ \sigma(f, \Pi) - J\ | + \sum_{\tau_i \in N} |\ f(\tau_i)\ |\,\mu\,(E_i) + \sum_{\tau_i \in N} |\ g(\tau_i)\ |\,\mu(E_i) \leq b_n + 2\frac{M}{n}$$

$$|\ \sigma(\varphi, \Pi) - J\ | \leq |\ \sigma(f, \Pi) - J\ | + |\ \sigma(g, \Pi) - J\ | \leq b_n + b_n + 2\frac{M}{n} = 2\left(b_n + \frac{M}{n}\right).$$

So $2(b_n + Mn^{-1})_n$ and $(\delta_n)_n$ are respectively the (o)-sequence and the corresponding sequence of gages proving that φ is (oM)- i integrable with integral J. Hence $\varphi = f - g : T \to X$ is(o)- McShane integrable and $(oM)\int_T \varphi = 0$ and $f = g\ \mu$ −almost everywhere in T.

Definition 1.3

A function $f: T \to X$ is said to be (o)- strongly McShane (Henstock-Kurzweil) integrable on T if there is an additive function : P(T) → X such that for every (o)- sequence $(b_n)_n$ in X, there is a corresponding sequence $(\gamma_n)_n$, of gauges $\gamma_n: T \to]0, +\infty[$ such that for every n and (γ_n) -fine M-partition (H-partition) $\{(E_i, t_i), i = 1, \ldots, p\}$ of T holds the inequality

$$\sum_{i=1}^{k} |f(t_i)\mu(E_i) - F(E_i)| \leq b_n \ (2)$$

Denote by o(SM) o(SH) the set of functions $f: T \to X$ which are (o)-strongly McShane (Henstock-Kurzweil) integrable on T .

Theorem 1.4

If for $f: T \to X$ we have $f = 0$ almost everywhere in T then is (o)- strongly McShane integrable and consequently also (o)-strongly -Henstock-Kurzweil integrable.

Proof. By Theorem 1.2 the real function $f: |T| \to \mathbb{R}$ is (o)-McShane integrable and $(oM)\int_T |f(t_i)| = 0.$

This means that for every (o)- sequence $(b_n)_n$ in X, there is a corresponding sequence $(\gamma_n)_n$, of gauges $\gamma_n: T \to]0, +\infty[$ such that for every n and (γ_n) -fine M-partition $\{(E_i, t_i), i = 1, \dots, p\}$ of T holds the inequality

$$\sum_{i=1}^{k} |f(t_i)\mu(E_i)| = \sum_{i=1}^{k} |f(t_i)|\mu(E_i) \leq b_n$$

And therefore $f: T \to X$ is (o)-strongly McShane integrable with the additive function for every: $F(E) = 0$ for every $E \in \Im$.

Theorem 1.5. [6].

Let $f: T \to X$ be any mapping. Then f is (o)– Henstock integrable ((o)- McShane integrable) if and only if there exist an (o) –sequence $(b_n)_n$ and a corresponding sequence $(\gamma_n)_n$ of gages, such that for every n, as soon as Π'',Π' are two -γ_n fine Henstock (McShane) partitions, the following holds true:

$$|\ \sigma(f, \Pi'') - \sigma(f, \Pi')\ | \leq b_n \quad (3)$$

Lemma 1.6. (Saks-Henstock).

Assume that $f: T \to X$ is (o) – McShane integrable. Given (o) - sequence $(b_n)_n$ assume that a corresponding sequence $(\gamma_n)_n$ of gauges $(\gamma_n(t): T \to]0, +\infty[$ on T such that for every n
and for every γ_n-fine M- partition $\Pi = \{(E_i, t_i): i = 1, \dots, k\}$, of T holds the inequality

$$\left|\sum_{i=1}^{k} f(t_i)\, \mu(E_i) - (oH) \int_T f\right| \leq b_n \quad (4)$$

Then if $\{(F_j, \tau_j): j = 1, \dots, m\}$ is an arbitrary γ_n-fine M-system we have

$$\left|\sum_{j=1}^{m} (f(\tau_j)\, \mu(F_j) - (oH) \int_{F_j} f)\right| \leq b_n \quad (5)$$

2. The property o(S^*M) o(S^*H)

Definition 2.1

A function $f: T \to X$ has the property $o(S^*M), o(S^*H)$ if for every (o)- sequence$(b_n)_n$ in X, there is a corresponding sequence $(\gamma_n)_n$, of gauges $\gamma_n: T \to]0, +\infty[$ such that for every n and (γ_n) -fine M-partition (H-partition) $\{(E_i, t_i), i = 1, \dots, p\}$ and $\{(L_j, s_j), j = 1, \dots l\}$ of T holds the inequality

$$\sum_{i=1}^{k} \sum_{j=1}^{l} |f(t_i) - f(s_j|\ \mu(E_i \cap L_j) \leq b_n \quad (4)$$

Proposition 2.2.

If $f: T \to X$ has the property o(S^*M) then it has the property o(S^*H).

In general the inclusion o$(S^*M) \subset o\ (S^*H)$ is proper ,i.e.for any infinite dimensional Banach lattices X there is a function $f: T \to X$ for which $f \in o(S^*H)$ but $f \notin o(S^*M)$

Theorem 2.3

If a function $f: T \to X$ has the property $o(S^*M)$, $o(S^*H)$ then f is oM (oH)– integrable.

Proof. If $\{(J_i, t_i), i = 1, \dots k\}$ and $\{(L_j, s_j), j = 1, \dots l\}$ are (γ_n) -fine M-partition (H-partition) of T we have

$$\mu(J_i) = \sum_{j=1}^{l} \mu(J_i \cap L_j)$$

and

$$\mu(L_j) = \sum_{i=1}^{k} \mu(J_i \cap L_j)$$

Hence

$$\left|\sum_{i=1}^{k} f(t_i)\mu(J_i) - \sum_{j=1}^{l} f(s_j)\mu(L_j)\right|$$
$$= \left|\sum_{j=1}^{l}\sum_{i=1}^{k} f(t_i)\mu\left(J_i \cap L_j\right) - \sum_{i=1}^{k}\sum_{j=1}^{l} f(s_j)\mu\left(J_i \cap L_j\right)\right|$$

$$= \left|\sum_{j=1}^{l}\sum_{i=1}^{k}(f(t_i) - f(s_j))\mu\left(J_i \cap L_j\right)\right|$$

$$\leq \sum_{j=1}^{l}\sum_{i=1}^{k}\left|f(t_i) - f(s_j)\right|\mu\left(J_i \cap L_j\right) \leq b_n.$$

And by Definition 2.1 and Cauchy criterion this yields the statement.

Corollary 2.4

If a function $f: T \to X$ has the property $o(S^*M)$, then function $|f|: T \to X$ is (oM)- integrable.

If a function $f: T \to X$ has the property $o(S^*H)$, then function $|f|: T \to X$ is (oH)- integrable.

Proof. Since

$$\left||f(t_i)| - |f(s_i)|\right| \leq |f(t_i) - f(s_i)|$$

For any choice of t_i, $s_i \in T$ we can see that the function, $|f|: T \to X$ has the property $o(S^*M)$ (S^*oH) and theorem 2.3 implies its oM (oH)- integrability.

Lemma 2.5

If $f: T \to X$ has the property $o(S^*M)$ then it is (o)- strongly M-integrable on T.

Proof. If $f: T \to X$ has the property $o(S^*M)$ then, by Definition 2.1 for every (o)- sequence $(b_n)_n$ in X, there is a corresponding sequence $(\gamma_n)_n$, of gauges γ_n: $T \to]0, +\infty[$, such that for every n holds the inequality

$$\sum_{i=1}^{k} \sum_{j=1}^{l} \left|f(t_i) - f(s_j)\right| \mu\left(J_i \cap L_j\right) \leq \frac{b_n}{2}$$

for any two (γ_n) -fine M-partition $\{(J_i, t_i), i = 1, \dots k\}$, and $\{(L_j, s_j), j = 1, \dots l\}$ of T.

Assume that $\{(J_i, t_i), i = 1, \dots k\}$ is an arbitrary (γ_n) -fine M-partition T. By Theorem [2.3] we have $f \in o\mathcal{M}$ and therefore f is (oM)- integrable on every interval $J_i, i = 1, \dots k$ by Cauchy criterion, for the existence of the -(oM)- integral. Hence for the given (o)-sequence $(b_n)_n$ there is a corresponding sequence $(\gamma'_n)_n$ of gauges $(\gamma'_n$: $T \to]0, +\infty[$, such that $(\gamma'_n)_n \leq (\gamma_n)_n$ for $t \in T$ and such that for any (γ'_n) -fine M-partition $\left\{\left(L^{(i)}{}_j, s^{(i)}{}_j\right), j = 1, \dots l^{(i)}, i = 1, \dots k\right\}$ of the J_i we have

$$\left|\sum_{j=1}^{l^{(i)}} f\left(s_j^{(i)}\right)\mu\left(L_j^{(i)}\right) - (oM)\int_{J_i} f\right|$$

$$= \left|\sum_{j=1}^{l^{(i)}} [f\left(s_j^{(i)}\right)\mu\left(L_j^{(i)}\right) - (oM)\int_{L_j^{(i)}} f]\right| \leq \frac{b_n}{2k}.$$

Note that $\left\{\left(L^{(i)}{}_j, s^{(i)}{}_j\right), j = 1, \dots l^{(i)}, i = 1, \dots k\right\}$ is a (γ_n) -fine M-partition of the interval T and that for any $i = 1, \dots k$ we have

$$f(t_i)\,\mu(J_i) = \sum_{j=1}^{l^{(i)}} f(t_i)\,\mu\left(J_i \cap L_j^{(i)}\right)$$

and, because of the additivity of the indefinite integral

$F(J) = oM)\int_J f$, also

$$F(J_i) = \sum_{j=1}^{l^{(i)}} F\left(J_i \cap L_j^{(i)}\right)$$

Hence

$$\sum_{i=1}^{k} |f(t_i)\mu(J_i) - F(J_i)| =$$

$$= \sum_{i=1}^{k} \left|\sum_{j=1}^{l^{(i)}} f(t_i)\mu\left(J_i \cap L_j^{(i)}\right) - \sum_{j=1}^{l^{(i)}} F\left(J_i \cap L_j^{(i)}\right)\right|$$

$$= \sum_{i=1}^{k} \left|\sum_{j=1}^{l^{(i)}} (f(t_i) - f\left(s_j^{(i)}\right))\mu\left(J_i \cap L_j^{(i)}\right) + \sum_{j=1}^{l^{(i)}} [f\left(s_j^{(i)}\right)\mu\left(J_i \cap L_j^{(i)}\right) - F\left(J_i \cap L_j^{(i)}\right)]\right|$$

$$\leq \sum_{i=1}^{k} \left| \sum_{j=1}^{l(i)} (f(t_i) - f\left(s_j^{(i)}\right)) \mu\left(J_i \cap L_j^{(i)}\right) \right|$$

$$+ \sum_{i=1}^{k} \left| \sum_{j=1}^{l(i)} [f\left(s_j^{(i)}\right) \mu\left(J_i \cap L_j^{(i)}\right) - F\left(J_i \cap L_j^{(i)}\right)] \right|$$

$$\leq \sum_{i=1}^{k} \sum_{j=1}^{l(i)} \left| f(t_i) - f\left(s_j^{(i)}\right) \right| \mu\left(J_i \cap L_j^{(i)}\right)$$

$$+ \sum_{i=1}^{k} \left| \sum_{j=1}^{l(i)} [f\left(s_j^{(i)}\right) \mu\left(J_i \cap L_j^{(i)}\right) - F\left(J_i \cap L_j^{(i)}\right)] \right|$$

$$< \frac{b_n}{2} + \sum_{i=1}^{k} \frac{b_n}{2k} = b_n$$

This shows f is $o(SM)$- integrable on T.

The same holds if (oM) is replaced by (oH) and H- partitions are used instead of M- partitions.

Lemma 2.6

If a function $f: T \to X$ is (o)- strongly McShane integrable on T, then it has the property o(S^*M).

Proof. By Definition 1.3 that for every (o)- sequence $(b_n)_n$ in X, there is a corresponding sequence $(\gamma_n)_n$, of gauges $\gamma_n: T \to]0, +\infty[$ such that for every n and (γ_n) -fine M-partition $\{,(J_i, t_i), i = 1, \dots k\}$ of T holds the inequality

$$\sum_{i=1}^{k} |f(t_i)\mu(J_i) - F(J_i)| < \frac{b_n}{2}$$

Where F is the additive interval function from Definition 1.3. If we have two (γ_n) -fine M-partition $\{(J_i, t_i), i = 1, \dots k\}$and $\{(L_j, s_j), j = 1, \dots l\}$ of T, then

$$\sum_{i=1}^{k} \sum_{j=1}^{l} |f(t_i) - f(s_j)| \mu(J_i \cap L_j)$$

$$= \sum_{i=1}^{k} \sum_{j=1}^{l} |f(t_i)\mu(J_i \cap L_j) - f(s_j)\mu(J_i \cap L_j)|$$

$$= \sum_{i=1}^{k} \sum_{j=1}^{l} |f(t_i)\mu(J_i \cap L_j) - F(J_i \cap L_j)$$

$$+ F(J_i \cap L_j) - f(s_j)\mu(J_i \cap L_j)|$$

$$\leq \sum_{i=1}^{k} \sum_{j=1}^{l} \left| f(t_i)\mu\big(J_i \cap L_j\big) - F\big(J_i \cap L_j\big) \right| +$$

$$\sum_{i=1}^{k} \sum_{j=1}^{l} \left| F\big(J_i \cap L_j\big) - f\big(s_j\big)\mu\big(J_i \cap L_j\big) \right| \leq b_n$$

Because evidently$\{\big(J_i \cap L_j, t_i\big), i = 1, \dots k, j = 1, \dots l\}$, dhe $\{\big(J_i \cap L_j, s_j\big), j = 1, \dots l, i = 1, \dots k\}$ are (γ_n) -fine M-partition of T. Hence f has the property $o(S^*M)$.

Note that reasoning use to prove Lemma 2.6 cannot be used for (o) strong Henstock-Kurzweil integrability and the property o(S^*H).

Using Lemmas 2.5 and 2.6 we obtain the following result.

Theorem 2.7

A function $f: T \to X$ has the property $o(S^*M)$ if and only if is (o)- strongly McShane integrable on T.

Theorem 2.8

If a function $f: T \to X$ has the property $o(S^*M)$ then

$$\left|(oM)\int_T f\right| \leq (oM)\int_T |f| \quad (5)$$

Proof. Theorems [2.7] and [2.3] imply the (o)-McShane integrability of f and Collollary 2.4 yields the (o)-McShane integrability of $|f|$. Let (o)- sequence $(b_n)_n$ be given .Then by Definition 1.1 there is a corresponding sequence $(\gamma_n)_n$, of gauges $\gamma_n: T \to]0, +\infty[$ such that for every n and (γ_n) -fine M-partition $\{,(I_i, t_i), i = 1, \dots p\}$ of T holds the inequalities

$$\left|\sum_{i=1}^{p} f(t_i)\mu(I_i) - (oM)\int_T f\right| \leq b_n$$

$$\left|\sum_{i=1}^{p} |f(t_i)|\mu(I_i) - (oM)\int_T |f|\right| \leq b_n$$

For a fixed (γ_n) -fine M-partition $\{(I_i, t_i), i = 1, \dots p\}$ of T we obtain

$$\left|(oM)\int_T f\right| \leq \left|\sum_{i=1}^{p} f(t_i)\mu(I_i) - (oM)\int_T f\right| + \left|\sum_{i=1}^{p} f(t_i)\mu(I_i)\right|$$

$$< b_n + \sum_{i=1}^{p} |f(t_i)|\mu(I_i) \leq b_n + \left|\sum_{i=1}^{p} |f(t_i)|\mu(I_i) - (oM)\int_T |f|\right| + (oM)\int_T |f|$$

$$\leq 2b_n + (oM)\int_T |f|.$$

So, $(b_n)_n$ is the (o)- sequence proving the statement of the theorem.
Proposition 2.9 Let X be Dedekind complete Riesz Space . A function $f: [a,b] \to X$ is (oM) integrable on $I \subset [a,b]$, if and only if, for every (o)- sequence$(b_n)_n$ in X, there is a corresponding sequence $(\gamma_n)_n$, of gauges γ_n:$[a,b] \to]0,+\infty[$ such that for every n and (γ_n) -fine M-partition $\{(I_i,t_i), i=1,\dots,n\}$ and $\{(E_j,s_j), j=1,\dots m\}$ of $[a,b]=I$ holds the inequality

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\left|f(t_i)-f(s_j)\right|\left|I_i\cap E_j\right| \le b_n$$

Proof. Suppose that the function f has the property $o(S^*M)$. Let we consider two (γ_n) -fine M-partition $\{(I_i,t_i), i=1,\dots,n\}$,$\{(E_j,s_j), j=1,\dots m\}$ of I.

We have $|I_i| = \sum_{j=1}^{m}|I_i\cap E_j|$ and $|E_j| = \sum_{i=1}^{n}|I_i\cap E_j|$

we observe that $\left|\sum_{i=1}^{n} f(t_i)\,|I_i| - \sum_{j=1}^{m} f(s_j)\,|E_j|\right|.$

$$= \left|\sum_{j=1}^{m}\sum_{i=1}^{n} f(t_i)|I_i\cap E_j| - \sum_{i=1}^{n}\sum_{j=1}^{m} f(s_j)|I_i\cap E_j|\right| \le$$

$$\left|\sum_{i=1}^{n}\sum_{j=1}^{m} f(t_i)-f(s_j)|I_i\cap E_j|\right| \le b_n$$

Since f has the property $o(S^*M)$ we have proved the conditions of necessary.

For the converse let we set $\mathcal{F} = \{K_{i,j} = I_i\cap E_j, I_i, E_j\}$ are respectively elements of partitions π_1,and π_2, , $\mathring{K}_{i,j}\cap\mathring{K}_{i',j'}$

We know that $|f(t_i)-f(s_j)| = f(t_i)\vee f(s_j) - f(t_i)\wedge f(s_j)$.Define the tags t_{ij} and s_{ij} as follows $f(t_{ij}) = f(t_i)\vee f(s_j)$ and $f(s_{ij}) = f(t_i)\wedge f(s_j)$.

We get $|f(t_i)-f(s_j)| = f(t_{ij}) - f(s_{ij})$.

Let $\pi' = \{(K_{i,j}; t_{ij}): K_{i,j}\in\mathcal{F}\}$ and $\pi'' = \{(K_{i,j}; s_{ij}): K_{i,j}\in\mathcal{F}\}$. By the Lemma Henstock $\pi'$and $\pi''$are (γ_n) -fine M-partition

of I, so by hypothesis we have

$|\sigma(f,\pi') - \sigma(f,\pi'')| \le \left|\sigma(f,\pi') - (oM)\int_a^b f\right| + \sigma(f,\pi')\left|-(oM)\int_a^b f\right| \le 2b_n$. On the other hand $\sum_{i=1}^{n}\sum_{j=1}^{m}|f(t_i)-f(s_j)||I_i\cap E_j| = \left|\sum_{K_{i,j}}[f(t_{ij})-f(s_{ij})]\,|I_i\cap E_j|\right|$

$= |\sigma(f,\pi') - \sigma(f,\pi'')|$ which proves the theorem.

Conclusion

In this article we define the property $o(S^{*}\mathrm{M})$, o(S^{*}H) and we compare the order type integrals, showing that order-strongly type integrals respect almost everywhere equality for order- bounded functions.We arrive new result in relation with (o)-strong Henstock ones. The order- strong McShane integrability of a function imply that it has the property $o(S^{*}\mathrm{M})$ but this condition cannot be used for the order-strong Henstock integrability and the property $o(S^{*}\mathrm{H})$.

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Solution for Second-Order Differential Equation Using Least Square Method

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Abstract

This paper study the numerical method for solving differential equation. The continuous least square method (CLSM) alonside with the L_2 norm are used to obtain explicit solution and the minimum approximation error respectively.

Keywords: Differential Equation, Continuous Least Square Method and L_2 norm.

1. Introduction

The CLSM is an important issue in solving ODEs, which play a great role in mathematical physics. The efforts of finding several methods for solving problems of ODEs has been practice by many researchers[1, 2]. The (CLSM) is use to solve complex problems involving ODEs, FDEs and PDEs [3, 4, 5]. The authors in [6, 7, 8] introduced numerical approximation approach that involve curves and surfaces which play a vital role in numerical analysis.

The aim of this paper to promote numerical technique for (ODEs). The L_2 norm alongside with the (CLSM) are used to obtain the minimum approximation error and numerical approximate solution, respectively.

2. Priliminaries

In this research, the CLSM for solving ODEs is considered.

$$\mathrm{L(y) = f(x)\ for\ \ x\ \in\ domain}\ \Omega$$

$$\mathrm{W(y) = g(x)\ for\ \ x\ \in\ domain}\ \ \delta\Omega.$$

Where L stand for differential operator and Ω indicates the domain in R^1 *or* R^2 *or* R^3, while W refers to the boundary operator. The approximate solution of ODEs can be written as

$$\tilde{y} = \sum_{i=1}^{n} q_i\, C^i(X). \tag{2.1}$$

$C^i(X)$ and q_i represent the weighted basis function and the coefficients (weights) respectively, the q_i is realize using the CLSM. Let the residual R_L(X) and R_W(X) be defined as

$$R_L(\text{x}, \tilde{\text{y}}) \; = \; \text{L}\,(\tilde{\text{y}}) - \text{f(x)} \quad \text{for} \quad \text{x} \in domain\,\Omega\,. \tag{2.2}$$

$$R_W(\text{x}, \tilde{\text{y}}) = w\,(\tilde{\text{y}}) - g(x) \quad for\;\; x \; \in boundary\; \delta\Omega. \tag{2.3}$$

Substituting y_{exact} into Eq. (2.2) and Eq. (2.3) leads to $R_L(\text{x}, y_{exact}) = 0$ and $R_W(\text{x}, y_{exact}) = 0$ respectively.

3. Continuous Least Square Method

The CLSM is an approximation process that involves the use of L_2 norm to solve ODEs, the q_i from Eq. (2.1) are obtain using the CLSM, considering the Minimize error function as

$$E = \int_{\Omega} R_L^2\,(x, \tilde{y})dx + \int_{\alpha\Omega} R_W^2(x, \tilde{y})dx. \tag{3.1}$$

The first derivative of Eq. (3.1) with respect to q_i and equating to zero leads to

$$\frac{\partial E}{\partial q_i} = 0, \qquad for\; i = 1, \ldots N,$$

which yields

$$\int_{\Omega} R_L(x, \tilde{y})\; \frac{\partial R_L}{\partial q_i} dx + \int_{\alpha\Omega} R_W(x, \tilde{y})\; \frac{\partial R_L}{\partial q_i} dx = 0 \quad i = 1, \ldots, N. \tag{3.2}$$

Eq. (3.2) is algebraic equation which can be written in the form of

$$Ma = b. \tag{3.3}$$

Note that *M is n x n* matrix, $a = [q_1, q_2\,, q_3, \ldots, q_n\;\;]^T$, and some column vector *b*.

4. Example

In this section, we make use of the results obtained from the CLSM and implement it by considering an example.

***4.1* Example 1:** Consider the 2^{nd}-order initial value problem.

$$\frac{d^2y}{dx^2} + (x+1)\frac{dy}{dx} + \sqrt{5}\,y = 0, \qquad y(0) = 0.3, \qquad y^I(0) = 0.4, \tag{4.1}$$

where $0 \leq x \leq 1$. Let

$$L(x,y) = \frac{d^2y}{dx^2} + (x+1)\frac{dy}{dx} + \sqrt{5}\,y \tag{4.2}$$

Step 1: Let the polynomial.

$$\tilde{y} = \sum_{i=1}^{N} q_i\,x^i + y_0. \tag{4.3}$$

Step 2: We set $y_0 = 0.3$ and $q_1 = 0.4$ in Eq. (4.3) to satisfy the boundary condition.

Step 3: The residual

$$R(x) = \frac{d^2\tilde{y}}{dx^2} + (x+1)\frac{d\tilde{y}}{dx} + \sqrt{5}\,\tilde{y}. \tag{4.4}$$

By replacing $\tilde{y}(x)$ from Eq. (4.3) into Eq. (4.4), we will get:

$$R(x) = \frac{d^2\left(\sum_{i=1}^{N} q_i\,x_i + 0.3\right)}{dx^2} + (x+1)\frac{d\left(\sum_{i=1}^{N} q_i\,x_i + 0.3\right)}{dx} + \sqrt{5}\left(\sum_{i=1}^{N} q_i x_i + 0.3\right) \tag{4.5}$$

Step 4: The minimum error is obtain by considering

$$E = \int_0^1 R^2(x)dx. \tag{4.6}$$

Step 5: The continous least square solution is obtain by solving Eq. (4.6).

$$\frac{\partial E}{\partial q_i} = 0, \qquad for \quad i = 1,..,N, \tag{4.7}$$

$$\int_0^1 R(x)\frac{\partial R}{\partial q_i}dx = 0, \qquad i = 1,\dots,N. \tag{4.8}$$

Substituting Eq. (4.5) into Eq. (4.8) for N = 3, we obtain the following matrices with the help of Matlab program

$$D = \begin{pmatrix} 45.6127 & 61.6099 \\ 61.6099 & 89.0385 \end{pmatrix}, \quad b = \begin{pmatrix} 16.5054 \\ 21.2004 \end{pmatrix}, \quad a = \begin{pmatrix} q_2 \\ q_3 \end{pmatrix}. \tag{4.9}$$

Solving Eq. (4.9) lead to

$q_1 = 0.4, q_2 = -0.615629, q_3 = 0.187878.$

And the approximate solution is given as

$$\tilde{y} = 0.187878\, x^3 - 0.615629\, x^2 + 0.4\ \ x + 0.3. \tag{4.10}$$

The exact solution is given by

$$\begin{aligned} y_{exact} &= 0.2560850487909125 e^{-x-\frac{x^2}{2}}(1.\mathrm{HermiteH}[-1+\sqrt{5}, \frac{1}{\sqrt{2}} + \frac{x}{\sqrt{2}}] \\ &- 0.5189851135455146\mathrm{Hypergeometric1F1}[\frac{1}{2}(1-\sqrt{5}), \frac{1}{2}, (\frac{1}{\sqrt{2}} + \frac{x}{\sqrt{2}})^2]).. \end{aligned} \tag{4.11}$$

The approximate with exact solutions and the error are depicted in figure 4.1 and figure 4.2 for $N = 3$

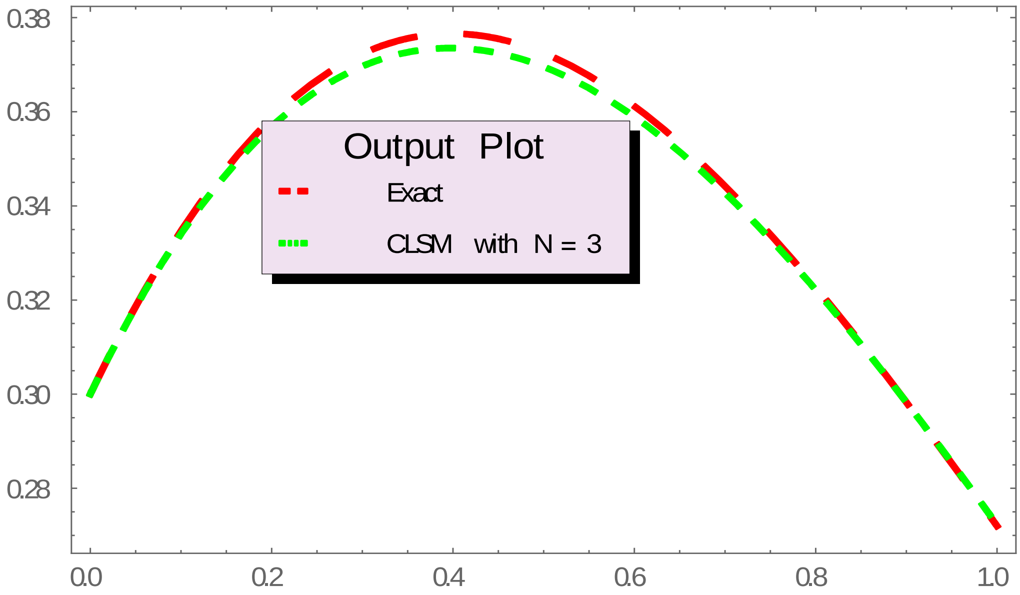

Figure 4.1: Showing the result ofExample 1 with N = 3.

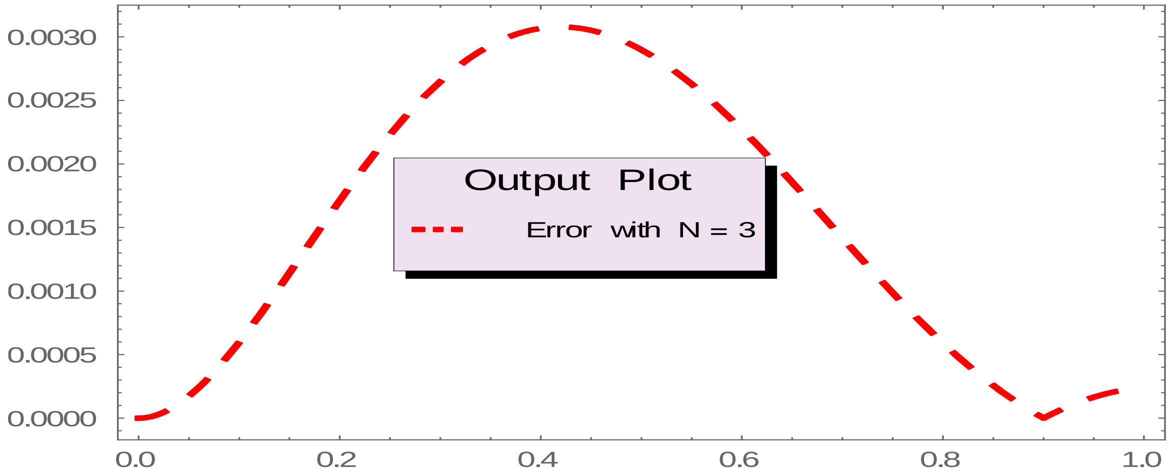

Figure 4.2: Showing the Error plots of Example 1 with N=3.

For $N = 5$, we obtain the following matrices

$$D = \begin{pmatrix} 45.61 & 61.61 & 77.88 & 94.25 \\ 61.61 & 89.03 & 116.91 & 145.00 \\ 77.88 & 116.91 & 158.05 & 200.27 \\ 94.25 & 145.00 & 200.27 & 258.15 \end{pmatrix}, \quad b = \begin{pmatrix} 16.51 \\ 21.20 \\ 25.91 \\ 30.62 \end{pmatrix}, \quad a = \begin{pmatrix} q_2 \\ q_3 \\ q_4 \\ q_5 \end{pmatrix}. \qquad (4.12)$$

And the approximate solution is

$$\tilde{y} = -0.0649883x^5 + 0.209667x^4 - 0.0367039x^3 - 0.535967x^2 + 0.4x + 0.3. \quad (4.12)$$

The approximate with exact solutions and the error are depicted in figure 4.3 and figure 4.4 for N =5

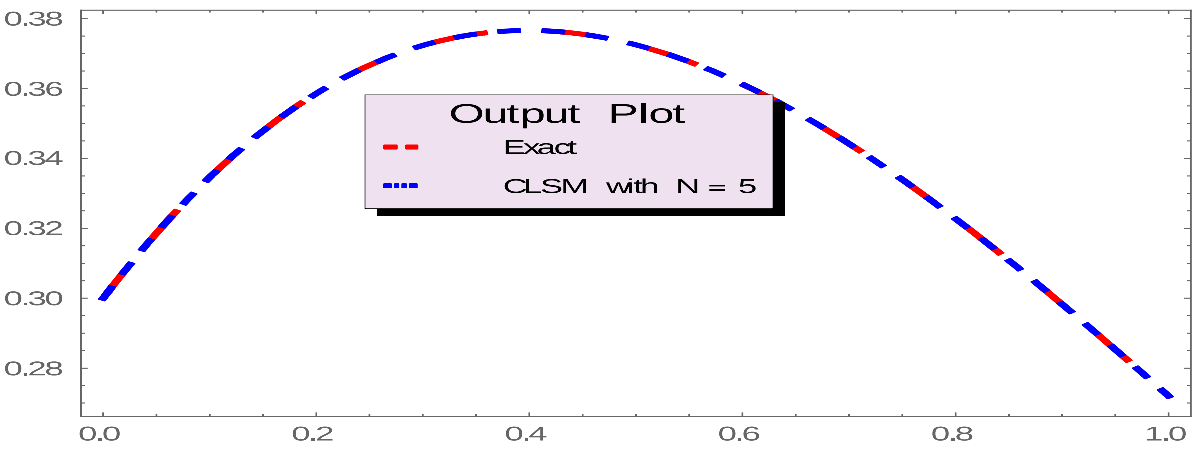

Figure 4.3: Showing the results of Example 1 with N = 5.

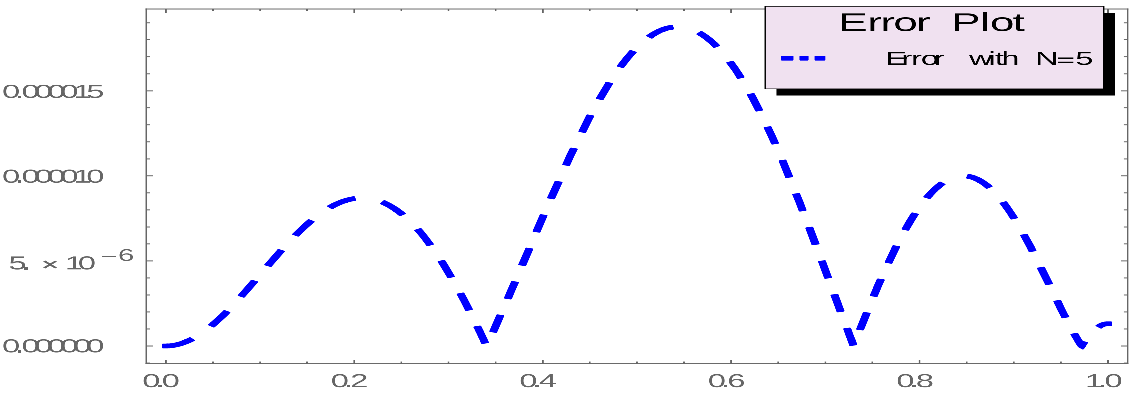

Figure 4.4: Showing the Error plots of Example 1 with N=6.

The comparison between the exact CLSM and error are depicted in figure 4.5 and figure 4.6 for $N = 3$ and that of N = 5.

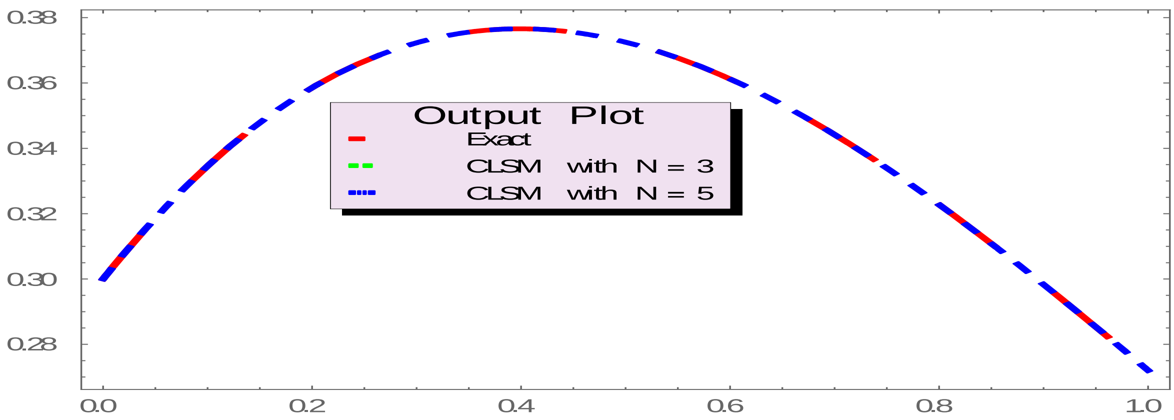

Figure 4.5: Showing the results of Example 1 with N = 3 and N=6

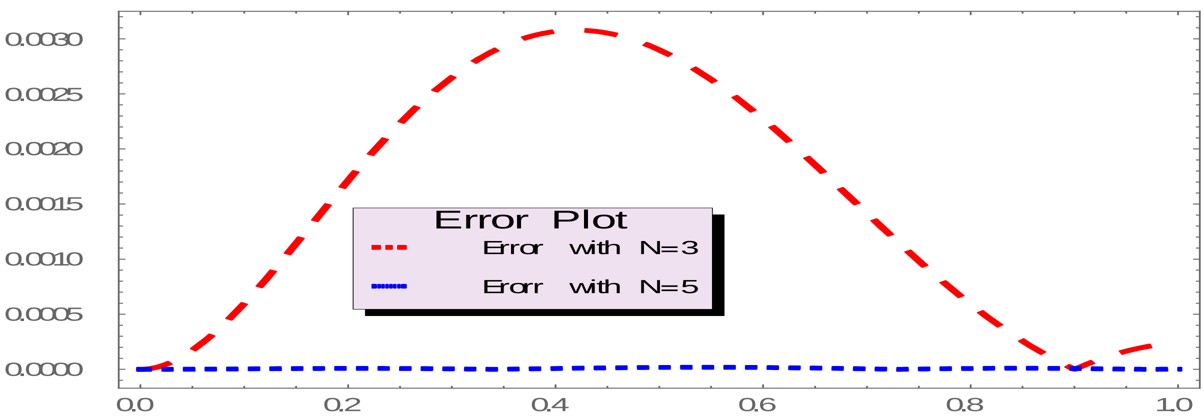

Figure 4.6: Showing the Error plots of Example 1 with N=3 and N=6.

x	Y exact	Y CLSM. with N = 3	y CLSM with N = 5	Errors with N = 3	Errors with N = 5
0	0.3	0.3	0.3	0	0
0.1	0.334628	0.334032	0.334624	0.000596543	4.18861×10^{-7}
0.2	0.358591	0.356878	0.358582	0.00171316	8.66736×10^{-6}
0.3	0.372317	0.36967	0.372312	0.00265057	4.31975×10^{-6}
0.4	0.376591	0.373524	0.376598	0.00306711	7.5654×10^{-6}
0.5	0.372476	0.369577	0.372494	0.00289852	0.0000175481
0.6	0.361227	0.358955	0.361243	0.00227139	0.0000166093
0.7	0.344201	0.342784	0.344205	0.00141683	4.45457×10^{-6}
0.8	0.322781	0.322191	0.322773	0.000590218	8.16031×10^{-6}
0.9	0.298305	0.298303	0.298297	1.2752×10^{-6}	7.54997×10^{-6}
1.0	0.272007	0.2722489	0.272008	0.000242287	1.30805×10^{-6}

Table 4.1: Data of errors analysis with N=3 and N=6 for 2nd - order ODE

5. Conclusion

This paper investigates numerical methods for solving differential equation. The (CLSM) together with the L_2 norm is used to find better approximation with minimal error by solving differential equations, The results obtained is shown to be equivalent with the exact solution with minimum error. The results are supported with MATLAB and Wolfram Mathematica 11.

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Statistical Physics Approach to Small Scale Artificial Neural Networks

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Abstract

We discuss the statistical physics approach to the application of small-scale artificial neural networks (ANNs) well-trained with data collected from the '*Ab Initio*' principle, as it was proposed by Wang, Jiang, and Zhou in 2020 for mimicking the microscopic statistical states of a quantum system. Such networks could be used for efficient numerical modeling of different statistical systems: spin structures, phase transitions, and others related statistical systems. We investigate the alternative network configuration based on the Hodgkin – Huxley elements and demonstrate that the reproduction of the macroscopic states for the Ising quantum system can be done with a sufficiently smaller number of neurons, and with a lower computational cost.

Keywords: Ising ferromagnetic model, data collection, Hodgkin-Huxley neuron.

1. Introduction

For the purpose of modeling microscopic states of quantum physical systems, a set of different approaches has been proposed in the literature: generic machines and quantum simulators [1], intricate neural networks [2], the wave-function *Ansatz* [3], and others. All those approaches could be applied for efficient numerical modeling of different physical systems: spin structures, phase transitions, and other related statistical systems.

In the Wang-Jiang-Zhou paradigm [4], the small-scale artificial neural networks (ANNs) have been trained with data collected from the '*Ab Initio*' principle. By this method, the quantum ensemble and the distribution of micro-states are built with an autoregressive neural network in the form of Masked Autoencoder for Distribution Estimation (MADE) [5]. The Wang-Jiang-Zhou network gets the data on the microscopic states of the quantum system from experiments or the first-principle calculations and then produces the probability distributions of these microscopic states. Particularly, in [4] the Ising ferromagnetic model has been studied. Thus, this well-trained MADE predicts the macroscopic phase state of the system based on the microscopic computations.

Here we discuss the alternative approach to construct small-scale artificial neural networks consisting of Hodgkin – Huxley (HH) elements [6] for modeling macroscopic states of the Ising quantum system. We demonstrate that in the frame of our model it can be done with a sufficiently smaller number of neurons, and with a lower computational cost.

2. Networks for the Ising Quantum System

Ising Ferromagnetic Model

The Hamiltonian of the Ising ferromagnetic model [7] has a form:

$$H(\mathbf{s}) = -\sum_{\substack{i,j \\ i \neq j}} J_{ij} s_i s_j - \sum_j h_j s_j \ , \tag{1}$$

where the spin variables s_i are equal to –1 or +1, and J_{ij} are positive (ferromagnetic). We consider here the simplified case, where all $J = 1$, and the parameters of the interaction with the external field $h = 0$.

Our example is represented with the cyclic Ising quantum model consisting of three interacting spins s_1, s_2, s_3 (Fig. 1).

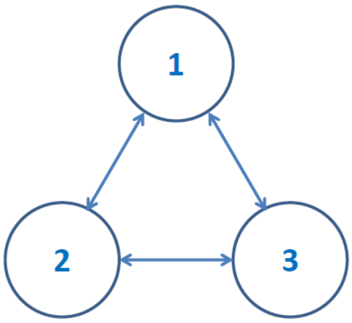

Figure 1. Configuration of the three spin cyclic Ising model.

The probabilities of the microscopic states are given by the equations of the type [4] (here i,j,k = 1,2,3):

$$p(s_1, s_2, s_3) \propto \exp\left\{-\frac{s_1 s_2 + s_2 s_3 + s_3 s_1}{T}\right\};$$
$$p(s_1, s_2, s_3) = p(s_i)\, p(s_i \mid s_j)\, p(s_k \mid s_i, s_j);$$
$$p(\mathbf{s}) = p(s_1 \mid s_2, s_3)\, p(s_3 \mid s_2)\, p(s_2)\ . \tag{2}$$

The computations in paradigm (2) consistently follow the principles "from bottom to top": one should start with the probabilities for the particular spins, and gradually end up with the characteristics of the whole statistical system.

Eqs. (2) for the microscopic probabilities serve for the computation of the macroscopic properties of the system:

$$H(\mathbf{s}) = -T \log\, p(\mathbf{s})\ . \tag{3}$$

Eq.(3) provides the information about the macroscopic characteristics of the quantum statistical system based on the computation of the microscopic state probabilities, like in (2).

Deep MADE Network for the Ising Model

Deep learning MADE network for the three spin Ising model is represented in Fig. 2.

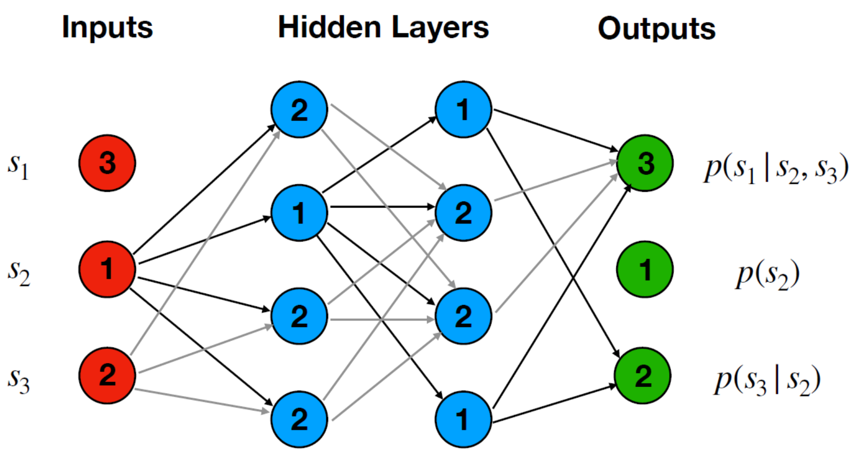

Figure 2. Deep learning MADE for the three spin cycle Ising model [4].

The MADE network in Fig. 2 must be trained at a fixed temperature *T*, and it reproduces the phase structure of the Ising system by the computation of the set for the probabilities (2), for further details see [4].

One can see in Fig. 2, that the architecture of the MADE network is quite complex even for a small number of spins in the Ising model. Such a network must have input (red) and output (green) layers, and also a few hidden (blue) layers to evaluate the full set of (2).

3. Small Scale Hodgkin – Huxley Network for the Modeling Macroscopic States

Ordinary Differential Equation Model for Hodgkin – Huxley Neuron

As an alternative approach, we propose here to mimic the system in Fig. 1 with the small-scale ANN consisting of Hodgkin – Huxley neurons based on the phenomenological model [6] for the membrane action potential produced in the axons of real cells under the external electrical or optogenetic stimulations. Each HH neuron is represented with the system of four non-linear ordinary differential equations:

$$\begin{aligned}
&C_M \cdot \frac{dv}{dt} = -g_{Na} m^3 h \cdot (v - E_{Na}) - g_K n^4 \cdot (v - E_K) - g_{Cl} \cdot (v - E_{Cl}) + I(t)\ ; \\
&\frac{dm}{dt} = \alpha_m(v) \cdot (1 - m) - \beta_m(v) \cdot m\ ; \\
&\frac{dn}{dt} = \alpha_n(v) \cdot (1 - n) - \beta_n(v) \cdot n\ ; \\
&\frac{dh}{dt} = \alpha_h(v) \cdot (1 - h) - \beta_h(v) \cdot h\ .
\end{aligned} \tag{4}$$

Here $v(t)$ stands for the axon membrane action potential, $m(t)$, $n(t)$, $h(t)$ are the membrane gate variables, and the control signal $I(t)$ is represented by the sum of external and synaptic currents entering the cell.

The nonlinear functions of the action potential v are given by:

$$\alpha_m(v)=\frac{0.1\cdot(25-v)}{\exp\left\{\frac{25-v}{10}\right\}-1}\;;\;\beta_m(v)=4\cdot\exp\left\{-\frac{v}{18}\right\}\;;$$
$$\alpha_n(v)=\frac{0.01\cdot(10-v)}{\exp\left\{\frac{10-v}{10}\right\}-1}\;;\;\beta_n(v)=0.125\cdot\exp\left\{-\frac{v}{80}\right\}\;; \tag{5}$$
$$\alpha_h(v)=0.07\cdot\exp\left\{-\frac{v}{20}\right\}\;;\;\beta_h(v)=\frac{1}{\exp\left\{\frac{30-v}{10}\right\}+1}\;.$$

The set of constants covers the potentials E_{Na} (the equilibrium potential at which the net flow of *Na* ions is zero), E_K (the equilibrium potential at which the net flow of *K* ions is zero), E_{Cl} (the equilibrium potential at which leakage is zero) in mV, the membrane capacitance C_M and the conductivities g_{Na} (the sodium channel conductivity), g_K (the potassium channel conductivity), g_{Cl} (the leakage channel conductivity) in mS/cm^2:

$$\begin{aligned} &g_{Na}=120;\; E_{Na}=115\;;\\ &g_K=36;\;\; E_K=-12\;;\\ &g_{Cl}=0.3;\; E_{Cl}=10.36. \end{aligned} \tag{6}$$

The important property of the model (4)-(6) is the existence of a ***threshold***: the control current *I* does not stimulate a spiking or bursting regime in the output *v* if it stays below a certain level. If the current overcome a minimum threshold level, the HH neuron produces a spike; for the current stimulus above a certain greater level, the outcome is a spike train, etc. The magnitude of the spikes almost does not change as the current magnitude increases.

Macroscopic States of the Ising System via the Resting / Spiking HH Neuron Regimes

Our approach is based on the following idea: the ANN should evaluate the Hamiltonian energy levels rather than the probabilities of the microscopic states. The network possesses an algorithm which computes in a simple and fast manner the Hamiltonian of our quantum system, and the ANN output following a set of microscopic inputs must reproduce the correct probabilities for the macroscopic states based on the all possible set of the microscopic states like in Table 1.

In Table 1 we collect all possible microscopic states of the three spin cyclic Ising model in Fig. 1 (2^3 = 8 microstates) versus the corresponding macroscopic states (they are marked in Table 1 with the blue and red color, with the probabilities 2/8 = 0.25 and 6/8 = 0.75).

Table 1. Macrostates vs microstates for the three spin cycle Ising model

s_1	-1	-1	-1	-1	$+1$	$+1$	$+1$	$+1$	**microstates**
s_2	-1	-1	$+1$	$+1$	-1	-1	$+1$	$+1$	
s_3	-1	$+1$	-1	$+1$	-1	$+1$	-1	$+1$	
s_1s_2	$+1$	$+1$	-1	-1	-1	-1	$+1$	$+1$	products of microstates
s_2s_3	$+1$	-1	-1	$+1$	$+1$	-1	-1	$+1$	
s_3s_1	$+1$	-1	$+1$	-1	-1	$+1$	-1	$+1$	
$s_1s_2 + s_2s_3 + s_3s_1$	**+3**	**−1**	**−1**	**−1**	**−1**	**−1**	**−1**	**+3**	**macrostates**

The set of possible macroscopic states in Table 1 defines the Hamiltonian of the system:

$$H = -J(s_1s_2 + s_2s_3 + s_3s_1)\ ;\ J = 1. \tag{7}$$

and, therefore, the probabilities, as in (2):

$$p(s_1, s_2, s_3) \propto e^{-\frac{H}{T}}. \tag{8}$$

Now let's demonstrate that a ***single*** HH neuron is enough to reproduce the probabilities of all macroscopic states of the cyclic Ising system of ***three*** spins, see Fig. 3.

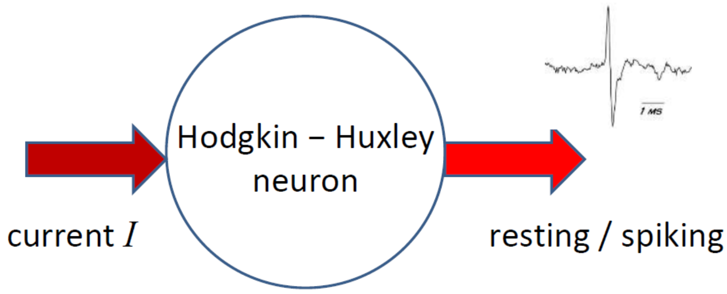

Figure 3. Single Hodgkin − Huxley neuron network for the evaluation of the macroscopic states via the resting / spiking regime.

Such a system of three spins has two principally different macroscopic configurations: a) all three spins are co-oriented, or b) one of them is oriented opposite towards the other two companions), see Table 1. By that, to evaluate them, we need two different dynamical regimes of the HH neuron, such as resting and spiking.

Let's define the current I entering to the HH neuron in Fig. 3 as:

$$I = \frac{(s_1 s_2 + s_2 s_3 + s_3 s_1) + 1}{4} \cdot I_{\text{threshold}}, \tag{9}$$

where $I_{\text{threshold}}$ is the threshold level creating a single spike. Thus, the resting outcome of the membrane action potential v of HH neuron corresponds to the macroscopic state $s_1s_2+s_2s_3+s_3s_1=-1$ with $I = 0$, while the single spike outcome has $s_1s_2+s_2s_3+s_3s_1=+3$ with $I = I_{\text{threshold}}$. The probabilities to get a resting / spiking outcome for the HH neuron follow exactly the probabilities of the corresponding macroscopic states of the Ising system (7).

4. Conclusions

The concept of an ANN dealing with the probabilities of macroscopic states of a statistical system rather than with the probabilities of microscopic states involves sufficiently less number of neurons.

Usage of neurons with the threshold input (i.e. with the different dynamical regimes as outcome properties), such as Hodgkin – Huxley neuron, allows decreasing the number of neurons in the network.

The computational costs to evaluate the spin products s_is_j is much less than the computational costs for the exponents ot chain products of the probabilities (2).

Particularly, for the cyclic system of three Ising elements, one HH neuron is enough to analyze all the probabilities of the macroscopic states. For a greater number of the Ising spins, the number of HH elements in the ANN is also sufficiently smaller to compare with MADE network.

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On Wijsman I_λ-Statistical Convergence for Sequences of Sets in Intuitionistic Fuzzy Normed Spaces

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Abstract

In this study, we examine the notions of Wijsman $I-[V,\lambda]-$summability and Wijsman $I_\lambda-$statistical convergence for sequences of sets with regards to (briefly, w.r.t) the intuitionistic fuzzy norm (briefly, IFN) (μ,ν), reveal their relationship, and make some observations about these classes. We mainly study the relation between these two new methods and the relation between Wijsman $I-\lambda-$statistical convergence and Wijsman $I-$statistical convergence for sequences of sets in the corresponding intuitionistic fuzzy normed space.

Keywords: Wijsman convergence, Wijsman $\mathrm{I}-\lambda-$statistical convergence, intuitionistic fuzzy normed space.

1. Introduction

Statistical convergence was firstly examined by Fast [1]. Some beneficial results on this topic can be found in [2-3]. As a consequence of the study of ideal convergence defined by Kostyrko et al [4], there has been valuable studies to discover summability works of the classical theories. Ideal convergence became a noteworthy topic in summability theory after the studies of [5-7]. In other direction, the idea of λ-statistical convergence was worked by Mursaleen [8] as an extension of the $[V,\lambda]$-summability of Leinder [9]. λ-statistical convergence is a special case of the more general A-statistical convergence studied in Ref. [10].

Theory of fuzzy sets (FSs) was firstly given by Zadeh [11]. This work affected deeply all the scientific fields. The Theory of FSs has submitted to employ imprecise, vagueness and inexact data [11]. FSs, have been widely implemented in different disciplines and technologies. The Theory of FSs cannot always cope with the lack of knowledge of membership degrees. That is why Atanassov [12] investigated the theory of IFS which is the extension of the theory of FSs. Kramosil and Michalek [13] investigated fuzzy metric space (FMS) utilizing the concepts fuzzy and probabilistic metric space. The FMS as a distance between two points to be a non-negative fuzzy number was examined by Kaleva and Seikkala [14]. George and Veeramani [15] gave some qualifications of FMS. Some basic features of FMS were given and significant theorems were proved in [16]. Moreover, FMS has used by practical researches as for example decision-making, fixed point theory, medical imaging. Park [17] generalized FMSs and defined

IF metric space (IFMS). Park utilized George and Veeramani's [15] opinion of using t-norm and t-conorm to the FMS meantime describing IFMS and investigating its fundamental properties. The concept of IF-normed spaces (IFNS for shortly) was given by Lael and Nourouzi [18]. Statistical convergence, ideal convergence and different features of sequences in INFS were examined by several authors [19-22].

Recently, convergence of sequences of sets was studied by several authors. Nuray and Rhoades [23] presented the idea of statistical convergence of set sequences and established some essential theorems. Convergence for sequences of sets became a notable topic in summability theory after the studies of (see, [24-27]).

2. Main Results

In this section we deal with the relation between these two new methods and with relations between Wijsman $I-\lambda-$statistical convergence and Wijsman $I-$statistical convergence for sequences of sets with regards to the IFN.

Definition 2.1. Let $(X,\phi,\rho,\Delta,\Theta)$ be an IFNS. Then, a sequence $\{A_k\}$ is named to be Wijsman $I-$statistically convergent to $A\in X$ w.r.t the IFN (ϕ,ρ) and is indicated by

$$A_k \xrightarrow{(\phi,\rho)} A\left(S^{(\phi,\rho)}(I_W)\right)$$

if for each $\varepsilon,\delta>0$ and $t>0$,

$$\left\{n\in\mathbb{N}:\frac{1}{n}\left|\{k\le n: |\phi(x,A_k,t)-\phi(x,A,t)|\le 1-\varepsilon \; or \; |\rho(x,A_k,t)-\rho(x,A,t)|\ge\varepsilon\}\right|\ge\delta\right\}\in I.$$

Let I_f be the family of all finite subsets of $\mathbb{N}$. Then, I_f is an admissible ideal in $\mathbb{N}$, and Wijsman $I-$statistical convergence coincides with the notion of Wijsman statistical convergence in IFNS.

Let $\lambda=(\lambda_n)$ be a non-decreasing sequence of positive numbers tending to ∞ such that $\lambda_{n+1}=\lambda_n+1,\lambda_1=1$. The collection of such a sequence λ will be demonstrated by Δ.
The generalized de Valée-Pousin mean is given by

$$t_n(x)=\frac{1}{\lambda_n}\sum_{k\in I_n}x_k,$$

where $I_n=[n-\lambda_n+1,\lambda_n]$. Now, we acquire our new definitions.

Definition 2.2. A sequence $\{A_k\}$ is named to be Wijsman $I-[V,\lambda]$ −summable to A ∈ X w.r.t the IFN (ϕ,ρ) and is signified by

$$I_W-[V,\lambda]^{(\phi,\rho)}-lim A_k=A,$$

if for each $\delta>0$ and $t>0$,

$$\{n\in\mathbb{N}:|\phi(x,t_n(A_k),t)-\phi(x,t_n(A),t)|\leq 1-\delta\ or\ |\rho(x,t_n(A_k),t)-\rho(x,t_n(A),t)|\geq\delta\}\in I.$$

Definition 2.3. A sequence $\{A_k\}$ is named to be Wijsman $I-\lambda$ −statistically convergent or I_W-S_λ convergent to A ∈ X w.r.t the IFN (ϕ,ρ) and is indicated by

$$I_W-S_\lambda^{(\phi,\rho)}-lim A_k=A\ or A_k\to A\left(I_W-S_\lambda^{(\phi,\rho)}\right)$$

if for every $\varepsilon,\delta>0$ and $t>0$,

$$\left\{n\in\mathbb{N}:\frac{1}{\lambda_n}|\{k\in I_n:|\phi(x,A_k,t)-\phi(x,A,t)|\leq 1-\varepsilon\ or|\rho(x,A_k,t)-\rho(x,A,t)|\geq\varepsilon\}|\geq\delta\right\}\in I.$$

We can indicate by $S^{(\phi,\rho)}(I_W), S_\lambda^{(\phi,\rho)}(I_W), [V,\lambda]^{(\phi,\rho)}(I_W)$ the collections of all I_W-statistically convergent, $I_W-S_\lambda^{(\phi,\rho)}$ −convergent and $I_W-[V,\lambda]^{(\phi,\rho)}$ −convergent sequences, respectively.

Theorem 2.1. *Let $(X,\phi,\rho,\Delta,\Theta)$ be an IFNS and let $\lambda=(\lambda_n)$ be a sequence in Δ.*

a. If $A_k\to A\left([V,\lambda]^{(\phi,\rho)}(I_W)\right)$, then $A_k\to A\left(S_\lambda^{(\phi,\rho)}(I_W)\right)$,

b. If $A_k\in L_\infty(X)$, the space of all bounded set sequences of X and $A_k\to A\left(S_\lambda^{(\phi,\rho)}(I_W)\right)$, then $A_k\to A\left([V,\lambda]^{(\phi,\rho)}(I_W)\right)$,

c. $S_\lambda^{(\phi,\rho)}(I_W)\cap L_\infty(X)=[V,\lambda]^{(\phi,\rho)}(I_W)\cap L_\infty(X)$.

Proof. a. By hypothesis, for every $\varepsilon,\delta>0$ and $t>0$, let $A_k\to A\left([V,\lambda]^{(\phi,\rho)}(I_W)\right)$. We acquire

$$\sum_{k\in I_n}(|\phi(x,A_k,t)-\phi(x,A,t)|\ or\ |\rho(x,A_k,t)-\rho(x,A,t)|)$$
$$\geq \sum_{\substack{k\in I_n, |\phi(x,A_k,t)-\phi(x,A,t)|\leq 1-\varepsilon \\ or |\rho(x,A_k,t)-\rho(x,A,t)|\geq\varepsilon}}(|\phi(x,A_k,t)-\phi(x,A,t)|\ or\ |\rho(x,A_k,t)-\rho(x,A,t)|)$$
$$\geq \varepsilon|\{k\in I_n: |\phi(x,A_k,t)-\phi(x,A,t)|\leq 1-\varepsilon\ or\ |\rho(x,A_k,t)-\rho(x,A,t)|\geq\varepsilon\}|.$$

Then, notice that

$$\frac{1}{\lambda_n}|\{k\in I_n: |\phi(x,A_k,t)-\phi(x,A,t)|\leq 1-\varepsilon\ or\ |\rho(x,A_k,t)-\rho(x,A,t)|\geq\varepsilon\}|\geq\delta$$

$$\Rightarrow \frac{1}{\lambda_n}\sum_{k\in I_n}|\phi(x,A_k,t)-\phi(x,A,t)|\ \leq (1-\varepsilon)\delta\ or|\rho(x,A_k,t)-\rho(x,A,t)|\geq\varepsilon\delta\,,$$

which means that

$$\left\{n\in\mathbb{N}: \frac{1}{\lambda_n}|\{k\in I_n: |\phi(x,A_k,t)-\phi(x,A,t)|\leq 1-\varepsilon\ or|\rho(x,A_k,t)-\rho(x,A,t)|\geq\varepsilon\}|\geq\delta\right\}$$
$$\subseteq \left\{n \in\mathbb{N}: \frac{1}{\lambda_n}\left\{\sum_{k\in I_n}|\phi(x,A_k,t)-\phi(x,A,t)|\ \leq 1-\varepsilon\ or\ \sum_{k\in I_n}|\rho(x,A_k,t)-\rho(x,A,t)|\geq\varepsilon\right\} \geq \varepsilon\delta\right\}.$$

Since $A_k\to A\left([V,\lambda]^{(\phi,\rho)}(I_W)\right)$, we directly see that $A_k\to A\left(S_\lambda^{(\phi,\rho)}(I_W)\right)$, consequently the result is obtained.

b. We assume that $A_k\in L_\infty(X)$ and $A_k\to A\left(S_\lambda^{(\phi,\rho)}(I_W)\right)$. The inequalities $|\phi(x,A_k,t)-\phi(x,A,t)|\geq 1-M$ or $|\rho(x,A_k,t)-\rho(x,A,t)|\leq M$ hold for all *k*. Let $\varepsilon>0$. Then, we get

$$\sum_{k\in I_n}(|\phi(x,A_k,t)-\phi(x,A,t)|\ or\ |\rho(x,A_k,t)-\rho(x,A,t)|)$$
$$= \sum_{\substack{k\in I_n, |\phi(x,A_k,t)-\phi(x,A,t)|\le 1-\varepsilon \\ or|\rho(x,A_k,t)-\rho(x,A,t)|\ge\varepsilon}} (|\phi(x,A_k,t)-\phi(x,A,t)|\ or\ |\rho(x,A_k,t)-\rho(x,A,t)|)$$
$$+ \sum_{\substack{k\in I_n |\phi(x,A_k,t)-\phi(x,A,t)|> 1-\varepsilon \\ or|\rho(x,A_k,t)-\rho(x,A,t)|<\varepsilon}} (|\phi(x,A_k,t)-\phi(x,A,t)|\ or\ |\rho(x,A_k,t)-\rho(x,A,t)|)$$
$$\le \frac{M}{\lambda_n}|\{k\in I_n: |\phi(x,A_k,t)-\phi(x,A,t)|\le 1-\varepsilon\ or|\rho(x,A_k,t)-\rho(x,A,t)|\ge\varepsilon\}|+\varepsilon.$$

Emphasize that

$$A_{(\phi,\rho)}(\varepsilon,t) := \left\{n\in\mathbb{N}: \frac{1}{\lambda_n}|\{k\in I_n: |\phi(x,A_k,t)-\phi(x,A,t)|\le 1-\varepsilon\ or|\rho(x,A_k,t)-\rho(x,A,t)|\ge\varepsilon\}| \ge \frac{\varepsilon}{M}\right\}\in I.$$

If $n\in A^c_{(\phi,\rho)}(\varepsilon,t)$, then we have

$$\frac{1}{\lambda_n}\sum_{k\in I_n}|\phi(x,A_k,t)-\phi(x,A,t)|>1-2\varepsilon\ or\ \frac{1}{\lambda_n}\sum_{k\in I_n}|\rho(x,A_k,t)-\rho(x,A,t)|<2\varepsilon.$$

Now

$$B_{(\phi,\rho)}(\varepsilon,t) = \left\{n \in\mathbb{N}: \frac{1}{\lambda_n}\sum_{k\in I_n}|\phi(x,A_k,t)-\phi(x,A,t)|\le 1-2\varepsilon\ or \frac{1}{\lambda_n}\sum_{k\in I_n}|\rho(x,A_k,t)-\rho(x,A,t)| \ge 2\varepsilon\right\}.$$

Hence, $B_{(\phi,\rho)}(\varepsilon,t)\subseteq A_{(\phi,\rho)}(\varepsilon,t)$ and so, according to definition of an ideal, $B_{(\phi,\rho)}(\varepsilon,t)\in I$. Hence, we conclude that $A_k\to A\left([V,\lambda]^{(\phi,\rho)}(I_W)\right)$.

(c). This easily follows from (a) and (b).

Theorem 2.2.

a. $S^{(\phi,\rho)}(I_W) \subseteq S_\lambda^{(\phi,\rho)}(I_W)$ *if* $liminf_{n\to\infty} \frac{\lambda_n}{n} > 0.$

b. *If* $liminf_{n\to\infty} \frac{\lambda_n}{n} = 0,\ I_W$ *−strongly (by that we mean that* $\exists$ *a subsequence* $\left(n_j\right)_{j=1}^{\infty}$*, for which* $\left(\frac{\lambda_{n_j}}{n_j}\right)\left(\frac{1}{j}\right)$*,* $\forall j$ *and* $\{n(j): j \in \mathbb{N}\} \notin I$*) then* $S^{(\phi,\rho)}(I_W) \subsetneq S_\lambda^{(\phi,\rho)}(I_W)$.

Proof. a) For given $\varepsilon, \delta > 0$ and $t > 0$, we get

$$\begin{aligned}
&\frac{1}{n}|\{k \le n: |\phi(x, A_k, t) - \phi(x, A, t)| \le 1 - \varepsilon \text{ or } |\rho(x, A_k, t) - \rho(x, A, t)| \ge \varepsilon\}| \\
&\qquad \ge \frac{1}{n}|\{k \in I_n: |\phi(x, A_k, t) - \phi(x, A, t)| \le 1 - \varepsilon \text{ or } |\rho(x, A_k, t) - \rho(x, A, t)| \ge \varepsilon\}| \\
&\qquad = \frac{\lambda_n}{n}\frac{1}{\lambda_n}|\{k \in I_n: |\phi(x, A_k, t) - \phi(x, A, t)| \le 1 - \varepsilon \text{ or } |\rho(x, A_k, t) - \rho(x, A, t)| \ge \varepsilon\}|.
\end{aligned}$$

If $liminf_{n\to\infty} \frac{\lambda_n}{n} = \alpha$ then from the definition $\left\{n \in \mathbb{N}: \frac{\lambda_n}{n} < \frac{1}{2}\alpha\right\}$ is finite. For every $\delta > 0$ and $t > 0$,

$$\begin{aligned}
&\left\{n \in \mathbb{N}: \frac{1}{\lambda_n}|\{k \in I_n: |\phi(x, A_k, t) - \phi(x, A, t)| \le 1 - \varepsilon \text{ or} |\rho(x, A_k, t) - \rho(x, A, t)| \ge \varepsilon\}| \ge \delta\right\} \\
&\qquad \subset \left\{n \in \mathbb{N}: \frac{1}{n}|\{k \in I_n: |\phi(x, A_k, t) - \phi(x, A, t)| \le 1 - \varepsilon \text{ or} |\rho(x, A_k, t) - \rho(x, A, t)| \ge \varepsilon\}| \right. \\
&\qquad \left. \ge \frac{\alpha}{2}\delta\right\} \cup \left\{n \in \mathbb{N}: \frac{\lambda_n}{n} < \frac{1}{2}\alpha\right\}
\end{aligned}$$

Since *I* is admissible, the set on the right-hand side belongs to *I* and this completed the proof of (a).
(b) The proof is standard.

Theorem 2.3. *If* $liminf_{n\to\infty} \frac{\lambda_n}{n} = 1$*, then* $S_\lambda^{(\phi,\rho)}(I_W) \subseteq S^{(\phi,\rho)}(I_W)$.

Proof. Let $\delta > 0$ be given. Since $liminf_{n\to\infty} \frac{\lambda_n}{n} = 1$, we can select $m \in \mathbb{N}$ such that $\phi\left(\frac{\lambda_n}{n} - 1, t\right) > 1 - \frac{1}{2}\delta$ or $\rho\left(\frac{\lambda_n}{n} - 1, t\right) < \frac{1}{2}\delta$, for all $n \ge m$. Now notice that, for every $\varepsilon > 0$, every $t > 0$ and $n \ge m$

$$\begin{aligned}
&\frac{1}{n}|\{k \le n: |\phi(x,A_k,t)-\phi(x,A,t)| \le 1-\varepsilon \; or \; |\rho(x,A_k,t)-\rho(x,A,t)| \ge \varepsilon\}| \\
&\qquad = \frac{1}{n}|\{k \le n-\lambda_n: |\phi(x,A_k,t)-\phi(x,A,t)| \le 1-\varepsilon \; or \; |\rho(x,A_k,t)-\rho(x,A,t)| \ge \varepsilon\}| \\
&\qquad + \frac{1}{n}|\{k \in I_n: |\phi(x,A_k,t)-\phi(x,A,t)| \le 1-\varepsilon \; or |\rho(x,A_k,t)-\rho(x,A,t)| \ge \varepsilon\}| \\
&\qquad \le \frac{n-\lambda_n}{n} + \frac{1}{n}|\{k \in I_n: |\phi(x,A_k,t)-\phi(x,A,t)| \le 1-\varepsilon \; or |\rho(x,A_k,t)-\rho(x,A,t)| \ge \varepsilon\}| \\
&\qquad \le 1-\left(1-\frac{\delta}{2}\right) + \frac{1}{n}|\{k \in I_n: |\phi(x,A_k,t)-\phi(x,A,t)| \le 1-\varepsilon \; or |\rho(x,A_k,t)-\rho(x,A,t)| \ge \varepsilon\}| \\
&\qquad = \frac{\delta}{2} + \frac{1}{n}|\{k \in I_n: |\phi(x,A_k,t)-\phi(x,A,t)| \le 1-\varepsilon \; or |\rho(x,A_k,t)-\rho(x,A,t)| \ge \varepsilon\}|.
\end{aligned}$$

Hence

$$\begin{aligned}
&\left\{n \in \mathbb{N}: \frac{1}{n}|\{k \in I_n: |\phi(x,A_k,t)-\phi(x,A,t)| \le 1-\varepsilon \; or |\rho(x,A_k,t)-\rho(x,A,t)| \ge \varepsilon\}| \ge \delta\right\} \\
&\qquad \subset \left\{n \in \mathbb{N}: \frac{1}{\lambda_n}|\{k \in I_n: |\phi(x,A_k,t)-\phi(x,A,t)| \le 1-\varepsilon \; or |\rho(x,A_k,t)-\rho(x,A,t)| \ge \varepsilon\}| \ge \frac{\delta}{2}\right\} \\
&\qquad \cup \{1,2,3,\dots,m\}.
\end{aligned}$$

If $I_W - S_\lambda^{(\phi,\rho)} - lim A_k = A$ then the set on the right-hand side belongs to I and so the set on the left-hand side also belongs to I. This shows that $\{A_k\}$ is Wijsman I −statistically convergent to $A \in X$ w.r.t the IFN (ϕ,ρ).

Theorem 2.4. *Let* $(X,\phi,\rho,\Delta,\Theta)$ *be an IFNS such that* $\frac{1}{4}\varepsilon_n \Delta \frac{1}{4}\varepsilon_n < \frac{1}{2}\varepsilon_n$ *and* $\left(1-\frac{1}{4}\varepsilon_n\right)\Theta\left(1-\frac{1}{4}\varepsilon_n\right) > 1-\frac{1}{2}\varepsilon_n$. *If X is a Banach space then* $S_\lambda^{(\phi,\rho)}(I_W) \cap m(X)$ *is a closed subset of* $m(X)$.

Proof. We first assume that $\{A^n\} \in S_\lambda^{(\phi,\rho)}(I_W) \cap m(X)$ is a convergent set sequence and it converges to $A \in m(X)$. We need to denote that $A \in S_\lambda^{(\phi,\rho)}(I_W) \cap m(X)$. Presume that $A^n \to \gamma_n \left(S_\lambda^{(\phi,\rho)}(I_W)\right)$ for all $n \in \mathbb{N}$. Let $\{\varepsilon_n\}$ be a strictly decreasing positive numbers converging to zero. We can identify an $n \in \mathbb{N}$ such that $sup_j(|\rho(x,A,t)-\rho(x,A^jA,t)|) < \frac{1}{4}\varepsilon_n$ for all $j \ge n$. Select $0 < \delta < \frac{1}{5}$. Now assume

$$\begin{aligned}
A_{(\phi,\rho)}(\varepsilon_n,t) = &\left\{m \in \mathbb{N}: \frac{1}{\lambda_m}\left|\left\{k \in I_m: \left|\phi(x,A_k^n,t)-\phi\left(x,\gamma_n,t\right)\right| \le 1-\frac{\varepsilon_n}{4} \; or \left|\rho(x,A_k^n,t)-\rho\left(x,\gamma_n,t\right)\right| \ge \frac{\varepsilon_n}{4}\right\}\right|\right. \\
&\left. < \delta\right\} \in F(I)
\end{aligned}$$

and

$$B_{(\phi,\rho)}(\varepsilon_n,t)=\Big\{m \in \mathbb{N}: \frac{1}{\lambda_m}\Big|\Big\{k\in I_m: |\phi(x,A_k^{n+1},t)-\phi(x,\gamma_{n+1},t)| \le 1-\frac{\varepsilon_n}{4}\ or |\rho(x,A_k^{n+1},t)-\rho(x,\gamma_{n+1},t)| \ge \frac{\varepsilon_n}{4}\Big\}\Big| < \delta\Big\} \in F(I)$$

Since $A_{(\phi,\rho)}(\varepsilon_n,t)\cap B_{(\phi,\rho)}(\varepsilon_n,t)\in F(I)$ and $\emptyset \notin F(I)$, we can select $m\in A_{(\phi,\rho)}(\varepsilon_n,t)\cap B_{(\phi,\rho)}(\varepsilon_n,t)$. Then

$$\frac{1}{\lambda_m}\Big|\Big\{k\in I_m: |\phi(x,A_k^n,t)-\phi(x,\gamma_n,t)| \le 1-\frac{\varepsilon_n}{4}\ or |\rho(x,A_k^n,t)-\rho(x,\gamma_n,t)| \ge \frac{\varepsilon_n}{4}\ \vee\ |\phi(x,A_k^{n+1},t)-\phi(x,\gamma_{n+1},t)| \le 1-\frac{\varepsilon_n}{4}\ or|\rho(x,A_k^{n+1},t)-\rho(x,\gamma_{n+1},t)| \ge \frac{\varepsilon_n}{4}\Big\}\Big| \le 2\delta < 1.$$

Since $\lambda_m \to \infty$ and $A_{(\phi,\rho)}(\varepsilon_n,t)\cap B_{(\phi,\rho)}(\varepsilon_n,t)\in F(I)$ is finite, we can select m so that $\lambda_m > 5$. Hence, there have to exist a $k\in I_m$ for which at the same time, $|\phi(x,A_k^n,t)-\phi(x,\gamma_n,t)| > 1-\frac{\varepsilon_n}{4}$ or $\left|\rho(x,A_k^n,t)-\rho(x,\gamma_n,t)\right| < \frac{\varepsilon_n}{4}$ and $|\phi(x,A_k^{n+1},t)-\phi(x,\gamma_{n+1},t)| > 1-\frac{\varepsilon_n}{4}$ or $\left|\rho(x,A_k^{n+1},t)-\rho(x,\gamma_{n+1},t) < \frac{\varepsilon_n}{4}\right|$. For a given $\varepsilon_n > 0$ select $\frac{1}{2}\varepsilon_n$ such that $\left(1-\frac{1}{2}\varepsilon_n\right)\Theta\left(1-\frac{1}{2}\varepsilon_n\right) > 1-\varepsilon_n$ and $\frac{1}{2}\varepsilon_n\Delta\frac{1}{2}\varepsilon_n < \varepsilon_n$. Then, we get

$$\left|\rho\left(x,\gamma_n,\frac{t}{2}\right)-\rho\left(x,A_k^n,\frac{t}{2}\right)\right|\Delta\left|\rho\left(x,\gamma_{n+1},\frac{t}{2}\right)-\rho\left(x,A_k^{n+1},\frac{t}{2}\right)\right| \le \frac{\varepsilon_n}{4}\Delta\frac{\varepsilon_n}{4} < \frac{\varepsilon_n}{2}$$

and

$$|\rho(x,A_k^n,t)-\rho(x,A_k^{n+1},t)| \le sup_n\left(\left|\rho\left(x,A,\frac{t}{2}\right)-\rho\left(x,A_k^n,\frac{t}{2}\right)\right|\right)\Delta sup_n\left(\left|\rho\left(x,A,\frac{t}{2}\right)-\rho\left(x,A_k^{n+1},\frac{t}{2}\right)\right|\right) \le \frac{\varepsilon_n}{4}\Delta\frac{\varepsilon_n}{4} < \frac{\varepsilon_n}{2}$$

Hence, we obtain

$$|\rho(x,\gamma_n,t)-\rho(x,\gamma_{n+1},t)| \le \left[\left|\rho\left(x,\gamma_n,\frac{t}{3}\right)-\rho\left(x,A_k^n,\frac{t}{3}\right)\right|\Delta\left|\rho\left(x,A_k^{n+1},\frac{t}{3}\right)-\rho\left(x,\gamma_{n+1},\frac{t}{3}\right)\right|\right]\Delta\left|\rho\left(x,A_k^n,\frac{t}{3}\right)-\rho\left(x,A_k^{n+1},\frac{t}{3}\right)\right| \le \frac{\varepsilon_n}{2}\Delta\frac{\varepsilon_n}{2} < \varepsilon_n$$

and similarly

$$|\rho(x,\gamma_n,t)-\rho(x,\gamma_{n+1},t)|>1-\varepsilon_n.$$

This gives that $\{\gamma_n\}$ is Cauchy sequence in $X$and let $\gamma_n\to\gamma\in X$ as $n\to\infty$. We have denote that $\gamma_n\to\gamma\left(S_\lambda^{(\phi,\rho)}(I_W)\right)$. For any $\varepsilon>0$ and $t>0$, select $n\in\mathbb{N}$ such that $\varepsilon_n<\frac{\varepsilon}{4}$,

$$sup_n(|\rho(x,A,t)-\rho(x,A^n,t)|)<\frac{\varepsilon}{4},$$
$$|\phi(x,\gamma_n,t)-\phi(x,\gamma,t)|>1-\frac{\varepsilon}{4}$$

or

$$|\rho(x,\gamma_n,t)-\rho(x,\gamma,t)|<\frac{\varepsilon}{4}.$$

Now

$$\frac{1}{\lambda_n}|\{k\in I_n:|\rho(x,A_k,t)-\rho(x,\gamma,t)|\geq\varepsilon\ \}|$$
$$\leq\frac{1}{\lambda_n}\left|\left\{k\in I_n:\left|\rho\left(x,A_k,\frac{t}{3}\right)-\rho\left(x,A_k^n,\frac{t}{3}\right)\right|\Delta\left[\left|\rho\left(x,A_k^n,\frac{t}{3}\right)-\rho\left(x,\gamma_n,\frac{t}{3}\right)\right|\Delta\left|\rho\left(x,\gamma_n,\frac{t}{3}\right)-\rho\left(x,\gamma,\frac{t}{3}\right)\right|\right]\geq\varepsilon\ \right\}\right|\leq\frac{1}{\lambda_n}\left|\left\{k\in I_n:\left|\rho\left(x,A_k^n,\frac{t}{3}\right)-\rho\left(x,\gamma_n,\frac{t}{3}\right)\right|\geq\frac{\varepsilon}{2}\ \right\}\right|$$

and similarly

$$\frac{1}{\lambda_n}|\{k\in I_n:|\phi(x,A_k,t)-\phi(x,\gamma,t)|\leq1-\varepsilon\ \}|>\frac{1}{\lambda_n}\left|\left\{k\in I_n:\left|\phi\left(x,A_k^n,\frac{t}{3}\right)-\phi\left(x,\gamma,\frac{t}{3}\right)\right|\leq1-\frac{\varepsilon}{2}\ \right\}\right|.$$

It follows that

$$\left\{n\in\mathbb{N}:\frac{1}{\lambda_n}|\{k\in I_n:|\phi(x,A_k,t)-\phi(x,\gamma,t)|\leq1-\varepsilon\ or\ |\rho(x,A_k,t)-\rho(x,\gamma,t)|\geq\varepsilon\}|\geq\delta\right\}$$
$$\subset\left\{n\in\mathbb{N}:\frac{1}{\lambda_n}\left|\left\{k\in I_n:\left|\phi\left(x,A_k^n,\frac{t}{3}\right)-\phi\left(x,\gamma,\frac{t}{3}\right)\right|\leq1-\frac{\varepsilon}{2}\ or\ \left|\rho\left(x,A_k^n,\frac{t}{3}\right)-\rho\left(x,\gamma,\frac{t}{3}\right)\right|\geq\frac{\varepsilon}{2}\right\}\right|\geq\delta\right\}$$

for any given $\delta>0$. Hence $A_k\to\gamma\left(S_\lambda^{(\phi,\rho)}(I_W)\right)$.

4. Conclusion

In this paper we examine the notions of Wijsman $I-[V,\lambda]-$summability and Wijsman $I_\lambda-$statistical convergence for sequences of sets w.r.t the IFN, investigate their relationship, and make some observations about these classes. We intend to unify these two approaches and use ideals to define the concept of Wijsman $I-\lambda-$statistical convergence w.r.t the IFN (ϕ,ρ). Our study of Wijsman $I-\lambda-$statistical convergence and Wijsman $I-$statistical convergence for sequences of sets in IFNS also provides a tool to deal with convergence problems of sequences of fuzzy real numbers. These results can be used to think the convergence problems of sequences of fuzzy numbers having a chaotic pattern in IFNS.

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Solving Lientof model with algebric methods and its implementation in R

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Abstract

In economics an important issue is the balance between the production of the main sectors of economy and the external demand of the production. Refereeing to the Leontief model, the structure of each industry's production activity is represented by appropriate structural coefficients that describe relationships between the inputs that the industry absorbs and the output that it produces. We study the economic development of Albania for ten-year period. We solve the problem in different algebraic ways using matrices theory. Then, we implement the solutions in R programming language and analyze the final results.

Key words: Leontief model, matrix solution, R programming language

1. Introduction

Leontief is well known for his work in economics. His theory called the Leontief Input-Output model serves as a simplified model to predict a production. Many other authors followed the research in this area based on his model [4], [9], [10], [11], [12]
The motivation for this paper steams from the role of matrices theory in solving economic problems as [6], [7], [8], [9]. Part of his work was to apply the basic concepts of linear algebra [1], [2], [3] to model supply and demand within simple economies. More cases from business, economic are treated in [5].
One of this model is Leontief Input-Output model. There are many sectors in economy that are related to each other in the way that interchange among them. An input-output table presents the correlation among the sectors. The rows of the table describe the delivers of the total amount of a product or primary input to all uses. The columns of the table describe the input requirements to produce the gross output totals.
In this paper, first we present Leontief model for the Albanian case in a ten -years period (2009-2018). Second, we solve algebraically the matrix equation in three different ways, and finally we implement the solutions in R programming language and analyzing the final results.

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2. Materials and methods

The input-output model is a linear model based on Leontief production functions and a given vector of final uses. In constructing an input-output (I-O) table [4], the entries can be in physical units or in terms of monetary values. We present here a general Input output table

Purchases by:	Intermediate Users Sectors			Final demand	Total demand
	Agriculture	Manufactory	Services		
Agriculture	X11	X12	X13	d1	X1
Manufactory	X21	X22	X23	d2	X2
Services	X31	X32	X33	d3	X3
Total Supply	X1	X2	X3		

where:

X_i value of the output of sector i (i=1,2,3)

X_{ij} the amount of the ith sector 'output used by the jth sector used to produce its output

$a_{ij} = \frac{X_{ij}}{X_j}$ the input output coefficients, can be interpreted as the amount of input i used per unit output of product j

d_i government purchases of the output of sector i+ personal consumption expenditures for the output of sector i+ export of the output of sector i(component of demand vector d)

Input-Output table can be described mathematically as a set of equations that must be satisfied simultaneously for the gross output of each sector to balance the intermediate and final demand for its product. Thus we have

$$X_i = \sum_{j=1}^{3} a_{ij} X_j + d_i,\ i = 1,2,3$$

The matrix equation with the Leontief Input Output model is:

$$X = AX + d \quad (*)$$

where

$$X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}, A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \text{ and } d = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}.$$

The Leontief model that we present here is based on input-output data for the Albania economy

for ten-years period (2009-2018), with data for 35 sectors grouped into three larger sectors: Agriculture, Manufacturing and Services. The data base used is from Institution of Statistic of Albania. We use the input output tables in million ALL.
By the data given we have the coefficient matrix of this model for the Albanian economy for a period of ten years.

Coefficient matrix 2009

	Agriculture	Manufacturing	Services	demand vector
Agriculture	0.22	0.07	0.01	171940.628
Manufacturing	0.05	0.46	0.28	481683.8785
Services	0.02	0.10	0.22	690636.1794

Coefficient matrix 2010

	Agriculture	Manufacturing	Services	demand vector
Agriculture	0.21	0.06	0.01	191957.8241
Manufacturing	0.05	0.44	0.30	584186.4459
Services	0.02	0.09	0.17	726615.1231

Coefficient matrix 2011

	Agriculture	Manufacturing	Services	demand vector
Agriculture	0.21	0.06	0.01	207056.7264
Manufacturing	0.05	0.46	0.30	611470.3872
Services	0.02	0.10	0.18	790696.6981

Coefficient matrix 2012

	Agriculture	Manufacturing	Services	demand vector
Agriculture	0.21	0.05	0.01	221704.8264
Manufacturing	0.06	0.48	0.24	618344.6566
Services	0.02	0.10	0.23	783256.1524

Coefficient matrix 2013

	Agriculture	Manufacturing	Services	demand vector
Agriculture	0.22	0.05	0.01	227416.6248
Manufacturing	0.05	0.46	0.24	593212.1841
Services	0.01	0.11	0.24	771085.3199

Coefficient matrix 2014

	Agriculture	Manufacturing	Services	demand vector
Agriculture	0.22	0.05	0.01	240331.827
Manufacturing	0.04	0.45	0.23	627624.8854

Services	0.02	0.11	0.23	818754.001

Coefficient matrix 2015				
	Agriculture	Manufacturing	Services	demand vector
Agriculture	0.22	0.05	0.01	251438.2512
Manufacturing	0.04	0.45	0.20	616490.6918
Services	0.01	0.12	0.26	834807.4808

Coefficient matrix 2016				
	Agriculture	Manufacturing	Services	demand vector
Agriculture	0.21	0.04	0.01	267756.8281
Manufacturing	0.05	0.49	0.22	621907.629
Services	0.02	0.10	0.25	888740.9504

Coefficient matrix 2017				
	Agriculture	Manufacturing	Services	demand vector
Agriculture	0.21	0.04	0.01	274718.8105
Manufacturing	0.06	0.47	0.23	646805.4327
Services	0.02	0.11	0.24	963626.073

Coefficient matrix 2018				
	Agriculture	Manufacturing	Services	demand vector
Agriculture	0.19	0.03	0.01	290331.0325
Manufacturing	0.06	0.47	0.23	674861.1705
Services	0.02	0.12	0.23	1020588.468

The goal is to find the production vector X that gives a perfectly balance economy. The question is:

How to solve the matrix equation $X = AX + d$?

For finding solution we use different algebraic ways [1], [2], [3] [5] some of these are the following:
The first method *is by using the inverse matrix:*
The solution of the matrix equation $(*)$ is

$$X = AX + d \Longrightarrow (\,I - A)X = \, d \Longrightarrow X = (I - A)^{-1}d$$

The second method *is Gaussian elimination*
Gaussian Elimination demonstrates the algorithm of row reduction used for solving systems of linear equations of the form $Ax \; = \; b$.

The third method *is QR-factorization*

The QR-factorization is used when the rank of the matrix $I - A$ is equal to the number of columns, for Q an orthogonal matrix ($Q^T Q = I$) and R a triangular matrix. The solution of the matrix equation $(*)$ is:

$$X = AX + d \Longrightarrow (\, I - A)X = \; d \Longrightarrow QRX = d \Longrightarrow Q^T QRX = Q^T d \Longrightarrow Rx = Q^T d$$

The last equation $Rx = Q^T d$ is solved by back substitution.
If the matrix $I - A$ is invertible, then the R matrix is invertible, consequently the QR- factorization is interpreted in another way:

$$X = AX + d \Longrightarrow (\, I - A)X = \; d \Longrightarrow QRX = d \Longrightarrow$$
$$X = (QR)^{-1} d = R^{-1} Q^{-1} d \Longrightarrow x = R^{-1} Q^T d$$

3. **Implementation of the solution of Leontief model by modeling in R program language**

We implement the algebraic solutions in R programming language. The packages solve () and matlib (). are used for modeling the solutions in R

- ***Inverse matrix***

$f_1 = function(A_i,\, d_i)$
$I = diag\big(1,\, ncol(A)\big)$
$s_i = solve(I - A_i,\, d_i)$
$print(\text{“}si\text{"})$
$print(system\ time(f_1(A, d_i))$

Arguments:
A_i coefficient matrix for years i
d_i right-hand side vector
I unitary matrix with dimension the same as matrix A_i
s_i the solution that we will find by using invers matrix
$print(\text{“}si\text{"})$ we have the result
$print(system\ time(f_1(A, d_i))$ execution time

- ***Gaussian elimination***

Usage :
$f2$ =aussianElimination($I - A_i, d_i, tol \;=\; sqrt(.Machine\$double.eps), verbose \;=\; FALSE,\; latex \;=\; FALSE\;, fractions \;=\; FALSE$)
$print(system\ time(f_2)$
Arguments
A_i coefficient matrix for years i
d_i right-hand side vector
tol tolerance for checking for 0 pivot
verbose logical if TRUE, print intermediate steps
latex logical if TRUE, and verbose is TRUE, print intermediate steps using LaTeX equation outputs rather than R output

fractions logical if TRUE, try to express non-integers as rational numbers
$print(system\ time(f_2)$ execution time

- ***QR factorization***

$f_3 = function(A_i,\ d_i)$
$I = diag\big(1,\ ncol(A)\big)$
$qr(I - A_i)$
$QRi = qr(I - A_i)$
$Si = solve.qr(QRi,\ d_i)$
$print(\text{“}Si\text{"})$
$print(system\ time(f_3(A, d_i)))$

Arguments

A_i	coefficient matrix for years i
d_i	right-hand side vector
$qr(I - A_i)$	the command for QR factorization of a matrix, show the rank of a matrix and the number of pivot
$solve.qr(QRi,\ d_i)$	solve the solution with QR factorization
$print(\text{“}Si\text{"})$	we have the result

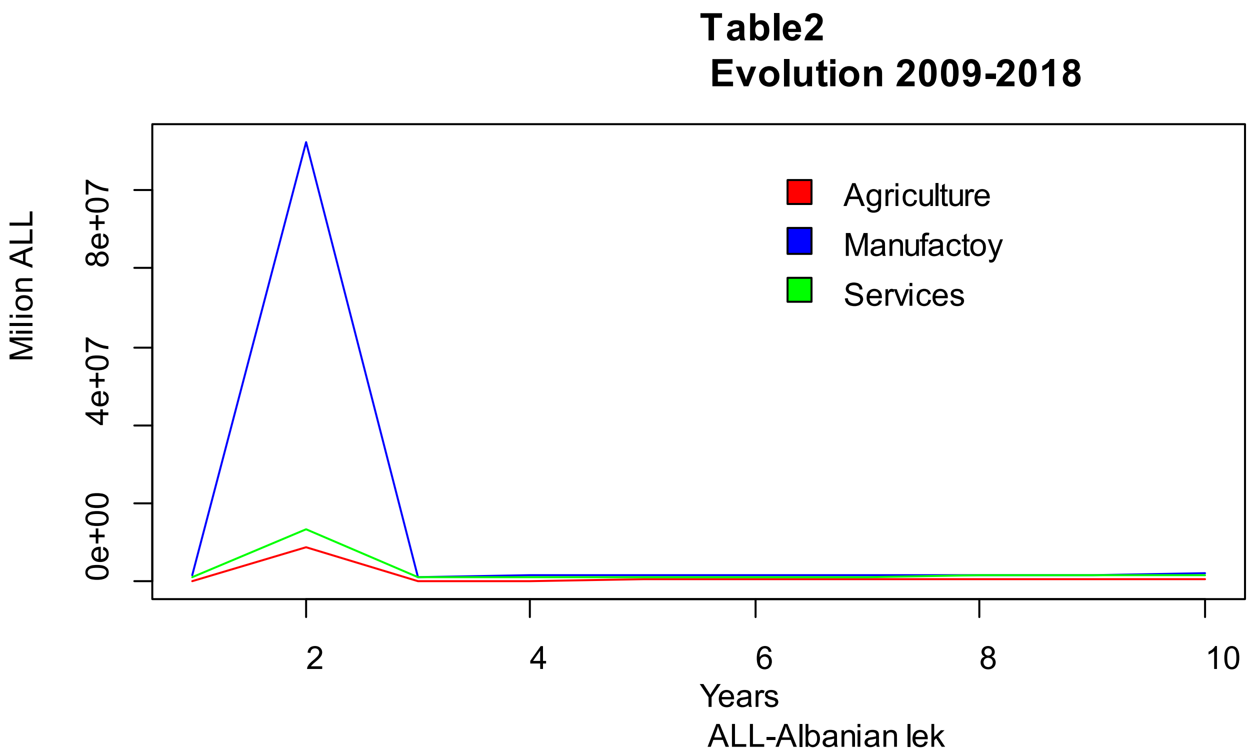
$print(system\ time(f_3(A, d_i)))$ execution time

During the execution, we obtained the solution of the problem by using rounding with two digits after the decimal point. The equation had only one solution. The results are in the following table

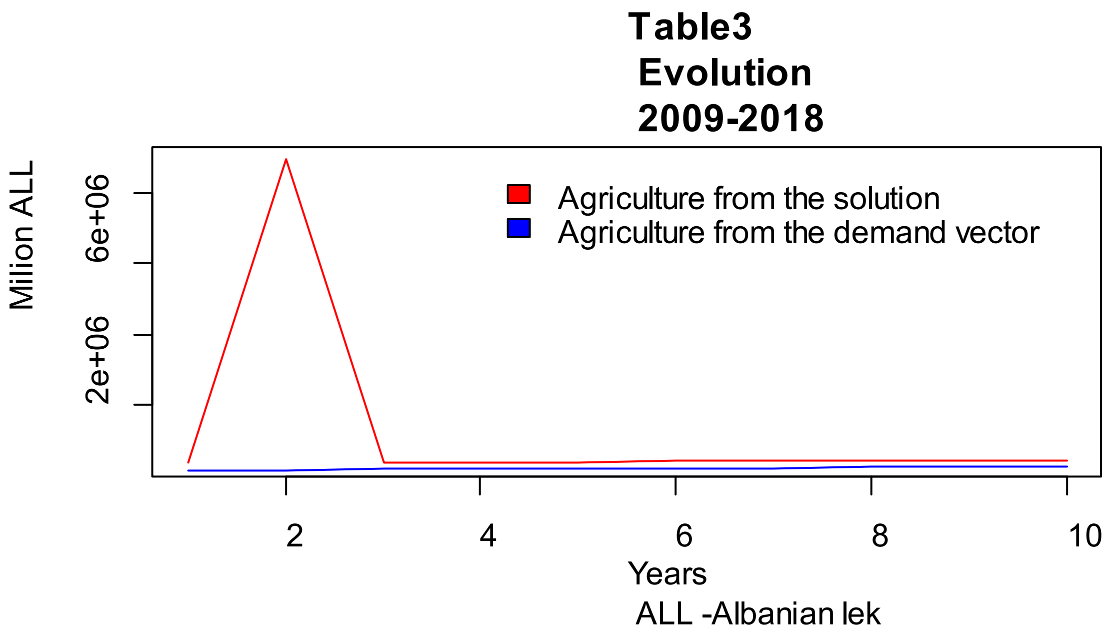
Table 1

	Agriculture	Manufactory	Services
s1	367992.2	1489074	1085773
s2	8933842.4	112219775	13259122
s3	369654.6	1228977	1123156
s4	411843.5	1820168	1264298
s5	416756.4	1699840	1266099
s6	435650	1725280	1321101
s7	446297.1	1656657	1384092
s8	452911.6	1888852	1448913
s9	452911.6	1888852	1448913
s10	454551	2043284	1655679

The solutions obtained from algebraic methods in the following tables are interpreted in economy.

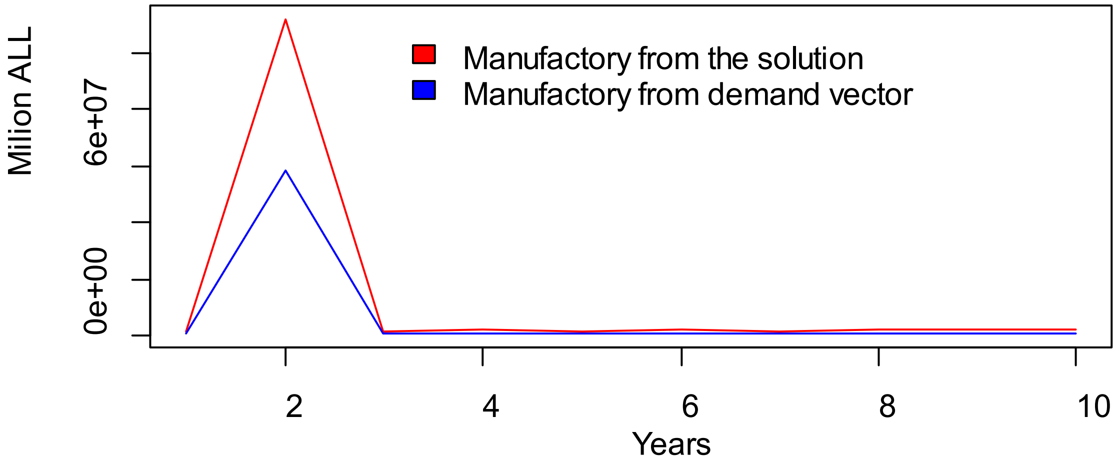


In table 2 its presented the product vector X, and is obviously seen that in the first two years the manufacturing sector was more developed than the other two sectors.



In the table 3 is presented the relations between the demand vector and product vector for agriculture sector. As it is seen in the first two years the agriculture is sufficit related to the demand vector.

Table 4
Evolution
2009-2018



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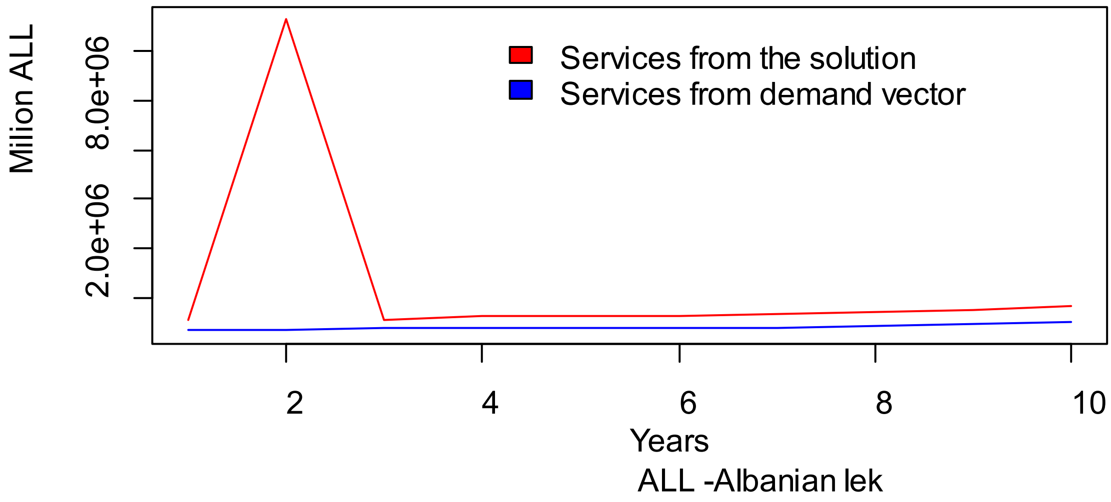
In the table 4 is presented the relations between the demand vector and product vector for manufacturing sector. As it is seen in the first two years the graphic presents a closer proximity of the manufactory sector to the demand vector.

Table 5
Evolution 2009-2018



In the table 5 is presented the relations between the demand vector and product vector for services sector. As it is seen in the first two years the service is sufficit related to the demand vector.

4. Results

The results obtained in this work are rounded with two digits after the decimal point**.** Using R programming language, we evaluate the time of execution for each algebraic method used, respectively the execution time is; **Gaussian Elimination** 0.1282439 secs, **Inverse matrix** 0.250715 secs and $\boldsymbol{QR}$**-factorization** 0.576961 secs. The best method to use for finding the solution of Leontief model is Gaussian Elimination. In R programing language there are two commands for Gaussian Elimination execution. The command gaussianElimination($I - A1, d1$) execute the solution immediately(for 0.1282439 secs), while the command gaussianElimination($I - A1, d1$, verbose=TRUE, fractions=TRUE, latex=TRUE) execute the solution in detailed steps (0.4631231 secs). Consequently, is recommended the first command of Gaussian Elimination.

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New Characterization of Schrödinger Flow with Bäcklund Transformations

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Abstract

In this paper, we characterize integrable geometric Schrödinger flow with differential geometry properties of surfaces We give some new solutions by using Bäcklund transformations. Finally, we obtain some solutions of mKdV system.

Keywords: Schrödinger flow, $\mathbb{E}^3$, extended Riccati mapping method, Bäcklund transformations.

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1. Introduction

Bäcklund transformations are a robust application to research numerous elements of integrable non linear partial differential equations [1,2]. They could be applied to obtain additional accurate solutions of integrable systems by a specific alternative. The common Bäcklund transformations are regional geometric transformations, which are accustomed to develop areas attached to continuous negative Gaussian curvature [1]. This gives a applied geometric development from different pseudospherical surfaces by a special solution of the integrable partial differential equation. Actually, solutions of the sine-Gordon equation explain pseudospherical areas. Making use of Bäcklund transformations n moments to a specific solution of sine-Gordon equation, you can get yourself a arranged category of solutions of sine-Gordon equation. All these solutions can be acquired applying the Bianchi's permutability method by using solely algebraic involves. [2].

In applied differential geometry, theory of curves in space is one of the significant study areas. In the theory of curves, helices, slant helices, and rectifying curves are the most fascinating curves. Flows of curves of a given curve are also widely studied, [7-9].

A particular nice feature of integrable systems is their relationship with invariant geometric flows of curves and surfaces in certain geometries. Those flows are invariant with respect to the symmetry groups of the geometries [13]. A number of integrable equations have been shown to be related to motions of curves in Euclidean geometry, centro-equiaffine geometry, affine geometry, homogeneous manifolds and other geometries etc., and many interesting results have been obtained.

This study is organised as follows: Firstly, we present a new approach for computing the differential geometry properties of surfaces by using Bäcklund transformations of integrable geometric curve flows. We give some new solutions by using the extended Riccati mapping method. Finally, we obtain figures of this solutions.

2. Preliminaries

In this paper, we are mainly concerned with Bäcklund transformations for integrable geometric curve flows in certain geometries.

Let $\gamma(s)$ be a smooth curve of constant torsion τ in R^3, parametrized by arclength s. Let **T**, **N** and **B** be a Frenet frame, and k(s) the curvature of γ. For any constant C, suppose $\beta = \beta$ (s; k(s);C) is a solution of the differential equation

$$\frac{d\beta}{ds} = C\sin\beta - k$$

then

$$\tilde{\gamma}(s) = \gamma(s) + \frac{2C}{C^2+\tau^2}(\cos\beta\mathbf{T} + \sin\beta\mathbf{N})$$

is a curve of constant torsion τ, also parametrized by arclength s.

Note that this transformation can be obtained by restricting the classical Bäcklund transformation for pseudospherical surfaces to the asymptotic lines of the surfaces with constant torsion.

We will restrict our attention to the geometric space curve flows

$$\gamma_t = f\mathbf{T} + g\mathbf{N} + h\mathbf{B},$$

where f, g and h depend on the curvatures of the curves and their derivatives with respect to the arclength parameter, namely, these geometric flows are invariant with respect to the symmetry groups of the geometries.

For a spacial curve $\gamma(s,t)$ in a given geometry, let $\tilde{\gamma}(s,t)$ be another curve related to through the following Bäcklund transformation

$$\tilde{\gamma}(s,t) = \gamma(s,t) + \alpha(s,t)\mathbf{T} + \beta(s,t)\mathbf{N} + \chi(s,t)\mathbf{B}$$

Throughout the paper, we assume that both curve flows for γ and $\tilde{\gamma}$ are governed by the same integrable system, that means the curvatures of the curves $\tilde{\gamma}$ determined by the flows (4) or (5) satisfy the integrable systems as for the curves

3. Bäcklund Transformations for Space Curve Flows in $\mathbf{R}^3$

In this section, we consider the integrable flows for space curves in R^3

$$\gamma_t = U\mathbf{n} + V\mathbf{b} + W\mathbf{t},$$

where $\mathbf{t}$, $\mathbf{n}$ and $\mathbf{b}$ are the tangent, normal and binormal vectors of the space curve γ, respectively. The velocities U, V and W depend on the curvature and torsion as well as their derivatives with respect to arclength s. It is well know that the vectors $\mathbf{t}$, $\mathbf{n}$ and $\mathbf{b}$ satisfy the Serret-Frenet formulae

$$\begin{aligned} \mathbf{t}_s &= k\mathbf{n} \\ \mathbf{n}_s &= -k\mathbf{t} + \tau\mathbf{b} \\ \mathbf{b}_s &= -\tau\mathbf{n}. \end{aligned}$$

4. Bonnet surfaces as geometric space curve flows

Let ϖ_γ be the standard unit normal vector field on a surface. Then, the first fundamental form $\mathbf{I}$ and the second fundamental form $\mathbf{II}$ of a surface γ are defined by, respectively,

$$\begin{aligned} \mathbf{I} &= \mathbf{E}ds^2 + 2\mathbf{F}dsdt + \mathbf{G}dt^2, \\ \mathbf{II} &= \mathbf{e}ds^2 + 2\mathbf{f}dsdt + \mathbf{g}dt^2, \end{aligned}$$

where $\mathbf{E}, \mathbf{F}, \mathbf{G}$ are the coefficients of the first fundamental form of the surface and $\mathbf{e}, \mathbf{f}, \mathbf{g}$ are the coefficients of the second fundamental form.

Definition 4.1. *A-net on a surface such that, when this net is parametrized, the conditions* $\mathbf{E} = \mathbf{G}$, $\mathbf{F} = 0$, $\mathbf{f} = c = \text{const.} \neq 0$ *are satisfied, is called an A-net, where* $\mathbf{E}, \mathbf{F}, \mathbf{G}$ *are the coefficients of the first fundamental form of the surface and* $\mathbf{e}, \mathbf{f}, \mathbf{g}$ *are the coefficients of the second fundamental form.*

Theorem 4.2. *Necessary and sufficient condition for a surface to be a Bonnet surface is that the surface can have an A-net.*

Lemma 4.3. *Let* β *be geometric space curve flows. Then,*

$$\begin{aligned}
\mathbf{E} &= g(\beta_s,\beta_s) = 1,\\
\mathbf{F} &= g(\beta_s,\beta_t) = W,\\
\mathbf{G} &= g(\beta_t,\beta_t) = (U)^2+(V)^2+(W)^2,
\end{aligned}$$

Lemma 4.4. *Let β be be geometric space curve flows. If β is regular surface, then*

$$U,V \neq 0.$$

Moreover, we have

$$\varpi_\beta(s,t) = U\mathbf{b} - V\mathbf{n}.$$

Theorem 4.5.

$$\begin{aligned}
\gamma_{tt} \;=\; & [U_t - V\mathbf{[}\frac{1}{k}\frac{\partial}{\partial s}(\frac{\partial V}{\partial s}+\tau U)+\frac{\tau}{k}(\frac{\partial U}{\partial s}-\tau V+kW)]+W(\frac{\partial U}{\partial s}-\tau V+kW)]\mathbf{n}\\
&+[V_t + U\mathbf{[}\frac{1}{k}\frac{\partial}{\partial s}(\frac{\partial V}{\partial s}+\tau U)+\frac{\tau}{k}(\frac{\partial U}{\partial s}-\tau V+kW)]+W\mathbf{(}\frac{\partial V}{\partial s}+\tau U)]\mathbf{b}\\
&+[W_t - U(\frac{\partial U}{\partial s}-\tau V+kW)-V\mathbf{(}\frac{\partial V}{\partial s}+\tau U)]\mathbf{t}
\end{aligned}$$

Lemma 4.6. *Let β be geometric space curve flows. Then,*

$$\begin{aligned}
\varpi_\gamma(s,t)_t \;=\; & [U_t - V\mathbf{[}\frac{1}{k}\frac{\partial}{\partial s}(\frac{\partial V}{\partial s}+\tau U)+\frac{\tau}{k}(\frac{\partial U}{\partial s}-\tau V+kW)]]\mathbf{b}\\
&+\mathbf{[}-V_t - U\mathbf{[}\frac{1}{k}\frac{\partial}{\partial s}(\frac{\partial V}{\partial s}+\tau U)+\frac{\tau}{k}(\frac{\partial U}{\partial s}-\tau V+kW)]]\mathbf{n}\\
&+[V(\frac{\partial U}{\partial s}-\tau V+kW)-U\mathbf{(}\frac{\partial V}{\partial s}+\tau U)]\mathbf{t}
\end{aligned}$$

Theorem 4.7. *Let β be geometric space curve flows. β is a Bonnet surface if and only if*

$$\begin{aligned}
W &= 0\\
(U)^2+(V)^2 &= 1,\\
V(\frac{\partial U}{\partial s}-\tau V+kW)-U\mathbf{(}\frac{\partial V}{\partial s}+\tau U) &= c.
\end{aligned}$$

5. The Schrödinger Flow

The Schrödinger flow is given by

$$\gamma_t = k\mathbf{b}$$

In this case, the time evolution of frame vectors is governed by

$$\begin{aligned}
\mathbf{t}_t &= -\tau k\mathbf{n} + k_s\mathbf{b} \\
\mathbf{n}_t &= -\left(\frac{k_{ss}}{k} - \tau^2\right)\mathbf{b} + \tau k\mathbf{t} \\
\mathbf{b}_t &= -k_s\mathbf{t} - \left(\frac{k_{ss}}{k} - \tau^2\right)\mathbf{n}
\end{aligned}$$

We now construct Bäcklund transformation of the Schrödinger flow

$$\tilde{\gamma}(s,t) = \gamma(s,t) + \alpha(s,t)\mathbf{T} + \beta(s,t)\mathbf{N} + \chi(s,t)\mathbf{B}.$$

Using above equations, a direct computation leads to

$$\begin{aligned}
\tilde{\gamma}_s &= (1+\alpha_s - \beta k)\mathbf{t} + (\beta_s + \alpha k - \chi\tau)\mathbf{n} + (\chi_s + \beta\tau)\mathbf{b} \\
\tilde{\gamma}_t &= (\alpha_t + \beta\tau k - \chi k_s)\mathbf{t} + \mathbf{(}\beta_t - \chi\left(\frac{k_{ss}}{k} - \tau^2\right) - \alpha\tau k)\mathbf{n} \\
&\quad + (\chi_t + k + \alpha k_s + \beta\left(\frac{k_{ss}}{k} - \tau^2\right))\mathbf{b}
\end{aligned}$$

Normal vector of Bäcklund transformation of the Schrödinger flow

$$\begin{aligned}
\varpi_{\tilde{\gamma}} &= [(\beta_s + \alpha k - \chi\tau)(\chi_t + k + \alpha k_s + \beta\left(\frac{k_{ss}}{k} - \tau^2\right)) \\
&\quad - (\chi_s + \beta\tau)\mathbf{(}\beta_t - \chi\left(\frac{k_{ss}}{k} - \tau^2\right) - \alpha\tau k)]\mathbf{t} \\
&\quad + [(\chi_s + \beta\tau)(\alpha_t + \beta\tau k - \chi k_s) - (1+\alpha_s - \beta k)(\chi_t \\
&\quad + k + \alpha k_s + \beta\left(\frac{k_{ss}}{k} - \tau^2\right))]\mathbf{n} + [(1+\alpha_s - \beta k)\mathbf{(}\beta_t \\
&\quad - \chi\left(\frac{k_{ss}}{k} - \tau^2\right) - \alpha\tau k) - (\beta_s + \alpha k - \chi\tau)(\alpha_t + \beta\tau k - \chi k_s)]\mathbf{b.}
\end{aligned}$$

From above equations we have

$$\begin{aligned}\tilde{\gamma}_{ss} &= [(1+\alpha_s-\beta k)_s-(\beta_s+\alpha k-\chi\tau)k]\mathbf{t}+[(\beta_s\\ &+\alpha k-\chi\tau)_s+(1+\alpha_s-\beta k)k-(\chi_s+\beta\tau)\tau]\mathbf{n}\\ &+[(\chi_s+\beta\tau)_s+(\beta_s+\alpha k-\chi\tau)\tau]\mathbf{b}\end{aligned}$$

$$\begin{aligned}\tilde{\gamma}_{ts} &= [(\alpha_t+\beta\tau k-\chi k_s)_s-(\beta_t-\chi(\frac{k_{ss}}{k}-\tau^2)-\alpha\tau k)k]\mathbf{t}\\ &+[(\beta_t-\chi(\frac{k_{ss}}{k}-\tau^2)-\alpha\tau k)_s+(\alpha_t+\beta\tau k-\chi k_s)k\\ &-(\chi_t+k+\alpha k_s+\beta(\frac{k_{ss}}{k}-\tau^2))\tau]\mathbf{n}+[(\chi_t+k\\ &+\alpha k_s+\beta(\frac{k_{ss}}{k}-\tau^2))_s+(\beta_t-\chi(\frac{k_{ss}}{k}-\tau^2)-\alpha\tau k)\tau]\mathbf{b}\end{aligned}$$

Theorem 5.1. *Let* β *be* Bäcklund transformation of the Schrödinger flow. β *is a Bonnet surface if and only if*

$$\begin{aligned}(1+\alpha_s-\beta k)^2+(\beta_s+\alpha k-\chi\tau)^2+(\chi_s+\beta\tau)^2 &= (\alpha_t+\beta\tau k-\chi k_s)^2+(\beta_t\\ -\chi(\frac{k_{ss}}{k}-\tau^2)-\alpha\tau k)^2+(\chi_t+k+\alpha k_s+\beta(\frac{k_{ss}}{k}-\tau^2))^2\end{aligned}$$

$$\begin{aligned}(1+\alpha_s-\beta k)(\alpha_t+\beta\tau k-\chi k_s)+(\beta_s+\alpha k-\chi\tau)(\beta_t-\chi(\frac{k_{ss}}{k}-\tau^2)-\alpha\tau k)\\ +(\chi_s+\beta\tau)(\chi_t+k+\alpha k_s+\beta(\frac{k_{ss}}{k}-\tau^2)) = 0,\end{aligned}$$

$$
\begin{aligned}
c \quad = \quad & [(\beta_s + \alpha k - \chi\tau)(\chi_t + k + \alpha k_s + \beta(\frac{k_{ss}}{k} - \tau^2)) - (\chi_s \\
& + \beta\tau)(\beta_t - \chi(\frac{k_{ss}}{k} - \tau^2) - \alpha\tau k)] \times (\alpha_t + \beta\tau k \\
& - \chi k_s)_s - (\beta_t - \chi(\frac{k_{ss}}{k} - \tau^2) - \alpha\tau k)k] + [(\chi_s + \beta\tau)(\alpha_t \\
& + \beta\tau k - \chi k_s) - (1 + \alpha_s - \beta k)(\chi_t + k + \alpha k_s + \beta(\frac{k_{ss}}{k} \\
& - \tau^2))] \times [(\beta_t - \chi(\frac{k_{ss}}{k} - \tau^2) - \alpha\tau k)_s + (\alpha_t + \beta\tau k \\
& - \chi k_s)k - (\chi_t + k + \alpha k_s + \beta(\frac{k_{ss}}{k} - \tau^2))\tau] + [(1 + \alpha_s \\
& - \beta k)(\beta_t - \chi(\frac{k_{ss}}{k} - \tau^2) - \alpha\tau k) - (\beta_s + \alpha k - \chi\tau)(\alpha_t \\
& + \beta\tau k - \chi k_s)] \times (\chi_t + k + \alpha k_s + \beta(\frac{k_{ss}}{k} - \tau^2))_s \\
& + (\beta_t - \chi(\frac{k_{ss}}{k} - \tau^2) - \alpha\tau k)\tau].
\end{aligned}
$$

Proof. First fundamental form of Bäcklund transformation of the Schrödinger flow

$$
\begin{aligned}
\mathbf{E} \quad & = (1 + \alpha_s - \beta k)^2 + (\beta_s + \alpha k - \chi\tau)^2 + (\chi_s + \beta\tau)^2, \\
\mathbf{F} \quad & = g(\beta_s, \beta_t) = (1 + \alpha_s - \beta k)(\alpha_t + \beta\tau k - \chi k_s) + (\beta_s + \alpha k - \chi\tau)(\beta_t \\
& - \chi(\frac{k_{ss}}{k} - \tau^2) - \alpha\tau k) + (\chi_s + \beta\tau)(\chi_t + k + \alpha k_s + \beta(\frac{k_{ss}}{k} - \tau^2)) \\
\mathbf{G} \quad & = g(\beta_t, \beta_t) = (\alpha_t + \beta\tau k - \chi k_s)^2 + (\beta_t - \chi(\frac{k_{ss}}{k} - \tau^2) - \alpha\tau k)^2 \\
& + (\chi_t + k + \alpha k_s + \beta(\frac{k_{ss}}{k} - \tau^2))^2.
\end{aligned}
$$

On the other hand we have

$$
\begin{aligned}
\mathbf{f} \;=\; & [(\beta_s+\alpha k-\chi\tau)(\chi_t+k+\alpha k_s+\beta(\frac{k_{ss}}{k}-\tau^2))-(\chi_s+\beta\tau)\boldsymbol{(}\beta_t \\
& -\chi(\frac{k_{ss}}{k}-\tau^2)-\alpha\tau k)]\times(\alpha_t+\beta\tau k-\chi k_s)_s-\boldsymbol{(}\beta_t-\chi(\frac{k_{ss}}{k} \\
& -\tau^2)-\alpha\tau k)k]+[(\chi_s+\beta\tau)(\alpha_t+\beta\tau k-\chi k_s)-(1+\alpha_s \\
& -\beta k)(\chi_t+k+\alpha k_s+\beta(\frac{k_{ss}}{k}-\tau^2))]\times\boldsymbol{[(}\beta_t-\chi(\frac{k_{ss}}{k}-\tau^2) \\
& -\alpha\tau k)_s+(\alpha_t+\beta\tau k-\chi k_s)k-(\chi_t+k+\alpha k_s+\beta(\frac{k_{ss}}{k} \\
& -\tau^2))\tau]+[(1+\alpha_s-\beta k)\boldsymbol{(}\beta_t-\chi(\frac{k_{ss}}{k}-\tau^2)-\alpha\tau k) \\
& -(\beta_s+\alpha k-\chi\tau)(\alpha_t+\beta\tau k-\chi k_s)]\times(\chi_t+k+\alpha k_s \\
& +\beta(\frac{k_{ss}}{k}-\tau^2))_s+\boldsymbol{(}\beta_t-\chi(\frac{k_{ss}}{k}-\tau^2)-\alpha\tau k)\tau].
\end{aligned}
$$

Application to Mathematica

The curvature k and torsion τ :

$$
\begin{aligned}
\frac{\partial\tau}{\partial t} &= \frac{\partial}{\partial s}[\frac{1}{k}\frac{\partial}{\partial s}(\frac{\partial V}{\partial s}+\tau U)+\frac{\tau}{k}(\frac{\partial U}{\partial s}-\tau V)+\tau\int kU ds]+k\tau U+k\frac{\partial V}{\partial s}, \\
\frac{\partial k}{\partial t} &= \frac{\partial^2 U}{\partial s^2}+(k^2-\tau^2)U+\frac{\partial k}{\partial s}\int kU ds-2\tau\frac{\partial V}{\partial s}-k\frac{\partial\tau}{\partial s}V.
\end{aligned}
$$

From above equations, using the following Hasimoto transformation

$$\phi=k\eta,\eta=\exp[i\tau(t,s)ds]$$

Let $U=-k_s$, $V=-k\tau$. Then $W=-\frac{1}{2}k^2$, and ϕ satisfies the mKdV system

$$\phi_t+\phi_{sss}+\frac{3}{2}|\phi|^2\phi_s=0. \tag{6.1}$$

Consider the traveling wave variable:

$$\phi(s,t)=q(\varsigma),\qquad \varsigma=s-Qt, \tag{6.2}$$

Then, using Eq. (6.2), Eq. (6.1) is changed into an ordinary differential equation for $q(\varsigma)$:

$$-Qq'(\varsigma)+\frac{3}{2}|q(\varsigma)|^2 q'(\varsigma)+q'''(\varsigma)=0. \tag{6.3}$$

We can give the extended generalized Riccati mapping method to obtain the solution of Eq.(6.1). By balancing $|q(\varsigma)|^2 q'(\varsigma)$ with $q'''(\varsigma)$ in Eq.(6.1), we yield $N=1$.

Then, the solution of Eq. (6.3) is as follows:

$$q(\varsigma)=a_1\left(\frac{G'(\varsigma)}{G(\varsigma)}\right)+a_0, \quad a_1\neq 0. \tag{6.4}$$

Eq. (6.4) can be rewritten as:

$$q(\varsigma)=a_1\left(hG^{-1}(\varsigma)+f+gG(\varsigma)\right)+a_0, \tag{6.5}$$

where f, g, h are arbitrary constants, $g\neq 0$ and $G'(\varsigma)=h+fG(\varsigma)+gG^2(\varsigma)$ is auxiliary equation.

By substituting Eq. (6.5) into Eq. (6.4), we find a set of algebraic equations for a_0, a_1, f, g, h and Q from coefficients of $G^k(\varsigma)$ and $G^{-k}(\varsigma)$ $(k=0,1,2,...)$. Solving the system of algebraic equations by using software Mathematica, we find the following solution,

$$a_0=if,\ a_1=-2i,\ Q=\frac{1}{2}(-f^2-8gh). \tag{6.6}$$

One of solutions of Eq.(6.5) is; $(f=3,\ g=2,\ h=1)$

$$q=-2i\left(\frac{\Delta \sec h^2\left(\frac{\sqrt{\Delta}}{2}\phi\right)}{2\left(f+\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}}{2}\phi\right)\right)}\right)+if, \tag{6.7}$$

where $\Delta=f^2-4gh.$

a) *b*) *c*)

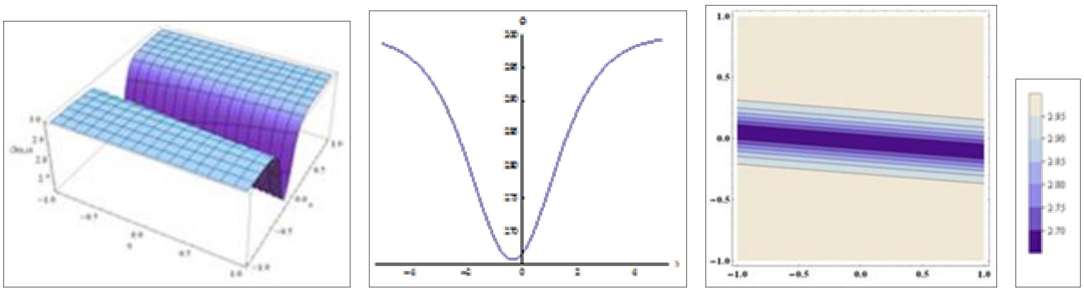

Fig. Shape of solution for imaginary part of Eq.(6.7) ,(a) in 3D, (b) in 2D ($t = 0$), (c) its contour.

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Some properties of Finite Generalized Groups

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Abstract

Finite generalized groups is the special case of generalized groups, which it was introduced by M.R. Molaei in 1998, as an extension of the groups. It has a background in Unified Gauge Theory. We will review of Generalized groups. In this article, we consider the Generalized groups in finite state, we have some interesting properties. We give some example and results.

Keywords: Generalized group, group, Generalized Lagrange Theorem.

1. Introduction

According to Araujo and Konieczny [2], generalized groups are equivalent to the notion of completely simple semigroups. In fact, a semigroup G is called a completely simple semigroup if for all $g \in G$, $GgG = G$, and if a and b are idempotents in G such that $ab = ba$ then $a = b$. Here we call them as generalized groups. Generalized groups was introduced by Molaei in [3]. It is as an extension of groups. A generalized group is a non-empty set G admitting an operation called multiplication subject to the set of rules given below:

1. $x(yz) = (xy)z$; for all $x, y, z \in G$; (associative low)
2. For each $x \in G$, there exists a unique $e(x) \in G$ such that $xe(x) = e(x)x = x$;
3. For each $x \in G$, there exists $x^{-1} \in G$ such that $xx^{-1} = x^{-1}x = e(x)$.

Some of the structures and properties of generalized groups have been studied by Vagner [6], Molaei [4], and Agboola [1]. Also, various applications of these algebraic structures are studied in some recent papers. In [5], Shajareh Poursalavati, introduced the concept of Molaeis generalized hyper- groups by using construction of the Rees matrix semigroup over a polygroup P and a matrix with entries in P.

2. Main Results

In this section, we recall some Definitions and Theorems, and we get some examples and we obtain some new Theorems and results.

Definition 2.1. Let G be a non-empty set, and "$\cdot$" be a binary operation on G, then the couple $(G, \cdot)$ is called a groupoid. If the equations $g \cdot x = h$ and $y \cdot g = h$ have an unique solutions relative to x and y respectively and for all $g, h \in G$, then the couple $(G, \cdot)$ is called a quasigroup. If the couple $(G, \cdot)$ be a groupoid, and for all g, h and $k \in G, (g \cdot h) \cdot k = g \cdot (h \cdot k)$, then the couple $(G, \cdot)$ is called a semigroup.

Definition 2.2. A generalized group *(G, ·)* is a semigroup, which is satisfy the following conditions:

1. For each $g \in G$ there exists a unique $e(g) \in G$ such that $g \cdot e(g) = e(g) \cdot g = g$;
2. For each $g \in G$, there exists $g^{-1} \in G$ such that $g \cdot g^{-1} = g^{-1} \cdot g = e(g)$.

Definition 2.3. Let *(G, ·)* be a generalized group. If, for all g and h in G, $e(g \cdot h) = e(g) \cdot e(h)$, then *(G, ·)* is called normal generalized group. If, for every elements g and h in G, g ·h = g · h;, then, G is called Abelian generalized group. If G be an Abelian generalized group, then G is an Abelian group.

Example 2.4. Let G be a group, then G is a normal generalized group.

Example 2.5. Assume that F be a field and let

$$H = \left\{ \begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix} \mid 0 \neq y, x \in F \right\},$$

then by the ordinary matrices product, H is a generalized group with. We can obtain:

$$e(\begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix}) = \begin{pmatrix} 0 & 0 \\ xy^{-1} & 1 \end{pmatrix} \ and \begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 0 \\ x^2y^{-1} & y^{-1} \end{pmatrix},$$

and

$$\begin{aligned} e(\begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix}\begin{pmatrix} 0 & 0 \\ z & t \end{pmatrix}) &= e(\begin{pmatrix} 0 & 0 \\ yz & yt \end{pmatrix}) = \begin{pmatrix} 0 & 0 \\ zt^{-1} & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ xy^{-1} & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ zt^{-1} & 1 \end{pmatrix} = e(\begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix})e(\begin{pmatrix} 0 & 0 \\ z & t \end{pmatrix}). \end{aligned}$$

Then H is a normal generalized group.

Example 2.6. Assume that Γ and I be nonempty sets and G be a group with the identity element e. Let $M = (g_{\gamma i})$ be a $\Gamma \times I$ matrix with entries in G. Define the operation " · " on the set $I \times G \times \Gamma$ by

$$(i, k, \gamma) \cdot (j, h, \mu) = (i, kg_{\gamma j}h, \mu).$$

for all, $i, j \in I$, and $\gamma, \mu \in \Gamma$, and $k, h \in G$. Therefore, $I \times G \times \Gamma$ is a generalized group.

We can obtain:

$$e((i, k, \gamma) \cdot (j, h, \mu)) = e((i, kg_{\gamma j}h, \mu)) = (i, g_{\mu i}^{-1}, \mu)$$

$$e((i, k, \gamma)) \cdot e((j, h, \mu)) = (i, g_{\gamma i}^{-1}, \gamma) \cdot (j, g_{\mu j}^{-1}, \mu) = (i, g_{\gamma i}^{-1} g_{\gamma j} g_{\mu j}^{-1}, \mu)$$

Then, in general, $I \times G \times \Gamma$ may be not a normal generalized group.

Theorem 2.7. Let *(G, ·)* be a generalized group and *g, h* $\in$ *G*. Then, *e(g)* is unique and *e(e(g)) = e(g),*

e(g)·e(g) = e(g) and g^{-1} is unique and for every integer number *n, e(*g^n*) = e(g).*

Theorem 2.8. Let *(G, ·)* be a normal finite generalized group then, for every elemnt *g* in *G*, there is a positive integer number *k*, such that, $\mathrm{e(g)} = g^k$, and the set $G_g := \{\, t \in G \mid e(t) = e(g) \,\}$, with the induced binary operation on *G* is a group.

Theorem 2.9. Let *(G, ·)* be a finite generalized group then, for every elemnt *g* in *G*, the cardinal number of the group G_g, divided the cardinal number of *G*. Also, we have:

$$G = \bigcup_{g \in A} G_g,$$

Where, *A* is a subset of *G* such that for all *x* and *y* in *A*, if $x \neq y$ then $G_x \neq G_y$. Then, the cardinal number of the set *A* divided the cardinal number of *G*.

Theorem 2.10. Let *(G, ·)* be a finite generalized group and *H* be a generalized subgroup of *G*. Then the generalized Lagrang Theorem may be not true for cardinal number of *H* and *G*.

and G, i.e., it may be card(H) not divided card(G).

3. Conclusion

In this study, we introduce and consider the Generalized groups in finite state, we have some interesting properties. We give some example and results. In special case, we demonstrate the generalized Lagrange Theorem not true for the finite generalized group.

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