4th International Conference on Mathematics*
"An Istanbul Meeting for World Mathematicians"

ICOM 2020 Conference Proceedings Book

ISBN: 978-605-67964-6-3

Editor Kenan Yıldırım

***Online conference due to pandemic**

ICOM 2020
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4th INTERNATIONAL CONFERENCE ON MATHEMATICS
"An Istanbul Meeting for World Mathematicians"
27-30 October 2020, Istanbul, Turkey
This conference is dedicated to 67th birthday of Prof. M. Mursaleen

Contents

A Class of Multivalent Harmonic Convex Functions Defined by Subordination

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Abstract

We have introduced a generalized class of complex-valued multivalent harmonic convex functions defined by subordination. We study some properties of our class. The results obtained here include a number of known and new results as their special cases.

Keywords: Harmonic multivalent functions, convex functions, subordination.

1. Introduction

A continuous complex-valued function $f = u + iv$ defined in a simply connected complex domain $D \subset \mathbb{C}$ is said to be harmonic in D if both u and v are real harmonic in D. Consider the function U and V analytic in D so that $u = \mathrm{Re}U$ and $v = \mathrm{Im}V$. Then the harmonic function f can be expresses by

$$f(z) = h(z) + \overline{g(z)}, \qquad z \in D,$$

where $h = (U + V)/2$ and $g = (U - V)/2$. We call h the analytic part and g co-analytic part of f. If g is identically zero then f reduces to the analytic case.

For a fixed positive integer $p \geq 1$, let $H(p)$ denote the class of all multivalent harmonic functions $f = h + \bar{g}$ which are sense-preserving in the open unit dist $\mathbb{D} = \{z: z \in \mathbb{C} \text{ and } |z| < 1\}$ and are of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n + \sum_{n=p}^{\infty} \overline{b_n z^n}, \qquad |b_p| < 1. \tag{1}$$

Recent interest in the study of multivalent harmonic function prompted the publication of several articles such as [1], [2], [7], [8], [9], [10].

It is known, (see Clunie and Sheil-Small [3]), that harmonic function is sense-preserving in $\mathbb{D}$ is that $|g'(z)| < |h'(z)|$, $z \in \mathbb{D}$. Note that the class $H(p)$ for $p = 1$ was defined and studied by Clunie and Sheil-Small [3]. We say that $f \in H(p)$ ia a multivalent harmonic convex of order α, $0 \leq \alpha < p$ if f satisfies the condition

$$Re\left\{\frac{z^2 f_{zz}(z) + z f_z(z) + \bar{z} f_{\bar{z}}(z) + \overline{z^2} f_{\overline{zz}}(z)}{z f_z(z) - \bar{z} f_{\bar{z}}(z)}\right\} > p\alpha,$$

for each $z \in \mathbb{D}$. Denote this class of multivalent harmonic convex functions of order α by $C_H(p,\alpha)$. The classes $C_H(p,\alpha)$ and $C_H(1,\alpha)$ were studied in [1] and [6].

We say that $f \in H(p)$ is subordinate to a function $F \in H(p)$, and write $f(z) \prec F(z)$ if there exists a complex-valued function w which maps $\mathbb{D}$ into oneself with $w(0) = 0$, such that $f(z) = F(w(z))$, $(z \in \mathbb{D})$.

Denote by $C_H(p,\alpha,A,B)$ the subclass of $H(p)$ consisting of functions f of the form (1) that satisfy the condition

$$\frac{z^2 h''(z) + z h'(z) + \overline{z^2 g''(z) + z g'(z)}}{z h'(z) - \overline{z g'(z)}} \prec \alpha + (p-\alpha)\frac{1+Az}{1+Bz}, \tag{2}$$

Where $-1 \le B \le -B < A \le 1$ and $0 \le \alpha < p$.

By suitably specializing the parameters, the classes $C_H(p,\alpha,A,B)$ reduces to the various subclasses of harmonic univalent functions. Such as,

(i) $C_H(1,0,A,B) = C_H(A,B)$, ([4]),

(ii) $C_H(p,\alpha,1,-1) = C_H(p,\alpha)$, ([1]),

(iii) $C_H(1,\alpha,1,-1) = C_H(\alpha)$, ([6]),

(iv) $C_H(1,0,1,-1) = C_H$, ([6]).

Finally, we define the family $TC_H(p,\alpha,A,B) \equiv C_H(p,\alpha,A,B) \cap TH(p)$, where $TH(p)$, $p \ge 1$ denote the class of functions $f = h + \bar{g}$ in $H(p)$ so that h and g are the form

$$h(z) = z^p - \sum_{n=p+1}^{\infty} |a_n| z^n \quad and \quad g(z) = -\sum_{n=p}^{\infty} |b_n| z^n, \ \ |b_p| < 1. \tag{3}$$

Making use of the techniques and methodology used by Dziok (see [4], [5]), in this paper, we find necessary and sufficient conditions, distortion bounds, compactness and extreme points for the above defined class $TC_H(p,\alpha,A,B)$.

2. Main Results

For functions f_1 and $f_2 \in H(p)$ of the form

$$f_k(z) = z^p + \sum_{n=p+1}^{\infty} a_{k,n} z^n + \sum_{n=p}^{\infty} \overline{b_{k,n} z^n}, \quad (k = 1,2),$$

we define the Hadamard product of f_1 and f_2 by

$$(f_1 * f_2)(z) = z^p + \sum_{n=p+1}^{\infty} a_{1,n} a_{2,n} z^n + \sum_{n=p}^{\infty} \overline{b_{1,n} b_{2,n} z^n}.$$

First we state and prove the necessary and sufficient conditions for harmonic functions in $C_H(p,\alpha,A,B)$.

Theorem 1. Let $f \in H(p)$. Then $f \in C_H(p,\alpha,A,B)$ if and only if

$$f(z) * \varphi(z;\zeta) \neq 0, \quad (\zeta \in \mathbb{C}, |\zeta| = 1, z \in \mathbb{D}\backslash\{1\}),$$

where

$$\varphi(z;\zeta) = z^p \left\{ \frac{p(p-\alpha)(B-A)\zeta + [(1+B\zeta)(1+p) + (1-2p)(p-\alpha)(B-A)\zeta]z}{(1-z)^3} \right.$$
$$\left. + \frac{(1-p)[1+B\zeta+(p-\alpha)(A-B)\zeta]}{(1-z)^3} z^2 \right\} + \overline{z^p} \left\{ \frac{p^2(2+(A+B)\zeta) + \alpha(B-A)p\zeta}{(1-\bar{z})^3} \right.$$
$$+ \frac{(3p-4p^2+1)(1+B\zeta) + (p-\alpha)(1-2p)(A-B)\zeta}{(1-\bar{z})^3} \bar{z}$$
$$\left. + \frac{(p-1)[(2p-1)(1+B\zeta) + (p-\alpha)(A-B)\zeta]}{(1-\bar{z})^3} \bar{z}^2 \right\}$$

Proof. Let $f \in H(p)$ be of the form (1). Then $f \in C_H(p,\alpha,A,B)$ if and only if it satisfies (2) or equivalently

$$\frac{z^2 h''(z) + z h'(z) + \overline{z^2 g''(z) + z g'(z)}}{z h'(z) - \overline{z g'(z)}} \neq \frac{p + [pB + (p-\alpha)(A-B)\zeta]}{1 + B\zeta}, \qquad (4)$$

where $\zeta \in \mathbb{C}, |\zeta| = 1, z \in \mathbb{D}\backslash\{1\}$. Since

$$z h'(z) = h(z) * \frac{z^p[p + (1-p)z]}{(1-z)^2}, \qquad z g'(z) = g(z) * \frac{z^p[p + (1-p)z]}{(1-z)^2}$$

and

$$z^2 h''(z) = h(z) * \frac{z^p[p(p-1) + 2p(2-p)z + (2-3p+p^2)z^2]}{(1-z)^3},$$

$$z^2 g''(z) = g(z) * \frac{z^p[p(p-1) + 2p(2-p)z + (2-3p+p^2)z^2]}{(1-z)^3},$$

the inequality (4) yields

$$(1+B\zeta)\left[z^2h''(z) + zh'(z) + \overline{z^2g''(z) + zg'(z)}\right] - \left[p + [pB + (p-\alpha)(A-B)\zeta]\right]\left(zh'(z) - \overline{zg'(z)}\right)$$

$$= h(z) * \left\{(1+B\zeta)\frac{z^p[p^2 + (2p-2p^2+1)z + (p-1)^2z^2]}{(1-z)^3} \right.$$
$$\left. - (p + [pB + (p-\alpha)(A-B)]\zeta)\frac{z^p[p + (1-p)z]}{(1-z)^2}\right\} + \overline{g(z)}$$
$$* \left\{(1+B\zeta)\frac{\overline{z^p}\left[p^2 + (2p-2p^2+1)\bar{z} + (p-1)^2\overline{z^2}\right]}{(1-\bar{z})^3}\right.$$
$$\left. + (p + [pB + (p-\alpha)(A-B)]\zeta)\frac{\overline{z^p}[p + (1-p)\bar{z}]}{(1-\bar{z})^2}\right\}$$

$$= h(z) * z^p\left\{\frac{p(p-\alpha)(B-A)\zeta}{(1-z)^3} + \frac{(1+B\zeta)(1+p) + (1-2p)(p-\alpha)(B-A)\zeta}{(1-z)^3}z\right.$$
$$\left. + \frac{(1-p)[1 + B\zeta + (p-\alpha)(A-B)\zeta]}{(1-z)^3}z^2\right\} + \overline{g(z)}$$
$$* \overline{z^p}\left\{\frac{p^2(2 + (A+B)\zeta) + \alpha(B-A)p\zeta}{(1-\bar{z})^3}\right.$$
$$+ \frac{(3p-4p^2+1)(1+B\zeta) + (p-\alpha)(1-2p)(A-B)\zeta}{(1-\bar{z})^3}\bar{z}$$
$$\left. + \frac{(p-1)[(2p-1)(1+B\zeta) + (p-\alpha)(A-B)\zeta]}{(1-\bar{z})^3}\overline{z^2}\right\}$$

$$= f(z) * \varphi(z;\zeta) \neq 0.$$

Next we give the sufficient coefficient bound for functions in $C_H(p,\alpha,A,B)$.

Theorem 2. Let f be of the form (1). If $-1 \leq B \leq -B < A \leq 1$, $0 \leq \alpha < p$, $p \geq 1$ and

$$\sum_{n=p}^{\infty}\left(\theta(p,\alpha,A,B)|a_n| + \Upsilon(p,\alpha,A,B)|b_n|\right) \leq 2p(p-\alpha)(A-B) \tag{5}$$

where

$$\theta(p,\alpha,A,B) = n[(1-B)(n-p) + (p-\alpha)(A-B)] \tag{6}$$

and

$$\Upsilon(p,\alpha,A,B)=n[(1-B)(n-p)+(p-\alpha)(A-B)] \tag{7}$$

then $f\in C_H(p,\alpha,A,B)$.

Proof. We need to show that if (2) holds then $f\in C_H(p,\alpha,A,B)$. By definition of subordination, $f\in C_H(p,\alpha,A,B)$ if and only if there exists a complex valued function w; $w(0)=0$, $|w(z)|<1(z\in\mathbb{D})$ such that

$$\frac{z^2h''(z)+zh'(z)+\overline{z^2g''(z)+zg'(z)}}{zh'(z)-\overline{zg'(z)}}=\frac{p+[pB+(p-\alpha)(A-B)]w(z)}{1+Bw(z)}$$

or equivalently

$$\left|\frac{z^2h''(z)+\overline{z^2g''(z)}+(1-p)zh'(z)+(1+p)\overline{zg'(z)}}{B\left(z^2h''(z)+\overline{z^2g''(z)}\right)+[(1-\alpha)B-(p-\alpha)A]zh'(z)+[(1+\alpha)B+(p-\alpha)A]\overline{zg'(z)}}\right|<1. \tag{8}$$

Substituting for $z^2h''(z)$, $z^2g''(z)$, $zh'(z)$ and $zg'(z)$ in (8), we obtain

$$\left|B\left(z^2h''(z)+\overline{z^2g''(z)}\right)+[(1-\alpha)B-(p-\alpha)A]zh'(z)+[(1+\alpha)B+(p-\alpha)A]\overline{zg'(z)}\right| \\ -\left|z^2h''(z)+\overline{z^2g''(z)}+(1-p)zh'(z)+(1+p)\overline{zg'(z)}\right|$$

$$=\left|p(p-\alpha)(A-B)z^p+\sum_{n=p+1}^{\infty}n[-B(n-\alpha)+A(p-\alpha)]a_nz^n\right. \\ \left.+\sum_{n=p}^{\infty}n[-B(n+\alpha)-A(p-\alpha)]\overline{b_nz^n}\right|-\left|\sum_{n=p+1}^{\infty}n(n-p)a_nz^n-\sum_{n=p}^{\infty}n(n+p)\overline{b_nz^n}\right|$$

$$\geq p(p-\alpha)(A-B)|z^p|-\sum_{n=p+1}^{\infty}n[(1-B)(n-p)+(p-\alpha)(A-B)]|a_n|\,|z|^n \\ -\sum_{n=p}^{\infty}n[(1-B)(n+p)-(p-\alpha)(A-B)]|b_n||z|^n$$

$$>|z|^p\left\{p(p-\alpha)(A-B)-\sum_{n=p+1}^{\infty}n[(1-B)(n-p)+(p-\alpha)(A-B)]|a_n|\right. \\ \left.-\sum_{n=p}^{\infty}n[(1-B)(n+p)-(p-\alpha)(A-B)]|b_n|\right\}\geq 0,$$

by (5).

The hormonic functions

$$f(z) = z + \sum_{n=p+1}^{\infty} \frac{p(p-\alpha)(A-B)}{\theta(p,\alpha,A,B)} x_n z^n + \sum_{n=p}^{\infty} \frac{p(p-\alpha)(A-B)}{\Upsilon(p,\alpha,A,B)} y_n \overline{z^n}, \tag{9}$$

where $\sum_{n=p+1}^{\infty}|x_n| + \sum_{n=p}^{\infty}|y_n| = 1$, show that the coefficient bound given by in Theorem 2 is sharp.

Since

$$\sum_{n=p+1}^{\infty} \theta(p,\alpha,A,B)|a_n| + \sum_{n=p}^{\infty} \Upsilon(p,\alpha,A,B)|b_n|$$

$$= p(p-\alpha)(A-B) \sum_{n=p+1}^{\infty} |x_n| + p(p-\alpha)(A-B) \sum_{n=p}^{\infty} |y_n|$$

$$= p(p-\alpha)(A-B),$$

the functions of the form (9) are in $C_H(p,\alpha,A,B)$.

Next we show that the bound (5) is also necessary for $TC_H(p,\alpha,A,B)$.

Theorem 3. Let $f = h + \bar{g}$ with h and g of the form (2). Then $f \in TC_H(p,\alpha,A,B)$ if and only if the condition (5) holds.

Proof. In view of Theorem 2, we only need to show that $f \notin TC_H(p,\alpha,A,B)$ if condition (5) does not hold. We note that a necessary and sufficient condition for $f = h + \bar{g}$ given by (2) to be in $TC_H(p,\alpha,A,B)$ is that the coefficient condition (5) to be satisfied. Equivalently, we must have

$$\left|\frac{H(z)}{G(z)}\right| < 1.$$

where

$$H(z) = \sum_{n=p+1}^{\infty} n(n-p)|a_n|\overline{z^n} + \sum_{n=p}^{\infty} n(n+p)|b_n|\overline{z^n}$$

and

$$G(z)=p(p-\alpha)(A-B)z^p-\sum_{n=p+1}^{\infty}n[-B(n-\alpha)+(p-\alpha)(A-B)]|a_n|z^n$$
$$-\sum_{n=p}^{\infty}n[-B(n+\alpha)-(p-\alpha)(A-B)]|b_n|\overline{z^n}.$$

For $z=r<1$ we obtain

$$\left[\sum_{n=p+1}^{\infty}n(n-p)|a_n|r^{n-1}+\sum_{n=p}^{\infty}n(n+p)|b_n|r^{n-1}\right]$$
$$\times\left[p(p-\alpha)(A-B)-\sum_{n=p+1}^{\infty}n[-B(n-\alpha)+(p-\alpha)(A-B)]|a_n|r^{n-1}\right.$$
$$\left.-\sum_{n=p}^{\infty}n[-B(n+\alpha)-(p-\alpha)(A-B)]|b_n|r^{n-1}\right]^{-1}<1. \qquad (10)$$

If condition (5) does not hold then condition (10) does not hold for r sufficiently close to 1. Thus there exists $z_0=r_0$ in (0,1) for which the quotient (10) is greater than 1. This contradicts the required condition for $f\in TC_H(p,\alpha,A,B)$ and so the proof is complete.

Theorem 4. Let $f\in TC_H(p,\alpha,A,B)$. Then for $|z|=r<1$, we have

$$|f(z)|\le(1+|b_p|)r^p+\left(\frac{p(p-\alpha)(A-B)-p[2p(1-B)-(p-\alpha)(A-B)]|b_p|}{(p+1)[1-B+(p+\alpha)(A-B)]}\right)r^{p+1},$$

and

$$|f(z)|\ge(1-|b_p|)r^p-\left(\frac{p(p-\alpha)(A-B)-p[2p(1-B)-(p-\alpha)(A-B)]|b_p|}{(p+1)[1-B+(p+\alpha)(A-B)]}\right)r^{p+1}.$$

Proof. We only prove the left hand inequality. The proof for the right hand inequality is similar and will be omitted. Let $f\in TC_H(p,\alpha,A,B)$. Taking the absolute value of f we have

$$|f(z)|\ge(1-|b_p|)r^p-\sum_{n=p+1}^{\infty}(|a_n|+|b_n|)r^n$$
$$\ge(1-|b_p|)r^p-\sum_{n=p+1}^{\infty}\frac{\theta(p,\alpha,A,B)|a_n|+\Upsilon(p,\alpha,A,B)|b_n|}{(p+1)[1-B+(p+\alpha)(A-B)]}r^{p+1}$$

$$\geq \left(1-\left|b_p\right|\right)r^p-\frac{p(p-\alpha)(A-B)-p[2p(1-B)-(p-\alpha)(A-B)]\left|b_p\right|}{(p+1)[1-B+(p+\alpha)(A-B)]}r^{p+1}.$$

The following covering result follows from the left hand inequality in Theorem 4.

Corollary 5. Let $f = h + \bar{g}$ with h and g of the form (3). If $f \in TC_H(p,\alpha,A,B)$ then

$$\left\{w:|w|<\frac{(p+1)(1-B)+(p-\alpha)(A-B)+(2p+1)[(p-1)(1-B)-(p-\alpha)(A-B)]\left|b_p\right|}{(p+1)[1-B+(p+\alpha)(A-B)]}\right\}\subset f(\mathbb{D}).$$

Theorem 6. Set

$$h_p(z)=z^p, h_p(z)=z^p-\frac{p(p-\alpha)(A-B)}{\theta(p,\alpha,A,B)}z^n, \qquad (n=p+1,p+2,\dots),$$

and

$$g_n(z)=z^p+\frac{p(p-\alpha)(A-B)}{\Upsilon(p,\alpha,A,B)}\overline{z^n}, \qquad (n=p,p+1,\dots).$$

Then $f \in TC_H(p,\alpha,A,B)$ if and only if it can be expressed as

$$f(z)=\sum_{n=p}^{\infty}\left(x_n h_n(z)+y_n g_n(z)\right)$$

where $x_n \geq 0$, $y_n \geq 0$ and $\sum_{n=p}^{\infty}(x_n+y_n)=1$. In particular, the extreme points of $f \in TC_H(p,\alpha,A,B)$ are $\{h_n\}$ and $\{g_n\}$.

Proof. Suppose

$$f(z)=\sum_{n=p}^{\infty}\left(x_n h_n(z)+y_n g_n(z)\right)$$

$$=\sum_{n=p}^{\infty}(x_n+y_n)z^p-\sum_{n=p+1}^{\infty}\frac{p(p-\alpha)(A-B)}{\theta(p,\alpha,A,B)}x_n z^n+\sum_{n=p}^{\infty}\frac{p(p-\alpha)(A-B)}{\Upsilon(p,\alpha,A,B)}y_n\overline{z^n}.$$

Then

$$\sum_{n=p+1}^{\infty}\theta(p,\alpha,A,B)|a_n|+\sum_{n=p}^{\infty}\Upsilon(p,\alpha,A,B)|b_n|$$

$$= p(p-\alpha)(A-B)\sum_{n=p}^{\infty} x_n + p(p-\alpha)(A-B)\sum_{n=p}^{\infty} y_n$$

$$= p(p-\alpha)(A-B)\left(1-x_p\right) \leq p(p-\alpha)(A-B)$$

and so $f \in TC_H(p,\alpha,A,B)$. Conversely, if $f \in TC_H(p,\alpha,A,B)$, then

$$|a_n| \leq \frac{p(p-\alpha)(A-B)}{\theta(p,\alpha,A,B)} \quad \text{and} \quad |b_n| \leq \frac{p(p-\alpha)(A-B)}{\Upsilon(p,\alpha,A,B)}.$$

Set

$$x_n = \frac{\theta(p,\alpha,A,B)}{p(p-\alpha)(A-B)}|a_n| \qquad (n = p+1, p+2, \dots)$$

and

$$y_n = \frac{\Upsilon(p,\alpha,A,B)}{p(p-\alpha)(A-B)}|b_n| \qquad (n = p, p+1, \dots).$$

Then note by Theorem 3,

$$0 \leq x_n \leq 1 \ \ (n = p+1, p+2, \dots) \ \text{ and } \ 0 \leq y_n \leq 1 \ \ (n = p, p+1, \dots).$$

We define

$$x_p = 1 - \sum_{n=p+1}^{\infty} x_n - \sum_{n=p}^{\infty} y_n$$

and note that by Theorem3, $x_p \geq 0$. Consequently, we obtain

$$f(z) = \sum_{n=p}^{\infty}\left(x_n h_n(z) + y_n g_n(z)\right)$$

as required.

Now we show that $TC_H(p,\alpha,A,B)$ is closed under convex combinations of its members.

Theorem 7. The class $TC_H(p,\alpha,A,B)$ is closed under convex combination.

Proof. For $k = 1,2,3, \dots$ let $f_k \in TC_H(p,\alpha,A,B)$, where f_k is given by

$$f_k(z) = z^p - \sum_{n=p+1}^{\infty} |a_{k,n}| z^n + \sum_{n=p}^{\infty} |b_{k,n}| \overline{z^n}.$$

Then by (6),

$$\sum_{n=p}^{\infty} \left(\theta(p,\alpha,A,B)|a_{k,n}| + \Upsilon(p,\alpha,A,B)|b_{k,n}|\right) \leq 2p(p-\alpha)(A-B). \tag{11}$$

For $\sum_{k=1}^{\infty} t_k = 1$, $0 \leq t_k \leq 1$, the convex combination of f_k may be written as

$$\sum_{k=1}^{\infty} t_k f_k(z) = z^p - \sum_{n=p+1}^{\infty} \left(\sum_{k=1}^{\infty} t_k |a_{k,n}|\right) z^n + \sum_{n=p}^{\infty} \left(\sum_{k=1}^{\infty} t_k |b_{k,n}|\right) \overline{z^n}.$$

Then by (11),

$$\sum_{n=p}^{\infty} \left(\theta(p,\alpha,A,B) \sum_{k=1}^{\infty} t_k\, |a_{k,n}| + \Upsilon(p,\alpha,A,B) \sum_{k=1}^{\infty} t_k\, |b_{k,n}|\right)$$

$$= \sum_{k=1}^{\infty} t_k \left(\sum_{k=1}^{\infty} \left[\theta(p,\alpha,A,B)|a_{k,n}| + \Upsilon(p,\alpha,A,B)|b_{k,n}|\right]\right)$$

$$\leq 2p(p-\alpha)(A-B) \sum_{k=1}^{\infty} t_k = 2p(p-\alpha)(A-B).$$

This is the condition required by (6) and so $\sum_{k=1}^{\infty} t_k f_k(z) \in TC_H(p,\alpha,A,B)$.

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A General Theorem Involving Quasi Power Increasing Sequences

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Abstract
In this paper, a known theorem dealing with absolute Cesàro summability of an infinite series is generalized for $\varphi-\left|C,\alpha;\beta\right|_k$ summability by using a quasi-*f*-power increasing sequence instead of an almost increasing sequence.
Keywords: δ **-**quasi-monotone sequences, Cesàro summability, almost increasing sequences, quasi power increasing sequences, summability factors, infinite series, Hölder's inequality, Minkowski's inequality.

1. Introduction

A sequence (A_n) is said to be δ-quasi-monotone if $A_n \to 0$, $A_n > 0$ ultimately and $\Delta A_n \geq -\delta_n$, where $\Delta A_n = A_n - A_{n+1}$ and $\delta = (\delta_n)$ is a sequence of positive numbers (see [2]). A positive sequence (b_n) is said to be almost increasing if there exist a positive increasing sequence (c_n) and two positive constants K and M such that $Kc_n \leq b_n \leq Mc_n$ (see [1]). Obviously, every increasing sequence is almost increasing, but the converse need not be true as can be seen from the example $b_n = ne^{(-1)^n}$. Let t_n^α be the nth Cesàro mean of order α, with $\alpha > -1$, of the sequence (na_n), that is (see [12])

$$t_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v$$

where

$$A_n^\alpha = O(n^\alpha),\ A_0^\alpha = 1 \text{ and } A_{-n}^\alpha = 0 \text{ for } n > 0.$$

Let $\left(w_n^\alpha\right)$ be a sequence defined by (see [22])

$$w_n^\alpha = \begin{cases} \left|t_n^\alpha\right|, & \alpha = 1 \\ \max_{1\leq v\leq n}\left|t_v^\alpha\right|, & 0 < \alpha < 1. \end{cases} \tag{1}$$

Let (φ_n) be a sequence of positive numbers. The series $\sum a_n$ is said to be summable $\varphi-\left|C,\alpha;\beta\right|_k$, $k\geq 1$, $\alpha>-1, \beta\geq 0$, if (see [24])

$$\sum_{n=1}^{\infty}\varphi_n^{\beta k+k-1}n^{-k}\left|t_n^{\alpha}\right|^k<\infty.$$

If we take $\varphi_n=n$ and $\beta=0$, then $\varphi-\left|C,\alpha;\beta\right|_k$ summability reduces to $\left|C,\alpha\right|_k$ summability (see [13]).

2. Known Result

The following theorem dealing with $\left|C,\alpha\right|_k$ summability has been proved via δ-quasi-monotone and almost increasing sequences.

Theorem 2.1 ([3]). Let (X_n) be an almost increasing sequence such that $\left|\Delta X_n\right|=O(X_n/n)$ and (λ_n) is a sequence such that

$$\left|\lambda_n\right|X_n=O(1)\ \ as\ \ n\to\infty. \tag{2}$$

Suppose also that there exists a sequence of numbers (A_n) such that it is δ-quasi-monotone with $\sum nX_n\delta_n<\infty$, $\sum A_nX_n$ is convergent, and $\left|\Delta\lambda_n\right|\leq\left|A_n\right|$ for all n. If the sequence $\left(w_n^{\alpha}\right)$ defined by (1) satisfies the condition

$$\sum_{n=1}^{m}\frac{\left(w_n^{\alpha}\right)^k}{nX_n^{k-1}}=O(X_m)\ \ as\ \ m\to\infty, \tag{3}$$

then the series $\sum a_n\lambda_n$ is summable $\left|C,\alpha\right|_k$, $0<\alpha\leq 1$ and $k\geq 1$.

3. Main Result

A positive sequence $X=(X_n)$ is said to be a quasi-*f*-power increasing sequence, if there exists a constant $K=K\left(X,f\right)\geq 1$ such that $Kf_nX_n\geq f_mX_m$ for all $n\geq m\geq 1$, where

$$f=\left\{f_n\left(\sigma,\gamma\right)\right\}=\left\{n^{\sigma}\left(\log n\right)^{\gamma},\gamma\geq 0, 0<\sigma<1\right\}$$

(see [25]). If we take $\gamma=0$, then we get a quasi σ-power increasing sequence. It should be noted that, every almost increasing sequence is a quasi σ-power increasing sequence for any nonnegative σ, but the converse is not true if $\sigma>0$ (see [14]). Absolute Cesàro summability methods have some applications on different sequences. For further details, see [5-10, 15-21, 23].

The object of this paper is to generalize Theorem 2.1 for $\varphi-\left|C,\alpha;\beta\right|_k$ summability method by using a quasi-*f*-power increasing sequence as in the following form.

Theorem 3.1. Let (X_n) be a quasi-*f*-power increasing sequence and (λ_n) be a sequence as in (2). Suppose that there exists a sequence of numbers (A_n) such that it is δ-quasi-monotone with $\Delta A_n \le \delta_n$, $\sum nX_n\delta_n < \infty$, $\sum A_n X_n$ is convergent, and $\left|\Delta\lambda_n\right| \le \left|A_n\right|$ for all n. Suppose also that there is an $\varepsilon > 0$ such that the sequence $\left(n^{\varepsilon-k}\varphi_n^{\beta k+k-1}\right)$ is non-increasing. If the sequence $\left(w_n^\alpha\right)$ defined by (1) satisfies the condition

$$\sum_{n=1}^{m}\varphi_n^{\beta k+k-1}n^{-k}\frac{\left(w_n^\alpha\right)^k}{X_n^{k-1}} = O(X_m) \quad as \quad m\to\infty, \tag{4}$$

then the series $\sum a_n\lambda_n$ is summable $\varphi-\left|C,\alpha;\beta\right|_k$, $\beta\ge 0$, $0<\alpha\le 1$, $\varepsilon+\left(\alpha-1\right)k>0$ and $k\ge 1$.

To prove Theorem 3.1, we need the following lemmas.

Lemma 3.2 ([11]). If $0<\alpha\le 1$ and $1\le v\le n$, then

$$\left|\sum_{p=0}^{v}A_{n-p}^{\alpha-1}a_p\right| \le \max_{1\le m\le v}\left|\sum_{p=0}^{m}A_{m-p}^{\alpha-1}a_p\right|. \tag{5}$$

Lemma 3.3 ([4]). Let (X_n) be a quasi-*f*-power increasing sequence. If (A_n) is a δ-quasi-monotone with $\Delta A_n \le \delta_n$ and $\sum nX_n\delta_n < \infty$, then

$$nX_nA_n = O(1) \quad as \quad n\to\infty, \tag{6}$$

$$\sum_{n=1}^{\infty} nX_n\left|\Delta A_n\right| < \infty. \tag{7}$$

4. Proof of Theorem 3.1

Let $0<\alpha\le 1$. Let (I_n^α) be the nth $\left(C,\alpha\right)$ mean of the sequence $\left(na_n\lambda_n\right)$. By Abel's transformation , we get

$$I_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=1}^{n}A_{n-v}^{\alpha-1}va_v\lambda_v = \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1}\Delta\lambda_v\sum_{p=1}^{v}A_{n-p}^{\alpha-1}pa_p + \frac{\lambda_n}{A_n^\alpha}\sum_{v=1}^{n}A_{n-v}^{\alpha-1}va_v\,.$$

Then, using Lemma 3.2, we have

$$\left|I_n^\alpha\right| \le \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1}\left|\Delta\lambda_v\right|\left|\sum_{p=1}^{v}A_{n-p}^{\alpha-1}pa_p\right| + \frac{\left|\lambda_n\right|}{A_n^\alpha}\left|\sum_{v=1}^{n}A_{n-v}^{\alpha-1}va_v\right| \le \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1}A_v^\alpha w_v^\alpha\left|\Delta\lambda_v\right| + \left|\lambda_n\right|w_n^\alpha = I_{n,1}^\alpha + I_{n,2}^\alpha.$$

First, we have

$$\sum_{n=2}^{m+1}\varphi_n^{\beta k+k-1}n^{-k}\left|I_{n,1}^{\alpha}\right|^k \le \sum_{n=2}^{m+1}\varphi_n^{\beta k+k-1}n^{-k}\left(A_n^{\alpha}\right)^{-k}\left(\sum_{v=1}^{n-1}A_v^{\alpha}w_v^{\alpha}\left|\Delta\lambda_v\right|\right)^k$$
$$\le \sum_{n=2}^{m+1}\varphi_n^{\beta k+k-1}n^{-k}\left(A_n^{\alpha}\right)^{-k}\left(\sum_{v=1}^{n-1}A_v^{\alpha}w_v^{\alpha}\left|A_v\right|\right)^k.$$

Using Hölder's inequality with indices k and k', where $k>1$ and $\frac{1}{k}+\frac{1}{k'}=1$, we get

$$\sum_{n=2}^{m+1}\varphi_n^{\beta k+k-1}n^{-k}\left|I_{n,1}^{\alpha}\right|^k \le \sum_{n=2}^{m+1}\varphi_n^{\beta k+k-1}n^{-k}\left(A_n^{\alpha}\right)^{-k}\sum_{v=1}^{n-1}\left(A_v^{\alpha}\right)^k\left(w_v^{\alpha}\right)^k\left|A_v\right|^k\left(\sum_{v=1}^{n-1}1\right)^{k-1}$$
$$=O(1)\sum_{v=1}^{m}v^{\alpha k}\left(w_v^{\alpha}\right)^k\left|A_v\right|\left|A_v\right|^{k-1}\sum_{n=v+1}^{m+1}\frac{\varphi_n^{\beta k+k-1}n^{\varepsilon-k}}{n^{1+\varepsilon+(\alpha-1)k}}.$$

Now, we get $\left|A_v\right|=O\left(\frac{1}{vX_x}\right)$ by (6), therefore

$$\sum_{n=2}^{m+1}\varphi_n^{\beta k+k-1}n^{-k}\left|I_{n,1}^{\alpha}\right|^k = O(1)\sum_{v=1}^{m}v^{\alpha k}\left(w_v^{\alpha}\right)^k\left|A_v\right|\frac{1}{\left(vX_v\right)^{k-1}}\varphi_v^{\beta k+k-1}v^{\varepsilon-k}\int_v^{\infty}\frac{dx}{x^{1+\varepsilon+(\alpha-1)k}}$$
$$=O(1)\sum_{v=1}^{m}v\left|A_v\right|\varphi_v^{\beta k+k-1}v^{-k}\frac{\left(w_v^{\alpha}\right)^k}{X_v^{k-1}}.$$

By applying Abel's transformation, and by using the hypotheses of Theorem 3.1 and Lemma 3.3, we have

$$\sum_{n=2}^{m+1}\varphi_n^{\beta k+k-1}n^{-k}\left|I_{n,1}^{\alpha}\right|^k = O(1)\sum_{v=1}^{m-1}\Delta\left(v\left|A_v\right|\right)\sum_{r=1}^{v}\varphi_r^{\beta k+k-1}r^{-k}\frac{\left(w_r^{\alpha}\right)^k}{X_r^{k-1}}+O(1)m\left|A_m\right|\sum_{v=1}^{m}\varphi_v^{\beta k+k-1}v^{-k}\frac{\left(w_v^{\alpha}\right)^k}{X_v^{k-1}}$$
$$=O(1)\sum_{v=1}^{m-1}v\left|\Delta A_v\right|X_v+O(1)\sum_{v=1}^{m-1}\left|A_v\right|X_v+O(1)m\left|A_m\right|X_m=O(1) \quad as \quad m\to\infty.$$

By (2), we get $\left|\lambda_n\right|=O\left(\frac{1}{X_n}\right)$. Therefore, we have

$$\sum_{n=1}^{m}\varphi_n^{\beta k+k-1}n^{-k}\left|I_{n,2}^{\alpha}\right|^k = \sum_{n=1}^{m}\varphi_n^{\beta k+k-1}n^{-k}\left|\lambda_n\right|\left|\lambda_n\right|^{k-1}\left(w_n^{\alpha}\right)^k = O(1)\sum_{n=1}^{m}\varphi_n^{\beta k+k-1}n^{-k}\left|\lambda_n\right|\frac{1}{X_n^{k-1}}\left(w_n^{\alpha}\right)^k.$$

Now, using Abel's transformation, and by using the hypotheses of Theorem 3.1, we get

$$\sum_{n=1}^{m}\varphi_n^{\beta k+k-1}n^{-k}\left|I_{n,2}^{\alpha}\right|^k = O(1)\sum_{n=1}^{m-1}\Delta\left|\lambda_n\right|\sum_{v=1}^{n}\varphi_v^{\beta k+k-1}v^{-k}\frac{\left(w_v^{\alpha}\right)^k}{X_v^{k-1}}+O(1)\left|\lambda_m\right|\sum_{v=1}^{m}\varphi_v^{\beta k+k-1}v^{-k}\frac{\left(w_v^{\alpha}\right)^k}{X_v^{k-1}}$$
$$=O(1)\sum_{n=1}^{m-1}\left|A_n\right|X_n+O(1)\left|\lambda_m\right|X_m=O(1) \quad as \quad m\to\infty.$$

Hence, we obtain $\sum_{n=1}^{\infty} \varphi_n^{\beta k+k-1} n^{-k} \left| I_{n,r}^{\alpha} \right|^k < \infty$ for $r=1$ and $r=2$, which complete the proof of Theorem 3.1.

If we take (X_n) as an almost increasing sequence such that $|\Delta X_n| = O(X_n / n)$, $\varphi_n = n$, $\beta = 0$, and $\varepsilon = 1$ in Theorem 3.1, the condition (4) reduces to the condition (3), then we obtain Theorem 2.1. In this case, the condition "$\Delta A_n \le \delta_n$" is not needed.

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A Mathematical Modelling of Tuberculosis infection Dynamics with Effects of Case Detection and Drug Resistance

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Abstract

A deterministic mathematical model of tuberculosis incorporating case detection and drug resistance with constant recruitment rate was developed. The population was subdivided into six compartments according to their disease status. The basic reproduction number of the model was obtained using the next generation matrix. The existence of disease free and endemics equilibrium points were shown and the conditions for their stability was also esterblished. The results show that the disease free equilibrium points are locally asymptotically stable if $R_0 < 1$ and globally stable if $R_0 \leq 1$. Also the results further show that the endemic equilibrium points are locally asymptotically stable if $R_0 > 1$ and globally stable if $R_0 \geq 1$.We obtain the approximate solution of the model using Homotopy Perturbation Methods. The graphical summaries of the solution were carried out and the result show that increase in case detection and sustained treatment can help to reduce transmission of tuberculosis disease.

Key words: Tuberculosis, Reproduction Number, Homotopy Perturbation Method, Next Generation Matrix

1. Introduction

Tuberculosis is a bacterial disease which attacks some part of the human body such as lungs, bones, lymph nodes and brain. This disease is caused by a known mycobacterium tuberculosis that looks like rod-shape bacterium. Some of the symptoms are in the form of cough, chest pains, shortness of breath, loss of appetite, weight loss, fever, chills and fatigue.

Tuberculosis is the second leading cause of death from infectious disease worldwide after those caused by Human Immune Deficiency Virus (HIV) [4]**.** This disease affect over 2 billion of the world population. Approximately over nine million people develop active tuberculosis and up to 2 million death cases is recorded from tuberculosis every year. Also over 480 thousand people developed drug resistance to tuberculosis with 210 thousand of those who developed multi drug resistance tuberculosis result to death [6].

Tuberculosis (TB) infection is of two type, namely, latent infection and active infection. The latent infection in the body system is a condition in which a patient holds dormant (sleeping) Tuberculosis bacteria in the body and they do not cause TB disease to the patient's body. However, in a certain period, the sleeping bacteria would be awake and become active. People infected latently are called latent TB patients and are unable to infect those vulnerable to TB disease. Actively infected is a state in which the bacteria causing tuberculosis in the patient's body are actively multiplying and the disease symptoms becomes very visible in the body system. Those patients that are infected actively are called active tuberculosis and they can transmit the disease to vulnerable people [17].

Patients of latent and active TB can be treated but they are not totally protected from the disease. Within a certain time, those who recovered can be re-infected again in case contact with TB infectious patient. Based upon the chain of actions of TB bacteria infection, the associated population can be grouped into several different sub-populations which are susceptible to the TB disease. The sub-population groups are latent infectious, active infectious and recovered infectious groups.

About one-third of the world population has latent TB, which means people have been infected by TB bacteria but are not (yet) ill with the disease and cannot transmit the disease. People infected with TB bacteria have a 10 % lifetime risk of falling ill with Tuberculosis disease. Latent Tuberculosis infection can progress to active tuberculosis when the immunity of the host decreases due to aging, stress, over use of immunosuppressant or co-infection with HIV [16].

The spread of tuberculosis diseases can be analysed through mathematical models. The models can help to predict and control the spread of tuberculosis outbreak in the future. The spread of tuberculosis disease can be modelled with some types of epidemics models which can come in different variance of SIR models; see [1,2,5,11,12,13,14]. Also, some author studied the effects of vaccination or otherwise on the overall dynamics of tuberculosis in the population,[7, 9,10].

In this current study, we are going to develop and analyze a model with six compartements and solve the model with a semi-analytuc method known as homotopy perturbation method.

2. Material and Methods

In this section, we are going to formulate the proposed model and conduct standard epidemiological analysis on it.

2.1 Model formulation

We first divide the human population at time $t \geq 0$, into six classes they are Susceptible $S(t)$, Expossed $E(t)$, Infected $I(t)$, Resistance to first line of treatment $R_1(t)$, Resistance to second line of treatmet $R_2(t)$, and the recorvered humans $R(t)$. The size of the human population is given by

$$N(t) = S(t) + E(t) + I(t) + R_1(t) + R_2(t) + R.$$

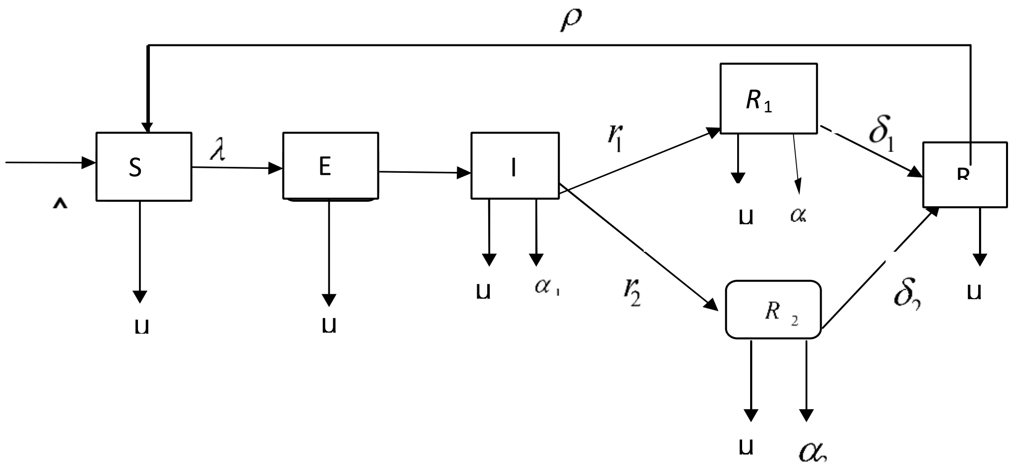

Figure 1 Schematic Diagram Showing the Flow of Tuberculosis Transmission Model

Table 1: Description of Parameters/Variables of the model

Parameter/Variables	Description
Λ	Recruitment rate
$S(t)$	Susceptible humans at time t
$E(t)$	Exposed human at time t
$I(t)$	Infected humans at time t
$R_1(t)$	Resistance class of individual to first line of treatment
$R_2(t)$	Resistance class of individual to second line of treatment
$R(t)$	Recovered humans at time t
η	Case detection rate
λ_1	Rate of transmission (detection)
λ_2	Rate of transmission (undetected)
μ	Natural death rate
γ	The rate at which the infected becomes infectious
ρ	Rate at which recovered individual loss their immunity
r_1	Resistance rate to first line of treatment
r_2	Resistance rate to second line of treatment

α_1	Diseases induced death rate
δ_1	Recovery rate after first line of treatment
δ_2	Recovery rate after second line of treatment
α_2	Diseases induced death rate after first line of treatment
α_3	Diseases induced death rate after second line of treatment

2.2 Basic assumptions of the model

1. The population is varying and homogenously mixed i.e. All people are equally likely to be infected by the infectious individual in case of contact.
2. Both detected and undetected case of individual transmit Tuberculosis at the different rate i.e. it is higher in undetected cases.
3. It is assumed that no permanent immunity to Tuberculosis.
4. Some infected individual delay treatment and moved to resistance classes.
5. Natural death occur in all the classes.

2.3 The model equation

$$\frac{dS}{dt} = \Lambda - \mu S - \left(\lambda_1\eta + \lambda_2(1-\eta)\right) IS + \rho R \tag{1}$$

$$\frac{dE}{dt} = \left(\lambda_1\eta + \lambda_2(1-\eta)\right) IS - \left(\mu + \gamma\right) E \tag{2}$$

$$\frac{dI}{dt} = \gamma E - \left(\mu + \alpha_1 + r_1 + r_2\right) I \tag{3}$$

$$\frac{dR_1}{dt} = r_1 I - \left(\mu + \alpha_2 + \delta_1\right) R_1 \tag{4}$$

$$\frac{dR_2}{dt} = r_2 I - \left(\mu + \alpha_3 + \delta_2\right) R_2 \tag{5}$$

$$\frac{dR}{dt} = \delta_1 R_1 + \delta_2 R_2 - \left(\mu + \rho\right) R \tag{6}$$

2.4 Analysis of the Model

We present standard epidemiological analysis of the model in the following subsections.

2.4.1 Invariant Region of the Model

The rate of total population is given by

$$\frac{dN}{dt} = \frac{dS}{dt} + \frac{dE}{dt} + \frac{dI}{dt} + \frac{dR_1}{dt} + \frac{dR_2}{dt} + \frac{dR}{dt} \tag{7}$$

$$\frac{dN}{dt} = \Lambda - \mu\left(S + E + I + R_1 + R_2 + R\right) - \left(\alpha_1 I + \alpha_2 R_1 + \alpha_3 R_2\right) \tag{8}$$

$$\frac{dN}{dt} \le \Lambda - \mu\left(S + E + I + R_1 + R_2 + R\right) \tag{9}$$

$$\frac{dN}{dt} \le \Lambda - \mu N \tag{10}$$

Where

$$N = S + E + I + R_1 + R_2 + R \tag{11}$$

Theorem 1: Model equations (1)-(6) has solutions which contain in the feasible region Ω for all $t \ge 0$.

Proof:

Let $\Omega = \left(S, E, I, R_1, R_2, R\right) \in \Re^6$ be any solution of the model 1-6 with non-negative initial conditions then from equation (10) we have

$$\frac{dN}{dt} \le \Lambda - \mu N \tag{12}$$

$$0 \le N \le \frac{\Lambda}{\mu} \tag{13}$$

We seek solution of the form

$$IF = e^{\int \mu dt} = e^{\mu t} \tag{14}$$

By multiplying through our equation (12) with the integrating factor we obtain

$$\left[Ne^{\mu t}\right]' \le \Lambda e^{\mu t} \tag{15}$$

$\Rightarrow$

$$\Lambda - \mu N \geq Ce^{\mu t} \tag{16}$$

Therefore, all feasible solution of the human population of the model is in the region

$$\Omega = \left\{ \left(S, E, I, R_1, R_{2,} R \right) \in \Re^6 : \left(S, E, I, R_1, R_{2,} R \right) \geq 0, N \leq \frac{\Lambda}{\mu} \right\} \tag{17}$$

2.4.2 Positivity of the solutions

Lemma 1: Let the initial solutions be $\left\{ S(0), E(0), I(0), R_1(0), R_2(0), R(0) \geq 0 \right\} \in \Omega$ then the solution $\left\{ S(t), E(t), I(t), R_1(t), R_2(t), R(t) \right\}$ of the model equations (1)-(6) is positive for all time $t \geq 0$.

Proof:

From equation (1) we have

$$\frac{dS}{dt} \geq -\mu S \tag{18}$$

By separating the variable and integrating equation (18) we have

$$\int \frac{dS}{S} \geq -\int \mu dt \tag{19}$$

$\Rightarrow$

$$S(t) \geq S(0)e^{-\mu t} \tag{20}$$

Similarly we can show that

$$E(t) \geq E(0)e^{-(\mu+\gamma)t} \tag{21}$$

$$I(t) \geq I(0)e^{-(\mu+\alpha_1+r_1+r_2)t} \tag{22}$$

$$R_1(t) \geq R_1(0)e^{-(\mu+\alpha_2+\delta_1)t} \tag{23}$$

$$R_2(t) \geq R_2(0)e^{-(\mu+\alpha_3+\delta_2)t} \tag{24}$$

$$R(t) \geq R(0)e^{-(\mu+\rho)t} \tag{25}$$

Therefore all the solution of the equations (1) –(6) are positive for all time $t \geq 0$.

2.5 Basic Reproduction Number R_0

The basic reproduction number (R_0) is the average number of new infections that one infected case will generate during their entire infection life time. It is an important tool in determining whether the diseases persist or die out in population.

We use the next generation matrix approach an in [15] to compute the basic reproduction number.

Basic reproduction number is the spectral radius $\rho\left(F_1V_1^{-1}\right)$ where the matrix F_i and V_i are the new infection terms and the remaining transfer terms respectively. The basic reproduction number is obtained as follow

Consider the following differential equations for the diseases compartment

$$\frac{dE}{dt}=\lambda IS-\left(\mu+\gamma\right)E \tag{26}$$

$$\frac{dI}{dt}=\gamma E-\left(\mu+\alpha_1+r_1+r_2\right)I \tag{27}$$

$$\frac{dR_1}{dt}=r_1I-\left(\mu+\alpha_2+\delta_1\right)R_1 \tag{28}$$

$$\frac{dR_2}{dt}=r_2I-\left(\mu+\alpha_3+\delta_2\right)R_2 \tag{29}$$

Where $\lambda=\lambda_1\eta+\lambda_2(1-\eta)$

Let $X=\left(E,I,R_1,R_2\right)^T$ then the above system can be represented in matrix form as shown below:

$$\frac{dX_i}{dt}=F_i\left(X\right)-V_i\left(X\right) \tag{30}$$

Where $F\left(X\right)_i=\begin{pmatrix}\lambda SI\\0\\0\\0\end{pmatrix},\qquad V_i\left(X\right)=\begin{pmatrix}\left(\mu+\gamma\right)E\\-\gamma E+\left(\mu+\alpha_1+r_1+r_2\right)I\\-r_1I+\left(\mu+\alpha_2+\delta_1\right)R_1\\-r_2I+\left(\mu+\alpha_3+\delta_2\right)R_2\end{pmatrix}$ (31)

The Jacobian matrix of $F_i\left(X\right)$ and $V_i\left(X\right)$ at the diseases free equilibrium X_0 are,

$$\frac{dF_i\left(X_0\right)}{d\left(X\right)}=F_1=\begin{pmatrix}0&\frac{\beta\Lambda}{\mu}&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix} \tag{32}$$

$$\frac{dV_i\left(X\right)}{d\left(X\right)}=V_1=\begin{pmatrix}\mu+\gamma&0&0&0\\-\gamma&\mu+\alpha_1+r_1+r_2&0&0\\0&-r_1&\mu+\alpha_2+\delta_1&0\\0\quad 0\quad 0\quad -r_2&0&\mu+\alpha_3+\delta_2\end{pmatrix} \tag{33}$$

Now

$$V_1^{-1}=\begin{pmatrix} \frac{1}{\mu+\gamma} & 0 & 0 & 0 \\ \frac{\gamma}{(\mu+\gamma)(\mu+\alpha_1+r_1+r_2)} & \frac{1}{\mu+\alpha_1+r_1+r_2} & 0 & 0 \\ \frac{r_1\gamma}{(\mu+\gamma)(\mu+\alpha_1+r_1+r_2)(\mu+\alpha_2+\delta_1)} & \frac{r_1}{(\mu+\alpha_1+r_1+r_2)(\mu+\alpha_2+\delta_1)} & \frac{1}{\mu+\alpha_2+\delta_1} & 0 \\ 0 & & & \\ \frac{r_2\gamma}{(\mu+\alpha_1+r_1+r_2)(\mu+\alpha_2+\delta_1)(\mu+\alpha_3+\delta_2)} & \frac{r_2}{(\mu+\alpha_1+r_1+r_2)(\mu+\alpha_3+\delta_2)} & 0 & \frac{1}{\mu+\alpha_3+\delta_2} \end{pmatrix} \tag{34}$$

The next generation matrix of the system is given by

$$F_1V_1^{-1}=\begin{pmatrix} \frac{\lambda\gamma\Lambda}{\mu(\mu+\gamma)(\mu+\alpha_1+r_1+r_2)} & \frac{\lambda\Lambda}{\mu(\mu+\alpha_1+r_1+r_2)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{35}$$

Thus our basic reproduction number is:

$$R_0=\frac{\lambda\gamma\Lambda}{\mu(\mu+\gamma)(\mu+\alpha_1+r_1+r_2)} \tag{36}$$

2.6 Equilibrium points of the model

We obtained two equilibrium point for our model, these are the Disease free Equilibrium point and Endemic Equilibrium point.

2.6.1 Disease Free Equilibrium of the Model

Let $X_0=\left(S^0,E^0,I^0,R_1^0,R_2^0,R^0\right)$ be the diseases free equilibrium points on equations (1) – (6) then the equation (1)- (6) above becomes

$$\Lambda - \mu S^0 - \left(\lambda_1\eta + \lambda_2(1-\eta)\right) I^0 S^0 + \rho R^0 = 0 \tag{37}$$

$$\left(\lambda_1\eta + \lambda_2(1-\eta)\right) I^0 S^0 - \left(\mu + \gamma\right) E^0 = 0 \tag{38}$$

$$\gamma E^0 - \left(\mu + \alpha_1 + r_1 + r_2\right) I^0 = 0 \tag{39}$$

$$r_1 I^0 - \left(\mu + \alpha_2 + \delta_1\right) R_1 = 0 \tag{40}$$

$$r_2 I^0 - \left(\mu + \alpha_3 + \delta_2\right) R^0{}_2 = 0 \tag{41}$$

$$\delta_1 R_1^0 + \delta_2 R^0{}_2 - \left(\mu + \rho\right) R^0 = 0 \tag{42}$$

From equation (38) we have

$$E^0 = \frac{\left(\lambda_1\eta + \lambda_2(1-\eta)\right) I^0 S^0}{\left(\mu + \gamma\right)} \tag{43}$$

Substituting (43) into (39) we have

$$\left(\frac{\gamma\left(\lambda_1\eta + \lambda_2(1-\eta)\right) S^0}{\left(\mu + \gamma\right)} - \left(\mu + \alpha_1 + r_1 + r_2\right)\right) I^0 = 0 \tag{44}$$

Which implies that

$$I^0 = 0 \tag{45}$$

or,

$$\left(\frac{\gamma\left(\lambda_1\eta + \lambda_2(1-\eta)\right) S^0}{\left(\mu + \gamma\right)} - \left(\mu + \alpha_1 + r_1 + r_2\right)\right) = 0 \tag{46}$$

Thus if $I^0 \neq 0$ we have

$$S^0 = \frac{\left(\mu + \gamma\right)\left(\mu + \alpha_1 + r_1 + r_2\right)}{\gamma\left(\lambda_1\eta + \lambda_2(1-\eta)\right)} \tag{47}$$

Putting equation (45) into (40) and (41) gives

$$R_1^0 = 0 \tag{48}$$

$$R_2^0 = 0 \tag{49}$$

Substituting equation (45) into (42)

$$R^0 = 0 \tag{50}$$

Putting (45) and (50) into (37) result in

$$S^0 = \frac{\Lambda}{\mu} \tag{51}$$

Thus diseases free equilibrium of the model is given by

$$X_0 = \left(S^0, E^0, I^0, R_1^0, R_2^0, R^0\right) = \left(\frac{\Lambda}{\mu}, 0, 0, 0, 0, 0\right) \tag{52}$$

2.6.2 Endemic equilibrium $\left(S^*, E^*, I^*, R_1^*, R_2^*, R^*\right)$

We have $I \neq 0, \quad E \neq 0, R_1 \neq 0 \quad and \quad R_2 \neq 0$

Thus from equation (47) we have

$$S^* = \frac{(\mu+\gamma)(\mu+\alpha_1+r_1+r_2)}{\gamma\left(\lambda_1\eta+\lambda_2(1-\eta)\right)} \tag{53}$$

From equation (1) we have

$$I^* = \frac{\Lambda-\mu S^*+\rho R^*}{\left(\lambda_1\eta+\lambda_2(1-\eta)\right)S^*} \tag{54}$$

From (2)

$$E^* = \frac{\left(\lambda_1\eta + \lambda_2\left(1-\eta\right)\right)S^*I^*}{\left(\mu+\gamma\right)} \tag{55}$$

From equation (4)

$$R_1^* = \frac{r_1 I^*}{\mu + \alpha_2 + \delta_1} \tag{56}$$

From (5) we have

$$R_2^* = \frac{r_2 I^*}{\mu + \alpha_3 + \delta_2} \tag{57}$$

From (6) we have

$$R^* = \frac{\delta_1 R^*{}_1 + \delta_2 R^*{}_2}{\mu + \rho} \tag{58}$$

On solving equations (53) to (58) we have the endemic equilibrium pointof the model as

$$S^* = \frac{\left(\mu+\gamma\right)\left(\mu+\alpha_1+r_1+r_2\right)}{\lambda\gamma} = \frac{\Lambda}{\mu R_0} \tag{59}$$

$$E^* = \frac{\lambda K\mu\left(\mu+\alpha_1+r_1+r_2\right)\left(R_0-1\right)}{\left(\lambda K-b\right)} \tag{60}$$

$$I^* = \frac{K\mu}{\lambda K - b}(R_0 - 1) \tag{61}$$

$$R_1^* = \frac{r_1 K(R_0 - 1)}{(\mu + \alpha_2 + \delta_1)(\lambda K - b)} \tag{62}$$

$$R_2^* = \frac{r_2 K(R_0 - 1)}{(\mu + \alpha_3 + \delta_2)(\lambda K - b)} \tag{63}$$

$$R^* = \frac{bK(R_0 - 1)}{\rho(\lambda K - b)} \tag{64}$$

Where $K = \dfrac{(\mu + \gamma)(\mu + \alpha_1 + r_1 + r_2)(\mu + \rho)}{\lambda\gamma}$ (65)

$$b = \rho\left(\frac{\delta_1 r_1}{\mu + \alpha_2 + \delta_1} + \frac{\delta_2 r_2}{\mu + \alpha_3 + \delta_2}\right) \tag{66}$$

$$\lambda = \lambda_1 \eta + \lambda_2 (1 - \eta) \tag{67}$$

2.6.3 Condition of existence and positivity of endemic equilibrium

The system will remain positive provided we have:

$$\frac{\Lambda-\mu\left(\frac{(\mu+\gamma)(\mu+\alpha_1+r_1+r_2)}{\lambda\gamma}\right)}{\lambda\left(\frac{(\mu+\gamma)(\mu+\alpha_1+r_1+r_2)}{\lambda\gamma}\right)(\mu+\rho)-b}>0 \tag{68}$$

$$\Leftrightarrow \Lambda-\mu\left(\frac{(\mu+\gamma)(\mu+\alpha_1+r_1+r_2)}{\lambda\gamma}\right)>0 \quad \text{and} \quad \lambda\left(\frac{(\mu+\gamma)(\mu+\alpha_1+r_1+r_2)}{\lambda\gamma}\right)(\mu+\rho)-b>0$$

$$\Leftrightarrow \lambda\gamma\Lambda>\mu(\mu+\gamma)(\mu+\alpha_1+r_1+r_2) \quad \text{and} \quad \lambda K-b>0$$

$$\Leftrightarrow \frac{\lambda\gamma\Lambda}{\mu(\mu+\gamma)(\mu+\alpha_1+r_1+r_2)}>1 \quad \text{and} \quad \lambda K-b>0$$

$$\Leftrightarrow R_0\succ 1 \quad \text{and} \quad \lambda K>b \tag{69}$$

This expression in equation (69) is the condition for existence and positivity of the endemic equilibrium solution.

2.6.4 Local Stability of Disease Free Equilibrium

Theorem 2. If $R_0<1$ then the diseases free equilibrium of the model is locally asymptotically stable and unstable if $R_0\geq 1$.

Proof: We use the jacobian stability approach to prove the stability of the diseases free equilibrium.

$$J(E)=\begin{pmatrix} -(\mu+\lambda I) & 0 & -\lambda S & 0 & 0 & \rho \\ \lambda I & -(\mu+\gamma) & \lambda S & 0 & 0 & 0 \\ 0 & \gamma & -(\mu+\alpha_1+r_1+r_2) & 0 & 0 & 0 \\ 0 & 0 & r_1 & -(\mu+\alpha_2+\delta_1) & 0 & 0 \\ 0 & 0 & r_2 & 0 & -(\mu+\alpha_3+\delta_2) & 0 \\ 0 & 0 & 0 & \delta_1 & \delta_2 & -(\mu+\rho) \end{pmatrix} \tag{70}$$

At the diseases free equilibrium we have $I=E=R_1=R_2=R=0 \ \text{and} \ S=\frac{\Lambda}{\mu}$

Thus the Jacobian matrix at disease free equilibrium is given by:

$$J(E_0)=\begin{pmatrix} -\mu & 0 & -\frac{\lambda\Lambda}{\mu} & 0 & 0 & \rho \\ 0 & -(\mu+\gamma) & \frac{\lambda\Lambda}{\mu} & 0 & 0 & 0 \\ 0 & \gamma & -(\mu+\alpha_1+r_1+r_2) & 0 & 0 & 0 \\ 0 & 0 & r_1 & -(\mu+\alpha_2+\delta_1) & 0 & 0 \\ 0 & 0 & r_2 & 0 & -(\mu+\alpha_3+\delta_2) & 0 \\ 0 & 0 & 0 & \delta_1 & \delta_2 & -(\mu+\rho) \end{pmatrix} \tag{71}$$

Let $K=\mu+\alpha_1+r_1+r_2$

$$J(E_0)=\begin{pmatrix} -\mu & 0 & -\frac{\lambda\Lambda}{\mu} & 0 & 0 & \rho \\ 0 & -(\mu+\gamma) & \frac{\lambda\Lambda}{\mu} & 0 & 0 & 0 \\ 0 & \gamma & -K & 0 & 0 & 0 \\ 0 & 0 & r_1 & -(\mu+\alpha_2+\delta_1) & 0 & 0 \\ 0 & 0 & r_2 & 0 & -(\mu+\alpha_3+\delta_2) & 0 \\ 0 & 0 & 0 & \delta_1 & \delta_2 & -(\mu+\rho) \end{pmatrix} \tag{72}$$

Thus the Eigenvalues of the jacobian matrix is given by

$$\lambda_1=-\mu<0 \tag{73}$$

$$\lambda_2=-(\mu+\alpha_2+\delta_1)<0 \tag{74}$$

$$\lambda_3=-(\mu+\alpha_3+\delta_2)<0 \tag{75}$$

$$\lambda_4=-(\mu+\rho)<0 \tag{76}$$

$$\lambda_5=-\frac{1}{2}\left(K+\mu+\gamma+\sqrt{\frac{4\lambda\gamma\Lambda}{\mu}+K^2-2K\mu-2K\gamma+\mu^2+2\mu\gamma+\gamma^2}\right)<0 \tag{77}$$

and

$$\lambda_6=\frac{1}{2}\left(-(K+\mu+\gamma)+\sqrt{\frac{4\lambda\gamma\Lambda}{\mu}+K^2-2K\mu-2K\gamma+\mu^2+2\mu\gamma+\gamma^2}\right) \tag{78}$$

Now for

$$\lambda_6 = \frac{1}{2}\left(-\left(K+\mu+\gamma\right) + \sqrt{\frac{4\lambda\gamma\Lambda}{\mu} + K^2 - 2K\mu - 2K\gamma + \mu^2 + 2\mu\gamma + \gamma^2} \right) < 0$$

implies that

$$\left(K+\mu+\gamma\right)^2 > \frac{4\lambda\gamma\Lambda}{\mu} + K\left(\left(K-2(\mu+\gamma)\right)\right) + \left(\mu+\gamma\right)^2 \tag{79}$$

Let $Z = \mu+\gamma$ (80)

$$\left(K+Z\right)^2 > \frac{4\lambda\gamma\Lambda}{\mu} + K\left(K-2Z\right) + Z^2 \tag{81}$$

$\Rightarrow$

$$\frac{4\lambda\gamma\Lambda}{\mu} + K\left(K-2Z\right) + Z^2 - \left(K+Z\right)^2 < 0 \tag{82}$$

$\Rightarrow$

$$\frac{4\lambda\gamma\Lambda}{\mu} + K^2 + Z^2 - 2KZ - K^2 - 2KZ - Z^2 < 0 \tag{83}$$

$\Rightarrow$

$$\frac{4\beta\gamma\Lambda}{\mu} - 4KZ < 0 \tag{84}$$

Putting $Z = \mu+\gamma$ and $K = \mu + \alpha_1 + r_1 + r_2$ into equation (84) we have

$$\frac{4\lambda\gamma\Lambda}{\mu} - 4\left(\mu+\alpha_1+r_1+r_2\right)\left(\mu+\gamma\right) < 0 \tag{85}$$

$\Rightarrow$

$$\frac{4}{\left(\mu+\alpha_1+r_1+r_2\right)\left(\mu+\gamma\right)}\left(\frac{\lambda\gamma\Lambda}{\mu\left(\mu+\alpha_1+r_1+r_2\right)\left(\mu+\gamma\right)} - 1\right) < 0 \tag{86}$$

But R_0 is

$$R_0 = \frac{\lambda\gamma\Lambda}{\mu\left(\mu+\gamma\right)\left(\mu+\alpha_1+r_1+r_2\right)}$$

Thus we have

$$\frac{4}{\left(\mu+\alpha_1+r_1+r_2\right)\left(\mu+\gamma\right)}\left(R_0 - 1\right) < 0 \tag{87}$$

Equation (87) will hold if $R_0 < 1$

Thus all eigenvalues of the Jacobian Matrix above has negative real part if $R_0 < 1$

Hence from Routh Hurwitz stability criterion we conclude that the diseases free equilibrium is locally asymptotically stable.

2.6.5 Global stability of the diseases free equilibrium.

Theorem 2: If $R_0 \le 1$ then the diseases free equilibrium of the system is globally asymptotically stable on Ω.

Proof. By constructing an appropriate Lyapunov function $V=(S,E,I,R_1,R_2,R)$ on the positively invariant compact set Ω.

Defined

$$V=(S,E,I,R_1,R_2,R)=\gamma E+(\mu+\gamma)I \tag{88}$$

Differentiate equation (88) by t we have

$$\frac{dV}{dt}=\gamma\frac{dE}{dt}+(\mu+\gamma)\frac{dI}{dt} \tag{89}$$

Substitute equation (2) and (3) into (89) we have

$$\frac{dV}{dt}=\gamma\left(\lambda IS-(\mu+\gamma)E\right)+(\mu+\gamma)\left(\gamma E-(\mu+\alpha_1+r_1+r_2)\right)I \tag{90}$$

which gives

$$\frac{dV}{dt}=(\mu+\gamma)(\mu+\alpha_1+r_1+r_2)\left(\frac{\lambda\gamma\Lambda}{\mu(\mu+\gamma)(\mu+\alpha_1+r_1+r_2)}-1\right)I \tag{91}$$

therefore

$$\frac{dV}{dt}=(\mu+\gamma)(\mu+\alpha_1+r_1+r_2)(R_0-1)I \tag{92}$$

Which is strictly decreasing when $R_0<1$

i.e

$$\frac{dV}{dt}<0 \ \ if \ \ R_0<1 \tag{93}$$

and

$\frac{dV}{dt}=0$ If and only if $E=0, I=0,\ R_1=0,\ R_2=0 \ and \ R_0=1$

Defining the set $E_0 = \left\{ \left(E, I, R_1, R_2 \right) \in \Omega : \dfrac{dL}{dt} = 0 \right\}$ the largest invariant set E_0 is contained in the set thus by LaSalle invariant principle [8] the diseases free equilibrium is globally asymptotically stable.

Hence the proof is complete.

2.6.6 Local stability of the endemic equilibrium

Theorem 4 The endemic equilibrium state of the system (1-6) is locally asymptotically stable if $R_0 > 1$.

Proof:
Using jacobian stability approach we consider the Jacobian matrix of (1-6) at endemic equilibrium points

$$J(E) = \begin{pmatrix} -\left(\mu+\lambda I^*\right) & 0 & -\lambda S* & 0 & 0 & \rho \\ \lambda I* & -\left(\mu+\gamma\right) & \lambda S* & 0 & 0 & 0 \\ 0 & \gamma & -\left(\mu+\alpha_1+r_1+r_2\right) & 0 & 0 & 0 \\ 0 & 0 & r_1 & -\left(\mu+\alpha_2+\delta_1\right) & 0 & 0 \\ 0 & 0 & r_2 & 0 & -\left(\mu+\alpha_3+\delta_2\right) & 0 \\ 0 & 0 & 0 & \delta_1 & \delta_2 & -\left(\mu+\rho\right) \end{pmatrix} \tag{94}$$

Let K1= $\mu + \alpha_1 + r_1 + r_2$, K2= $\mu + \alpha_2 + \delta_1$, K3= $\mu + \alpha_3 + \delta_2$, K4= $\mu + \rho$, K5= $\mu + \gamma$ Also at the endemic equilibrium we have

$$S^* = \frac{(\mu+\gamma)(\mu+\alpha_1+r_1+r_2)}{\lambda\gamma} = \frac{\Lambda}{\mu R_0} \tag{95}$$

$$I^* = \frac{K\mu}{\lambda K - b}(R_0 - 1) = A(R_0 - 1) \tag{96}$$

Where $A = \dfrac{K\mu}{\lambda K - b}$

We are sure that A is always positive since both K and μ are positive parameter and from equation (69)

We have that $R_0 > 1$ implies $\lambda K - b > 0$

Putting equation (95) and (96) into (94) we obtain

$$J(E)=\begin{pmatrix} -(\mu+\lambda A(R_0-1)) & 0 & \frac{-\lambda\Lambda}{\mu R_0} & 0 & 0 & \rho \\ \lambda A(R_0-1) & -K_5 & \frac{\lambda\Lambda}{\mu R_0} & 0 & 0 & 0 \\ 0 & \gamma & -K_1 & 0 & 0 & 0 \\ 0 & 0 & r_1 & -K_2 & 0 & 0 \\ 0 & 0 & r_2 & 0 & -K_3 & 0 \\ 0 & 0 & 0 & \delta_1 & \delta_2 & -K_4 \end{pmatrix} \tag{97}$$

By reducing equation (97) to upper triangular matrix using Gaussian elimination method we have

$$J(E)=\begin{pmatrix} -(\mu+\lambda A(R_0-1)) & 0 & \frac{-\lambda\Lambda}{\mu R_0} & 0 & 0 & \rho \\ 0 & -K_5 & \frac{\lambda\Lambda}{(A\lambda(R_0-1)+\mu)R_0} & 0 & 0 & \frac{\lambda A(R_0-1)\rho}{(A\lambda(R_0-1)+\mu)} \\ 0 & 0 & -\frac{ZR_0(R_0-1)}{K_5(A\lambda(R_0-1)+\mu)R_0} & 0 & 0 & \frac{\lambda A\gamma(R_0-1)\rho}{K_5(A\lambda(R_0-1)+\mu)} \\ 0 & 0 & 0 & -K_2 & 0 & \frac{\lambda AR_0 r_1\gamma(R_0-1)\rho}{ZR_0(R_0-1)} \\ 0 & 0 & 0 & 0 & -K_3 & \frac{\lambda AR_0 r_2\gamma(R_0-1)\rho}{Z(R_0-1)} \\ 0 & 0 & 0 & 0 & 0 & -\frac{R_0(K_3K_4K_2Z-(M_1+M_2))(R_0-1)}{K_2K_3(Z(R_0-1))} \end{pmatrix} \tag{98}$$

Where $Z = AK_1K_5\lambda$ *and* $M_1+M_2=\left(K_3\delta_1 r_1+K_2\delta_2 r_2\right)\gamma\lambda A\rho$

Thus the Eigenvalues of the reduced Jacobian matrix is

$$\lambda_1=-K_5<0 \tag{99}$$

$$\lambda_2=-K_3<0 \tag{100}$$

$$\lambda_3=-K_2<0 \tag{101}$$

$$\lambda_4=-(\mu+\lambda A(R_0-1)<0 \tag{102}$$

$If \quad R_0 > 1$

$$\lambda_5 = -\frac{Z(R_0 - 1)}{K_5\left(A\lambda(R_0 - 1) + \mu\right)R_0} < 0 \tag{103}$$

$$\lambda_6 = -\frac{R_0\left(K_3K_4K_2Z - (M_1 + M_2)\right)(R_0 - 1)}{K_2K_3\left(Z(R_0 - 1)\right)} < 0 \tag{104}$$

$if \quad R_0 > 1 \quad and \quad K_3K_4K_5Z > M_1 + M_2$

Which verified the local stability of the endemic equilibrium $if \quad R_0 > 1$.

The epidemiological implication of this is that the diseases persist in the population $if \quad R_0 > 1$

2.6.7 Global Stability of Diseases Endemic Equilibrium

Theorem 5 The endemic equilibrium $\Phi = \left(S^*, E^*, I^*, R_1^*, R_1^*, R^*\right)$ is globally asymptotically stable Ω $If \quad R_0 > 1$.

Proof:

We establish the stability of endemic equilibrium by constructing Lyapunuv function

$$V\left(S, E, I, R_1, R_2, R\right) = \begin{bmatrix} \lambda_1\left[S - S^* - S^*\ln\left(\frac{S}{S^*}\right)\right] + \lambda_2\left[E - E^* - E^*\ln\left(\frac{E}{E^*}\right)\right] + \\ \lambda_3\left[I - I^* - I^*\ln\left(\frac{I}{I^*}\right)\right] + \lambda_4\left[R_1 - R_1^* - R_1^*\ln\left(\frac{R_1}{R_1^*}\right)\right] + \\ \lambda_5\left[R_2 - R_2^* - R_2^*\ln\left(\frac{R_2}{R_2^*}\right)\right] + \lambda_6\left[R - R^* - R^*\ln\left(\frac{R}{R^*}\right)\right] \end{bmatrix} \tag{105}$$

Where $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6$ are positive constant.

Taking the derivative of the Lyapunov function V above we have

$$\frac{dV}{dt} = \begin{bmatrix} \lambda_1\left(1-\frac{S^*}{S}\right)\frac{dS}{dt} + \lambda_2\left(1-\frac{E^*}{E}\right)\frac{dE}{dt} + \lambda_3\left(1-\frac{I^*}{I}\right)\frac{dE}{dt} + \\ \lambda_4\left(1-\frac{R_1^*}{R_1}\right)\frac{dR_1}{dt} + \lambda_5\left(1-\frac{R_2^*}{R_2}\right)\frac{dR_2}{dt} + \lambda_6\left(1-\frac{R^*}{R}\right)\frac{dR}{dt} \end{bmatrix} \tag{106}$$

By substituting equation 1-6 into (98) and using the relation obtain from (1-6) as

$$\Lambda + \rho R^* = \left(\lambda I^* + \mu\right) S^* \tag{107}$$

$$\lambda I^* S^* = \left(\mu + \gamma\right) E^* \tag{108}$$

$$\gamma E^* = \left(\mu + \alpha_1 + r_1 + r_2\right) I^* \tag{109}$$

$$r_1 I^* = \left(\mu + \alpha_2 + \delta_1\right) R_1^* \tag{110}$$

$$r_2 I^* = \left(\mu + \alpha_3 + \delta_2\right) R_2^* \tag{111}$$

$$\delta_1 R_1^* + \delta_2 R_2^* = \left(\mu + \rho\right) R^* \tag{112}$$

We obtain

$$\frac{dV}{dt} = -\begin{bmatrix} \frac{\lambda_1\left(S-S^*\right)^2\left(\lambda I^* + \mu\right)}{S} + \frac{\lambda_2\left(E-E^*\right)^2\left(\mu+\gamma\right)}{E} + \frac{\lambda_3\left(I-I^*\right)^2\left(\mu+\alpha_1+r_1+r_2\right)}{I} + \\ \frac{\lambda_4\left(R_1-R_1^*\right)^2\left(\mu+\alpha_2+\delta_1\right)}{R_1} + \frac{\lambda_5\left(R_2-R_2^*\right)^2\left(\mu+\alpha_3+\delta_2\right)}{R_2} + \frac{\lambda_6\left(R-R^*\right)^2\left(\mu+\rho\right)}{R} \end{bmatrix} \tag{113}$$

But $I^* = A(R_0 - 1)$, thus equation (113) becomes

$$\frac{dV}{dt} = -\begin{bmatrix} \frac{\lambda_1\left(S-S^*\right)^2\left(\lambda A(R_0-1) + \mu\right)}{S} + \frac{\lambda_2\left(E-E^*\right)^2\left(\mu+\gamma\right)}{E} + \frac{\lambda_3\left(I-I^*\right)^2\left(\mu+\alpha_1+r_1+r_2\right)}{I} + \\ \frac{\lambda_4\left(R_1-R_1^*\right)^2\left(\mu+\alpha_2+\delta_1\right)}{R_1} + \frac{\lambda_5\left(R_2-R_2^*\right)^2\left(\mu+\alpha_3+\delta_2\right)}{R_2} + \frac{\lambda_6\left(R-R^*\right)^2\left(\mu+\rho\right)}{R} \end{bmatrix} \tag{114}$$

$\Rightarrow$

$$\frac{dV}{dt} < 0 \ \ if \ \ R_0 > 1$$

and

$$\frac{dV}{dt} = 0 \ \ iff \ \ S = S^*,\ E = E^*,\ I = I^*,\ \ R_1 = R_1^*,\ R_2 = R_2^*,\ and \ \ R = R^*$$

Thus the largest compact invariant set in $\Phi = \left(S^*, E^*, I^*, R_1^*, R_1^*, R^*\right) \in \Omega : \frac{dV}{dt} = 0$ is the singleton set Φ where Φ is the endemic equilibrium. Thus Φ is globally asymptotically stable in the interior of the region Ω.

Hence the proof is complete.

3.0 Results and Discussion

3.1 Solutions of the model

We solve the model using homotopy perturbation method; see [3] for details on applications of homotopy perturbation method.

The solutions to equations 1-6 are

$$S(t) = S_0 + (\Lambda + \rho z_0 - \lambda I_0 S_0 - \mu S_0)t + \begin{bmatrix} \rho\left(\delta_1 (R_1)_0 + \delta_2 (R_2)_0 - (\mu+\rho) R_0\right) - \lambda I_0 (\Lambda + \rho z_0 - \lambda I_0 S_0 - \mu S_0) \\ -\lambda\left(\gamma E_0 - K_1 I_0\right) S_0 - \mu\left(\Lambda + \rho z_0 - \lambda I_0 S_0 - \mu S_0\right) \end{bmatrix} \frac{t^2}{2} \tag{115}$$

$$E(t) = E_0 + \left(\lambda I_0 S_0 - (\mu+\gamma) E_0\right)t + \begin{bmatrix} \lambda w_0\left(\Lambda + \rho z_0 - \lambda I_0 S_0 - \mu S\right) - (\mu+\gamma)\left(\lambda I_0 S_0 - (\mu+\gamma) E_0\right) \\ -S_0\left(\gamma E_0 - K_1 I_0\right) \end{bmatrix} \frac{t^2}{2} \tag{116}$$

$$I(t) = I_0 + \left(\gamma E_0 - K_1 I_0\right)t + \left[\gamma\left(\lambda I_0 S_0 - (\mu+\gamma) E_0\right) - K_1\left(\gamma E_0 - K_1 I_0\right)\right] \frac{t^2}{2} \tag{117}$$

$$R_1(t)=(R_1)_0+\left(r_1I_0-K_2(R_1)_0\right)t+\left[r_1\left(\gamma E_0-K_1I_0\right)-K_2\left(r_1I_0-K_2(R_1)_0\right)\right]\frac{t^2}{2} \quad (118)$$

$$R_2(t)=(R_2)_0+\left(r_2I_0-K_3(R_2)_0\right)t+\left[r_2\left(\gamma E_0-K_1I_0\right)-K_3\left(r_2I_0-K_3(R_2)_0\right)\right]\frac{t^2}{2} \quad (119)$$

$$R(t)=R_0+\left(\delta_1(R_1)_0+\delta_2(R_2)_0-(\mu+\rho)R_0\right)t+\begin{bmatrix}\delta_1\left(r_2I_0-K_3(R_2)_0\right)+\delta_2\left(r_2I_0-K_3(R_2)_0\right)\\ -(\mu+\rho)\left(\delta_1(R_1)_0+\delta_2(R_2)_0\right)\end{bmatrix}\frac{t^2}{2} \quad (120)$$

3.2 Numerical Simulation

The parameter and variable values of six compartments model are assumed and estimated from the population of interest and also on Tuberculosis disease epidemiology. The description of the parameters of the model is show in the table below.

Table 2 Variable and Parameter Values and Estimations

Parameters	**Descriptions**	**Values (yr^{-1})**	**References**
Λ	Recruitment rate	15 .00	Estimated
η	Case detection rate	0.570	Arthitian (2013)
r_1	Resistance to first line of treatment rate	0.400	Kumar Gupta et al (2018)
r_2	Resistance to second line of treatment rate	0.500	Kumar Gupta et al (2018)
δ_1	Recovery due to first line of treatment rate	0.800	Estimated
δ_2	Recovery due to second line of treatment rate	0.300	Estimated
ρ	Rate at which individual losses their immunity	0.400	Kumar Gupta et al (2018)

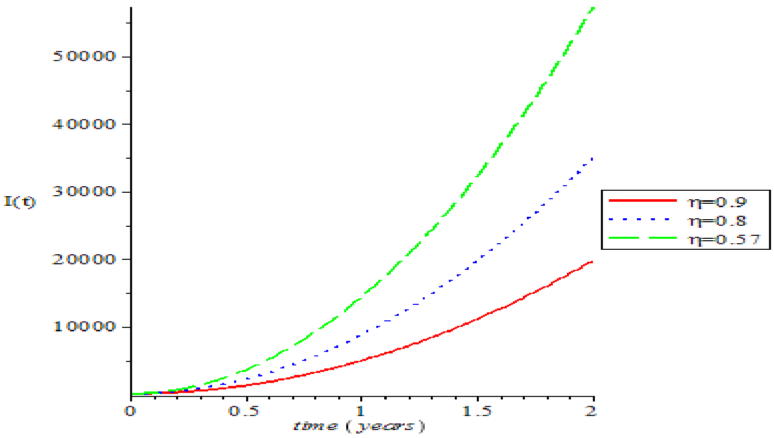

Figure 2 Graph of Infected Individuals against time for different case detection Rate η.

It was observed that the population of infected individuals decreases as case detection rate increases. This means that if early case detection is high then fewer people will be infected compare to when the case detection rate is low as we will have high contact rate between the susceptible and infected individuals as a results of unidentified cases of Tuberculosis.

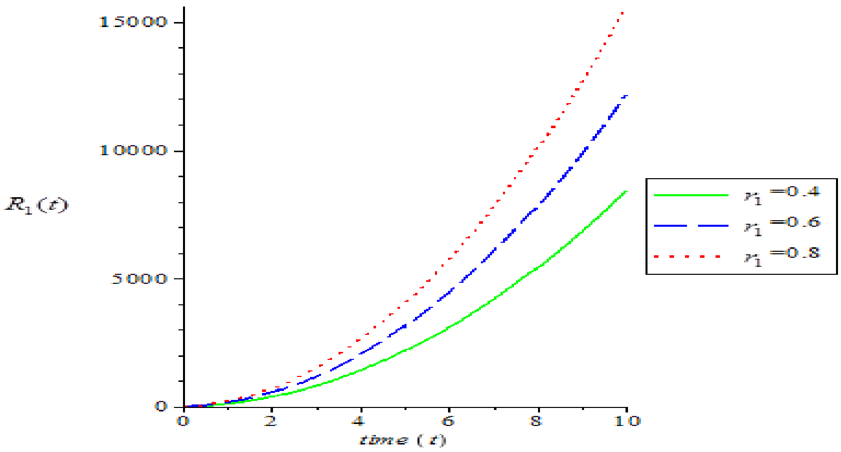

Figure 3 Graph of Individuals who are resistance to first line of treatment against time for different resistance rate to first line of treatment r_1 .

We see from the graph that the population of resistant individuals increases as the resistance rate of first line of treatment increases. This shows that more people will move from infected class to resistance class of first line of treatment as resistance rate due to first line of treatment increases.

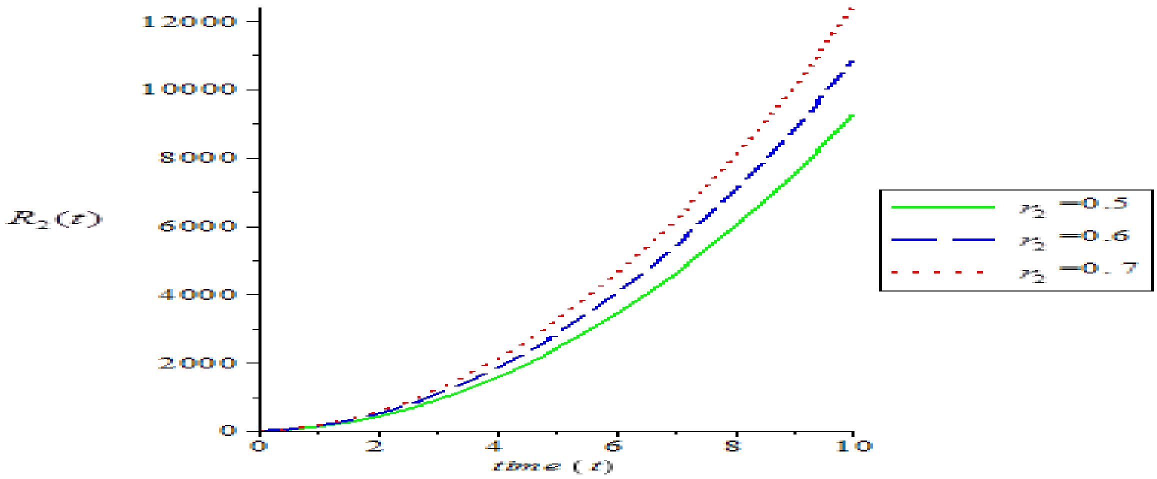

Figure 4 Graph of Individuals who are resistance to second line of treatment against time for different resistance rate to second line of treatment r_2 .

We see from the graph that the population of resistant individuals of second line of treatment class increases as the resistance rate of second line of treatment increases. This shows that more people will move from resistance class of first line of treatment to resistance class of second line of treatment as resistance rate due to second line of treatment increases.

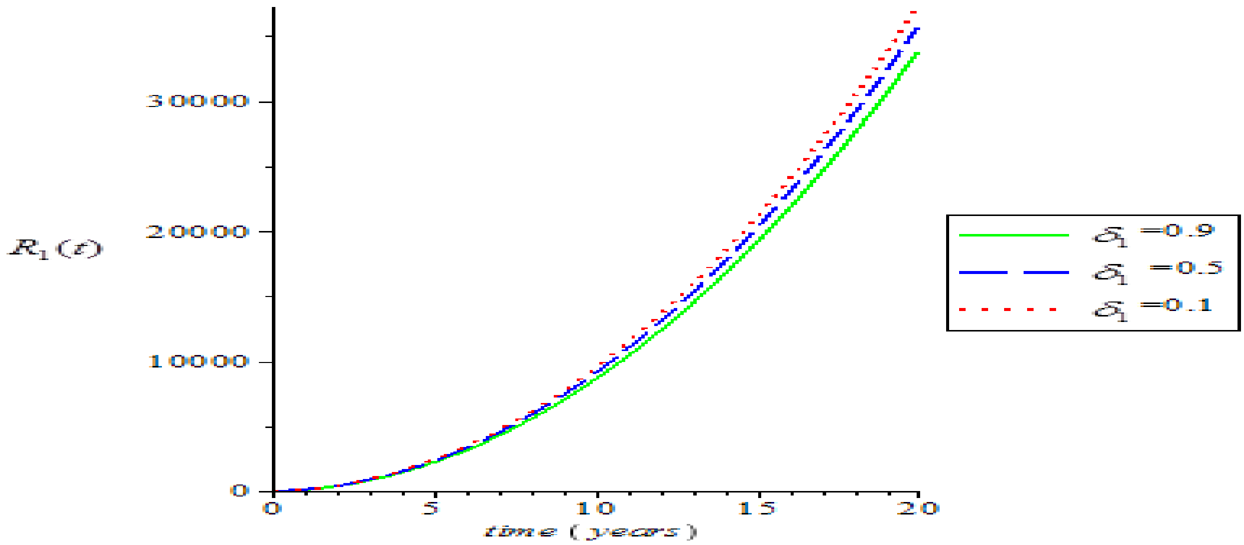

Figure 5 Graph of Individuals who are resistance to first line of treatment against time for different recovery rate due to first line treatment δ_1.

We observed from the graph that the population of resistant individuals decreases as the recovery rate due to first line of treatment increases. This shows that more people will move from resistance class of first line of treatment to recovered class as recovery rate due to first line of treatment increases.

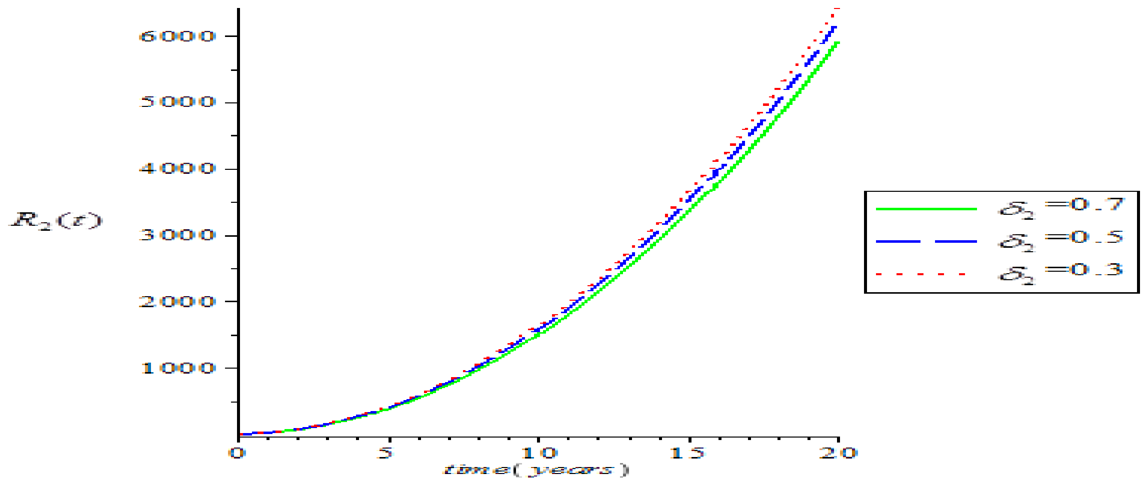

Figure 6 Graph of Individuals who are resistance to second line of treatment against time for different recovery rate due to second line treatment δ_2.

We observed from the graph that the population of resistant individuals decreases as the recovery rate due to second line of treatment increases. This shows that more people will move from resistance class of second line of treatment to recovered class as recovery rate due to second line of treatment increases.

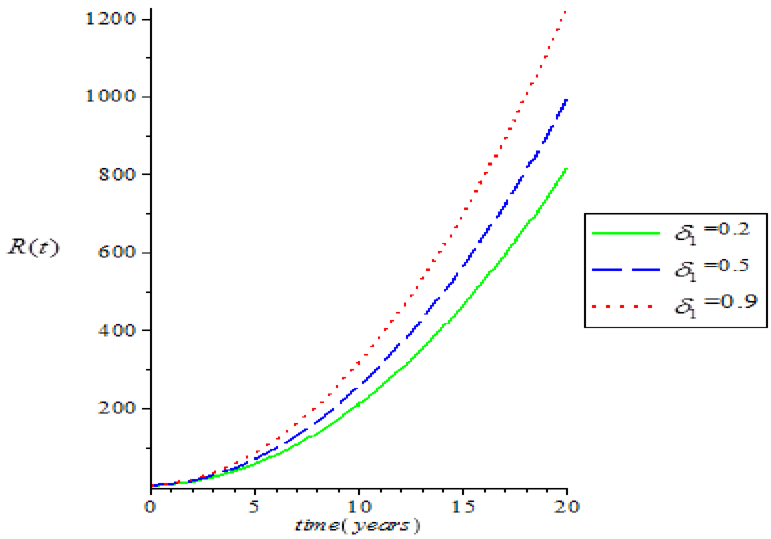

Figure 7 Graph of Recovered Individuals against time for different recovery rate due to first line treatment δ_1.

It was noticed that the population of recovered class increases as the rate of recovery due to first line of treatment increases. This shows that more people will move to recovered class if adequate treatment is administered early.

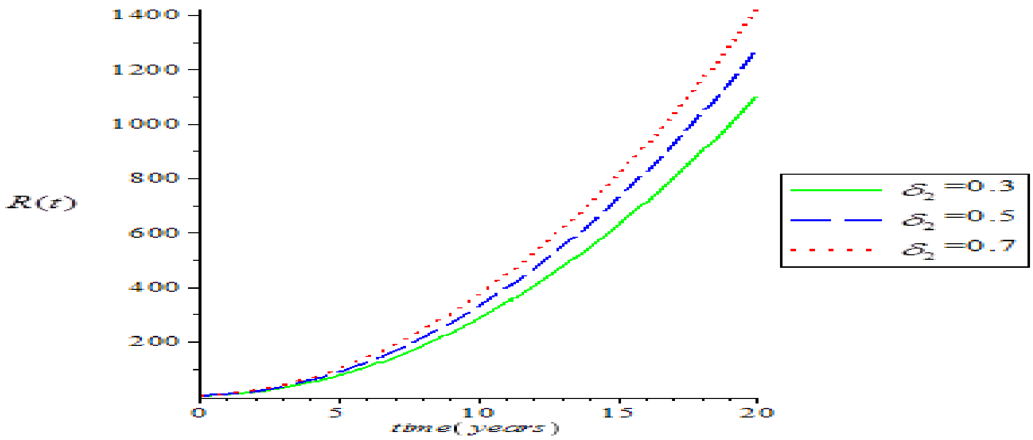

Figure 8 Graph of Recovered Individuals against time for different recovery rate due to second line treatment δ_2 .

It was observed that the population of recovered class increases as the rate of recovery due to second line of treatment increases. This shows that more people will move to recovered class from Resistance class due to second line of treatment.

4. Conclusion

This study presents a deterministic model for the effects of case detection and Resistance to tuberculosis diseases. It was shown that the model is mathematically and epidemiologically meaningful in the feasible region. The positivity of the solution was established, equilibrium points were obtained, and their stability analysis was performed. The conditions for local and global stability of both disease free equilibrium point and endemic equilibrium point were also established. The basic reproduction number using the next generation matrix was obtained was used to form the bases for stability of the equilibrium points. The analysis revealed that diseases free equilibrium is locally asymptotically stable if $R_0 < 1$ and globally stable if $R_0 \leq 1$. Also, the endemic equilibrium point is locally asymptotically stable if $R_0 > 1$ and globally asymptotically stable if $R_0 \geq 1$. Semi-Analytical solutions of the model using Homotopy Perturbation Method (HPM) were obtained graphical profiles of the solutions were presented.

From the results obtained, it was observed that when the case detection rate is high the infected population reduced drastically due to low possibility of contacts between the susceptible population and infectious individuals. Also the result revealed that resistant individuals to first and second line of treatment increase as resistance rate of both classes increases respectively mainly due to treatment failure. The result further revealed that recovered class increases as recovery rate of first and second line of treatment increases.

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A New Characterization of Dual General Helices

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Abstract

We derive a general differential equation satisfied by the distance function of every Frenet curve in dual space D^3. By using this differential equation, we get a new characterization of dual general helices.

Key words: Dual space, Dual general helices.

1. Introduction

Helices are among the simplest shapes that are observed in the filamentary and molecular structures of nature. The local mechanical properties of such structures are often modeled by a uniform elastic potential energy dependent on bending and twist, which is what we term a rod model. Also, helices arise in nanosprings, carbon nanotubes, α-helices, DNA double and collagen triple helices, lipid bilayers, bacterial flagella, aerial hyphae in actynomycetes, bacterial shape in spirochetes, horns, tendrils, vines, screws, springs, and helical staircases [2]. The study of helices by scientists from different fields due to their presence in these structures becomes very attractive.

In the Euclidean 3-space, a general helix is defined by the property that its tangent vector makes a constant angle with a fixed straight line (the axis of the general helix) and a general helix is characterized by its curvature and torsion by means of a moving frame along the curve. Lancret's theorem states that a space curve is a general helix if and only if its torsion τ and its curvature κ satisfy $\tau = c\kappa$ for some $c\epsilon R$ [1,5]. Recently, Deshmukh et. al., give a new characterization for helix in a Euclidean 3-space by using 4-th order differential equation satisfied by the distance function of every Frenet curves [3].

It is also very interesting to study the theory of curves on Dual space since a differentiable curve on dual unit sphere in D^3 represents a ruled surface in Euclidean 3-space R^3 with the aid of the E. Study mapping. According to E. Study mapping, oriented lines in the Euclidean three-space R^3 can be represented by unit dual vectors with three component of the ring of dual numbers and a differentiable curve on the dual unit sphere in D^3 corresponds to a ruled surface in R^3.

In this work, we firstly prove that the distance function of every Frenet curve in D^3 satisfies differential equation. By using this differential equation, we give a new characterization for dual general helices. Also, we obtain a characterization of dual general helices with respect to dual centrode and co-centrode.

2. Preliminaries

Dual numbers are ordered paid of real numbers (a, a^*) and may be formally expressed as $\hat{a} = a + \varepsilon a^*$, where ε is the dual operator with the property $\varepsilon^2 = 0,\ \varepsilon \neq 0.$ a is called real part and a^* is called dual part of $\hat{a}.$ The set of all dual numbers D is a commutative ring with the following addition and multiplication operations:

$$\hat{a} + \hat{b} = (a + \varepsilon a^*) + (b + \varepsilon b^*) = (a + b) + \varepsilon(a^* + b^*)$$

and

$$\hat{a}.\hat{b} = (a + \varepsilon a^*)(b + \varepsilon b^*) = ab + \varepsilon(ab^* + a^*b).$$

for $\hat{a} = a + \varepsilon a^*$, $\hat{b} = b + \varepsilon b^* \in D$. Also, the division is

$$\frac{\hat{a}}{\hat{b}} = \frac{a}{b} + \varepsilon \frac{a^*b - ab^*}{b^2}, b \neq 0.$$

In all other cases division is either impossible or ambiguous. If f is a differentiable function of x, its value for a dual argument $x = x + \varepsilon x^*$ may be separated into parts by a Taylor expansion:

$$f(x) = f(x + \varepsilon x^*) = f(x) + \varepsilon x^* \dot{f}(x), \tag{2.1}$$

where $\dot{f}$ is the derivative of f with respect to x. By using (2.1), we can define sine and cosine functions as follows:

$$sin\hat{x} = sin(x + \varepsilon x^*) = sinx + \varepsilon x^* cosx$$

and

$$cos\hat{x} = cos(x + \varepsilon x^*) = cosx - \varepsilon x^* sinx.$$

A dual vector is an ordered triple of dual numbers $(\hat{x}_1, \hat{x}_2, \hat{x}_3)$ and denoted by $\vec{\hat{x}} = (\hat{x}_1, \hat{x}_2, \hat{x}_3)$. Then, the set of all dual vector

$$D^3 = \left\{\vec{\hat{x}} \middle| \vec{\hat{x}} = (\hat{x}_1, \hat{x}_2, \hat{x}_3) = (x_1 + \varepsilon x_1{}^*, x_2 + \varepsilon x_2{}^*, x_3 + \varepsilon x_3{}^*) = \vec{x} + \varepsilon \overrightarrow{x^*}, \vec{x}, \overrightarrow{x^*} \in R^3\right\}.$$

Multiplication by a dual scalar of a dual vector $\vec{\hat{x}}$ is defined by

$$\hat{\lambda}\vec{\hat{x}} = \left(\hat{\lambda}\hat{x}_1, \hat{\lambda}\hat{x}_2, \hat{\lambda}\hat{x}_3\right),$$

scalar (or inner) and cross product of dual vectors $\vec{\hat{x}}\ and\ \vec{\hat{y}}$ are defined by

$$< \vec{\hat{x}}, \vec{\hat{y}} >=< \vec{x}, \vec{y} > +\varepsilon\left(< \vec{x}, \overrightarrow{y^*} > +< \overrightarrow{x^*}, \vec{y} >\right)$$

and

$$\vec{\hat{x}} \times \vec{\hat{y}} = \vec{x} \times \vec{y} + \varepsilon(\vec{x} \times \overrightarrow{y^*} + \overrightarrow{x^*} \times \vec{y}).$$

If $x \neq 0$, the norm $\|\hat{x}\|$ of $\hat{x}$ is defined by

$$\|\hat{x}\| = \sqrt{< \vec{\hat{x}}, \vec{\hat{x}} >}=\|\vec{x}\| + \varepsilon\frac{<\vec{x},\overrightarrow{x^*}>}{\|\vec{x}\|}.$$

A dual vector $\hat{x}$ with norm 1 (or (1,0)) is called a dual unit vector and $< \vec{x}, \vec{x} >= 1$, $< \vec{x}, \overrightarrow{x^*} >= 0$ for a dual unit vector. Also, the set of all unit dual vector is called as unit dual sphere [6,7].

Let $\hat{\gamma}(t) = \gamma(t) + \varepsilon\gamma^*(t)$ be a dual curve with parameter $t \in I \subset R$ in the dual space D^3. The real curve $\gamma(t)$ is called the (real) indicatrix of $\hat{\gamma}(t)$. If every $\gamma_i(t)$ and $\gamma_i^*(t)$ are differentiable, then $\hat{\gamma}(t)$ is differentiable in D^3. The dual arclength of the dual curve $\hat{\gamma}$ is defined as

$$\hat{s} = \int_0^s \left\|\dot{\hat{\gamma}}(t)\right\| dt = \int_0^s \|\dot{\gamma}(t)\| dt + \varepsilon \int_0^s < \vec{T}, \dot{\gamma}^*(t) > dt = s + \varepsilon s^*, \tag{2.2}$$

where s and $\vec{T}$ is arclength and the unit tangent vector to γ, respectively. Now we will give equations relative to derivatives of dual Frenet vectors along the dual curve $\hat{\gamma}$ in D^3. Assume that $\hat{\gamma}$ is a reparametrization curve with the parametrization s of the indicatrix. Then,

$$\hat{\gamma}' = \dot{\hat{\gamma}}\frac{ds}{d\hat{s}} = \vec{\hat{T}}$$

is called the dual unit vector to $\hat{\gamma}(s)$, where $\hat{\gamma}' = \frac{d\hat{\gamma}}{d\hat{s}}$ and $\dot{\hat{\gamma}} = \frac{d\hat{\gamma}}{ds}$. On the other hand we have from (2.2) $\frac{d\hat{s}}{ds} = 1 + \varepsilon\nabla$ such that $\nabla = < \vec{T}, \dot{\gamma}^*(t) >$. The dual unit vectors $\vec{\hat{N}}$ and $\vec{\hat{B}}$ are respectively called the principle normal and the binormal of $\hat{\gamma}$ at the point $\hat{\gamma}(s)$. For the dual Frenet trihedron $\left\{\vec{\hat{T}}, \vec{\hat{N}}, \vec{\hat{B}}\right\}$ along $\hat{\gamma}$, we have the dual Frenet formulas

$$\frac{d}{d\hat{s}}\begin{pmatrix} \vec{\hat{T}} \\ \vec{\hat{N}} \\ \vec{\hat{B}} \end{pmatrix} = \begin{pmatrix} 0 & \hat{\kappa} & 0 \\ -\hat{\kappa} & 0 & \hat{\tau} \\ 0 & -\hat{\tau} & 0 \end{pmatrix}\begin{pmatrix} \vec{\hat{T}} \\ \vec{\hat{N}} \\ \vec{\hat{B}} \end{pmatrix}, \tag{2.3}$$

where $\hat{\kappa} = \kappa + \varepsilon\kappa^*$ and $\hat{\tau} = \tau + \varepsilon\tau^*$ are nowhere pure dual curvature and dual torsion functions of $\hat{\gamma}$ [6,7,8].

Similar to the definition in Euclidean space R^3, a general helix in dual space is defined as follows: Let $\hat{\gamma}$ be a dual Frenet curve in D^3 with non-zero dual curvature $\hat{\kappa}$. The curve $\hat{\gamma}(s)$ is called a general dual helix if its tangent vector field makes a constant dual angle with a dual fixed line $\hat{u}$.

Theorem 2.1 (Lancret Theorem in D^3) (see, [4]). A dual Frenet curve in D^3 is a dual general helix if and only if there exists a constant dual number $\hat{c}$ such that $\hat{\tau} = \hat{c}\hat{\kappa}$.

3. A New Characterization of Dual General Helices

In this section containing the main results, we firstly prove that the distance function of every Frenet curve in D^3 satisfies a differential equation. Then, we give a new characterization of dual general helices by using this equation.

Proposition 3.1. Any dual unit speed Frenet curve $\hat{\gamma}(s) = \gamma(s) + \varepsilon\gamma^*(s)$ in D^3 satisfies the following differential equation

$$\hat{\rho}\hat{\sigma}\frac{d^3\hat{h}}{d\hat{s}^3} + \left(2\frac{d\hat{\rho}}{d\hat{s}}\hat{\sigma} + \hat{\rho}\frac{d\hat{\sigma}}{d\hat{s}}\right)\frac{d^2\hat{h}}{d\hat{s}^2} + \frac{d}{d\hat{s}}\left(\frac{d\hat{\rho}}{d\hat{s}}\hat{\sigma} + \frac{\hat{\rho}}{\hat{\sigma}} + \frac{\hat{\sigma}}{\hat{\rho}}\right)\frac{d\hat{h}}{d\hat{s}} + \frac{d}{d\hat{s}}\left(\frac{\hat{\rho}}{\hat{\sigma}}\right)\hat{h} = \frac{d}{d\hat{s}}\left(\frac{d\hat{\rho}}{d\hat{s}}\hat{\sigma}\right) + \frac{\hat{\rho}}{\hat{\sigma}} \tag{3.1}$$

where $\hat{\kappa}^{-1} = \hat{\rho} = \rho + \varepsilon\rho^*$, $\hat{\tau}^{-1} = \hat{\sigma} = \sigma + \varepsilon\sigma^*$ and $\hat{h}(s) = \hat{f}(s)\frac{d\hat{f}}{d\hat{s}}(s)$.

Proof. Taking derivative of $\hat{f}(s) = \|\hat{\gamma}(s)\| = \sqrt{< \hat{\gamma}(s), \hat{\gamma}(s) >}$ with respect to the dual arclength parameter $\hat{s}$:

$$\hat{f}(s)\frac{d\hat{f}}{d\hat{s}}(s) = < \hat{\gamma}(s), \frac{d\hat{\gamma}(s)}{d\hat{s}} >.$$

We can write $\hat{h}(s)$ and $\hat{T}$ instead of $\hat{f}(s)\frac{d\hat{f}}{d\hat{s}}(s)$ and $\frac{d\hat{\gamma}(s)}{d\hat{s}}$, respectively. Then we have

$$\hat{h}(s) =< \hat{\gamma}(s), \hat{T} >. \tag{3.2}$$

Taking derivative of (3.2) with respect to $\hat{s}$ and using dual Frenet formulas (2.3), we arrive at

$$\frac{d\hat{h}}{d\hat{s}}(s) = \frac{d}{d\hat{s}} < \hat{\gamma}(s), \hat{T} >= 1 + \hat{\kappa} < \hat{\gamma}(s), \hat{N} >$$

or

$$\hat{\rho}\left(\frac{d\hat{h}(s)}{d\hat{s}} - 1\right) =< \hat{\gamma}(s), \hat{N} >. \tag{3.3}$$

Similarly, taking derivative of (3.3) with respect to $\hat{s}$ and using dual Frenet formulas (2.3), we obtain

$$\hat{\rho}\frac{d^2\hat{h}(s)}{d\hat{s}^2} + \frac{d\hat{\rho}}{d\hat{s}}\left(\frac{d\hat{h}(s)}{d\hat{s}} - 1\right) = -\hat{\kappa} < \hat{\gamma}(s), \hat{T} > +\hat{\tau} < \hat{\gamma}(s), \hat{B} > \tag{3.4}$$

Substituting (3.2) into (3.4), we get

$$\hat{\rho}\hat{\sigma}\frac{d^2\hat{h}(s)}{d\hat{s}^2} + \frac{d\hat{\rho}}{d\hat{s}}\hat{\sigma}\frac{d\hat{h}(s)}{d\hat{s}} + \frac{\hat{\sigma}}{\hat{\rho}}\hat{h}(s) - \frac{d\hat{\rho}}{d\hat{s}}\hat{\sigma} =< \hat{\gamma}(s), \hat{B} >. \tag{3.5}$$

Finally, taking derivative of (3.5) with respect to $\hat{s}$, using (2.3) and (3.3), we have

$$\begin{aligned}\frac{d(\hat{\rho}\hat{\sigma})}{d\hat{s}}\frac{d^2\hat{h}(s)}{d\hat{s}^2} + \hat{\rho}\hat{\sigma}\frac{d^3\hat{h}(s)}{d\hat{s}^3} + \frac{d}{d\hat{s}}\left(\frac{d\hat{\rho}}{d\hat{s}}\hat{\sigma}\right)\frac{d\hat{h}(s)}{d\hat{s}} + \frac{d\hat{\rho}}{d\hat{s}}\hat{\sigma}\frac{d^2\hat{h}(s)}{d\hat{s}^2}\\ + \frac{d}{d\hat{s}}\left(\frac{\hat{\sigma}}{\hat{\rho}}\right)\hat{h}(s) + \frac{\hat{\sigma}}{\hat{\rho}}\frac{d\hat{h}(s)}{d\hat{s}} - \frac{d}{d\hat{s}}\left(\frac{d\hat{\rho}}{d\hat{s}}\hat{\sigma}\right) = -\frac{\hat{\rho}}{\hat{\sigma}}\left(\frac{d\hat{h}(s)}{d\hat{s}} - 1\right).\end{aligned} \tag{3.6}$$

If Eq. (3.6) is edited, Eq. (3.1) is obtained.

By using Proposition 3.1, dual rectifying curves, dual spherical curves can also be characterized, but in this work, we will give a new characterization of only dual general helices.

Theorem 3.2. A dual unit speed Frenet curve $\hat{\gamma}(s)$ in D^3 is a dual general helix if and only if according to dual arclength parameter $\hat{s}$, the function $\hat{h}(s) = \hat{f}(s)\frac{d\hat{f}}{d\hat{s}}(s)$ satisfies the following differential equation

$$\frac{d}{d\hat{s}}\left(\hat{\rho}\frac{d\hat{h}}{d\hat{s}}\right) + \left(\frac{\hat{\rho}}{\hat{\sigma}^2} + \frac{1}{\hat{\rho}}\right)\hat{h} = \frac{d\hat{\rho}}{d\hat{s}} + \frac{\hat{\rho}}{\hat{\sigma}^2}\left(\hat{s} + \hat{b}\right), \tag{3.7}$$

where $\hat{f}(s) = \|\hat{\gamma}(s)\|$, $\hat{\rho} = \hat{\kappa}^{-1}$ and $\hat{\sigma} = \hat{\tau}^{-1}$.

Proof. We suppose that $\hat{\gamma}$ is a dual general helix with an axis parallel to the dual unit vector. Then, for a dual constant $\hat{c}$, we have

$$< \hat{T}, \hat{u} >= cos\hat{\theta} = \hat{c}$$

and

$$<\widehat{N},\hat{u}>=0,$$

where the dual number $\hat{\theta}=\theta+\varepsilon\theta^{*}$ is the dual angle between $\hat{T}$ and $\hat{u}$. On the other hand, because of

$$sin\hat{\theta}=<\hat{u},\hat{B}>,$$

we have

$$\sqrt{1-cos^{2}\hat{\theta}}=<\hat{u},\hat{B}>.$$

Since the unit dual vector $\hat{u}$ can be written as $\hat{u}=<\hat{u},\hat{T}>\hat{T}+<\hat{u},\widehat{N}>\widehat{N}+<\hat{u},\hat{B}>\hat{B}$, we can write

$$\hat{u}=\hat{c}\hat{T}+\sqrt{1-\hat{c}^{2}}\hat{B}. \tag{3.8}$$

Taking derivative of (3.8) with respect to $\hat{s}$, we get

$$0=\hat{c}\frac{d\hat{T}}{d\hat{s}}+\sqrt{1-\hat{c}^{2}}\frac{d\hat{B}}{d\hat{s}}.$$

Taking into consideration dual Frenet formulas (2.3), we obtain

$$0=\hat{\kappa}\hat{c}\widehat{N}-\hat{\tau}\sqrt{1-\hat{c}^{2}}\widehat{N}$$

or

$$\hat{\kappa}\hat{c}=\hat{\tau}\sqrt{1-\hat{c}^{2}}. \tag{3.9}$$

Because of

$$\frac{d}{d\hat{s}}(<\hat{\gamma}(s),\hat{u}>)=\hat{c},$$

we get

$$<\hat{\gamma}(s),\hat{u}>=\hat{c}\hat{s}+\hat{d}. \tag{3.10}$$

By using (3.8), we have

$$\hat{c}\hat{s}+\hat{d}=\hat{c}<\hat{\gamma}(s),\hat{T}>+\sqrt{1-\hat{c}^{2}}<\hat{\gamma}(s),\hat{B}>$$

or

$$<\hat{\gamma}(s),\hat{T}>=\frac{1}{\hat{c}}\left(\hat{c}\hat{s}+\hat{d}\right)-\frac{\sqrt{1-\hat{c}^{2}}}{\hat{c}}<\hat{\gamma}(s),\hat{B}>.$$

Taking into consideration (3.2), we obtain

$$\hat{h}(s)=\hat{s}+\hat{b}-\frac{\sqrt{1-\hat{c}^{2}}}{\hat{c}}<\hat{\gamma}(s),\hat{B}> \tag{3.11}$$

where $\hat{b}=\frac{\hat{d}}{\hat{c}}$. Taking derivative of (3.11) with respect to $\hat{s}$, using (2.3) and (3.9), we get

$$\frac{d\hat{h}(s)}{d\hat{s}} = 1 + \frac{1}{\hat{\rho}} < \hat{\gamma}(s), \widehat{N} >$$

or

$$\hat{\rho}\left(\frac{d\hat{h}(s)}{d\hat{s}} - 1\right) =< \hat{\gamma}(s), \widehat{N} >. \tag{3.12}$$

Taking derivative of (3.11) with respect to $\hat{s}$ and using dual Frenet formulas (2.3), we have

$$\hat{\rho}\frac{d^2\hat{h}(s)}{d\hat{s}^2} + \frac{d\hat{\rho}}{d\hat{s}}\frac{d\hat{h}(s)}{d\hat{s}} - \frac{d\hat{\rho}}{d\hat{s}} = \frac{\hat{c}(\hat{s}+\hat{b})\hat{\tau}}{\sqrt{1-\hat{c}^2}} - \left(\frac{\hat{c}}{\sqrt{1-\hat{c}^2}} + \frac{\sqrt{1-\hat{c}^2}}{\hat{c}}\right)\hat{\tau}\hat{h}(s). \tag{3.13}$$

By using (3.9), we get

$$\frac{\hat{\rho}}{\hat{\sigma}} = \frac{\hat{c}}{\sqrt{1-\hat{c}^2}} \quad \text{and} \quad \frac{\hat{\sigma}}{\hat{\rho}} = \frac{\sqrt{1-\hat{c}^2}}{\hat{c}}. \tag{3.14}$$

Substituting (3.14) into (3.13), we obtain (3.7).

4. A characterization of dual general helices according to dual centrode and co-centrode

Let $\hat{\gamma}: I \longrightarrow D^3$ be a dual unit speed Frenet curve in dual space D^3. Suppose that $\hat{\delta}(s)$ and $\hat{\delta}^*(s)$

$$\hat{\delta} = \hat{\tau}\widehat{T} + \hat{\kappa}\widehat{\mathrm{B}} \text{ and } \hat{\delta}^* = -\hat{\kappa}\widehat{\mathrm{T}} + \hat{\tau}\widehat{\mathrm{B}}$$

are denoted as dual centrode and co-centrode of dual Frenet curve $\hat{\gamma}$, respectively. Then, $\{\widehat{N}, \hat{\delta}, \hat{\delta}^*\}$ is a moving orthogonal frame along dual Frenet curve $\hat{\gamma}$.

We can see that Frenet curve $\hat{\gamma}$ in dual space D^3 is a dual general helix if and only if $\frac{d\hat{\delta}}{d\hat{s}}$ and dual co-centrode $\hat{\delta}^*$ are always orthogonal for all dual arclength parameter $\hat{s}$. This fact can be easily see from the derivative of $\hat{\delta}$ and dual Frenet formulas:

$$\frac{d\hat{\delta}}{d\hat{s}} = \frac{d\hat{\tau}}{d\hat{s}}\widehat{T} + \frac{d\hat{\kappa}}{d\hat{s}}\widehat{\mathrm{B}}.$$

So, $\frac{d\hat{\delta}}{d\hat{s}}$ and dual co-centrode $\hat{\delta}^*$ are orthogonal if and only if $\frac{d\hat{\kappa}}{d\hat{s}}\hat{\tau} = \hat{\kappa}\frac{d\hat{\tau}}{d\hat{s}}$. Thus, dual Frenet curve $\hat{\gamma}$ is dual general helix if and only if $\frac{d\hat{\delta}}{d\hat{s}}$ and $\hat{\delta}^*$ are orthogonal for all points.

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4th INTERNATIONAL CONFERENCE ON MATHEMATICS
"An Istanbul Meeting for World Mathematicians"
27-30 October 2020, Istanbul, Turkey
This conference is dedicated to 67th birthday of Prof. M. Mursaleen

A NOTE ON EULER TOTIENT PARANORMED SEQUENCE SPACES

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Abstract

Let $\lambda \in \{c, c_0, \ell_\infty\}$. In this study, we introduce the new nonabsolute type paranormed sequence space $\lambda(\varphi, p)$ and show that $\lambda(\phi, p)$ and $\lambda(p)$ linearly isomorphic using the regular matrix given by Euler Totient function Φ. Further, we give a number of results concerning inclusion relations of the sequence space $\lambda(\phi, p)$. Then, we investigate some topological properties. Finally, we compute alfa, beta and gamma duals of this space and characterize certain matrix transformations on this sequence space.

1. Introduction

Assume that the set of all complex termed sequences is represented by ω. In this case, ω is a linear space under co-ordinatewise addition and scalar multiplication and a sequence space is any subspace of ω. For example c the set of all convergent complex sequences is a sequence space.

First, we point out the concept of a paranorm. Let U be a linear topological space over the real field $\mathbb{R}$. By a paranorm on U, we mean a mapping $p : U \to \mathbb{R}$ such that

(i) $p(\theta) = 0$, where θ is the zero vector in the linear space U.

(ii) $p(u) = p(-u)$, for all $u \in U$.

(iii) $p(u + v) \leq p(u) + p(v)$, for all $u, v \in U$.

(iv) if (λ_n) is in $\mathbb{R}$ with $\lambda \to \lambda_0$ and (u_n) is in U with $p(u - u_0) \to 0$ imply $p(\lambda_n u_n - \lambda u) \to 0$.

Let denote (p_j) a bounded sequences of strictly positive real numbers with $\sup p_j = H$ and $L = \max\{1, H\}$. The paranormed sequence spaces $c_0(p), c(p), \ell_\infty(p)$ and $\ell(p)$ were defined by Maddox [20] as follows:

$$\begin{aligned} c_0(p) &= \{a = (a_j) \in w : \lim_{j\to\infty} |a_j|^{p_j} = 0\}, \\ c(p) &= \{a = (a_j) \in w : \lim_{j\to\infty} |a_j - l|^{p_j} = 0 \text{ for some } l \in \mathbb{R}\}, \\ \ell_\infty(p) &= \{a = (a_j) \in w : \sup_{j\in\mathbb{N}} |a_j|^{p_j} < \infty\}, \end{aligned}$$

and

$$\ell(p) = \left\{ a = (a_j) \in w : \sum_k |a_j|^{p_k} < \infty \right\},$$

which are the complete spaces paranormed by

$$g_1(a) = \sup_{k\in\mathbb{N}} |a_k|^{p_k/L} \iff \inf p_k > 0 \text{ and } g_2(a) = \left(\sum_k |a_k|^{p_k} \right)^{1/L},$$

respectively. For convenience in notation, here and in what follows, the summation without limits runs from 0 to ∞. By $\mathcal{F}$ and $\mathbb{N}_k$, we shall denote the collection of all finite subsets of $\mathbb{N}$ and the set of all $n \in \mathbb{N}$ such that $n \geq k$, respectively. We shall assume throughout that $p_k^{-1} + (p'_k)^{-1} = 1$ provided $1 < \inf p_k \leq H < \infty$.

Let $A = (A_{nk})$ be an infinite matrix of complex entries, X and Y be subsets of ω and $x \in \omega$. We write A_n and A^k for the sequences in the $n-$th row and $k-$th column of A, $A_n x = \sum_{k=0}^{\infty} a_{nk} x_k$ and $A_x = (A_n x)$ (provided all the series $A_n x$ converge). The sets

$$X_A = \{x \in \omega : Ax \in X\} \tag{1.1}$$

and

$$M(X, Y) = \{a \in \omega : a.x = (a_k x_k) \in Y \text{ for all } x \in X\} \tag{1.2}$$

are called the matrix domain of A in X and the multiplier space of X in Y; in particular, $X^\alpha = M(X, \ell_1), X^\beta = M(X, cs)$ and $X^\gamma = M(X, bs)$ are called the $\alpha-, \beta-$ and $\gamma-$duals of X. Finally, (X, Y) is the class of all matrices A such that $X \subset Y_A$; so $A \in (X, Y)$ if and only if $A_n \in X$ for all $n \in \mathbb{N}$ and $Ax \in Y$ for all $x \in X$.

Let (X, g) be a paranormed space. A sequence (c_k) in X is called a Schauder basis for X if and only if for each $x \in X$, there exists a unique sequence (λ_k) of scalars such that $g(x - \sum_{k=0}^{n} \lambda_k c_k) \to 0$ as $n \to \infty$. In this case, we can write $x = \sum_k \lambda_k c_k$.

In the literature, there are many paper [1–4, 6–10, 16, 17] about paranormed sequence spaces obtained by matrix domain of infinite triangular matrices. Also, for more details about matrix domains of infinite triangular matrices, one can see [5, 11, 12, 18].

Throughout the paper, φ and μ denote the Euler Totient function and Möbius function, respectively. If n is a pozitif integer, then $\varphi(n)$ denotes the number of elements from $\{1, 2, ..., n\}$ coprime to n. It is well known that if $n = p_1^{a_1} p_2^{a_2} ... p_m^{a_m}$ is the prime number decomposition of $n > 1$, then

$$\varphi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)...\left(1 - \frac{1}{p_m}\right)$$

and $\varphi(1) = 1$. It is also well known that the function φ is multiplicative, i.e., if $n_1, n_2 \in \mathbb{N}$ are coprime, then

$$\varphi(m_1 m_2) = \varphi(m_1)\varphi(m_2).$$

Another important property of the Euler function is the equality

$$n = \sum_{d|n} \varphi(d).$$

For any positive integer n, Möbius function μ is defined as

$$\mu(n) = \begin{cases} 1 & \text{if n=1,} \\ 0 & \text{if } p^2|n, \text{ for some prime number } p, \\ (-1)^r & \text{if } n = q_1 q_2 ... q_r, \text{ where } q_1, q_2, ..., q_r \text{ are pairwise different prime numbers.} \end{cases}$$

Obviously, the function μ is multiplicative and

$$\sum_{d|n} d\mu(d) = (1 - q_1)(1 - q_2)...(1 - q_r),$$

where $n = p_1^{a_1} p_2^{a_2} ... p_l^{a_l}$ is the prime number decomposition of n[19].

Also, function $\mu(n)$ is multiplicative and

$$\sum_{d|n} \mu(d) = \begin{cases} 1, & n = 1 \\ 0, & n > 1 \end{cases} \tag{1.3}$$

One can consult to [21] for more details related to these functions. The Euler totient matrix $\Phi = (\phi_{mn})$ is defined by

$$\phi_{mn} = \begin{cases} \frac{\varphi(n)}{m} & n|m \\ 0 & \text{otherwise} \end{cases}$$

for all $m, n \in \mathbb{N}$. The inverse $\Phi^{-1} = (\phi^{-1}_{mn})$ of the matrix Φ is computed in [22] as

$$\phi^{-1}_{mn} = \begin{cases} \frac{\mu(\frac{m}{n})}{\varphi(m)} n & n|m \\ 0 & \text{otherwise} \end{cases}$$

for all $m, n \in \mathbb{N}$. Quite recently, [15] and [14] have introduced new Banach sequence spaces

$$\ell_p(\Phi) = \left\{ u = (u_i) \in \omega : \sum_i \left| \frac{1}{i} \sum_{i|j} \varphi(i) u_i \right|^p < \infty, \ (1 \le p < \infty) \right\}$$

$$\ell_\infty(\Phi) = \left\{ u = (u_i) \in \omega : \sup_i \left| \frac{1}{i} \sum_{i|j} \varphi(i) u_i \right| < \infty, \right\}$$

$$c(\Phi) = \left\{ u = (u_i) \in \omega : \lim_{i \to \infty} \frac{1}{i} \sum_{i|j} \varphi(i) u_i = L \text{ for some } L \in \mathbb{R} \right\}$$

$$c_0(\Phi) = \left\{ u = (u_i) \in \omega : \lim_{i \to \infty} \frac{1}{i} \sum_{i|j} \varphi(i) u_i = 0 \right\}$$

2. The Paranormed Sequence Space $\lambda(\Phi, p)$

The main purpose of the present section, is to introduce the sequence spaces $\lambda(\Phi, p)$ by using Euler totient matrix Φ and is to derive some linear and topological properties. For $\lambda \in \{c_0, c, \ell_\infty\}$, we define the sequence spaces $\lambda(\Phi, p)$ by

$$\lambda(\Phi, p) = \left\{ u = (x_i) \in w : \left(\frac{1}{i} \sum_{j|i} \varphi(j) u_j \right) \in \lambda(p) \right\} \tag{2.1}$$

In the case $(p_n) = e = (1, 1, 1, ...)$, the sequence space $\lambda(\Phi, p)$ is reduced to the sequence space $\lambda(\Phi)$ which are introduced by [14, 15]. With the notation of (1.1), we may redefine the space $\lambda(\Phi, p)$ as follows:

$$\lambda(\Phi, p) = \{\lambda(p)\}_\Phi.$$

Theorem 2.1. *Consider the function g such that*

$$g(u) = \sup_{i \in \mathbb{N}} \left| \frac{1}{i} \sum_{j|i} \varphi(j) u_j \right|^{p_i/M}$$

for all $u = (u_i) \in \lambda(\Phi, p)$. Then $\lambda(\Phi, p)$ sequence space is the complete linear metric spaces by the paranorm g.

Theorem 2.2. *The sequence space $\lambda(\phi, p)$ for $\lambda \in \{c_0, c, \ell_\infty\}$ is linearly isomorphic to the space $\lambda(p)$ where $0 < p_k \le H < \infty$.*

3. Dual spaces and matrix transformations

Now, we state and prove the theorems determining the alpha-,beta- and gamma-duals of the sequence space $\lambda(\Phi, p)$ for $\lambda \in \{c_0, c, \ell_\infty\}$. In addition to that, we also characterize the matrix transformations between the spaces $\lambda(\Phi, p)$ and $\lambda(q)$ for $\lambda \in \{c_0, c, \ell_\infty\}$.

In the following lemmas, let L and M denote natural numbers, N and K finite subsets of $\mathbb{N}$ and α and α_k complex numbers. Put $K_1 = \{k \in \mathbb{N} : p_k \le 1\}, K_2 = \{k \in \mathbb{N} : p_k > 1\}, p'_k = \frac{p_k}{p_k - 1}$ for $k \in K_2$ and $q_n \ge 1$ for all $n \in \mathbb{N}$.

Lemma 3.1. *[13]*

(i) $A \in (c_0(p) : \ell(q))$ *if and only if*

$$\exists M \in \mathbb{N}, \sup_{K \in \mathcal{F}} \sum_n \left| \sum_{k \in K} a_{nk} M^{1/p_k} \right|^{q_n} < \infty, \tag{3.1}$$

(ii) $A \in (c(p) : \ell(q))$ *if and only if* (3.1) *holds and*

$$\sum_n \left| \sum_k a_{nk} \right|^{q_n} < \infty, \tag{3.2}$$

(iii) $A \in (\ell_\infty(p) : \ell(q))$ *if and only if*

$$\forall M \in \mathbb{N}, \sup_{K \in \mathcal{F}} \sum_n \left| \sum_{k \in K} a_{nk} M^{1/p_k} \right|^{q_n} < \infty. \tag{3.3}$$

Theorem 3.1. *Define the matrix $C = (c_{nk})$ and the sets $T_1(p)$, T_2 and $T_3(p)$ as follows*

$$c_{nk} := \begin{cases} \dfrac{\mu(\frac{k}{n})}{\varphi(k)} k a_n & , \quad k|n, \\ \\ 0 & , \quad \text{otherwise.} \end{cases} \tag{3.4}$$

for all $n, k \in \mathbb{N}$.

$$\begin{aligned} T_1(p) : &= \bigcup_{M>1} \left\{ a = (a_k) \in \omega : \sup_{K \in \mathcal{F}} \sum_n \left| \frac{\mu(\frac{n}{k})}{\varphi(n)} k a_n M^{-1/p_k} \right| < \infty \right\}, \\ T_2 : &= \left\{ a = (a_k) \in w : \sum_n \left| \frac{\mu(\frac{n}{k})}{\varphi(n)} k a_n \right| < \infty \right\}, \\ T_3(p) : &= \bigcap_{M>1} \left\{ a = (a_k) \in \omega : \sup_{K \in \mathcal{F}} \sum_n \left| \frac{\mu(\frac{n}{k})}{\varphi(n)} k a_n M^{-1/p_k} \right| < \infty \right\}. \end{aligned}$$

Then,

$$\begin{aligned} \{c_0(\Phi, p)\}^\alpha &= T_1(p), \\ \{c(\Phi, p)\}^\alpha &= T_1(p) \cap T_2, \\ \{\ell_\infty(\Phi, p)\}^\alpha &= T_3(p). \end{aligned}$$

Lemma 3.2. *[13]*

(i) $A \in (c_0(p) : c(q))$ *if and only if*

$$\exists M, \sup_n \sum_k |a_{nk}| M^{-1/p_k} < \infty, \tag{3.5}$$

$$\lim_n |a_{nk} - \alpha_k|^{q_n} = 0 \text{ for all } k, \tag{3.6}$$

$$\forall L, \exists M, \sup_n L^{1/q_n} \sum_k |a_{nk} - \alpha_k| M^{-1/p_k} < \infty. \tag{3.7}$$

(ii) $A \in (c(p) : c(q))$ *if and only if* (3.5), (3.6), (3.7) *hold and*

$$\exists \alpha, \lim_n \left| \sum_k a_{nk} - \alpha \right|^{q_n} = 0, \tag{3.8}$$

(iii) *$A \in (\ell_\infty(p) : c(q))$ if and only if*

$$\forall M, \sup_n \sum_k |a_{nk}| M^{1/p_k} < \infty, \tag{3.9}$$

$$\exists \alpha_k, \forall M, \lim_n \left(\sum_k |a_{nk} - \alpha_k| M^{1/p_k} \right)^{q_n} = 0, \tag{3.10}$$

Lemma 3.3. *[13]*

(i) *$A \in (c_0(p) : \ell_\infty(q))$ if and only if*

$$\exists M, \sup_n \left(\sum_k |a_{nk}| M^{-1/p_k} \right)^{q_n} < \infty. \tag{3.11}$$

(ii) *$A \in (c(p) : \ell_\infty(q))$ if and only if (3.11) holds and*

$$\sup_n \left| \sum_k a_{nk} \right|^{q_n} < \infty. \tag{3.12}$$

(iii) *$A \in (\ell_\infty(p) : \ell_\infty(q))$ if and only if*

$$\forall M, \sup_n \left(\sum_k |a_{nk}| M^{1/p_k} \right)^{q_n} < \infty. \tag{3.13}$$

Theorem 3.2. *Define the matrix $D = (d_{nk})$ and the sets $T_4(p)$, T_5, $T_6(p)$, T_7, $T_8(p)$, $T_9(p)$, T_{10} and $T_{11}(p)$ as follows*

$$d_{nk} := \begin{cases} \displaystyle\sum_{j=k, k|j}^{n} \frac{\mu(\frac{j}{k})}{\varphi(j)} k a_j & , \quad 1 \le k \le n \\ \\ 0 & , \quad n > k \end{cases} \tag{3.14}$$

for all $n, k \in \mathbb{N}$.

$$\begin{aligned}
T_4(p) : &= \bigcup_{M>1} \left\{ a = (a_k) \in w : \sup_{n \in \mathbb{N}} \sum_{k=0}^{n} |d_{nk}| \, M^{-1/p_k} < \infty \right\}, \\
T_5 : &= \left\{ a = (a_k) \in w : \lim_{n \to \infty} |d_{nk}| \text{ exists for each } k \in \mathbb{N} \right\}, \\
T_6(p) : &= \bigcup_{M>1} \left\{ a = (a_k) \in w : \exists (\alpha_k) \in \mathbb{R} \ni \sup_{n \in \mathbb{N}} \sum_{k=0}^{n} |d_{nk} - \alpha_k| \, M^{-1/p_k} < \infty \right\}, \\
T_7 : &= \left\{ a = (a_k) \in w : \exists \alpha \in \mathbb{R} \ni \lim_{n \to \infty} \left| \sum_{k=0}^{n} d_{nk} - \alpha \right| = 0 \right\}, \\
T_8(p) : &= \bigcap_{M>1} \left\{ a = (a_k) \in w : \sup_{n \in \mathbb{N}} \sum_{k=0}^{n} |d_{nk}| \, M^{-1/p_k} < \infty \right\}, \\
T_9(p) : &= \bigcap_{M>1} \left\{ a = (a_k) \in w : \exists (\alpha_k) \in \mathbb{R} \ni \lim_{n \to \infty} \sum_{k=0}^{n} |d_{nk} - \alpha_k| \, M^{1/p_k} = 0 \right\}, \\
T_{10} : &= \left\{ a = (a_k) \in w : \sup_{n \in \mathbb{N}} \left| \sum_{k=0}^{n} d_{nk} \right| < \infty \right\}, \\
T_{11}(p) : &= \bigcap_{M>1} \left\{ a = (a_k) \in w : \sup_{n \in \mathbb{N}} \sum_{k=0}^{n} |d_{nk}| \, M^{1/p_k} < \infty \right\}.
\end{aligned}$$

Then,

(i) $\{c_0(\Phi,p)\}^\beta = T_4(p)\cap T_5\cap T_6(p)$, $\{c(\Phi,p)\}^\beta = \{c_0(\Phi,p)\}^\beta\cap T_7$, $\{\ell_\infty(\Phi,p)\}^\beta = T_8(p)\cap T_9(p)$,

(ii) $\{c_0(\Phi,p)\}^\gamma = T_4(p)$, $\{c(\Phi,p)\}^\gamma = T_4(p)\cap T_{10}$, $\{\ell_\infty(\Phi,p)\}^\gamma = T_{11}(p)$.

4. Certain Matrix Transformation Related to the Spaces $\lambda(\Phi,p)$

In this section, we characterize the matrix transformations from the sequence space $\lambda(\Phi,p)$ into any given sequence space. For an infinite matrix $A=(a_{nk})$, we write for brevity that

$$\begin{aligned}\hat{a}_{nk}(m) &= \sum_{j=k,k|j}^{m}\frac{\mu(\frac{j}{k})}{\varphi(j)}ka_{nj}\ (k<m),\\ \hat{a}_{nk} &= \sum_{j=k,k|j}^{\infty}\frac{\mu(\frac{j}{k})}{\varphi(j)}ka_{nj}\ (k<m),\end{aligned}$$

for all $k,n,m\in\mathbb{N}$ provided the convergence of series. Now, we give the characterization of the classes $(\lambda(\Phi,p),\delta)$ and $(\delta,\lambda(\Phi,p))$, where δ is any given sequence space.

Theorem 4.1. *Suppose that the entries of the infinite matrices $A=(a_{nk})$ and $E=(e_{nk})$ are connected with the relation*

$$e_{nk} = \sum_{j=k,k|j}^{m}\frac{\mu(\frac{j}{k})}{\varphi(j)}ka_{nj}\ (k<m) \tag{4.1}$$

for all $k,n,m\in\mathbb{N}$ and δ be any given sequence spaces. Then, $A\in(\lambda(\Phi,p),\delta)$ if and only if $(a_{nk})\in\{\lambda(\Phi,p)\}^\beta$ for all $n\in\mathbb{N}$ and $E\in(\lambda(p),\delta)$.

Theorem 4.2. *Suppose that the entries of the infinite matrices $\Theta=(\theta_{ij})$ and $\Sigma=(\sigma_{ij})$ are connected with the relation*

$$\sigma_{ij} = \frac{1}{i}\sum_{k|i}\varphi(k)\theta_{kj}$$

for all $i,j\in\mathbb{N}$ and δ be any given sequence space. Then, $\Theta\in(\delta:\lambda(\Phi,p))$ if and only if $\Sigma\in(\delta:\lambda(p))$.

Corollary 4.1. *The following statements hold:*

(i) *$A\in(\ell_\infty(\hat{L},p):\ell_\infty(p))$ if and only if (3.13) holds with $a_{nk}=e_{nk}$.*

(ii) *$A\in(\ell_\infty(\hat{L},p):c(p))$ if and only if (3.9) and (3.10) hold with $a_{nk}=e_{nk}$.*

(iii) *$A\in(\ell_\infty(\hat{L},p):c_0(p))$ if and only if (3.11) holds with $a_{nk}=e_{nk}$.*

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A P-adic Analytic Proof of Reflectivity of Twisted Finite Sums of Continous Functions

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Abstract

The twisted sums of powers which is a generalization of the alternating sums of powers are studied through a p-adic integral transform. The reflectivity of twisted sums of powers is proved by the reflectivity of the corresponding p-adic integrals. In particular the polynomial expressions for the alternating sums of powers follows as a special case.

1. Introduction

The sum of powers of integers of the form

$$\sum_{a=0}^{x} a^m$$

for $x \in \mathbb{N}$ and a fixed $m \in \mathbb{N}$ has drawn considerable attention starting with Faulhaber. Faulhaber proved that these sums can be expressed as polynomials in $x(x+1)$ or a multiple of such polynomail depending on the parity of m. One can also consider the r-fold sum of powers of integers defined as

$$\begin{aligned} &\sum\nolimits^{1} f(x) = \sum_{a=0}^{x} f(a) \\ &\sum\nolimits^{r} f(x) = \sum\nolimits^{r-1} f(1) + \sum\nolimits^{r-1} f(2) + \ldots + \sum\nolimits^{r-1} f(x), \quad r \geq 2 \end{aligned} \tag{1}$$

where $f(x) = x^m$ for some fixed $m \geq 1$. Convenient references for Faulhaber's theorem for r-fold sums are [9, 7, 8].

We may also consider the alternating r-fold sums defined as

$$\sum\nolimits^{r} (-1)^x x^m \tag{2}$$

These sums are are studied in detail in [11] and [10] (for general $r \geq 1$). It is a common phenomena that the sums (1) and (2) can be expressed in terms of Bernoulli and Euler polynomials respectively. Also the explicit expressions of (alternating) r-fold sums involve polynomials in $x(x+r)$ possibly multiplied by some linear factors and the exponential term $(-1)^x$ for the alternating case depending on the parities of m and r. A comprehensive list of polynomial expressions of these sums in terms of Bernoulli and euler polynomials with proofs can be found in [10] (See also the references therein).

In this paper we aim to explore twisted sums of powers of the form

$$\sum_{a=0}^{x} w^a a^m$$

where w is to be specified in a p-adic setting. Indeed there will not be any loss if we consider

$$\sum_{a=0}^{x} f(a)w^a \tag{3}$$

for an arbitaray even or odd continuous function $f(x)$ on $\mathbb{Z}_p$ (the ring of p-adic integers). The method we use is based on the p-adic integral equation

$$w^x \int_{thbbZ_p} f(x+t)\ \mu_w(t) = \int_{\mathbb{Z}_p} f(t)\ \mu_w(t) + \sum_{a=0}^{x-1} w^a f(a)$$

for $x \in \mathbb{Z}_{\geq 1}$ where μ_w is a p-adic measure specified below. We prove a reflectivity property of the integral transform

$$\int_{\mathbb{Z}_p} f(x+t)\ \mu_w(t)$$

and then this naturally gives an analogues result for the sum $\sum_{a=0}^{x-1} w^a f(a)$.

2. Reflectivity of Twisted Sums

We start with the basic definitions and facts required in the main result. We use the convention of [9] for reflective functions.

Definition. *Let f be a function on $\mathbb{C}_p$, and $r \in C_p$. We say that f is r-reflective (respectively anti r-reflective) if*

$$f(x) = f(-r-x) \quad (\textit{respectively } f(x) = -f(-r-x)).$$

Definition. *Let $w \in \mathbb{C}_p^*$ with $|w-1|_p \geq 1$. We define μ_w to be the p-adic measure defined as*

$$\mu_w(a+(p^N)) = \frac{w^a}{w^{p^N}-1}$$

on the compact-open subsets $a+(p^N)$ of $\mathbb{Z}_p$ with $0 \leq a \leq p^N-1$ and extended to all compact-open subsets of $\mathbb{Z}_p$ linearly.

Boundedness of μ_w easily follows, indeed $\sup |\mu_w(a+(p^N))|_p \leq 1$. For basic facts on p-adic measures and p-adic integration the reader is referred to [2]. The measure μ_w had been defined and studied to p-adically interpolate L-functions by N. Koblitz in [2] and [3]. Note that the version we use here differs just by a minus sign due to notational conventions. It also plays am improtant role in the theory of p-adic polylogarithms ([4]).

In the sequel we set

$$H_w(f)(x) = \int_{\mathbb{Z}_p} f(x+t)\ \mu_w(t)$$

to simplify the notation. We will use the following identity which follows by definition of p-adic integration;

$$wH_w(f)(x) = H_w(f)(x-1) + f(x-1) \tag{4}$$

(We may also refer to [2]).

Proposition 1. *Let $w \in \mathbb{C}_p^*$ with $|w-1|_p \geq 1$ and $f \in C(\mathbb{Z}_p)$. Then for $x \in \mathbb{Z}_{\geq 1}$ we have*

$$w^x H_w(f)(x) = H_w(f)(0) + \sum_{a=0}^{x-1} w^a f(a). \tag{5}$$

Proof. The claim is clear for $x = 1$ by (4). Then the result follows by indcution on x. $\square$

Note that for $f(x) = x^{k-1}$, $H_w(f)(x)$ turns out to be the so-called k-th Apostol-Bernoulli polynomial. But we will not restirct to polynomials (We refer to [5, 1] for properties of Apostol-Bernoulli polynomials).

We also have another interesting fact about (5). It is well known that the discrete convolution of two continous functions on $\mathbb{Z}_p$ is also continuous [6]. Now consider the discrete convolution of f and the exponential function $x \mapsto (1/w)^x$. Note that $x \mapsto (1/w)^x$ is discontinous since $|w-1|_p \geq 1$. We can write (5) as

$$H_w(f)(x) = w^{-x} H_w(f)(0) + \sum_{a=0}^{x-1} (1/w)^{a-x} f(a)$$

Note that the above sum is the discrete convolution of f and the function $x \mapsto (1/w)^x$. But $H_w(f)$ is continuous and so the discrete convolution of f and the function $x \mapsto (1/w)^x$ is continous if and only if $H_w(f)(0) = 0$.

Now we state and prove the main theorem.

Theorem 1. *Let $w \in \mathbb{C}_p^*$ with $|w-1|_p \geq 1$ and f be a continous function on $\mathbb{Z}_p$. If f is an even function, then*

$$wH_w f(1-x) = -H_{1/w} f(x).$$

Similarly if f is an odd function then

$$wH_w f(1-x) = H_{1/w} f(x)$$

.

Proof. Before the proof note that for any continuous function $f(t)$ and any p-adic measure μ on $\mathbb{Z}_p$ we have that

$$\int_{\mathbb{Z}_p} f(t)\ \mu = \lim_{N\to\infty} \sum_{a=0}^{p^N-1} f(\tilde{a})\mu(a+(p^N))$$

where $\tilde{a}$ is any representative in the subset $a+(p^N)$. First let f be even. Then we have

$$\begin{aligned}
H_{1/w}f(x) &= \int_{\mathbb{Z}_p} f(x+t)\ \mu_{1/w}(t) = \lim_{N\to\infty} \sum_{a=0}^{p^N-1} \frac{f(x+a-p^N)w^{-a}}{w^{-p^N}-1} \\
&= \lim_{N\to\infty} \sum_{a=0}^{p^N-1} \frac{f(-x-a+p^N)w^{p^N-a}}{1-w^{p^N}} = -\lim_{N\to\infty} \sum_{b=1}^{p^N} \frac{f(b-x)w^b}{w^{p^N}-1} \\
&= -\lim_{N\to\infty} \left[\sum_{b=0}^{p^N-1} \frac{f(b-x)w^b}{w^{p^N}-1} + \frac{f(p^N-x)w^{p^N}}{w^{p^N}-1} - \frac{f(-x)}{w^{p^N}-1} \right] \\
&= -[H_w f(-x) + f(-x)] = -[H_w f(-x) + f(x)]
\end{aligned}$$

Now we replace x by $1-x$ in (4) and obtain

$$wH_w f(1-x) = H_w f(-x) + f(-x) = H_w f(-x) + f(x).$$

Combining these equalities we see that

$$wH_w f(1-x) = -H_{1/w} f(x)$$

as desired. If f is odd then in a similar way first we obtain

$$H_{1/w} f(x) = H_w f(-x) - f(x)$$

and then $wH_w f(1-x) = H_{1/w} f(x)$. □

As a consequence if we consider $H_w f(x)$ and $H_{1/w} f(x)$ simultaneously then we can obtain a reflectivity result for the twisted sums (3). First by Theorem 1 we observe that

$$H_w f(x).H_{1/w} f(x) = \pm w H_w f(x) H_w f(1-x)$$

where the + sign (respectively − sign) occurs if f is odd (respectively even). But the right hand side is clearly -1-reflective. Also by (5) we have that

$$\left[H_w f(0) + \sum_{a=0}^{x-1} w^a f(a)\right]\left[H_{1/w} f(0) + \sum_{a=0}^{x-1} w^{-a} f(a)\right]$$
$$= \left[w^x H_w f(x)\right]\left[w^{-x} H_{1/w} f(x)\right] = H_w f(x) H_{1/w} f(x) = \pm w H_w f(x) H_w f(1-x)$$

which gives the following result.

Corollary 1. *Let $w \in \mathbb{C}_p^*$ with $|w-1|_p \geq 1$ and $f \in C(\mathbb{Z}_p)$. Suppose that f is an even or an odd function and that $H_w f(0) = H_{1/w} f(0) = 0$. Then the product*

$$\left[\sum_{a=0}^{x-1} w^a f(a)\right]\left[\sum_{a=0}^{x-1} w^{-a} f(a)\right]$$

is -1-reflective.

Now we restrict to the special case $w = -1$. Accordingly we may take any prime $p > 2$, so that $|w-1|_p = 1$. In this case Theorem 1 reads as

$$H_{-1} f(1-x) = \pm H_{-1} f(x)$$

where the positive (respectively negative) sign occurs if f is odd (respectively even). So $H_{-1} f(1-x)$ is -1-reflective (respectively anti -1 reflective) if f is odd (respectively even). Combining with (5) we obtain the following reflectivity result for the alternating sums.

Corollary 2. *Let p be an odd prime and $f \in C(\mathbb{Z}_p)$. Suppose that f is an odd (respectively even) function. Then*

$$-H_{-1}f(0) + \sum_{a=0}^{x-1} w^a f(a)$$

is equal to the product of a -1-reflective (respectively anti -1-reflective) function and $(-1)^x$.

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A STUDY OF STATISTICAL REASONING ABILITIES USING COOPERATIVE LEARNING FOR MATHAYOMSUKSA IV STUDENTS

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Abstract

The purpose of this research was to study the statistical reasoning abilities in students from grade 10 (Mathayomsuksa IV) by using cooperative learning. The study was conducted with 40 students, working in groups of 3-4 with different relative ability. The participants were assessed in statistical reasoning by comparing the scores from a designed pre-test and post-test. It was hypothesized that the statistical reasoning abilities scores of the students would increase after participating in the study project.

The result shows that

1. Statistical reasoning was higher than the criterion of 60% at the .05 level of significance.
2. Satisfaction of Mathayomsuksa IV students toward cooperative learning was at a high level.

Keywords: statistical reasoning, cooperative learning, statistics

1. Introduction

The human lives are surrounded with data and decision-making. Using the intuition or previous experience may not be enough to make sense of the data. In addition, a person’s life is inevitably related to the use of statistical concepts and methods, directly and indirectly, including biology, physics, engineering, economics, sociology, psychology, business, and others. [1] Garfield defines statistical reasoning as the way statistical knowledge is used to make sense of data. [2] Therefore, statistical reasoning is the tool necessary for this digital age and it should begin with the right educational foundation, school. The study of statistics provides students with tools and ideas to use in order to react intelligently to quantitative information in the world around them [2] According to Jone’s [3], the statistical reasoning composes of four processes: describing data, organizing data, representing data and analyzing data. However, Garfield [3] found that most students performed well in the statistics classroom but lacking statistical reasoning ability and skills. Result of this study point out that statistics teachers do not teach students how to apply statistics, but most teachers tend to teach statistics in a way that the formulas are applied and can be calculated accurately. Therefore, teaching statistics should be in a way that enhances students’ thinking more than focusing on the calculation and formulas in statistics classroom. Cooperative learning is the type of learning when small groups of students work to enhance their own and their peers’ learning in their own group.[4] The cooperative learning results in higher achievement than lecture instruction for all students and is especially beneficial for those least prepared for statistics. Due to this, cooperative learning is suitable for enhancing the statistics, especially the statistical reasoning.

2. Statistical Reasoning

Garfield defines statistical reasoning as the way statistical knowledge is used to make sense of data. [2] To demonstrate a statistical reasoning, a person needs to be able to give the explanation of the occurred results which require the understanding of the data. Galotti [4] found that statistical reasoning is occurring in a person's thinking after being assigned to state the implications.

Referring to Jones et al. and Mooney statistical reasoning framework, the four key statistical processes consist of describe data, organizing data, representing data, and analyzing data. The four statistical processes are defined as follows.

1. Describe data is the first step of interpreting and data analysis. This step involves the explicit reading of the given raw data or data presented in the form of tables, charts, or graphical representations. There are two subprocesses of describing data: showing awareness of display features and identifying units of data values.

2. Organizing data refers to the process of arranging, categorizing, or consolidating data into a term of center and the spread of data. To organize the data, there are three subprocesses includes grouping data, summarizing data in terms of center, and describing the spread of data.

3. Representing data refers to the process of displaying data in a graphical form. Students need to think about what is involved in constructing graphs and optimize the choice of the graph in the given situation. Data representation has two subprocesses which includes completing or constructing a data display for a provided set of data and giving the effectiveness evaluation for the data displays.

4. Analyzing data is considered as a core of statistical reasoning. This process involves recognizing patterns and trends in the data, making inferences and predictions from data. Curcio [5] defines two subprocesses of analyzing data, reading between the data, and reading beyond the data.

Based on Jones et al. and Mooney statistical reasoning framework [2], there are four levels of statistical reasoning: The four levels consist of idiosyncratic, transitional, quantitative, and analytical. 1st level is when the response of the students is limited to the unrelated data or when they are using their personal experiences and beliefs to answer the questions. At the 2nd level, transitional, students notice the importance of reasoning but still not consistent and tend to focus only one aspect of the statistical problems. On the 3rd level, quantitative, students are capable to identify the mathematical ideas of the problem situation and are not distracted or misled by the irrelevant aspects. The highest level of statistical reasoning, analytical, the reasoning towards statistical reasoning has developed to the task integration of the relevant aspects into a meaningful structure.

Referring to the level of statistical reasoning and subprocess of each statistical reasoning, this research has created the rubric score that corelate to the students statistical reasoning abilities and their responds in the classroom. The table 1 is developed rubric score by the researcher.

Process	Subprocess	Score	Respond
Describing data	1.Showing awareness of the data	0	Student does not show any awareness of the data, give the answer that unrelated to the context
		1	Show some awareness but easily to get distracted
		2	Show well awareness of the data, not easily to be distracted
		3	Provide correct answer with the supporting reasons and the discussion beyond the context.
	2. Identifying units of data values.	0	Students give the unrelated respond to the context
		1	Show some awareness in data values units but easily to get distracted
		2	Show well awareness of the data, not easily to be distracted
		3	Provide correct respond with the supporting reasons and the discussion beyond the context.
Organizing data	1. Summarizing data in the measures of centre and spread of data	0	Students give the unrelated respond to the context
		1	Student just only provide the number but does not show the process of calculation or misunderstand in the use of statistical formula.
		2	Show well awareness and right statistical computation of the measures of centre and spread of data , not easily to be distracted.
		3	Provide correct respond with the supporting reasons and the discussion beyond the context.
	2. Provide the reason for each selected measures of centre	0	Students give the unrelated respond to the context
		1	Show some awareness for each selected measures of centre but easily to get distracted
		2	Show well awareness of the data, not easily to be distracted
		3	Provide correct respond with the supporting reasons and the discussion beyond the context.
Representation data	1. Completing or constructing a data display for a provided set of data	0	Students give the unrelated respond to the context
		1	Student just only provide the number but does not show the process of calculation or misunderstand in the use of statistical formula.

		2	Show well awareness of the data, not easily to be distracted
		3	Provide correct respond with the supporting reasons and the discussion beyond the context.
	2. Giving the effectiveness evaluation for the data displays.	0	Students give the unrelated respond to the context
		1	Show some awareness for each selected measures of centre but easily to get distracted
		2	Show well awareness of the data, not easily to be distracted
		3	Provide correct respond with the supporting reasons and the discussion beyond the context.
Analysing data	1. Reading between the data, recognize the patterns and trends in the data	0	Students give the unrelated respond to the context
		1	Show some awareness of patterns and trends in the data but easily to get distracted
		2	Show well awareness of patterns and trends in the data, not easily to be distracted
		3	Provide correct respond with the supporting reasons and the discussion beyond the context.
	2. Reading beyond the data, making inferences and predictions from data	0	Students give the unrelated respond to the context
		1	Show some awareness of data interpretation but easily to get distracted
		2	Show well awareness in making inferences, not easily to be distracted
		3	Provide correct respond with the supporting reasons and the discussion beyond the context.

Table 1 rubric score according to the level of statistical reasoning

3. Cooperative Learning

Johnson and Johnson [6] states that cooperative learning exists when small groups of students work to enhance their own and their groupmates' learning. Several studies confirm that cooperative learning has provided the learners with positive results. Giraud [7] compared the effective of cooperative learning vs. lecture methods of instruction in the undergraduate statistics course. The cooperative learning results in higher achievement than lecture instruction for all students and is especially beneficial for those least prepared for statistics. As well as, Keeler and Steinhorst [8] have also found that working in cooperative groups resulted the increase in students' final scores in two experimental sections than in a comparison course section , more percentage of completing the course, and the students had positive attitudes toward the study project. Johnson and Johnson [6] found that the cooperative learners have better performance on questions comparing to their peers in traditional classrooms and also have a higher level of thinking compare to the non-cooperative classroom.

Consequently, cooperative learning enhances the student in many ways, it increases the higher achievement, productivity, self-esteem, self-confidence, the acceptance and support from groupmates, and independence. Moreover, the cooperative learners are more likely to generate new ideas and solutions, more higher-level reasoning, than the non-cooperative learners. Therefore, employing a cooperative learning strategy in statistical reasoning was expected to improve the statistical reasoning scores of students after the participation.

4.Method

This research is the experimental study, which aims to find out the statistical reasoning abilities in students grade 10 by using cooperative learning. The subjects were 40 students from grade 10 at Puranawat School, Bangkok, Thailand. The data collection was performed in the second semester of the academic year 2020, which took a total of 7 periods, 100 minutes each. To construct the statistical instrument, there were experts in the statistical reasoning field had validated the content of this instrument. The researcher construct it based on the statistical reasoning framework. The participants were individually given the pre-test and post-test along with the satisfaction survey. This pre-test contained 16 questions and the posttest contained 15 questions. The students were given 40 minutes to complete each test, the tests were done individually. The hypothesis of this research is after participating in the study program statistical reasoning of the students will be higher than the criterion of 60% at the .05 level of significance.

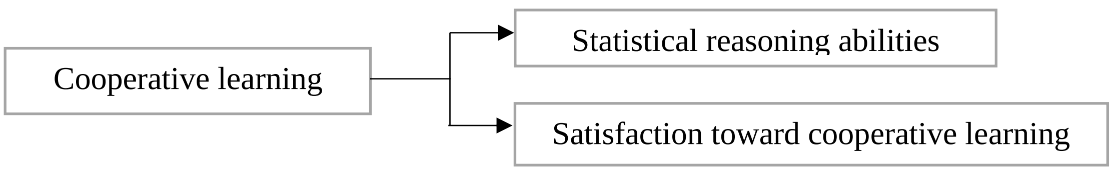
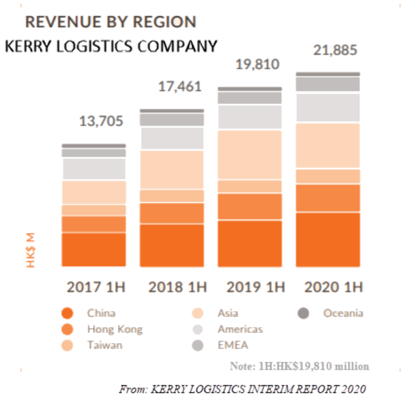


Figure 1. The research framework



Process	Example questions on each statistical processes
Describing data	1.What is this data about? 2. What is the unit of 19,810? 3. Which year does Kerry have the highest revenue?
Organizing data	1. Find mean, median and mode. 2. Which measure of center represent data the best? Choose one mean, median and mode. Justify your reason.
Representation data	1. If the journalist would like to draw the graph to present the growth of this company, which type of graph represent data the best out of the given context? Justify your reason.
Analyzing data	1. Based on the revenue of the previous year, would the revenue of Kerry Logistics increase or decrease in the year 2021? Justify your reason.

Figure 2. Example statistics problem in the research

5. Result

The data obtained during the study of the statistical reasoning ability of Mathayomsuksa 4 students before and after participating in a cooperative learning.

	$\overline{X}$	S.D	Total score	z
Pre-test	9	3.06	20	-8.19*
Post-test	16.5	2.80	24	

*Note*p<.05*

Table 2 The comparison of pre-test and post-test scores

The table 1 shows the improvement in statistical reasoning of the students at the .05 level of significance. The students have improved from earning 45% on their pre-test to 68.75% on their posttest or after the completion of conducting the experiment. A term "improving statistical reasoning" refers to the actual improvement of the students based on their participating in the cooperative learning using designed statistical reasoning lesson plans.

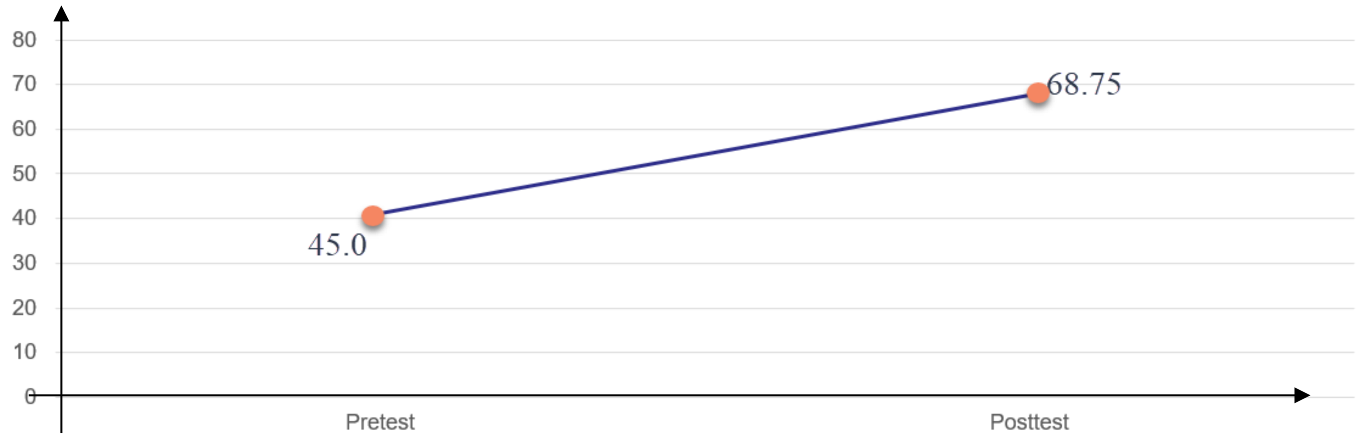


Figure 3. The improvement of the students statistical reasoning comparing between the pretest and posttest

After the participation in the project, students were given 20 survey questions of their satisfaction toward cooperative learning. The result found that Satisfaction of Mathayomsuksa IV students toward cooperative learning was at a high level.

6. Discussion

This research promotes statistical reasoning in all four areas. The study of statistical reasoning ability of Mathayomsuksa 4 students by using cooperative learning found that Mathayomsuksa IV students that participated in the cooperative learning activities, have higher statistical reasoning abilities than the students in the normal classroom at the criterion of 60% at the .05 level of significance because of the following:

1. Cooperative learning is a learning management that focuses on the students to work together as a group where all members participate in the group assignment with the mutual common goal. The researcher divided the group of students into 10 groups, 4 students each group with different learning abilities using the scores from the pre-test. Throughout the experiment, it was found that the group learning process allows students to exchange ideas with each other, explain knowledge to one another and ask more questions. This can be seen from the satisfaction questionnaire, students mostly agreed that the cooperative learning gives students the courage to ask questions with their teacher and peers. Students feel more engaged and can understand the content better. This is in line with the research of Somdej

Boonprajak [6] which found that collaborative learning gives students the potential of mathematics in problem solving and reasoning at the 0.01 level of significance, in the same way that Ibán and Marcos [7]found a significant increase in creativity scores in the experimental group and a moderate positive correlation between creative thinking and academic achievement.

2. Statistical reasoning is the student's ability in understanding the statistical data through four statistical processes: describing data, organizing data, representing data, and analyzing data.

2.1 Describing data is the first step to understand statistical data and see the overview of the given context. It provides students a preliminary understanding to prepare them for the next step in statistical reasoning. The researcher has designed a lesson plan with list of questions that practice statistical reasoning in describing data, see figure 2. Each series of problems feature various real-life statistical data. The study found that students were more likely to pay attention and participate in the discussion more with the real-life data. Similarly, Svetlana and Gillian [8] states that teacher should allow students to solve real-world problems in a variety of contexts and the discuss up-to-date examples can help stimulate student interest. In the early stages of the experiment, researcher also found that students were likely to use their personal experience to answer the questions. Moreover, the students were unable to provide the support reasoning of their peers' responses. However, after the end of the lesson plan most students had better score in the describing data comprehensive test.

2.2 Organizing data is the second process of statistical reasoning. In this process, researcher found that students were quite familiar with finding arithmetic mean or the calculation aspect of statistics. There were still some groups that had difficulty in finding the median. In addition, they are unable to justify selecting a suitable intermediate representative of the data. However, after the end of lesson plan, many of the students performed better in the organizing data comprehensive test. There was better awareness in arranging the data before the calculation was performed. In addition, students reason better toward the use of measures of center.

2.3 Representing data is the third area of statistical reasoning. This process involves the data representation in various types, tables, charts, bar charts, linear graph, and histogram. It is also including the optimal choice of data representation toward the given context. The researcher found that in the early stages of learning activity, there were some groups of students struggled with determining the X and Y axes, and students were likely to scale their graph incorrectly. However, after the end of lesson plan, many of the students performed better in the data representation comprehensive test. There were less mistakes found in constructing statistical graph comparing to the early stage. Student were able to provide reason toward the given data representation. Jane and Helen's [9] has also found that cooperative learning enhance the learning in data representation, students were likely to have a better concentration in the classroom compare to the normal class.

2.4 Analyzing data is the fourth process of statistical reasoning. It refers to students' ability in recognizing the patterns and trends of data and use given information to make inferences and predictions. The researcher found that in the early stages majority of the students were struggling in providing reasons towards the given data. There were some number of students that acknowledge in the trends of the data but still could not explain why. Interestingly, this study also found that the students tend to make less mistake in this process from the early stage. This might because they were familiar with the real-life data and trend seen in the news. After the end of lesson plan, many of the students performed better in the analyzing data comprehensive test.

7. Conclusion

In conclusion, the study of statistical reasoning using cooperative learning in the classroom has improved the students' statistical reasoning abilities. Statistical reasoning was higher than the criterion of 60% at the .05 level of significance. The researcher also found that students were more engaged in the classroom when they exposed the real data. For future study, teacher can consider incorporating more of the real-world example in the statistics classroom and let the students work together in group using different active learning techniques. It can also be developed into a statistical reasoning course in the school when the students will spend the entire semester engaging in the statistical reasoning processes.

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[1-4, 10-15]

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Açık B-Spline Eğrileri Üzerine

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Abstract

Bu çalışmada açık B-spline eğrileri hakkında bilgiler verilmiş, onların uç knot vektörlerinde yani t_d ve t_{m-d} noktalarında bu eğrilerin birinci ikinci ve üçüncü türevleri bulunmuştur. Daha sonra aynı noktalarda bu eğriler üzeinde $\{T, N, B\}$ Frenet- Serret çatısı oluyşturularak eğrinin belirtilen noktalarda eğrilikleri ifade edilmiştir.

1. Giriş

B-spline eğrileri ve NURBS eğriler kontrol noktaları denilen bir grup noktanın oluşturduğu konvexhull'uniçinde kalma özelliğine sahiptirler. Ayrıca bu spline eğrilerinin kontrol noktalarından herhangi birinin değiştirilmesinde lokal olarak o noktanın bir komşuluğunda eğri değişim gösterir. Spline eğrilerinde kontrol noktaları sayısı ne kadar azalırsa kontrol noktasının koordinatındaki değişimin eğrinin diğer noktalarını etkilemesi o kadar artmaktadır. örneğin bir data setindeki noktalardan elde edilecek kübikspline eğrilerinde datadan herhangi biri değiştiğinde tüm noktalarda eğrilerin tamamı etkilenebilmektedir (Yükselen,2018).

B-spline eğrileri ve NURBS eğrilerle ilgili yapılan çalışmalara (Tiller,1992), (Hoschek,1992), (Meek ve Walton,1993), (Neamtu ve ark.,1998), (Juhasz,1999), (Piegl ve Tiller, 1999,2002), (Liu ve Wang, 2002), (Selimovic,2006), (Samancı,2018) çalışmalar örnek olarak verilebilir.

2. B-Spline Eğrileri

Derecesi d olan bir B- spline eğrisi, b_i ile gösterilen ve eğriyi kontrol eden bir takım kontrol noktaları ile yine derecesi d olan B-spline taban fonksiyonları diye adlandırılan fonksiyonlar ve bu fonksiyonların tanımlandığı aralıkları belirleyen $t_0, t_1, \dots, t_m$ ile gösterilen knot vektörler yardımıyla tanımlanan parçalı eğrilerdir.

Tanım 1. Knot vektörü olarak adlandırılan ve $t_0, t_1, \dots, t_m$ ile tanımlanan, $N_{i,d}(t)$ ile belirtilen d. dereceden B-spline taban fonksiyonları i=0,1,...,n ve $d \geq 1$ için aşağıdaki gibi tanımlanır:

$$N_{i,d}(t) = \frac{t-t_i}{t_{i+d}-t_i} N_{i,d-1}(t) + \frac{t_{i+d+1}-t}{t_{i+d+1}-t_{i+1}} N_{i+1,d-1}(t). \tag{1}$$

burada

$$N_{i,0}(t) = \begin{cases} 1, eğer\ t \epsilon [t_i, t_{i+1}] \\ 0, aksi\ taktirde \end{cases}$$

dır. Eğer knot vektörler yeterli sayıda tekrarlandığında yani ardışık t_i değerlerinin eşit olması durumunda, tekrarlamanın gerçekleştirilmesi sırasında (bazı i için) $\frac{N_{i,d-1}(t)}{t_{i+d}-t_i} = 0/0$ değeri ile karşılaşılabilir. Bu meydana geldiğinde $\frac{0}{0} = 0$ olduğu varsayılır.

Tanım 2. Knot vektörleri $t_0, t_1, \dots, t_m$ ve kontrol noktaları $b_0, b_1, \dots, b_n$ ile d. dereceden bir B-spline eğrisi

$$B(t) = \sum_{i=0}^{n} b_i N_{i,d}(t) \tag{2}$$

Olarak tanımlanır. Burada tanım aralığı $[a, b] = [t_d, t_{m-d}]$ dır. Burada $N_{i,d}(t)$, d. dereceden B-spline temel fonksiyonudur.

Örnek 1. Düzlemde kontrol noktaları $b_0(1,2)$, $b_1(3,5)$, $b_2(6,2)$, $b_3(9,4)$ ve knot vektörleri $t_0 = 2, t_1 = 4, t_2 = 5, t_3 = 7, t_4 = 8, t_5 = 10, t_6 = 11$ olarak verilsin. Derecesi d=2 olsun. Bu durumda B-spline eğrisini aşağıdaki şekilde bulalım: Bunun için d=2 oluncaya kadar $N_{i,d}(t)$ taban fonksiyonlarını bulalım.

d=0 için:

$$N_{0,0}(t) = \begin{cases} 1, & eğer\ t \in [2,4) \\ 0, & aksi\ \ taktirde \end{cases}, \qquad N_{1,0}(t) = \begin{cases} 1, & eğer\ t \in [4,5) \\ 0, & aksi\ \ taktirde \end{cases},$$

$$N_{2,0}(t) = \begin{cases} 1, & eğer\ t \in [5,7) \\ 0, & aksi\ \ taktirde \end{cases}, \qquad N_{3,0}(t) = \begin{cases} 1, & eğer\ t \in [7,8) \\ 0, & aksi\ \ taktirde \end{cases},$$

$$N_{4,0}(t) = \begin{cases} 1, & eğer\ t \in [8,10) \\ 0, & aksi\ \ taktirde \end{cases}, \qquad N_{5,0}(t) = \begin{cases} 1, & eğer\ t \in [10,11) \\ 0, & aksi\ \ taktirde \end{cases}.$$

d=1 için:

$$N_{0,1}(t) = \frac{t-t_0}{t_1-t_0} N_{0,0}(t) + \frac{t_2-t}{t_2-t_1} N_{1,0}(t) = \frac{t-2}{4-2} N_{0,0}(t) + \frac{5-t}{5-4} N_{1,0}(t) = \frac{t-2}{2} N_{0,0}(t) + (5-t) N_{1,0}(t),$$

$$N_{1,1}(t) = \frac{t-t_1}{t_2-t_1} N_{1,0}(t) + \frac{t_3-t}{t_3-t_2} N_{2,0}(t) = \frac{t-4}{5-4} N_{1,0}(t) + \frac{7-t}{7-5} N_{2,0}(t) = (t-4)N_{1,0}(t) + \frac{1}{2}(7-t)N_{2,0}(t),$$

$$N_{2,1}(t) = \frac{t-t_2}{t_3-t_2} N_{2,0}(t) + \frac{t_4-t}{t_4-t_3} N_{3,0}(t) = \frac{1}{2}(t-5)N_{2,0}(t) + (8-t)N_{3,0}(t),$$

$$N_{3,1}(t) = \frac{t-t_3}{t_4-t_3} N_{3,0}(t) + \frac{t_5-t}{t_5-t_4} N_{4,0}(t) = (t-7)N_{3,0}(t) + \frac{1}{2}(10-t)N_{4,0}(t)$$

$$N_{4,1}(t) = \frac{t-t_4}{t_5-t_4} N_{4,0}(t) + \frac{t_6-t}{t_6-t_5} N_{5,0}(t) = \frac{1}{2}(t-8)N_{4,0}(t) + (11-t)N_{5,0}(t).$$

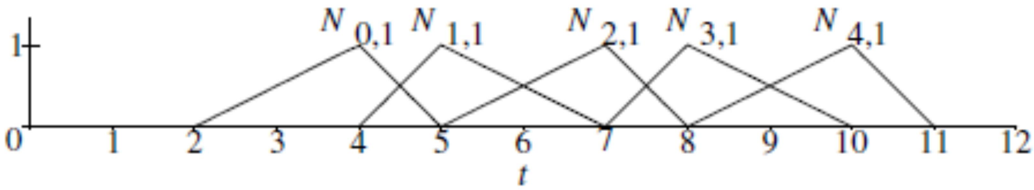
Yazılabilir. Buna göre,

$$N_{0,1}(t) = \begin{cases} \frac{t-2}{2}, & t \in [2,4) \\ 5-t, & t \in [4,5) \\ 0, & aksi\ \ taktirde \end{cases}, \qquad N_{1,1}(t) = \begin{cases} t-4, & t \in [4,5) \\ \frac{7-t}{2}, & t \in [5,7) \\ 0, & aksi\ \ taktirde \end{cases},$$

$$N_{2,1}(t) = \begin{cases} \frac{t-5}{2}, & t \in [5,7) \\ 8-t, & t \in [7,8) \\ 0, & aksi\ \ taktirde \end{cases}, \qquad N_{3,1}(t) = \begin{cases} t-7, & t \in [7,8) \\ \frac{10-t}{2}, & t \in [8,10) \\ 0, & aksi\ \ taktirde \end{cases},$$

$$N_{4,1}(t) = \begin{cases} \frac{t-8}{2}, & t \in [8,10) \\ 11-t, & t \in [10,11) \\ 0, & aksi\ \ taktirde \end{cases},$$

olarak bulunur.



Şekil 7. Knot vektörleri $t_0 = 2, t_1 = 2, t_2 = 5, t_3 = 7, t_4 = 8, t_5 = 10, t_6 = 11$ olan birinci dereden taban fonksiyonları

Son olarak, d=2 için

$$N_{0,2}(t) = \frac{t-t_0}{t_2-t_0} N_{0,1}(t) + \frac{t_3-t}{t_3-t_1} N_{1,1}(t) = \frac{t-2}{3} N_{0,1}(t) + \frac{7-t}{3} N_{1,1}(t)$$

$$= \frac{t-2}{3}\left(\frac{t-2}{2} N_{0,0}(t) + (5-t)N_{1,0}(t)\right) + \frac{7-t}{3}\left((t-4)N_{1,0}(t) + \frac{7-t}{2} N_{2,0}(t)\right)$$

$$=\frac{1}{6}(t-2)^2 N_{0,0}(t) + \frac{1}{3}(-2t^2+18t-38)N_{1,0}(t) + \frac{1}{6}(7-t)^2 N_{2,0}(t),$$

$$N_{1,2}(t)=\frac{t-t_1}{t_3-t_1}N_{1,1}(t)+\frac{t_4-t}{t_4-t_2}N_{2,1}(t)=\frac{t-4}{3}N_{1,1}(t)+\frac{8-t}{3}N_{2,1}(t)$$

$$=\frac{t-4}{3}\left((t-4)N_{1,0}(t)+\frac{7-t}{2}N_{2,0}(t)\right)+\frac{8-t}{3}\left(\frac{t-5}{2}N_{2,0}(t)+(8-t)N_{3,0}(t)\right)$$

$$=\tfrac{1}{3}(t-4)^2N_{1,0}(t)+\tfrac{1}{3}(-t^2+12t-34)N_{2,0}(t)+\tfrac{1}{3}(8-t)^2N_{3,0}(t),$$

$$N_{2,2}(t)=\frac{t-t_2}{t_4-t_2}N_{2,1}(t)+\frac{t_5-t}{t_5-t_3}N_{3,1}(t)=\frac{t-5}{3}N_{2,1}(t)+\frac{10-t}{3}N_{3,1}(t)$$

$$=\frac{t-5}{3}\left(\frac{t-5}{2}N_{2,0}(t)+(8-t)N_{3,0}(t)\right)+\frac{10-t}{3}\left((t-7)N_{3,0}(t)+\frac{10-t}{2}N_{4,0}(t)\right)$$

$$=\tfrac{1}{6}(t-5)^2N_{2,0}(t)+\tfrac{1}{3}(-2t^2+30t-110)N_{3,0}(t)+\tfrac{1}{6}(10-t)^2N_{4,0}(t),$$

$$N_{3,2}(t)=\frac{t-t_3}{t_5-t_3}N_{3,1}(t)+\frac{t_6-t}{t_6-t_4}N_{4,1}(t)=\frac{t-7}{3}N_{3,1}(t)+\frac{11-t}{3}N_{4,1}(t)$$

$$=\frac{t-7}{3}\left((t-7)N_{3,0}(t)+\frac{10-t}{2}N_{4,0}(t)\right)+\frac{11-t}{3}\left(\frac{t-8}{2}N_{4,0}(t)+(11-t)N_{5,0}(t)\right)$$

$$=\tfrac{1}{3}(t-7)^2N_{3,0}(t)+\tfrac{1}{3}(-t^2+18t-79)N_{4,0}(t)+\tfrac{1}{3}(11-t)^2N_{5,0}(t).$$

Buna göre d=2 için

$$N_{0,2}(t)=\begin{cases}0, & t<2\\ \frac{1}{6}(t-2)^2, & 2\le t<4\\ \frac{1}{3}(-2t^2+18t-38), & 4\le t<5\\ \frac{1}{6}(7-t)^2, & 5\le t<7\\ 0, & 7\le t\end{cases}$$

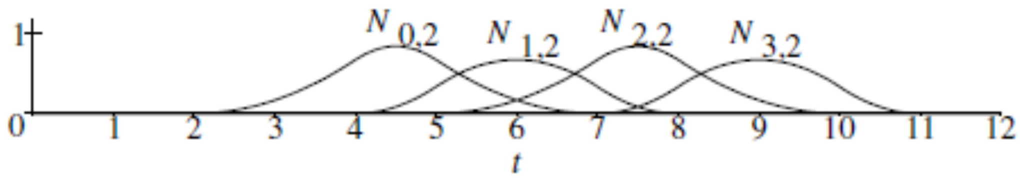
$$N_{1,2}(t)=\begin{cases}0, & t<4\\ \frac{1}{3}(t-4)^2, & 4\le t<5\\ \frac{1}{3}(-t^2+12t-34), & 5\le t<7\\ \frac{1}{3}(8-t)^2, & 7\le t<8\\ 0, & 8\le t\end{cases}$$

$$N_{2,2}(t) = \begin{cases} 0, & t < 5 \\ \frac{1}{6}(t-5)^2, & 5 \le t < 7 \\ \frac{1}{3}(-2t^2 + 30t - 110), & 7 \le t < 8 \\ \frac{1}{6}(10-t)^2, & 8 \le t < 10 \\ 0, & 10 \le t \end{cases}$$

ve

$$N_{3,2}(t) = \begin{cases} 0, & t < 7 \\ \frac{1}{3}(t-7)^2, & 7 \le t < 8 \\ \frac{1}{3}(-t^2 + 18t - 79), & 8 \le t < 10 \\ \frac{1}{3}(11-t)^2, & 10 \le t < 11 \\ 0, & 11 \le t \end{cases}$$

biçiminde ifade edilebilir. 1. ve 2. dereceden olan temel fonksiyonun alanları şekil 1 ve 2 de gösterilmiştir.



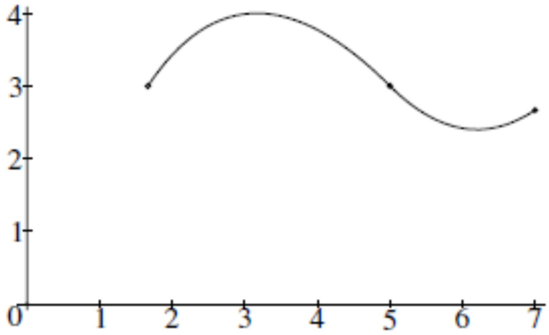
Şekil 2. Knot vektörleri $t_0 = 2, t_1 = 2, t_2 = 5, t_3 = 7, t_4 = 8, t_5 = 10, t_6 = 11$ olan ikinci dereden taban fonksiyonları

Bunların sonucunda istenilen B-spline eğrisi [5,8] aralığında aşağıdaki şekilde tanımlanır:

$$B(t) = (1,2)N_{0,2}(t) + (3,5)N_{1,2}(t) + (6,2)N_{2,2}(t) + (9,4)N_{3,2}(t)$$

$$= \begin{cases} \frac{1}{6}(7-t)^2(1,2) + \frac{1}{3}(-t^2 + 12t - 34)(3,5) + \frac{1}{6}(t-5)^2(6,2), \ eğer\ 5 \le t < 7, \\ \frac{1}{3}(8-t)^2(3,5) + \frac{1}{3}(-2t^2 + 30t - 110)(6,2) + \frac{1}{3}(t-7)^2(9,4), \ eğer\ 7 \le t \le 8. \end{cases}$$

Şekil 3' da gösterilen, B-spline eğrisi, iki polinom eğri bölümünün birleşimidir.



Şekil 3. Örnek 1 de verilen B-spline eğrisi

B-spline eğrilerinde d. Nci dereceden bir B-spline eğrisi için m tane knot vektörü ve n tane de kontrol noktası verildiğinde bunlar arasında m=n+d+1 bağıntısı sağlanmalıdır.

Teorem 1. B-spline taban fonksiyonları $N_{i,k}(t)$ aşağıdaki özellikleri sağlar.

i) **Pozitiflik:** $t \in (t_i, t_{i+k+1})$ için $N_{i,k}(t) > 0$

ii) **Bölgesel Destek:** $t \notin (t_i, t_{i+k+1})$ için $N_{i,k}(t) = 0$

iii) **Parçalı Polinom:** $N_{i,k}(t)$ k. dereceden parçalı polinom fonksiyonlardır.

iv) **Birliğin Bölümü:** $t \in [t_r, t_{r+1})$ için $\sum_{j=r-k}^{r} N_{j,k}(t) = 1$

Teorem 2. Knot vektörleri $t_0, t_1, \dots, t_m$ üzerinde tanımlanmış olan d. dereceden (2) ile verilen bir B-spline eğrisi aşağıdaki özellikleri sağlar.

i) **Bölgesel Kontrol:** Her bölüm $d + 1$ kontrol noktaları ile belirlenir. Yani, $t \in [t_r, t_{r+1})$ $(d \le r \le m - d - 1)$ ise o zaman

$$B(t) = \sum_{i=r-d}^{r} b_i N_{i,d}(t).$$

Böylece $B(t)$'yi değerlendirmek için, $N_{r-d,d}(t), \dots, N_{r,d}(t)$ değerini değerlendirmek yeterlidir.

ii) **Konveks Hull:** $t \in [t_r, t_{r+1})$ $(d \le r \le m - d - 1)$ ise, o zaman

$$B(t) \in CH\{b_{r-d}, \dots, b_r\}.$$

iii) **Afin Dönüşümleri Altındaki Değişmezlik:** T bir afin dönüşüm olsun. O zaman

$$T\left(\sum_{i=0}^{n} b_i N_{i,d}(t)\right) = \sum_{i=0}^{n} T(b_i) N_{i,d}(t).$$

2.1. Açık B-spline Eğrileri

Genel olarak, B-spline eğrileri ilk ve son kontrol noktası b_0 ve b_n'i interpolasyona uğratmaz. d. dereceden eğriler için, son nokta interpolasyonu ve bir son nokta teğet durumu, ancak açık B- spline eğrilerinde geçerlidir. Açık B-spline eğrileri, dış knot vektörleri olarak adlandırılan ilk ve son d tane knot vektörlerinin sırasıyla t_0 ve t_m değerlerine eşit olduğu B-spline eğrileridir. Yani, $t_0 = t_1 = \cdots = t_d$ ve $t_{m-d} = t_{m-d+1} = \cdots = t_m$ dir.

Böylece b_0, b_1 başlangıç teğet yönünü tanımlar ve b_{n-1}, b_n açık bir B-spline eğrisinin son teğet yönünü tanımlar. Son nokta interpolasyonu ve son nokta teğet özellikleri, açık B-spline'larının Bezier eğrilerine benzer şekilde davrandığını gösterir.

3. Temel Sonuçlar: Açık B-Spline Eğrilerinde Frenet Çatısı

Teorem 3. $\mathfrak{B}$, kontrol noktaları $b_0, b_1, \ldots, b_n$ olan ve knot vektörleri $t_0 = t_1 = \cdots = t_d; t_{d+1}, t_{d+2}, \ldots t_{m-d-1}; t_{m-d} = t_{m-d+1} = \cdots = t_m$olacak biçimde d. dereceden bir açık B-spline eğrisi olsun. Bu taktirde

$$\mathfrak{B}(t_d) = b_0 \text{ve} \mathfrak{B}(t_{m-d}) = b_n \text{dir.} \tag{3}$$

İspat: (Marsh, 1999)

Teorem 4. $\mathfrak{B}$, kontrol noktaları $b_0, b_1, \ldots, b_n$ olan ve knot vektörleri $t_0 = t_1 = \cdots = t_d; t_{d+1}, t_{d+2}, \ldots t_{m-d-1}; t_{m-d} = t_{m-d+1} = \cdots = t_m$olacak biçimde d. dereceden bir açık B-spline eğrisi olsun. Bu taktirde

$$\mathfrak{B}'(t_d) = \frac{d}{t_{d+1} - t_1}(b_1 - b_0) \tag{4}$$

$$\mathfrak{B}'(t_{m-d}) = \frac{d}{t_{m-1} - t_{m-d-1}}(b_n - b_{n-1}) \tag{5}$$

dir.(Marsh, 1999)

Sonuç 1. $\mathfrak{B}$, kontrol noktaları $b_0, b_1, \ldots, b_n$ olan ve knot vektörleri $t_0 = t_1 = \cdots = t_d; t_{d+1}, t_{d+2}, \ldots t_{m-d-1}; t_{m-d} = t_{m-d+1} = \cdots = t_m$olacak biçimde d. dereceden bir açık B-spline eğrisi verilsin. Eğer $t_0 = t_1 = \cdots = t_d = 0$ ve $t_{m-d} = t_{m-d+1} = \cdots = t_m = 1$ bu taktirde

$$\mathfrak{B}'(0) = \frac{d}{t_{d+1}}(b_1 - b_0) \tag{6}$$

$$\mathfrak{B}'(1) = \frac{d}{1 - t_{m-d-1}}(b_n - b_{n-1}) \tag{7}$$

dir.

Teorem 5. $\mathfrak{B}$, kontrol noktaları $b_0, b_1, \dots, b_n$ olan ve knot vektörleri $t_0 = t_1 = \cdots = t_d; t_{d+1}, t_{d+2}, \dots t_{m-d-1}; t_{m-d} = t_{m-d+1} = \cdots = t_m$ olacak biçimde d. dereceden bir açık B-spline eğrisi olsun. Bu taktirde

$$\mathfrak{B}''(t_d) = \frac{d(d-1)}{(t_{d+1}-t_2)(t_{d+2}-t_2)}(b_2 - b_1) - \frac{d(d-1)}{(t_{d+1}-t_2)(t_{d+1}-t_1)}(b_1 - b_0) \tag{8}$$

ve

$$\mathfrak{B}''(t_{m-d}) = \frac{d(d-1)}{(t_{m-2}-t_{m-d-1})(t_{m-1}-t_{m-d-1})}(b_n - b_{n-1}) - \frac{d(d-1)}{(t_{m-2}-t_{m-d-1})(t_{m-2}-t_{m-d-2})}(b_{n-1} - b_{n-2}) \tag{9}$$

dir.

Teorem 6. $\mathfrak{B}$, kontrol noktaları $b_0, b_1, \dots, b_n$ olan ve knot vektörleri $t_0 = t_1 = \cdots = t_d; t_{d+1}, t_{d+2}, \dots t_{m-d-1}; t_{m-d} = t_{m-d+1} = \cdots = t_m$olacak biçimde d. dereceden bir açık B-spline eğrisi olsun. Bu taktirde

$$\mathfrak{B}'''(t_d) = \frac{d(d-1)(d-2)(b_3 - b_2)}{(t_{d+1} - t_3)(t_{d+2} - t_3)(t_{d+3} - t_3)} - \frac{d(d-1)(d-2)[t_{d+1} + t_{d+2} - t_2 - t_3](b_2 - b_1)}{(t_{d+1} - t_3)(t_{d+2} - t_2)(t_{d+2} - t_3)(t_{d+1} - t_2)} + \frac{d(d-1)(d-2)(b_1-b_0)}{(t_{d+1}-t_3)(t_{d+1}-t_2)(t_{d+1}-t_1)} \tag{10}$$

ve

$$\mathfrak{B}'''(t_{m-d}) = \frac{d(d-1)(d-2)(b_n - b_{n-1})}{(t_{m-3} - t_{m-d-1})(t_{m-2} - t_{m-d-1})(t_{m-1} - t_{m-d-1})} - \frac{d(d-1)(d-2)[t_{m-3}+t_{m-2}-t_{m-d-2}-t_{m-d-1}](b_{n-1}-b_{n-2})}{(t_{m-3}-t_{m-d-1})(t_{m-2}-t_{m-d-2})(t_{m-2}-t_{m-d-1})(t_{m-3}-t_{m-d-2})} + \frac{d(d-1)(d-2)(b_{n-2}-b_{n-3})}{(t_{m-3}-t_{m-d-1})(t_{m-3}-t_{m-d-2})(t_{m-3}-t_{m-d-3})} \tag{11}$$

dir.

Teorem 7. Kontrol noktaları $b_0, b_1, \dots, b_n$ ve knot vektörleri $t_0 = t_1 = \cdots = t_d; t_{d+1}, t_{d+2}, \dots t_{m-d-1}; t_{m-d} = t_{m-d+1} = \cdots = t_m$ olan d.nci dereceden bir açık B-spline eğrisinin Frenet vektör alanları; t_d noktalarında;

$t = t_d$'de

$$T|_{t=t_d} = \frac{b_1-b_0}{\|b_1-b_0\|}; \qquad B|_{t=t_d} = \frac{(b_1-b_0)\times(b_2-b_1)}{\|(b_1-b_0)\times(b_2-b_1)\|}; \tag{12}$$

$$N|_{t=t_d} = \frac{(b_2-b_1)\csc\Phi}{\|b_2-b_1\|} - \frac{(b_1-b_0)}{\|b_1-b_0\|}\cot\Phi \tag{13}$$

$$\kappa|_{t=t_d} = \frac{(t_{d+1}-t_1)^2(d-1)}{d(t_{d+1}-t_2)(t_{d+2}-t_2)}\frac{\|b_2-b_1\|}{\|b_1-b_0\|^2}\sin\Phi \tag{14}$$

(Φ: $b_1 - b_0$ ile $b_2 - b_1$ arasındaki açıdır).

$$\tau|_{t=t_d} = \frac{(d-2)(t_{d+1}-t_1)(t_{d+1}-t_2)(t_{d+2}-t_2)}{d(t_{d+1}-t_3)(t_{d+2}-t_3)(t_{d+3}-t_3)}\frac{\|b_3-b_2\|\cos\varphi}{\|b_1-b_0\|\|b_2-b_1\|\sin\Phi} \tag{15}$$

Burada φ; $[(b_1 - b_0) \times (b_2 - b_1)]$ vektörü ile $b_3 - b_2$ vektörleri arasındaki açıdır.

Teorem 4.7. Kontrol noktaları $b_0, b_1, \dots, b_n$ ve knot vektörleri

$t_0 = t_1 = \cdots = t_d; t_{d+1}, t_{d+2}, \dots t_{m-d-1}; t_{m-d} = t_{m-d+1} = \cdots = t_m$oland.nci dereceden bir açık B-spline eğrisinin Frenet vektör alanları; $t = t_{m-d}$ noktalarında;

$$T|_{t=t_{m-d}} = \frac{b_n-b_{n-1}}{\|b_n-b_{n-1}\|} \tag{16}$$

$$B|_{t=t_{m-d}} = \frac{-(b_n-b_{n-1})\times(b_{n-1}-b_{n-2})}{\|(b_n-b_{n-1})\times(b_{n-1}-b_{n-2})\|} \tag{17}$$

$$N|_{t=t_{m-d}} = -\frac{b_{n-1}-b_{n-2}}{\|b_{n-1}-b_{n-2}\|}\csc\vartheta + \frac{b_n-b_{n-1}}{\|b_n-b_{n-1}\|}\cot\vartheta \tag{18}$$

dir. ϑ, $(b_{n-1} - b_{n-2})$ ile $(b_n - b_{n-1})$ vektörleri arasındaki açıdır.

$$\kappa|_{t=t_{m-d}} = \frac{d-1}{d}\frac{(t_{m-1}-t_{m-d-1})^2}{(t_{m-2}-t_{m-d-1})(t_{m-2}-t_{m-d-2})}\frac{\|b_{n-1}-b_{n-2}\|\sin\vartheta}{\|b_n-b_{n-1}\|^2} \tag{19}$$

$$\tau|_{t=t_{m-d}} = -\frac{(d-2)}{d}\frac{\|b_{n-2}-b_{n-3}\|\csc\varphi}{\|(b_n-b_{n-1})\times(b_{n-1}-b_{n-2})\|(t_{m-3}-t_{m-d-1})(t_{m-3}-t_{m-d-2})(t_{m-3}-t_{m-d-3})} \tag{20}$$

Burada $\varphi, (b_n - b_{n-1}) \times (b_{n-1} - b_{n-2})$ vektörü ile $(b_{n-2} - b_{n-3})$ vektörleri arasındaki açıdır.

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An investigation of exact traveling wave solutions of the nonlinear partial differential equation arising in plasma physics using two different methods

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Abstract

In this work, a variety of solitary wave solutions of the nonlinear partial differential equation arising in plasma physics with the help of two efficient and reliable methods is investigated. Many explicit wave solutions are found by using the analytical technique. These solutions consisting of trigonometric, hyperbolic and rational functions allow studying the physical properties of underlying model. Moreover, the graphical demonstrations for some of the obtained solutions are given.

1. Introduction

The third-order nonlinear partial differential equation (NLPDE) was introduced in [1] by Gilson and Pickering as

$$\frac{\partial u(x,t)}{\partial t}-\epsilon\frac{\partial^3 u(x,t)}{\partial x^2\partial t}+2k\frac{\partial u(x,t)}{\partial x}-u(x,t)\frac{\partial^3 u(x,t)}{\partial x^3}-\alpha u(x,t)\frac{\partial u(x,t)}{\partial x}-\beta\left(\frac{\partial u(x,t)}{\partial x}\right)\frac{\partial^2 u(x,t)}{\partial x^2}=0, \tag{1}$$

where $\epsilon,\alpha,k,and\beta$ are non-zero real numbers. This equation considered is reduced to equations well known in the literature for specific values of the coefficients. For example;
for $\epsilon=1,\alpha=-3,\beta=2$, Eq. (1) gives the Fuchssteiner-Fokas-Camassa-Holm equation,
for $\epsilon=0,\alpha=1,k=0$, and $\beta=3$, Eq. (1) reduces to the Rosenau-Hyman equation (RH),
for $\epsilon=1,\alpha=-1,k=0.5$, and $\beta=3$, Eq. (1) gives the Fronberg-Whitham (FW).
These equations arise at different levels of approximation in shallow water theory [2,3], in the study of the influence of nonlinear dispersion on the structure of patterns in liquid drops [4]. and analyzing the qualitative characteristics of wave breakage and admits a wave of the highest height [5,6,7], respectively. the Gilson-Pickering equation has attracted the attention of researchers recently. Numerous studies have been done on this equation such as the anstaz method [9], the Bernoulli sub-equation model [11], the G'/G-expansion method to tanh, the coth, cot, and the logical forms under certain conditions [10], the G'/G-expansion method [8], a not a knot meshless method [12], and the symmetry method [13].

2. Description of $exp(-w(\zeta))$-expansion method

In this section, the basic steps of the method to be applied will be given. Let us consider the following partial differential equation

$$Q(u, u_t, u_x, u_{tt}, u_{tt}, \ldots) = 0, \tag{2}$$

where Q is a polynomial in $u(x,t)$and its derivatives in which higher order derivatives and nonlinear terms are involved. In virtue of the traveling wave transformation $u = u(\zeta), \zeta = x - ct$, where c is a constant to be determined later, we can reduce (2) to an ordinary differential equation (ode)

$$P(u, u', u'', \ldots) = 0, \tag{3}$$

where $(.)' = \frac{d}{d\zeta}(.)$. Assume that the solutions of (3) are given as follows

$$u = a_0 + a_1 exp(-w(\zeta)) + \ldots + a_n(exp(-w(\zeta)))^n, \tag{4}$$

where n is determined by balancing principle and $w = w(\zeta)$ satisfies the following ode

$$w'(\zeta) = exp(-w(\zeta)) + \mu exp(w(\zeta)) + \lambda. \tag{5}$$

Equation (5) has the following analytical solutions:

Case 1: $\mu \neq 0, \lambda^2 - \mu > 0$, $w_1(\zeta) = ln\left(\frac{-\lambda - \sqrt{\lambda^2 - 4\mu} tanh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\zeta\right)}{2\mu}\right)$,

Case 2: $\mu \neq 0, \lambda^2 - \mu < 0$,$w_2(\zeta) = ln\left(\frac{-\lambda + \sqrt{-\lambda^2 + 4\mu} tan\left(\frac{\sqrt{-\lambda^2 + 4\mu}}{2}\zeta\right)}{2\mu}\right)$,

Case 3: $\mu \neq 0, \lambda^2 - \mu = 0$,$w_3(\zeta) = ln\left(-\frac{2\lambda\zeta + 4}{\lambda^2\zeta}\right)$,

Case 4: $\lambda \neq 0, \mu = 0$, $w_4(\zeta) = -ln\left(\frac{\lambda}{exp(\lambda\zeta) - 1}\right)$,

Case 5: $\mu = 0, \lambda = 0$, $w_5(\zeta) = ln(\zeta)$.

Here, for simplicity, in $w_i(\zeta)$,$1 \leq i \leq 5$, we have replaced all of $\zeta + \zeta_0$ with ζ since (5) is an autonomous ode. Plugging (4) along with (5) into (3) gives a polynomial in $exp(-w(\zeta))$. Setting each coefficient of this polynomial to zero, a set of algebraic equations in terms of $a_0, a_1, \ldots, a_n, c, \lambda, \mu$ is obtained. With the help of computer programming, the system of algebraic equations can be solved and later substituting these results and analytical solutions $w_1(\zeta),\ldots,w_5(\zeta)$ into (4) give solutions of (3).

3. Description of the modified auxiliary equation (MAE) method

The basics for the MAE method is given in this section. Consider the partial differential equation given in (2) and ordinary differential equation given in (3). To obtain the solution of Eq. (3) via the MAE method, the general solution has the form:

$$u(\zeta) = a_0 + \sum_{j=1}^{N} a_j K^{jf(\zeta)} + \sum_{j=1}^{N} b_j K^{-jf(\zeta)}\,, \tag{6}$$

where a_j, K, b_j are constants to be determined and $f(\zeta)$ satisfy the following auxiliary equation

$$f^{'}(\zeta) = \frac{1}{lnK}\left(AK^{(-f(\zeta))} + B + CK^{(f(\zeta))}\right), \tag{7}$$

where A,B,C are parameters to be determined and $K > 0, K \neq 1$. Here, homogenous balance principle will be used to find the integer N. Plugging Eq. (6) along with Eq. (7) into Eq. (3) and summing all the terms of the same power $K^{jf(\zeta)}$ where $j=-N,\ldots,N$ and equating them to zero, we obtain a system of algebraic equations. This system can be solved to achieve the values of a_j, b_j and A,B,C. Using these values and the solutions of Eq. (6) with Eq. (3), the exact solutions of Eq. (2) can be obtained. The solution of Eq. (7) subject to the couple of cases is given as:

If $B^2 - 4AC < 0$ and $C \neq 0$, then

$$K^{f(\zeta)} = \frac{1}{2}\frac{-B + \sqrt{4AC - B^2}tan\left(\frac{1}{2}\sqrt{4AC - B^2}\zeta\right)}{C}, \tag{8}$$

or

$$K^{f(\zeta)} = -\frac{1}{2}\frac{B + \sqrt{4AC - B^2}cot\left(\frac{1}{2}\sqrt{4AC - B^2}\zeta\right)}{C}. \tag{9}$$

If $B^2 - 4AC > 0$ and $C \neq 0$, then

$$K^{f(\zeta)} = \frac{1}{2}\frac{-B + \sqrt{-4AC + B^2}tanh\left(\frac{1}{2}\sqrt{-4AC + B^2}\zeta\right)}{C} \tag{10}$$

or

$$K^{f(\zeta)} = -\frac{1}{2}\frac{B + \sqrt{-4AC + B^2}coth\left(\frac{1}{2}\sqrt{-4AC + B^2}\zeta\right)}{C}. \tag{11}$$

If $B^2 - 4AC > 0$ and $C \neq 0$, then

$$K^{f(\zeta)} = -\frac{1}{2}\frac{2 + B\zeta}{C\zeta}. \tag{12}$$

4. Application of $exp(-w(\zeta))$-expansion method

Consider the Gilson-Pickering equation

$$\frac{\partial u(x,t)}{\partial t} - \epsilon \frac{\partial^3 u(x,t)}{\partial x^2 \partial t} + 2k\frac{\partial u(x,t)}{\partial x} - u(x,t)\frac{\partial^3 u(x,t)}{\partial x^3} - \alpha u(x,t)\frac{\partial u(x,t)}{\partial x} - \beta\left(\frac{\partial u(x,t)}{\partial x}\right)\frac{\partial^2 u(x,t)}{\partial x^2} = 0, \tag{13}$$

where ϵ, α, k and β are nonzero real numbers. We use $u = V(\zeta), \zeta = x - ht$, we can convert (13) into an ode.

$$(2k - h)V' + \epsilon h V'' - VV' - \beta V'V' - \alpha VV' = 0, \tag{14}$$

where ϵ, α, k and β are nonzero constants and the prime denotes the derivative with respect to ζ. Now integrating (14) and assuming that the integration constant to be zero, we obtain,

$$(2k - h)V + (\epsilon h - V)V'' + \tfrac{1}{2}(1 - \beta)(V')^2 - \tfrac{1}{2}\alpha V^2 = 0. \tag{15}$$

Using homogenous principle, balancing V'' and V^2 we have *2m=m+2, m=2.* So, the trial solution of Eq. (15) can be stated as,

$$V(\zeta) = A + A_1 exp(-w(\zeta)) + A_2(exp(-w(\zeta)))^2, \tag{16}$$

where $A_2 \neq 0, A_1, A$ are constants. Putting V, V', V'', V^2 in (15) and comparing, we get,

$$\begin{aligned}
&-8{A_2}^2 - 4\beta {A_2}^2 = 0,\\
&-4AA_2\mu^2 + 2\epsilon h A_1\mu\lambda - \beta\mu^2{A_1}^2 + 4kA + \mu^2{A_1}^2 - \alpha A^2 - 2hA + 4\epsilon h A_2\mu^2 - 2AA_1\mu\lambda = 0,\\
&8\beta\mu\lambda A_1A_2 - 6AA_1\lambda - 2{A_1}^2\mu - 2\beta\mu{A_1}^2 - 4\beta\mu^2{A_2}^2 + 4kA_2 - \beta\lambda^2{A_1}^2 - 6A_1A_2\mu\lambda + 8\epsilon h A_2\lambda^2\\
&\qquad + 6\epsilon h A_1\lambda - 16AA_2\mu - 8AA_2\lambda^2 + 16\epsilon h A_2\mu - 2\alpha AA_2 - \lambda^2{A_1}^2 - \alpha{A_1}^2 - 2hA_2 = 0,\\
&-8{A_2}^2\mu - 3{A_1}^2 - 12AA_2 - 4{A_2}^2\lambda^2 - 8\beta\lambda A_1A_2 - \alpha{A_2}^2 + 12\epsilon h A_2 - \beta{A_1}^2 - 4\beta\lambda^2{A_2}^2 - 8\beta\mu{A_2}^2 - 18A_1A_2\lambda = 0,\\
&20\epsilon h A_2\lambda - 12A_1A_2\mu - 4{A_2}^2\mu\lambda - 8\beta\mu\lambda{A_2}^2 + 4\epsilon h A_1 - 4AA_1 - 8\beta\mu A_1A_2 - 20AA_2\lambda - 4\beta\lambda^2A_1A_2\\
&-6A_1A_2\lambda^2 - 2\beta\lambda{A_1}^2 - 2\alpha A_1A_2 - 4{A_1}^2\lambda = 0,\\
&-4\beta A_1A_2 - 8\beta\lambda{A_2}^2 - 12A_1A_2 - 12{A_2}^2\lambda = 0,\\
&-4AA_1\mu - 4\beta\mu^2A_1A_2 - 2AA_1\lambda^2 - 2hA_1 + 2\epsilon h A_1\lambda^2 - 12AA_2\mu\lambda - 2\beta\mu\lambda{A_1}^2 + 4kA_1 + 4\epsilon h A_1\mu\\
&-2\alpha AA_1 + 12\epsilon h A_2\mu\lambda = 0.
\end{aligned} \tag{17}$$

By solving the algebraic equations with the help of Maple, we obtain two sets of coefficients for the solutions of (15).

$$\textbf{\textit{Set 1}}: \{\mu = \mu, \alpha = \alpha, \beta = -2, \epsilon = -\frac{(32\mu^2 + 4\alpha\mu - 16\mu\lambda^2 + 2\lambda^4 - \alpha^2 - \alpha\lambda^2)A_1}{12\alpha h\lambda}, A_2 = \frac{A_1}{\lambda},$$

$$A = -\frac{1}{6}\frac{A_1(-2\alpha\mu - \alpha\lambda^2 - 8\mu\lambda^2 + \lambda^4 + 16\mu^2)}{\alpha\lambda}, k = -\frac{1}{24}\frac{16\mu^2 A_1 + 4\alpha\mu A_1 - 8A_1\mu\lambda^2 - \alpha A_1\lambda^2 + A_1\lambda^4 - 12h\lambda}{\lambda}.\}$$

For Set 1, we obtained the desired solutions as

Case 1: $\mu \neq 0, \lambda^2 - \mu > 0$,

$$u_{1.1}(x,t) = -\frac{1}{6}\frac{A_1(-2\alpha\mu - \alpha\lambda^2 - 8\mu\lambda^2 + \lambda^4 + 16\mu^2)}{\alpha\lambda} + 2\frac{A_1\mu}{-\lambda - \sqrt{\lambda^2 - 4\mu}tanh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\zeta\right)} + 4\frac{\mu^2 A_1}{\lambda\left(-\lambda - \sqrt{\lambda^2 - 4\mu}tanh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\zeta\right)\right)^2}$$

where $\zeta = x - ht$.

Case 2: $\mu \neq 0, \lambda^2 - \mu < 0$,

$$u_{1.2}(x,t) = -\frac{1}{6}\frac{A_1(-2\alpha\mu - \alpha\lambda^2 - 8\mu\lambda^2 + \lambda^4 + 16\mu^2)}{\alpha\lambda} + 2\frac{A_1\mu}{-\lambda + \sqrt{-\lambda^2 + 4\mu}tan\left(\frac{1}{2}\sqrt{-\lambda^2 + 4\mu}\zeta\right)} + 4\frac{\mu^2 A_1}{\lambda\left(-\lambda + \sqrt{-\lambda^2 + 4\mu}tan\left(\frac{1}{2}\sqrt{-\lambda^2 + 4\mu}\zeta\right)\right)^2},$$

where $\zeta = x - ht$.

Case 3: $\mu \neq 0, \lambda^2 - \mu = 0$,

$$u_{1.3}(x,t) = \frac{A_1\lambda}{(\lambda\zeta + 2)^2},$$

where $\zeta = x - ht$.

Case 4: $\lambda \neq 0, \mu = 0$,

$$u_{1.4}(x,t) = 1/6\frac{A_1\lambda\left(\alpha e^{2\lambda\zeta} - \lambda^2 e^{2\lambda\zeta} + 4\alpha e^{\lambda\zeta} + 2\lambda^2 e^{\lambda\zeta} + \alpha - \lambda^2\right)}{\alpha(e^{\lambda\zeta} - 1)^2},$$

where $\zeta = x - ht$.

***Set* 2**: $A = 12\dfrac{\epsilon h\mu}{\alpha - \lambda^2 + 4\mu}, A_1 = 12\dfrac{\epsilon h\lambda}{\alpha - \lambda^2 + 4\mu}, A_2 = 12\dfrac{h\epsilon}{\alpha - \lambda^2 + 4\mu}, k = \dfrac{1}{2}h(4\epsilon\mu + 1 - \epsilon\lambda^2), \beta = -2.$

For Set 2, we obtain solutions of (15) as follows:

Case 1: $\mu \neq 0, \lambda^2 - \mu > 0,$

$$u_{2.1}(x,t) = 12\frac{\epsilon h\mu}{\alpha - \lambda^2 + 4\mu} + 24\frac{\epsilon h\lambda\mu}{(\alpha - \lambda^2 + 4\mu)\left(-\lambda - \sqrt{\lambda^2 - 4\mu}tanh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\zeta\right)\right)}$$
$$+ 48\frac{\epsilon h\mu^2}{(\alpha - \lambda^2 + 4\mu)\left(-\lambda - \sqrt{\lambda^2 - 4\mu}tanh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\zeta\right)\right)^2},$$

where $\zeta = x - ht$.

Case 2: $\mu \neq 0, \lambda^2 - \mu < 0,$

$$u_{2.2}(x,t) = 12\frac{\epsilon h\mu}{\alpha - \lambda^2 + 4\mu} + 24\frac{\epsilon h\lambda\mu}{(\alpha - \lambda^2 + 4\mu)\left(-\lambda + \sqrt{-\lambda^2 + 4\mu}tan\left(\frac{1}{2}\sqrt{-\lambda^2 + 4\mu}\zeta\right)\right)}$$
$$+48\frac{\epsilon h\mu^2}{(\alpha - \lambda^2 + 4\mu)\left(-\lambda + \sqrt{-\lambda^2 + 4\mu}tan\left(\frac{1}{2}\sqrt{-\lambda^2 + 4\mu}\zeta\right)\right)^2},$$

where $\zeta = x - ht$.

Case 3: $\mu \neq 0, \lambda^2 - \mu = 0,$

$$u_{2.3}(x,t) = 3\frac{\epsilon h\lambda^2}{\alpha} - 12\frac{\epsilon h\lambda^3\zeta}{\alpha(2\lambda\zeta + 4)} + 12\frac{\epsilon h\lambda^4\zeta^2}{\alpha(2\lambda\zeta + 4)^2},$$

where $\zeta = x - ht$.

Case 4: $\lambda \neq 0, \mu = 0,$

$$u_{2.4}(x,t) = 12\frac{\epsilon h\lambda^2 e^{\lambda\zeta}}{(\alpha-\lambda^2)(e^{\lambda\zeta}-1)^2},$$

where $\zeta = x - ht.$

Case 5: $\mu = 0, \lambda = 0,$

$$u_{2.5}(x,t) = 12\frac{\epsilon h}{\alpha(x-ht)^2}.$$

5. Application of the modified auxiliary equation method

We consider following reduced equation:

$$(2k-h)V + (\epsilon h - V)V'' + \tfrac{1}{2}(1-\beta)(V')^2 - \tfrac{1}{2}\alpha V^2 = 0. \quad (18)$$

Using homogenous principle, balancing V'' and V^2 we have 2N=N+2, N=2. So, the trial solution of Eq. (18) can be stated as,

$$V(\zeta) = a_0 + K^{f(\zeta)}a_1 + K^{2f(\zeta)}a_2 + K^{-f(\zeta)}b_1 + K^{-2f(\zeta)}b_2, \quad (19)$$

Plugging Eq. (19) into Eq. (18) and picking all terms with the same powers of $K^{f(\zeta)}$ and equalizing to zero all the coefficients of $K^{f(\zeta)}$, the following algebraic equations are obtained:

$$-4\beta A^2 b_1 b_2 - 8\beta AB{b_2}^2 - 12{b_2}^2AB - 12b_1b_2A^2 = 0$$
$$-8{b_2}^2AC - \beta A^2{b_1}^2 - 4{b_2}^2B^2 - 3{b_1}^2A^2 - 8\beta AC{b_2}^2 - 4\beta B^2{b_2}^2 - \alpha{b_2}^2$$
$$-18b_1b_2AB - 8\beta ABb_1b_2 - 12a_0b_2A^2 + 12\epsilon hb_2A^2 = 0$$
$$-12{a_2}^2CB - 8\beta BC{a_2}^2 - 4\beta C^2a_1a_2 - 12a_1a_2C^2 = 0$$
⋮

After solving these algebraic equations, the following solutions are obtained:

Case 1:

$$\{A = \frac{a_0C}{a_2}, B = B, C = C, a_0 = a_0, a_1 = \frac{a_2B}{C}, a_2 = a_2, b_1 = 0, b_2 = 0, h = h,$$

$$\alpha = \frac{-8a_0a_2C^2B^2 + {a_2}^2B^4 + 16{a_0}^2C^4 - 24ka_2C^2 + 12a_2C^2h}{(-4a_0C^2 + a_2B^2)a_2}, \beta = -2, \epsilon = \frac{a_2(-2k+h)}{h(-4a_0C^2 + a_2B^2)}.\}$$

If $B^2 - 4AC < 0$ and $C \neq 0$, then

$$V_1(\zeta) = a_0 + \frac{1}{2}\left(-B + \sqrt{4\frac{a_0C^2}{a_2} - B^2}tan\left(\frac{1}{2}\sqrt{4\frac{a_0C^2}{a_2} - B^2}\zeta\right)\right)a_2BC^{-2}$$

$$+\frac{1}{4}\left(-B + \sqrt{4\frac{a_0C^2}{a_2} - B^2}tan\left(1/2\sqrt{4\frac{a_0C^2}{a_2} - B^2}\zeta\right)\right)^2 a_2C^{-2}, \qquad (20)$$

or

$$V_2(\zeta) = a_0 - \frac{1}{2}\left(B + \sqrt{4\frac{a_0C^2}{a_2} - B^2}cot\left(\frac{1}{2}\sqrt{4\frac{a_0C^2}{a_2} - B^2}\zeta\right)\right)a_2BC^{-2} + \frac{1}{4}\left(B + \sqrt{4\frac{a_0C^2}{a_2} - B^2}cot\left(1/2\sqrt{4\frac{a_0C^2}{a_2} - B^2}\zeta\right)\right)^2 a_2C^{-2}. \qquad (21)$$

If $B^2 - 4AC > 0$ and $C \neq 0$, then

$$V_3(\zeta) = a_0 + \frac{1}{2}\left(-B + \sqrt{-4\frac{a_0C^2}{a_2} + B^2}tanh\left(\frac{1}{2}\sqrt{-4\frac{a_0C^2}{a_2} + B^2}\zeta\right)\right)a_2BC^{-2}$$

$$+\frac{1}{4}\left(-B + \sqrt{-4\frac{a_0C^2}{a_2} + B^2}tanh\left(\frac{1}{2}\sqrt{-4\frac{a_0C^2}{a_2} + B^2}\zeta\right)\right)^2 a_2C^{-2}, \qquad (22)$$

or

$$V_4(\zeta) = a_0 - \frac{1}{2}\left(B + \sqrt{-4\frac{a_0C^2}{a_2} + B^2}coth\left(\frac{1}{2}\sqrt{-4\frac{a_0C^2}{a_2} + B^2}\zeta\right)\right)a_2BC^{-2}$$

$$+\frac{1}{4}\left(B + \sqrt{-4\frac{a_0C^2}{a_2} + B^2}coth\left(\frac{1}{2}\sqrt{-4\frac{a_0C^2}{a_2} + B^2}\zeta\right)\right)^2 a_2C^{-2}. \qquad (23)$$

If $B^2 - 4AC = 0$, then

$$V_5(\zeta) = -\frac{-a_0C^2\zeta^2-2a_0C^2\zeta^3\sqrt{AC}-C^3\zeta^4Aa_0-2a_2\sqrt{AC}\zeta+2a_2AC\sqrt{AC}\zeta^3+C^2\zeta^4A^2a_2-a_2}{C^2\zeta^2\left(1+\sqrt{AC}\zeta\right)^2}. \tag{24}$$

Case 2:

$$\{A = \frac{1}{2}\frac{b_1B^3}{2b_1CB+h-2k}, B = B, C = C, a_0 = 2\frac{2b_1CB+h-2k}{B^2}, a_1 = 2\frac{C(2b_1CB+h-2k)}{B^3},$$

$$h = h, \alpha = -\frac{(6b_1CB+h-2k)B^2}{2b_1CB+h-2k}, \beta = -3, \epsilon = \frac{2b_1CB+h-2k}{B^2h}, a_2 = 0, b_1 = b_1, b_2 = 0.\}$$

If $B^2 - 4AC < 0$and $C \neq 0$, then

$$V_6(\zeta) = \left(-B+\sqrt{2\frac{b_1B^3C}{2b_1CB+h-2k}-B^2}tan\left(\frac{1}{2}\sqrt{2\frac{b_1B^3C}{2b_1CB+h-2k}-B^2}\zeta\right)\right)\frac{2b_1CB+h-2k}{B^3}+\frac{2(2b_1CB+h-2k)}{B^2}+$$
$$2b_1C\left(-B+\sqrt{2\frac{b_1B^3C}{2b_1CB+h-2k}-B^2}tan\left(\frac{1}{2}\sqrt{2\frac{b_1B^3C}{2b_1CB+h-2k}-B^2}\zeta\right)\right)^{-1}, \tag{25}$$

or

$$V_7(\zeta) = -\left(B+\sqrt{2\frac{b_1B^3C}{2b_1CB+h-2k}-B^2}cot\left(\frac{1}{2}\sqrt{2\frac{b_1B^3C}{2b_1CB+h-2k}-B^2}\zeta\right)\right)\frac{2b_1CB+h-2k}{B^3}+\frac{2(2b_1CB+h-2k)}{B^2}-$$
$$2b_1C\left(B+\sqrt{2\frac{b_1B^3C}{2b_1CB+h-2k}-B^2}cot\left(\frac{1}{2}\sqrt{2\frac{b_1B^3C}{2b_1CB+h-2k}-B^2}\zeta\right)\right)^{-1}. \tag{26}$$

If $B^2 - 4AC > 0$ and $C \neq 0$, then

$$V_8(\zeta) = \left(-B+\sqrt{-2\frac{b_1B^3C}{2b_1CB+h-2k}+B^2}tanh\left(\frac{1}{2}\sqrt{-2\frac{b_1B^3C}{2b_1CB+h-2k}+B^2}\zeta\right)\right)\frac{2b_1CB+h-2k}{B^3}+\frac{2(2b_1CB+h-2k)}{B^2}+$$
$$2b_1C\left(-B+\sqrt{-2\frac{b_1B^3C}{2b_1CB+h-2k}+B^2}tanh\left(\frac{1}{2}\sqrt{-2\frac{b_1B^3C}{2b_1CB+h-2k}+B^2}\zeta\right)\right)^{-1}, \tag{27}$$

Or

$$V_9(\zeta) = -\left(B+\sqrt{-2\frac{b_1B^3C}{2b_1CB+h-2k}+B^2}coth\left(\frac{1}{2}\sqrt{-2\frac{b_1B^3C}{2b_1CB+h-2k}+B^2}\zeta\right)\right)\frac{2b_1CB+h-2k}{B^3}$$
$$+\frac{2(2b_1CB+h-2k)}{B^2}-2b_1C\left(B+\sqrt{-2\frac{b_1B^3C}{2b_1CB+h-2k}+B^2}coth\left(\frac{1}{2}\sqrt{-2\frac{b_1B^3C}{2b_1CB+h-2k}+B^2}\zeta\right)\right)^{-1}. \tag{28}$$

6. Conclusion

A direct and systematic solution procedure for constructing traveling wave solutions to nonlinear partial differential equations is proposed. The presented method is oriented towards the ease of use and capability of computer algebra systems, allowing us to carry out the involved computations conveniently through powerful computer algebra systems. We construct several new solutions. We have obtained periodic, hyperbolic, rational and exponential solutions, and some of them are physically meaningful. Moreover, we got 3D graphic representations. We hope that the results found will serve as a guide for further studies.

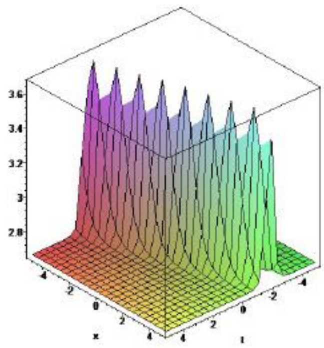

Fig. 1: Profile of $u_{1.1}$ fo,
$A_1 = -2.4, \epsilon = 2, \mu = 1, \alpha = 1, \lambda = 3, h = -3,$

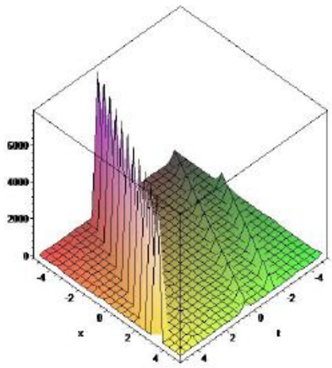

Fig. 2: Profile of $u_{1.2}$ for,
$A_1 = 2, \mu = 1, \alpha = 1, \lambda = 1, h = 2, \epsilon = 5,$

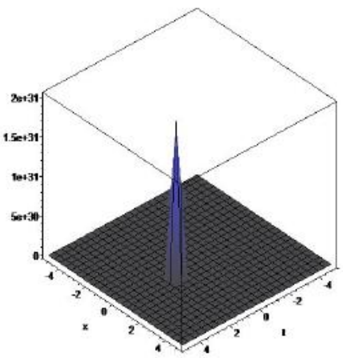

Fig. 3: Profile of $u_{1.3}$ for,
$A_1 = 2, h = 1.1, \lambda = 2,$

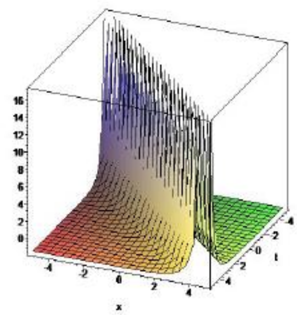

Fig. 4: Profile of $u_{1.4}$ for,
$A_1 = 5, \mu = 0, \alpha = 1, \beta = 4, h = 1, \epsilon = 3, \lambda = 1.5,$

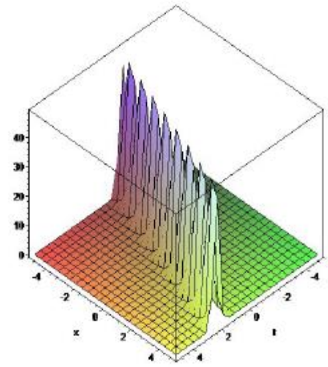

Fig. 5: Profile of $u_{2.1}$ for, $\epsilon = 3, \mu = .1, \alpha = 4, \lambda = 3, h = 3,$

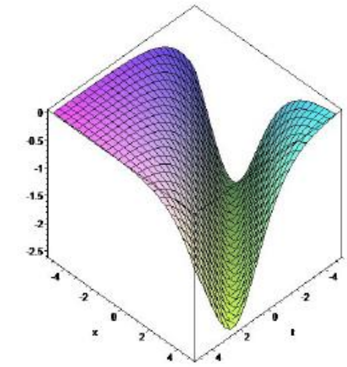

Fig. 6: Profile of $u_{2.2}$ for , $\mu = .1, \alpha = 2, \lambda = -1, h = 1, \epsilon = 2,$

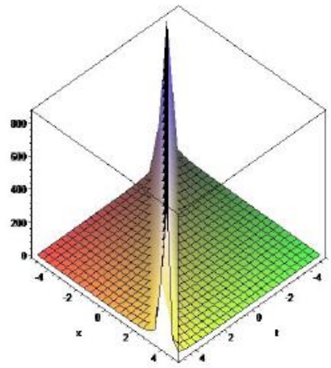

Fig. 7: Profile of $u_{2.3}$ for, $h = 1, \mu = 1, \alpha = 2, \lambda = 2, \epsilon = 4,$

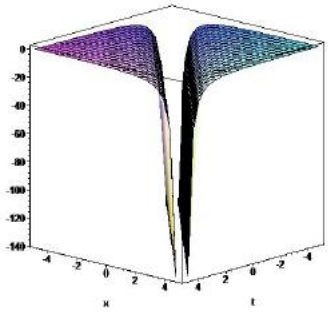

Fig. 8: Profile of $u_{2.4}$ for , $\alpha = .2, \lambda = .5, h = 1, \epsilon = .1, \mu = 0,$

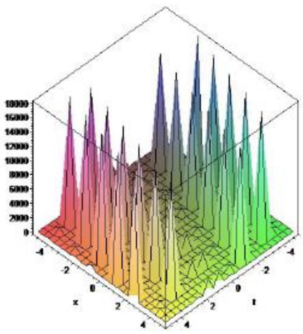

Fig. 9: Profile of V_1 for, $A = 4, B = 1, C = 2,$ $\epsilon = 3, \alpha = 4, a_2 = 5, a_1 = 1, a_0 = 1, h = 4,$

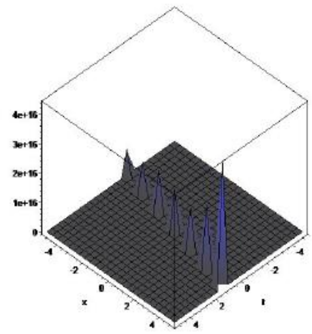

Fig. 10: Profile of V_2 for, $A = 4, B = 1, C = 2,$ $\epsilon = 3, \alpha = 4, a_2 = 5, a_1 = 1, a_0 = 1, h = 4,$

7. References

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Applications of a Pascal distribution series on the certain subclasses of analytic functions

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Abstract

In the present paper, we consider a generalized distribution with the Pascal model defined by

$$P(\mathcal{X}=j)=\binom{j+t-1}{t-1}p^j(1-p)^t, \quad j\in\{0,1,2,3,\dots\}$$

for the analytic function classes $D(\lambda,\alpha)$ and $S^*C(\alpha,\delta;\lambda)$. Furthermore, we derive some conditions for functions in these classes.

Keywords: Analytic functions, univalent functions, the Pascal distribution.

1. Introduction

The Poisson distribution is one of the most well-utilized discrete distributions in multivariate data research fields. However, nowadays, the elementary distributions such as the Poisson, the Pascal, the Logarithmic, the Binomial have been partially studied in the Geometric Function Theory from a theoretical point of view (see [2], [3], [4], [5], [6], [7], [8], [9]).

Let us consider a non-negative discrete random variable $\mathcal{X}$ with a Pascal probability generating function

$$P(\mathcal{X}=j)=\binom{j+t-1}{t-1}p^j(1-p)^t, \quad j\in\{0,1,2,3,\dots\}$$

where p, t are called the parameters.
Let A represent the class of functions f of the form

$$f(z)=z+\sum_{j=2}^{\infty}a_jz^j, \tag{1}$$

which are analytic in the open unit disk

$$\mathbb{U}=\{z: z\in\mathbb{C} \text{ and } |z|<1\}$$

Let S be the subclass of A consisting of functions which are univalent in $\mathbb{U}$.

2. Main Results

We will give some definitions and lemmas from [1] and [10].

2.1 Definition. A function $f\in A$ is said to be in the class $S^*C(\alpha,\delta;\lambda)$ if it satisfies the following condition

$$Re\left\{\frac{zf'(z)+(1+2\lambda)z^2f''(z)+\lambda z^3f'''(z)}{(1-\lambda)f(z)+\lambda zf'(z)}\right\}>\alpha$$

where $0\le\alpha<1$, $0\le\lambda$, $z\in\mathbb{U}$ [1].

2.2 Definition. A function $f\in A$ is said to be in the class $S^*C(\alpha,\delta;\lambda)$ if it satisfies the following condition

$$Re\left\{\frac{zf'(z)+\lambda z^2f''(z)}{\lambda z(f'(z)+\delta zf''(z))+(1-\lambda)(\delta zf'(z)+(1-\delta)f(z))}\right\}>\alpha$$

where $0\le\alpha,\lambda<1$, $0\le\delta\le1$, $z\in\mathbb{U}$ [10].

We recall here the following proved lemmas.

2.3 Lemma. Let $f\in A$ be of the form (1), then $f\in D(\lambda,\alpha)$, if

$$\sum_{j=2}^{\infty}(j\lambda-\lambda+1)(j^2-\alpha)\left|a_j\right|\le1-\alpha. \tag{2}$$

Moreover, Altıntaş proved that

2.4 Lemma. Let $f\in A$ be of the form (1), then $f\in S^*C(\alpha,\delta;\lambda)$, if

$$\sum_{j=2}^{\infty}(j\lambda-\lambda+1)[j-\alpha-(j-1)\alpha\delta]\left|a_j\right|\le1-\alpha. \tag{3}$$

Now, based upon the Pascal distribution, consider the following power series:

$$K(t,p,z)=z+\sum_{j=2}^{\infty}\binom{j+t-2}{t-1}p^{j-1}(1-p)^tz^j \qquad (t\ge1,0\le p\le1,z\in\mathbb{U}). \tag{4}$$

Note that, by using ratio test we conclude that the radius of convergence of the above power series is infinity.

By considering above definition and lemmas, we have the following sufficient conditions fort he function K.

2.5 Theorem. A sufficient condition for the function K given by (4) to be in the class $D(\lambda,\alpha)$ is

$$\frac{\lambda t(t+1)(t+2)p^3}{(1-p)^3}+\frac{(1+5\lambda)t(t+1)p^2}{(1-p)^2}+\frac{(3-\alpha\lambda+4\lambda)tp}{1-p}+(1-\alpha)-(1-\alpha)(1-p)^t$$
$$\le1-\alpha \quad ,0\le p<1 \tag{5}$$

Proof. According to Lemma 2.3, we must show that

$$\sum_{j=2}^{\infty}\binom{j+t-2}{t-1}(j^2-\alpha)\,(1-\lambda+j\lambda)p^{j-1}(1-p)^t\le1-\alpha.$$

Therefore, by combining the relation (4) and the implication (5), we have the equality

$$\sum_{j=2}^{\infty}\binom{j+t-2}{t-1}(j^2-\alpha)\,(1-\lambda+j\lambda)p^{j-1}(1-p)^t$$
$$=\lambda(1-p)^t\sum_{j=2}^{\infty}\binom{j+t-2}{t-1}(j-1)(j-2)(j-3)p^{j-1}$$
$$+(1+5\lambda)(1-p)^t\sum_{j=2}^{\infty}\binom{j+t-2}{t-1}(j-1)(j-2)p^{j-1}$$
$$+(3-\alpha\lambda+4\lambda)(1-p)^t\sum_{j=2}^{\infty}\binom{j+t-2}{t-1}(j-1)p^{j-1}$$
$$+(1-\alpha)(1-p)^t\sum_{j=2}^{\infty}\binom{j+t-2}{t-1}p^{j-1}$$
$$=\lambda(1-p)^t\sum_{j=4}^{\infty}\binom{j+t-2}{t-1}t(t+1)(t+2)p^{j-4}p^3$$
$$+(1+5\lambda)(1-p)^t\sum_{j=3}^{\infty}\binom{j+t-2}{t-1}t(t+1)p^{j-3}p^2$$
$$+(3-\alpha\lambda+4\lambda)(1-p)^t\sum_{j=2}^{\infty}\binom{j+t-2}{t-1}tp^{j-2}p$$
$$+(1-\alpha)(1-p)^t\sum_{j=2}^{\infty}\binom{j+t-2}{t-1}p^{j-1}$$
$$=\lambda t(t+1)(t+2)p^3(1-p)^t\sum_{j=0}^{\infty}\binom{j+t+2}{t+2}p^j$$
$$+(1+5\lambda)t(t+1)p^2(1-p)^t\sum_{j=0}^{\infty}\binom{j+t+1}{t+1}p^j$$
$$+(3-\alpha\lambda+4\lambda)tp(1-p)^t\sum_{j=0}^{\infty}\binom{j+t}{t}p^j$$
$$+(1-\alpha)(1-p)^t\sum_{j=0}^{\infty}\binom{j+t-1}{t-1}p^j-(1-\alpha)(1-p)^t$$
$$=\frac{\lambda t(t+1)(t+2)p^3}{(1-p)^3}+\frac{(1+5\lambda)t(t+1)p^2}{(1-p)^2}+\frac{(3-\alpha\lambda+4\lambda)tp}{1-p}$$
$$+(1-\alpha)-(1-\alpha)(1-p)^t\leq 1-\alpha$$

Thus the proof of Theorem 2.5 is now completed.

2.6 Theorem. A sufficient condition for the function K given by (4) to be in the class $S^*C(\alpha,\delta;\lambda)$ is

$$\frac{\lambda(1-\alpha\delta)t(t+1)p^2}{(1-p)^2}+\frac{(1-\alpha\delta+[2-(1+\delta)\alpha]\lambda)tp}{1-p}+(1-\alpha)-(1-\alpha)(1-p)^t\leq 1-\alpha \quad (6)$$

where $0\leq p<1$.

Proof. According to Lemma 2.4, we must show that

$$\sum_{j=2}^{\infty}\binom{j+t-2}{t-1}(j\lambda-\lambda+1)[j-\alpha-(j-1)\alpha\delta]p^{j-1}(1-p)^t\leq 1-\alpha.$$

From the relation (4) and the implication (6), we get

$$\begin{aligned}
&\sum_{j=2}^{\infty}\binom{j+t-2}{t-1}(j\lambda-\lambda+1)[j-\alpha-(j-1)\alpha\delta]p^{j-1}(1-p)^t\\
&=\lambda(1-\alpha\delta)(1-p)^t\sum_{j=2}^{\infty}\binom{j+t-2}{t-1}(j-1)(j-2)p^{j-1}\\
&\quad+(1-\alpha\delta+[2-(1+\delta)\alpha]\lambda)(1-p)^t\sum_{j=2}^{\infty}\binom{j+t-2}{t-1}(j-1)p^{j-1}\\
&\quad+(1-\alpha)(1-p)^t\sum_{j=2}^{\infty}\binom{j+t-2}{t-1}p^{j-1}\\
&=\lambda(1-\alpha\delta)(1-p)^t\sum_{j=3}^{\infty}\binom{j+t-2}{t+1}t(t+1)p^{j-3}p^2\\
&\quad+(1-\alpha\delta+[2-(1+\delta)\alpha]\lambda)(1-p)^t\sum_{j=2}^{\infty}\binom{j+t-2}{t}tp^{j-2}p\\
&\quad+(1-\alpha)(1-p)^t\sum_{j=2}^{\infty}\binom{j+t-2}{t-1}p^{j-1}\\
&=\lambda(1-\alpha\delta)t(t+1)p^2(1-p)^t\sum_{j=0}^{\infty}\binom{j+t+1}{t+1}p^j\\
&\quad+(1-\alpha\delta+[2-(1+\delta)\alpha]\lambda)tp(1-p)^t\sum_{j=0}^{\infty}\binom{j+t}{t}p^j\\
&\quad+(1-\alpha)(1-p)^t\sum_{j=0}^{\infty}\binom{j+t-1}{t-1}p^j-(1-\alpha)(1-p)^t\\
&=\frac{\lambda(1-\alpha\delta)t(t+1)p^2}{(1-p)^2}+\frac{(1-\alpha\delta+[2-(1+\delta)\alpha]\lambda)tp}{1-p}\\
&\quad+(1-\alpha)-(1-\alpha)(1-p)^t\leq 1-\alpha
\end{aligned}$$

and this completes the proof of Theorem 2.6.

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Applications of Almost Increasing Sequences to Infinite Series

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Abstract

In the present paper, two theorems on $|A, p_n|_k$ summability of infinite series, which generalize the theorems on absolute Riesz summability method, are obtained.

Keywords: Absolute matrix summability, almost increasing sequences, infinite series, Hölder's inequality, Minkowski's inequality.

1. Introduction

A positive sequence (b_n) is said to be almost increasing if there exist a positive increasing sequence (c_n) and two positive constants K and M such that $Kc_n \leq b_n \leq Mc_n$ (see [1]). Obviously, every increasing sequence is almost increasing, but the converse need not be true as can be seen from the example $b_n = ne^{(-1)^n}$. Let $\sum a_n$ be an infinite series with its partial sums (s_n). Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \ as \ n \to \infty, \ (P_{-i} = p_{-i} = 0, \ i \geq 1).$$

Let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of non-zero diagonal entries. Then A defines the sequence-to sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \qquad n = 0,1,...$$

The series $\sum a_n$ is said to be summable $|A, p_n|_k$, $k \geq 1$, if (see [11])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{k-1} | A_n(s) - A_{n-1}(s) |^k < \infty.$$

If we take $a_{nv} = \dfrac{p_v}{P_n}$, then we get $|\bar{N}, p_n|_k$ summability method (see [2]).

Given a normal matrix $A = (a_{nv})$, two lower semimatrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ are defined as follows:

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \ n, v = 0,1,... \tag{1}$$

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \ \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \ n = 1,2,... \tag{2}$$

and

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v = \sum_{v=0}^{n} \bar{a}_{nv} a_v \tag{3}$$

$$\bar{\Delta} A_n(s) = \sum_{v=0}^{n} \hat{a}_{nv} a_v . \tag{4}$$

2. Known Results

In [3], Mazhar has proved the following two theorems dealing with $\left|\bar{N}, p_n\right|_k$ summability factors of infinite series via almost increasing sequences.

Theorem 2.1. Let (X_n) be an almost increasing sequence, and let there be sequences (β_n) and (λ_n) such that

$$\left|\Delta\lambda_n\right| \le \beta_n, \tag{5}$$

$$\beta_n \to 0 \quad as \quad n \to \infty, \tag{6}$$

$$\sum_{n=1}^{\infty} n\left|\Delta\beta_n\right| X_n < \infty, \tag{7}$$

$$|\lambda_n| X_n = O(1) \quad as \quad n \to \infty. \tag{8}$$

If

$$\sum_{v=1}^{n} \frac{\left|s_v\right|^k}{v} = O(X_n) \quad as \quad n \to \infty, \tag{9}$$

$$\sum_{v=1}^{n} \frac{p_v}{P_v}\left|s_v\right|^k = O(X_n) \quad as \quad n \to \infty, \tag{10}$$

then the series $\sum a_n \lambda_n$ is summable $\left|\bar{N}, p_n\right|_k$, $k \ge 1$.

Theorem 2.2. Let (X_n) be an almost increasing sequence, and let the sequences (β_n) and (λ_n) satisfy the conditions (5)-(8) and (10). If the conditions

$$\sum_{n=1}^{\infty} P_n\left|\Delta\beta_n\right| X_n < \infty, \tag{11}$$

$$\sum_{n=1}^{m} \frac{\left|s_n\right|^k}{P_n} = O(X_m) \quad as \quad m \to \infty \tag{12}$$

are satisfied, then the series $\sum a_n \lambda_n$ is summable $\left|\bar{N}, p_n\right|_k$, $k \ge 1$.

3. Main Results

One can find some different applications dealing with absolute matrix summability of infinite series (see [4-10]). The aim of this paper is to generalize Theorem 2.1 and Theorem 2.2 for $|A, p_n|_k$ summability method.

Theorem 3.1. Let $A = (a_{nv})$ be a positive normal matrix such that

$$\overline{a}_{n0} = 1, \quad n = 0,1,..., \tag{13}$$

$$a_{n-1,v} \geq a_{nv} \text{ for } n \geq v+1, \tag{14}$$

$$a_{nn} = O\left(\frac{p_n}{P_n}\right). \tag{15}$$

If all conditions of Theorem 2.1 are satisfied, then the series $\sum a_n \lambda_n$ is summable $|A, p_n|_k$, $k \geq 1$.

Theorem 3.2. Let $A = (a_{nv})$ be a positive normal matrix as in Theorem 3.1. If all conditions of Theorem 2.2 are satisfied, then the series $\sum a_n \lambda_n$ is summable $|A, p_n|_k$, $k \geq 1$.

We need the following lemmas for the proofs of Theorem 3.1 and Theorem 3.2.

Lemma 3.1 ([3]). Let (X_n) be an almost increasing sequence. If conditions (6) and (7) are satisfied, then

$$nX_n \beta_n = O(1) \quad as \quad n \to \infty, \tag{16}$$

$$\sum_{n=1}^{\infty} X_n \beta_n < \infty. \tag{17}$$

Lemma 3.2 ([3]). Let (X_n) be an almost increasing sequence. If conditions (6) and (11) are satisfied, then

$$P_n X_n \beta_n = O(1) \quad as \quad n \to \infty, \tag{18}$$

$$\sum_{n=1}^{\infty} p_n X_n \beta_n < \infty. \tag{19}$$

4. Proof of Theorem 3.1

Let (Υ_n) denotes A-transform of the series $\sum a_n\lambda_n$. Then, by (3) and (4), we have $\bar{\Delta}\Upsilon_n = \sum_{v=1}^{n} \hat{a}_{nv}\lambda_v a_v$. By Abel's transformation, we have

$$\bar{\Delta}\Upsilon_n = \sum_{v=1}^{n-1} \Delta_v\left(\hat{a}_{nv}\lambda_v\right) \sum_{k=1}^{v} a_k + \hat{a}_{nn}\lambda_n \sum_{v=1}^{n} a_v$$

$$= \sum_{v=1}^{n-1} (\hat{a}_{nv}\lambda_v - \hat{a}_{n,v+1}\lambda_{v+1})\, s_v + \hat{a}_{nn}\lambda_n s_n$$

$$= \sum_{v=1}^{n-1} (\hat{a}_{nv}\lambda_v - \hat{a}_{n,v+1}\lambda_{v+1} - \hat{a}_{n,v+1}\lambda_v + \hat{a}_{n,v+1}\lambda_v)\, s_v + a_{nn}\lambda_n s_n$$

$$= \sum_{v=1}^{n-1} \Delta_v\left(\hat{a}_{nv}\right)\lambda_v s_v + \sum_{v=1}^{n-1} \hat{a}_{n,v+1}\Delta\lambda_v s_v + a_{nn}\lambda_n s_n = \Upsilon_{n,1} + \Upsilon_{n,2} + \Upsilon_{n,3}.$$

To prove Theorem 3.1, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} \left|\Upsilon_{n,r}\right|^k < \infty \quad for \quad r = 1,2,3.$$

First, applying Hölder's inequality with indices k and k', where $k > 1$ and $\frac{1}{k} + \frac{1}{k'} = 1,$ we get

$$\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} \left|\Upsilon_{n,1}\right|^k \le \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} \left(\sum_{v=1}^{n-1} \left|\Delta_v(\hat{a}_{nv})\right|\left|\lambda_v\right|\left|s_v\right|\right)^k$$

$$\le \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} \left(\sum_{v=1}^{n-1} \left|\Delta_v(\hat{a}_{nv})\right|\left|\lambda_v\right|^k \left|s_v\right|^k\right)\left(\sum_{v=1}^{n-1} \left|\Delta_v(\hat{a}_{nv})\right|\right)^{k-1}.$$

Here

$$\Delta_v(\hat{a}_{nv}) = \hat{a}_{nv} - \hat{a}_{n,v+1} = \bar{a}_{nv} - \bar{a}_{n-1,v} - \bar{a}_{n,v+1} + \bar{a}_{n-1,v+1} = a_{nv} - a_{n-1,v} \tag{20}$$

by (2) and (1). Then,

$$\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| = \sum_{v=1}^{n-1} (a_{n-1,v} - a_{nv}) = \sum_{v=0}^{n-1} a_{n-1,v} - a_{n-1,0} - \sum_{v=0}^{n} a_{nv} + a_{n0} + a_{nn} = \bar{a}_{n-1,0} - a_{n-1,0} - \bar{a}_{n0} + a_{n0} + a_{nn} \le a_{nn}$$

by (14), (1) and (13). Thus, we have

$$\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} \left|\Upsilon_{n,1}\right|^k = O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n} a_{nn}\right)^{k-1} \left(\sum_{v=1}^{n-1} \left|\Delta_v(\hat{a}_{nv})\right|\left|\lambda_v\right|^k \left|s_v\right|^k\right) = O(1) \sum_{v=1}^{m} \left|\lambda_v\right|^k \left|s_v\right|^k \sum_{n=v+1}^{m+1} \left|\Delta_v(\hat{a}_{nv})\right|.$$

Now, by (20) and (14), we get $\sum_{n=v+1}^{m+1} |\Delta_v(\hat{a}_{nv})| \leq a_{vv}$. Then, using (15) and (8), we have

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1} |\Upsilon_{n,1}|^k = O(1)\sum_{v=1}^{m}\frac{p_v}{P_v}|\lambda_v||\lambda_v|^{k-1}|s_v|^k = O(1)\sum_{v=1}^{m}\frac{p_v}{P_v}|\lambda_v||s_v|^k .$$

By using Abel's transformation, and using the conditions (5), (10), (17) and (8), we get

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1} |\Upsilon_{n,1}|^k = O(1)\sum_{v=1}^{m-1}\Delta|\lambda_v|\sum_{r=1}^{v}\frac{p_r}{P_r}|s_r|^k + O(1)|\lambda_m|\sum_{r=1}^{m}\frac{p_r}{P_r}|s_r|^k$$

$$= O(1)\sum_{v=1}^{m-1}\beta_v X_v + O(1)|\lambda_m|X_m = O(1) \quad as \quad m\to\infty.$$

Again, applying Hölder's inequality with the same indices as those above, we get

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1} |\Upsilon_{n,2}|^k \leq \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left(\sum_{v=1}^{n-1}|\Delta\lambda_v||\hat{a}_{n,v+1}||s_v|\right)^k$$

$$\leq \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left(\sum_{v=1}^{n-1}\beta_v|\hat{a}_{n,v+1}||s_v|^k\right)\left(\sum_{v=1}^{n-1}\beta_v|\hat{a}_{n,v+1}|\right)^{k-1}.$$

By using the conditions (2), (1), (14), we have

$$|\hat{a}_{n,v+1}| = \bar{a}_{n,v+1} - \bar{a}_{n-1,v+1} = \sum_{i=v+1}^{n} a_{ni} - \sum_{i=v+1}^{n-1} a_{n-1,i} = a_{nn} + \sum_{i=v+1}^{n-1}(a_{ni} - a_{n-1,i}) \leq a_{nn}.$$

Thus, using (15) and (17), we have

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1} |\Upsilon_{n,2}|^k = O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}a_{nn}\right)^{k-1}\left(\sum_{v=1}^{n-1}\beta_v|\hat{a}_{n,v+1}||s_v|^k\right)\left(\sum_{v=1}^{n-1}\beta_v\right)^{k-1}$$

$$= O(1)\sum_{v=1}^{m}\beta_v|s_v|^k\sum_{n=v+1}^{m+1}|\hat{a}_{n,v+1}|.$$

By (2), (1), (13) and (14), we obtain

$$|\hat{a}_{n,v+1}| = \sum_{i=0}^{v}(a_{n-1,i} - a_{ni}).$$

Thus, using (1) and (13), we have

$$\sum_{n=v+1}^{m+1}|\hat{a}_{n,v+1}| = \sum_{n=v+1}^{m+1}\sum_{i=0}^{v}(a_{n-1,i} - a_{ni}) \leq 1.$$

Then, we get

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1} |\Upsilon_{n,2}|^k = O(1)\sum_{v=1}^{m}\beta_v|s_v|^k = O(1)\sum_{v=1}^{m}(v\beta_v)\frac{|s_v|^k}{v}.$$

Here, using Abel's transformation, and using the conditions (9), (7), (17) and (16), we obtain

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1} |\Upsilon_{n,2}|^k = O(1)\sum_{v=1}^{m-1}\Delta(v\beta_v)\sum_{r=1}^{v}\frac{|s_r|^k}{r} + O(1)m\beta_m\sum_{r=1}^{m}\frac{|s_r|^k}{r}$$

$$= O(1)\sum_{v=1}^{m-1} v\,|\Delta\beta_v|\, X_v + O(1)\sum_{v=1}^{m-1}\beta_{v+1}X_{v+1} + O(1)m\beta_m X_m = O(1) \quad as \quad m\to\infty.$$

Finally, using (15) and (8), we have

$$\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1} \left|\Upsilon_{n,3}\right|^k = \sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1} a_{nn}^k\,|\lambda_n|^k |s_n|^k = O(1)\sum_{n=1}^{m}\frac{p_n}{P_n}|\lambda_n||s_n|^k .$$

Then, as in $\Upsilon_{n,1}$, we have

$$\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1} \left|\Upsilon_{n,3}\right|^k = O(1) \quad as \quad m\to\infty.$$

This completes the proof of Theorem 3.1.

5. Proof of Theorem 3.2

Since the proof of Theorem 3.2 for $r=1$ and $r=3$ is the same, let us prove

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1} \left|\Upsilon_{n,r}\right|^k < \infty \quad for \quad r=2.$$

As in the proof of Theorem 3.1, we obtain

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1} \left|\Upsilon_{n,2}\right|^k = O(1)\sum_{v=1}^{m}\beta_v\,|s_v|^k .$$

Then,

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1} \left|\Upsilon_{n,2}\right|^k = O(1)\sum_{v=1}^{m}P_v\beta_v\frac{|s_v|^k}{P_v}$$

$$= O(1)\sum_{v=1}^{m-1}\Delta(P_v\beta_v)\sum_{r=1}^{v}\frac{|s_r|^k}{P_r} + O(1)P_m\beta_m\sum_{r=1}^{m}\frac{|s_r|^k}{P_r}$$

$$= O(1)\sum_{v=1}^{m-1}P_v\,|\Delta\beta_v|X_v + O(1)\sum_{v=1}^{m-1}p_{v+1}\beta_{v+1}X_{v+1} + O(1)P_m\beta_m X_m = O(1) \quad as \quad m\to\infty,$$

by using (12), (11), (19) and (18).

If we take $a_{nv} = \dfrac{p_v}{P_n}$ in Theorem 3.1 and Theorem 3.2, then we get Theorem 2.1 and Theorem 2.2.

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Arithmetic Development in Problem-Solving Among Primary and Secondary School Age Children

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Abstract

Basic arithmetics skills, which are necessary for mathematical problem-solving, are expected to develop faster after starting school. These basic skills help to improve the level of mathematical operations during the classes. When this development is insufficient, there may occur learning difficulties in mathematics or dyscalculia. This research has been conducted to examine these basic calculating skills in problem situations from a developmental perspective. The Cognitive Developmental aRithmetics (CDR) tests, was used for this aim. These tests have 3 different levels based on the grade level. In this research, the problems section in the tests was compared in terms of different grade levels. For this purpose, the CDR1 test was applied to the 2nd and 3rd grade, the CDR2 test to the 4th and 5th grade, and the CDR3 test to the 6th and 7th-grade levels. There are 1203 participants in total. In the statistical analysis, a significant difference was found regarding the classes in the CDR1 test concerning the development of arithmetics in short problem situations. There is not any significant difference in other grade levels for CDR2 and CDR3.

Keywords: arithmetic development, problem-solving, mathematics, calculating skills

1. Introduction

It is important to plan activities that form the basis of mathematical thinking and have a very important effect on the development of mathematics success and arithmetic skills in the following years, including problem situations in which counting will be used [1]. In order to develop calculating skills, starting from the preschool period, it is aimed to develop skills such as verbal counting, recognition, mental calculation, and problem-solving [2]. Students' meaningful learning is closely related to their ability to apply knowledge in different environments, to establish relationships between concepts, to associate conceptual and operational knowledge, to establish relationships between learning areas, and to transform knowledge into various forms of representation [3]. In cases where meaningful learning is not achieved, students cannot go beyond the patterns they memorize. Soylu and Soylu [4] attributed the students to fail in problem-solving requiring operational and conceptual knowledge and learning difficulties which has the main reason for the students' inability to learn meaningfully the concepts of addition, subtraction, and multiplication with learning processes based on rote. Knowing that numbers represent multiplicities and deciding what to do accordingly, students become able to do basic operations consciously by controlling certain dimensions of their thinking processes. In the study conducted by Olkun et al. [1] on the acquisition of counting principles starting in pre-school, the development of students' skills related to the concept of numbers was examined and it was found that all skills developed with age. For example; Okamato et al.[6], who developed the scale that measures the size comparison between numbers and calculating skills of kindergarten students, applied this scale to children at the end of the first grade, and found the correlation between the tests strong and significant. Various studies have determined that children who do not develop the concept of the number will have difficulties in certain areas of mathematics (arithmetic operations, flexible mental calculations, prediction skills, etc.) and their academic success will be significantly affected [7]. Because students who lack basic calculating skills cannot be successful problem solvers, and those who fail to solve problems cannot be successful problem-makers [4]. Considering that the basis of problem-solving is basic calculating skills and calculation ability, it is necessary to determine in advance the slowness or difficulties in the development of these skills. Calculation difficulty in arithmetics has been found to be the result of a neurological condition that is either innate or subsequently developed in the brain by researchers [8]. It would be wrong to expect students with these situations to be at the same level as their peers in mathematics class. The difficulty or deficiency in calculating skills is also a sign of mathematics learning difficulties [9]. The results in the achievement and intelligence tests are not sufficient to conclude that the student has difficulty in mathematics [10]. Therefore, this study was carried out to determine the developmental differences in problem-solving that require the use of four basic operations and at the same time to determine whether problem-solving were made whether consciously or unconsciously. For this aim, The Cognitive Developmental aRithmetics test (CDR) [11] which is developed for the assessment of arithmetics, was used.

Materials and Participants

A total of 1203 students from different public schools participated in this study. There are 334 students from the 2nd and 3rd grades, 411 from the 4th and 5th grades, and 458 from the 6th and 7th grades. The tests applied at three different levels based on the grades.

In the CDR tests, it is aimed to measure the use of four basic operations (addition, subtraction, multiplication, division) and cognitive sub-skills related to them. In the problems section of CDR tests, there are 2 parts and 20 questions in total. In the first part of the problems (P1), the students were asked 10 questions such that they could reach a result by doing only one of the four basic operations. In the second part of the problems (P2), the students were given extra information and 10 problem situations were asked to measure their attention whether they did the problem consciously or unconsciously.

Results

P1 Section

CDR1 test, P1 part's t-test results are shown in Table1 ($t = 3.27$ and $P < .05$). A statistically significant difference was found between the mean at the second grade ($M = 5.71$ $SD = 2.98$) and the third grade level ($M = 6.75$ $SD = 3.35$). Third graders were 10% more successful than the second graders regarding the mean.

Table 1

t-test results of the CDR1 test in P1

	N	X	SS	Sd	t	p
2. grade	162	5.71	2.98	332	3.27	.001
3. grade	172	6.75	3.35			

According to Table2 in which P1 part's t-test results ($t = 0.65$ $p > .05$) of the CDR2 test, are given. There is not any significant differences between the fourth grade ($M = 6.63$ $SD = 2.37$) and the fifth grade levels results ($M = 6.80$ $SD = 2.82$).

Table2

t-test results of the CDR2 Test in P1

	N	X	SS	Sd	t	p
4. grade	222	6.63	2.37	409	.65	.51
5. grade	189	6.80	2.82			

P1 t-test results (t = .20 p>.05) of the CDR3 test are given at the Table3 and according to the results there is not any significant difference between two grades regarding the mean.

Table 3

t-test results of the CDR2 Test in P1

	N	X	SS	Sd	t	p
6. grade	234	6.85	2.34	456	.20	.83
7. grade	224	6.89	2.55			

Problems P2 Section

According to Table4 just like in the P1 t-test results, P2 t-test (t = 0.20; p> .05) results have a significant difference between the mean of the second grades (M = 5.71 SD = 2.98) and the third grades (M= 6.75; SD = 3.35). The percentage of correct answers in both grades is 57% in the second grade and 67% in the third grade.

Table 4

t-test results of the CDR1 Test in P2

	N	X	SS	Sd	t	p
2. grade	162	5.71	2.98	332	3.27	.001
3. grade	172	6.75	3.35			

In the Table 5 in which the P2 part t-test results (t = 0.67 ; p> .05) are given, no statistically significant difference was found between the fourth (M = 4.65; SD = 3.03) and the fifth grades (M = 4.77; SD =2.97) regarding the means.

Table 5

t-test results of the CDR2 Test in P2

	N	X	SS	Sd	t	p
4. grade	222	4.65	3.03	409	.67	.41
5. grade	189	4.77	2.97			

The t-test results (t = 1.63; p> .05) for the CDR3 test P2 is presented in Table 6 and there was not found any statistically significant difference between the sixth grade (M = 4.72 and SD = 2.70) and the seventh grade results (M= 5.14 and SD = 2.87).

Table 6

t-test Results of the CDR3 Test in P2

	N	X	SS	Sd	t	p
6. grade	234	4.72	2.70	456	1.63	.10
7. grade	224	5.14	2.87			

Conclusion

Basic calculating skills are very important for the development of arithmetic success during academic life and so on. This development's first steps start at a primary school. Based on the results in this research, at 2nd and 3rd-grade students have significant differences regarding the mean. One of the reasons for this result can be the slower arithmetic development in the early grades compared to others. The results of the other grades demonstrate there are not any significant differences regarding the means. Therefore, it can be said that the development of calculating skills becomes faster after learning the basic four operations (addition, subtraction, division, and multiplication) in primary school. Based on the results in this research, the P1 parts, ' mean was significantly higher for all students, but this means decreased significantly in problem P2 where extra information was given. Most of the students used the extra information in the solution, and this situation affected the result of the total mean. This indicates that meaningfully learning, and the ability to use arithmetic to consciously solving problems is weaker just as the Floyd et al.'s [5] research; where the students performed more memorization on basic calculating skills, and in problem situations, there were no significant differences between a primary school student and a secondary school student regarding strategies used. Considering the differences in the mean between the grades, it indicates there is a significant difference between the 2nd and 3rd grades of the primary school where arithmetic is started and that these skills are developing. At other grade levels, it is seen that the mean of two grades in which a test is applied are close to each other. Accordingly, these comparisons are useful at the individual level, concerning showing students who are behind their peers or their grade level in arithmetic terms. It cannot be expected for students with these situations to have the same level of success in math as their peers. In cases where calculation skills cannot be provided, problems will arise especially in mental or estimation calculations. Such inadequacies and problem situations are considered under the name of calculation difficulties. The difficulty or deficiency in calculating skills is also a sign of mathematics learning difficulties (Jordan, 2007). Therefore, these tests which are applied at three different levels were important to identify students who have calculating difficulties regarding problem-solving or who are cognitively behind their peers by solving short problems that require basic four operations.

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ASYMPTOTIC FORMULA OF EIGENVALUES AND EIGENFUNCTIONS OF A BOUNDARY VALUE PROBLEM

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Abstract

In this work, we consider a Sturm-Liouville problem with spectral parameter on boundary condition. We investigate spectral properties of eigenvalue and eigenfuntion for the problem where the potential *q(x)* is a real valued function. With the help of the properties, asymptotic formula is obtained for the problem.

Keywords: Eigenvalue, eigenfunction, spectral parameter, asymptotic formula.

1. Introduction

We consider the following boundary value problem with a spectral parameter in the equation and the boundary conditions:

$$-u'' + q(x)u = \lambda^2 u, \qquad 0 < x < 1, \tag{1}$$

$$(\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2)u(0) + u'(0) = 0, \tag{2}$$

$$(\beta_0 + \beta_1\lambda)u(1) + u'(1) = 0. \tag{3}$$

Here λ is spectral parameter, $q(x) \geq 0$ C [0,1] is a real valued function, α_i and β_i are real constants $(i = 0,1,2)$ and satisfy $\alpha_2 > 0$, $\alpha_0 < 0$, $\beta_0 > 0$, $\beta_1 < 0$.

In recent years, in many fields of physical interpretation of mathematics, Sturm-Liouville problems involving spectral parameters at the boundary have been encountered. The spectral properties of the Sturm-Liouville equation generated by the boundary condition are investigated by many researchers. These types of the problems were studied in [1-13]. The boundary value problem under boundary conditions that do not include the spectral parameter in the linear state is discussed in [11]. However, unlike this thesis, no conditions are placed in α_1 and β_1 constants.

Problems with similar situations were studied in [3], [4], [15]. However, there is an essential difference between the problem. Namely, $\alpha_1 = 0$ and $\beta_1 = 0$ are accepted by authors. Thus, the boundary conditions are transformed into linear form. Different from other works, the boundary conditions include λ and λ^2 in this paper. Therefore, it provides the opportunity to comment on the second order operators. A.G. Kostyuchenko and A.A. Shkalikov's work is an example of this [14].

2. Some Properties of the Eigenvalues and Eigenfunctions of the Boundary Value Problems (1)-(3)

In this study, we examined the spectral properties of the eigenvalues of the (1) -(3) boundary value problem. Important main results for the eigenvalues of the problem which is the basic structure of this study are obtained as follows:

Lemma 1. All eigenvalues of the boundary value problem (1)-(3) are real.

Proof. Let $u(x,\lambda)$ be an eigenfunction of the boundary problem (1)-(3) which correspond to the eigenvalue λ. Multiply both sides of (1)-(3) by $\overline{u(x,\lambda)}$ and integrate the result with respect to x from 0 to 1 and using the boundary conditions (2)-(3), the conclusion gives us quadratic equation depends on λ . From the existence of real roots, Lemma is proved.

The other lemma is obtained as a result of the Lemma 1.

Lemma 2. All eigenvalues of the boundary value problem (1)-(3) are simple.

The following oscillatory theorem proves existence of a countable set of eigenvalues of the boundary value problem (1)-(3).

Lemma 3. Suppose that *u(x)* is a solution of the equation

$$u'' + g(x)u = 0, \tag{4}$$

with the initial value problem

$$u(0) = 1, \quad u'(0) = -\alpha_0 - \alpha_1\lambda_1 - \alpha_2{\lambda_1}^2, \tag{5}$$

and

$$v'' + h(x)v = 0, \tag{6}$$

$$v(0) = 1, \quad v'(0) = -\alpha_0 - \alpha_1\lambda_2 - \alpha_2{\lambda_2}^2, \tag{7}$$

v(x) is a solution to the initial value problem (6)-(7). Moreover, assume that *g(x)<h(x)* and *0<x<1.*

If it is $\lambda_2 > \lambda_1 > \max\{0, -\frac{\alpha_1}{2\alpha_2}\}$ or $\lambda_2 < \lambda_1 < \min\{0, -\frac{\alpha_1}{2\alpha_2}\}$, then when *u(x)* has m zeros then *v(x)* has at least m zeros in in this interval. Additionally *kth* zero of *v(x)* is less than *kth* zero of *u(x)*.

Theorem1. (Oscillation Theorem) There are an unboundedly decreasing sequence of negative eigenvalues $\{\lambda_{-n}\}_{n=1}^{\infty}$ and an unboundedly increasing sequence of positive eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$

$$\ldots\lambda_{-n} < \lambda_{-n+1} \ldots < \lambda_{-2} < \lambda_{-1} < \lambda_1 < \lambda_2 < \cdots < \lambda_{n-1} < \lambda_n \ldots$$

$$\ldots\lambda_{-n} < \lambda_{-n+1} \ldots < \lambda_{-2} < \lambda_{-1} < \lambda_1 < \lambda_2 < \cdots < \lambda_{n-1} < \lambda_n \ldots$$

Moreover, there exist numbers $n_* \in N$, $n^* \in N$ and $k_*, k^* \in N \cup \{0\}$. Then, for eigenvalues λ_{-n} ($n \geq n_*$) and λ_n ($n \geq n^*$) have zeros respectively ($n+k_*-n_*$) and ($n+k^*-n^*$) in (0,1).

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BAYESIAN TWO-STAGE DESIGNS WITH FREQUENTIST TEST IN EXPERIMENTAL TRIALS

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Abstract

Prediction provides discipline and pragmatic importance to empirical research. The design with the predictive probability approach provides an excellent alternative for conducting multi-stage phase II trials; it is efficient and flexible and possesses desirable statistical properties. . In this paper we consider the Bayesian predictive procedures within the experimental design, for this, we define indices of satisfaction related to a test as a decreasing function of the p-value and satisfaction is higher than the null hypothesis is rejected wider. This design possesses good frequentist properties and allows early termination of the trial. We treated our applications in experimental planning and sequential designs with binary outcomes.

Keywords: Bayesian approach, Prediction, Binomial model, interim monitoring

1. Introduction

A major strength of the Bayesian paradigm is the ease with which one can make predictions about future observations. The predictive idea is central in experimental investigations. Furthermore Bayesian predictive probabilities are efficient tools for designing and monitoring experiments. Bayesian predictive procedures give the applied researcher a very appealing method to evaluate the chances that an experiment will end up showing a conclusive result, or on the contrary a non-conclusive result. These procedures are far more intuitive and much closer to the thinking of scientists than frequentist procedures [1].

Experimental (or interventional) studies, as the name implies, these are studies in which the participants undergo some kind of intervention in order to evaluate its impact. An intervention could include a medical or surgical intervention, a new drug, or an intervention to change lifestyle. Because they are the most methodologically rigorous design, experiments are the default choice for providing evidence for best practice in patient management, so this discussion will begin with them. The experimental researcher has control over the intervention, its timing, and dose or intensity. In its simplest form, an experimental study to test the effect of a treatment requires that the researcher formally states the hypothesis to be tested where he must selects people eligible for the treatment, in which the sample is divided into two groups and each group (the experimental, or intervention group) is given the intervention while the other (the control group) is not, and the outcomes of interest are recorded over time, and the results compared between the two groups [2].

Prediction models will become more relevant in the medical field with the increase in knowledge on potential predictors of outcome, e.g. from genetics. Also, the number of applications will increase, e.g. with targeted early detection of disease, and individualized approaches to diagnostic testing and treatment. The current era of evidence-based medicine asks for an individualized approach to medical decision-making. Evidence-based medicine has a central place for meta-analysis to summarize results from randomized controlled trials; similarly prediction models may summarize the effects of predictors to provide individualized predictions of a diagnostic or prognostic outcome [3].

In this paper we consider the Bayesian predictive procedures within the experimental design, for this, we define a hypothesis is defined indices of satisfaction and anticipation of satisfaction related to a test as a decreasing function of the p-value, satisfaction is higher than the null hypothesis is rejected wider, that is to say, the p-value is small. We consider the case of a two-step procedure, which is often done in the case of experimental trials where these satisfaction indices are interesting protocols and when the inference concerns an effect evaluated from the future sample. We treated our applications in experimental planning and sequential designs with binary outcomes. The computations and the simulation results concern an inferential problem are given by software: Matlab and R.

2. STATISTICAL METHODOLOGY

2.1. Experimental designs

2.1.1. Bayesian predictive design

We use the Bayesian framework as a tool to design clinical trials with desirable frequentist properties. Taking the Bayesian approach, we derive an efficient and flexible design. Statistical methodology has already been used. Remember that, the Bayesian model was introduced in the context of [4] and after improved in [5, 6, and 7].

We specify the experimental context by choosing $(P_\theta)_{\theta\in\Theta}$ a family of probability observations on a space

Ω and where Θ is the space of unknown parameters and is to test the null hypothesis Θ_0 against the alternative hypothesis Θ_1. In classical asymmetric design test problems, such a situation is generally in the experimenter, a desire to put in evidence a significant result, that is to say, to conclude the rejection of the null hypothesis.

- Step 1: If $p_\theta < \theta_L$, stop the trial and reject the alternative hypothesis;
- Step 2: If $p_\theta < \theta_U$, stop the trial and reject the null hypothesis;
- Step 3: Otherwise continue to the next stage until reaching the number maximum of patients.

Typically, we choose θ_L as a small positive number and θ_U as a large positive number, both between 0 and 1 (inclusive). $p_\theta < \theta_L$ indicates that it is unlikely the response rate will be larger than θ_0 at the end of the trial given the current information. When this happens, we may as well stop the trial and reject the alternative hypothesis at that point. On the other hand, when $p_\theta < \theta_L$, the current data suggest that, if the same trend continues, we will have a high probability of concluding that the treatment is efficacious at the end of the study we can see [8]. This result, then, provides evidence to stop the trial early due to efficacy. By choosing $\theta_L > 0$ and $\theta_U < 1$, the trial can terminate early due to either futility or efficacy. For phase IIA trials, we often prefer to choose $\theta_L > 0$ and $\theta_U = 1$ to allow early stopping due to futility, but not due to efficacy.

2.1.2. Satisfaction index

If adopted a procedure deterministic test , relative to a level α , leading to partition Ω in a region of not-rejection $\Omega_0^{(\alpha)}$ and a rejection region $\Omega_1^{(\alpha)}$, an index particularly simple satisfaction is the indicator function of $\Omega_0^{(\alpha)}$. It is satisfied if the result is significant at α, if dissatisfied. But very often users want rather face an outcome that seems likely to lead to their rejection of Θ_0, know what its degree of significance; that is to say, know how far the results appear significant.

Using the fact that any reasonable test technique leads to a family of not -rejection regions $\Omega_0^{(\alpha)}$ in the sense of decreasing inclusion when α increases, that is to say, when our precautions s' mitigate , then use a new index of satisfaction , a little less rough than the previous one , denoted $\Phi^{(\alpha)}$, and defined as a function of p- value , the simplest as zero on the region of not- rejection $\Omega_0^{(\alpha)}$ and equal to (1 -p) , or more generally

$(1\text{-p})^l$ with $l > 0$ otherwise. In other words it offers a satisfaction index which is zero if it is not detected significant and otherwise a decreasing function of the p- value, and therefore, the more p is small and the experimenter believes that the result is significant [9] remind that the p- value is considered as a measure of credibility to attach to the null hypothesis that practitioners often use to meet several critical and disadvantages of approach Neymann of Pearson. The value of this index satisfaction and an extended family of indices in the concept of predicting satisfaction of a sample future as a first sample.

2.1.3. Prediction of satisfaction

Experimental contexts that we have mentioned in the introduction often lend themselves to analysis in several phases, and we limit ourselves to two phases and the situation , which corresponds to the requirements in the experimental trials, where the first phase is that indicative and is intended only to consider whether to resume testing for a second phase , conducted independently of the first and of which only the test result based on the conclusion of which is the ultimate purpose of the study . We note here Ω, Ω' and Ω '' sets complete results, the results of the first phase and the results of the second phase

$(\Omega = \Omega' + \Omega'')$. It is in this context that has proposed to introduce a Bayesian model with a prior distribution on Θ and the family of probabilities $(P_\theta)_{\theta\in\Theta}$ on Ω. He sees in this model the probability of

Ω '' , influenced by the outcome of the first phase ω', which we denote by $P_{\Omega''}^{\omega'}$, recall that, according to the usual Bayesian terminology , the term predictive probability , the probability P_Ω on the space of complete results , which is used here is the probability on Ω'' which is deduced by conditioning by ω' . We find as a prediction on the view of the first phase of a significant result in the second phase, the value $P_{\Omega''}^{\omega'}(\Omega_1^{''(\alpha)})$, where $\Omega_1^{''(\alpha)}$ is the rejection region of the classical test made on the basis of the results of the second phase. It is in this sense that here is practice both classical statistics and Bayesian statistics. We propose here, more typically associate with any satisfaction index on the second phase index forecasting is the mathematical expectation with respect to $P_{\Omega''}^{\omega'}$, satisfaction provided by consider the second phase of

the experiment and the predicted using the first.

Consider a single-arm phase II clinical trial in which all patients are assigned to experimental treatment E. The dose level of E is determined by a previous phase I trial and, thus, is considered to be fixed. Patients are recruited, treated, and evaluated group sequentially, but the size of a group may vary because the group size after the first interim look is determined based on the most up-to-date observations during the trial. The total number of interim looks may also vary from trial to trial, depending on the observations and defined loss function, but we assume an upper bound on the number of patients that could be accrued to the phase II trial. The decision space for the whole phase II clinical trial is defined as $D = \{d_1, d_2, d_3\}$. At each interim evaluation, three possible decisions could be made:

- d_1 : Stop the phase II trial and recommend no further study of E.
- d_2 : Stop the phase II trial and recommend E to a subsequent phase III trial.
- d_3 : Continue to enroll patients to the phase II trial.

It is shown elementarily that the value ω' an index of prediction can also be obtained as the expectation with respect to the posterior distribution based on ω', the average value of the index of satisfaction related to the law sampling the second phase. The problem that arises is that of the calculation of this hope in situations of tests, for a choice of prior distribution. Several models are considered to illustrate the Bayesian predictive procedure proposed.

2.1.4. Statistical inference for the design

We define the indices of satisfaction and anticipation of satisfaction related to a decreasing hypothesis test as a function of the p-value, satisfaction is higher than the null hypothesis is rejected more broadly, that is to say that p-value is small. We consider the case of a two-step procedure, which is often done in the case of clinical trials where these satisfaction indices are interesting protocols and when the inference concerns an effect evaluated from the future sample only.

Being fixed α, a level α test defined by the critical first satisfaction index region $\Omega_1''^{(\alpha)}$:

$$\phi(\omega'') = 1_{\Omega''(\alpha)}(\omega'') \qquad (1)$$

The default of the above rudimentary index is that it expresses a satisfaction in "all or nothing". It is interesting to take into account to what level will the result always appears significant. It thus appears natural to consider satisfaction indexes that are null if a significant effect is not detected, and in the opposite case are an increasing function of the classical indicator of significance that is in theory of tests, the *p*-value. One thus uses a new index of satisfaction defined by:

$$\phi(\omega'') = 0 \text{ if } p(\omega'') \geq 1 - \alpha \qquad (2)$$

$$= L(\mathrm{p}(\omega'')) \quad \text{else}.$$

Where L is a decreasing function. We can generalize this procedure to a family of limited indices defined by:

$$L(p) = (1-p)^l \quad where\ l \geq 0 \qquad (3)$$

It is preferable to choose limited indexes because of their easier interpretation. In the case where *l=1*, $1-\phi(\omega'')$ is the p-value and in the case where *l=0*, one finds the indicator function of the critical region.

For the sequel, we choose *l=1*, L (p) = (1 – p) therefore

$$\phi(\omega'') = 0 \quad \text{if} \quad p(\omega'') \geq 1-\alpha$$

$$= (1-p) \quad else. \qquad (4)$$

Based on the fact that most clinical trials meeting "legal" requirements (imposed by the control authorities for the authorization of placing drugs on the drug market) use as primary criterion of evaluation the significance level of a frequentist test, which is no else than the p-value. May we recall for this purpose that the p-value is always regarded as a measure of credibility to be attached to the null hypothesis that practitioners often use to answer several criticisms and disadvantages of the Neymann Pearson approach

Recall that p= $\inf\{\beta, \omega'' \in \Omega_1^{''(\beta)}\}$ is what practitioners note the associated ω" and is called the p-value, it is considered a measure of credibility to be attached to the null hypothesis and practitioners often use to meet several critical and disadvantages of the approach Neymann -Pearson, you can see why. Therefore, the more that p is, the more the practitioner Considers that the results significant.

An indicator of prediction is given by:

$$\pi(\omega') = \int_{\Omega_1^{''(\alpha)}} \phi(\omega'') P_{\Omega''}^{\omega'}(d\omega'')$$

$$= \int_{\{\omega''; p(\omega'' \geq 1-\alpha)\}} L(p(\omega''))P_{\Omega''}^{\omega'}(d\omega'')$$

$$= \int_{\Theta} \left(\int_{\Omega_1''(\alpha)} \phi(\omega'') P_{\Omega''}^{\theta}(d\omega'') \right) P_{\Theta}^{\omega'}(d\theta') \tag{5}$$

It is noticed that $\int_{\Omega_1''(\alpha)} \phi(\omega'') P_{\Omega''}^{\theta}(d\omega'')$ generalizes the power of the test in the logic of the index of satisfaction proposed. Therefore, this index of prediction can be used to determine whether the trial should be stopped early due to efficacy/futility or continued because the current data are not yet conclusive; it is the experimenter to take the final decision.

2.1.5 Application

It is proposed to calculate the prediction of satisfaction *in this Bayesian design with two kinds of prior distributions* in the binomial model, where the law of the unknown parameter θ is a conjugate prior or non-informative.

Several phase II clinical trial designs are proposed in the statistical literature anmost of them are conducted in two stages. Let X_i be the dichotomous response variable, which assumes value 1 if the clinicians classify the patient i as responder to the treatment t and 0 otherwise. In a typical two-stage design, T patients are accrued and treated at the first stage and *T' additional patients* in the second stage.

We denote θ the probability that an individual suffering from a disease is cured with the treatment t. It is also considered that the medication (treatment related t) may be marketed only if $\theta \geq \theta_0$. From a statistical point of view, we can formulate the problem using the following test:

$H_0 : \theta \leq \theta_0$.

We work in the framework of the sampling model where we assume that are realizations of independent random variables X_i and even Bernoulli parameter θ, again for the sake of completeness we take $\omega' = \sum_{i=1}^{T} X_i$. If we choose as prior distribution for θ a beta law $B(a,b)$ is then known that the posterior distribution of θ/ω' is still a beta law $B(\overline{\alpha}, \overline{\delta})$ with $\overline{\alpha} = a + \omega'$ and $\overline{\delta} = b + T - \omega'$.

The satisfaction index for observation $\omega'' = \sum_{i=1}^{T'} X_i$ is:

$$\phi(\omega'') = 0 \text{ if } \omega'' < q_0 \tag{6}$$

$$= \sum_{t=0}^{\omega''-1} C_N^t \, \theta_0^t \, (1-\theta_0)^{N-t} \text{ if } \omega'' \geq q_0 \,, \tag{7}$$

where

$$q_0 = \inf\left\{u;\ \sum_{t=u}^{T'} C_{T'}^t \, \theta_0^t \, (1-\theta_0)^{T'-t} \leq \alpha\right\}. \tag{8}$$

Then the Bayesian prediction distribution of ω''/ω'

$$f(\omega''/\omega') = \int_0^1 f(\omega''/\theta) f(\theta/\omega') d\theta \tag{9}$$

$$= \int_0^1 C_{T'}^{\omega''} \theta^{\omega''} (1-\theta)^{(T'-\omega'')} \left[\beta(\overline{\alpha},)\overline{\beta}\right]^{-1} \times \theta^{\overline{\alpha}-1} (1-\theta)^{\overline{\delta}-1} \tag{10}$$

$$= C_{T'}^{\omega''} \frac{\left[\beta(\overline{\alpha}+\omega'', \overline{\delta}+T'-\omega'')\right]}{\left[\beta(\overline{\alpha},\overline{\delta})\right]}$$

From here on, observed the response of the first step ω', the prediction $\pi(\omega')$ is

$$\pi(\omega') = \sum_{\omega''=q_0}^{T'} \sum_{t=0}^{\omega''-1} C_{T'}^t \, \theta_0^t \, (1-\theta_0)^{T'-t} \, \frac{C_{T'}^{\omega''} \, \beta\,(a+\omega'+\omega'', b+T+T'-\omega'-\omega'')}{\beta\,(a+\omega', b+T+-\omega')} \tag{11}$$

Moreover, due to ethical considerations, the phase II clinical studies are planned as a multi-stage design to ensure that the trials do not last too long if the treatment shows a clear inadequateness.

Simulation sittings

The aim of exploratory clinical trials, such as phase II trials and proof-of-concept studies, is to determine whether a new intervention is promising for further testing in confirmatory clinical trials, such as phase III randomized controlled trials. The clinical trial, a prospective study to evaluate the effect of interventions in humans under prespecified conditions, is a standard and integral part of modern medicine. Many adaptive and sequential approaches have been proposed for use in clinical trials to allow adaptations or modifications to aspects of a trial after its initiation without undermining the validity and integrity of the trial. In all rigor, the Jeffreys rule gives different priors for the different designs, since it is based on the Fisher information, showd that Jeffreys prior offers new perspectives for the development of Bayesian procedures with good frequentist properties in hypothesis testing procedures.

Suppose two imaging modalities (e.g., CT vs.MRI) for diagnosing lung cancer are to be compared on the basis of test accuracy (sensitivity, specificity, and the area under the ROC curve).

We note in results 1 with:

- The prior Beta (1,1) where ω' is included in [0, 17], the result of θ_0 = 0.6 varies from 0.00000004 to 0.6694. Therefore we conclude H_0 for ω' <17. On the other hand, when ω' is included in [18, 20] the result $\pi_1(\omega')$ varies from 0.8057 to 0.9763.
- The prior Beta (0.5,0.5) where ω' is included in [0, 17], the result of θ_0 = 0.6 varies from 0.0000007 to 0.7161. Therefore we conclude H_0 for ω' <17. On the other hand, when ω' is included in [18, 20] the result $\pi_2(\omega')$ varies from 0.8490 to 0.9914.
- The real data with prior Beta (0.6,0.4) where ω' is included in [0, 17], the result of θ_0 = 0.6 varies from 0.00000002 to 0.7306. Therefore we conclude H_0 for ω' <17. On the other hand, when ω' is included in [18, 20] the result $\pi_3(\omega')$ varies from 0.8604 to 0.9938.

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2.2 Sequential designs

The concept of sequential statistical methods was originally motivated by the need to obtain clinical benefits under certain economic or ethical constraints. For a trial with a positive result, early stopping means that a new product can be exploited sooner. If a negative result is indicated, early stopping ensures that resources are not wasted. [11]

We propose in the sequential designs, a satisfaction made by both the first and the second phase of the experiment (ω', ω'') in the case of a classical test study, and is predicted using the first phase unlike previous work where only the result of the second phase ω'' is to establish the formal conclusion of the study (2.1), with the precedent Binomial model. We use a usual test on the results z of the first and second phase defined by : $z = \omega' + \omega''$.

Group sequential methods are used routinely to monitor clinical trials and to provide early stopping when there is evidence of a treatment effect, a lack of an effect or concerns about patient safety. In many studies, the response of clinical interest is measured some time after the start of treatment and there are subjects at each interim analysis who have been treated but are yet to respond.
The satisfaction index for observation z is

$$\phi(z) = 0 \text{ if } z < q_0 \tag{12}$$

$$= \sum_{t=0}^{z-1} C_{T+T'}^{t}\, \theta_0^t\, (1-\theta_0)^{T+T'-t} \text{ if } z \geq q_0\,, \tag{13}$$

Where

$$q_0 = \inf\left\{u;\ \sum_{t=u}^{T+T'} C_{T+T'}^{t}\, \theta_0^t\, (1-\theta_0)^{T+T'-t} \leq \alpha\right\}. \tag{14}$$

Then the Bayesian prediction distribution of z/ω'

$$f(z/\omega') = \int_0^1 f(z/\theta) f(\theta/\omega') d\theta \tag{15}$$

$$= \int_0^1 C_{T+T'}^{z} \theta^{z} (1-\theta)^{(T+T'-z)} \frac{\theta^{a+\omega'-1}(1-\theta)^{T+b-\omega'-1}}{B(a+\omega', T+b-\omega')} d\theta \tag{16}$$

$$= C_{T+T'}^{z} \frac{B(a+\omega'+z, 2T+T'+b-z-\omega')}{B(a+\omega', T+b-\omega')} \tag{17}$$

From here on, observed the response of the first step ω', the prediction $\Pi(\omega')$ is

$$\Pi(\omega') = \sum_{z=q_0}^{T+T'} \sum_{t=0}^{z-1} C_{T+T'}^{t} \theta^{t} (1-\theta)^{T+T'-t} C_{T'}^{\omega''} \frac{B(a+\omega'+\omega'', T+T'+b-\omega'-\omega'')}{B(a+\omega', b+T-\omega')} \tag{18}$$

Simulation sittings

Group sequential design is probably one of the most commonly used clinical trial designs in clinical research and development, the primary reasons for conducting interim analyses of accrued data.

Suppose are the sample sizes of the two groups and the prior probability of the null hypotheses is θ_0 =0.6, the predictive probability at each point $\Pi(\omega')$ is calculated via simulation in table II, where $\Pi_1(\omega')$ represented the uniform prior beta(1,1), $\Pi_2(\omega')$ represented the non informative prior beta(0.5,0.5) and the other case $\Pi_3(\omega')$ is the real data with a vague prior beta (0.6,0.4).

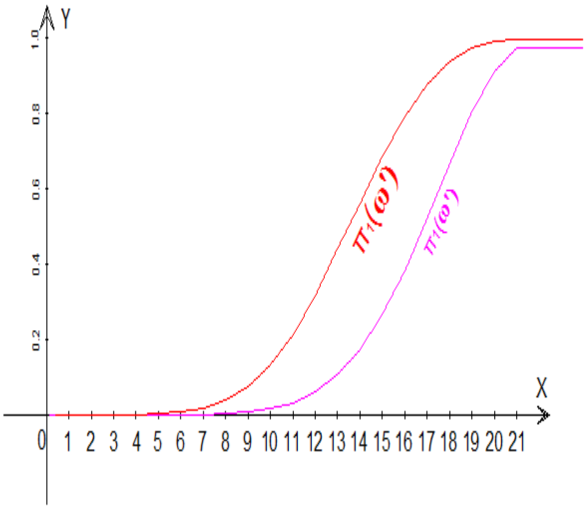
We note in results 2 with:

- The prior B(1,1) where ω' is included in [0, 15], the result of θ_0 = 0.6 varies from 0.0000001 to 0.7932. Therefore we conclude H_0 for ω' <15. On the other hand, when ω' is included in [16, 20] the result $\Pi_1(\omega')$ varies from 0.8789 to 0.9986.
- The prior B(0.5,0.5) where ω' is included in [0, 15], the result of θ_0 = 0.6 varies from 0.00000007 to 0.8136. Therefore we conclude H_0 for ω' <15. On the other hand, when ω' is included in [16, 20] the result $\Pi_2(\omega')$ varies from 0.0.8964 to 0.9995.
- The real data with prior B(0.6,0.4) where ω' is included in [0, 15], the result of θ_0 = 0.6 varies from 0.00000005 to 0.8231. Therefore we conclude H_0 for ω' <15. On the other hand, when ω' is included in [16, 20] the result $\Pi_3(\omega')$ varies from 0.9032 to 0.9997.

Comparison of prevision indices

If we denote by $\pi_1(\omega')$, $\pi_2(\omega')$ and $\pi_3(\omega')$, the prevision index in the case of experimental design, and $\Pi_1(\omega')$, $\Pi_2(\omega')$ and $\Pi_3(\omega')$, the prevision index in the case of sequential design, figures 1, 2, and 3 show compared curves $\pi_1(\omega')$ and $\Pi_1(\omega')$, $\pi_2(\omega')$and $\Pi_2(\omega')$, $\pi_3(\omega')$ and $\Pi_3(\omega')$ of the prevision indices in a study of frequentist test respectively in experimental design and in sequential analysis.

One can see that when we choose the sequential design the prediction is better.



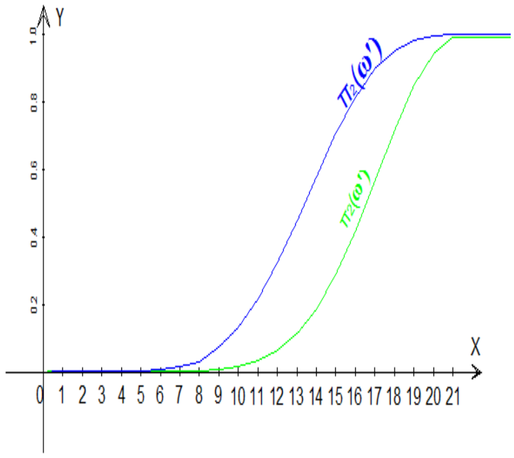


Figure1-Prediction in experimental and sequential designs with uniform prior Beta(1,1)

Figure2- Prediction in experimental and sequential designs with non-informative prior Beta (0.5,0.5)

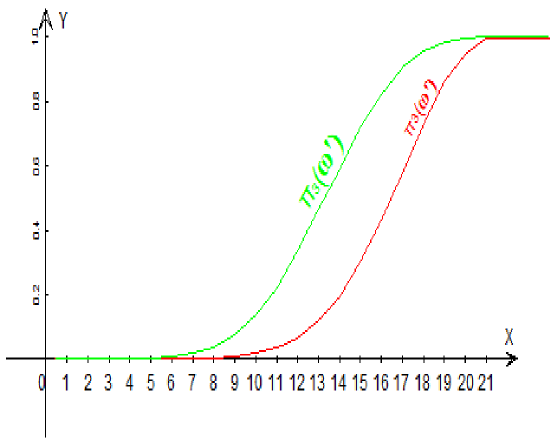


Figure3- Prediction in experimental and sequential designs with a vague prior Beta (0.6,0.4)

3. CONCLUSION

The aim of our work was to propose a Bayesian experimental designs and sequential adaptation design in experimental trials, considered the procedure based on satisfaction of the prediction index concept design, we believe that we can say that the Bayesian predictive approach proposed can be used to predict on the basis of statements frequentist results, we nevertheless believe that the frequentist approach sheds a different light on data and should not be excluded from further our goal is to develop an effective and flexible design that has desirable statistical properties . We have improved the methodology in the design of clinical trials by providing a prediction indices in a Bayesian framework, as is always the case in the experimental trial protocol and we illustrate our results using the binomial model.

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Blow up of Solution for a Viscoelastic Wave Equation with m-Laplacian and Delay Terms

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Abstract

In this paper, we deal with a viscoelastic wave equation with delay terms. We study the blow up of solutions with positive initial energy.

Keywords: Blow up, Delay term, m-Laplacian.

1. Introduction

We consider the following problem

$$\begin{cases} |u_t|^\rho u_{tt} - \Delta u - div(|\nabla u|^{m-2}\nabla u) + \int_0^t g(t-s)\Delta u(s)ds & \\ -\Delta u_{tt} + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) & \\ = b|u|^{p-2}u, & in\ \Omega \times (0,\infty), \\ u_t(x,t-\tau) = f_0(x,t-\tau), & x \in \Omega,\ t \in (0,\tau), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & x \in \Omega, \\ u(x,t) = 0, & x \in \partial\Omega, t \geq 0, \end{cases} \tag{1}$$

where Ω is a bounded domain in R^n, $n \geq 1$, with a smooth boundary $\partial\Omega$, $\rho > 0$, b, μ_1 are positive constants, μ_2 is a real number, $\tau > 0$ represents the time delay, the term $\Delta_m u = div(|\nabla u|^{m-2}\nabla u)$ is called m -Laplacian, the kernel g is satisfying some conditions to be specified later. $u_{0,} u_{1,}\ f_0$ are the initial data in a suitable function space.

Some other authors studied the related problems (see [Messaoudi and Kafini 2016, Wu 2019]).

2. Preliminaries

In this part, we give some material needed for the proof of our results. We use the standard Lebesgue space $L^p(\Omega)$ and Sobolev space $H_0^1(\Omega)$ with their usual norms $\|\cdot\|_p$ and $\|\cdot\|_{H_0^1(\Omega)}$.

Lemma 1. [Adams and Fournier 2003, Pişkin 2017] Let $2 \leq p \leq \frac{2n}{n-2}$, the inequality

$$\|u\|_p \leq c_s \|\nabla u\|_2 \ \ for\ u \in H_0^1(\Omega),$$

holds with some positive constants c_s.

Related to $g(t)$ kernel function, we suppose that

(A1) $g: R^+ \to R^+$, and

$$g(0) > 0,\ g' \leq 0\ and\ 1 - \int_0^\infty g(s)ds = l > 0, \tag{2}$$

satisfies.

Also, from Messaoudi and Kafini (2016), we get the following lemma.

Lemma 2. Assume that

$$\|u\|_p^s \leq C\left(\|\nabla u\|_2^2 + \|\nabla u\|_m^m + \|u\|_p^p\right),$$

where C is a positive constant, satisfies for any $u \in H_0^1(\Omega)$ and $2 \leq s \leq p$.

Now we introduce, similar to the work of Nicaise and Pignotti (2006), the new variable

$$z(x,\kappa,t) = u_t(x, t-\tau\kappa)\ x \in \Omega, \kappa(0,1),$$

thus, it is easy to see that

$$\tau z_t(x,\kappa,t) + z_\kappa(x,\kappa,t) = 0\ in\ \Omega \times (0,1) \times (0,\infty).$$

Then, problem (1) takes the form

$$\begin{cases} |u_t|^\rho u_{tt} - \Delta u - div(|\nabla u|^{m-2}\nabla u) + \int_0^t g(t-s)\Delta u(s)ds & \\ -\Delta u_{tt} + \mu_1 u_t(x,t) + \mu_2 z(x,1,t) & \\ = b|u|^{p-2}u, & in\ \Omega \times (0,\infty), \\ \tau z_t(x,\kappa,t) + z_\kappa(x,\kappa,t) = 0, & x \in \Omega,\ \kappa \in (0,1), t > 0, \\ z(x,0,t) = u_t(x,t), & x \in \Omega,\ t > 0, \\ z(x,\kappa,0) = f_0(x,-\tau\kappa), & x \in \Omega, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & x \in \Omega, \\ u(x,t) = 0, & x \in \partial\Omega, t \geq 0. \end{cases} \tag{3}$$

Next, by combining the arguments Kirane and Belkacem (2011) and Wu (2019), we give the local existence theorem of problem (3).

Theorem 3. Suppose that $\mu_2 < \mu_1$, (A1) and assume that $u_0, u_1 \in H_0^1(\Omega)$ and $f_0 \in L^2\left(\Omega \times (0,1)\right)$. Then, for $T > 0$, there exists a unique solution (u, z) satisfies

$$u, u_t \in C\left([0,T); H_0^1(\Omega)\right),$$

$$z \in C\left([0,T); L^2\big(\Omega \times (0,1)\big)\right).$$

3. Blow up result

In this part, we get the blow up result with positive initial energy. Firstly, we define the energy functional of the problem (3) as follows

$$E(t) = \frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} + \frac{1}{m}\|\nabla u\|_m^m + \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u\|^2$$

$$+\frac{1}{2}(go\nabla u)(t) + \frac{1}{2}\|\nabla u_t\|^2 + \frac{\xi}{2}\int_\Omega \int_0^1 z^2\,(x,\kappa,t)d\kappa dx - \frac{b}{p}\|u\|_p^p, \qquad (4)$$

We can get the following Lemma similar to Pişkin and Yüksekkaya (2020).

Lemma 4. $\mathrm{E}(t)$ is a decreasing function, such that

$$E'(t) \leq -\alpha\left(\|u_t\|^2 + \int_\Omega z^2(x,\kappa,t)\,dx\right) + \frac{1}{2}(g'o\nabla u)(t) - \frac{1}{2}g(t)\|\nabla u\|^2$$
$$\leq -\alpha\left(\|u_t\|^2 + \int_\Omega z^2(x,\kappa,t)\,dx\right) \leq 0, \qquad for\ t \geq 0.$$

Lemma 5. [Wu 2019] Suppose that $l\|\nabla u_0\|^2 > \lambda_1^2$ and $E(0) < E_1$, then there exists $\lambda_2 > \lambda_1$, such that

$$l\|\nabla u\|^2 + (go\nabla u)(t) \geq \lambda_2^2, \qquad (5)$$

for all $t \in [0,T)$, and

$$\|u\|_p^p \geq \frac{bB_1^p}{p}\lambda_2^p. \qquad (6)$$

Theorem 7. Assume that $u_0, u_1 \in H_0^1(\Omega)$ with $l\|\nabla u_0\|^2 > \lambda_1^2$ and $E(0) < \beta E_1$. Suppose further that $\rho < p-2$. Then the solution of (3) blows up in finite time.

Proof. By contradiction, we assume that the solution of problem (3) is global, such that

$$\|u_t\|_{\rho+2}^{\rho+2} + \|\nabla u\|_m^m + \|u\|_p^p + \|\nabla u_t\|^2 + \|\nabla u\|^2 \leq K_1, \forall t \geq 0, \qquad (8)$$

where $K_1 > 0$.

We set, $E_2 \in (E(0), \beta E_1)$, such that

$$H(t) = E_2 - E(t).$$

From Lemma 5, (5) and $E_1 = \frac{p-2}{2p}\lambda_1^2$, we get

$$H(t) \geq H(0) = E_2 - E(0) > 0. \quad (9)$$

We define

$$L(t) = H^{1-\sigma}(t) + \frac{\varepsilon}{\rho+1}\int_\Omega |u_t|^\rho u_t u dx + \frac{\mu_1 \varepsilon}{2}\int_\Omega u^2 dx + \varepsilon \int_\Omega \nabla u_t \nabla u dx. \quad (10)$$

There exists $K > 0$, such that

$$L'(t) \geq \varepsilon K \begin{pmatrix} \|u_t\|_{\rho+2}^{\rho+2} + \|\nabla u\|^2 + \|\nabla u\|_m^m + (go\nabla u)(t) \\ + \|\nabla u_t\|^2 + H(t) + \|u\|_p^p + \int_\Omega \int_0^1 z^2(x,\kappa,t) d\kappa dx \end{pmatrix}, \quad (11)$$

Utilizing Hölder and Young's inequalities, we get

$$\begin{aligned} \left(\left|\int_\Omega |u_t|^\rho u_t u dx\right|\right)^{\frac{1}{1-\sigma}} &\leq \|u_t\|_{\rho+2}^{\frac{\rho+1}{1-\sigma}} \|u\|_{\rho+2}^{\frac{1}{1-\sigma}} \\ &\leq c_5 \|u_t\|_{\rho+2}^{\frac{\rho+1}{1-\sigma}} \|u\|_\rho^{\frac{1}{1-\sigma}} \leq c_1 \left(\|u_t\|_{\rho+2}^{\frac{\rho+1}{1-\sigma}\mu} + \|u\|_\rho^{\frac{1}{1-\sigma}\theta} \right), \end{aligned} \quad (12)$$

where $\frac{1}{\mu} + \frac{1}{\theta} = 1$.

From Lemma 2 and (12), we obtain

$$\left(\left|\int_\Omega |u_t|^\rho u_t u dx\right|\right)^{\frac{1}{1-\sigma}} \leq c_2\left(\|u_t\|_{\rho+2}^{\rho+2} + \|\nabla u\|_m^m + \|\nabla u\|^2 + \|u\|_p^p\right), \quad (13)$$

with $c_2 > 0$. In a similar way, as in deriving (12), we also get

$$\left(\left|\int_\Omega \nabla u_t \nabla u dx\right|\right)^{\frac{1}{1-\sigma}} \leq c_3 \left(\|\nabla u_t\|^2 + \|\nabla u\|_2^{\frac{2}{1-2\sigma}} \right), \quad (14)$$

for $c_3 > 0$. Combining (10), (13) and (14) to satisfy

$$L(t)^{\frac{1}{1-\sigma}} \leq c_4 \begin{pmatrix} H(t) + \|u_t\|_{\rho+2}^{\rho+2} + \|\nabla u\|^2 + \|\nabla u\|_m^m \\ +\|\nabla u_t\|^2 + H(t) + \|u\|_p^p + \|u\|_\rho^{\frac{2}{1-\sigma}} + \|u\|_2^{\frac{2}{1-2\sigma}} \end{pmatrix}, \tag{15}$$

for $t \geq 0$ and $c_4 > 0$. From (11) and (15), we get

$$L(t)^{\frac{1}{1-\sigma}} \leq c_5\left(H(t) + \|u_t\|_{\rho+2}^{\rho+2} + \|\nabla u\|^2 + \|\nabla u\|_m^m + \|\nabla u_t\|^2 + \|u\|_p^p\right), \tag{16}$$

with $c_5 > 0$. Combining (11) and (16), we obtain

$$L'(t) \geq c_6 L(t)^{\frac{1}{1-\sigma}}, t \geq 0, \tag{17}$$

here $c_6 = \frac{\varepsilon K}{c_5}$. A simple integration of (17) over $(0, t)$, we have

$$L(t) \geq \left(L(0)^{-\frac{\sigma}{1-\sigma}} - \frac{\sigma c_{11}}{1-\sigma} t\right)^{-\frac{1-\sigma}{\sigma}}. \tag{18}$$

As we know, $L(0) > 0$, (18) indicates that L becomes infinite in a finite time T with $0 < T \leq \frac{1-\sigma}{c_6 \sigma L(0)^{\frac{\sigma}{1-\sigma}}}$.

4. Conclusion

In recent years, there has been published much work concerning the wave equation with constant delay or time-varying delay. However, to the best of our knowledge, there was no blow-up result for a viscoelastic wave equation with m-Laplacian and delay terms. We have been proved that blow-up of solutions with positive initial energy for problem (1) under the sufficient conditions in a bounded domain.

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Bounds for a new subclass of bi-univalent functions with respect to symmetric conjugate points related to Fibonacci numbers

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Abstract

In the present investigation, we use the Fibonacci numbers to derive estimates on the initial coefficients for a new subclass of bi-univalent functions with respect to symmetric conjugate points. Also, we derive Fekete-Szegö inequalities for functions belonging to the newly-defined class.

1. Introduction

Let $\mathbb{C}$ be the set of complex numbers and $\mathbb{D} = \{z \in \mathbb{C}: |z| < 1\}$ be the open unit disc in $\mathbb{C}$. Further, let $\mathcal{A}$ represent the class of functions f of the from

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots = z + \sum_{n=2}^{\infty} a_n z^n \ , \tag{1}$$

which are analytic in $\mathbb{D}$ and normalized by $f(0) = f'(0) - 1 = 0$. The subclass of $\mathcal{A}$ consisting of univalent functions is denoted by $\mathcal{S}$. Next, the well known class of functions with positive real part, consisting of all functions p analytic in $\mathbb{D}$ satisfying $\mathrm{p}(0) = 1$ and $\Re \mathrm{p}(z) > 0$, is usually denoted by P and called the Carathéodory class. Each $\mathrm{p} \in P$ has a Taylor series expansion

$$\mathrm{p}(z) = 1 + x_1 z + x_2 z^2 + x_3 z^3 + \cdots (x_1 > 0)$$

with coefficients satisfying $|x_n| \leq 2$ for $n \in \mathbb{N}$ (see [7], [11]).

With a view to remanding the rule of subordination between analytic functions, let the functions f, g be analytic in $\mathbb{D}$, we say that the function f is subordinate to g, indicated as $f \prec g$ $\left(\text{or } f(z) \prec g(z)\right)$ $(z \in \mathbb{D})$, if there exists a Schwarz function $\mathfrak{w} \in \Lambda$, where

$$\Lambda = \{\mathfrak{w} : \ \mathfrak{w}(0) = 0, |\mathfrak{w}(z)| < 1, z \in \mathbb{D}\} \ .$$

such that

$$f(z) = g(\mathfrak{w}(z)) \qquad (z \in \mathbb{D}).$$

The *Koebe-One Quarter Theorem* [4] provides that the image of $\mathbb{D}$ under every univalent function $f \in \mathcal{A}$ contains a disc of radius 1/4. Thus every univalent function $f \in \mathcal{A}$ has an inverse f^{-1} satisfying

$$f^{-1}(f(z)) = z, \text{ and } f(f^{-1}(w)) = w \qquad \left(|w| < r_0(f),\ r_0(f) \geq \tfrac{1}{4}\right),$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots.$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{D}$ if both f and f^{-1} are univalent in $\mathbb{D}$. Let Σ represent the class of bi-univalent functions in $\mathbb{D}$ given by (1).

Historically, motivated substantially by the aferomentioned pioneering work on this subject by Srivastava [13], many authors investigated the coefficient bounds for various subclasses of bi-univalent functions (see e.g., [2], [9], [11] and see also the references cited therein). Until now, not much is known about the bounds on the general coefficient bounds for $|a_n|$ for bi-univalent functions (see, for example, [1], [12], [14]). The coefficient estimate problem for each of the coefficients

$$|a_n| \ (n \in \mathbb{N} \setminus \{1,2\})$$

is still an open problem.

The classical Fekete-Szegö inequality, presented by means of Loewner's method, for the coefficients of $f \in \mathcal{S}$ is

$$|a_3 - \varrho a_2^2| \leq 1 + 2exp(-2\varrho/(1-\vartheta)) \quad \text{for } \varrho \in [0,1) .$$

As $\varrho \to 1^-$, we have the elementary inequality $|a_3 - a_2^2| \leq 1$. Moreover, the coefficient functional $F_\varrho(f) = a_3 - \varrho a_2^2$ on the normalized analytic functions f in the unit disc $\mathbb{D}$ plays an important role in function theory. The problem of maximizing the absolute value of the functional $F_\vartheta(f)$ is called the Fekete-Szegö problem (see [6]).

By $S^*(\varphi)$ and $C(\varphi)$ we denote the following classes of functions:

$$S^*(\varphi) = \left\{ f\colon f \in \mathcal{A},\ \frac{zf'(z)}{f(z)} \prec \varphi(z), \qquad z \in \mathbb{D} \right\},$$

$$C(\varphi) = \left\{ f\colon f \in \mathcal{A},\ 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z), \qquad z \in \mathbb{D} \right\}.$$

The classes $S^*(\varphi)$ and $C(\varphi)$ are the extensions of classical sets of starlike and convex functions and in such form were defined and studied by Ma and Minda [9]. Afterwards, El-Ashwah and Thomas [5] introduced the class $S_c^*(\varphi)$ of functions called starlike with respect to symmetric conjugate points, they are the functions $f \in \mathcal{S}$ satisfy the condition

$$\Re\left(\frac{zf'(z)}{f(z) - \overline{f(-\overline{z})}}\right) > 0 \qquad (z \in \mathbb{D}).$$

A function $f \in \mathcal{S}$ is called convex with respect to symmetric conjugate points, if

$$\Re\left(\frac{(zf'(z))'}{(f(z)-\overline{f(-\overline{z})})'}\right) > 0 \qquad (z \in \mathbb{D}).$$

The class of all convex functions with respect to symmetric conjugate points is denoted by $C_c(\varphi)$.

Now, we will introduce the family $\mathcal{N}_\Sigma(\eta; \tilde{\mathrm{p}})$ as follows:

Definition 1. A function $f \in \Sigma$ is said to be in the family $\mathcal{N}_\Sigma(\eta; \tilde{\mathrm{p}})$ $(0 \le \eta \le 1)$ if it fulfills the subordinations:

$$\left(\frac{2[\eta z^3 f'''(z) + (\eta+1)z^2 f''(z) + zf'(z)]}{\eta\left(z^2\big(f(z)-\overline{f(-\overline{z})}\big)'' + \big(f(z)-\overline{f(-\overline{z})}\big)\right) + (1-\eta)z\big(f(z)-\overline{f(-\overline{z})}\big)'}\right) \prec \tilde{\mathrm{p}}(z) = \frac{\tau^2 z^2}{1-\tau z-\tau^2 z^2}$$

and

$$\left(\frac{2[\eta w^3 g'''(w) + (\eta+1)w^2 g''(w) + wg'(w)]}{\eta\left(w^2\big(g(w)-\overline{g(-\overline{w})}\big)'' + \big(g(w)-\overline{g(-\overline{w})}\big)\right) + (1-\eta)w\big(g(w)-\overline{g(-\overline{w})}\big)'}\right) \prec \tilde{\mathrm{p}}(w) = \frac{\tau^2 w^2}{1-\tau w-\tau^2 w^2},$$

where $g = f^{-1}$ and $\tau = \frac{1-\sqrt{5}}{2} \approx -0.618$.

Remark 1. The function $\tilde{\mathrm{p}}(z)$ is not univalent in $\mathbb{D}$, but it is univalent in the disc $|z| < \frac{3-\sqrt{5}}{2} \approx 0.38$. For example, $\tilde{\mathrm{p}}(0) = \tilde{\mathrm{p}}\left(-\frac{1}{2\tau}\right)$ and $\tilde{\mathrm{p}}\left(e^{\pm i\, arccos(1/4)}\right) = \frac{\sqrt{5}}{5}$. Also, it can be written as

$$\frac{1}{|\tau|} = \frac{|\tau|}{1-|\tau|},$$

which indicates that the number $|\tau|$ divides $[0,1]$ such that it fulfills the golden section (see for details Dziok et al. [3]).

Additionally, Dziok et al. [3] indicate a useful connection between the function $\tilde{\mathrm{p}}(z)$ and the Fibonacci numbers. Let $\{\mathrm{F}_n\}$ be the sequence of Fibonacci numbers

$$\mathrm{F}_0 = 0, \mathrm{F}_1 = 1,\ \mathrm{F}_{n+2} = \mathrm{F}_n + \mathrm{F}_{n+1} \quad (n \in \mathbb{N}_0).$$

Then

$$\mathrm{F}_n = \frac{(1-\tau)^n - \tau^n}{\sqrt{5}}, \quad \tau = \frac{1-\sqrt{5}}{2}.$$

If we set

$$\tilde{\mathrm{p}}(z) = 1 + \sum_{n=1}^{\infty} \tilde{\mathrm{p}}_n z^n = 1 + (\mathrm{F}_0 + \mathrm{F}_2)\tau z + (\mathrm{F}_1 + \mathrm{F}_3)\,\tau^2 z^2$$

$$+ \sum_{n=3}^{\infty} (\mathrm{F}_{n-3} + \mathrm{F}_{n-2} + \mathrm{F}_{n-1} + \mathrm{F}_n)\tau^n z^n,$$

then the coefficients $\tilde{\mathrm{p}}_n$ satisfy

$$\tilde{\mathrm{p}}_n = \begin{cases} \tau & (n = 1) \\ 3\tau^2 & (n = 2) \\ \tau\tilde{\mathrm{p}}_{n-1} + \tau^2\tilde{\mathrm{p}}_{n-2} & (n = 3, 4, \cdots). \end{cases} \tag{2}$$

2. Main Results

In this part, we offer to get the estimates on the Taylor-Maclaurin coefficients and derive the Fekete-Szegö inequalities for functions in the subclass $\mathcal{N}_\Sigma(\eta, r)$ of bi-univalent functions.

Theorem 1. Let the function *f* given by (1) be in the class $\mathcal{N}_\Sigma(\eta; \tilde{\mathrm{p}})$. Then

$$|a_2| \le \frac{|\tau|}{\sqrt{2|(4\eta + 3)\tau - 2(\eta + 2)^2(3\tau - 1)|}},$$

$$|a_3| \le \frac{|\tau|}{2(4\eta + 3)} + \frac{\tau^2}{8(\eta + 2)^2}$$

and for any real number ϱ

$$|a_3 - \varrho a_2^2| \le \begin{cases} \dfrac{|\tau|}{2(4\eta + 3)}, & |1 - \varrho| \le \dfrac{|(4\eta + 3)\tau - 2(\eta + 2)^2(3\tau - 1)|}{(4\eta + 3)|\tau|} \\ \dfrac{|1 - \varrho|\tau^2}{2|(4\eta + 3)\tau - 2(\eta + 2)^2(3\tau - 1)|}, & |1 - \varrho| \ge \dfrac{|(4\eta + 3)\tau - 2(\eta + 2)^2(3\tau - 1)|}{(4\eta + 3)|\tau|}. \end{cases}$$

Proof. Suppose that $f \in \mathcal{N}_\Sigma(\eta; \tilde{\mathrm{p}})$. Firstly, let $\mathrm{p} \prec \tilde{\mathrm{p}}$ Then, by the definition of subordination, for two analytic functions u, υ such that $\mathrm{u}(0) = \upsilon(0) = 0, |\mathrm{u}(z)| < 1, |\upsilon(w)| < 1 \quad (z, w \in \mathbb{D})$, we can write

$$\left(\frac{2[\eta z^3 f'''(z) + (\eta + 1)z^2 f''(z) + z f'(z)]}{\eta\left(z^2\left(f(z) - \overline{f(-\overline{z})}\right)'' + \left(f(z) - \overline{f(-\overline{z})}\right)\right) + (1 - \eta)z\left(f(z) - \overline{f(-\overline{z})}\right)'} \right) = \tilde{\mathrm{p}}(\mathrm{u}(z)) \tag{3}$$

and

$$\left(\frac{2[\eta w^3 g'''(w)+(\eta+1)w^2 g''(w)+wg'(w)]}{\eta\left(w^2\left(g(w)-\overline{g(-\overline{w})}\right)''+\left(g(w)-\overline{g(-\overline{w})}\right)\right)+(1-\eta)w\left(g(w)-\overline{g(-\overline{w})}\right)'}\right)=\tilde{\mathrm{p}}(\upsilon(w)). \quad (4)$$

Next, define the functions p_1 and p_2 by

$$\mathrm{p}_1(z)=\frac{1+\mathrm{u}(z)}{1-\mathrm{u}(z)}=1+x_1z+x_2z^2+\cdots,$$

$$\mathrm{p}_2(w)=\frac{1+\upsilon(w)}{1-\upsilon(w)}=1+y_1w+y_2w^2+\cdots.$$

Since u and υ are Schwarz functions, p_1 and p_2 are analytic functions in $\mathbb{D}$, with $\mathrm{p}_1(0)=\mathrm{p}_2(0)=1$ and which have positive real part in $\mathbb{D}$, we obtain the equations

$$\mathrm{u}(z)=\frac{\mathrm{p}_1(z)-1}{\mathrm{p}_1(z)+1}=\frac{1}{2}\left[x_1z+\left(x_2-\frac{x_1^2}{2}\right)z^2\right]+\cdots,$$

$$\upsilon(w)=\frac{\mathrm{p}_2(w)-1}{\mathrm{p}_2(w)+1}=\frac{1}{2}\left[y_1w+\left(y_2-\frac{y_1^2}{2}\right)w^2\right]+\cdots$$

lead to

$$\tilde{\mathrm{p}}(\mathrm{u}(z))=1+\frac{\tilde{\mathrm{p}}_1x_1}{2}z+\left[\frac{1}{2}\left(x_2-\frac{x_1^2}{2}\right)\tilde{\mathrm{p}}_1+\frac{x_1^2}{4}\tilde{\mathrm{p}}_2\right]z^2+\cdots,$$

$$\tilde{\mathrm{p}}(\upsilon(\mathrm{w}))=1+\frac{\tilde{\mathrm{p}}_1y_1}{2}w+\left[\frac{1}{2}\left(y_2-\frac{y_1^2}{2}\right)\tilde{\mathrm{p}}_1+\frac{y_1^2}{4}\tilde{\mathrm{p}}_2\right]w^2+\cdots.$$

Now, upon comparing the corresponding coefficients in (3) and (4), we get

$$2(\eta+2)a_2=\frac{\tilde{\mathrm{p}}_1x_1}{2}, \quad (5)$$

$$2(4\eta+3)a_3=\frac{1}{2}\left(x_2-\frac{x_1^2}{2}\right)\tilde{\mathrm{p}}_1+\frac{x_1^2}{4}\tilde{\mathrm{p}}_2, \quad (6)$$

$$-2(\eta+2)a_2=\frac{\tilde{\mathrm{p}}_1y_1}{2}, \quad (7)$$

$$2(4\eta+3)(2a_2^2-a_3)=\frac{1}{2}\left(y_2-\frac{y_1^2}{2}\right)\tilde{\mathrm{p}}_1+\frac{y_1^2}{4}\tilde{\mathrm{p}}_2. \quad (8)$$

From the equations (5) and (7), one can easily find that

$$x_1=-y_1, \quad (9)$$

$$8(\eta+2)^2 a_2^2=\frac{\tilde{\mathrm{p}}_1^2}{4}(x_1^2+y_1^2). \tag{10}$$

If we add (6) to (8), we obtain

$$4(4\eta+3)a_2^2=\frac{\tilde{\mathrm{p}}_1}{2}(x_2+y_2)+\frac{(\tilde{\mathrm{p}}_2-\tilde{\mathrm{p}}_1)}{4}(x_1^2+y_1^2). \tag{11}$$

By making the use of (10) in (11), we have

$$a_2^2=\frac{\tilde{\mathrm{p}}_1^3(x_2+y_2)}{8[(4\eta+3)\tilde{\mathrm{p}}_1^2-2(\eta+2)^2(\tilde{\tilde{\mathrm{p}}}_2-\tilde{\mathrm{p}}_1)]} \tag{12}$$

which yields

$$|a_2|\leq\frac{|\tau|}{\sqrt{2|[(4\eta+3)\tau-2(\eta+2)^2(3\tau-1)]|}}.$$

Next, if we subtract (8) from (6), we obtain

$$4(4\eta+3)(a_3-a_2^2)=\frac{\tilde{\mathrm{p}}_1}{2}(x_2-y_2) \tag{13}$$

Then, in view of (10), the equation (13) becomes

$$a_3=\frac{\tilde{\mathrm{p}}_1(x_2-y_2)}{8(4\eta+3)}+\frac{\tilde{\mathrm{p}}_1^2(x_1^2+y_1^2)}{32(\eta+2)^2}$$

Thus, by virtue of (2), we get the bound for $|a_3|$.
From (12) and (13), we find that

$$a_3-\varrho a_2^2=\frac{(1-\varrho)\tilde{\mathrm{p}}_1^3(x_2+y_2)}{8\left[(4\eta+3)\tilde{\mathrm{p}}_1^2-2(\eta+2)^2(\tilde{\mathrm{p}}_2-\tilde{\mathrm{p}}_1)\right]}+\frac{\tilde{\mathrm{p}}_1(x_2-y_2)}{8(4\eta+3)}$$

$$=\frac{\tilde{\mathrm{p}}_1}{4}\left[\left(h(\varrho)+\frac{1}{2(4\eta+3)}\right)x_2+\left(h(\varrho)-\frac{1}{2(4\eta+3)}\right)y_2\right],$$

where

$$h(\varrho)=\frac{(1-\varrho)\tilde{\mathrm{p}}_1^2}{2\left[(4\eta+3)\tilde{\mathrm{p}}_1^2-2(\eta+2)^2(\tilde{\mathrm{p}}_2-\tilde{\mathrm{p}}_1)\right]}.$$

Thus, in view of (2), we get

$$|a_3-\varrho a_2^2|\leq\begin{cases}\dfrac{|\tilde{\mathrm{p}}_1|}{2(4\eta+3)}, & 0\leq|h(\varrho)|\leq\dfrac{1}{2(4\eta+3)}\\ |\tilde{\mathrm{p}}_1||h(\varrho)|, & |h(\varrho)|\geq\dfrac{1}{2(4\eta+3)}\end{cases}.$$

Acknowledgement. This work is supported by the Scientific and Technological Research Council of Turkey (TUBITAK), Project Number: 118F543.

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Comparing the efficiency of hierarchical cluster algorithms

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Abstract

In this paper we use some known hierarchical methods in order to organize observations (in our case, students) method in order to cluster cases (in our case, students) into homogeneous groups based on some criteria (in our case, their preferences). A student can be classified in one of favourite studies programs according to the results in State Matura Exams. The results of analysis are compared with the real results.

Most of the clustering algorithms are very sensitive to their input parameters. For example, by the hierarchical clustering, the number of clusters is defined a priory or at the end of the algorithms based on some criteria. Therefore, it is very important to evaluate the result of the clustering algorithms. It is difficult to define when a clustering result is acceptable. Thus, in this paper the most commonly used validity internal and external measures are compared to each other in order to determine the most appropriate method and an optimal number of clusters for this dataset.

The analysis is based on a sample taken from the database of the MSH 2013 (State Matura). All analysis is performed using SPSS statistics 20.

Keywords: Hierarchical cluster analysis, external, internal and distance measures..

1. Introduction

The objective of cluster analysis (CA) is to assign observations (e.g. people, things, events, companies) into groups (clusters) so that observations within each group are similar to one another with respect to variables or attributes of interest, and the groups them-selves stand apart from one another, (Anderberg M. R., 1973). Objects in the formed groups could be cases or variables, data set can be numeric or categorical. Categorical data can be derived from either quantitative or qualitative data where observations are directly observed from counts.

Cluster analysis is supported by a number of software packages like SAS, SPSS, BMDP, which are often available in academic and other computing environments. A brief descriptions and sources for these and other packages are provided by Romesburg (Romesburg, H. C., 1984).

Another issue in CA is the interpretation of groups. In fact, interpretation of groups that result from arithmetic procedure is not very clear. An important role in the identification of group, play the distance between objects. Usually, a graphical representation helps in distinguishing before applying any method. We must be careful in case when there are no real groups for the data sets but method contains a partition.

Because we don’t know the number of clusters that will emerge in our sample, a two- stage sequence of analysis occurs as follows:

- We carry out some of the hierarchical cluster algorithms for example average, complete linkage, single linkage, Ward method and k-means algorithm applying square Euclidean distance as the distance measure for a given number of cluster K=2:6.
- The next stage we will calculate three internal and three external measures which are available on cvClust package in R. These calculations are done for each clustering method and for each pro defined number of clusters. Then the optimal number of clustering correspond to the optimal score for each measure.

2. Measures of distance for observations and Cluster methods

There are some techniques CA, which are based on the transformation of the matrix with observation into another matrix which is called distance matrix, similarity or non-similarity matrix between variables or observations. The Euclidean distance, the weighted Euclidean distance, Mahalanobis distance are some measures of distance for numerical variables. For other data types (nominal, ordinal) are used others measures of distance (Holmes F., 2005).
In this paper, the Squares Euclidean distance between two points, a and b, with k dimensions is calculated as:

$$d(P_k, P_l) = \sqrt{\sum_{i=1}^{q} (x_i - x_i^{'})^2} \text{ , k, l=1:n,}$$

where n=the number of individuals and q=the number of factors

Squared Euclidean distances is usually computed from raw data, and not from standardized data. The distance between any two objects is not affected by outliers’ values, but it can be greatly affected by differences in scale among the dimensions from which the distances are computed. In our application, the last one is not a problem after all of the dimensions denotes a measured evaluation in points, and you don’t need to transform the dimensions. So that they have similar scales.

3. Some algorithms in hierarchical cluster analysis

In the hierarchical method, one method (agglomerative), for example, begins with as many groups as there are observations, and then systematically merges observations to reduce the number of groups by one, two,..., k, until a single group containing all observations is formed. Another method (partition) begins with a given number of groups and an arbitrary assignment of the observations to the groups, and then reassigns the observations one by one so that ultimately each observation belongs to the nearest group.

One characteristic of hierarchical methods is that agglomeration which happened along procedure cannot be rearranged. This means that, from the moment that two or more objects join into a cluster, they cannot be separated after they are joined (until the procedure ends). Of course, this is a disadvantage of these methods.

After having chosen the distance or similarity measure, we need to decide which clustering algorithm to apply. There are several agglomerative procedures and they can be distinguished by the way they define the distance from a newly formed cluster to a certain object, or to other clusters in the solution. Nearest neighbor, furthest neighbor, Average linkage, Centroid linkage and Ward's methods are the most popular agglomerative clustering procedures available in most common statistical packages. All these differ in the definition of distance. So, each of inter-group proximity measures giving rise to a different agglomerative method (Brian S. Everitt. et. al., 2011).

4. Decide on the Number of Clusters

An important question we haven't yet addressed is how to decide on the number of clusters to retain from the data. Unfortunately, hierarchical methods provide only very limited guidance for making this decision. The only meaningful indicator relates to the distances at which the objects are combined.
One way to solve this problem is to plot the number of clusters on the x-axis against the distance at which objects or clusters are combined on the y-axis. Using this plot, we then search for the distinctive break (elbow) or we can make use of the dendrogram which essentially carries the same information.

As an example of how to determine the optimum number of clusters we work with, suppose we have several objects (10 students) and each student have 10 features (total points that students have gathered in each studies programs) as show in in Table 1.

Table 1. A sample from the data set

Point_1	**Point_2**	**Point_3**	**Point_4**	**Point_5**	**Point_6**	**Point_7**	**Point_8**	**Point_9**	**Point_10**
6178.3	6334.1	6334.1	6178.3	6178.3	6178.3	6334.1	6334.1	6334.1	6334.1
5267.4	5007	5267.4	5007	5093.8	5267.4	5007	5007	5267.4	5267.4
6386.6	6386.6	6386.6	6386.6	6095.3	6386.6	6386.6	6095.3	6095.3	6386.6
6297.2	6297.2	6297.2	6297.2	6297.2	6297.2	6134.8	6297.2	6297.2	6297.2
4222.2	4222.2	4010.4	4169.4	4010.4	4222.2	4010.4	4248.9	4169.4	4151.6
6070.3	6070.3	6070.3	6070.3	6070.3	5775.1	5775.1	5971.9	5971.9	5775.1
6525.4	6525.4	6525.4	6286.9	6286.9	6286.9	6525.4	6525.4	6525.4	6445.9
5833	5833	5833	5833	5641	5545	5833	5545	5545	5737
5012.1	5012.1	5012.1	4934.2	5012.1	4934.2	4934.2	4934.2	4934.2	4934.2
3746	3681	3681	3995.8	3746	3681	3681	3746	3681	3681

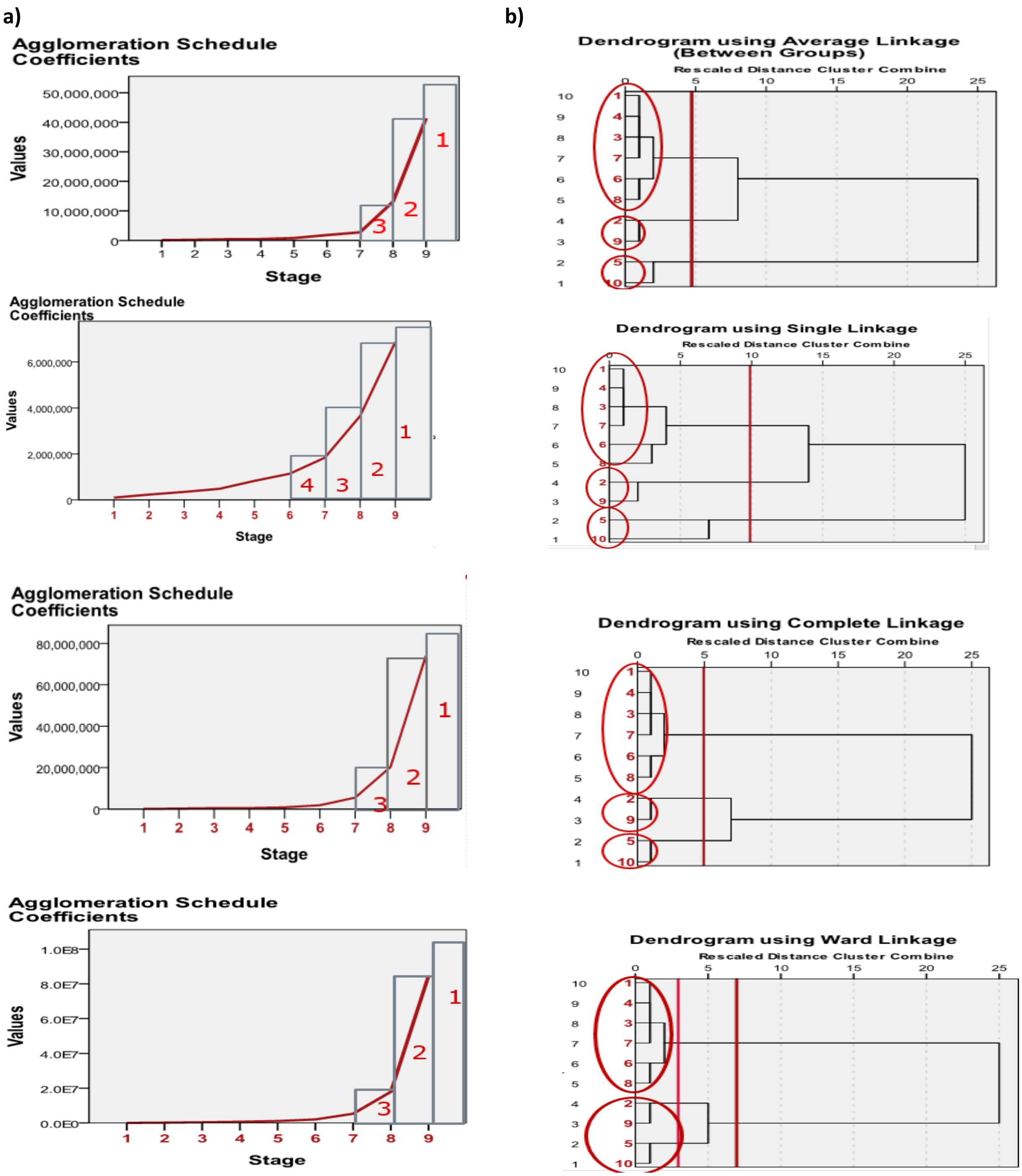

Figure 1. Validation cluster for average, single linkage, complete and Ward.

We carry out a hierarchical cluster analysis using four methods; **average, complete linkage, single linkage, and Ward applying square Euclidean distance** as the distance measure. All these processes can be represented on a diagram known as a dendrogram (see figure 1a). Also, we draw a graphic of pair (j, c_j), the "the scree plot", and we observe the position in which this graphic begin to become "flat" (see Figure 1b).

From Figure 1, we see that the number of clusters once is two and once is three. Also, these two methods cannot be used when the number of observations is large, because this not easily distinguished. For these reasons it is good to use internal and external indicators.

5. Internal and External measures

In this section, we illustrate the validation measures. There are two types of clustering validation techniques (Ashish Jaiswal and Nitin Janwe, 2011; Rousseeuw P. J., 1987; Duda P. E. et al, 2001) which are based on external criteria and internal criteria, respectively. The focus of this paper is on the evaluation of internal clustering validation measures including **connectivity Silhouette Width, Dunn Index (**Maria Halkidi et al, 2002; A.K. Jain and R.C. Dubes, 1998, Michael Steinbach, 2000) and external clustering validation measures including the average proportion of non-overlap (APN), the average distance (AD) and the average distance between means (ADM). The average proportion of non-overlap (APN), the average distance (AD) and the average distance between means (ADM). The APN, AD, and ADM are all based on the cross-classification table of the original clustering with the clustering based on the removal of one column

The connectivity indicates the degree of connectedness of the clusters, as determined by the k-nearest neighbours. where the number of neighbours to use will be specified. The connectivity has a value between 0 and infinity and should be minimized.
Let define $nn_{i,(j)}$ as the j-th nearest neighbour of observation i, and let

$$x_{i,nn_{i,(j)}} = \begin{cases} 0 & \text{if i and j are in the same cluster} \\ 1/j & \text{otherwise} \end{cases}$$

Then for a particular clustering partition $C = \{C_1, ..., C_k\}$ of the N observation into k-clusters, it is defined as

$$\text{Connectivity}(C) = \sum_{i=1}^{N} \sum_{j=1}^{L} x_{i,nn_{i,(j)}}$$

where L is a parameter giving the number of nearest neighbours to use.

The Silhouette Width is the average of each observation's Silhouette value. It is a quantity between -1 and 1, with well-clustered observations having values near 1 and poorly clustered observations having values near -1 and should be maximised (Chen G. et al., 2002, Rousseeuw, 2006). The silhouette index value to detect if we have to do with an appropriate clustering or not. For each observation i, it is defined as

$$s(i) = \frac{b(i) - a(i)}{\max(a(i), b(i))}$$

where a_i is the average distance between i and all other observation in the same cluster, and b_i is the average distance between i and the observations in the nearest neighboring cluster (nn_i)

The Dunn Index is computed as the ratio between the minimum distance between two clusters and the size of the largest cluster. It has a value between 0 and infinity and should be maximized (Bolshakova N. & Azuaje F., 2003). So, we locate our optimal number of clusters looking for the maximum value of this index. It is defined as (J. Dunn 1974)

$$D(C) = \frac{\min\limits_{C_k, C_l \in C, C_k \neq C_l} \left(\min\limits_{i \in C_k, j \in C_l} dist(i, j) \right)}{\max\limits_{C_m \in C} dist(C_m)}$$

where $dist(C_m)$ is the maximum distance between observation in the cluster C_m

The APN measures the average proportion of observations not placed in the same cluster under both cases, while the ***AD*** measures the average distance between observations placed in the same cluster under both cases and the ***ADM*** measures the average distance between cluster centers for observations placed in the same cluster under both cases. In all cases the average is taken over all the deleted columns, and all measures should be minimized.

The average proportion of non-overlap (APN), the average distance (AD) and the average distance between means (ADM) are defined respectively as

$$APN(k) = \frac{1}{MN} \sum_{i=1}^{N} \sum_{j=1}^{M} \left(1 - \frac{n(C_{i,o} \cap C_{i,l})}{n(C_{i,o})}\right), \quad AD(k) = \frac{1}{MN} \sum_{i=1}^{N} \sum_{j=1}^{M} \frac{1}{n(C_{i,o}) n(C_{i,l})} \sum_{i \in C_{i,o}, j \in C_{i,l}} dist(i, j)$$

$$ADM(k) = \frac{1}{MN} \sum_{i=1}^{N} \sum_{j=1}^{M} dist\_Euclidean(\bar{x}_{C_{i,l}}, \bar{x}_{Ci,o})$$

where $n(C_{i,o})$ is the cluster containing observation i using original cluster, $n(C_{i,l})$ is the cluster containing observations i where the cluster is based on the dataset with column l removed, $\bar{x}_{C_{i,l}}$ is the mean of the observation in the cluster which contain observation i.

6. Data set information. Experimental results

We will use a sample retrieved from the database of results in year 2013, in different Universities of Tirana, to determine the best grouping of students with regard to their total points they have for each preference. This sample consist of n=400 admitted students. To illustrate, we cluster the above data (a sample taken from the database of the MSH 2013, State Matura) using the hierarchical and K-means algorithms with two to six clusters. Both internal and external measures are used for validation.

First, we carry out a hierarchical cluster analysis using four methods; **complete linkage, single linkage, average and Ward applying square Euclidean distance** as the distance measure and then we use two validates methods; a dendrogram and "scree plot", which will helps us to determine the optimum number of clusters we work with. From the example above we see that the number of clusters once is two and once is three. Also, these two methods cannot be used when the number of observations is large, because the distinction is not done easily. For these reasons it is good to use internal and external indicators. The results are shown in table 2 and in figure 2.

Table 2. Internal measures for hierarchical methods and k-means

Number of clusters	**2**	**3**	**4**	**5**	**6**
Connectivity	**5.1837**	10.0417	11.0417	12.8401	18.2964
Dunn	**0.4035**	0.4035	0.4035	0.4035	0.0454
Silhouette	**0.6225**	0.6005	0.5300	0.5256	0.4893
Connectivity_K-means	**5.1837**	16.1726	21.0306	22.0306	23.8290
Dunn	**0.4035**	0.4035	0.0244	0.0303	0.0303
Silhouette	**0.6225**	0.5305	0.5339	0.5301	0.5364

From table 2 (column 2, with red colour), we see that hierarchical clustering and k-means with two clusters performs the best in each case. The plots of the connectivity, Dunn Index, and Silhouette Width are given in Figure 2. Recall that the **connectivity** should be **minimized**, while both the **Dunn Index** and the **Silhouette Width** should be **maximized**. Thus, it appears that hierarchical clustering outperforms the other clustering algorithms under each validation measure. For hierarchical clustering and k-means clustering the optimal number of clusters is clearly two.

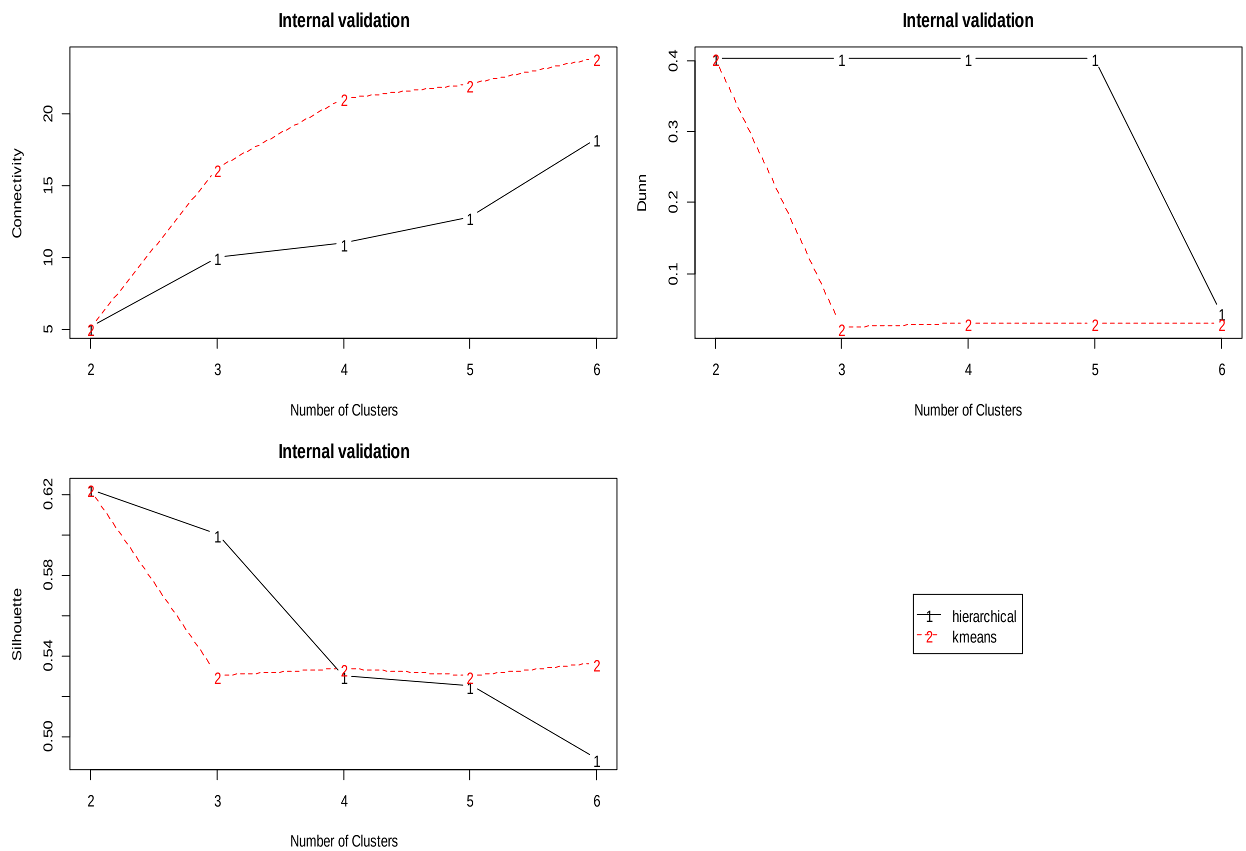

Figure 2. Plots of the connectivity measure, the Dunn Index, and the Silhouette Width

Table 3. Optimal score value for external measures

	Score	**Methods**	**cluster**
APN	1.89E-03	hierarchical	**3**
AD	1.67E+03	Kmeans	6
ADM	2.06E+01	hierarchical	2

Instead of viewing all the validation measures, we can instead just view the optimal values as shown in Table 3. From table 3, we see that, for the APN measures, hierarchical clustering with three clusters while for ADM measures, hierarchical clustering with two clusters again gives the best score. It is illustrative to graphically visualize each of the validation measures.

The plots of the APN, AD, and ADM are given in Figure 3. The APN measure shows an interesting trend, in that it initially increases from two to four clusters (for both algorithm) but subsequently decreases afterwards (for hierarchical). Though hierarchical clustering with two clusters has the best score, K-means with four clusters is a close second. The AD measures tend to decrease as the number of clusters increases (for both algorithm). Here both algorithm with six cluster have the best overall score. K-means with six clusters has better overall score than the hierarchical algorithms. For the ADM measure both algorithm with four clusters has the best score, though the hierarchal methods outperform K-means for smaller numbers of clusters.

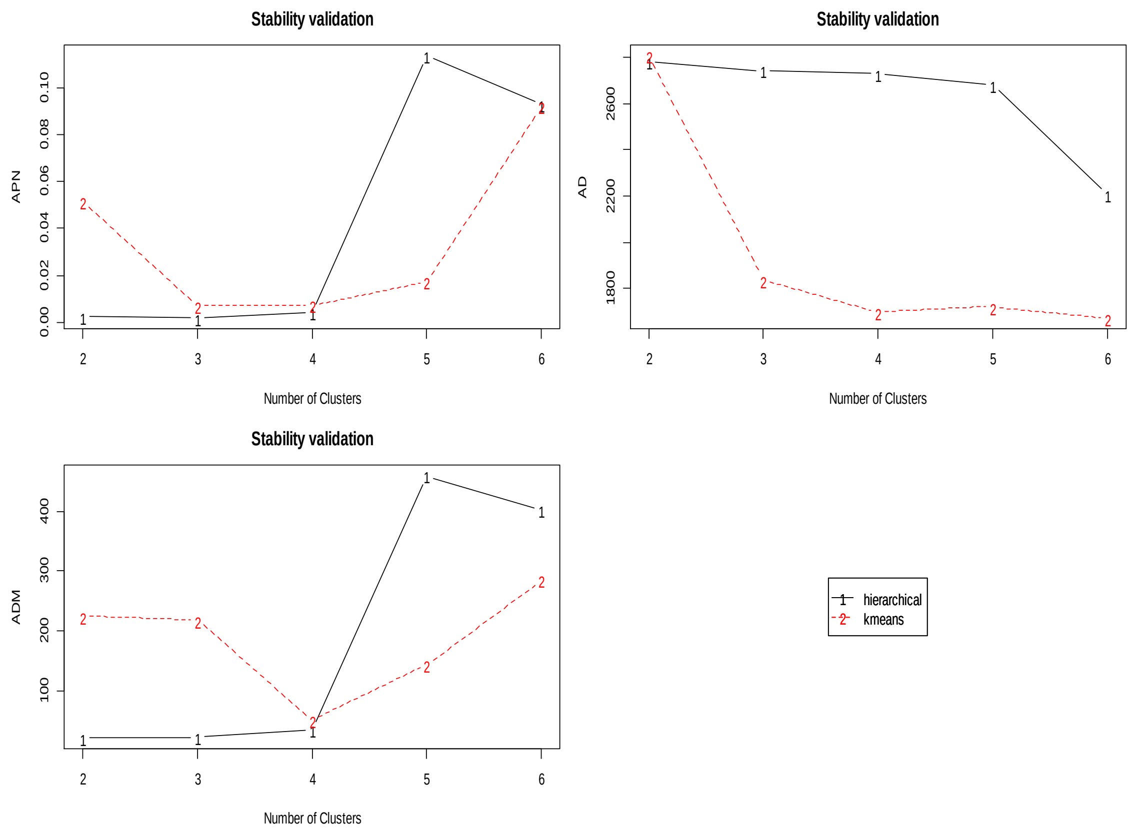

Figure 3. Plot of the APN, AD, and AMD measures.

Table 4. The top three ranking algorithms for each measure.

RANK	**1**	**2**	**3**
APN	hierarchical_3	hierarchical_2	hierarchical_4
AD	K-means_6	K-means_4	K-means_5
ADM	hierarchical_2	hierarchical_3	hierarchical_4
Connectivity	hierarchical_2	K-means-2	hierarchical_3
Dunn	hierarchical_2	hierarchical_3	hierarchical_4
Silhouette	hierarchical_2	K-means_2	hierarchical_3

From Table 4, we see that hierarchical with two clusters performs best on four of the six measures, so picking the best algorithm is relatively simple in this case. However, in many cases there is no clearly best performing algorithm. Also, it would be rather difficult to give the overall ordered list, and we may be interested in, say, the top two or more performing algorithms instead of restricting ourselves to a single set of results.

7. Conclusions

HCA and K-means (Johnson R. A. and Wichern D. W., 2002) are used for reducing of dimension in a data set and the results show the efficiency of multivariate statistical analysis.

HCA and K-means are run and for 2 to 6 clusters stand out on two diagrams: *the dendrogram* and *the scree plot*. We see that the number of clusters once is two and once is three. Also, these two methods cannot be used when the number of observations is large, because this not easily distinguished. For these reasons it is good to use internal and external indicators.

HCA with two clusters performs best on four of the six measures, so picking the best algorithm is relatively simple in this case. Also, it would be rather difficult to give the overall ordered list, and we may be interested in, say, the top two or more performing algorithms instead of restricting ourselves to a single set of results.

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$(\mathcal{F}, h)$ cone upper class on Fixed Point Results in Quasi–Cone Metric Space for Generalized Contractive Mappings Using Diameter of Orbits

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Abstract

In this paper is defined cone upper class. Some fixed point theorems related to (F, h) - cone upper class in quasi - cone metric space are given. As applications to illustrate these results are provided some examples.

1. Introduction

In 2007, Huang and Zhang [13] introduced the concept of cone metric space replacing the real axis in distance function by an ordered Banach space. They proved some fixed point theorems generalizing the result of Kannan [11], Chatterja [12]. There are many authors like Binayak et al. [10], Raja and Vaezpour [14], Ding [18] have studied fixed point in cone metric spaces.

Later, in 2009 Abdeljawad and Karapinar [15] generalized cone metric spaces into quasi cone metric spaces. Using this generalization many authors like Sila et.al [16], Shaddad and Noorani [7], ect have given their contribution fixed points for contractions given in [8].

In 2012, Karapinar and Samet [1] proved some fixed point theorems related $\alpha - \psi$ contractive mappings in metric space and then Bilgili et al. [2] extended these result to quasi-cone metric space.

Later, Sila et al. [17] generalized these results in quasi-cone metric space using the diameter of orbits.

Recently A. H. Ansari [21] introduced the concept of (F, h) upper class of type I in metric spaces. In this paper is defined (F, h) upper class of type I and a class of $\alpha - \mu - \psi$ contractive mappings in quasi cone metric space and is proved some fixed point theorems which generalize the results of [2], [17].

Below there are given some preliminaries that are used for proving the new results.

Definition 1. 1 *[13] Let E be a real Banach space and P be a subset of E. P is called a cone if and only if:*

1. P is closed, $P \neq \emptyset, P \neq \{0\}$;
2. For all $x, y \in P, \alpha x + \beta y \in P$, where $\alpha, \beta \in \mathbb{R}^+$;
3. if $x \in P$ and $-x \in P$ implies $x = 0$.

For a given cone $P \subset E$, we can define a partial ordering "$\leq$" with respect to P by $x \leq y$ if and only if $y - x \in P$. $x < y$ will stand $x \leq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in \operatorname{int} P$, where $\operatorname{int} P$ denotes the interior of P.

The cone P is called normal if there is a number $k > 0$ such that $0 \leq x \leq y \Rightarrow \|x\| \leq k\|y\|$, for all $x, y \in P$. The least positive k satisfying this, is called the normal constant of P.

Definition 1.2 *[13] Let X be a nonempty set. Suppose the mapping $d: X \times X \to E$ satisfies following conditions:*

1. $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$ for all $x, y \in X$;
3. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then, d is called a *cone metric* on X and (X, d) is called a *cone metric space*.

Definition 1.3 *[15] Let X be a nonempty set. Suppose the mapping $q: X \times X \longrightarrow P$ satisfies following conditions:*

1. $0 \leq q(x, y)$ for all $x, y \in X$;
2. $q(x, y) = 0$ if and only if $x = y$;
3. $q(x, y) \leq q(x, z) + q(z, y)$ for all $x, y, z \in X$.

Then, q is called a *quasi–cone metric* on X and (X, q) is called a *quasi–cone metric space*.

Remark 1.4 *Note that any cone metric space is a quasi–cone metric space.*

Shaddad and Noorani [7] introduced the appropriate generalization in quasi–cone metric spaces by considering the established notions in quasi–metric spaces.

Definition 1. 5 [7] *Let (X, q) be a quasi–cone metric space.*

A sequence $\{x_n\}$ in X is called right (left) Cauchy if for each $c \in \operatorname{int} P$, there is $n_0 \in \mathbb{N}$ such that $q(x_n, x_m) \ll c$ ($q(x_m, x_n) \ll c$ resp.) for all $n \geq m \geq n_0$.

The sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is called Cauchy if and only if it is both left and right Cauchy.

Definition 1. 6 [7] *Let (X, q) be a quasi–cone metric space.* Let $\{x_n\}_{n\in\mathbb{N}}$ in X. We say that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is right convergent to $x \in X$ if $q(x, x_n) \to 0$. We say that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is convergent to $x \in X$ if the sequence is right and left convergent to x. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.

Definition 1. 7 [7] *A quasi–cone metric space (X, q) is called complete if every Cauchy sequence in X converges.*

Definition 1. 8 [10] *Let $O(x) = \{x, Tx, T^2x, \dots\}$ where $x \in X$. The set $O(x)$ is called orbit of x.*

Using the same method as in [5], there is defined diameter of a set in quasi - cone metric space.

Definition 1. 9 ([5], [9]) *Let $M \subseteq X$, where X is a quasi-cone metric space.*

$\delta(M) = \sup\{q(x, y), q(y, x), x, y \in M\}$ *is called diameter of M.*

Define $\delta(O(x) \cup O(y)) = \max\{q(T^i x, T^j x), q(T^k y, T^m y), q(T^i x, T^k y)\}$ for $i, j, k, m \in \mathbb{N}_0$.

The orbit $O(x)$ is called bounded if there exist a $c \in P$, $\delta(O(x)) \leq c$.

Definition 1. 10 [4]*The continuous function $\psi: P \to P$ which satisfies the following conditions:*

1. $\forall t \in P, \psi(t) < t$;
2. $\forall t_1, t_2 \in P$, for $t_1 < t_2$ implies $\psi(t_1) < \psi(t_2)$;
3. $\lim_{n\to\infty} \|\psi^n(c)\| = 0$

is called a *comparison function in cone P.*

Definition 1. 11 [3] *Let (X, q) be a quasi--cone metric space and $T: X \to X$ be a given function.*

The map T is $\alpha - \psi$ contractive mapping if there exist two functions $\alpha: X \times X \to [0, +\infty)$ and ψ a comparison function which satisfy the nonlinear contraction condition:

$$\alpha(x, y) q(T(x), T(y)) \leq \psi(q(x, y)).$$

Definition 1. 12 [23] *Let $T: X \to X$ and $\alpha: X \times X \to [0, +\infty)$. The mapping T is α –admissible if for all $x, y \in X$ the following implication is true*

$$\alpha(x,y) \geq 1 \text{ implies } \alpha(Tx,Ty) \geq 1.$$

Definition 1. 13 *[6] Let* $T: X \to X$ *and* $\mu: X \times X \to [0,+\infty)$. *The mapping* T *is* $\mu - subadmissible$ *if for all* $x, y \in X$ *and* $\mu(x,y) \leq 1$ *the inequality* $\mu(Tx,Ty) \leq 1$ *holds*.

Definition 1. 14 *[19, 20]A mapping* $F: P^2 \to P$ *is called cone* C*-class function if it is continuous and satisfies following axioms:*

(1) $F(s,t) \leq s$;

(2) $F(s,t) = s$ *implies that either* $s = \theta$ *or* $t = \theta$*; for all* $s, t \in P$.

We denote C - class functions as $\mathcal{C}$.

In 2014 the concept of pair $(\mathcal{F}, h)$ is an upper class was introduced by A. H. Ansari in [21]

Definition 1. 15 *[21, 22] The function* $h: \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ *is a function of subclass of type I, if* $x \geq 1 \Rightarrow h(1,y) \leq h(x,y)$ *for all* $y \in \mathbb{R}^+$.

Definition 1. 16 *[21, 22] Let* $h, \mathcal{F}: \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$, *then we say that the pair* $(\mathcal{F}, h)$ *is an upper class of type I, if* h *is a function of subclass of type I and:*

(i) $0 \leq s \leq 1 \Rightarrow \mathcal{F}(s,t) \leq \mathcal{F}(1,t)$,

(ii) $h(1,y) \leq \mathcal{F}(s,t) \Rightarrow y \leq st$ *for all* $t, y \in \mathbb{R}^+$.

2. Main Results

Definition 2. 1 *The function* $h: \mathbb{R}^+ \times \mathbb{P} \to \mathbb{P}$ *is a cone function of subclass of type I, if* $x \geq 1 \Rightarrow h(1,y) \leq h(x,y)$ *for all* $y \in$ P.

Example 2. 2 *Define* $h: \mathbb{R}^+ \times \mathbb{P} \to \mathbb{P}$, *where* $P = \{(m,n) \in R^2, m,n \geq 0\}$, *by:*

1. $h(x,(m,n)) = x(m,n)$;
2. $h(x,(m,n)) = e^x(m,n)$,
3. $h(x,(s,t)) = (s,t)$;
4. $h(x,(s,t)) = \frac{1}{n+1}(\sum_{i=0}^{n} x^i)(s,t), n \in \mathbb{N}$;

for all $x \in \mathbb{R}^+$. Then h is a cone function of subclass of type I.

Definition 2. 3 *Let* $h, \mathcal{F}: \mathbb{R}^+ \times \mathbb{P} \to \mathbb{P}$. *The pair* $(\mathcal{F}, h)$ *is an upper class of type I, if* h *is a cone function of subclass of type I and:*

(i) $for\ 0 \leq s \leq 1\ implies\ \mathcal{F}(s,t) \leq \mathcal{F}(1,t)$,
(ii) $if\ h(1,y) \leq \mathcal{F}(s,t)\ then\ y \leq st$ *for all* $t, y \in \mathbb{R}^+$.

Example 2. 4 *Define* $h, \mathcal{F}: \mathbb{R}^+ \times \mathbb{P} \to \mathbb{P}$ *by:*

1. $h(x,(m,n)) = x(m,n)$ and $\mathcal{F}(s,(u,w)) = s(u,w)$;

2. $h(x,y) = \frac{1}{n+1}(\sum_{i=0}^{n} x^i)y,\ n \in \mathbb{N}$ and $\mathcal{F}(s,t) = st$;

where $P = \{(m,n) \in R^2, m,n \geq 0\}$. Then the pair $(\mathcal{F}, h)$ is an cone upper class of type I.

Theorem 2. 5 *Let* (X,q) *be a complete Hausdorff quasi-cone metric space and let* $T: X \to X$ *be a continuous function that satisfies the nonlinear contraction condition:*

$$h(\alpha(x,y), q(T(x),T(y))) \leq \mathcal{F}(\mu(x,y), \psi(M(x,y)).$$

$M(x,y) = \max\{q(x,y), q(Tx,x), q(Ty,y), \frac{1}{2}[q(Tx,y) + q(x,Ty)]\}$ for all $x, y \in X$, the pair $(\mathcal{F}, h)$ is an cone upper class of type I, and $\psi: P \to P$ is a continuous function that for every $t \in P, \psi(t) < t$ and $\forall t_1, t_2 \in P$, for $t_1 < t_2$ implies $\psi(t_1) < \psi(t_2)$.

Suppose that

1. $\alpha: X \times X \to [0,\infty)$ satisfies the property that exists $x_0 \in X$ such that $\alpha(T^n x_0, T^m x_0) \geq 1, \mu(T^n x_0, T^m x_0) \leq 1$ for every $n, m \in \mathbb{N}$.

Then T has a fixed point $x^* \in X$ and the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ is convergent to x^*.

Proof. Define the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$. The sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ is right Cauchy.

Indeed, firstly, is proved that the sequence $\{q(T^{n+1}x_0, T^n x_0)\}_{n\in\mathbb{N}}$ is monotonic decreasing as follows:

$$\begin{aligned} h(1, q(T^{n+1}x_0, T^n x_0)) &\leq h(\alpha(T^n x_0, T^{n-1}x_0), q(T^{n+1}x_0, T^n x_0)) \\ &\leq \mathcal{F}(\mu(T^n x_0, T^{n-1}x_0), \psi(M(T^n x_0, T^{n-1}x_0))) \\ &\leq \mathcal{F}(1, \psi(M(T^n x_0, T^{n-1}x_0))) \\ q(T^{n+1}x_0, T^n x_0) &\leq \psi(M(T^n x_0, T^{n-1}x_0)) \end{aligned}$$

where

$$M(T^n x_0, T^{n-1} x_0) = \left\{ q(T^n x_0, T^{n-1} x_0), q(T^{n+1} x_0, T^n x_0), q(T^n x_0, T^{n-1} x_0), \frac{1}{2}[q(T^{n+1} x_0, T^{n-1} x_0) + q(T^n x_0, T^n x_0)] \right\} = \max\{q(T^n x_0, T^{n-1} x_0), q(T^{n+1} x_0, T^n x_0)\}$$

Case 1. $M(T^n x_0, T^{n-1} x_0) = q(T^{n+1} x_0, T^n x_0)$, so $q(T^{n+1} x_0, T^n x_0) \leq \psi(q(T^{n+1} x_0, T^n x_0))$ which is a contradiction.

Case 2. $M(T^n x_0, T^{n-1} x_0) = q(T^n x_0, T^{n-1} x_0)$, so

$$q(T^{n+1} x_0, T^n x_0) \leq \psi(q(T^n x_0, T^{n-1} x_0)) \leq q(T^n x_0, T^{n-1} x_0) \text{ for all } n \geq 1.$$

Let $C_k = \sup\{q(T^i x_0, T^j x_0), i > j > k\}$. The sequence $\{C_k\}$ is monotonic decreasing and lower bound, so it converges to $C_0 \in P$. As a result, it yields

$$\forall p \in \mathbb{N}, i_p > j_p > p, C_p - \frac{C_0}{p} \leq q\left(T^{i_p} x_0, T^{j_p} x_0\right) \leq C_p \; implies \; q(T^{i_p} x_0, T^{j_p} x_0) \to C_0 \text{ as } p \to \infty$$

Below is proved that $C_0 = 0$.

$$q(T^{i_p+1} x_0, T^{j_p+1} x_0) \leq \alpha(T^{i_p} x_0, T^{j_p} x_0) q(T^{i_p+1} x_0, T^{j_p+1} x_0) \leq \psi(M(T^{i_p} x_0, T^{j_p} x_0))$$

Taking the limit when $p \to +\infty$, is taken $C_0 \leq \psi(C_0)$, so $C_0 = 0$.

Using the same method, it can be proved that the sequence $\{T^n x_0\}$ is left Cauchy. So it is a Cauchy sequence and since the space is complete, it is convergent to x^*.

$$\lim_{n\to\infty} q(T^n x_0, x^*) = \lim_{n\to\infty} q(x^*, T^n x_0) = 0.$$

Since T is continuous $\lim_{n\to\infty} q(T^n x_0, Tx^*) = \lim_{n\to\infty} q(T(T^{n-1} x_0), Tx^*)) = 0.$

Furthermore, $\lim_{n\to\infty} q(Tx^*, T^n x_0) = \lim_{n\to\infty} q(Tx^*, T(T^{n-1} x_0)) = 0.$

Thus, $\lim_{n\to\infty} q(T^n x_0, Tx^*) = \lim_{n\to\infty} q(Tx^*, T^n x_0) = 0.$

As X is Hausdorff, it is true that $x^* = Tx^*$ and x^* is a fixed point of T.

In the following theorem the function $T: X \to X$ is non-continuous.

Theorem 2. 6 *Let (X, q) be a complete, Hausdorff quasi–cone metric space where cone is normal with constant of normality K and $T: X \to X$ be a function that satisfies the nonlinear contraction condition:*

$$h(\alpha(x,y),q(Tx,Ty))) \leq \mathcal{F}(\mu(x,y),\psi(N(x,y))$$

where $N(x,y)) = \max\{q(x,y),\frac{1}{a}[q(Tx,x)+q(Ty,y)],\frac{1}{a}[q(Tx,y)+q(x,Ty)]\}$ for all $x,y \in X, a \geq K, \psi: P \to P$ is a continuous function that for every $t \in P, \psi(t) < t$ and $\forall t_1, t_2 \in P$, for $t_1 < t_2$ implies $\psi(t_1) < \psi(t_2)$.

Suppose that

1. $\alpha: X \times X \to [0,+\infty)$ satisfies the property that exists $x_0 \in X$ such that $\alpha(T^n x_0, T^m x_0) \geq 1, \mu(T^n x_0, T^m x_0) \leq 1$, for every $n,m \in \mathbb{N}$.

2. If $\{T^n x\}$ is a sequence such that $\alpha(T^{n+1}x_0, T^n x_0) \geq 1, \mu(T^{n+1}x_0, T^n x_0) \leq 1$ for all n and $T^n x \to x^*$ as $n \to \infty$, then there exists a subsequence $\{T^{n_k}x\}$ of $\{T^n x\}$ such that $\alpha(T^{n_k}x, x^*) \geq 1, \alpha(x^*, T^{n_k}x) \geq 1, \mu(T^{n_k}x, x^*) \leq 1, \mu(x^*, T^{n_k}x) \leq 1$.

Then T has a fixed point $x^* \in X$ and the sequence $\{T^n x\}_{n\in\mathbb{N}}$ is convergent to x^*.

Proof. Define the sequence $\{T^n x_0)\}_{n\in\mathbb{N}}$. Continuing the same procedure as Theorem 2. 5, we prove that the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ is Cauchy and so it converges to x^*.

From Condition 2 of theorem, there exist a subsequence $\{T^{n_k}x_0\}$ of $\{T^n x_0\}$, such that $\alpha(T^{n_k}x, x^*) \geq 1$, $\alpha(x^*, T^{n_k}x) \geq 1$ for all $k \in N$.

$$\begin{aligned} h(1,q(Tx^*,T^{n_k+1}x_0)) &\leq h(\alpha(x^*,T^{n_k}x_0),q(Tx^*,T^{n_k+1}x_0)) \\ &\leq \mathcal{F}(\mu(x^*,T^{n_k}x_0),\psi(N(Tx^*,T^{n_k+1}x_0))) \\ &\leq \mathcal{F}(1,\psi(N(Tx^*,T^{n_k+1}x_0))) \end{aligned}$$

$$\begin{aligned} q(Tx^*,T^{n_k+1}x_0) &\leq \psi(N(Tx^*,T^{n_k+1}x_0) < N(Tx^*,T^{n_k}x_0) \\ &= \max\{q(x^*,T^{n_k}x_0),\frac{1}{a}[q(Tx^*,x^*)+q(T^{n_k+1}x_0,T^{n_k}x_0)], \\ &\quad \frac{1}{a}[q(Tx^*,T^{n_k}x_0)+q(T^{n_k+1}x_0,x^*)]\} \end{aligned}$$

Taking the limit when $n_k \to +\infty$, $N(Tx^*,T^{n_k}x_0)$ converges to $\frac{1}{a}q(Tx^*,x^*)$.

From $q(Tx^*,T^{n_k+1}x_0) \leq \psi(N(Tx^*,T^{n_k}x_0)) < N(Tx^*,T^{n_k}x_0)$ and taking the limit when $n_k \to +\infty$, is taken the following inequality

$$q(Tx^*,x^*) < \frac{1}{a}q(Tx^*,x^*),$$

which is a contradiction. Consequently, $q(Tx^*,x^*) = 0$ and $Tx^* = x^*$ and x^* is a fixed point of T.

In the following theorem the function T is taken non-continuous.

Theorem 2. 7 *Let* (X,q) *be a complete Hausdorff quasi–cone metric space and let* $T: X \to X$ *be a continuous function that satisfies the nonlinear contraction condition:*

$$h(\alpha(x,y)q(T(x),T(y))) \leq \mathcal{F}(\mu(x,y),\psi(\delta(O(x)\cup O(y))))$$

for all $x,y \in X$, where $\psi: P \to P$ is a comparison function and the pair $(\mathcal{F},h)$ is a cone upper class of type *I*. Suppose that $\alpha,\mu: X\times X \to [0,+\infty)$ satisfies the property that exists $x_0 \in X$ such that $\alpha(T^n x_0, T^m x_0) \geq 1$ and $\mu(T^n x_0, T^m x_0) \leq 1$ for every $n,m \in N$.

Moreover for $x_0 \in X$, the orbit $O(x_0)$ is bounded.

Then T has a fixed point $x^* \in X$ and the sequence $\{T^n x_0\}_{n\in N}$ is convergent to x^*.

Proof. From condition of theorem, there exists $x_0 \in X$ such that $\alpha(T^n x_0, T^m x_0) \geq 1$ and $\mu(T^n x_0, T^m x_0) \leq 1$ for every $n,m \in \mathbb{N}$. Define the sequence $\{T^n x_0\}_{n\in N}$.

Below is proved that the sequence $\{T^n x_0\}_{n\in N}$ is right Cauchy.

Taking $x = T^{n+i}x_0, y = T^{n+j}x_0$, where $i,j,n \in N$ and $i > j$.

$$\begin{aligned} h(1,q(Tx,Ty)) &\leq h(\alpha(x,y),q(Tx,Ty)) \\ &\leq \mathcal{F}(\mu(x,y),\psi(\delta(O(x)\cup O(y)))) \\ &\leq \mathcal{F}(1,\psi(\delta(O(x)\cup O(y)))) \end{aligned}$$

As a result

$$\begin{aligned} q(Tx,Ty) &= q(T^{n+i+1}x_0, T^{n+j+1}x_0) \\ &\leq \psi(\delta(O(x)\cup O(y))) \\ &= \psi(\delta(O(T^{n+i}x_0)\cup O(T^{n+j}x_0))) \\ &\leq \psi(\delta(O(T^n x_0)) \end{aligned}$$

So, it is true that $(T^{n+i+1}x_0, T^{n+j+1}x_0) \leq \psi(\delta(O(T^n x_0))$

for every $i,j,n \in N$ and $i > j$.

Furthermore

$$\begin{aligned} q(T^{n+i+1}x_0, T^{n+j+1}x_0) &\leq \delta(O(T^{n+1}x_0)) \\ &= \max\{q(T^{n+i+1}x_0, T^{n+j+1}x_0), i,j \in \mathbb{N}\} \\ &\leq \psi(\delta(O(T^n x_0))). \end{aligned}$$

From this, it yields

$$\begin{aligned} q(T^{n+i+1}x_0, T^{n+j+1}x_0) &\leq \psi(\delta(O(T^n x_0))) \\ &\leq \psi^2(\delta(O(T^{n-1}x_0))) \leq \cdots \\ &\leq \psi^n(\delta(O(x_0))) \leq \psi^n(c) \end{aligned}$$

Due to $\lim_{n\to\infty} \|\psi^n(c)\| = 0 \Leftrightarrow (\forall \frac{\varepsilon}{K} > 0)(\exists n_0 \in N)(\forall n > n_0 \Rightarrow \|\psi^n(c)\| < \frac{\varepsilon}{K})$, where K is the constant of normality of cone, we have

$$\left\|q(T^{n+i+1}x_0, T^{n+j+1}x_0)\right\| \leq K\|\psi^n(c)\| < K \cdot \frac{\varepsilon}{K} = \varepsilon, \text{ for } n > n_0, \text{ and } i > j.$$

Consequently, the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ is right Cauchy. In the same manner it can be proved that the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ is left Cauchy. Since (X, q) is complete, there exists a point $x^* \in X$ such that the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ is convergent to $x^* \in X$, so $q(T^n x_0, x^*) \to 0$ as $n \to \infty$ and $q(x^*, T^n x_0) \to 0$ as $n \to +\infty$.

The point x^* is a fixed point of T, $Tx^* = x^*$. Indeed, it is true that $\lim_{n\to+\infty} q(T^n x_0, x^*) = 0$ and $\lim_{n\to+\infty} q(x^*, T^n x_0) = 0$. By using the continuity of T, we have $\lim_{n\to+\infty} q(T(T^n x_0), Tx^*) = 0$ and $\lim_{n\to+\infty} q(Tx^*, T(T^n x_0)) = 0$. By uniqueness of the limit, it is concluded that $Tx^* = x^*$.

Theorem 2. 8 *Let (X, q) be a complete Hausdorff quasi–cone metric space and let $T: X \to X$ be a function that satisfies the nonlinear contraction condition:*

$$h(\alpha(x,y)q(T(x),T(y))) \leq \mathcal{F}(\mu(x,y), \psi(\frac{1}{a}\delta(O(x) \cup O(y))))$$

for $x, y \in X$ and $a \geq K$, where $\psi: P \to P$ is a comparison function. Suppose that

1. $\alpha, \mu: X \times X \to [0, \infty)$ satisfies the property that exists $x_0 \in X$ such that $\alpha(T^n x_0, T^m x_0) \geq 1$ and $\mu(T^n x_0, T^m x_0) \leq 1$ for every $n, m \in \mathbb{N}$.

2. If $\{T^n x\}_{n\in\mathbb{N}}$ is a sequence such that for $T^n x \to x^* \in X$ as $n \to \infty$, then there exists a subsequence $\{T^{n_k}x\}$ of $\{T^n x\}_{n\in\mathbb{N}}$ such that $\alpha(T^{n_k}x, T^q x^*) \geq 1$, $\alpha(T^{n_k}x, x^*) \geq 1, \alpha(x^*, T^{n_k}x) \geq 1$ and $\mu(T^{n_k}x, T^q x^*) \leq 1$, $\mu(T^{n_k}x, x^*) \leq 1$, $\mu(x^*, T^{n_k}x) \leq 1$.

Moreover for every $z \in X$, the orbit $O(z)$ is bounded. Then T has a fixed point $x^* \in X$.

Proof. Let $x_0 \in X$ such that $\alpha(T^n x_0, T^m x_0) \geq 1$. Define the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$. Using the same method as in Theorem 2. 7 it can be proved that the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ is Cauchy. Since (X, q) is complete, there exists a point $x^* \in X$ such that the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ converges to $x^* \in X$, so $q(T^n x_0, x^*) \to 0$ as $n \to +\infty$ and $q(x^*, T^n x_0) \to 0$ as $n \to +\infty$.

The next step is to prove that x^* is a fixed point of T, $Tx^* = x^*$. For this, it is needed to prove that the sequence $\{T^m x_0\}_{m\in\mathbb{N}}$ converges to x^*. Suppose that this sequence converges to $l \in X$.

$$h(1, q(T^{n_k+1}x^*, T^{n_k+1}x_0)) \leq h(\alpha(T^{n_k}x^*, T^{n_k}x_0), q(T^{n_k+1}x^*, T^{n_k+1}x_0))$$
$$\leq \mathcal{F}(\mu(T^{n_k}x^*, T^{n_k}x_0), \psi(\frac{1}{a}\delta(O(T^{n_k}x^*) \cup O(T^{n_k}x_0)))$$
$$\leq \mathcal{F}(1, \psi(\frac{1}{a}\delta(O(T^{n_k}x^*) \cup O(T^{n_k}x_0))))$$

So, the following inequality is true.

$$q(T^{n_k+1}x^*, T^{n_k+1}x_0) \leq \psi(\frac{1}{a}\delta(O(T^{n_k}x^*) \cup O(T^{n_k}x_0)))$$
$$\leq \frac{1}{a}\delta(O(T^{n_k}x^*) \cup O(T^{n_k}x_0))$$
$$= \frac{1}{a}\max\{q^{n_k+i}x^*, T^{n_k+j}x^*), q(T^{n_k+p}x_0, T^{n_k+r}x_0), q(T^{n_k+i}x^*, T^{n_k+p}x_0)\}$$

for $i, j, p, r \in \mathbb{N}_0$.

Taking the limit of both sides when $k \to \infty$, $q(l, x^*) < \frac{1}{a}q(l, x^*)$.

So this is a contradiction, and it yields $q(l, x^*) = 0$, $l = x^*$.

Now we prove that x^* is a fixed point of T.

$$h(1, q(T^{n_k+1}x^*, Tx^*)) \leq h(\alpha(T^{n_k}x^*, Tx^*), q(T^{n_k+1}x^*, Tx^*))$$
$$\leq \mathcal{F}(\mu(T^{n_k}x^*, Tx^*), (\frac{1}{a}\delta(O(T^{n_k}x^*) \cup O(x^*))))$$
$$\leq \mathcal{F}(1, (\frac{1}{a}(O(T^{n_k}x^*)\ O(x^*))))$$

Corollary 2. 9 (X, q) *a quasi–cone metric space and* $T: X \to X$ *a* α*-admissible and* μ*-subadmissible such that* $\alpha(x, y)d(Tx, Ty) \leq \mu(x, y)\psi(M(x, y))$,*where* $\psi: P \to P$ *is a continuous function that satisfies the conditions* $\psi(0) = 0, \forall t \in P, \psi(t) < t$ *and* $\forall t_1, t_2 \in P$, for $t_1 < t_2$ implies $\psi(t_1) < \psi(t_2)$ and

1. $\alpha: X \times X \to [0, +\infty)$ satisfies the property that exists $x_0 \in X$ such that $\alpha(T^n x_0, T^m x_0) \geq 1, \mu(T^n x_0, T^m x_0) \leq 1$ for every $n, m \in \mathbb{N}$.

2. If $\{T^n x\}$ is a sequence such that $\alpha(T^{n+1}x_0, T^n x_0) \geq 1, \mu(T^{n+1}x_0, T^n x_0) \leq 1$ for all $n \in N$ and $T^n x \to x^*$ as $n \to +\infty$, then there exists a subsequence $\{T^{n_k}x\}$ of $\{T^n x\}$, such that $\alpha(T^{n_k}x, x^*) \geq 1, \alpha(x^*, T^{n_k}x) \geq 1, \mu(T^{n_k}x, x^*) \leq 1, \mu(x^*, T^{n_k}x) \leq 1$.

Then T has a fixed point $x^* \in X$ and the sequence $\{T^n x\}_{n\in\mathbb{N}}$ is convergent to x^*.

Proof. Taking $h(s,t) = st$ and $F(x,y) = xy$ in Theorem 2. 7, the corollary is true.

Corollary 2. 10 *Let (X,q) a quasi–cone metric space and and $T: X \to X$ a α-admissible and such that $\alpha(x,y)d(Tx,Ty) \leq \psi(M(x,y))$, $\varphi, \psi: P \to P$ satisfies the conditions $\psi(0) = 0, \forall t \in P, \psi(t) < t$ and ψ is semi–lower continuous and*

1. $\alpha: X \times X \to [0,\infty)$ satisfies the property that exists $x_0 \in X$ such that $\alpha(T^n x_0, T^m x_0) \geq 1$for every $n, m \in \mathbb{N}$.

2. If $\{T^n x\}$ is a sequence such that $\alpha(T^{n+1}x_0, T^n x_0) \geq 1$for all $n \in N$ and $T^n x \to x^*$ as $n \to \infty$, then there exists a subsequence $\{T^{n_k}x\}$ of $\{T^n x\}$ such that

$\alpha(T^{n_k}x, x^*) \geq 1, \alpha(x^*, T^{n_k}x) \geq 1$.

Then T has a fixed point $x^* \in X$ and the sequence $\{T^n x\}_{n\in\mathbb{N}}$ is convergent to x^*.

Proof. Taking $h(s,t) = st$ and $F(x,y) = xy$ and $\mu(x,y) = 1$ in Theorem 2. 7, the corollary is proved.

Corollary 2. 11 *Let (X,q) a quasi-cone metric space and $T: X \to X$such that $d(Tx,Ty) \leq \psi(M(x,y))$, $\psi: P \to P$ satisfies the conditions $\psi(0) = 0, \forall t \in P, \psi(t) < t$ and ψ is increasing.*

Then T has a fixed point $x^* \in X$ and the sequence $\{T^n x\}_{n\in\mathbb{N}}$ is convergent to x^*.

Proof. Taking $h(s,t) = st$ and $F(x,y) = xy$ and $\alpha(x,y) = 1, \mu(x,y) = 1$ in Theorem 2. 7, the corollary is proved.

3. Examples

Example 3. 1 *Let $X = [0,1], E = \mathbb{R}^2$, and P a normal cone with constant of normality K, $P = \{(x,y) \in \mathbb{R}^2, x,y \geq 0\}$. Determine $q: X \times X \to P$,*

$$q(x,y) = \begin{cases} (\frac{y}{3}, y), & x < y \\ (0,0), & x = y \\ (x, 3x), & x > y \end{cases}$$

is a quasi–cone metric and (X,q) is quasi-cone metric space.

Taking $h(x,y) = xy$ and $\mathcal{F}(x,y) = xy$ we have that the pair $(\mathcal{F}, h)$ is an cone upper class of type I.

Let $T: X \to X, T(x) = x^2$ be a continuous function, $\psi: P \to P, \psi(x,y) = (\frac{x}{3}, \frac{y}{3})$ be a comparison function and $\alpha: X \times X \to [0,\infty), \alpha(x,y) = \begin{cases} \frac{1}{2}\max(x,y), x \neq y \\ a, x = y \end{cases}$, where $a \geq 1$, $\mu: X \times X \to [0,\infty), \mu(x,y) = \begin{cases} 1, & x \neq y \\ b, & x = y \end{cases}$, where $b < 1$. The function T satisfies the condition of Theorem 2. 7.

Indeed, taking $x_0 = 0$ it is true that $\alpha(T^n x_0, T^m x_0) = a \geq 1$ and $\mu(T^n x_0, T^m x_0) = b < 1$.

Moreover, T satisfies the nonlinear contraction of Theorem 2. 7.

Case 1. $x = y$

This case is trivial because $q(Tx, Ty) = 0$.

Case 2. $x = y$

In this case

$$q(Tx,Ty) = q(x^2,y^2) = (\frac{y^2}{3}, y), \alpha(x,y) = \frac{y}{3}$$
$$\mu(x,y) = 1, h(\alpha(x,y), q(Tx,Ty)) = (\frac{y^3}{3}, y^3)$$
$$\begin{aligned}\delta(O(x) \cup O(y)) = \max\{&q(x,y), q(x,Tx), q(y,Ty), q(x,T^i x),\\ &q(y,T^j y), q(T^i x, T^j y), q(T^i x, T^k x), q(T^j y, T^p y)\}\end{aligned}$$
$$\delta(O(x) \cup O(y)) = (y, 3y), \psi(\delta(O(x) \cup O(y)) = (\frac{y}{3}, y)$$

Consequently, the nonlinear contraction of Theorem 2.7 is taken.

$$h(\alpha(x,y), q(Tx,Ty)) \leq \mathcal{F}(\mu(x,y), \psi(\delta(O(x) \cup O(y)))).$$

Case 3. $x > y$

$$q(Tx,Ty) = q(x^2,y^2) = (x^2, 3x^2), \alpha(x,y) = \frac{x}{3},$$

$$\mu(x,y) = 1, h(\alpha(x,y), q(Tx,Ty)) = (\frac{x^3}{3}, x^3)$$

$$\delta(O(x) \cup O(y)) = \max\{q(x,y), q(x,Tx), q(y,Ty), q(x,T^i x), q(y,T^j y),$$

$$q(T^i x, T^j y), q(T^i x, T^k x), q(T^j y, T^p y)\}$$

$$\delta(O(x) \cup O(y)) = (x, 3x), \psi(\delta(O(x) \cup O(y)) = (\frac{x}{3}, x)$$

As a result, the nonlinear contraction of Theorem 2.7 is completed.

$$h(\alpha(x,y), q(Tx,Ty)) \leq \mathcal{F}(\mu(x,y), \psi(\delta(O(x) \cup O(y)))).$$

So the function T has fixed points. The points $x = 0$ and $x = 1$ are the fixed points of T.

Example 3. 2 *Let* $X = [0,1], E = \mathbb{R}^2$, *and* P *be a normal cone,* $P = \{(x,y) \in \mathbb{R}^2, x, y \geq 0\}$. *Determine* $q: X \times X \to P$, *such that*

$$q(x,y) = \begin{cases} (\frac{y}{2}, y), & x < y \\ (0,0), & x = y \\ (x, 2x) & x > y \end{cases}$$

is a quasi–cone metric and (X, q) is quasi–cone metric space.

Let

$$T: X \to X, T(x) = \begin{cases} \frac{x^4}{3}, & 0 \leq x < \frac{1}{9} \\ \frac{1}{9}, & \frac{1}{9} \leq x \leq 1 \end{cases}$$

be a non–continuous function, $\psi: P \to P, \psi(x,y) = (\frac{x}{2}, \frac{y}{2})$ be a comparison function and $\alpha: X \times X \to [0, \infty)$,

$$\alpha(x,y) = \begin{cases} 0, & (x,y) \in [0, \frac{1}{9}) \times [\frac{1}{9}, 1] \cup [\frac{1}{9}, 1] \times [0, \frac{1}{9}) \\ b, & x = y \\ \frac{1}{3}\max\{x,y\}, & \text{otherwise} \end{cases}$$

where $b \geq 1$, $\mu: X \times X \to [0, \infty)$,

$$\mu(x,y) = \begin{cases} \frac{1}{2}, & (x,y) \in [0, \frac{1}{9}) \times [\frac{1}{9}, 1] \cup [\frac{1}{9}, 1] \times [0, \frac{1}{9}) \\ c, & x = y \\ \max\{x,y\}, & \text{otherwise} \end{cases}$$

where $0 \leq c < 1$.

The function satisfies the conditions of Theorem 2. 8.

Indeed, taking $x_0 = 0$ the following inequalities are true $\alpha(T^n x_0, T^m x_0) \geq, \mu(T^n x_0, T^m x_0) \leq 1$.

Also $T^n 0 \to 0$ when $n \to +\infty$, so $\alpha(T^n 0,0) \geq 1$, $\mu(T^n 0,0) \leq 1$.

Below is shown that the function T satisfies the nonlinear contraction of Theorem 2. 8.

Consequently, it needs to prove that for every $(x, y) \in X \times X$, the inequality

$$\frac{1}{n+1}(\sum_{i=0}^{n}(\alpha(x,y))^i)q(Tx,Ty) \leq \mu(x,y)\psi(\delta(O(x) \cup O(y))$$

holds.

Case 1. $x = y$. This case is trivial because $q(Tx, Ty) = 0$.

Case 2. 1. $x, y \in [0, \frac{1}{9}), x < y$. In this case $Tx = \frac{x^4}{3}, Ty = \frac{y^4}{3}, \alpha(x,y) = \frac{y}{3}, \mu(x,y) = y$,

$$\frac{1}{n+1}(\sum_{i=0}^{n}(\frac{y}{3})^i)q(\frac{x^4}{3},\frac{y^4}{3}) = \frac{1}{n+1}\frac{1-(\frac{y}{3})^n}{1-\frac{y}{3}}(\frac{y^4}{6},\frac{y^4}{3})$$

$$\leq \frac{1}{n+1}\frac{3}{3-y}(\frac{y^4}{6},\frac{y^4}{3}) \leq \frac{3}{4}(\frac{y^4}{6},\frac{y^4}{3}) = (\frac{y^4}{8},\frac{y^4}{4}) \leq y(\frac{y}{2},y)$$

$$= \mu(x,y)\psi(\delta(O(x) \cup O(y))$$

$$\delta(O(x) \cup O(y)) = \max\{q(x,y), q(x,Tx), q(y,Ty), q(x,T^i x), q(y,T^j y), q(T^i x, T^j y), q(T^i x, T^k x), q(T^j y, T^p y)\}$$

$$\delta(O(x) \cup O(y)) = (y, 2y), \psi(\delta(O(x) \cup O(y)) = (\frac{y}{2}, y)$$

Case 2. 2. $x, y \in [0, \frac{1}{9}), x > y$ In this case $Tx = \frac{x^4}{3}, Ty = \frac{y^4}{3}, \alpha(x,y) = \frac{x}{3}, \mu(x,y) = x$, .

$$\frac{1}{n+1}(\sum_{i=0}^{n}(\frac{x}{3})^i)q(\frac{x^4}{3},\frac{y^4}{3}) = \frac{1}{n+1}\frac{1-(\frac{x}{3})^n}{1-\frac{x}{3}}(\frac{x^4}{3},\frac{2y^4}{3}) \leq \frac{1}{n+1}\cdot\frac{3}{3-x}(\frac{x^4}{3},\frac{2y^4}{3}) \leq \frac{3}{4}(\frac{x^4}{3},\frac{2y^4}{3})$$

$$= (\frac{x^4}{4}, \frac{y^4}{2}) \leq (\frac{x}{2}, y) = \mu(x,y)\psi(\delta(O(x) \cup O(y))\ \delta(O(x) \cup O(y))$$
$$= \max\{q(x,y), q(x,Tx), q(y,Ty), q(x,T^i x), q(y,T^j y), q(T^i x, T^j y), q(T^i x, T^k x), q(T^j y, T^p y)\}$$

$$\delta(O(x) \cup O(y)) = (x, 2y), \qquad \psi(\delta(O(x) \cup O(y)) = (\frac{x}{2}, y)$$

Case 3. 1. $x, y \in [\frac{1}{9}, 1], x < y$ or $x > y$. So, $Tx = \frac{1}{9}, Ty = \frac{1}{9}, q(Tx, Ty) = 0$, so

$$\frac{1}{n+1}(\sum_{i=0}^{n}(\alpha(x,y))^i)q(Tx,Ty) \leq \mu(x,y)\psi(\delta(O(x) \cup O(y))$$

Case 4. 1. $x \in [0, \frac{1}{9}), y \in [\frac{1}{9}, 1]$. Consequently, $Tx = \frac{x^4}{3}, Ty = \frac{1}{9}, \alpha(x,y) = 0, \mu(x,y) = \frac{1}{2}$, so

$$\frac{1}{n+1}(\sum_{i=0}^{n}(\alpha(x,y))^i)q(Tx,Ty) \leq \mu(x,y)\psi(\delta(O(x) \cup O(y))$$

Case 4. 2. $y \in [0, \frac{1}{9}), x \in [\frac{1}{9}, 1]$. In this case $Ty = \frac{y^4}{3}, Tx = \frac{1}{9}, q(Tx, Ty) = 0$, so

$$\frac{1}{n+1}(\sum_{i=0}^{n}(\alpha(x,y))^i)q(Tx,Ty) \leq \mu(x,y)\psi(\delta(O(x) \cup O(y))$$

So the function T has fixed points. The points $x = 0$ and $x = \frac{1}{9}$ are the fixed points of T.

4. Conclusion

Conclusion 4. 1 For $h(x,y) = xy$ and $\mathcal{F}(s,t) = st$ and $\mu(x,y) = 1$ in Theorem 2.7 and Theorem 2.8, there are taken the conditions of results of [17]. So, these results generalize the results of [17].
Conclusion 4. 2 Corollary 2.10 is a generalization of Theorem 3.4 of [1] and Theorem 14 of [2].

Conclusion 4. 3 Every result mentioned in corollaries above are true in case when it is replaced $M(x,y)$ by $\delta\big(O(x) \cup O(y)\big)$, which are generalizations of results in [9].

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Configurations of SDM Methods Proposed between 1999 and 2012: A Follow-up Study

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Abstract

Recently, the concept of fuzzy parameterized fuzzy soft matrices (*fpfs*-matrices) has become a prominent mathematical tool to cope with decision-making problems, where both parameters and alternatives are fuzzy. Therefore, many soft decision-making (SDM) methods, constructed by the substructures of this concept, have been configured faithfully to the original to render them operable in *fpfs*-matrices space and successfully applied to decision-making problems. In this study, we complete the configurations of the SDM methods having been constructed with soft sets, soft matrices, and their fuzzy hybrid versions and proposed between 1999 and 2012. Afterwards, we apply the configured methods herein to a performance based-value assignment (PVA) intended for the known filters used in image denoising, so that we can compare their ranking performances. Finally, we discuss the need for further research.

Keywords – Fuzzy sets, soft sets, soft matrices, *fpfs*-matrices, soft decision-making, PVA problems

1. Introduction

The concepts of soft sets [1], fuzzy soft sets [2,3], fuzzy parameterized soft sets [4], and fuzzy parameterized fuzzy soft sets (*fpfs*-sets) [5] have been propounded to model uncertainties in which alternatives or parameters are fuzzy. Moreover, to allow for processing soft decision-making (SDM) methods constructed by these concepts in a computer environment, the concepts of soft matrices [6], fuzzy soft matrices (*fs*-matrices) [7], and fuzzy parameterized fuzzy soft matrices (*fpfs*-matrices) [8] have been introduced. So far, many theoretical and applied studies in certain areas ranging from algebra to machine learning have been conducted on the concepts [9-25].

Recently, 43 SDM methods, constructed by the aforesaid concepts, have been configured in [26-28] to operate them in *fpfs*-matrices space and successfully applied to performance-based value assignment (PVA) problems. Moreover, several methods in [29-34] have been simplified and improved to apply them to PVA problems. The results show that the improvements have provided a crucial advantage in the presence of PVA problems.

The primary motivation of this study is to complete the configurations of the SDM methods having been constructed with soft sets, soft matrices, and their fuzzy hybrid versions and proposed between 1999 and 2012. Therefore, we consider the methods provided in [35-43]. In [35], the authors have availed of fuzzy soft sets to model a problem concerning object recognition. After that, [36] has revised the method provided in [35] and applied it to the same problem. In [37], the authors have developed a method constructed by soft sets and based on the grey relational analysis. Moreover, they have applied this method to a problem concerning the eligibility of candidates in an online job advertisement and compared it with the uni-int SDM method [10]. In [38], the authors have applied *fs*-matrices to the problem of finding a healthy environment. [39] has used *fs*-matrices to arrive at a decision on a purchasing problem. Similarly, [40] has suggested two SDM methods using the soft matrices to decide upon a dress and factory purchase problem. In [41], the authors have attempted to overcome a city selection problem by employing fuzzy parameterized soft sets. [42] has studied a house purchase problem through soft sets. In [43], the authors have proposed a method by revising the algorithm given in [35].

In Section 2 of the present paper, we give the concepts of *fs*-matrices and *fpfs*-matrices and some of their basic definitions. Besides, we present three of the configured SDM methods in [26,28] to be employed in the next sections. In Section 3, to operate in *fpfs*-matrices, we configure 12 SDM methods, constructed by soft sets, fuzzy soft sets, fuzzy parameterized soft sets, soft matrices, and *fs*-matrices. In Section 4, we apply six of the configured methods to a PVA problem with the known filters used in image denoising. In Section 5, we discuss the need for further research.

2. Preliminaries

In this section, firstly, we present the concepts of *fs*-matrices [7] and *fpfs*-matrices [8] and some of their basic definitions. Throughout this study, let E be a parameter set, $F(E)$ be the set of all fuzzy sets over E, and $\mu \in F(E)$. Here, a fuzzy set is denoted by $\left\{{}^{\mu(x)}x \mid x \in E\right\}$.

Definition 1. [2,3] *Let U be a universal set, E be a parameter set, and α be a function from E to $F(U)$. Then, the set $\left\{\left(x, \alpha(x)\right) \mid x \in E\right\}$, being the graphic of α, is called a fuzzy soft set (fs-set) parameterized via E over U (or briefly over U).*

In the present paper, the set of all the *fs*-sets over U is denoted by $FS_E(U)$. In $FS_E(U)$, since the $graph(\alpha)$ and α generate each other uniquely, the notations are interchangeable. Therefore, as long as it causes no confusion, we denote an *fs*-set $graph(\alpha)$ by α.

Example 1. *Let $E = \{x_1, x_2, x_3\}$ and $U = \{u_1, u_2, u_3, u_4\}$. Then,*

$$\alpha = \{(x_1, \{{}^{0.6}u_1, {}^{0.3}u_3, {}^{0.5}u_4\}), (x_2, \{{}^{0.7}u_3, {}^{0.1}u_4\}), (x_3, \{{}^{0.2}u_2, {}^{0.9}u_3, {}^{0.5}u_4\})\}$$

is an *fs*-set over U.

Definition 2. [7] *Let $\alpha \in FS_E(U)$. Then, $[a_{ij}]$ is called fs-matrix of α and is defined by*

$$[a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \end{bmatrix}$$

such that $a_{ij} := \alpha(x_j)(u_i)$, for $i, j \in \{1, 2, \cdots\}$. Here, if $|U| = m$ and $|E| = n$, then $[a_{ij}]$ has order $m \times n$.

From now on, the set of all the *fs*-matrices parameterized via E over U is denoted by $FS_E[U]$.

Example 2. *The fs-matrix of α provided in Example 1 is as follows:*

$$[a_{ij}] = \begin{bmatrix} 0.6 & 0 & 0 \\ 0 & 0 & 0.2 \\ 0.3 & 0.7 & 0.9 \\ 0.5 & 0.1 & 0.5 \end{bmatrix}$$

Definition 3. [7] *Let $[a_{ij}]_{m\times n_1} \in FS_{E_1}[U]$, $[b_{ik}]_{m\times n_2} \in FS_{E_2}[U]$, and $[c_{ip}]_{m\times n_1 n_2} \in FS_{E_1\times E_2}[U]$ such that $p = n_2(j-1)+k$. For all i and p, if $c_{ip} := \min\{a_{ij}, b_{ik}\}$, then $[c_{ip}]$ is called and-product of $[a_{ij}]$ and $[b_{ik}]$ and is denoted by $[a_{ij}] \wedge [b_{ik}]$.*

Definition 4. [5] *Let U be a universal set, $\mu \in F(E)$, and α be a function from μ to $F(U)$. Then, the set $\left\{\left({}^{\mu(x)}x, \alpha\left({}^{\mu(x)}x\right)\right) \mid x \in E\right\}$, being the graphic of α, is called a fuzzy parameterized fuzzy soft set (fpfs-set) parameterized via E over U (or briefly over U).*

In this paper, the set of all the *fpfs*-sets over U is denoted by $FPFS_E(U)$. In $FPFS_E(U)$, since the $graph(\alpha)$ and α generate each other uniquely, the notations are interchangeable. Therefore, as long as it causes no confusion, we denote an *fpfs*-set $graph(\alpha)$ by α.

Example 3. *Let $E = \{x_1, x_2, x_3, x_4\}$ and $U = \{u_1, u_2, u_3, u_4, u_5\}$. Then,*

$$\alpha = \{({}^{0.3}x_1, \{{}^{0.4}u_1, {}^{0.8}u_2, {}^{0.2}u_4, {}^{0.9}u_5\}), ({}^{0.1}x_2, \{{}^{0.7}u_3, {}^{0.5}u_5\}), ({}^{0.3}x_3, \{{}^{0.6}u_2, {}^{0.1}u_3, {}^{0.5}u_4\}), ({}^{0.7}x_4, \{{}^{0.3}u_2, {}^{0.8}u_4, {}^{1}u_5\})\}$$

is an *fpfs*-set over U.

Definition 5. [8] *Let $\alpha \in FPFS_E(U)$. Then, $[a_{ij}]$ is called the fpfs-matrix of α and is defined by*

$$[a_{ij}] = \begin{bmatrix} a_{01} & a_{02} & a_{03} & \cdots & a_{0n} & \cdots \\ a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \end{bmatrix}$$

such that for $i \in \{0,1,2,\cdots\}$ and $j \in \{1,2,\cdots\}$,

$$a_{ij} := \begin{cases} \mu(x_j), & i = 0 \\ \alpha\left({}^{\mu(x_j)}x_j\right)(u_i), & i \neq 0 \end{cases}$$

Here, if $|U| = m-1$ and $|E| = n$, then $[a_{ij}]$ has order $m \times n$.

Hereinafter, the set of all the *fpfs*-matrices parameterized via E over U is denoted by $FPFS_E[U]$.

Example 4. *The fpfs-matrix of α provided in Example 3 is as follows:*

$$[a_{ij}] = \begin{bmatrix} 0.3 & 0.1 & 0.3 & 0.7 \\ 0.4 & 0 & 0 & 0 \\ 0.8 & 0 & 0.6 & 0.3 \\ 0 & 0.7 & 0.1 & 0 \\ 0.2 & 0 & 0.5 & 0.8 \\ 0.9 & 0.5 & 0 & 1 \end{bmatrix}$$

Definition 6. [8] *Let $[a_{ij}], [b_{ij}], [c_{ij}] \in FPFS_E[U]$. For all i and j, if $c_{ij} := \max\{a_{ij}, b_{ij}\}$, then $[c_{ij}]$ is called union of $[a_{ij}]$ and $[b_{ij}]$ and is denoted by $[a_{ij}] \tilde{\cup} [b_{ij}]$.*

Definition 7. [8] *Let $[a_{ij}], [b_{ij}] \in FPFS_E[U]$. For all i and j, if $b_{ij} := 1 - a_{ij}$, then $[b_{ij}]$ is called complement of $[a_{ij}]$ and is denoted by $[a_{ij}]^{\tilde{c}}$ or $[a_{ij}^{\tilde{c}}]$.*

To obtain an increasing sequence consisting of all the elements of an index set, being a subset of $\mathbb{N}^n$, we define a linear ordering relation over $\mathbb{N}^n$ as follows:

Definition 8. *Let $(j_1, j_2, \dots, j_n), (k_1, k_2, \dots, k_n) \in \mathbb{N}^n$. Then, the relation "$\leq$" is called linear ordering relation and is defined by*

$$(j_1, j_2, \dots, j_n) \leq (k_1, k_2, \dots, k_n) \Leftrightarrow [j_1 < k_1 \vee (j_1 = k_1 \wedge j_2 < k_2) \vee \dots \vee (j_1 = k_1 \wedge j_2 = k_2 \wedge \dots \wedge j_{n-1} = k_{n-1} \wedge j_n \leq k_n)]$$

Secondly, we give some of the configured algorithms, provided in [26,28], to be required in the next section of the paper. Henceforth, $I_n = \{1,2,\cdots,n\}$ and $I_n^* = \{0,1,2,\cdots,n\}$.

MBR01 [26]

Step 1. Construct an *fpfs*-matrix $[a_{ij}]_{m\times n}$

Step 2. Obtain $[b_{ik}]_{(m-1)\times(m-1)}$ defined by $b_{ik} := \sum_{j=1}^{n} a_{0j}\chi(a_{ij}, a_{kj}), \quad i,k \in I_{m-1}$ such that

$$\chi(a_{ij}, a_{kj}) := \begin{cases} 1, & a_{ij} \geq a_{kj} \\ 0, & a_{ij} < a_{kj} \end{cases}$$

Step 3. Obtain $[c_{i1}]_{(m-1)\times 1}$ defined by $c_{i1} := \sum_{k=1}^{m-1} b_{ik}, \quad i \in I_{m-1}$

Step 4. Obtain $[d_{i1}]_{(m-1)\times 1}$ defined by $d_{i1} := \sum_{k=1}^{m-1} b_{ki}, \quad i \in I_{m-1}$

Step 5. Obtain the score matrix $[s_{i1}]_{(m-1)\times 1}$ defined by $s_{i1} := c_{i1} - d_{i1}, \quad i \in I_{m-1}$

Step 6. Obtain the decision set $\{^{\mu(u_k)}u_k | u_k \in U\}$ such that $\mu(u_k) = \frac{s_{k1} + |\min_i s_{i1}|}{\max_i s_{i1} + |\min_i s_{i1}|}$

MRB02 [26]

Step 1. Construct an *fpfs*-matrix $[a_{ij}]_{m\times n}$

Step 2. Obtain the score matrix $[s_{i1}]_{(m-1)\times 1}$ defined by $s_{i1} := \sum_{j=1}^{n} a_{0j}a_{ij}, \quad i \in I_{m-1}$

Step 3. Obtain the decision set $\{^{\mu(u_k)}u_k | u_k \in U\}$ such that $\mu(u_k) = \frac{s_{k1}}{\max_i s_{i1}}$

M11 [28]

Step 1. Construct an *fpfs*-matrix $[a_{ij}]_{m\times n}$

Step 2. Obtain $[b_{ik}]_{(m-1)\times(m-1)}$ defined by $b_{ik} \coloneqq \sum_{j=1}^{n} a_{0j}(a_{ij} - a_{kj}), \quad i,k \in I_{m-1}$

Step 3. Obtain the score matrix $[s_{i1}]_{(m-1)\times 1}$ defined by $s_{i1} \coloneqq \sum_{k=1}^{m-1} b_{ik}, \quad i \in I_{m-1}$

Step 4. Obtain the decision set $\{^{\mu(u_k)}u_k | u_k \in U\}$ such that $\mu(u_k) = \frac{s_{k1} + |\min_i s_{i1}|}{\max_i s_{i1} + |\min_i s_{i1}|}$

3. Configurations of 12 SDM Methods

In this section, we configure 12 SDM methods constructed by soft sets [37,42], fuzzy soft sets [35,36,43], fuzzy parameterized soft sets [41], soft matrices [40], and *fs*-matrices [38,39]. Thus, we complete the configurations of the SDM methods having been constructed with soft sets, soft matrices, and their fuzzy hybrid versions and proposed between 1999 and 2012. In [26-28], the authors have propounded a notation consisting of the first letters of the authors' surnames of the papers and the last two digits of their publication year to refer to each method. We too use this notation style in the present study.

In [35], the authors have benefitted from fuzzy soft sets to model a problem concerning object recognition. We configure the proposed methods therein as follows:

Algorithm 1 (RM07a)

Step 1. Construct three *fpfs*-matrices $[a_{ij}]_{m\times n_1}$, $[b_{ik}]_{m\times n_2}$, and $[c_{il}]_{m\times n_3}$

Step 2. Determine an indices set R such that $R \subseteq I_{n_1} \times I_{n_2} \times I_{n_3}$

Step 3. Obtain increasing sequence (r_t) consisting of all elements of R such that $r_t \coloneqq (u_t, v_t, w_t)$

Step 4. Obtain $[d_{it}]_{m\times|R|}$ defined by

$$d_{it} \coloneqq \min_{r_t\in R}\{a_{iu_t}, b_{iv_t}, c_{iw_t}\}, \quad i \in I_{m-1}^*$$

Here, $|R|$ denotes the cardinality of R.

Step 5. Apply MBR01 to $[d_{it}]$

Algorithm 2 (RM07o)

Step 1. Construct three *fpfs*-matrices $[a_{ij}]_{m\times n_1}$, $[b_{ik}]_{m\times n_2}$, and $[c_{il}]_{m\times n_3}$

Step 2. Determine an indices set R such that $R \subseteq I_{n_1} \times I_{n_2} \times I_{n_3}$

Step 3. Obtain increasing sequence (r_t) consisting of all elements of R such that $r_t \coloneqq (u_t, v_t, w_t)$

Step 4. Obtain $[d_{it}]_{m\times|R|}$ defined by

$$d_{it} \coloneqq \max_{r_t\in R}\{a_{iu_t}, b_{iv_t}, c_{iw_t}\}, \quad i \in I_{m-1}^*$$

Here, $|R|$ denotes the cardinality of R.

Step 5. Apply MBR01 to $[d_{it}]$

In [36], the researchers have revised the method provided in [35]. We configure the proposed method therein as follows:

Algorithm 3 (KGW09)

Step 1. Apply RM07a except its Step 5
Step 2. Apply M11 to $[d_{it}]$

[37] has developed a new method constructed by soft sets and based on the grey relational analysis. We configure the proposed method therein as follows:

Algorithm 4 (KWW11/2(w,z))

Step 1. Construct two *fpfs*-matrices $\left[a_{ij}\right]_{m\times n}$ and $\left[b_{ij}\right]_{m\times n}$
Step 2. Apply MRB02 to the matrices $\left[a_{ij}\right]_{m\times n}$ and $\left[b_{ij}\right]_{m\times n}$ and obtain the score matrices $[s_{i1}^1]_{(m-1)\times 1}$ and $[s_{i1}^2]_{(m-1)\times 1}$, respectively
Step 3. Obtain $[c_{i1}]_{(m-1)\times 1}$ and $[d_{i1}]_{(m-1)\times 1}$ defined by

$$c_{i1} := \frac{s_{i1}^1 - \min\limits_{k\in I_{m-1}} s_{k1}^1}{\max\limits_{k\in I_{m-1}} s_{k1}^1 - \min\limits_{k\in I_{m-1}} s_{k1}^1}, \quad i \in I_{m-1}$$

and

$$d_{i1} := \frac{s_{i1}^2 - \min\limits_{k\in I_{m-1}} s_{k1}^2}{\max\limits_{k\in I_{m-1}} s_{k1}^2 - \min\limits_{k\in I_{m-1}} s_{k1}^2}, \quad i \in I_{m-1}$$

Step 4. Obtain $[e_{i1}]_{(m-1)\times 1}$ and $[f_{i1}]_{(m-1)\times 1}$ defined by

$$e_{i1} := \max_{k\in I_{m-1}} c_{k1} - c_{i1}, \quad i \in I_{m-1}$$

and

$$f_{i1} := \max_{k\in I_{m-1}} d_{k1} - d_{i1}, \quad i \in I_{m-1}$$

Step 5. For $w \in [0,1]$, obtain $[g_{i1}]_{(m-1)\times 1}$ and $[h_{i1}]_{(m-1)\times 1}$ defined by

$$g_{i1} := \frac{\min\limits_{k\in I_{m-1}} \{e_{k1}, f_{k1}\} + w \max\limits_{k\in I_{m-1}} \{e_{k1}, f_{k1}\}}{e_{i1} + w \max\limits_{k\in I_{m-1}} \{e_{k1}, f_{k1}\}}, \quad i \in I_{m-1}$$

and

$$h_{i1} := \frac{\min\limits_{k\in I_{m-1}} \{e_{k1}, f_{k1}\} + w \max\limits_{k\in I_{m-1}} \{e_{k1}, f_{k1}\}}{f_{i1} + w \max\limits_{k\in I_{m-1}} \{e_{k1}, f_{k1}\}}, \quad i \in I_{m-1}$$

Step 6. For $z \in [0,1]$, obtain the score matrix $[s_{i1}]_{(m-1)\times 1}$ defined by

$$s_{i1} := z g_{i1} + (1-z) h_{i1}, \quad i \in I_{m-1}$$

Step 7. Obtain the (w,z)-decision set $\{^{\mu(u_k)}u_k | u_k \in U\}$ such that $\mu(u_k) = \frac{s_{k1}}{\max_i s_{i1}}$

In [38], the authors have applied *fs*-matrices to the problem of finding a healthy environment. We configure the proposed method therein as follows:

Algorithm 5 (NS11)

Step 1. Construct two *fpfs*-matrices $[a_{ij}]_{m\times n}$ and $[b_{ij}]_{m\times n}$

Step 2. Obtain *fpfs*-matrices $[a_{ij}]_{m\times n}^{\tilde{c}}$ and $[b_{ij}]_{m\times n}^{\tilde{c}}$

Step 3. Obtain union *fpfs*-matrix $[c_{ij}]_{m\times n}$ of $[a_{ij}]_{m\times n}$ and $[b_{ij}]_{m\times n}$

Step 4. Obtain union *fpfs*-matrix $[d_{ij}]_{m\times n}$ of $[a_{ij}]_{m\times n}^{\tilde{c}}$ and $[b_{ij}]_{m\times n}^{\tilde{c}}$

Step 5. Obtain $[e_{ij}]_{(m-1)\times n}$ defined by

$$e_{ij} \coloneqq (c_{0j} - d_{0j})(c_{ij} - d_{ij}),\ i \in I_{m-1} \text{ and } j \in I_n$$

Step 6. Obtain the score matrix $[s_{i1}]_{(m-1)\times 1}$ defined by

$$s_{i1} \coloneqq \sum_{j=1}^{n} e_{ij}, \quad i \in I_{m-1}$$

Step 7. Obtain the decision set $\{^{\mu(u_k)}u_k | u_k \in U\}$ such that $\mu(u_k) = \frac{s_{k1} + |\min_i s_{i1}|}{\max_i s_{i1} + |\min_i s_{i1}|}$

[39] has used *fs*-matrices to make a decision on a purchasing problem. We configure the proposed method therein as follows:

Algorithm 6 (BMM12)

Step 1. Construct two *fpfs*-matrices $[a_{ij}]_{m\times n}$ and $[b_{ij}]_{m\times n}$

Step 2. Obtain $[c_{kj}]_{n\times n}$ and $[d_{kj}]_{n\times n}$ defined by

$$c_{kj} \coloneqq \min\{a_{0j}, \mathrm{b}_{0k}\}, \quad k,j \in I_n$$

and

$$d_{kj} \coloneqq c_{jk}, \quad k,j \in I_n$$

Step 3. Obtain $[e_{ij}]_{(m-1)\times n}$ and $[f_{ij}]_{(m-1)\times n}$ defined by

$$e_{ij} \coloneqq \sum_{k=1}^{n} \min\{a_{ik}, d_{kj}\}, \quad i \in I_{m-1} \text{ and } j \in I_n$$

and

$$f_{ij} \coloneqq \sum_{k=1}^{n} \min\{b_{ik}, c_{kj}\}, \quad i \in I_{m-1} \text{ and } j \in I_n$$

Step 4. Obtain $[g_{ij}]_{(m-1)\times n}$ and $[h_{ij}]_{(m-1)\times n}$ defined by

$$g_{ij} \coloneqq \frac{e_{ij}}{\sum_{i=1}^{m-1}\sum_{j=1}^{n} e_{ij}}, \quad i \in I_{m-1} \text{ and } j \in I_n$$

and

$$h_{ij} \coloneqq \frac{f_{ij}}{\sum_{i=1}^{m-1}\sum_{j=1}^{n} f_{ij}}, \quad i \in I_{m-1} \text{ and } j \in I_n$$

Step 5. Obtain $[u_{ij}]_{(m-1)\times n}$

$$u_{ij} \coloneqq \max\{g_{ij}, h_{ij}\}, \quad i \in I_{m-1} \text{ and } j \in I_n$$

Step 6. Obtain the score matrix $[s_{i1}]_{(m-1)\times 1}$ defined by

$$s_{i1} \coloneqq \sum_{j=1}^{n} u_{ij}, \quad i \in I_{m-1}$$

Step 7. Obtain the decision set $\{^{\mu(u_k)}u_k | u_k \in U\}$ such that $\mu(u_k) = \frac{s_{k1}}{\max_i s_{i1}}$

The authors in [40] have modelled the problems of buying a dress and factory through soft matrices. We configure the proposed methods therein as follows:

Algorithm 7 (BMM12/2)

Step 1. Construct three *fpfs*-matrices $[a_{ij}]_{m\times n}$, $[b_{ij}]_{m\times n}$, and $[c_{ij}]_{m\times n}$

Step 2. Obtain $[d_{kj}]_{n\times n}$, $[e_{kj}]_{n\times n}$, and $[f_{kj}]_{n\times n}$ defined by

$$d_{kj} \coloneqq \min\{\mathrm{a}_{0j}, b_{0k}, \mathrm{c}_{0k}\}, \quad k, j \in I_n$$

$$e_{kj} \coloneqq \min\{\mathrm{a}_{0k}, b_{0j}, \mathrm{c}_{0k}\}, \quad k, j \in I_n$$

and

$$f_{kj} \coloneqq \min\{\mathrm{a}_{0k}, b_{0k}, \mathrm{c}_{0j}\}, \quad k, j \in I_n$$

Step 3. Obtain $[u_{ij}]_{(m-1)\times n}$, $[v_{ij}]_{(m-1)\times n}$ and $[w_{ij}]_{(m-1)\times n}$ defined by

$$u_{ij} \coloneqq \sum_{k=1}^{n} \min\{a_{ik}, d_{kj}\}, \quad i \in I_{m-1} \text{ and } j \in I_n$$

$$v_{ij} \coloneqq \sum_{k=1}^{n} \min\{b_{ik}, e_{kj}\}, \quad i \in I_{m-1} \text{ and } j \in I_n$$

and

$$w_{ij} \coloneqq \sum_{k=1}^{n} \min\{c_{ik}, f_{kj}\}, \quad i \in I_{m-1} \text{ and } j \in I_n$$

Step 4. Obtain $[g_{ij}]_{(m-1)\times n}$

$$g_{ij} := \max\{u_{ij}, v_{ij}, w_{ij}\}, \quad i \in I_{m-1} \text{ and } j \in I_n$$

Step 5. Obtain $[s_{i1}]_{(m-1)\times 1}$ defined by

$$s_{i1} := \sum_{j=1}^{n} g_{ij}, \quad i \in I_{m-1}$$

Step 6. Obtain the decision set $\{^{\mu(u_k)}u_k | u_k \in U\}$ such that $\mu(u_k) = \frac{s_{k1}}{\max_i s_{i1}}$

Algorithm 8 (BMM12/3)

Step 1. Apply BMM12 except its Step 4.

In [41], the authors have attempted to overcome a city selection problem employing fuzzy parameterized soft sets. We configure the proposed methods therein as follows:

Algorithm 9 (CD12/3)

Step 1. Construct two *fpfs*-matrices $[a_{ij}]_{m\times n}$ and $[b_{ij}]_{m\times n}$

Step 2. Obtain $[c_{ij}]_{m\times n}$ defined by

$$c_{0j} := a_{0j} + b_{0j} - a_{0j}b_{0j}, \ j \in I_n$$

and

$$c_{ij} := \max\{a_{ij}, b_{ij}\}, \ i \in I_{m-1} \text{ and } j \in I_n$$

Step 3. Obtain $[s_{i1}]_{(m-1)\times 1}$ defined by

$$s_{i1} := \frac{1}{m-1}\sum_{j=1}^{n} c_{0j}c_{ij}, \quad i \in I_{m-1}$$

Step 4. Obtain the decision set $\left\{^{\mu(u_k)}u_k | u_k \in U\right\}$ such that $\mu(u_k) = \frac{s_{k1}}{\max_i s_{i1}}$

Algorithm 10 (CD12/4)

Step 1. Construct two *fpfs*-matrices $[a_{ij}]_{m\times n}$ and $[b_{ij}]_{m\times n}$

Step 2. Obtain $[c_{ij}]_{m\times n}$ defined by

$$c_{0j} := a_{0j}b_{0j} \quad \text{and} \quad c_{ij} := \min\{a_{ij}, b_{ij}\}, \ i \in I_{m-1} \text{ and } j \in I_n$$

Step 3. Obtain $[s_{i1}]_{(m-1)\times 1}$ defined by

$$s_{i1} := \frac{1}{m-1}\sum_{j=1}^{n} c_{0j}c_{ij}, \quad i \in I_{m-1}$$

Step 4. Obtain the decision set $\left\{^{\mu(u_k)}u_k | u_k \in U\right\}$ such that $\mu(u_k) = \frac{s_{k1}}{\max_i s_{i1}}$

In [42], the authors have studied a house purchase problem through soft sets. We configure the proposed method therein as follows:

Algorithm 11 (FLC12)

Step 1. Construct two *fpfs*-matrices $\left[a_{ij}\right]_{m\times n}$ and $[b_{ik}]_{m\times n}$

Step 2. Obtain two *fs*-matrices $\left[c_{ij}\right]_{(m-1)\times n}$ and $\left[d_{ij}\right]_{(m-1)\times n}$ defined by

$$c_{ij} := \begin{cases} a_{0j}, & \sum_{t=1}^{n} a_{0t}a_{it} \geq \sum_{t=1}^{j} a_{0t} \\ 0, & otherwise \end{cases}$$

and

$$d_{ij} := \begin{cases} b_{0j}, & \sum_{t=1}^{n} b_{0t}b_{it} \geq \sum_{t=1}^{j} b_{0t} \\ 0, & otherwise \end{cases}$$

such that $i \in I_{m-1}$ and $j \in I_n$

Step 3. Obtain and-product *fs*-matrix $\left[e_{ip}\right]_{m\times n^2}$ of $\left[c_{ij}\right]_{m\times n}$ and $[d_{ik}]_{m\times n}$

Step 4. Obtain $[s_{i1}]_{(m-1)\times 1}$ defined by $s_{i1} := e_{il}$ such that $i \in I_{m-1}$ and $l := \max\left\{p \middle| \exists i \in I_{m-1}, e_{ip} \neq 0\right\}$

Step 5. Obtain the decision set $\{{}^{s_{k1}}u_k | u_k \in U\}$

[43] has proposed a method by revising the algorithm provided in [35]. We configure the proposed method therein as follows:

Algorithm 12 (QYZ12)

Step 1. Apply RM07a except its Step 5

Step 2. Obtain $\left[e_{it}^{p}\right]_{(m-1)\times|R|}$ defined by

$$e_{it}^{p} := \begin{cases} d_{ot}, & d_{pt} > d_{it} \\ 0, & d_{pt} = d_{it} \\ -d_{ot}, & d_{pt} < d_{it} \end{cases}$$

such that $i, p \in I_{m-1}$ and $t \in I_{|R|}$

Step 3. Obtain the score matrix $\left[s_{p1}\right]_{(m-1)\times 1}$ defined by

$$s_{p1} := \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}_{1\times(m-1)} \cdot \left[e_{it}^{p}\right]_{(m-1)\times|R|} \cdot \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{|R|\times 1}, \quad p \in I_{m-1}$$

Step 4. Obtain the decision set $\left\{{}^{\mu(u_k)}u_k \mid u_k \in U\right\}$ such that $\mu(u_k) = \frac{s_{k1} + \left|\min_i s_{i1}\right|}{\max_i s_{i1} + \left|\min_i s_{i1}\right|}$

4. An Application of Some of the Configured Methods to PVA Problem

In this section, we apply KWW11/2(w,z), NS11, BMM12, BMM12/3, CD12/3, and CD12/4 to a PVA problem intended for the known filters used in image denoising. To this end, we present the results of the filters in [44], harvested with Structural Similarity (SSIM) [45] and Visual Information Fidelity (VIF) [46], for the 20 traditional images at noise densities ranging from 10% to 90% in Table 1 and 2, respectively. The results herein were obtained by MATLAB R2020b.

Table 1. Mean-SSIM results of the filters for the 20 traditional images

Filters/Noise Density	10%	20%	30%	40%	50%	60%	70%	80%	90%
DBA	0.9796	0.9584	0.9315	0.8968	0.8520	0.7949	0.7213	0.6265	0.4966
MDBUTMF	0.9774	0.9197	0.8117	0.7973	0.8399	0.8410	0.8025	0.7023	0.3566
BPDF	0.9783	0.9536	0.9229	0.8838	0.8323	0.7634	0.6680	0.5096	0.2585
NAFSMF	0.9748	0.9504	0.9248	0.8973	0.8666	0.8320	0.7910	0.7357	0.6190
AWMF	0.9728	0.9622	0.9484	0.9315	0.9098	0.8816	0.8437	0.7904	0.7028
DAMF	0.9854	0.9699	0.9516	0.9303	0.9051	0.8748	0.8368	0.7846	0.6964
ARmF	**0.9868**	**0.9735**	**0.9581**	**0.9400**	**0.9173**	**0.8880**	**0.8491**	**0.7947**	**0.7056**

Table 2. Mean-VIF results of the filters for the 20 traditional images

Filters/Noise Density	10%	20%	30%	40%	50%	60%	70%	80%	90%
DBA	0.8548	0.7319	0.6179	0.5119	0.4095	0.3128	0.2229	0.1365	0.0635
MDBUTMF	0.8272	0.6713	0.5044	0.4420	0.4310	0.3978	0.3302	0.2212	0.0730
BPDF	0.8188	0.6858	0.5659	0.4564	0.3529	0.2541	0.1614	0.0783	0.0334
NAFSMF	0.7902	0.6751	0.5828	0.5030	0.4307	0.3604	0.2897	0.2129	0.1226
AWMF	0.7896	0.7366	0.6789	0.6181	0.5533	0.4833	0.4066	0.3129	0.1928
DAMF	0.8787	0.7816	0.6943	0.6162	0.5437	0.4731	0.3998	0.3096	0.1913
ARmF	**0.8832**	**0.7975**	**0.7210**	**0.6474**	**0.5741**	**0.4974**	**0.4158**	**0.3182**	**0.1955**

Suppose that the success of the seven filters at high noise densities is more important than at other densities. In that case, the values in Table 1 and 2 can be represented with two *fpfs*-matrices, constructed in the first steps of the algorithms as follows:

$$[a_{ij}] := \begin{bmatrix} 0.1 & 0.2 & 0.3 & 0.4 & 0.5 & 0.6 & 0.7 & 0.8 & 0.9 \\ 0.9796 & 0.9584 & 0.9315 & 0.8968 & 0.8520 & 0.7949 & 0.7213 & 0.6265 & 0.4966 \\ 0.9774 & 0.9197 & 0.8117 & 0.7973 & 0.8399 & 0.8410 & 0.8025 & 0.7023 & 0.3566 \\ 0.9783 & 0.9536 & 0.9229 & 0.8838 & 0.8323 & 0.7634 & 0.6680 & 0.5096 & 0.2585 \\ 0.9748 & 0.9504 & 0.9248 & 0.8973 & 0.8666 & 0.8320 & 0.7910 & 0.7357 & 0.6190 \\ 0.9728 & 0.9622 & 0.9484 & 0.9315 & 0.9098 & 0.8816 & 0.8437 & 0.7904 & 0.7028 \\ 0.9854 & 0.9699 & 0.9516 & 0.9303 & 0.9051 & 0.8748 & 0.8368 & 0.7846 & 0.6964 \\ 0.9868 & 0.9735 & 0.9581 & 0.9400 & 0.9173 & 0.8880 & 0.8491 & 0.7947 & 0.7056 \end{bmatrix}$$

and

$$[b_{ij}] := \begin{bmatrix} 0.1 & 0.2 & 0.3 & 0.4 & 0.5 & 0.6 & 0.7 & 0.8 & 0.9 \\ 0.8548 & 0.7319 & 0.6179 & 0.5119 & 0.4095 & 0.3128 & 0.2229 & 0.1365 & 0.0635 \\ 0.8272 & 0.6713 & 0.5044 & 0.4420 & 0.4310 & 0.3978 & 0.3302 & 0.2212 & 0.0730 \\ 0.8188 & 0.6858 & 0.5659 & 0.4564 & 0.3529 & 0.2541 & 0.1614 & 0.0783 & 0.0334 \\ 0.7902 & 0.6751 & 0.5828 & 0.5030 & 0.4307 & 0.3604 & 0.2897 & 0.2129 & 0.1226 \\ 0.7896 & 0.7366 & 0.6789 & 0.6181 & 0.5533 & 0.4833 & 0.4066 & 0.3129 & 0.1928 \\ 0.8787 & 0.7816 & 0.6943 & 0.6162 & 0.5437 & 0.4731 & 0.3998 & 0.3096 & 0.1913 \\ 0.8832 & 0.7975 & 0.7210 & 0.6474 & 0.5741 & 0.4974 & 0.4158 & 0.3182 & 0.1955 \end{bmatrix}$$

Performance Ranking of the Filters via KWW11/2(w,z)

Step 2. If MRB02 to $[a_{ij}]$ and $[b_{ij}]$ is applied, then the score matrices $[s_{i1}^1]$ and $[s_{i1}^2]$ are as follows:

$$[s_{i1}^1] = [3.2838 \quad 3.2132 \quad 2.9011 \quad 3.5588 \quad 3.7861 \quad 3.7678 \quad 3.8134]^T$$

$$[s_{i1}^2] = [1.3368 \quad 1.4731 \quad 1.1060 \quad 1.5051 \quad 1.9523 \quad 1.9544 \quad 2.0301]^T$$

Step 3. The matrices $[c_{i1}]$ and $[d_{i1}]$ are as follows:

$$[c_{i1}] = [0.4195 \quad 0.3421 \quad 0 \quad 0.7176 \quad 0.9701 \quad 0.9499 \quad 1]^T$$

$$[d_{i1}] = [0.2498 \quad 0.3972 \quad 0 \quad 0.4319 \quad 0.9158 \quad 0.9180 \quad 1]^T$$

Step 4. The matrices $[e_{i1}]$ and $[f_{i1}]$ are as follows:

$$[e_{i1}] = [0.5805 \quad 0.6579 \quad 1 \quad 0.2824 \quad 0.0299 \quad 0.0501 \quad 0]^T$$

$$[f_{i1}] = [0.7502 \quad 0.6028 \quad 1 \quad 0.5681 \quad 0.0842 \quad 0.0820 \quad 0]^T$$

Step 5. For $w = 0.5$, the matrices $[g_{i1}]$ and $[h_{i1}]$ are as follows:

$$[g_{i1}] = [0.4627 \quad 0.4318 \quad 0.3333 \quad 0.6391 \quad 0.9435 \quad 0.9090 \quad 1]^T$$

$$[h_{i1}] = [0.3999 \quad 0.4534 \quad 0.3333 \quad 0.4681 \quad 0.8558 \quad 0.8591 \quad 1]^T$$

Step 6. For $z = 0.5$, the score matrix is as follows:

$$[s_{i1}] = [0.4313 \quad 0.4426 \quad 0.3333 \quad 0.5536 \quad 0.8997 \quad 0.8841 \quad 1]^T$$

Step 7. The $(0.5,0.5)$-decision set is as follows:

$$\{{}^{0.4313}\text{DBA}, {}^{0.4426}\text{MDBUTMF}, {}^{0.3333}\text{BPDF}, {}^{0.5536}\text{NAFSMF}, {}^{0.8997}\text{AWMF}, {}^{0.8841}\text{DAMF}, {}^{1}\text{ARmF}\}$$

Performance Ranking of the Filters via NS11

Step 2. The *fpfs*-matrices $[a_{ij}]^{\tilde{c}}$ and $[b_{ij}]^{\tilde{c}}$ are as follows:

$$[a_{ij}]^{\tilde{c}} = \begin{bmatrix} 0.9 & 0.8 & 0.7 & 0.6 & 0.5 & 0.4 & 0.3 & 0.2 & 0.1 \\ 0.0204 & 0.0416 & 0.0685 & 0.1032 & 0.1480 & 0.2051 & 0.2787 & 0.3735 & 0.5034 \\ 0.0226 & 0.0803 & 0.1883 & 0.2027 & 0.1601 & 0.1590 & 0.1975 & 0.2977 & 0.6434 \\ 0.0217 & 0.0464 & 0.0771 & 0.1162 & 0.1677 & 0.2366 & 0.3320 & 0.4904 & 0.7415 \\ 0.0252 & 0.0496 & 0.0752 & 0.1027 & 0.1334 & 0.1680 & 0.2090 & 0.2643 & 0.3810 \\ 0.0272 & 0.0378 & 0.0516 & 0.0685 & 0.0902 & 0.1184 & 0.1563 & 0.2096 & 0.2972 \\ 0.0146 & 0.0301 & 0.0484 & 0.0697 & 0.0949 & 0.1252 & 0.1632 & 0.2154 & 0.3036 \\ 0.0132 & 0.0265 & 0.0419 & 0.0600 & 0.0827 & 0.1120 & 0.1509 & 0.2053 & 0.2944 \end{bmatrix}$$

and

$$[b_{\mathrm{ij}}]^{\tilde{c}} = \begin{bmatrix} 0.9 & 0.8 & 0.7 & 0.6 & 0.5 & 0.4 & 0.3 & 0.2 & 0.1 \\ 0.1452 & 0.2681 & 0.3821 & 0.4881 & 0.5905 & 0.6872 & 0.7771 & 0.8635 & 0.9365 \\ 0.1728 & 0.3287 & 0.4956 & 0.5580 & 0.5690 & 0.6022 & 0.6698 & 0.7788 & 0.9270 \\ 0.1812 & 0.3142 & 0.4341 & 0.5436 & 0.6471 & 0.7459 & 0.8386 & 0.9217 & 0.9666 \\ 0.2098 & 0.3249 & 0.4172 & 0.4970 & 0.5693 & 0.6396 & 0.7103 & 0.7871 & 0.8774 \\ 0.2104 & 0.2634 & 0.3211 & 0.3819 & 0.4467 & 0.5167 & 0.5934 & 0.6871 & 0.8072 \\ 0.1213 & 0.2184 & 0.3057 & 0.3838 & 0.4563 & 0.5269 & 0.6002 & 0.6904 & 0.8087 \\ 0.1168 & 0.2025 & 0.2790 & 0.3526 & 0.4259 & 0.5026 & 0.5842 & 0.6818 & 0.8045 \end{bmatrix}$$

Step 3. The *fpfs*-matrix $[c_{ij}]$ is as follows:

$$[c_{ij}] = \begin{bmatrix} 0.1 & 0.2 & 0.3 & 0.4 & 0.5 & 0.6 & 0.7 & 0.8 & 0.9 \\ 0.9796 & 0.9584 & 0.9315 & 0.8968 & 0.8520 & 0.7949 & 0.7213 & 0.6265 & 0.4966 \\ 0.9774 & 0.9197 & 0.8117 & 0.7973 & 0.8399 & 0.8410 & 0.8025 & 0.7023 & 0.3566 \\ 0.9783 & 0.9536 & 0.9229 & 0.8838 & 0.8323 & 0.7634 & 0.6680 & 0.5096 & 0.2585 \\ 0.9748 & 0.9504 & 0.9248 & 0.8973 & 0.8666 & 0.8320 & 0.7910 & 0.7357 & 0.6190 \\ 0.9728 & 0.9622 & 0.9484 & 0.9315 & 0.9098 & 0.8816 & 0.8437 & 0.7904 & 0.7028 \\ 0.9854 & 0.9699 & 0.9516 & 0.9303 & 0.9051 & 0.8748 & 0.8368 & 0.7846 & 0.6964 \\ 0.9868 & 0.9735 & 0.9581 & 0.9400 & 0.9173 & 0.8880 & 0.8491 & 0.7947 & 0.7056 \end{bmatrix}$$

Step 4. The *fpfs*-matrix $[d_{ij}]$ is as follows:

$$[d_{ij}] = \begin{bmatrix} 0.9 & 0.8 & 0.7 & 0.6 & 0.5 & 0.4 & 0.3 & 0.2 & 0.1 \\ 0.1452 & 0.2681 & 0.3821 & 0.4881 & 0.5905 & 0.6872 & 0.7771 & 0.8635 & 0.9365 \\ 0.1728 & 0.3287 & 0.4956 & 0.5580 & 0.5690 & 0.6022 & 0.6698 & 0.7788 & 0.9270 \\ 0.1812 & 0.3142 & 0.4341 & 0.5436 & 0.6471 & 0.7459 & 0.8386 & 0.9217 & 0.9666 \\ 0.2098 & 0.3249 & 0.4172 & 0.4970 & 0.5693 & 0.6396 & 0.7103 & 0.7871 & 0.8774 \\ 0.2104 & 0.2634 & 0.3211 & 0.3819 & 0.4467 & 0.5167 & 0.5934 & 0.6871 & 0.8072 \\ 0.1213 & 0.2184 & 0.3057 & 0.3838 & 0.4563 & 0.5269 & 0.6002 & 0.6904 & 0.8087 \\ 0.1168 & 0.2025 & 0.2790 & 0.3526 & 0.4259 & 0.5026 & 0.5842 & 0.6818 & 0.8045 \end{bmatrix}$$

Step 5. The matrix $[e_{ij}]$ is as follows:

$$[e_{ij}] = \begin{bmatrix} -0.6675 & -0.4142 & -0.2198 & -0.0817 & 0 & 0.0215 & -0.0223 & -0.1422 & -0.3519 \\ -0.6437 & -0.3546 & -0.1264 & -0.0479 & 0 & 0.0478 & 0.0531 & -0.0459 & -0.4563 \\ -0.6377 & -0.3836 & -0.1955 & -0.0680 & 0 & 0.0035 & -0.0682 & -0.2473 & -0.5665 \\ -0.6120 & -0.3753 & -0.2030 & -0.0801 & 0 & 0.0385 & 0.0323 & -0.0308 & -0.2067 \\ -0.6099 & -0.4193 & -0.2509 & -0.1099 & 0 & 0.0730 & 0.1001 & 0.0620 & -0.0835 \\ -0.6913 & -0.4509 & -0.2584 & -0.1093 & 0 & 0.0696 & 0.0946 & 0.0565 & -0.0898 \\ -0.6960 & -0.4626 & -0.2716 & -0.1175 & 0 & 0.0771 & 0.1060 & 0.0677 & -0.0791 \end{bmatrix}$$

Step 6. The score matrix is as follows:

$$[s_{i1}] = [-1.8781 \quad -1.5740 \quad -2.1634 \quad -1.4372 \quad -1.2385 \quad -1.3789 \quad -1.3761]^T$$

Step 7. The decision set is as follows:

$$\{{}^{0.3084}\mathrm{DBA},\ {}^{0.6373}\mathrm{MDBUTMF},\ {}^{0}\mathrm{BPDF},\ {}^{0.7851}\mathrm{NAFSMF},\ {}^{1}\mathrm{AWMF},\ {}^{0.8481}\mathrm{DAMF},\ {}^{0.8512}\mathrm{ARmF}\}$$

Performance Ranking of the Filters via BMM12

Step 2. The matrices $[c_{kj}]$ and $[d_{kj}]$ are as follows:

$$[c_{kj}]=\begin{bmatrix}0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1\\0.1&0.2&0.2&0.2&0.2&0.2&0.2&0.2&0.2\\0.1&0.2&0.3&0.3&0.3&0.3&0.3&0.3&0.3\\0.1&0.2&0.3&0.4&0.4&0.4&0.4&0.4&0.4\\0.1&0.2&0.3&0.4&0.5&0.5&0.5&0.5&0.5\\0.1&0.2&0.3&0.4&0.5&0.6&0.6&0.6&0.6\\0.1&0.2&0.3&0.4&0.5&0.6&0.7&0.7&0.7\\0.1&0.2&0.3&0.4&0.5&0.6&0.7&0.8&0.8\\0.1&02&0.3&0.4&0.5&0.6&0.7&0.8&0.9\end{bmatrix}$$

and

$$[d_{kj}]=\begin{bmatrix}0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1\\0.1&0.2&0.2&0.2&0.2&0.2&0.2&0.2&0.2\\0.1&0.2&0.3&0.3&0.3&0.3&0.3&0.3&0.3\\0.1&0.2&0.3&0.4&0.4&0.4&0.4&0.4&0.4\\0.1&0.2&0.3&0.4&0.5&0.5&0.5&0.5&0.5\\0.1&0.2&0.3&0.4&0.5&0.6&0.6&0.6&0.6\\0.1&0.2&0.3&0.4&0.5&0.6&0.7&0.7&0.7\\0.1&0.2&0.3&0.4&0.5&0.6&0.7&0.8&0.8\\0.1&0.2&0.3&0.4&0.5&0.6&0.7&0.8&0.9\end{bmatrix}$$

Step 3. The matrices $[e_{kj}]$ and $[f_{kj}]$ are as follows:

$$[e_{kj}]=\begin{bmatrix}0.9000&1.7000&2.4000&3.0000&3.4966&3.7966&3.9231&3.9231&3.9231\\0.9000&1.7000&2.4000&2.9566&3.3566&3.6566&3.8566&3.8589&3.8589\\0.9000&1.7000&2.3585&2.8585&3.2585&3.4681&3.5361&3.5361&3.5361\\0.9000&1.7000&2.4000&3.0000&3.5000&3.9000&4.1190&4.1547&4.1547\\0.9000&1.7000&2.4000&3.0000&3.5000&3.9000&4.2000&4.2932&4.2932\\0.9000&1.7000&2.4000&3.0000&3.5000&3.9000&4.1964&4.2810&4.2810\\0.9000&1.7000&2.4000&3.0000&3.5000&3.9000&4.2000&4.3003&4.3003\end{bmatrix}$$

and

$$[f_{kj}]=\begin{bmatrix}0.8635&1.5000&1.9229&2.1357&2.1452&2.1452&2.1452&2.1452&2.1452\\0.8730&1.5730&2.0942&2.4222&2.4532&2.4532&2.4532&2.4532&2.4532\\0.8117&1.3731&1.7272&1.8801&1.8801&1.8801&1.8801&1.8801&1.8801\\0.9000&1.6226&2.1252&2.3856&2.4163&2.4163&2.4163&2.4163&2.4163\\0.9000&1.6928&2.2928&2.7057&2.8956&2.8956&2.8956&2.8956&2.8956\\0.9000&1.6913&2.2913&2.7007&2.8738&2.8738&2.8738&2.8738&2.8738\\0.9000&1.6955&2.2955&2.7137&2.9269&2.9269&2.9269&2.9269&2.9269\end{bmatrix}$$

Step 4. The matrices $[g_{kj}]$ and $[h_{kj}]$ are as follows:

$$[g_{kj}] = \begin{bmatrix} 0.0047 & 0.0089 & 0.0126 & 0.0157 & 0.0183 & 0.0199 & 0.0205 & 0.0205 & 0.0205 \\ 0.0047 & 0.0089 & 0.0126 & 0.0155 & 0.0176 & 0.0191 & 0.0202 & 0.0202 & 0.0202 \\ 0.0047 & 0.0089 & 0.0123 & 0.0150 & 0.0170 & 0.0181 & 0.0185 & 0.0185 & 0.0185 \\ 0.0047 & 0.0089 & 0.0126 & 0.0157 & 0.0183 & 0.0204 & 0.0216 & 0.0217 & 0.0217 \\ 0.0047 & 0.0089 & 0.0126 & 0.0157 & 0.0183 & 0.0204 & 0.0220 & 0.0225 & 0.0225 \\ 0.0047 & 0.0089 & 0.0126 & 0.0157 & 0.0183 & 0.0204 & 0.0220 & 0.0224 & 0.0224 \\ 0.0047 & 0.0089 & 0.0126 & 0.0157 & 0.0183 & 0.0204 & 0.0220 & 0.0225 & 0.0225 \end{bmatrix}$$

and

$$[h_{kj}] = \begin{bmatrix} 0.0063 & 0.0110 & 0.0140 & 0.0156 & 0.0157 & 0.0157 & 0.0157 & 0.0157 & 0.0157 \\ 0.0064 & 0.0115 & 0.0153 & 0.0177 & 0.0179 & 0.0179 & 0.0179 & 0.0179 & 0.0179 \\ 0.0059 & 0.0100 & 0.0126 & 0.0137 & 0.0137 & 0.0137 & 0.0137 & 0.0137 & 0.0137 \\ 0.0066 & 0.0118 & 0.0155 & 0.0174 & 0.0176 & 0.0176 & 0.0176 & 0.0176 & 0.0176 \\ 0.0066 & 0.0124 & 0.0167 & 0.0198 & 0.0211 & 0.0211 & 0.0211 & 0.0211 & 0.0211 \\ 0.0066 & 0.0124 & 0.0167 & 0.0197 & 0.0210 & 0.0210 & 0.0210 & 0.0210 & 0.0210 \\ 0.0066 & 0.0124 & 0.0168 & 0.0198 & 0.0214 & 0.0214 & 0.0214 & 0.0214 & 0.0214 \end{bmatrix}$$

Step 5. The matrix $[u_{ij}]$ is as follows:

$$[u_{ij}] = \begin{bmatrix} 0.0063 & 0.0110 & 0.0140 & 0.0157 & 0.0183 & 0.0199 & 0.0205 & 0.0205 & 0.0205 \\ 0.0064 & 0.0115 & 0.0153 & 0.0177 & 0.0179 & 0.0191 & 0.0202 & 0.0202 & 0.0202 \\ 0.0059 & 0.0100 & 0.0126 & 0.0150 & 0.0170 & 0.0181 & 0.0185 & 0.0185 & 0.0185 \\ 0.0066 & 0.0118 & 0.0155 & 0.0174 & 0.0183 & 0.0204 & 0.0216 & 0.0217 & 0.0217 \\ 0.0066 & 0.0124 & 0.0167 & 0.0198 & 0.0211 & 0.0211 & 0.0220 & 0.0225 & 0.0225 \\ 0.0066 & 0.0124 & 0.0167 & 0.0197 & 0.0210 & 0.0210 & 0.0220 & 0.0224 & 0.0224 \\ 0.0066 & 0.0124 & 0.0168 & 0.0198 & 0.0214 & 0.0214 & 0.0220 & 0.0225 & 0.0225 \end{bmatrix}$$

Step 6. The score matrix is as follows:

$$[s_{i1}] = [0.1467 \quad 0.1484 \quad 0.1342 \quad 0.1551 \quad 0.1646 \quad 0.1641 \quad 0.1652]^T$$

Step 7. The decision set is as follows:

$$\{{}^{0.8879}\text{DBA}, {}^{0.8983}\text{MDBUTMF}, {}^{0.8122}\text{BPDF}, {}^{0.9386}\text{NAFSMF}, {}^{0.9962}\text{AWMF}, {}^{0.9930}\text{DAMF}, {}^{1}\text{ARmF}\}$$

Performance Ranking of the Filters via BMM12/3

Step 4. Let us consider $[e_{kj}]$ and $[f_{kj}]$ provided in BMM12, then the matrix $[g_{ij}]$ is as follows:

$$[g_{ij}] = \begin{bmatrix} 0.9000 & 1.7000 & 2.4000 & 3.0000 & 3.4966 & 3.7966 & 3.9231 & 3.9231 & 3.9231 \\ 0.9000 & 1.7000 & 2.4000 & 2.9566 & 3.3566 & 3.6566 & 3.8566 & 3.8589 & 3.8589 \\ 0.9000 & 1.7000 & 2.3585 & 2.8585 & 3.2585 & 3.4681 & 3.5361 & 3.5361 & 3.5361 \\ 0.9000 & 1.7000 & 2.4000 & 3.0000 & 3.5000 & 3.9000 & 4.1190 & 4.1547 & 4.1547 \\ 0.9000 & 1.7000 & 2.4000 & 3.0000 & 3.5000 & 3.9000 & 4.2000 & 4.2932 & 4.2932 \\ 0.9000 & 1.7000 & 2.4000 & 3.0000 & 3.5000 & 3.9000 & 4.1964 & 4.2810 & 4.2810 \\ 0.9000 & 1.7000 & 2.4000 & 3.0000 & 3.5000 & 3.9000 & 4.2000 & 4.3003 & 4.3003 \end{bmatrix}$$

Step 5. The score matrix is as follows:

$$[s_{i1}] = [27.0625 \quad 26.5442 \quad 25.1519 \quad 27.8284 \quad 28.1864 \quad 28.1584 \quad 28.2006]^T$$

Step 6. The decision set is as follows:

$$\{^{0.9596}\text{DBA}, {}^{0.9413}\text{MDBUTMF}, {}^{0.8919}\text{BPDF}, {}^{0.9868}\text{NAFSMF}, {}^{0.9995}\text{AWMF}, {}^{0.9985}\text{DAMF}, {}^{1}\text{ARmF}\}$$

Performance Ranking of the Filters via CD12/3

Step 2. The matrix $[b_{ij}]$ is as follows:

$$[b_{ij}] = \begin{bmatrix} 0.19 & 0.36 & 0.51 & 0.64 & 0.75 & 0.84 & 0.91 & 0.96 & 0.99 \\ 0.9796 & 0.9584 & 0.9315 & 0.8968 & 0.8520 & 0.7949 & 0.7213 & 0.6265 & 0.4966 \\ 0.9774 & 0.9197 & 0.8117 & 0.7973 & 0.8399 & 0.8410 & 0.8025 & 0.7023 & 0.3566 \\ 0.9783 & 0.9536 & 0.9229 & 0.8838 & 0.8323 & 0.7634 & 0.6680 & 0.5096 & 0.2585 \\ 0.9748 & 0.9504 & 0.9248 & 0.8973 & 0.8666 & 0.8320 & 0.7910 & 0.7357 & 0.6190 \\ 0.9728 & 0.9622 & 0.9484 & 0.9315 & 0.9098 & 0.8816 & 0.8437 & 0.7904 & 0.7028 \\ 0.9854 & 0.9699 & 0.9516 & 0.9303 & 0.9051 & 0.8748 & 0.8368 & 0.7846 & 0.6964 \\ 0.9868 & 0.9735 & 0.9581 & 0.9400 & 0.9173 & 0.8880 & 0.8491 & 0.7947 & 0.7056 \end{bmatrix}$$

Step 3. The score matrix is as follows:

$$[s_{i1}] = [0.6623 \quad 0.6478 \quad 0.5977 \quad 0.7087 \quad 0.7509 \quad 0.7478 \quad 0.7566]^T$$

Step 4. The decision set is as follows:

$$\{^{0.8754}\text{DBA}, {}^{0.8563}\text{MDBUTMF}, {}^{0.7900}\text{BPDF}, {}^{0.9367}\text{NAFSMF}, {}^{0.9925}\text{AWMF}, {}^{0.9884}\text{DAMF}, {}^{1}\text{ARmF}\}$$

Performance Ranking of the Filters via CD12/4

Step 2. The matrix $[b_{ij}]$ is as follows:

$$[b_{ij}] = \begin{bmatrix} 0.01 & 0.04 & 0.09 & 0.16 & 0.25 & 0.36 & 0.49 & 0.64 & 0.81 \\ 0.8548 & 0.7319 & 0.6179 & 0.5119 & 0.4095 & 0.3128 & 0.2229 & 0.1365 & 0.0635 \\ 0.8272 & 0.6713 & 0.5044 & 0.4420 & 0.4310 & 0.3978 & 0.3302 & 0.2212 & 0.0730 \\ 0.8188 & 0.6858 & 0.5659 & 0.4564 & 0.3529 & 0.2541 & 0.1614 & 0.0783 & 0.0334 \\ 0.7902 & 0.6751 & 0.5828 & 0.5030 & 0.4307 & 0.3604 & 0.2897 & 0.2129 & 0.1226 \\ 0.7896 & 0.7366 & 0.6789 & 0.6181 & 0.5533 & 0.4833 & 0.4066 & 0.3129 & 0.1928 \\ 0.8787 & 0.7816 & 0.6943 & 0.6162 & 0.5437 & 0.4731 & 0.3998 & 0.3096 & 0.1913 \\ 0.8832 & 0.7975 & 0.7210 & 0.6474 & 0.5741 & 0.4974 & 0.4158 & 0.3182 & 0.1955 \end{bmatrix}$$

Step 3. The score matrix is as follows:

$$[s_{i1}] = [0.0912 \quad 0.1092 \quad 0.0708 \quad 0.1118 \quad 0.1522 \quad 0.1509 \quad 0.1568]^T$$

Step 4. The decision set is as follows:

$$\{^{0.5816}\text{DBA}, {}^{0.6967}\text{MDBUTMF}, {}^{0.4515}\text{BPDF}, {}^{0.7132}\text{NAFSMF}, {}^{0.9707}\text{AWMF}, {}^{0.9625}\text{DAMF}, {}^{1}\text{ARmF}\}$$

In Table 3 and 4, we present the decision sets and the ranking orders of the filters obtained by the six SDM methods, respectively. The results show that KWW11/2(0.5,0.5), BMM12, and CD12/4 produce the same ranking order. Moreover, they also confirm both the ranking orders provided in [27,33] and the opinions of the experts about these ranking orders. Therefore, these methods can be applied to the PVA problem of image-denoising filters. Similarly, the results of BMM12/3 and CD12/3 tend to confirm the aforesaid ranking orders except those of MDBUTMF and DBA. Consequently, the considered methods herein except NS11 agree that ARmF outperforms the other filters, whereas all the methods reveal that BPDF exhibits minimum performance.

Table 3. Decision sets of the filters for the state-of-art methods

Methods	Ranking Orders
KWW11/2(0.5,0.5)	$\{{}^{0.4313}\text{DBA}, {}^{0.4426}\text{MDBUTMF}, {}^{0.3333}\text{BPDF}, {}^{0.5536}\text{NAFSMF}, {}^{0.8997}\text{AWMF}, {}^{0.8841}\text{DAMF}, {}^{1}\text{ARmF}\}$
NS11	$\{{}^{0.3084}\text{DBA}, {}^{0.6373}\text{MDBUTMF}, {}^{0}\text{BPDF}, {}^{0.7851}\text{NAFSMF}, {}^{1}\text{AWMF}, {}^{0.8481}\text{DAMF}, {}^{0.8512}\text{ARmF}\}$
BMM12	$\{{}^{0.8879}\text{DBA}, {}^{0.8983}\text{MDBUTMF}, {}^{0.8122}\text{BPDF}, {}^{0.9386}\text{NAFSMF}, {}^{0.9962}\text{AWMF}, {}^{0.9930}\text{DAMF}, {}^{1}\text{ARmF}\}$
BMM12/3	$\{{}^{0.9596}\text{DBA}, {}^{0.9413}\text{MDBUTMF}, {}^{0.8919}\text{BPDF}, {}^{0.9868}\text{NAFSMF}, {}^{0.9995}\text{AWMF}, {}^{0.9985}\text{DAMF}, {}^{1}\text{ARmF}\}$
CD12/3	$\{{}^{0.8754}\text{DBA}, {}^{0.8563}\text{MDBUTMF}, {}^{0.7900}\text{BPDF}, {}^{0.9367}\text{NAFSMF}, {}^{0.9925}\text{AWMF}, {}^{0.9884}\text{DAMF}, {}^{1}\text{ARmF}\}$
CD12/4	$\{{}^{0.5816}\text{DBA}, {}^{0.6967}\text{MDBUTMF}, {}^{0.4515}\text{BPDF}, {}^{0.7132}\text{NAFSMF}, {}^{0.9707}\text{AWMF}, {}^{0.9625}\text{DAMF}, {}^{1}\text{ARmF}\}$

Table 4. Ranking orders of the filters for the state-of-art methods

Methods	Ranking Orders
KWW11/2(0.5,0.5)	**BPDF≺DBA≺MDBUTMF≺NAFSMF≺DAMF≺AWMF≺ARmF**
NS11	BPDF≺DBA≺MDBUTMF≺NAFSMF≺DAMF≺ARmF≺AWMF
BMM12	**BPDF≺DBA≺MDBUTMF≺NAFSMF≺DAMF≺AWMF≺ARmF**
BMM12/3	BPDF≺MDBUTMF≺DBA≺NAFSMF≺DAMF≺AWMF≺ARmF
CD12/3	BPDF≺MDBUTMF≺DBA≺NAFSMF≺DAMF≺AWMF≺ARmF
CD12/4	**BPDF≺DBA≺MDBUTMF≺NAFSMF≺DAMF≺AWMF≺ARmF**

5. Conclusion

In this study, we configured 12 SDM methods to operate them in *fpfs*-matrices space, faithfully to the original. Thus, we completed the configurations of the methods constructed by soft sets, fuzzy soft sets, fuzzy parameterized soft sets, fuzzy parameterized fuzzy soft sets, and their matrix representations and proposed between 1999 and 2012. Besides, we applied some of them to a PVA problem for the known filters used in image denoising.

The methods incorporating parameter reduction were excluded from the present study. In the future, researchers can focus on these methods and configure them to *fpfs*-matrices space. Moreover, they can compare the methods by applying them to different decision-making problems. This study encourages researchers to compare all the configured methods, which were proposed between 1999 and 2012, in the presence of a PVA problem.

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Controlled Hodgkin – Huxley Neuron vs Controlled Qubit: Pros and Cons of Their Applications

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Abstract

Nonlinear multidimensional dynamical systems presented in the form of ordinary differential equations with free control parameters cover the variety of regular and chaotic regimes and can be used for computational purposes and data analysis. To demonstrate the chaotic regime, for instance, the dimension of the phase space for the deterministic system should be at least 3. Here we compare two multidimensional systems: 4-dimensional Hodgkin-Huxley neuron and 3-dimensional quantum bit (in its real ODE representation) in the external field. Both systems can be driven via the free parameters towards the necessary dynamical state (stabilization or tracking goal). Both systems could be used for the realization of complex computational algorithms. We compare the efficiency of qubit mathematical model with the classical model of mathematical neuron to analyze pros and cons of both systems to be applied for the computational purposes.

1. Introduction

Nonlinear multidimensional dynamical systems presented in the form of ordinary differential equations with free control parameters cover the variety of regular and chaotic regimes and can be used for computational purposes and data analysis. To demonstrate the chaotic regime, for instance, the dimension of the phase space for the deterministic system should be at least 3.

Here we compare two multidimensional systems: classical 4-dimensional Hodgkin-Huxley (HH) neuron and 3-dimensional quantum bit (in its real ODE representation) in the external field. Both systems can be driven via the free parameters towards the necessary dynamical state (stabilization or tracking goal).

To demonstrate the possibility of using a multi-dimensional classical system for the realization of quantum algorithms we describe the HH-neuron based adaptation of Deutsch algorithm with Fradkov's speed gradient feedback control and discuss its pros and cons of such approach.

2. Qubits and Deutsch algorithm

Quantum bits (qubits) could be considered as basic units of quantum information processes. In the ideal case a qubit can be represented as a real-variable 3-dimansional dynamical system making its evolution on the surface of 2d-sphere (the Bloch sphere).

The non-classical properties of qubits serve for the purpose of specific quantum algorithms. As an example let's describe the quantum algorithm for a simple searching problem, i.e. the Deutsch algorithm proposed in 1985 and generalized for an arbitrary positive integer n as the Deutsch – Jozsa algorithm in 1992 [1]. Here for simplicity we will discuss the case $n = 1$.

Assume that we are given a function $f : \{0,1\} \to \{0,1\}$, and we are promised that one of two possible holds:

1. f is **constant**: that means that either $f(x) = 0$ for all x from $\{0,1\}$ or $f(x) = 1$ all x from $\{0,1\}$.
2. f is **balanced**: i.e., the number of inputs x from $\{0,1\}$ for which the function takes values 0 and 1 are the same.

The goal is to determine which of the two possibilities holds.

The classical solution to this searching problem demands $2^{n-1} - 1$ evaluations. The quantum Deutsch algorithm demands only one 'measurement' and involves three Hadamard's gates H and one operator U_f, see the circuit implementation in Fig 1. The symbol $\oplus$ denotes addition mod 2 .

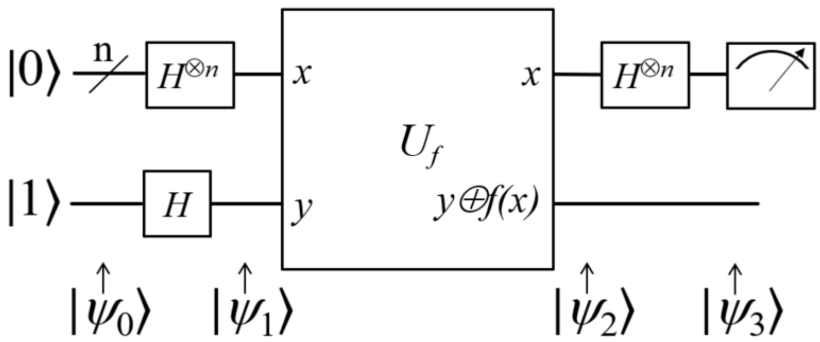

Figure 1. Circuit for implementing the Deutsch algorithm; based on [2].

The result of measurement on Fig. 1 is given by the combination:

$$M = \frac{1}{2}\left[(1+(-1)^{f(0)\oplus f(1)})\left|0\right\rangle + (1-(-1)^{f(0)\oplus f(1)})\left|1\right\rangle\right] \quad . \tag{1}$$

From (1) one can see that the single measurement is enough to define the type of the function f. Indeed,

- $f(0) \oplus f(1) = 0$ if and only if we measure 0, therefore f is constant;
- $f(0) \oplus f(1) = 1$ if and only if we measure 1, therefore f is balanced.

The similar phase kick-backe ffect is observed for the case of n bits resulting from the measurements. If all n measurement results are 0, we conclude that the function was constant. Otherwise, if at least one of the measurement outcomes is 1, we conclude that the function was balanced.

The Deutsch-based family of quantum algorithms has a wide spectrum of applications: quantum computation, quantum cryptography, quantum data mining, simulation of quantum systems, modeling of formal languages [3].

3. Hodgkin – Huxley mathematical neuron

Here we present a simple classical realization of Deutsch algorithm based on the Hodgkin – Huxley mathematical neuron [4]. HH model is based on the system of four differential equations:

$$\begin{aligned} &C_M \cdot \frac{dv}{dt} = -g_{Na} m^3 h \cdot (v - E_{Na}) - g_K n^4 \cdot (v - E_K) - g_{Cl} \cdot (v - E_{Cl}) + I(t) \; ; \\ &\frac{dm}{dt} = \alpha_m(v) \cdot (1-m) - \beta_m(v) \cdot m \; ; \\ &\frac{dn}{dt} = \alpha_n(v) \cdot (1-n) - \beta_n(v) \cdot n \; ; \\ &\frac{dh}{dt} = \alpha_h(v) \cdot (1-h) - \beta_h(v) \cdot h \; . \end{aligned} \tag{2}$$

In (2) $v(t)$ stands for the membrane potential, $m(t)$, $n(t)$, $h(t)$ are the membrane gate variables, and the control signal $I(t)$ is the net current stimulation the axon. The suitable rate functions are given by:

$$\begin{aligned} &\alpha_m(v) = \frac{0.1 \cdot (25-v)}{\exp\left\{\frac{25-v}{10}\right\} - 1} \; ; \; \beta_m(v) = 4 \cdot \exp\left\{-\frac{v}{18}\right\} \; ; \\ &\alpha_n(v) = \frac{0.01 \cdot (10-v)}{\exp\left\{\frac{10-v}{10}\right\} - 1} \; ; \; \beta_n(v) = 0.125 \cdot \exp\left\{-\frac{v}{80}\right\} \; ; \\ &\alpha_h(v) = 0.07 \cdot \exp\left\{-\frac{v}{20}\right\} \; ; \; \beta_h(v) = \frac{1}{\exp\left\{\frac{30-v}{10}\right\} + 1} \; . \end{aligned} \tag{3}$$

The set of constants includes the potentials E_{Na} (equilibrium potential at which the net flow of *Na* ions is zero), E_K (equilibrium potential at which the net flow of *K* ions is zero), E_{Cl} (equilibrium potential at which leakage is zero) in mV, the membrane capacitance C_M and the conductivities g_{Na} (sodium channel conductivity), g_K (potassium channel conductivity), g_{Cl} (leakage channel conductivity) in mS/cm^2:

$$\begin{aligned} &g_{Na} = 120; \; E_{Na} = 115 \; ; \\ &g_K = 36; \quad E_K = -12 \; ; \\ &g_{Cl} = 0.3; \; E_{Cl} = 10.36. \end{aligned} \tag{4}$$

The HH system demonstrates the variety of dynamical regimes, including resting (no spikes come), spiking (producing a single output voltage spike) and bursting (the series of chaotic spikes). The spiking regime has a threshold: one should apply a certain level of the input current to produce a single spike.

For transferring the action potential in the axon of the k-th cell via its synapse towards the dendrite/soma input of the l-th neuron we use our gain input-output model [5]:

$$I_l(t) = \alpha \cdot [v_k(t) - v_{\text{rest}}] \; ; \; \alpha = \text{const} > 0, \tag{5}$$

with the constant gain coefficient α and the phenomenological constant reference rest potential of the HH neuron.

Correspondingly, for the threshold (tr) level:

$$I_{\text{tr}} = \alpha \cdot (v_{\text{tr}} - v_{\text{rest}}) \; . \tag{6}$$

Such a threshold action potential v_{tr} stimulates the necessary treshold current I_{tr} entering the next cell to produce a single spike. If the current coming to the next neuron is below the threshold level, the neuron stays at the resting regime.

4. Deutsch algorithm with the Hodgkin – Huxley neuron

Now we are ready to formulate the classical HH-based analog of the Deutsch algorithm (1). At first, let's define the HH neuron 'quasi-quantum states':

$$\begin{aligned} |0\rangle &= 0 \cdot I_{\text{tr}} \quad (\text{resting}); \\ |1\rangle &= 1 \cdot I_{\text{tr}} \quad (\text{spiking}), \end{aligned} \tag{7}$$

and by (6) the corresponding potentials:

$$\begin{aligned} v_{|0\rangle} &= v_{\text{rest}} + \frac{0 \cdot I_{\text{tr}}}{\alpha} = v_{\text{rest}} \; ; \\ v_{|1\rangle} &= v_{\text{rest}} + \frac{1 \cdot I_{\text{tr}}}{\alpha} = v_{\text{rest}} + \frac{I_{\text{tr}}}{\alpha} \; . \end{aligned} \tag{8}$$

We will apply a control algorithm to drive the dynamical system (2) towards the goal potential:

$$v_* = v_{\text{rest}} + \left| f(0) - f(1) \right| \cdot \frac{I_{\text{tr}}}{\alpha} \, . \tag{9}$$

Now in (9) we get the following options:

- If $f(0) = f(1)$, then $v_* = v_{\text{rest}} = v_{|0\rangle}$, and f is constant;
- If $f(0) = f(1)$, then $v_* = v_{\text{rest}} + 1 \cdot \frac{I_{\text{tr}}}{\alpha} = v_{|1\rangle}$, and f is balanced.

To achieve the goal (9), let's use Fradkov's speed gradient algorithm [6]. We define the non-negative differentiable goal function:

$$G(t) = \frac{1}{2}\left[v(t) - v_*(t)\right]^2 . \tag{10}$$

It is dynamical (i.e. time-dependent), by that the control goal is tracking. The control signal in the HH neuron is one-dimensional, for that reason the gradient is reduced to the partial derivative:

$$I_{SG} = -\Gamma \frac{\partial}{\partial I}\left(\frac{dG}{dt}\right) = -\frac{\Gamma}{C_M}(v - v_*)\ ;\ \Gamma = \text{const} > 0. \tag{11}$$

This control current (11) drives the system (2) towards the minimization of the control goal (10). The achievability of the stabilization/tracking goal in HH dynamical system for the speed gradient algorithm has been proved in [7].

Now the differential equation for the action potential v looks like:

$$C_M \cdot \frac{dv}{dt} = -g_{Na} m^3 h \cdot (v - E_{Na}) - g_K n^4 \cdot (v - E_K) - g_{Cl} \cdot (v - E_{Cl}) - \frac{\Gamma}{C_M}(v - v_*) . \tag{12}$$

The produced action potential v enters to the next HH neuron, in which we perform a 'measurement' by checking if it demonstrates the resting or spiking.

Thus, the finalized algorithm follows the chain:

$$f \Rightarrow v_* \Rightarrow I \Rightarrow v = v_{|0\rangle} \text{ or } v = v_{|1\rangle} .$$

If we observe the resting in the measuring HH neuron (i.e. if it does not spike), the function f must be constant, if the measuring neuron produces a spike, the function f is balanced.

Our algorithm, as in the Deutch approach (1), demands only one measurement.

5. Conclusion

The given multidimensional classical system in the form of controlled 4d HH system is capable to imitate the effects similar to the quantum phase contributions to the computational process. Thus, we provide the simple examples of the HH-based computational algorithms following the quantum paradigm.

The algorithm proposed here could be easily generalized for the case of few HH neurons (the analog of multi-qubit system).

Of course, from the purpose of time optimization quantum algorithms work faster than their classical emulations. Nevertheless, the HH-based classical algorithms are easier for the practical implementation and do not demand very sophisticated software.

Thus, in summary, our approach can open a new gate for the practical realization of quantum algorithms and provide new perspectives for the computational properties of artificial neural networks.

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Controller Design of Haptic-Teleoperation System using PID and FLC methods to Compensate for Uncertain Dynamics

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Abstract

Haptic-Teleoperation systems are systems that play an important role in human-robot interaction. The main goal of these systems is seamless bilateral interaction. However, these systems have fundamental problems. One of them is the dynamic uncertainties of the robots that make up the system, and this problem significantly affects the performance of the systems. In this study, firstly the motion equations of the system are obtained. Secondly, Fuzzy Logic Control(FLC) and PID control methods are used to overcome this problem. The studies were carried out in a simulation environment and the results were compared with each other.

1. Introduction

Haptic-Teleoperation systems are systems that ensure human-robot interaction. The main purpose of such systems is to extend the user's manipulation capabilities to a remote environment. The main goal of these systems is a seamless two-way interaction. In the literature, various control methods have been proposed and applied to solve the problem of uncertainty of the parameters of these robots. Jiang and Praly have tried to overcome dynamic uncertainties by an adaptive controller design [1]. Hung et al. proposed a nonlinear control method for the teleoperation system in which master and slave robots are not fully known [2]. Nuno et al. proposed an adaptive controller for nonlinear teleoperators and realized it in a simulation environment [3]. In their study, Passenberg and others wrote a review article on the adaptive controllers of the environment and the operator working in the task space [4]. Liu and Tavakoli used adaptive bilateral controllers with nonlinear, ambiguous dynamics, and multi-degree of freedom based on inverse dynamics [5]. Mellah et. al. have created two adaptive rules to set up fuzzy logic membership functions. These rules were applied to the two-way teleoperation system [6]. Abut and Soyguder used the adaptive calculated torque method to deal with kinematic and dynamic uncertainties and interaction forces between the operator and the environment [7]. Yang et al. used an adaptive approach for monitoring control of a teleoperation system with kinematic and dynamic uncertainties. In this approach, they used the adaptive Fuzzy control method to estimate the parameters of the system and to control the system [8]. One of the most important sources of uncertainty in haptic-teleoperation systems is the dynamic models of robots due to their complex nonlinear structures. Therefore, the kinematic and dynamic robot model reveals parametric uncertainties that lead researchers to use adaptive algorithms in control design. Based on this observation in the literature, the fuzzy logic control method was preferred and tried to overcome the control problems arising due to the uncertainties of dynamic and kinematic connections. To purpose to overcome the parameter uncertainty problem of the haptic-teleoperation system consisting of a single degree of freedom master and slave robot, and the control of the system was performed using Fuzzy Logic and PID control methods.

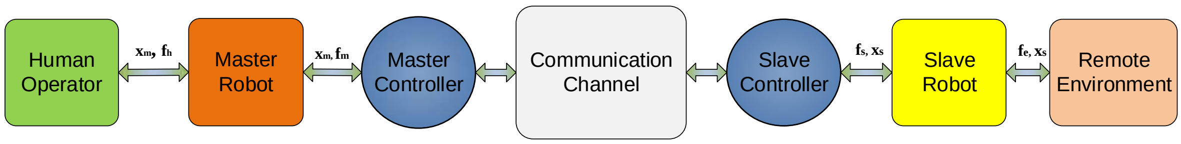


Figure 1. Block diagram of the bilateral teleoperation system

2. Modeling and Control Design of the System

The equations of motion of the haptic-teleoperation robotic system acquired using the Lagrange-Euler method is given below.

$$I_m\ddot{q}_m + b_m\dot{q}_m = \tau_m + \tau_h \quad (1)$$

$$I_s\ddot{q}_s + b_s\dot{q}_s = \tau_s - \tau_e \quad (2)$$

$$f_h = k_h(x_{md} - x_m) + b_h(\dot{x}_{md} - \dot{x}_m) \quad (3)$$

$$f_e = k_e(x_b - x_s) + b_e(\dot{x}_b - \dot{x}_s) \quad (4)$$

The equations of motion of the master and slave robot are given below. $q_i, \dot{q}_i, \ddot{q}_i$ and τ_i respectively represent the position, speed, acceleration, and control torque. The $i \in \{m, s\}$ indices represent the master and slave robots respectively. I_m and I_s represent the moment of inertia and b_m and b_s show the damping coefficients of the robots. f_m, f_s refer to force acting on the master and slave robots respectively. f_h and f_e represent the force corresponding to the disturbing forces exerted by the user and acting on the environment from the system. τ_m, τ_s refers to the torques affecting the master and slave robots, respectively. τ_h and τ_e represents the torques that are applied by the user and correspond to the disruptive forces that affect the system from the environment. $\tau_h = J_m{}^T F_h$ and $\tau_e = J_s{}^T F_e$. The relationship between the human operator and the environment the contacts between a virtual wall and the end of the robot is modeled as a spring-damper system. b_e and b_h represent damping coefficients, k_e and k_h represent spring constants. x_{md}, x_b indicate the reference positions of the robot while x_m and x_m represent its actual positions. Fuzzy Logic (FLC) and classical PID (proportional integral derivative) control methods are used in the control of the haptic-teleoperation system. With these controllers applied to the system, this error is tried to be minimized. Fuzzy logic (Fuzzy Logic) and fuzzy set theory were developed in 1965 by the Azerbaijani Prof. It was proposed by Lotfi A. Zadeh (California University, Berkeley) [9]. In this study, he attributed the reason why people can control some systems better than machines because they have the ability to make decisions by using some information that cannot be expressed with certainty (ambiguous). Fuzzy logic operations consist of the analysis and definition of a problem, the transformation of the information found by developing variable sets and logic relationships, and the interpretation of the model . The block diagram of the Fuzzy Logic Controller(FLC) is shown in Figure 2.

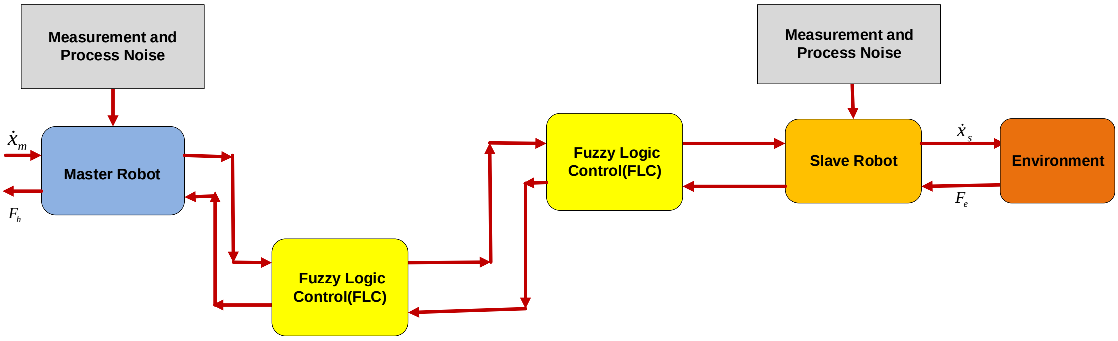
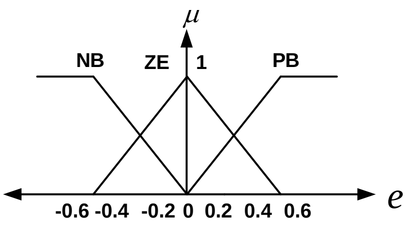


Figure 2. Block diagram of the FLC method used in the control of the system

The controller uses the error and the rate of change of errors as input and performs the control of the system based on time-varying error (e) and error derivative ($\dot{e}$). The proposed fuzzy logic controller aims to improve the control performance provided by a PID controller. The rule base of an FLC (Fuzzy logic controller) consists of a group of IF-THEN rules generally obtained from the verbal expressions of experts who have knowledge about the system to be controlled [10-12]. The rule base is described as the heart of an FLC. Because all other units and components are used to realize these rules in a reasonable and efficient way. The rule table and membership functions created for this system were created in order to obtain the minimum angle error values of the main and dependent robots. The rule base created for the control of this system consists of 9 rules, as seen in Table 1 below. In Figures 3 input values of the system e and $\dot{e}$ membership functions defined for it are shown. The membership functions defined for the output value τ are shown in Figure 4.

Table 1. Rule Base Created for FLC

	τ	ė: NB	ė: ZE	ė: PB
e	NB	NB	NB	ZE
e	ZE	NB	ZE	PB
e	PB	ZE	PB	PB



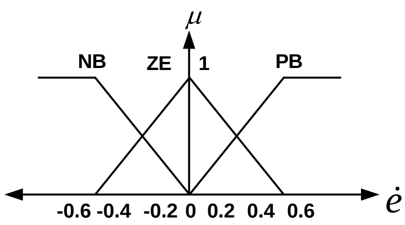


Figure 3. Membership functions defined for input value e and $\dot{e}$ ˙

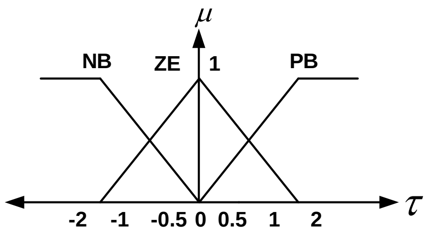


Figure 4. Membership functions defined for ouput value τ

In the control algorithms, the inputs of the system e are the error and the rate of change of error $\dot{e}$. Here, the error is the difference between the outputs of the system and the desired values. The outputs of the system are the appropriate torque values applied to the motors that drive the system in a way that minimizes the error and the instantaneous change values of the error over time. Here, Mamdani method is used in Fuzzy Logic type control. The system control structure consists of a total of one entry and one exit. This value is the position information of the system. Firstly, the graphs of the triangular type membership function used for fuzzy control are shown in Figures 3 and 4 for the input values and {-0.6, -0.4, -0.2, 0, 0.2, 0.4, 0.6}, respectively. and {-0.6, -0.4, -0.2, 0, 0.2, 0.4, 0.6} values in this range were used. At the same time, values in this range {-2, -1, -0.5, 0, 0.5, 1, 2} were used for the output value of the fuzzy control. The fuzzy control variables that form the rule bases shown in the tables above were formed as follows. e , $(\dot{e})$,τ = {error, error change, control variable torque {NB (Negative Large), ZE (Zero), PB (Positive Large)}. Input values of FLC are specified in the range of {-0.6, 0.6} and {-2, 2} respectively. All entry and exit membership functions are taken as triangle type.

3. Simulation Results

In this section, simulation studies have been done by using the motion equations of the system. The control variables of the system are position and force parameters. In this study, simulation studies of the single degree of freedom haptic-teleoperation system were carried out. The single degree of freedom haptic-teleoperation is the basis of teleoperation studies performed with multi-degree of freedom haptic robots. Figure 5 shows the simulation model of the interaction of the master and slave robot with the environment.

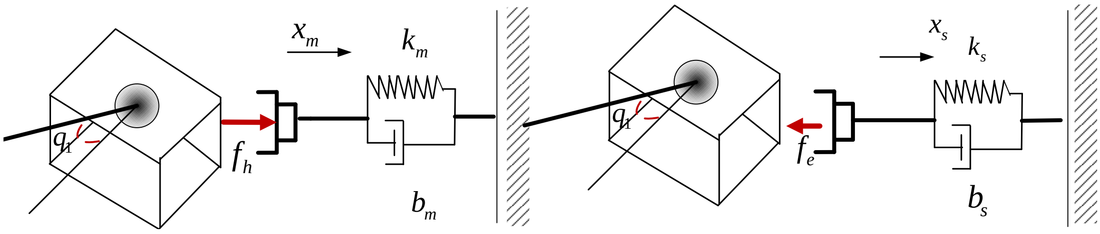


Figure 5. The interaction model of master and slave robots with the environment

Figures 6-a and 6-b are shown the angular position and angular position error graphs of the master and slave robots obtained by the PID control method, respectively. Figures 7-a and 7-b are shown the torque and torque error graphs of the master and slave robots obtained by the PID control method, respectively. Figures 8-a and 8-b are shown the angular position and angular position error graphs obtained by the FLC control method of master and slave robots respectively. Figures 9-a and 9-b are shown the torque and torque error graphs of the master and slave robots obtained by the FLC control method, respectively.

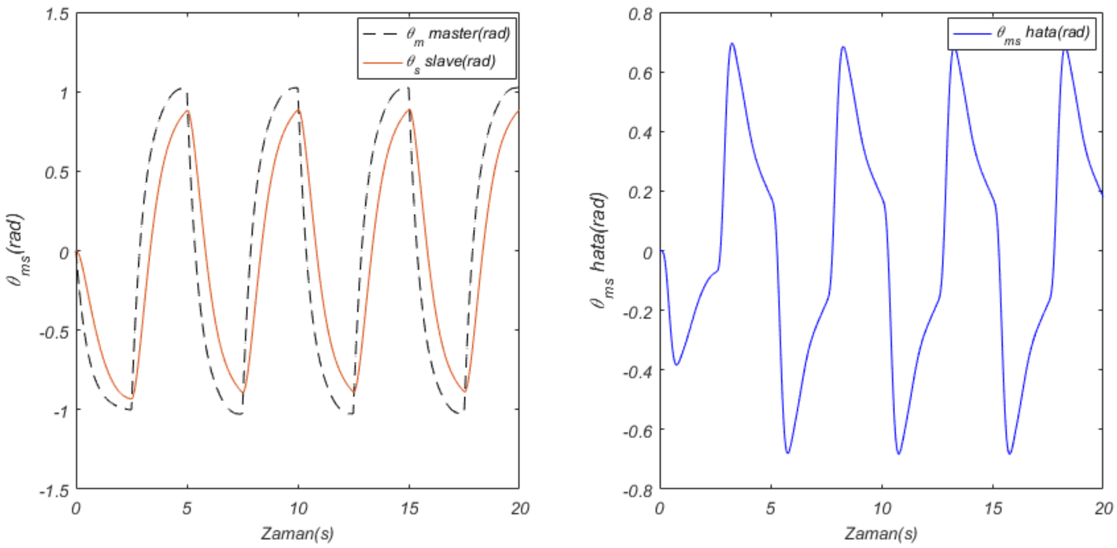


Figure 6. a) Angular position and b) Angular position error response of joint angle obtained by the PID control method

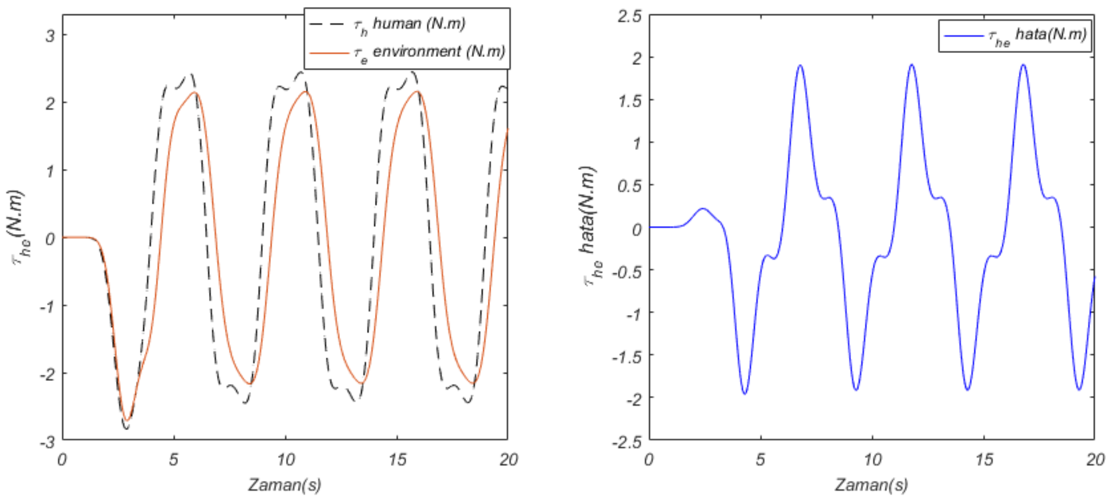


Figure 7. Human and environment torque response obtained a) torque and b) torque error by the PID control method

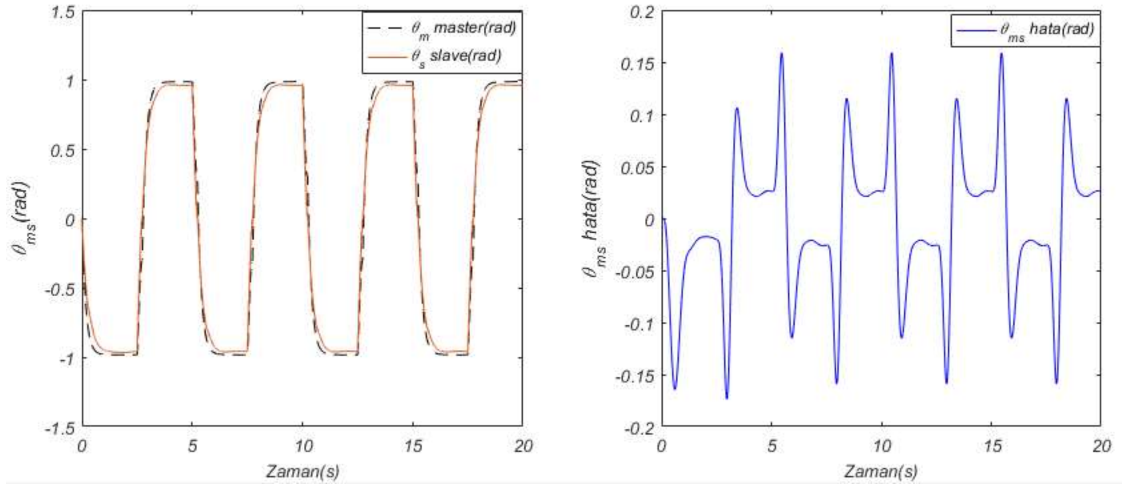


Figure 8. a) Angular position and b) Angular position error response of joint angle obtained by the FLC control method

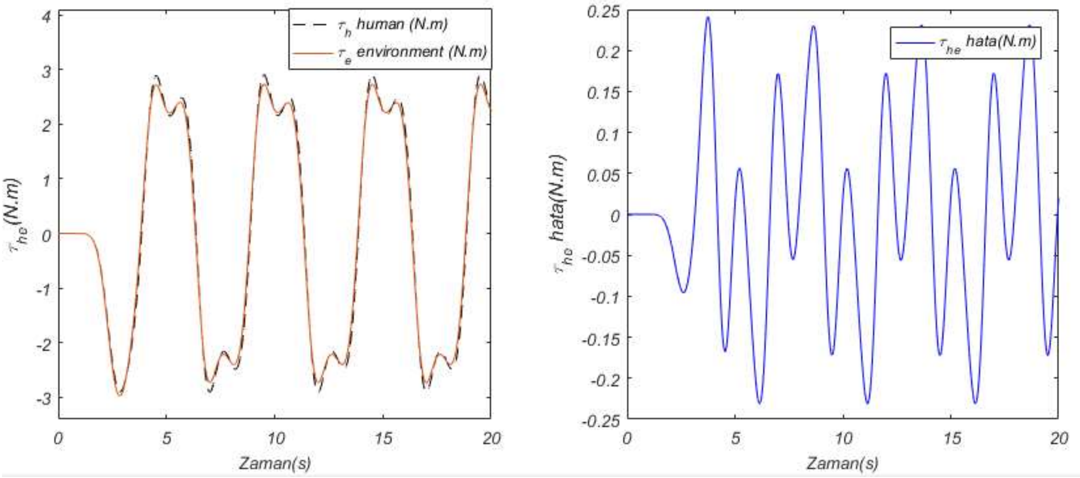


Figure 9. Human and environment torque response obtained a) torque and b) torque error by the FLC control method

The simulation run time was taken as 20 seconds. The forces generated as a result of the interaction of the dependent robot with the environment are the forces physically felt by the human-operator as a result of the torque-force sensors being transmitted to the main robot (Figure 7 and 9). As can be seen in the graphs obtained by the FLC method, a lower position error has occurred compared to the classical PID method. It was seen that the PID control method has a larger error in the compared position error responses. It has been observed that the human and environmental torque graphs obtained by the FLC method show the low amplitude and low continuous regime error performance compared to the classical PID method. FLC is seen to be the method that best follows the reference value generated by the master robot. The best torque follow-up was also obtained by the FLC method. The simulation result graphs showed that the haptic-teleoperation system consisting of main and dependent robots was found to be more feasible with the FLC method.

4. Conclusion

In this study, the control of a bilateral single degree of freedom teleoperation system was carried out. The best torque tracking performance was obtained with the fuzzy logic control(FLC) method as in position responses. Among the methods, it has been seen that the FLC method is more applicable for this teleoperation system. In addition, the PID control method showed the worst performance as can be seen in the graphs. The obtained results are given in the form of graphics, tables, and analyses. As a result of the simulations, useful prospective information about the movement of the system was obtained. In the future, real-time studies will be carried out in the laboratory using this method.

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Curvatures of the Translation Hypersurface in 4-Space

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Abstract

We study curvatures $\mathfrak{C}_{i=1,2,3}$ of translation hypersurface in the four dimensional Euclidean space. We also give some relations on $\mathfrak{C}_i$ of translation hypersurface.

1. Introduction

We see some nice papers about translation surfaces (TS) and translation hypersurfaces (TH) in the literature such as [1-17].

Arslan et al [1] worked TS in 4-dimensional Euclidean space; Chen, Sun, and Tang [2] studied TH with constant mean curvature (CMC) in $(n+1)$-dimensional spaces; Dillen, Verstraelen, and Zafindratafa [3] focused a generalization of the TS of Scherk; Inoguchi, Lopez, and Munteanu [4] gave minimal TS in the Heisenberg group Nil_3; Lima, Santos, and Sousa [5] introduced TH with constant scalar curvature into the Euclidean space; Lima, Santos, and Sousa [6] studied generalized TH in Euclidean space; Liu [7] considered TS with CMC in 3-dimensional spaces; Lopez [8] worked minimal TS in hyperbolic space; Lopez and Moruz [9] studied translation and homothetical surfaces with constant curvature in Euclidean space; Lopez and Munteanu [10] worked minimal TS in Sol_3; Moruz and Munteanu [11] obtained hypersurfaces in the Euclidean space $\mathbb{E}^4$ defined as the sum of a curve and a surface whose mean curvature vanishes; Munteanu, Palmas, and Ruiz-Hernandez [12] worked minimal TH in Euclidean space; Scherk [13] found his classical minimal TS; Seo studied [14] TH with constant curvature in space forms; Verstraelen, Walrave, and Yaprak [15] studied on the minimal TS in $\mathbb{E}^n$; Yang and Fu [16] considered affine TS in affine space; Yoon [17] focused the Gauss map of TS in Minkowski 3-space.

A translation surface in $\mathbb{E}^3$ is a surface generated by translations. For two space curves α, β with a common point P, the curve α is shifted such that point P is moving on β. Then the curve α generates a TS

$$\mathbf{x}(u,v) = \alpha(u) + \beta(v).$$

A translation hypersurface in the four dimensional Euclidean space $\mathbb{E}^4$ is a hypersurface generated by translations: for three space curves α, β, γ with a common point P, the curve α is shifted such that point P is moving on β and γ, respectively. Then, the curve α generates a TH in $\mathbb{E}^4$. TH is parametrized by

$$\mathbf{x}(u,v,w) = \big(u, v, w, f(u) + g(v) + h(w)\big). \tag{1.1}$$

where $f(u), g(v), h(w)$ are differentiable functions for all $u, v, w \in I \subset \mathbb{R}$. More clear form of it as follows

In this work, we obtain curvatures of hypersurfaces in $\mathbb{E}^4$. We present basic elements of the four dimensional Euclidean geometry. Moreover, we compute curvatures $\mathfrak{C}_{i=1,2,3}$ of TH.

2. Preliminaries

To find the i-th curvature formulas $\mathfrak{C}_{i=0,1,\dots n}$ in $\mathbb{E}^{n+1}$, we get characteristic polynomial of shape operator $\mathbf{S}$:

$$P_{\mathbf{S}}(\lambda) = 0 = \det(\mathbf{S} - \lambda I_n) = \sum_{k=0}^{n} (-1)^k s_k \lambda^{n-k}, \tag{2.1}$$

Here, I_n shows identity matrix. Then, we get curvature formulas $\binom{n}{i}\mathfrak{C}_i = s_i$, where $\binom{n}{0}\mathfrak{C}_0 = s_0 = 1$ by definition. k-th fundamental form of hypersurface M^n is given by $\mathrm{I}(\mathbf{S}^{k-1}(X), Y) = \langle \mathbf{S}^{k-1}(X), Y \rangle$. So, we get

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} \mathfrak{C}_i \, \mathrm{I}(\mathbf{S}^{k-1}(X), Y) = 0. \tag{2.2}$$

Throughout the work, we will identify a vector $(\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d})$ with its transpose.

One can assume $\mathbf{M} = \mathbf{M}(u, v, w)$ be an isometric immersion of a hypersurface M^3 in $\mathbb{E}^4$. Dot product of $\vec{x} = (x_1, x_2, x_3, x_4)$ and $\vec{y} = (y_1, y_2, y_3, y_4)$ in $\mathbb{E}^4$ is defined by $\langle \vec{x}, \vec{y} \rangle = \sum_{i=1}^{4} x_i y_i$. Vector product in $\mathbb{E}^4$ is given by

$$\vec{x} \times \vec{y} \times \vec{z} = \det \begin{pmatrix} e_1 & e_2 & e_3 & e_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{pmatrix}.$$

The Gauss map of a hypersurface $\mathbf{M}$ is defined by

$$e = \frac{\mathbf{M}_u \times \mathbf{M}_v \times \mathbf{M}_w}{\|\mathbf{M}_u \times \mathbf{M}_v \times \mathbf{M}_w\|},$$

where $\mathbf{M}_u = d\mathbf{M}/du$. We obtain following matrices for a hypersurface $\mathbf{M}$ in $\mathbb{E}^4$,

$$\mathrm{I} = \begin{pmatrix} E & F & A \\ F & G & B \\ A & B & C \end{pmatrix}, \quad \mathrm{II} = \begin{pmatrix} L & M & P \\ M & N & T \\ P & T & V \end{pmatrix}, \quad \mathrm{III} = \begin{pmatrix} X & Y & O \\ Y & Z & R \\ O & R & S \end{pmatrix},$$

where the coefficients are defined by

$$\begin{array}{llllll} E = \langle \mathbf{M}_u, \mathbf{M}_u \rangle, & F = \langle \mathbf{M}_u, \mathbf{M}_v \rangle, & G = \langle \mathbf{M}_v, \mathbf{M}_v \rangle, & A = \langle \mathbf{M}_u, \mathbf{M}_w \rangle, & B = \langle \mathbf{M}_v, \mathbf{M}_w \rangle, & C = \langle \mathbf{M}_w, \mathbf{M}_w \rangle, \\ L = \langle \mathbf{M}_{uu}, e \rangle, & M = \langle \mathbf{M}_{uv}, e \rangle, & N = \langle \mathbf{M}_{vv}, e \rangle, & P = \langle \mathbf{M}_{uw}, e \rangle, & T = \langle \mathbf{M}_{vw}, e \rangle, & V = \langle \mathbf{M}_{ww}, e \rangle, \\ X = \langle e_u, e_u \rangle, & Y = \langle e_u, e_v \rangle, & Z = \langle e_v, e_v \rangle, & O = \langle e_u, e_w \rangle, & R = \langle e_v, e_w \rangle, & S = \langle e_w, e_w \rangle. \end{array}$$

3. Curvatures

Next, we will obtain curvatures for a hypersurface $\mathbf{M}(u, v, w)$ in $\mathbb{E}^4$.

Using characteristic polynomial $P_{\mathbf{S}}(\lambda) = a\lambda^3 + b\lambda^2 + c\lambda + d = 0$, we obtain curvature formulas: $\mathfrak{C}_0 = 1$ (by definition),

$$\binom{3}{1}\mathfrak{C}_1 = -\frac{b}{a}, \quad \binom{3}{2}\mathfrak{C}_2 = \frac{c}{a}, \quad \binom{3}{3}\mathfrak{C}_3 = -\frac{d}{a}.$$

Therefore, we get curvature folmulas:

Theorem 3.1. *Any hypersurface* M^3 *in* $\mathbb{E}^4$ *has following curvature formulas,* $\mathfrak{C}_0 = 1$ *(by definition),*

$$\mathfrak{C}_1 = \frac{(EN+GL-2FM)C+(EG-F^2)V-LB^2-NA^2-2(APG-BPF-ATF+BTE-ABM)}{3[(EG-F^2)C-EB^2+2FAB-GA^2]}, \tag{3.1}$$

$$\mathfrak{C}_2 = \frac{(EN+GL-2FM)V+(LN-M^2)C-ET^2-GP^2-2(APN-BPM-ATM+BTL-PTF)}{3[(EG-F^2)C-EB^2+2FAB-GA^2]}, \tag{3.2}$$

$$\mathfrak{C}_3 = \frac{(LN-M^2)V-LT^2+2MPT-NP^2}{(EG-F^2)C-EB^2+2FAB-GA^2}. \tag{3.3}$$

Proof. Solving $\det(\mathbf{S}-\lambda I_3)=0$ with some calculations, we get coefficients of polynomial $P_{\mathbf{S}}(\lambda)$.

Theorem 3.2. For *any hypersurface* M^3 *in* $\mathbb{E}^4$, *curvatures are related by following formula*

$$\mathfrak{C}_0 IV - 3\mathfrak{C}_1 III + 3\mathfrak{C}_2 \mathrm{II} - \mathfrak{C}_3 I = 0. \tag{3.4}$$

4. Curvatures of TH

With the first differentials of (1.1) depends on u, v, w, we get the Gauss map of (1.1):

$$e = \frac{1}{(\det \mathrm{I})^{1/2}}(f', g', h', -1). \tag{4.1}$$

Here, $\det \mathrm{I} = f'^2+g'^2+h'^2+1$, and $f' = df/du,\ g' = dg/dv,\ h' = dh/dw$. We have the first and the second fundamental form matrices of (1.1), respectively,

$$\mathrm{I} = \begin{pmatrix} 1+f'^2 & f'g' & f'h' \\ f'g' & 1+g'^2 & g'h' \\ f'h' & g'h' & 1+h'^2 \end{pmatrix},\quad \mathrm{II} = \begin{pmatrix} -f''/(\det \mathrm{I})^{1/2} & 0 & 0 \\ 0 & -g''/(\det \mathrm{I})^{1/2} & 0 \\ 0 & 0 & -h''/(\det \mathrm{I})^{1/2} \end{pmatrix}.$$

Then, we obtain the third fundamental form matrix

$$\mathrm{III} = \begin{pmatrix} f'^2(g'^2+h'^2+1)/(\det \mathrm{I})^2 & -f'g'f''g''/(\det \mathrm{I})^2 & -f'h'f''h''/(\det \mathrm{I})^2 \\ -f'g'f''g''/(\det \mathrm{I})^2 & g'^2(f'^2+h'^2+1)/(\det \mathrm{I})^2 & -g'h'g''h''/(\det \mathrm{I})^2 \\ -f'h'f''h''/(\det \mathrm{I})^2 & -g'h'g''h''/(\det \mathrm{I})^2 & h'^2(f'^2+g'^2+1)/(\det \mathrm{I})^2 \end{pmatrix}.$$

Theorem 4.1. *TH hypersurface* (1.1) *in* $\mathbb{E}^4$ *has following curvature formulas,* $\mathfrak{C}_0 = 1$ *(by definition),*

$$\mathfrak{C}_1 = \frac{2(\det \mathrm{I})^{3/2}(f'^2 f''+g'^2 g'')h'^2 h'' - (\det \mathrm{I})^3 \left\{ \begin{matrix} (g'^2+h'^2+1)f''+(f'^2+h'^2+1)g'' \\ +(f'^2+g'^2+1)h'' \end{matrix} \right\}}{3(\det \mathrm{I})^{9/2}},$$

$$\mathfrak{C}_2 = \frac{\left\{ \begin{matrix} -\left\{ \begin{matrix} f'^2(g'^2+1)f''^2 - 2f'^2 g'^2 f'' g'' \\ +(f'^2+1)g'^2 g''^2 \end{matrix} \right\} h'^2 h''^2 \\ -2(\det \mathrm{I})^{3/2}(f'^2+g'^2)h'^2 f'' g'' h'' \end{matrix} \right\} - (\det \mathrm{I})^3 \left\{ \begin{matrix} (g'^2+1)f''h'' \\ +(f'^2+1)g''h'' \\ +(h'^2+1)f''g'' \end{matrix} \right\}}{3(\det \mathrm{I})^5},$$

$$\mathfrak{C}_3 = \frac{[(f'^2 f''+g'^2 g'')h'^2 h'' - (\det \mathrm{I})^3] f'' g'' h''}{(\det \mathrm{I})^{11/2}},$$

where $\det \mathrm{I} = f'^2+g'^2+h'^2+1$.

Proof. Computing (3.1), (3.2), and (3.3) of (1.1), we have the curvatures.

Corollary 4.1. *The curves* $\alpha(u), \beta(v)$ *and* $\gamma(w)$ *on translation hypersurface* (1.1) *have unit speed if and only if the curvatures of* (1.1) *are as follows*

$$\mathfrak{C}_1 = 0, \ \ \mathfrak{C}_2 = 0, \ \ \mathfrak{C}_3 = 0,$$

i.e., hypersurface (1.1) *is* i*-minimal translation hypersurface, where* $i = 1,2,3$.

5. Conclusion

Translation hypersurfaces have been studied by some authors. We have expanded well-known results of the translation hypersurfaces by using its curvatures in $\mathbb{E}^4$. In addition, we give minimality conditions of it.

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Determining the Relation Between Reasoning Skills and Critical Thinking Disposition of Pre-Service Science Teachers

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Abstract

In this study, it was aimed to determine the relation between reasoning skills and critical thinking dispositions of pre-service science teachers. Correlational research design was used in the study. The sample of the study consists of 50 teacher candidates studying at 3rd grade in science teaching program at Atatürk University Kazım Karabekir Education Faculty. The Hypothetico-creative Reasoning Skills Inventory (HRSI) developed by Duran (2014) and the Critical Thinking Disposition Scale (CTDS) developed by Semerci (2016) were used to collect data. The reliability of the scales was determined as $\alpha = .89$ and $\alpha = .96$, respectively. HRSI consists of 23 items and five sub-factors, and CTDS consists of 49 items and five sub-factors. During the analysis of the data, first of all, normality analyzes were made for each scale and sub-factors in the scales. Pearson's product moment correlation coefficients was calculated to determine the relation between reasoning skills and critical thinking disposition. Accordingly, no significant was found between the reasoning skills and critical thinking disposition of prospective science teachers ($p > .05$).

Keywords: Critical thinking disposition, pre-service science teachers, reasoning skills.

1. Introduction

Education programs in the world and in our country aim to bring individuals who can cope with real life problems, have the ability to question, discuss, and make decisions, think scientifically, and access information themselves. In other words, today's societies need individuals with 21st century skills.

Critical thinking, which is accepted as a factor that facilitates access to information and comprehension, is one of the desired achievements of education (Semerci, 2016). Stating that critical thinking is basically based on the ability and tendency to obtain, evaluate and use information effectively, Demirel (1999) explained the five dimensions of critical thinking as follows: Consistency; a critical thinker should be able to eliminate contradictions in thought. Combining; a critical thinker should be able to deal with all dimensions of thought. Applicability; a critical thinker should be able to apply his thoughts to a model. Competence; the critical thinker should base his experiences and results on a solid foundation. Communicating; the critical thinker should be able to share what they think effectively. One of the most important goals of the science program, which was organized and put into practice in the

2004-2005 academic year in our country and which is based on the constructivist approach, is to develop students' critical thinking skills (MEB, 2006). Teachers who lack critical thinking skills and disposition are seen as an important obstacle in achieving this goal. In this context, teachers who are trained in critical thinking and have critical thinking skills and disposition are needed to improve students' critical thinking skills.

Reasoning has defined it as a method at the heart of scientific knowledge production by which evidence is collected, analyzed, and links between concepts and theories are established (Schen, 2007). Reasoning skills are central to learning science by doing and experiencing. However, these skills prepare individuals to solve socio-scientific problems they may encounter in their lives and to be good citizens (Osborne, 2013) and assist in decision-making processes. For example, we use the proportional reasoning skill to calculate how much fuel we will consume based on the distance of the road we will travel, and the probabilistic reasoning skill when predicting the weather. For this reason, the contribution of reasoning skill to the formation of a scientifically literate individual profile is of undeniable importance. In addition to the benefits it provides in daily life, reasoning skills have an important place in academic terms. Chen and She (2015) found that 5th grade students who received explicit reasoning training produced significantly more testable and correct hypotheses, evidence-based explanations, and their level of reasoning increased compared to students who did not receive this training. Despite its importance, many studies focus on students' inadequacy in reasoning skills (Croker, 2012).

In this context, many studies conducted in recent years refer to the necessity of science education based on scientific process skills, which form the basis of reasoning skills and critical thinking, especially in university education (Quitadamo & Kurtz, 2007). Accordingly, it was aimed to determine the relation between reasoning skills and critical thinking dispositions of pre-service science teachers.

2. Method

Correlational research design was used in quantitative research methods.

2.1.Samples

The sample of the study consists of 50 teacher candidates studying at 3rd grade in science teaching program at Atatürk University Kazım Karabekir Education Faculty. Appropriate sampling method was used in determining the sample.

2.2.Data Collection Tool

The Hypothetico-creative Reasoning Skills Inventory (HRSI) is a likert-type scale consisting of 23 items and five sub-factors (Hypothetical Thinking and Creativity-HTC; Proportional Thinking-PT; Identifying and Controlling Variables-Combined Thinking-ICVCT; Probabilistic Thinking-PRT and

Correlational Thinking-CT). Items in the scale were scored as "1 = 20%", "2 = 40%", "3 = 60%", "4 = 80% and 5 = 100%". The highest score that can be obtained from the inventory is 115 and the lowest score is 23. The reliability of the scale was determined as $\alpha = .89$.

Critical Thinking Disposition Scale (CTDS) is a Likert-type scale consisting of 49 items and five sub-factors (Open Mindedness-OM; Systematicity-S; Flexibility-F; Perseverance and Patience-PP and Metacognition-M). The items in the scale were scored as "1 = strongly disagree", "2 = mostly disagree", "3 = partially agree", "4 = mostly agree and 5 = completely agree". The highest score that can be obtained from the scale is 245 and the lowest score is 49. The reliability of the scale was determined as $\alpha = .96$.

2.3.Data Analysis

During the analysis of the data, first of all, normality analyzes were made for each scale and sub-factors in the scales. Since the number of teacher candidates is more than 30, the normality of the data was determined with the help of the Kolmogorov-Smirnov Test. Pearson's product moment correlation (simple linear correlation) was used in correlation analysis, since the HRSI and CTDS data were normally distributed ($p = .200$; $p > .05$).

3. Findings

In this title, descriptive statistics obtained from each scale are presented first. Then, the result of correlation analysis was displayed.

Table 1. Descriptive statistics of HRSI and CTDS

Scales	Sub-factors	N	X	Median	SS	Min	Max	Ranj
HRSI	HTC	50	16,46	16,00	4,568	5	24	19
	PT	50	17,46	17,00	2,991	10	23	13
	ICVCT	50	22,24	23,00	3,734	14	28	14
	PRT	50	10,40	10,00	2,441	5	15	10
	CT	50	13,94	15,00	3,235	6	20	14
Full scale		50	80,48	80,00	13,995	48	107	59
CTDS	OM	50	12,38	12,00	1,576	8	15	7
	S	50	52,18	53,00	6,026	38	65	27
	F	50	43,50	43,50	5,697	24	54	30
	PP	50	32,30	32,00	4,273	23	39	16
	M	50	40,64	52,00	23,988	3	68	65
Full scale		50	181,00	183,00	30,103	128	236	108

According to Table 1, it is seen that teacher candidates received the lowest 48 and the highest 107 points on the hypothetico-creative reasoning skills inventory. The highest score that can be obtained from the scale is 115 and the lowest score is 23. The average of the scores the teacher candidates got from the scale is 80.48 standard deviation 13.9. It can be said that the reasoning skills of the teacher candidates are at an average value, since the average of the participants' reasoning skills scores ($\bar{X}$= 80.48) is very close to 80, which is the middle score value of the scale.

It is seen that the pre-service teachers received the lowest 128 and the highest 236 points on the critical thinking disposition scale. The highest score that can be obtained from the scale is 245 and the lowest score is 49. The average of the pre-service teachers' scores from the scale is 181 standard deviation of 30. It can be said that pre-service teachers 'critical thinking dispositions are low since the average score of the participants' critical thinking dispositions ($\bar{X}$ = 181) is less than 183, which is the midpoint value of the scale.

Correlation analysis was conducted to determine the relationship between hypothetico-creative reasoning skills and critical thinking dispositions of pre-service science teachers. Pearson product-moment correlation was calculated during the correlation analysis process, since the HRSI and CTDS data were normally distributed.

Table 2. Correlation between hypothetico-creative reasoning skills and critical thinking disposition

		HRSI	CTDS
HRSI	Pearson Correlation	1	,258
	Sig. (2-tailed)		,070
	N	50	50
CTDS	Pearson Correlation	,258	1
	Sig. (2-tailed)	,070	
	N	50	50

According to Table 2, no significant was found between the hypothetico-creative reasoning skills and critical thinking disposition of prospective science teachers (r=.258; p>.05).

4. Conclusion and Recommendations

Prospective teachers' hypothetico-creative reasoning skills were average, and critical thinking disposition were below average. The reason why teacher candidates do not have these skills at the desired level may be their inability to provide theory-evidence coordination and their acceptance of what they believe as certain truths. However, these skills are skills that can be developed later. As a matter of fact, Schwartz, Lederman, and Crawford (2004) reported in their study with pre-service teachers that providing direct opportunities for reasoning and critical thinking, as well as giving them enough time for reflection and discussion, reinforced the pre-service teachers' views on these skills. In this direction, it is thought that an effective science teaching can improve the deficiencies in students' skills.

In line with the research results, it is recommended to examine the sub-dimensions of the hypothetico-creative reasoning skills and critical thinking dispositions of teacher candidates, to examine them using different scales and inventories, and to reveal the reasons for the results with qualitative studies.

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ECC in special ring and cryptographic application

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Abstract

Let F_{3^d} is the finite field of order 3^d with d be a positive integer, we consider $A_4 := \mathrm{F}_{3^d}[\varepsilon] = \mathrm{F}_{3^d}[X]/(X^4)$ is a finite quotient ring, where $\varepsilon^4 = 0$ [5]. In this paper, we will show an example of encryption and decryption. The motivation for this paper came from the observation that communications, industrial automation and many more. On the other hand, cryptography is the study of mathematical techniques related to aspects of information security [6]. Firstly, we study the elliptic curve over this ring. Furthermore, we study the algorithmic properties by proposing effective implementations for representing the elements and the group law. Finally, we give an example cryptographic (encryption and decryption) with a secret key.

Keywords : Cryptography, Elliptic Curves, Finite Rings, Finite Field.

2010 Mathematics Subject Classification: 11T71, 14H52, 11T55, 94A60.

1. Introduction

Elliptic curves have been playing an increasingly important role both in number theory and in related fields such as cryptography [6]. Cryptography is the science of using mathematics to encrypt and decrypt data. Cryptography enables you to store sensitive information or transmit it across insecure networks (like the Internet) so that it cannot be read by anyone except the intended recipient.

In 2015 My Hachem Hassib, Abdelhakim Chillali and Mohamed Abdou Elomary [5] had constructed a local ring $\mathrm{F}_{3^d}[\varepsilon] = \mathrm{F}_{3^d}[X]/(X^4)$, $\varepsilon^4 = 0$, defined an elliptic curve over $\mathrm{F}_{3^d}[\varepsilon]$ and they had given the classification of elements in $E_{a,b}(\mathrm{F}_{3^d}[\varepsilon])$.

In this work, we will show cryptographic application over this ring (encryption, decryption) by using the public key scheme. In the first time, we show the fundamental results in the paper [5].

In section 3 we give a cryptosystem for ecryption and decryption. Finally, we present an application on cryptography by using code of elements of subgroup G generator by the point P.

2. Preliminaries

In this section, we will show some the fundamental results in the paper [5].

Let d be a positive integer. We consider the quotient ring $A_4=\mathbb{F}_{3^d}[X]/(X^4)$ where $\mathbb{F}_{3^d}$ is the finite field of order 3^d. Then the ring A_4 is identified to the ring $\mathbb{F}_{3^d}[\varepsilon]$, $\varepsilon^4=0$.

So we have:

$$A_4 = \{ \textstyle\sum_{i=0}^{3} x_i \varepsilon^i \mid (x_i)_{0\le i\le 3} \in \mathbb{F}_{3^d} \}$$

We have the following lemmas:

Lemma 2.1. *Let* $X = \sum_{i=0}^{3} x_i\varepsilon^i$. *X is invertible in* A_4 *if and only if* $x_0 \neq 0$.

Lemma 2.2. A_4 *is a local ring, it's maximal ideal is* $I_4=(\varepsilon)$.

Lemma 2.3. A_4 *is a vector space over* F_{3^d} *and have* $(1,\varepsilon,\varepsilon^2,\varepsilon^3)$ *as basis.*

Remark 2.1. We denote by π the canonical projection defined by:

$$\begin{array}{rcl} \pi:\ A_4 & \to & \mathbb{F}_{3^d} \\ \sum_{i=0}^{3} x_i\varepsilon^i & \mapsto & x_0 \end{array}$$

2.1. Elliptic curve over A_4

Definition 2.1. We consider the elliptic curve over the ring A_4 which is given by the equation: $Y^2Z = X^3 + aX^2Z + bZ^3$, where $a, b \in A_4$ and $-a^3b$ is invertible in A_4, and denoted by $E^4_{a,b}$. So we have: $E^4_{a,b} = \{[X:Y:Z] \in P_2(A_4) \mid Y^2Z = X^3 + aX^2Z + bZ^3\}$.

2.1.1. Classification of elements of $E^4_{a,b}$

Proposition 2.1. *Every element in* $E^4_{a,b}$ *is of the form* $[X : Y : 1]$ *(where* X *or* $Y \in A_4\backslash I_4$*), or* $[X : 1 : Z]$ *where* $X, Z \in I_4$ *and we write:*

$$E^4_{a,b} = \{[X:Y:1] \mid Y^2 = X^3 + aX^2 + b, and\ X\ or\ Y \in A_4 \backslash I_4\} \cup$$

$$\{[X:1:Z] \mid Z = X^3 + aX^2Z + bZ^3, and\ X, Z \in I_4\}.$$

Proof. See [5]. □

Lemma 2.4. *Let* $[X:1:Z] \in E^4_{a,b}$ *where* $X, Z \in (\varepsilon)$.

If $X = x_1\varepsilon + x_2\varepsilon^2 + x_3\varepsilon^3$, *then* $[X:1:Z] = [X:1:x_1^3\varepsilon^3]$

Proof. See [5]. □

2.2. The group law over $E^4_{a,b}$

After classifying the elements of $E^4_{a,b}$, we will define the group law over it. We consider firstly the mapping σ:

$$\begin{aligned} \sigma : E^4_{a,b} &\rightarrow E^4_{\pi(a),\pi(b)} \\ [X:Y:Z] &\mapsto [\pi(X):\pi(Y):\pi(Z)] \end{aligned}$$

Theorem 2.1. *Let* $P = [X_1 : Y_1 : Z_1]$ *and* $Q = [X_2 : Y_2 : Z_2]$ *two points in* $E^4_{a,b}$, *and* $P + Q = [X_3 : Y_3 : Z_3]$.

- *If* $\sigma(P) = \sigma(Q)$ *then:*

  $X_3 = Y_1 Y_2^2 X_1 + Y_1^2 Y_2 X_2 + 2a X_1^2 X_2 Y_2 + 2a X_1 X_2^2 Y_1 + 2 Z_1 Z_2^2 ab Y_1 + 2 Z_1^2 Z_2 ab Y_2.$

  $Y_3 = Y_1^2 Y_2^2 + 2a^2 X_1^2 X_2^2 + a^2 b X_1 Z_1 Z_2^2 + a^2 b X_2 Z_1^2 Z_2.$

  $Z_3 = aX_1X_2(Y_1Z_2 + Y_2Z_1) + a(X_1Y_2 + X_2Y_1)(X_1Z_2 + X_2Z_1) + Y_1Y_2(Y_1Z_2 + Y_2Z_1).$

- *If* $\sigma(P) \neq \sigma(Q)$ *then:*

  $X_3 = 2X_1Y_2Y_1Z_2 + X_1 Y_2^2 Z_1 + 2X_2 Y_1^2 Z_2 + X_2Y_1Y_2Z_1 + 2a X_1^2 X_2 Z_2 + aX_1 X_2^2 Z_1.$

  $Y_3 = 2 Y_1^2 Y_2 Z_2 + Y_1 Y_2^2 Z_1 + 2aX_1X_2Y_1Z_2 + aX_1X_2Y_2Z_1 + 2a X_1^2 Y_2 Z_2 + a X_2^2 Y_1 Z_1.$

  $Z_3 = 2 Y_1^2 Z_2^2 + Y_2^2 Z_1^2 + a X_1^2 Z_2^2 + 2a X_2^2 Z_1^2.$

Lemma 2.5. *σ is a surjective homomorphism of groups .*

3. Cryptographic protocols

In this section we describes some, public-key encryption, key generation, and the encryption and decryption method.

3.1. Exchange of secret key

Diffie-Hellman key exchange is a method of securely exchanging cryptographic keys over public channel [6].

The Diffie-Hellman key exchange is the following protocol:

- Alice and Bob agree on a prime number p, a point P generating subgroup of known order of the elliptic curve $E_{a,b}(A_4)$ and $(a,b) \in A_4 \times A_4$.
- Alice chooses a private random integer $K_A < ord(P)$ and sends $Y_A = K_AP$ to Bob.
- Bob chooses a private random integer $K_B < ord(P)$ *and sends* $Y_B = K_BP$ to Alice.
- Alice computes K_AY_B.
- Bob computes K_BY_A.
- *The common secret key :* $K = K_AK_BP = K_BK_AP$.

3.2. Cryptosystem for encryption and decryption

We will consider two-party key agreement protocols derived from the basic Diffie-Hellman protocol. Alice and Bob chooses elliptic curve over ring A_4 and a secret key*: K in $E_{a,b}^4$.* To encrypt a message M in $E_{a,b}^4$, Alice calculate

$$C = E_{nc}(M) = M + K$$

To decrypt a cipher text C, Bob calculate

$$M = D_{ec}(C) = C - K$$

Cryptosystem Based on A_4

- Space of lights: $L = E^4_{a,b}$.
- Space of quantified: $F = E^4_{a,b}$.
- Space of keys: $E^4_{a,b}$.
- Function of encryption: $\forall K \in E^4_{a,b}$

$$\begin{array}{rcl} E_{nc} : L & \to & F \\ M & \mapsto & M + K \end{array}$$

- Function of decryption: $\forall K \in E^4_{a,b}$

$$\begin{array}{rcl} D_{ec} : F & \to & L \\ C & \mapsto & C - K \end{array}$$

We have: $D_{ec} \circ E_{nc}(M) = M.$

4. Cryptographic application

Let $E^4_{a,b}$an elliptic curve over A_4 and an irreductible polynomial $H(X) = X^3 + 2X^2 + 1$ *in* $F_3[X]$. $H(X)$ have not roots in the F_3 because $H(0)$=1, $H(1)$=1, $H(2)$=2 but there exist an α where $H(\alpha) = 0$ in $\mathrm{F}_{27} = \frac{F_3[X]}{(H(X))}$ so $(1,\alpha,\alpha^2)$ is a basis of the vector space F_{27} over F_3

F_{27}= $\{0, 1, 2, 2\alpha^2 + 2, 2\alpha + 1, \alpha^2 + 2\alpha + 2, 2\alpha^2 + \alpha, 2\alpha^2 + \alpha + 1, 2\alpha^2 + 2\alpha, \alpha^2 + \alpha, \alpha^2 + 1,$
$\alpha + 2, \alpha^2 + \alpha + 1, \alpha, 2\alpha^2 + 2\alpha + 2, 2\alpha^2 + 2\alpha + 1, \alpha^2, 2\alpha^2 + \alpha + 2, 2\alpha, 2\alpha + 2, 2\alpha^2 + 1,$
$\alpha^2 + 2, \alpha^2 + 2\alpha, 2\alpha^2, \alpha + 1, \alpha^2 + \alpha + 2, \alpha^2 + 2\alpha + 1\}$

Let a and b in $\mathrm{F}_{27}[\varepsilon]$ so we have:

$$E_{a,b}(\mathrm{F}_{27}[\varepsilon]) : Y^2 = X^3 + aX^2 + b$$

Let $P \in E_{a,b}(\mathrm{F}_{27}[\varepsilon])$ of order n, we will use the subgroup $< P >$ *of* $E_{a,b}(\mathrm{F}_{27}[\varepsilon])$ to encrypt messages, and we denote $G =< P >$.

4.1. Coding of elements of *G*

We will give a code to each element $Q = mP$ where $m \in \{1,2,...,n\}$ defined as following:

Assume $Q = [x_0 + x_1\varepsilon + x_2\varepsilon^2 + x_3\varepsilon^3 : y_0 + y_1\varepsilon + y_2\varepsilon^2 + y_3\varepsilon^3 : Z]$ *where* x_i, $y_i \in \mathrm{F}_{27}$ for $i \in \{0,1,2,3\}$ and $Z = 0$ or $Z = 1$, we set:

$$x_i = e_{0i} + e_{1i}\alpha + e_{2i}\alpha^2$$

$$y_i = r_{0i} + r_{1i}\alpha + r_{2i}\alpha^2$$

where α is a primitive root of an irreducible polynomial of degree *3* over F_3 and $e_{ij}, r_{ij} \in \mathrm{F}_3$ then we code Q as it follows:

- If $Z = 1$ then:

$$Q = e_{00}e_{10}e_{20}e_{01}e_{11}e_{21}e_{02}e_{12}e_{22}e_{03}e_{13}e_{23}r_{00}r_{10}r_{20}r_{01}r_{11}r_{21}r_{02}r_{12}r_{22}r_{03}r_{13}r_{23}1$$

- If $Z = 0$ then:

$$Q = e_{00}e_{10}e_{20}e_{01}e_{11}e_{21}e_{02}e_{12}e_{22}e_{03}e_{13}e_{23}r_{00}r_{10}r_{20}r_{01}r_{11}r_{21}r_{02}r_{12}r_{22}r_{03}r_{13}r_{23}0$$

4.2. Example for cryptography

Let

$$a = 1 + \alpha + \alpha\varepsilon + \varepsilon^2 + 2\varepsilon^3$$

$$b = 1 + \alpha^2 + \alpha^2\varepsilon + \varepsilon^2 + (1 + \alpha + \alpha^2)\varepsilon^3$$

two elements in $\mathrm{F}_{27}[\varepsilon]$

We consider the point $P = [2+\varepsilon+\varepsilon^2+2\varepsilon^3 : 1+2\alpha+(2\alpha^2+\alpha)\varepsilon+(2\alpha^2+1)\varepsilon^2+(2\alpha^2+1)\varepsilon^3 : 1]$, we have

$G = \langle P \rangle$ is the subgroup of order 63 so, for $Q \in G$, $\exists m \in \{1,2,...,63\}: Q = mP$.
The points of G are:

$P = [2 + \varepsilon + \varepsilon^2 + 2\varepsilon^3 : 1 + 2\alpha + (2\alpha^2 + \alpha)\varepsilon + (2\alpha^2 + 1)\varepsilon^2 + (2\alpha^2 + 1)\varepsilon^3 : 1]$

$2P = [\alpha + 1 + 2\varepsilon + (\alpha + 2)\varepsilon^2 + \alpha^2\varepsilon^3 : 1 + \alpha^2\varepsilon + \alpha^2\varepsilon^2 + (\alpha + 1)\varepsilon^3 : 1]$

$3P = [2 + 2\alpha^2 + (2\alpha^2 + 2)\varepsilon + (\alpha + 1)\varepsilon^2 : 2\alpha + (\alpha + 1)\varepsilon + (2\alpha + 2\alpha^2)\varepsilon^2 + \varepsilon^3 : 1]$

$4P = [2+2\alpha^2 +(\alpha^2 +1)\varepsilon+(2\alpha+2)\varepsilon^2 +(\alpha^2 +\alpha+1)\varepsilon^3 : \alpha+(\alpha+1)\varepsilon+(2\alpha^2 +2)\varepsilon^2 +(2\alpha+1)\varepsilon^3 : 1]$

$5P = [\alpha+1+(2\alpha^2 +2\alpha+2)\varepsilon+(\alpha^2 +2)\varepsilon^2 +(2\alpha^2 +1)\varepsilon^3 : 2+2\varepsilon+(\alpha^2 +\alpha)\varepsilon^2 +(\alpha^2 +2\alpha+2)\varepsilon^3 : 1]$

$6P = [2 + (\alpha^2 + 2\alpha)\varepsilon + (2\alpha^2 + \alpha + 2)\varepsilon^2 : \alpha + 2 + (\alpha^2 + \alpha + 1)\varepsilon + (\alpha^2 + 2\alpha)\varepsilon^2 + \varepsilon^3 : 1]$

$7P = [(\alpha^2 + \alpha)\varepsilon + (2\alpha^2 + \alpha + 1)\varepsilon^2 + 2\alpha^2\varepsilon^3 : 1 : 0]$

$8P = [2 + (2\alpha^2 + \alpha + 2)\varepsilon + \varepsilon^2 + (\alpha^2 + 2\alpha + 2)\varepsilon^3 : 2\alpha + 1 + (2\alpha^2 + 1)\varepsilon + (2\alpha + 2)\varepsilon^2 + (\alpha^2 + 2)\varepsilon^3 : 1]$

$9P = [1 + \alpha + (\alpha^2 + \alpha + 2)\varepsilon + (2\alpha + 1)\varepsilon^3 : 1 + (2\alpha^2 + 2)\varepsilon + (\alpha^2 + 2)\varepsilon^2 + \alpha\varepsilon^3 : 1]$

$10P = [2\alpha^2 + 2 + (2\alpha + 1)\varepsilon^2 + (2\alpha^2 + 2)\varepsilon^3 : 2\alpha + 2\alpha\varepsilon^2 + (2\alpha^2 + \alpha)\varepsilon^3 : 1]$

$11P = [2 + 2\alpha^2 + (2\alpha + 1)\varepsilon^2 + (\alpha + 2)\varepsilon^3 : \alpha + \alpha\varepsilon^2 + (2\alpha^2 + \alpha + 2)\varepsilon^3 : 1]$

$12P = [\alpha + 1 + (\alpha^2 + \alpha + 2)\varepsilon + \alpha\varepsilon^3 : 2 + (\alpha^2 + 1)\varepsilon + (2\alpha^2 + 1)\varepsilon^2 + (\alpha^2 + 2\alpha)\varepsilon^3 : 1]$

$13P = [2 + (2\alpha^2 + \alpha + 2)\varepsilon + \varepsilon^2 + \varepsilon^3 : 2 + \alpha + (\alpha^2 + 2)\varepsilon + (\alpha + 1)\varepsilon^2 + (\alpha^2 + 2\alpha + 2)\varepsilon^3 : 1]$

$14P = [(2\alpha^2 + 2\alpha)\varepsilon + (\alpha^2 + 2\alpha + 2)\varepsilon^2 + (2\alpha^2 + \alpha + 2)\varepsilon^3 : 1 : 0]$

$15P = [2+(\alpha^2+2\alpha)\varepsilon+(2\alpha^2+\alpha+2)\varepsilon^2+(2\alpha+2)\varepsilon^3 : 1+2\alpha+(2\alpha^2+2\alpha+2)\varepsilon+(2\alpha^2+\alpha)\varepsilon^2+(\alpha^2+\alpha+2)\varepsilon^3 :1]$

$16P = [1+\alpha+(2\alpha^2 +2\alpha+2)\varepsilon+(\alpha^2 +2)\varepsilon^2 +(2\alpha+2)\varepsilon^3 : 1+\varepsilon+(2\alpha^2 +2\alpha)\varepsilon^2 +(\alpha^2 +\alpha+2)\varepsilon^3 : 1]$

$17P = [2+2\alpha^2+(\alpha^2+1)\varepsilon+(2\alpha+2)\varepsilon^2+(2\alpha^2+2\alpha)\varepsilon^3 : 2\alpha+(2\alpha+2)\varepsilon+(\alpha^2+1)\varepsilon^2+(2\alpha^2+\alpha+1)\varepsilon^3 : 1]$

$18P = [2 + 2\alpha^2 + (2\alpha^2 + 2)\varepsilon + (\alpha + 1)\varepsilon^2 + (\alpha^2 + 1)\varepsilon^3 : \alpha + (2\alpha + 2)\varepsilon + (\alpha^2 + \alpha)\varepsilon^2 + \varepsilon^3 : 1]$

$19P = [\alpha + 1 + 2\varepsilon + (\alpha + 2)\varepsilon^2 + (\alpha^2 + 2)\varepsilon^3 : 2 + 2\alpha^2\varepsilon + 2\alpha^2\varepsilon^2 + (\alpha^2 + \alpha + 1)\varepsilon^3 : 1]$

$20P = [2 + \varepsilon + \varepsilon^2 + (\alpha^2 + \alpha + 1)\varepsilon^3 : \alpha + 2 + (\alpha^2 + 2\alpha)\varepsilon + (\alpha^2 + 2)\varepsilon^2 + (2\alpha^2 + 2)\varepsilon^3 : 1]$

$21P = [(2\alpha^2 + \alpha + 2)\varepsilon^3 : 1 : 0]$

$22P = [2 + \varepsilon + \varepsilon^2 + (2\alpha^2 + 2\alpha)\varepsilon^3 : 1 + 2\alpha + (2\alpha^2 + \alpha)\varepsilon + (2\alpha^2 + 1)\varepsilon^2 + \varepsilon^3 : 1]$

$23P = [1 + \alpha + 2\varepsilon + (\alpha + 2)\varepsilon^2 + (\alpha + 2)\varepsilon^3 : 1 + \alpha^2\varepsilon + \alpha^2\varepsilon^2 + (\alpha^2 + \alpha + 2)\varepsilon^3 : 1]$

$24P = [2\alpha^2 +2+(2\alpha^2 +2)\varepsilon+(\alpha+1)\varepsilon^2 +(\alpha+2)\varepsilon^3 : 2\alpha+(\alpha+1)\varepsilon+(2\alpha^2 +2\alpha)\varepsilon^2 +(2\alpha^2 +2\alpha)\varepsilon^3 : 1]$

$25P = [2\alpha^2+2+(\alpha^2+1)\varepsilon+(2\alpha+2)\varepsilon^2+(\alpha^2+2)\varepsilon^3 : \alpha+(\alpha+1)\varepsilon+(2\alpha^2+2)\varepsilon^2+(2\alpha^2+\alpha)\varepsilon^3 : 1]$

$26P = [1 + \alpha + (2\alpha^2 + 2\alpha + 2)\varepsilon + (\alpha^2 + 2)\varepsilon^2 + (2\alpha + 2)\varepsilon^3 : 2 + 2\varepsilon + (\alpha^2 + \alpha)\varepsilon^2 + (2\alpha^2 + 2\alpha)\varepsilon^3 : 1]$

$27P = [2+(\alpha^2+2\alpha)\varepsilon+(2\alpha^2+\alpha+2)\varepsilon^2+(\alpha^2+\alpha+2)\varepsilon^3 : \alpha+2+(\alpha^2+\alpha+1)\varepsilon+(\alpha^2+2\alpha)\varepsilon^2+(\alpha^2+1)\varepsilon^3 : 1]$

$28P = [(\alpha^2 + \alpha)\varepsilon + (2\alpha^2 + \alpha + 1)\varepsilon^2 + (\alpha^2 + \alpha + 2)\varepsilon^3 : 1 : 0]$

$29P = [2 + (2\alpha^2 + \alpha + 2)\varepsilon + \varepsilon^2 + \alpha\varepsilon^3 : 2\alpha + 1 + (2\alpha^2 + 1)\varepsilon + (2\alpha + 2)\varepsilon^2 + (2\alpha^2 + 2)\varepsilon^3 : 1]$

$30P = [\alpha + 1 + (\alpha^2 + \alpha + 2)\varepsilon + 2\alpha^2\varepsilon^3 : 1 + (2\alpha^2 + 2)\varepsilon + (\alpha^2 + 2)\varepsilon^2 + (\alpha^2 + \alpha + 1)\varepsilon^3 : 1]$

$31P = [2\alpha^2 + 2 + (2\alpha + 1)\varepsilon^2 + (2\alpha^2 + \alpha + 1)\varepsilon^3 : 2\alpha + 2\alpha\varepsilon^2 + (\alpha^2 + 2)\varepsilon^3 : 1]$

$32P = [2\alpha^2 + 2 + (2\alpha + 1)\varepsilon^2 : \alpha + \alpha\varepsilon^2 + (\alpha^2 + 1)\varepsilon^3 : 1]$

$33P = [\alpha + 1 + (\alpha^2 + \alpha + 2)\varepsilon + (\alpha^2 + 1)\varepsilon^3 : 2 + (\alpha^2 + 1)\varepsilon + (2\alpha^2 + 1)\varepsilon^2 + (2\alpha^2 + 2\alpha + 1)\varepsilon^3 : 1]$

$34P = [2 + (2\alpha^2 + \alpha + 2)\varepsilon + \varepsilon^2 + (\alpha^2 + \alpha)\varepsilon^3 : 2 + \alpha + (\alpha^2 + 2)\varepsilon + (\alpha + 1)\varepsilon^2 + (2\alpha^2 + 2\alpha + 2)\varepsilon^3 : 1]$

$35P = [(2\alpha^2 + 2\alpha)\varepsilon + (\alpha^2 + 2\alpha + 2)\varepsilon^2 + (\alpha^2 + 2\alpha + 1)\varepsilon^3 : 1 : 0]$

$$36P = [2 + (\alpha^2 + 2\alpha)\varepsilon + (2\alpha^2 + \alpha + 2)\varepsilon^2 + (2\alpha^2 + \alpha)\varepsilon^3 : 2\alpha + 1 + (2\alpha^2 + 2\alpha + 2)\varepsilon + (2\alpha^2 + \alpha)\varepsilon^2 + (2\alpha^2 + \alpha + 2)\varepsilon^3 : 1]$$

$37P = [\alpha + 1 + (2\alpha^2 + 2\alpha + 2)\varepsilon + (\alpha^2 + 2)\varepsilon^2 + (2\alpha + 1)\varepsilon^3 : 1 + \varepsilon + (2\alpha^2 + 2\alpha)\varepsilon^2 + (2\alpha^2 + \alpha)\varepsilon^3 : 1]$

$38P = [2\alpha^2 + 2 + (\alpha^2 + 1)\varepsilon + (2\alpha + 2)\varepsilon^2 + (2\alpha^2 + 2)\varepsilon^3 : 2\alpha + (2\alpha + 2)\varepsilon + (\alpha^2 + 1)\varepsilon^2 + \alpha^2\varepsilon^3 : 1]$

$39P = [2\alpha^2+2+(2\alpha^2+2)\varepsilon+(\alpha+1)\varepsilon^2+(\alpha^2+2\alpha+2)\varepsilon^3 : \alpha+(2\alpha+2)\varepsilon+(\alpha^2+\alpha)\varepsilon^2+(2\alpha^2+2\alpha)\varepsilon^3 : 1]$

$40P = [\alpha + 1 + 2\varepsilon + (\alpha + 2)\varepsilon^2 + (2\alpha^2 + 2\alpha)\varepsilon^3 : 2 + 2\alpha^2\varepsilon + 2\alpha^2\varepsilon^2 + (2\alpha^2 + \alpha + 2)\varepsilon^3 : 1]$

$41P = [2 + \varepsilon + \varepsilon^2 + (2\alpha^2 + 2\alpha)\varepsilon^3 : \alpha + 2 + (\alpha^2 + 2\alpha)\varepsilon + (\alpha^2 + 2)\varepsilon^2 + 2\varepsilon^3 : 1]$

$42P = [(\alpha^2 + 2\alpha + 1)\varepsilon^3 : 1 : 0]$

$43P = [2 + \varepsilon + \varepsilon^2 + (\alpha^2 + \alpha + 1)\varepsilon^3 : 1 + 2\alpha + (2\alpha^2 + \alpha)\varepsilon + (2\alpha^2 + 1)\varepsilon^2 + (\alpha^2 + 1)\varepsilon^3 : 1]$

$44P = [\alpha + 1 + 2\varepsilon + (\alpha + 2)\varepsilon^2 + (2\alpha^2 + 2\alpha + 1)\varepsilon^3 : 1 + \alpha^2\varepsilon + \alpha^2\varepsilon^2 + (2\alpha^2 + \alpha)\varepsilon^3 : 1]$

$45P = [2\alpha^2+2+(2\alpha^2+2)\varepsilon+(\alpha+1)\varepsilon^2+(2\alpha+1)\varepsilon^3 : 2\alpha+(\alpha+1)\varepsilon+(2\alpha^2+2\alpha)\varepsilon^2+(\alpha^2+\alpha+2)\varepsilon^3 : 1]$

$46P = [2\alpha^2+2+(\alpha^2+1)\varepsilon+(2\alpha+2)\varepsilon^2+(\alpha^2+2\alpha)\varepsilon^3 : \alpha+(\alpha+1)\varepsilon+(2\alpha^2+2)\varepsilon^2+(\alpha^2+2)\varepsilon^3 : 1]$

$47P = [1 + \alpha + (2\alpha^2 + 2\alpha + 2)\varepsilon + (\alpha^2 + 2)\varepsilon^2 + (\alpha^2 + \alpha)\varepsilon^3 : 2 + 2\varepsilon + (\alpha^2 + \alpha)\varepsilon^2 + (2\alpha + 1)\varepsilon^3 : 1]$

$48P = [2+(\alpha^2+2\alpha)\varepsilon+(2\alpha^2+\alpha+2)\varepsilon^2+(2\alpha^2+2\alpha+1)\varepsilon^3 : \alpha+2+(\alpha^2+\alpha+1)\varepsilon+(\alpha^2+2\alpha)\varepsilon^2+(2\alpha^2+1)\varepsilon^3 :1]$

$49P = [(\alpha^2 + \alpha)\varepsilon + (2\alpha^2 + \alpha + 1)\varepsilon^2 + (2\alpha + 1)\varepsilon^3 : 1 : 0]$

$50P = [2 + (2\alpha^2 + \alpha + 2)\varepsilon + \varepsilon^2 + (2\alpha^2 + 1)\varepsilon^3 : 2\alpha + 1 + (2\alpha^2 + 1)\varepsilon + (2\alpha + 2)\varepsilon^2 + 2\varepsilon^3 : 1]$

$51P = [1 + \alpha + (\alpha^2 + \alpha + 2)\varepsilon + (\alpha^2 + \alpha + 2)\varepsilon^3 : 1 + (2\alpha^2 + 2)\varepsilon + (\alpha^2 + 2)\varepsilon^2 + (2\alpha^2 + \alpha + 2)\varepsilon^3 : 1]$

$52P = [2\alpha^2 + 2 + (2\alpha + 1)\varepsilon^2 + (2\alpha^2 + 2\alpha)\varepsilon^3 : 2\alpha + 2\alpha\varepsilon^2 + (2\alpha + 1)\varepsilon^3 : 1]$

$53P = [2 + 2\alpha^2 + (2\alpha + 1)\varepsilon^2 + (2\alpha + 1)\varepsilon^3 : \alpha + \alpha\varepsilon^2 + 2\alpha\varepsilon^3 : 1]$

$54P = [1 + \alpha + (\alpha^2 + \alpha + 2)\varepsilon + (2\alpha^2 + 2\alpha + 2)\varepsilon^3 : 2 + (\alpha^2 + 1)\varepsilon + (2\alpha^2 + 1)\varepsilon^2 + (2\alpha + 2)\varepsilon^3 : 1]$

$55P = [2 + (2\alpha^2 + \alpha + 2)\varepsilon + \varepsilon^2 + (2\alpha^2 + 2\alpha + 2)\varepsilon^3 : 2 + \alpha + (\alpha^2 + 2)\varepsilon + (\alpha + 1)\varepsilon^2 + (2\alpha + 2)\varepsilon^3 : 1]$

$56P = [(2\alpha^2 + 2\alpha)\varepsilon + (\alpha^2 + 2\alpha + 2)\varepsilon^2 : 1 : 0]$

$57P = [2+(\alpha^2+2\alpha)\varepsilon+(2\alpha^2+\alpha+2)\varepsilon^2+(\alpha^2+1)\varepsilon^3 : 2\alpha+1+(2\alpha^2+2\alpha+2)\varepsilon+(2\alpha^2+\alpha)\varepsilon^2+(\alpha+2)\varepsilon^3 : 1]$

$58P = [1 + \alpha + (2\alpha^2 + 2\alpha + 2)\varepsilon + (\alpha^2 + 2)\varepsilon^2 + (\alpha^2 + \alpha)\varepsilon^3 : 1 + \varepsilon + (2\alpha^2 + 2\alpha)\varepsilon^2 + (\alpha + 1)\varepsilon^3 : 1]$

$59P = [2+2\alpha^2 +(\alpha^2 +1)\varepsilon+(2\alpha+2)\varepsilon^2 +(2\alpha^2 +\alpha+1)\varepsilon^3 : 2\alpha+(2\alpha+2)\varepsilon+(\alpha^2 +1)\varepsilon^2 +(2\alpha+2)\varepsilon^3 : 1]$

$60P = [2+2\alpha^2 +(2\alpha^2 +2)\varepsilon+(\alpha+1)\varepsilon^2 +(\alpha^2 +\alpha)\varepsilon^3 : \alpha+(2\alpha+2)\varepsilon+(\alpha^2 +\alpha)\varepsilon^2 +(\alpha^2 +\alpha+2)\varepsilon^3 : 1]$

$61P = [\alpha + 1 + 2\varepsilon + (\alpha + 2)\varepsilon^2 + (\alpha + 1)\varepsilon^3 : 2 + 2\alpha^2\varepsilon + 2\alpha^2\varepsilon^2 + \alpha\varepsilon^3 : 1]$

$62P = [2 + \varepsilon + \varepsilon^2 + 2\varepsilon^3 : 2 + \alpha + (\alpha^2 + 2\alpha)\varepsilon + (\alpha^2 + 2)\varepsilon^2 + (\alpha^2 + 2)\varepsilon^3 : 1]$

$63P = [0 : 1 : 0]$.

Coding the elements of G, we use english letters for this application.

The coding are as follows:

$P = 2001001002001200121021021 = A$

$2P = 1102002100011000010011101 = B$

$3P = 2022021100000201100221001 = C$

$4P = 2021012201110101102021201 = D$

$5P = 1102222011022002000112211 = E$

$6P = 2000212120002101110211001 = F$

$7P = 0000111120021000000000000 = G$

$8P = 2002121002211201022202011 = H$

$9P = 1102110001201002022010101 = I$

$10P = 2020001202020200000200121 = J$

$11P = 2020001202100100000102121 = K$

$12P = 1102110000102001011020211 = L$

$13P = 2002121001002102011102211 = M$

$14P = 0000222212121000000000000 = N$

$15P = 2000212122201202220122111 = O$

$16P = 1102222012201001000222111 = P$

$17P = 2021012200220202201011121 = Q$

$18P = 2022021101010102200111001 = R$

$19P = 1102002102012000020021111 = S$

$20P = 2001001001112100212012021 = T$

$21P = 0000000002121000000000000 = U$

$22P = 2001001000221200121021001 = V$

$23P = 1102002102101000010012111 = W$

$24P = 2022021102100201100220221 = X$

$25P = 2021012202010101102020121 = Y$

$26P = 1102222012202002000110221 = Z$

$27P = 2000212122112101110211011 = a$

$28P = 0000111122111000000000000 = b$

$29P = 2002121000101201022202021 = c$

$30P = 1102110000021002022011111 = d$

$31P = 2020001201120200000202011 = e$

$32P = 2020001200000100000101011 = f$

$33P = 1102110001012001011021221 = g$

$34P = 2002121000112102011102221 = h$

$35P = 0000222211211000000000000 = i$

$36P = 2000212120121202220122121 = j$

$37P = 1102222011021001000220121 = k$

$38P = 2021012202020202201010011 = l$

$39P = 2022021102210102200110221 = m$

$40P = 1102002100222000020022121 = n$

$41P = 2001001000222100212012001 = o$

$42P = 0000000001211000000000000 = p$

$43P = 2001001001111200121021011 = q$

$44P = 1102002101221000010010121 = r$

$45P = 2022021101200201100222111 = s$

$46P = 2021012200210101102022011 = t$

$47P = 1102222010112002000111201 = u$

$48P = 2000212121222101110211021 = v$

$49P = 0000111121201000000000000 = w$

$50P = 2002121001021201022202001 = x$

$51P = 1102110002111002022012121 = y$

$52P = 2020001200220200000201201 = z$

$53P = 2020001201200100000100201 = -$

$54P = 1102110002222001011022201 = .$

$55P = 2002121002222102011102201 = [$

$56P = 0000222210001000000000000 =]$

$57P = 2000212121011202220122101 = /$

$58P = 1102222010111001000221101 = ($

$59P = 2021012201120202201012201 =)$

$60P = 2022021100110102200112111 = !$

$61P = 1102002101102000020020101 = ?$

$62P = 2001001002002100212012011 = :$

$63P = 0000000000001000000000000 = @$

4.2.1. Encryption and decryption

Exchange of secret key

- Alice chooses a random number integer $K_A = 7 < ord(P) = 63$ and computes $Y_A = 7P$.
- Alice sends Y_A to Bob, but keep K_A.
- Bob chooses a random number $K_B = 5 < ord(P) = 63$ and computes $Y_B = 5P$.
- Bob sends Y_B to Alice, but keep K_B.
- Alice computes $K_A Y_B = 35P$.
- Bob computes $K_B Y_A = 35P$.
- The secret key between Alice and Bob is $K = 35P$.

1. To encrypt the following message "Meet Me In The Garden". We follow these steps:
   a) Remove the white space, the message becomes "MeetMeInTheGarden" .
   b) Transmutation every letter in the message into a point of the subgroup $G =< P >$.
   c) To encrypt every point mP by using the secret key and computes $mP + 35P$.
   d) By the elements of G we get "vCCRvCrL[FCp:PBCL" .
   e) Build their codes according to elements of G.

   - **Its encryption is:**
     20002121212221011102110212022021100000201100221001
     20220211000002011002210012022021101010102200111001
     20002121212221011102110212022021100000201100221001
     11020021012210000100101211102110000102001011020211
     20021210022221020111022012000212120002101110211001
     20220211000002011002210010000000001211000000000000
     20010010020021002120120111102222012201001000222111
     11020021000110000100111012022021100000201100221001
     1102110000102001011020211

2. To decrypt the following message:

2022021100110102200112111202202110000020110022100
2021012200220202201011121200212100102120102220200
2021012201110101102021201202000120112020000020201
2002121001002102011102211110200210201200002002111
1102222012201001000222111202101220022020220101112
2022021100000201100221001

We follow these steps:

a) Gather the bits in the blocks of 25 bits.

b) Replace the blocks code with points of $G =< P >$ and Computes $mP - 35P$.

c) Transmutation the points on symbol letters.

d) We get the resulte after completing the space.

- **Its decryption is:** Yes Of Course

Remark 4.1.

1. With this application, we can encrypt and decrypt any message of any length.

2. The motivation of this work is that decryption is difficult for an interceptor who cannot solve the discret logarithm problem.

5. Conclusion

In this work, we showed the elliptic curve over special ring A_4=$\mathbb{F}_{3^d}[\varepsilon]$, where $\varepsilon^4 = 0$, and we have given an example of cryptography (encryption-decryption) with a secret key. We have calculated this cryptographic application with the help of Maple.

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Effects of water waves.

Study of water wave breaking through equations and experiments

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Abstract

This work describes water wave breaking. We introduce a class of nonlinear water waves. We deal with the space-time measurements study of solution in the absorption regions combined with the estimates of the solutions in the smooth regions.
Our work presents an experimental study on the propagation of nonlinear water waves. In shallow water, the damping of water waves is highly influenced by the bottom friction. Moreover, the contact line and perfect wave absorption play a significant role on the shallow water and water wave breaking. The final part of our work covers the measurement of wave absorption. We conclude with experiments and some important conclusions.

Key words: shallow water waves, wave breaking, wetting, absorption, nonlinearity problem

1. Introduction

The study explains some theories of progressive waves, it describes water wave breaking and their interaction with shorelines and coastal structures. Nonlinear behaviors are compared with theoretical results. During the years, scientists obtain a great deal of the water wave breaking theory. They use by observing natural and experimentally generated objects and effects.
Related to the water wave breaking theory our study takes an important role. We study a sinusoidal wave of wavelength L, height H and period T, propagating on water with undisturbed depth h. The variation of surface elevation with time, from the still water level, is denoted by η. We can explain that in shallow water, the damping of water waves is highly influenced by the bottom friction. Thus, shallow water may be characterized as irregular, short crested and long steep containing a large range of frequencies and directions. We study more specifically the propagation of nonlinear water waves.
The study of surface waves and its interaction with structures has substantial importance, permitting us to solve some important problems like wave amplification in harbors due to resonance, excitation of trapped modes or scattering of waves through an obstacle.
The comparison between theory and experiments represents the opportunity to reveal the experimental constraints (Farhat [2], Porter and Newman [3]). These subjects are of great interest for engineers, mathematicians and physicist, as well as the theoretical assumptions make models different from reality. In addition, the models are tested against experiments concerning periodic wave transformation.

2. Main theory

2.1 Surface wave theory

It is a reasonable starting point for shallow water waves, which are greatly influenced by viscosity, surface tension or turbulence.

Such a theory is based on some basic assumptions that establish the foundation of the following development. Therefore, let us consider an irrotational flow, with velocity u, which can be expressed as: $u = \nabla\varphi$, where ϕ represents a scalar potential. If we consider mass conservation, the scalar potential satisfies Laplace's equation: $\Delta\varphi = 0$.

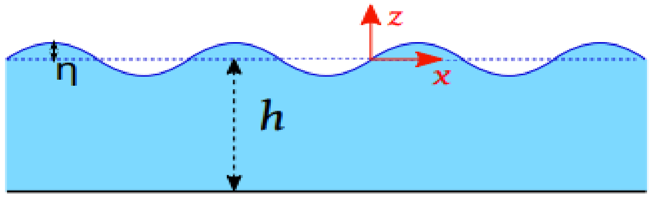

Figure 2.1: Basic variables in the surface waves theory

Let us consider in figure 2.1 a surface S (x, y, t) = z – η (x, y, t) = 0 describing surface of the fluid. Then, considering that the surface S moves with the fluid and always contains the same particles, we can express the zero exchange of particles by means of the material derivative:

$$\frac{DS}{Dt} = \left(\frac{\partial S}{\partial t} + \mathbf{u} \cdot \nabla S\right) = 0 \qquad 2.1$$

2.2 The propagation of the shallow water waves problem

Our interesting problem is the propagation of the shallow water waves in a coastal region with some obstacles inside it. In this context, water wave breaking is an important phenomenon in shallow water waves theory. Two main facts are very important in our study: the phenomena of the contact line and the fact of the wave absorption. These play a significant role on the shallow water studies and water wave breaking theory. At the scale of the water wave, the continental shelf behaves like a submerged step that changes the wave profile, increasing the steepness and wave height. Figure 2.2 shows a deformation of a water wave passing over a continental shelf where an obstacle increases the steepness of wave close to the breaking limit.

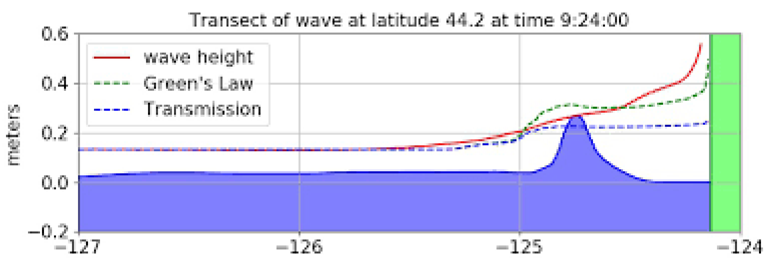

Figure 2.2 The deformation of a water wave passing over a continental shelf.

The experiments presented in this work were carried out using the facilities developed by Fourier Transform Profilometry previously in the laboratory, which permit the measurement of the water surface deformation. This technique developed initially by Takeda and Mutoh [4] for the scan of 3D surfaces, was adapted later by Cobelli [1] for the study of surface waves.

2.3 The mathematical models of surface waves

In this article, we mention the model developed by Massel [5], the mathematical models of surface waves that gives results for nonlinear conditions. We mention the fact that the application was study by Ohyama [6].
Among the earliest models, which solve the linearized equations, one can mention Mei and Black [7], who presents a variation of formulation, and Miles [13], where the scattering matrix with the transmission and reflection coefficients in both directions is calculated. The model proposed by Newman [8], considered an infinite depth before the step, which can be a good approximation for certain configurations.
The equation (2.1) corresponds to a classical wave equation including a loss term with a constant β which transforms the equation in a dispersive problem. In addition, a nonlinear source term in the right hand yields a non-homogeneous partial differential equation. Note that, this source term was especially chosen with a temporal derivative, to get rid of constant terms, as will appear in the following. Thus, we propose a model of nonlinear propagation expressed as a nonlinear wave equation:

$$\frac{\partial^2 \phi}{\partial t^2} + \beta \frac{\partial \phi}{\partial t} - c^2(x) \frac{\partial^2 \phi}{\partial x^2} = \alpha(x) \frac{\partial}{\partial t}\left(\phi \frac{\partial \phi}{\partial x}\right) \qquad 2.2$$

where β is constant and α(x) and c(x) are piecewise functions representing the nonlinearity and the phase velocity, respectively. The nonlinear problem can be solved starting with the perturbation method. Considering a small parameter ε, we develop the wave function **φ** as a power series:

$$\varphi(x,t) = \varepsilon\, \varphi_1(x,t) + \varepsilon^2\, \varphi_2(x,t) + \varepsilon^3\, \varphi_3(x,t) + \ldots \qquad 2.3$$

2.3.1 Solving the nonlinear propagation water wave equation

We continue solving the nonlinear propagation water wave equation (2.2). By separating the terms at this order ε^2, we get a homogeneous equation in the form:

$$\frac{\partial^2 \phi_1}{\partial t^2} + \beta \frac{\partial \phi_1}{\partial t} - c^2(x) \frac{\partial^2 \phi_1}{\partial x^2} = 0 \qquad 2.4$$

We state the problem with the following boundary conditions:
- In x = −∞ there are one incident wave and one reflected wave from the discontinuity.
- In x = +∞ there is one transmitted wave (only outgoing)

Note that the first order problem should satisfy continuity of the function and its partial derivative. Thus, we have $[\varphi_1] = 0$ and $[\frac{\partial}{\partial x}\varphi_1] = 0$ at $x = 0$.

2.3.2 Method of solution for the first order

Taking the proposed solution developed by Massel [5], according to the boundary conditions at the infinity as:

$$\varphi_1 = \alpha\, e^{i(kx-\omega t)} - \alpha\, Re^{-i(kx+\omega t)}, \qquad x < 0 \qquad 2.5$$
$$\varphi_1 = \alpha\, T\, e^{i(kx-\omega t)}, \qquad x > 0 \qquad 2.6$$

Besides, we get two equations from the continuity conditions at x = 0. Thus, at x = 0 we have

$$\varphi_1\big|_{x=0^-} = \varphi_1\big|_{x=0^+}$$
$$[\frac{\partial\varphi 1}{\partial x}]\big|_{x=0^-} = [\frac{\partial\varphi 1}{\partial x}]\big|_{x=0^+}$$
2.7

Now we replace the proposed solution for $x < 0$ in equation (2.4) (we omitted the case $x > 0$ because it is the same procedure as for $x < 0$). This gives us:

$$(-\omega^2 - i\omega\beta + c_1^2k^2)\,\alpha e^{i(kx-\omega t)} - (-\omega^2 - i\omega\beta + c_1^2k^2)\,\alpha\, Re^{-i(kx+\omega t)} = 0. \qquad 2.8$$

Both terms are linearly independent, so the equation is solved, if and only if, each term is equal to zero. This implies that:

$$\omega^2 + i\omega\beta = c_1^2k^2,\ x < 0 \qquad 2.9$$

In the same way, for $x > 0$ we have the same condition:

$$\omega^2 + i\omega\beta = c_2^2k^2,\ x > 0 \qquad 2.10$$

The equations (2.9) and (2.10) are the dispersion relation between ω and k, which will be referred afterwards as the function $k = D(\omega)$. As it will be useful in the next section, the dispersive wavenumbers obtained for the multiplies of the frequencies, will be denoted by:

$$D(n\omega) = \begin{Bmatrix} k_n, x < 0 \\ l_n, x > 0, \end{Bmatrix} \qquad n = 1, 2, \ldots$$
2.11

By replacing equations (2.5) and (2.6) in the continuity conditions (2.7) and changing the generic wavenumber k for the dispersive notation k_1 and l_1 from system (2.11), we get two equations:

$$1 - R = T \qquad 2.12$$
$$k_1 + k_1R = l_1T \qquad 2.13$$

Solving for the unknowns R and T we get:

$$R = \frac{l_1 - k_1}{l_1 + k_1} \qquad 2.14$$

$$T = \frac{2\,k_1}{l_1 + k_1} \qquad 2.15$$

3. Analysis of the reflection and transmission of a water wave over an obstacle

3.1 Statement of the problem

We consider the surface gravity wave problem in two dimensions. The geometry of the system can be described from left to right starting with a wave coming from x = − ∞ and including a submerged step located at x = 0. Thus, the depth h(x) is a piecewise function expressed as:

$$h(x) = \begin{cases} h & ,x < 0 \\ h_s < h & ,x > 0 \end{cases} , \qquad 3.1$$

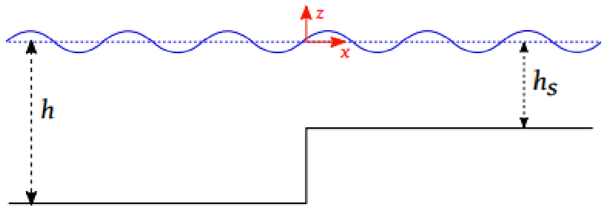

Figure 3.1: Scheme of the propagation problem of a nonlinear wave over a submerged rectangular obstacle. The incident wave comes from the left side

In both sides of the step, the velocity potential function **φ** and the surface displacement **η** should satisfy the Laplace equation (volume conservation):

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)\varphi = 0 \qquad 3.2$$

and the free surface boundary conditions expressed in the following equations:

$$\frac{\partial\varphi}{\partial z} = \frac{\partial \eta}{\partial t} + \frac{\partial\varphi}{\partial x}\frac{\partial \eta}{\partial x} , \qquad z = \eta$$

$$\frac{\partial\varphi}{\partial t} + \left[\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial z}\right)^2\right] g\eta = 0 , \qquad z = \eta \qquad 3.3$$

On the bottom and on the step wall, we impose impervious boundary:

$$\frac{\partial\varphi}{\partial z} = 0 , \qquad z = -h(x) \qquad 3.4$$

$$\frac{\partial \varphi}{\partial x} = 0 \ , \qquad -h(x) < z < -h_s \ , \ \ x = 0$$

3.2 Solution of the problem

The problem can be solved starting with the perturbation method, well explained by Hsu [9], Ohyama [6] and Mei [10]. Considering a small parameter ε (here we consider in dimensional form ε = a), we develop the wave potential **φ** and the surface displacement **η** into a power series expansion:

$$\varphi(x,z,t) = \varepsilon\, \varphi_1(x,z,t) + \varepsilon^2\varphi_2(x,z,t) + \varepsilon^3\varphi_3(x,z,t) + \ldots \qquad 3.5$$

$$\eta(x,t) = \varepsilon\, \eta_1(x,t) + \varepsilon^2\, \eta_2(x,t) + \varepsilon^3\, \eta_3(x,t) + \ldots \qquad 3.6$$

Additionally, the total potential can be expressed in a Taylor series expansion, around z = 0, thus we will use expansion in the form:

$$\varphi(x, \eta, t) = \varphi_{z=0} + \ \eta \frac{\partial \varphi}{\partial z}]|_{\,z=0} + \eta^{\,2} \frac{\partial^2 \varphi}{\partial z^2}]|_{\,z=0} + \ldots \qquad 3.7$$

We replace the power series (3.5) and (3.6), and the Taylor expansion (3.7) in the Laplace equation (3.2) and in the free surface and bottom boundary conditions (equations (3.3) to (3.4)). We truncate the expressions at the order ε^2:
After some calculation we get a nonlinear problem of surface waves over a submerged step expressing as follows.

$$\frac{\partial}{\partial x} \ (\varepsilon\, \varphi_1 + \ \varepsilon^2\, \varphi_2) + O(\,\varepsilon^3) = 0\,, \qquad -h(x) < z < -h_s \ , \ \ x = 0 \qquad 3.8$$

4. Numerical example

4.1 Numerical examples of solution for the first order equation discussed in the section 2.

In this section, we compute a particular example of the previous method of solution for the first order explained at section 2. We chose the following parameters that determine the conditions of the problem:

- $\omega = \pi$
- $\alpha_1 = 0.2$, $x < 0$
- $\alpha_2 = 0.4$, $x > 0$
- $\beta = 0.02$
- $c_1 = 0.1$, $x < 0$
- $c_2 = 0.3$, $x > 0$
- $a = 0.02$

with this condition, the steepness of the incident wave is $k\alpha = 0.08$, which indicates the weak nonlinearity of the incident wave.

Therefore, a consistent weakly nonlinear model can be applied. Figure 4.1 shows the real part of each component of the first order. The obtained coefficients are R = 1/3, T = 2/3. The continuity of φ_1 and $\partial_x\varphi_1$ imposed in equations above is observable in figure 4.1, where the addition of the incident and reflected wave in the left region match in amplitude and slope with the transmitted wave.

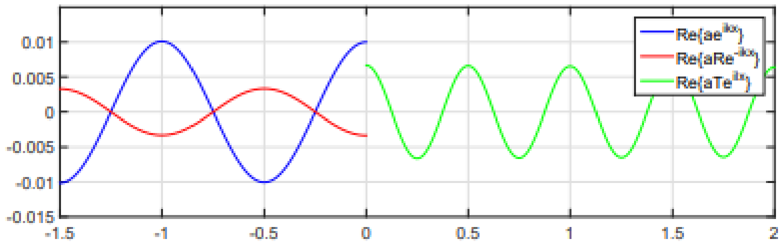

Figure 4.1: Real part of the first order waves. In x < 0 there are an incident and a reflected wave. In x > 0 there is a transmitted wave

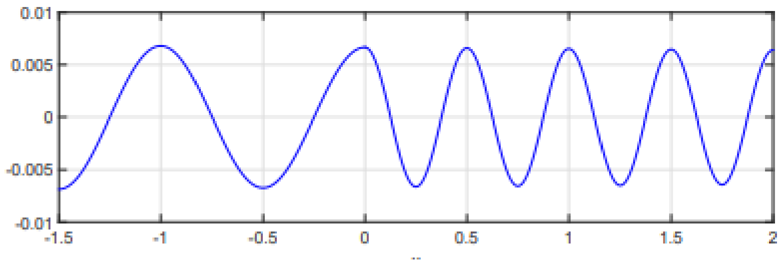

Figure 4.2: Real part of the first order waves φ_1, solution of the linear problem. There is continuity of the wave and its derivative at x = 0.

4.2 Influence of wetting conditions in an absorbing beach

The absorption of waves at the end of the channel is reached by installing a beach with weak slope. The quality of absorption is mainly driven by the beach slope, however, at small scale the contact line at the end of the wet surface has a substantial contribution.
In figure 4.3(*a* and *b*) two configurations are represented, corresponding to different wetting conditions of the beach surface, as is described by Qu´er´e [12]. The first case in figure 4.3-a, shows the change of the contact angle when the beach is made of plastic material, which is quite hydrophobic. In this case, when a small wave arrives, the contact line is pinned because of the high energy required to move the contact line upwards. Consequently, the contact angle increases, and the wave is partially reflected by the negative meniscus. This enhanced reflection by a pinned contact line was recently reported by Michel [11].

The opposite case is illustrated in figure 4.3-b, where the periodic structure represents the mesh located on the beach surface. The mesh spacing is 500 µm, which is sufficiently large to be filled with water and to keep the fluid trapped in the gaps. This configuration changes completely the wetting of the beach surface, creating a contact line that moves above a surface that is permanently wet, without the cost of energy required to wet the surface.

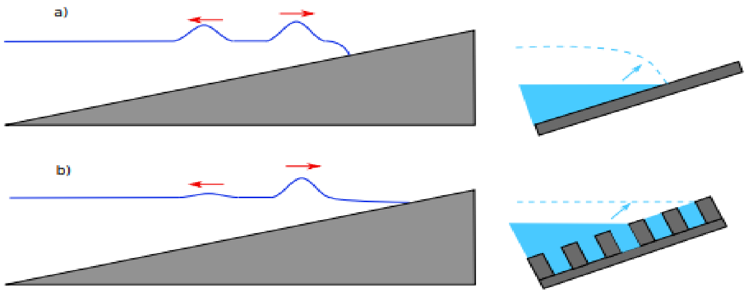

Figure 4.3: The wave reflection in a beach with different wetting properties. (a) The surface of the beach is not wet surface (b) The surface of the beach is covered with a mesh that keep the surface wet.

4.3 Experimental measurement of the beach absorption

Due to the experiments, two configurations were considered, focused on the experiments of nonlinear wave propagation over a step. Therefore, we compared in figures 4.5 and 4.6, the absorption of the beach for $\omega = 4\pi\ s{-1}$ and $\omega = 6\pi\ s{-1}$ respectively. In both configurations, the water depth is h = 2 cm, and the slope of the beach was estimated at 8%.
In figure 4.4 the experimental set-up is shown. We analyzed the influence of the wetting of the beach by adding a nylon mesh with a spacing of 500 µm. This mesh can trap the water and keep the surface permanently wet, changing the movement of the water-solid contact line.

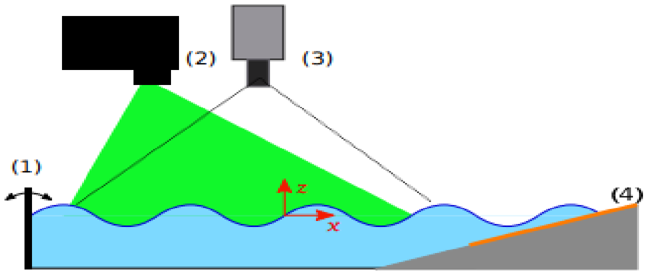

Figure 4.4: Experimental set-up for the measurement of the beach absorption.

In the first configuration, at $\omega = 4\pi\ s^{-1}$, the absorbing beach with and without mesh are compared and results are reported in figure 4.5. The reflection without mesh is 22% whereas the reflection coefficient with mesh is 6%. We observe in this case that the wetting reduces the beach reflection in a substantial fraction.

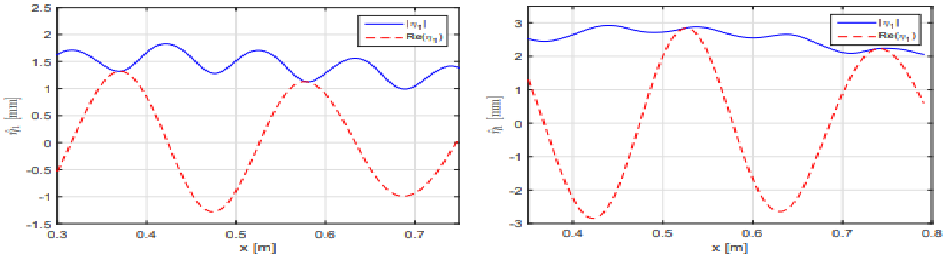

Figure 4.5: The fundamental mode. The depth is h = 2 cm, the beach start at x = 0.6 m. (Left) The surface is made of simple plastic. (Right) The surface is covered with a material that keeps a wet surface

At higher frequency, $\omega = 6\pi s^{-1}$, the difference is smaller but is still significant. In figure 4.6, the reflection from the beach without mesh is 6%, while the beach with mesh has a reflection coefficient of 1.8%. In this case, considering that high frequencies usually have smaller reflection the improvement due to the wetting is less marked. With such a weak reflection, we must consider that the linear fit performed on the fundamental mode is not an accurate tool to observe the difference in reflection.

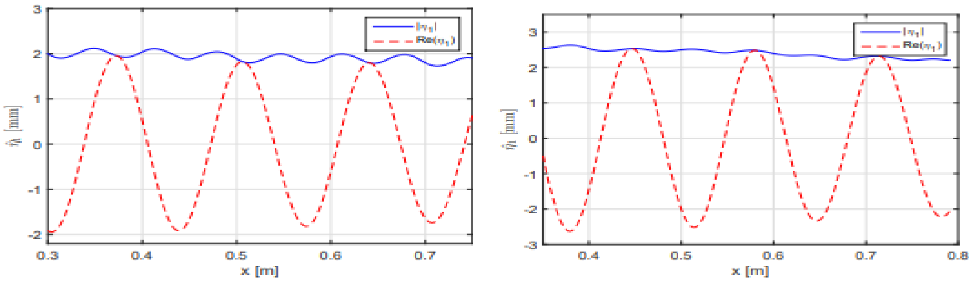

Figure 4.6: The fundamental mode. The depth is h = 2 cm, the beach start at x = 0.6 m. (Left) The surface is made of simple plastic. (Right) The surface is covered with a material that keeps a wet surface.

5. Conclusion

In conclusion, we analyzed the contribute of the wetting in an absorbing beach, with the objective of quantifying the role of the contact angle to the remaining reflection not absorbed by the beach. The wetting properties of the beach surface influence the absorption. Throw the facts, we compared the absorption of the beach for some different values of ω, respectively. In our configurations, the water depth is h = 2 cm, and the slope of the beach was estimated at 6%.
We practiced the water obstacle that modify the wave field, when the field size and the wave-amplitude are sufficiently small. In this context, we saw the influence of lateral walls(obstacles) on the propagation of water waves breaking. The great advantage of our technique is the possibility to measure a two-dimensional field η (x, y, z) with good temporal resolution.
In this article we have observed the importance of the wetting properties. We considered two samples; The first, when the surface of the beach is made of simple plastic. The second, when the surface is covered with a material that keeps a wet surface, generating smaller reflection.
in the experimental set-up of surface waves at small scale. Two problems were considered: the wetting properties of an absorbing beach and the wetting of the lateral walls in a narrow channel. In the first case, the addition of a mesh keeps the beach surface permanently wet, saving the energy consumed by the wetting

process and keeping the contact angle small and constant. In contrast, the beach without mesh has a pinned contact line that changes the contact angle creating a locally vertical surface that can reflect partially the capillary wave.
At low frequencies and sufficiently small amplitude, the reflection can be reduced 6 times by means of this mechanism. In the second problem, the wetting properties of lateral walls in a narrow waveguide were analyzed. When the walls have a simple plastic surface, the meniscus changes its contact angle when a wave is passing, and the contact line can be pinned if the wave-amplitude is small.
In contrast, when lateral mesh is added, the positive meniscus moves vertically with the amplitude of the wave, and the measurement of the relative deformation by Fourier Transform Profilometry gives a field without any modulation in the transverse direction, as it is expected theoretically with Neumann boundary conditions on the lateral walls.
In conclusion, the fact of a wet surface generates the quality of the measurement of the beach absorption.

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Evaluation of family tourism services in the old quartiers of Berat city, Albania, using AHP and VIKOR

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Abstract

The tourism industry has become one of Albania's most important and profitable industries. From a country where tourism almost did not exist, Albania has emerged as an attractive country, where tourism is booming, contributing to the growth of the economy and the benefits to its citizens. The annual increase in the number of visitors and revenues from tourism is about 10-20%, (INSTAT 2003 - 2018). That is, more tourists are coming, more customers are to be satisfied and there is more pressure to better performance and better quality services. The tourism industry as a service industry is always looking for improvements to become more competitive and to respond to the expectations of local and foreign customers. The city of Berat is more than 2500 years old, it is declared a museum city in 1961 and it is included in the 2008 UNESCO World Heritage List. During the last decade, we have seen a very successful activity of family tourism businesses, which are specially established in the three old quartiers of Berat; Kala, Gorica, and Mangalam, which are the main quartiers visited by tourists. These quartiers include dozens of old and characteristic houses, over 100 years old, some of which have been transformed into small guesthouses, hotels, restaurants, bars, and resorts for domestic and foreign tourists with remarkable success. These transformations from old houses to tourist businesses are created by the investments of their owners, immigrants, city businessmen, and foreigners. This study aims to analyze, study, and compare the family tourism businesses, created and managed by family businesses. We will compare the quality of family tourism services in these three quartiers, based on the ratings of foreign and domestic clients who have visited them. The data are obtained from Trip Advisor and local data. The criteria that will evaluate the quality of tourist business services of these three neighborhoods are generally location, cleanliness, rooms, service, price, value, distance, performance, atmosphere, etc. The AHP method will be used to calculate the weight of the criteria and the ranking of the three alternatives will be performed with the VIKOR method, [13].

Keywords: mcda, tourism, vikor, ahp, evaluation, rating.

1. Introduction

In recent years, which has been accompanied by a greater opening of Albania in Europe and the world, there has been a boom in the country's tourism development. This has happened not only on the coast, where the number of foreign tourists has increased greatly, this has also happened in the domestic cities of Albania, which have attractive characteristics for foreign tourists. Cultural tourism is the main focus of

attracting foreign visitors to these cities such as Berat, Kruja, Gjirokastra, which are very old cities. Berat is an old Illyrian city, with an uninterrupted life of more than 2400 years. It is built in its beginnings inside the old Castle, which is very well preserved and has the rare feature of being fully inhabited, with old houses and very well preserved. Inside the Castle, there were once over 40 Byzantine-Orthodox churches, of which only 8 remain, which is a special attraction for foreign and domestic tourists.
Other cities, especially Gjirokastra have similar characteristics to Berat, have an old and well-kept castle, which is not inhabited. The old houses are revered on the hillside on which the castle stands. Kruja is famous for its Castle and the Historical Museum located inside the castle.
In the three old neighborhoods of the city of Berat, important transformations have been carried out, cultural and archeological sites have been restored, which have highlighted the cultural and historical values of the old city. This has led to a very large increase in tourists in our city. In the service of tourists, many old houses have been transformed into three neighborhoods, which have coped with quality, cultured, successful service, domestic tourism, and especially foreign tourism. Many UNESCO projects are under development as well as other projects of the European Union and the Ministry of Culture to improve the infrastructure of the city in the service of citizens and tourists.
The number of foreign tourists entering Albania has been increased rapidly, figure 1. Data are comparing the number of tourists entering Albania from 2001-2019.
Along with the quantitative development of tourism, the increase in the number of domestic and foreign tourists, there is great importance in improving the quality of tourist services. quality service is closely related to the evaluation of guests, tourists who will always make their evaluation. All family tourism businesses of these neighborhoods are included in tourist portals and sites as trip advisor and others.
Data for evaluations of family tourism services are taken from trip advisor and local data, figure 1,2,3.

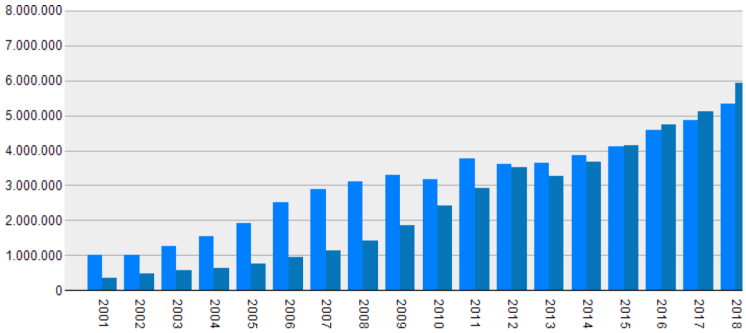

Figure 1. Entry and exit of Albanian and foreigners, 2011- 2019

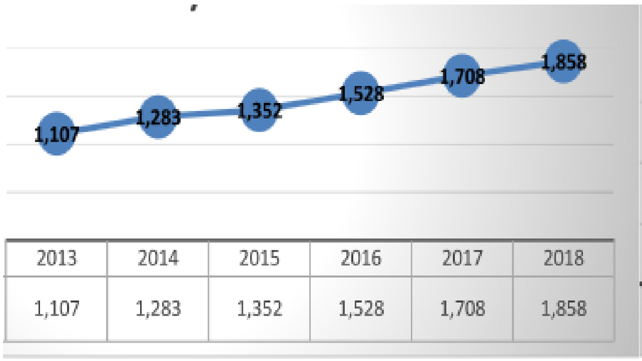

Figure 2. Profits, (million $), 2013-2018.

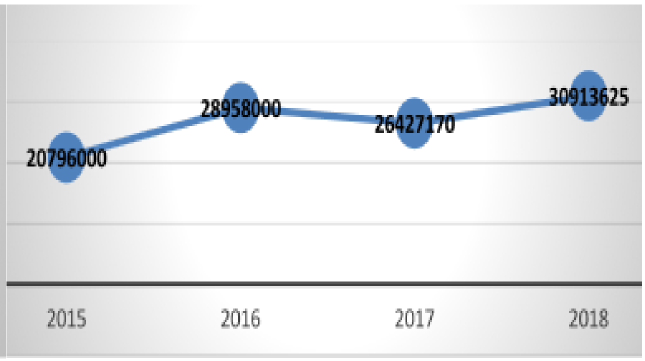

Figure 3. Overnights of tourists, 2015- 2018

2. Main results:

Multicriteria Decision Analyses (MCDA) and AHP.

Multiple-criteria decision analysis (MCDA) or Multiple- criteria decision making (MCDM) is a powerful tool of operations research which is used for selecting a set of alternatives and prioritizing, ranking them under given criteria, (Hwang & Yoon, 1981). MCDA is a valuable tool that can be applied to many complex decisions. It is most applicable to solving problems that are characterized as a choice among alternatives. It helps us focus on what is important, is logical and consistent, and is easy to use. This tool is used by practically everyone every day because there is always some decision, we need to analyze and make among many alternatives which are evaluated under some criteria, conditions, constraints. People make thousands of decisions, but this same process also occurs in the corporate world, government organs, engineering, constructions, oil industry, whenever you have a decision process. A Multiple Criteria Decision Analysis (MCDA) resembles a cost-benefit analysis, but with the notable advantage of not being solely limited to monetary units for its comparisons. When making important decisions, multiple criteria and levels of scale need to be accounted for. Taking decisions based on multiple different criteria with help from the Multiple Criteria Decision Analysis (MCDA) tool can then make things clear, [5]. By structuring complex problems and analyzing multiple sets of criteria, informed, more justifiable decisions can be made.

MCDA steps are:
Define the context,
Identify the options available,
Decide the objectives and select the right criteria that represent the value
Measure out each of the criteria to discern their relative importance.
The use of a Multi-criteria analysis comes with various advantages when compared to a decision-making tool not based on specific criteria:

- It's open and explicit
- The chosen criteria can be adjusted
- Many different actors can be compared with one another
- A Multiple Criteria Decision Analysis (MCDA) grants insight into different judgments of value
- Performance measurements can be left to experts
- Scores and weights can be used as reference

- It's an important means of communication between the different parties involved in the decision-making process

İt should be a Decision Analyses group of specialists, stakeholders, analysts, and persons of interest to analyze and decide the alternatives to work with, the criteria, and the importance or weight of criteria, by comparing them with the best tool available and the best method of comparison.
Selecting the criteria is very crucial because they influence the decision-making process of MCDA/MCDM methods. There are a lot of MCDM available techniques and, as it has been proved by literature, different techniques provide different results for the same problem, [8]. A combination of AHP and TOPSIS, VIKOR, or other methods, ELECTRE, PROMETHEE, etc, has been used in many evaluation problems, figure 4.

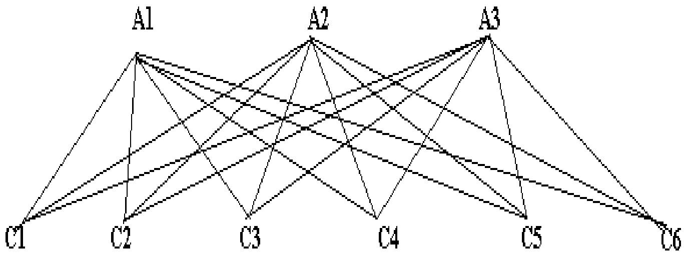

Figure 4. MCDA evaluation process.

The AHP was developed by T. Saaty in 1980 as a mathematical technique for solving complex and unstructured decision problems. The objective of AHP is the comparative evaluation and hierarchical ranking of the alternative solutions, concerning a group of compound criteria, (Saaty, 1996; Doumpos & Zopounidis, 2001), [13]. We will use AHP for the computation of the criteria weights we need for the VIKOR method of ranking alternatives. The mathematical approach used by AHP is based on the pairwise comparison technique through a numerical value scale from 1 to 9, (Saaty, 1996; Vaidya & Kuman, 2006), [15], [16]. Those scalar values express the subjective opinion of the decision-maker about how many times more important is one criterion in comparison to another. Table 1 exhibits the pairwise comparison scalar values and their linguistic equivalents (Saaty, 2008), [16].

Table 1. Pairwise comparison values of criteria a,b.

Value	Reciprocal value	Criterion a in comparison to b
1	1	Equally important
2	1/2	Slightly more important
3	1/3	Moderately more important
4	1/4	Moderately more important
5	1/5	Strongly more important
6	1/6	Strongly plus more important
7	1/7	Very strongly more important
8	1/8	Very, very strongly more important
9	1/9	Extremely more important

AHP steps are:
Step1. The identification of evaluation criteria which are considered to be the most important performance measures for the performance evaluation problem,
Step 2. The construction of a hierarchy of the evaluation criteria,
Step 3. Calculation of weights of chosen criteria using AHP method of MCDA,
Step 4. Calculate values for alternatives and get the final ranking results.

The process; Matrix of comparison values:
Let's assuming n attributes, then the pairwise comparison of attribute i with attribute j yields a square matrix $A_{n\,\mathrm{x}\,n}$ where the term $a_{i,j}$ denotes the comparative importance of attributes i concerning attribute j. In the comparison matrice $A_{n\,\mathrm{x}\,n}$ we have; $a_{i,j} = 1, for\ i = j,$ and$a_{i,j} = \frac{1}{a_{j,i}}$.

$$A_{nxn} = \overbrace{\begin{pmatrix} 1 \\ 2 \\ \dots \\ \dots \\ n \end{pmatrix}}^{Attributes} \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n-1} & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n-1} & a_{2n} \\ \dots & \dots. & \dots. & \dots\dots & \dots.. \\ \dots. & \dots. & \dots. & \dots\dots & \dots\dots \\ a_{n1} & a_n 2 & \dots & a_{nn-1} & a_{nn} \end{bmatrix}$$

Step 3. Construct normalized decision matrix,

$$c_{i,j} = \frac{a_{i,j}}{\sum_{j=1}^{n} a_{i,j}},\ i = \overline{1,n};\ j = \overline{1,n}. \qquad (1)$$

Step 4. Construct the weighted normalized matrix,

$$w_i = \sum_{j=1}^{n} \frac{c_{i,j}}{n},\ i = \overline{1,n} \qquad (2)$$

$$W = \begin{bmatrix} w_1 \\ w_2 \\ \dots \\ \dots \\ w_n \end{bmatrix} \qquad (3)$$

Step 5. Calculate eigenvectors and Row matrix

$$E = \frac{N^{th} rootvalue}{\sum N^{th} rootvalue} \qquad (4)$$

$$Rowmatrix = \sum_{j=1}^{n} a_{ij} * e_{j1} \qquad (5)$$

Step 6. Calculate the maximum Eigenvalue

$$\lambda_{max} = \frac{Rowmatrix}{E} \qquad (6)$$

Step 7. Calculate the consistency index and consistency ratio,

$$CI = (\lambda_{max} - n)/(n-1)\ \ and\ \ CR = \frac{CI}{RI} \qquad (7)$$

Where n and RI denote the order of matrix and randomly generated Consistency Index respectively.

Vikor method

In some cases, the heterogeneity of criteria might make it impossible to produce a solution that satisfies them all simultaneously. To confront such situations first Yu (1973) and almost a decade later Zeleny (1982) introduced the notion of 'compromise solution' which is defined as the feasible solution "closest" to the "ideal" one and compromise means an established agreement by mutual concessions (Opricovic & Tzeng, 2004). The VIKOR method is an implementation of the notion of a compromise solution. The method (Visekriterijumska Optimizacija I Kompromisno Resenje in Serbian) was developed by S. Opricovic in 1990 as a multicriteria optimization method to solve complex decision problems, which have several possible solutions. VIKOR aims to rank the set of alternatives concerning a set of conflicting evaluation criteria and suggest the solution that is "closest" to the "ideal" solution. (Yazdani & Graeml, 2014), [14].

3. Case study

We have taken under consideration three main quartiers, (three Alternatives) of Berat city, Albania, concerning their tourism services. The purpose is to rank the three of them by evaluating the service quality estimated by customers- tourists. The data are provided by TripAdvisor and local data.

1. Quartier of Kala- Castle. All the hotels, restaurants are inside old houses of the castle. The castle is more than 2400 years old and very well preserved. There are more than 60 familiar tourism businesses inside the castle.
2. Quartier of Gorica. It is positioned on the left side of the river, which the oldest quartier of Berat. The old houses are very well preserved, most of them are restored and changed to rooms to let, restaurants, family small hotels, there are more than 100 familiar tourism businesses.
3. Quartier of Mangalem. It's very picturesque and beautiful. Most of the old houses are transformed into restaurants, rooms, hotels, bars and other tourism services mostly frequented and visited by foreign tourists. More than 150 familiar tourism businesses are working there.

The process of AHP calculating criteria weights is done by using AHP online. The matrix of comparison scalars was built by comparing all the criteria using the method of pairwise comparison. The test of consistency is 4.8 % which is a very satisfactory, (< 10% is very good). The criteria and their weights resulted from using AHP are in tables 2,3. The alternative evaluation was made by calculating the evaluation of customers and criteria weights, the results are in figure 5, and the SQR ranking, figure 6.

Table 2. AHP criteria for Alternatives

Nr	Criteria	Symbols
1	Location	C1
2	Cleanness	C2
3	Price	C3
4	Atmosphere	C4
5	Service	C5
6	Value	C6

Number of comparisons = 15
Consistency Ratio CR = 4.8%
Principal eigenvalue = 6.300
Eigenvector solution: 4 iterations, delta = 5.2E-8

Table 3. Criteria weights, AHP.

Criteria	Weights
C1	12%
C2	18%
C3	15%
C4	16%
C5	22%
C6	17%

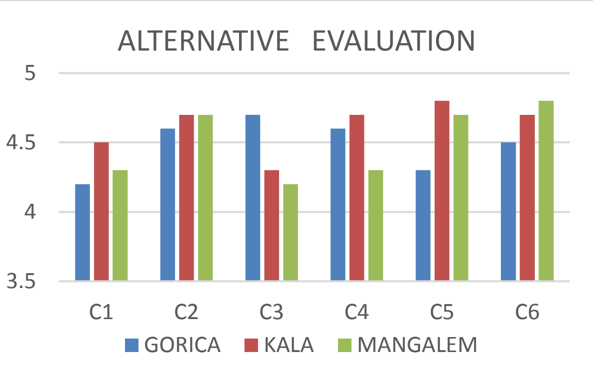

Figure 5. Alternative Evalution.

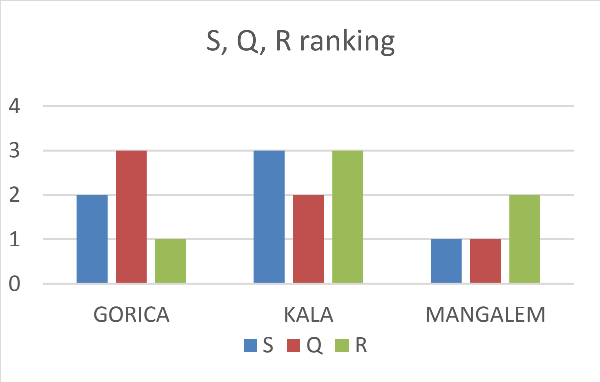

Figure 6. Vikor S, Q, R ranking.

4. Conclusions:

The purpose of this study was to evaluate the familiar tourism business and services that are working in three old quartiers of Berat. After many years of working, having an important contribution to the local economy, what they need is a process of evalution of their quality of services, to make improvements, to have a better understanding of their services, and ensuring a better future for their businesses. The guest's evaluation and customer satisfaction are of great importance for any business in the tourism industry, especially for small family businesses. Such businesses are having a great estimation from foreign tourists, as we have seen from tourist sites and the data provided by local offices. Still, there are many rooms for improvements, not only from the business but, from local and central governments, regarding the performance of the city, transportations, noise, organizing cultural events, activities such visiting cultural and historical sites, hiking and excursions, etc.

The best performance in almost all the criteria is the quartier of Mangalam.

They need to improve in some aspects such as price, value, etc.

The quartier of Gorica needs improvement in the quality of hotels, food, atmosphere.

The quartier of Kala needs to offer more to cover the problem of location. They offer the best atmosphere.

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Evaluation on the parameters of PSO algorithm using analytic tools

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Abstract

The PSO algorithm is used nowadays in nonlinear optimization problems. It is an evolutionary heuristic algorithm which aims to a global solution. Numerically, it is one of the most successful method, but there is still to work on the theorical part of the algorithm. In this work we deal with the convergence of the algorithm based on the matrices and differential equations. We perturbate the dynamical system that models the particle movement and we evaluate the parameters of perturbation based on theorical restrictions.

Keywords: PSO algorithm, parameters, convergence, differential equation

Introduction. The PSO algorithm we consider has n particles. Each one of the particles has an initial position in a dimensional space and is a potential candidate for the solution needed. How does a particle move? To answer we take in consideration three rules (1) the particle should keep its inertia (2) change the condition according to its most optimist position (individual rule) (3) change the condition according to the swarm's most optimist position (social rule). It is clear that the position of each particle is governed by rule 2 and 3, so by the most optimist position during its movement (individual experience) and the position of the most optimist particle in its neighborhood (near experience). As in the bird flock, the position of the particles nearby the one with the most optimist position coincides and this is called whole PSO. When the whole particle swarm is surrounding the particle, the most optimist position of the surrounding is equal to the one of the whole most optimist particle. A particle is identified by its current speed and its position, the most optimal position of each particle and the most optimal position of the neighborhood. We focus our work in the study of the behavior of one particle, of one individual. The speed and the velocity of a particle is given by the formula

$$(1) v_{id}^{k+1} = v_{id}^{k} + c_1 r_1^k \left(pbest_{id}^{k} - x_{id}^{k}\right) + c_2 r_2^k \left(gbest_{d}^{k} - x_{id}^{k}\right)$$

$$(2) \mathrm{x}_{id}^{k+1} = \mathrm{x}_{id}^{k} + v_{id}^{k+1}$$

v_{id}^{k} represents the speed of the particle i in the k time and x_{id}^{k} represents the d-dimension quantity of its position or more simply the current position of the particle $pbest_{id}^{k}$ represents the d dimension quantity of the particle *i* at its most optimal position at its *k* times and $gbest_{d}^{k}$ is the d-dimension quantity of the swarm at its most optimal position. The speed is always between two boundaries, $-v_{dmax}$ and v_{dmax} , in order to avoid the wandering of the point away from the operational searching space. c_1, c_2 are non-negative constants, called cognitive learning rate or the acceleration coefficients and play an important role in the algorithm process. They represent the particle stochastic acceleration weight toward the personal best (pbest) and the global best (gbest). We should avoid small accelerate constant since the particle can move away from the goal area and large accelerate constant values since the particle can move very quickly to the goal area and leave it totally. How do we determine these acceleration coefficients? If we take $c_1 = 0$ then the particle behaves social only and the particles cooperate with each other leaving apart the pbest and focusing only in the gbest tending into a local optima. If $c_2 = 0$ the particles have no interaction so the chances to get the optimal solution is very small. In previous work of Kennedy and Eberhart, also Clerc later was stated that high values of the cognitive component c_1 compared to the social component c_2 will bring to extra wandering of the search area (space). If there is a high value of the social component compared to the cognitive one then the particles may hurry prematurely to the local optimum. In many papers the values of the acceleration coefficients are equal to 2.05, other researchers recommend to not take the same value of the coefficients. All of this discussion is based more heuristically because still there is no mathematical assurance on the convergence of PSO. In this paper we try to give some values for the acceleration coefficients based on analytic tools. Our research is based on the well-known paper of Clerc and Kennedy.

2. Modelling the problem in a dynamical system The relations (1) and (2) can be modelled as a dynamical system proposed by Clerc and Kennedy. First, we denote $\varphi_1 = c_1 r_1^k,\ \varphi_2 = c_2 r_2^k,$ and $\varphi = \varphi_1 + \varphi_2$. Then we redefine

$$pbest_{id} \coloneqq \frac{\varphi_1 pbest_{id} + \varphi_2 gbest_d}{\varphi_1 + \varphi_2}$$

The dynamical system obtained by (1) and (2)

$$(3) \quad \begin{cases} v_{t+1} = w v_t + \varphi y_t \\ y_{t+1} = -w v_t + (1 - \varphi) y_t \end{cases}$$

where $y_t = p - x_t$, where p is the best position found so far. From three in an iterating step we have

$$\begin{cases} v_{t+2} = wv_{t+1} + \varphi y_{t+1} \\ y_{t+1} = -wv_t + (1-\varphi)y_t \end{cases}$$

substituting y_{t+1} at the first equation and adding the first identity of (3) we have

$$v_{t+2} + (\varphi - 1 - w)v_{t+1} + \omega v_t = 0$$

which is a second order difference equation. Using the Lagrange interpolation, we have a continuous solution in order to study the convergence so the respective differential equation is

$$v_{tt} + \ln(e_1 e_2)\, v_t + \ln(e_1)\ln(e_2) = 0$$

where $e_1,\ e_2$ are the roots of the characteristic equation

$$\lambda^2 + (\varphi - 1 - \omega)\lambda + \omega = 0,$$

respectively

$$(*)\ \ e_1 = \frac{\omega+1-\varphi+\sqrt{(\omega+1-\varphi)^2-4\omega}}{2},\quad e_2 = \frac{\omega+1-\varphi-\sqrt{(\omega+1-\varphi)^2-4\omega}}{2}$$

and the solution of the second order differential equation is $v(t) = l_1 {e_1}^t + l_2 {e_2}^t$.

From

$$v_{t+1} = wv_t + \varphi y_t$$

we can derive

$$y(t) = \frac{l_1 {e_1}^t(e_1-\omega) + l_2 {e_2}^t(e_2-\omega)}{\varphi}.$$

To estimate the parameters l_1, l_2 for $t = 0$

$$\begin{cases} v(0) = l_1 + l_2 \\ y(0) = \dfrac{l_1(e_1-\omega) + l_2(e_2-\omega)}{\varphi} \end{cases}$$

and

$$\begin{cases} v(0) = l_1 + l_2 \\ y(0) = \dfrac{l_1(e_1-\omega) + l_2(e_2-\omega)}{\varphi} \end{cases}$$

3. The perturbated dynamical system. We continue our study of convergence taking the inertia weight $\omega = 1$, and perturbate the dynamical system introducing the parameters $\alpha, \beta. \gamma, \delta, \eta$ as follows

$$(4)\begin{cases} v_{t+1} = \alpha v_t + \beta\varphi y_t \\ y_{t+1} = -\gamma v_t + (\delta - \eta\varphi) y_t \end{cases}$$

where $\varphi \in R^{+*}, \forall t \in N, (y_t, v_t) \in R^2$

In this case we operate with matrix theory as the eigenvalues of the coefficients of a linear dynamical system determine the solution of that system as well.

$$A = \begin{bmatrix} \alpha & \beta\varphi \\ -\gamma & \delta - \eta\varphi \end{bmatrix}$$

If we denote the eigenvalues of the matrix $\bar{e}_1 \ \bar{e}_2$ then the solution of (4) is

$$v(t) = l_1 {\bar{e}_1}^t + l_2 {\bar{e}_2}^t, \quad y(t) = \frac{l_1 {\bar{e}_1}^t (\bar{e}_1 - \alpha) + l_2 {\bar{e}_2}^t (\bar{e}_2 - \alpha)}{\beta\varphi}.$$

As in the previous section we evaluate the parameters l_1, l_2 for the initial time $t = 0$

$$\begin{cases} l_1 = \dfrac{-\beta\varphi y(0) - (\alpha - \bar{e}_2) v(0)}{\bar{e}_2 - \bar{e}_1} \\ l_2 = \dfrac{\beta\varphi y(0) + (\alpha - \bar{e}_1) v(0)}{\bar{e}_2 - \bar{e}_1} \end{cases}$$

And also from (*) for $\omega = 1$

$$e_1 = \frac{2 - \varphi + \sqrt{(2-\varphi)^2 - 4}}{2}, \quad e_2 = \frac{2 - \varphi - \sqrt{(2-\varphi)^2 - 4}}{2}$$

$$(**) \quad e_1 = 1 - \frac{\varphi}{2} + \frac{\sqrt{\varphi^2 - 4\varphi}}{2}, \qquad e_2 = 1 - \frac{\varphi}{2} - \frac{\sqrt{\varphi^2 - 4\varphi}}{2}$$

We have the constriction coefficients k_1, k_2 defined by

$$\begin{cases} \bar{e}_1 = k_1 e_1 \\ \bar{e}_2 = k_2 e_2 \end{cases}$$

From direct computation,

$$\begin{cases} k_1 = \dfrac{\alpha + \delta - \eta\varphi + \sqrt{(\eta\varphi)^2 + 2\varphi(\alpha\eta - \delta\eta - 2\beta\gamma) + (\alpha - \delta)^2}}{2 - \varphi + \sqrt{\varphi^2 - 4\varphi}} \\ k_2 = \dfrac{\alpha + \delta - \eta\varphi - \sqrt{(\eta\varphi)^2 + 2\varphi(\alpha\eta - \delta\eta - 2\beta\gamma) + (\alpha - \delta)^2}}{2 - \varphi - \sqrt{\varphi^2 - 4\varphi}} \end{cases}$$

4. Choosing the coefficients. For $k_1$and k_2 to be real for a given value of φ subtracting and adding the two above identities we obtain

$$2(\alpha+\delta-\eta\varphi) = (k_1 + k_2)(2-\varphi) + (k_1 - k_2)\sqrt{\varphi^2-4\varphi}$$

$$2\sqrt{(\eta\varphi)^2+2\varphi(\alpha\eta-\delta\eta-2\beta\gamma)+(\alpha-\delta)^2} = (k_1 + k_2)\sqrt{\varphi^2-4\varphi}(k_1-k_2)(2-\varphi)$$

or posing

- $A = sgn(\varphi^2-4\varphi)$
- $B = |\varphi^2-4\varphi|$
- $C = (\eta\varphi)^2+2\varphi(\alpha\eta-\delta\eta-2\beta\gamma)+(\alpha-\delta)^2$

we have

$$\sqrt{C}(1-sgn(C)(2-\varphi)-(\alpha+\delta-\eta\varphi)\sqrt{B}\,(1-A)=0$$

$$\sqrt{(|C|\,)}\,\sqrt{B}\,sgn(C)(1+A)=0$$

It is seen the presence of φ so the solution depends on it. As it is obvious there are many ways for the two equations to be zero. A possible choice is

$$\begin{cases} C>0 \\ A=-1,(\varphi<4) \\ \alpha+\delta-\eta\varphi=0 \end{cases}$$

First set of choice $\alpha=\delta,\ \beta\gamma=\eta^2$.

Then

$$\alpha = \frac{1}{4}(2(k_1+k_2)+(k_1-k_2)(\sqrt{\varphi^2-4\varphi}+\varphi\frac{2-\varphi}{\sqrt{\varphi^2-4\varphi}}))$$

$$\eta = \frac{1}{2}(k_1+k_2+\frac{2-\varphi}{\sqrt{\varphi^2-4\varphi}}(k_1-k_2))$$

Since we want real coefficients and from $A=-1$ then we choose

$$k_1 = k_2 = \mathrm{k} \in R$$

And in order for the five coefficients to satisfy the conditions $\alpha=\delta,\ \beta\gamma=\eta^2$, we choose

$\alpha=\beta=\gamma=\delta=\eta=k$

In this case the parameters do not depend on φ.

Second set of choice (depend on φ) $\alpha=\beta,\ \gamma=\delta=\eta,$ after the calculations

$$\alpha = \frac{(k_1 + k_2)(2-\varphi) + (k_1 - k_2)\sqrt{\varphi^2 - 4\varphi}}{2} + (\varphi - 1)$$

$$\gamma = \frac{1}{4(\varphi-1)} \frac{(k_1 + k_2)(\varphi-2) - (k_1 - k_2)\sqrt{\varphi^2 - 4\varphi}}{\sqrt{k_1^2(\varphi^2 - 4\varphi + 2 - \varphi\sqrt{\varphi^2 - 4\varphi} + k_2^2(\varphi^2 - 4\varphi + 2 + \varphi\sqrt{\varphi^2 - 4\varphi} + 8k_1 k_2(2\varphi - 1)}}$$

To simplify the result, we take $k_1 = k_2 = \mathrm{k} \in R$ and

$$\alpha = (2-\varphi)k + \varphi - 1$$

$$\gamma = k \ \ or \ \ \gamma = \frac{k}{\varphi - 1}$$

For the convergence of the first choice of parameters, with the condition $k_1 = k_2 = \mathrm{k}$,

$$(5)\begin{cases} |\bar{e}_1| = k|e_1| \\ |\bar{e}_2| = k|e_2| \end{cases}$$

And for the second choice of coefficients and for the parameter α to be positive, $\varphi \le 2$

$$(6)\begin{cases} |\bar{e}_1| = \left| k\left(1 - \frac{\varphi}{2}\right) + \frac{\sqrt{k^2(2-\varphi)^2 + 4k(\varphi-2) + 4(\varphi-1)}}{2} \right| \le k|e_1| = k \\ |\bar{e}_2| = \left| k\left(1 - \frac{\varphi}{2}\right) - \frac{\sqrt{k^2(2-\varphi)^2 + 4k(\varphi-2) + 4(\varphi-1)}}{2} \right| \le k|e_2| = k \end{cases}$$

(5) and (6) guarantee that the system is stable.

Conclusion. Since the coefficients k_1, k_2 which means also φ depend on the parameters α, β, γ, δ, η we should study if there are other restrictions needed to ensure the continuity of the system and to advance in the research of the converge of PSO.

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Exact and Numerical Solution for Pseudo- Parabolic Differential Equation Defined by Atangana-Baleanu Fractional Derivative

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Abstract

In this paper, the finite difference scheme method is applied to Pseudo-parabolic differential equation. Stability estimates are given for this method. The exact solution is obtained for this equation. Error analysis is calculated by comparing the exact and approximate solution for this problem.

1. Introduction

Fractional differential equations are applied many phenomena in various fields of engeneering and scientific displines such as control theory, physics,chemistry, biology, economic, mechanics and electromagnetic. In theoretical physics, it is usually very important to seek and construct explicit solutions of linear and nonlinear partial differential equations (PDEs). Therefore, the solution helps the researchers to understand the physical phenomena. The parabolic equation occurs in several areas of applied mathematics, such as heat conduction, the phenomenea of turbulence and flow through a shock wave traveling in a viscous fluid such as the modeling of dynamics. In recent years, several studies for the linear and nonlinear initial value of fractionnals problems arise in the literature.

The pseudo-parabolic equation models a variety of physical processes. The one dimensional pseudo-parabolic equation was derived in [3]. In general, some of the nonlinear models of real-life problems are still very difficult to solve either theoretically or numerically. Recently, many authors have proposed analytical solution to one dimensional system of parabolic equation (Burgers equation) eg.[4, 5] using a domain decomposition method. In [2] the author used Laplace transform and homotopy perturbation method to obtain approximate solutions of homogeneous and inhomogeneous coupleg Berger's equation. The author in [1] used a modified double Laplace decomposition method to solve coupled pseudo-parabolic equations. In [6], Akgül and Modanli studied Crank–Nicholson difference method and reproducing kernel function for third order fractional differential equations in the sense of Atangana–Baleanu Caputo derivative.

In this work, we shall use difference scheme method for pseudo-parabolic fractional partial differential equation. For this, following farctional order initial-boundary value problem

$$\begin{cases} {}^{ABC}_{\ \ 0}D_t^{\alpha}(u_t(t,x)) = \lambda u_{txx}(t,x) + u_{xx}(t,x) + f(t,x) \ = 0, x\epsilon[0,X], t\epsilon[0,T] \\ u(x,0) = g(x),\ x\epsilon[0,X], \\ u(t,0) = u(t,L) = 0,\ \ t\epsilon[0,T], \end{cases} \tag{1.1}$$

is investigated. Where, $g(x)$, $f(t,x)$ are known functions, $u(t,x)$ is unknown function and $\lambda > 0$. ${}^{ABC}_{\ \ 0}D_t^{\alpha}$ is Atangana-Baleanu Caputo(ABC) derivative that defined by the following definition.

Definition 1.1. Suppose that $f \epsilon H^1(a,b), b > a$ and $\sigma \epsilon [0,1]$. Then ABC fractional derivative is

$$ {}^{ABC}_{a}D_t^{\alpha}(f(t)) = \frac{B(\alpha)}{1-\alpha}\int_a^t f'(x) E_\alpha\left[-\alpha\frac{(t-x)^\alpha}{1-\alpha}\right]dx, \tag{1.2}$$

Here,

$$E_\alpha\left[-\alpha\frac{(t-x)^\alpha}{1-\alpha}\right] = \sum_{k=0}^{\infty}\frac{(-\alpha\frac{(t-x)^\alpha}{1-\alpha})^k}{\Gamma(\alpha k+1)}, \tag{1.3}$$

is Mittag-Leffler function, and $B(\alpha) = 1-\alpha+\frac{\alpha}{\Gamma(\alpha)}$.

Using Laplace transform method for the equation (1.2), it is written as

$$\mathcal{L}[{}^{ABC}_{a}D_t^{\alpha}(u(t,x)] = \frac{B(\alpha)}{1-\alpha}\frac{s^a u(s,x)-s^{a-1}u(0,x)}{s^a+\frac{\alpha}{1-\alpha}} \tag{1.4}$$

Next section, we will construct difference scheme for the equation (1.1). Then we will prove stability estimates for this difference scheme.

2. Finite Difference Scheme Method

We introduce grids with uniform steps

$$W^h = \{x_n : x_n = nh, n=0,1,\ldots, M\},\ h = \frac{X}{M}$$

$$W^\tau = \{t_k : t_k = k\tau, k = 0,1,\ldots,N\}, \tau = \frac{T}{N}.$$

From [7], we can write first order difference scheme for the equation (1.2) as

$$ {}^{ABC}_{a}D_t^{\alpha}(u(t_k,x_n) = \frac{1}{\Gamma(\alpha)}\sum_{j=0}^{k}(\frac{u_n^{k+1}-u_n^{k)}}{\tau}d_{j,k}), \tag{2.1}$$

where

$$d_{j,k} = (t_j - t_{k+1})^{1-\alpha} - (t_j - t_k)^{1-\alpha}.$$

Using Taylor expansion, the following formula

$$u_{xx}(t_k,x_n) \cong \frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{h^2} \tag{2.2}$$

is writen. From [8], $u_{txx}(t_k,x_n)$ can be found as:

$$u_{txx}(t_k,x_n) \cong \frac{1}{\tau}\left(\frac{u_{n+1}^k - 2u_n^{k+1} + u_{n-1}^k}{h^2} - \frac{u_{n+1}^{k-1} - 2u_n^k + u_{n-1}^{k-1}}{h^2}\right). \tag{2.3}$$

Using the formulas (2.1), (2.2) and (2.3), we obtain finite difference scheme for the equation (1.1) following as:

$$\begin{cases} \frac{1}{\Gamma(\alpha)}\sum_{j=0}^{k}\left(\frac{u_n^{k+1}-u_n^{k})}{\tau}d_{j,k}\right) = \frac{1}{\tau}\left(\frac{u_{n+1}^{k}-2u_n^{k+1}+u_{n-1}^{k}}{h^2} - \frac{u_{n+1}^{k-1}-2u_n^{k}+u_{n-1}^{k-1}}{h^2}\right) \\ +\frac{u_{n+1}^{k}-2u_n^{k}+u_{n-1}^{k}}{h^2} + f_n^k, 1 \le k \le N-1, \ \ 1 \le n \le M-1 \\ , \\ u_n^0 = g(x_n), \ \ 0 \le n \le M, \ \ u_0^k = \ u_M^k = 0, \ \ 0 \le k \le N, \ \ 0 < \alpha < 1. \end{cases} \tag{2.4}$$

The stability estimates can be proved similiarity of the procedure [6].

Next section, we will apply one test problem for the approximation solution of the problem (1.1).

3. Numerical Examples

In this section, we shall test one example for the pseudo-parabolic partial differential equations by finite difference method. For this, we utilize a procedure of modified Gauss elimination method for difference equation (2.4). To obtain error analysis table, the following maximum norm is used for the numerical solution by given

$$\varepsilon = \max_{\substack{i=0,1,\cdots,M \\ j=0,1,\cdots,N}} |u(x,t) - u(t_k, x_n)|.$$

Where $u_n^k = u(t_k, x_n)$ is the approximate solution, $u(x,t)$ is the exact solution and ε is the error analysis.

Example 3.1. The fractional order the pseudo-parabolic partial differential equations defined by ABC derivative operator with inital-boundary value condition

$$\begin{cases} {}^{ABC}_{\ \ 0}Du(t,x) = u_{txx}(t,x) + u_{xx}(t,x) + f(t,x), \ \ 0 < x < \pi, \ \ 0 < t < 1, \\ f(t,x) = \left(t^3 + \frac{3(1-\alpha)t^2}{B(\alpha)} + \frac{6\alpha t^{2+\alpha}}{B(\alpha)\Gamma(3+\alpha)}\frac{(1-\alpha)t^3}{B(\alpha)} + \frac{6\alpha t^{3+\alpha}}{B(\alpha)\Gamma(4+\alpha)}\right) sinx, \\ u(0,x) = 0, \ \ 0 \le x \le \pi, \\ u(t,0) = u(t,\pi) = 0, \ \ 0 \le t \le 1, \end{cases} \tag{3.1}$$

is tested.

In the following Table 1., the approximate solution and exact solution of the following example are given for different value α.

Table 1.

α	**N, M mesh grid**	**ε − Error Analysis**
1/100	N=M=25	0.1266
	N=M=50	0.0634
	N=M=100	0.0317
	N=M=200	0.0159
50/100	N=M=25	0.1215
	N=M=50	0.0608
	N=M=100	0.0304
	N=M=200	0.0152
99/100	N=M=25	0.0329
	N=M=50	0.0165
	N=M=100	0.0083
	N=M=200	0.0043
1	N=M=200	0.0039

Remark 3.1 If the following condition is satisfied $h^2 < \frac{2(\tau+1)\Gamma(2-\alpha)}{\tau^{1-\alpha}}$, then numerical results are showed stability estimates for the above Table 1.

4. Conclusion

In this paper, the exact solution and stability estimates are showed for Pseudo-parabolic partial differential equation with sense of ABC derivative. First order finite difference schemes are constructed. Stability inequalities are proved for given difference schemes. Approximate solutions for numerical experiment were found by the finite difference-method. MATLAB programs are used for the numerical computations.

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Extended *q*-Daehee Polynomials and Its Applications via Mahler Expansion

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Abstract

In the present paper, we firstly consider novel q-extensions of the Daehee polynomials named as the generalized twisted q-Daehee polynomials with weight (α,β) and generalized twisted q-Daehee polynomials of the second kind with weight (α,β). For the aforementioned polynomials, we derive various interesting and new formulas and identities covering correlations and recurrence relations associated with the Apostol type Stirling numbers of the second kind, the generalized twisted q-Bernoulli polynomials with weight (α,β) and Stirling numbers of both kinds. Moreover, we provide two relationships for the two types of generalized twisted q-Daehee polynomials with weight (α,β). Then, we examine some connections with the p-adic gamma function with its derivative via Mahler expansion for the aforesaid polynomials. Furthermore, we prove an exciting explicit formula for the p-adic Euler constant thanks to the generalized twisted q-Daehee polynomials with weight (α,β).

Keywords: p-adic gamma function, Mahler expansion, Bernoulli polynomials, Daehee polynomials, p-adic numbers.

1. Introduction

In the science of mathematics, *special polynomials and special functions* have been debated and used for centuries. The subjects of special polynomials and functions have been in uninterrupted development from then till now, *cf.* [1]. In recent years, diverse new special polynomials and functions in conjunction with some applications have been considered by many mathematicians (see [1-28]). These treatises provide an overview of the field of the diverse special polynomials and functions, centering mainly on the Hermite polynomials, hypergeometric functions, Bernoulli polynomials, gamma function, Daehee polynomials, and the related certain polynomials and functions.
The Daehee polynomials $D_n(x)$ are defined by Kim et al. [10] via the following exponential generating function:

$$\sum_{n\geq 0} D_n(x)\frac{t^n}{n!}=\frac{\log(1+t)}{t}(1+t)^x. \tag{1.1}$$

Upon setting $x=0$ in (1.1), we obtain $D_n(0):=D_n$ standing for n-th Daehee number, *cf.* [2,4,10-

12,14,17,20,22,23,25-27] and see also the related references cited therein.

The p-adic numbers are a counterintuitive arithmetic system, which was initially considered by the Kummer in circa 1850, *cf.* [16,24]. With the consideration of these numbers, many mathematicians and physicists started to develop new scientific tools using their available, useful, and applicable properties. Firstly, Kurt Hensel who is a German mathematician (1861-1941) advanced the p-adic numbers in a study related to the improvement of algebraic numbers in formal power series in about 1897. Several effects of these researches have emerged in mathematics and physics such as p-adic analysis, string theory, p-adic quantum mechanics, quantum field theory, representation theory, algebraic geometry, complex systems, dynamical systems, genetic codes, and so on (*cf.* [2-4,6-11,13,16,17,19-23,25-27]; also see the related references cited in each of these earlier studies). The one important tool of the mentioned advancements is p-adic gamma function which is introduced by Yasou Morita [19] in 1975. Intense research activities in such an area as p-adic gamma function is principally motivated by their significance in p-adic analysis. Hereby, in recent more than forty years, the p-adic gamma function and its diverse extensions have been studied and progressed broadly by many mathematicians, *cf.* [5-8,15,16,19,24]; see also the related references cited therein.

Here are some definitions and notations, which will be useful for the next sections.

Imagine that p be a fixed prime number. In the course of this paper, we use the following notations: $\mathbb{Q}$ denotes the field of all rational numbers, $\mathbb{Z}_p$ denotes the ring of all p-adic rational integers, $\mathbb{Q}_p$ denotes the field of all p-adic rational numbers, and $\mathbb{C}_p$ denotes the completion of the algebraic closure of $\mathbb{Q}_p$. Let the notation $\mathbb{N}$ indicates the set of all natural numbers and set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

For d a fixed positive number, let

$$X^* = \bigcup_{\substack{0<a<dp^N \\ (a,p)=1}} \left(a + dp\mathbb{Z}_p\right) \text{ and } X := X_d = \varprojlim_n \mathbb{Z}/dp^N\mathbb{Z},$$

and

$$\left\{x \in X \mid x \equiv a\left(\mathrm{mod}\, dp^N\right)\right\} = a + dp^N\mathbb{Z}_p$$

in which $a \in \mathbb{Z}$ lies in $0 \le a < dp^N$. See [2-4,6-11,13,16,17,20-23,25-27] for a detailed systematic study.

The normalized absolute value with respect to the p-adic analysis is known as $|p|_p = p^{-1}$. The symbol "q" can be chosen as a complex number $q \in \mathbb{C}$ with $|q| < 1$, an indeterminate, or a p-adic number $q \in \mathbb{C}_p$ with $|q-1|_p < p^{-\frac{1}{p-1}}$ and $q^x = \exp(x \log q)$ for $|x|_p \le 1$. This is every time obvious in the content of the paper.

The q-extension of any number x, denoted by $[x]_q$, is given by $[x]_q = \dfrac{1-q^x}{1-q}$. It is easily seen that $\lim_{q\to 1}[x]_q = x$, *cf.* [2-4,6,7,9,12-15,17,20-22].

For $f \in UD(\mathbb{Z}_p) = \{ f \mid f$ is uniformly differentiable function at a point $a \in \mathbb{Z}_p \}$, consider the following expression:

$$\frac{1}{[p^N]_q} \sum_{0 \le j < p^N} q^j f(j) = \sum_{0 \le j < p^N} f(j) \mu_q(j + p^N \mathbb{Z}_p) \tag{1.2}$$

indicating the p-adic q-generalization of Riemman sums for a function f. The integral of $f \in UD(\mathbb{Z}_p)$ will be introduced as a limit ($N \to \infty$) of these sums when it exists. The bosonic p-adic q-integral on $\mathbb{Z}_p$ (or sometimes called q-Volkenborn integral) of a function $f \in UD(\mathbb{Z}_p)$ is introduced as (*cf.* [2,4,6,7,9,12-14,17,20-22])

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x. \tag{1.3}$$

For $f_1(x) = f(x+1)$, in terms of (1.3), Kim [9] attained the following quirky integral formula

$$qI_q(f_1) - I_q(f) = (q-1) f(0) + \frac{q-1}{\log q} f'(0), \tag{1.4}$$

which densely preserves usable for introducing multifarious extensions of several special polynomials such as Daehee and Bernoulli polynomials with their varied generalizations. Kim [9] provided a more general integral formula for the equation (1.4) as follows:

$$q^m I_q(f_n) = I_q(f) + (q-1) \sum_{u=0}^{m-1} q^u f(u) + \frac{q-1}{\log q} \sum_{u=0}^{m-1} f'(u) \text{ for } f_n(x) = f(x+m). \tag{1.5}$$

Let $m \in \mathbb{N}$. The falling factorial or known as lower factorial is defined as

$$(x)_m := x(x-1)(x-2)\cdots(x-m+1) \tag{1.6}$$

and satisfies the following summation formula

$$(x)_m = \sum_{u=0}^{m} S_1(m,u) x^u, \tag{1.7}$$

where $S_1(m,u)$ is Stirling number of the first kind, *cf.* [1-4,6,7,10-12,14,17,20,22,23,25-28].

Kim et al. [12] introduced q-extension of the Daehee polynomials of both kinds as follows:

$$D_{n,q}(x) = \int_X (x+\omega)_n \, d\mu_q(\omega) \text{ and } D_{n,q}(x) = \int_X (-x-\omega)_n \, d\mu_q(\omega) \, (n \ge 0).$$

In [22], Park defined the modified q-Daehee polynomials of both kinds via q-Volkenborn integrals:

$$D_n(x:q) = \int_X q^{-\omega} (x+\omega)_n \, d\mu_q(\omega) \text{and } D_{n,q}(x:q) = \int_X q^{-\omega} (-x-\omega)_n \, d\mu_q(\omega) \, (n \ge 0). \tag{1.8}$$

Setting $x = 0$ in (1.8) yields $D_n(0:q) := D_{n:q}$ named as n-th modified q-Daehee number. It is clear to see that $\lim_{q\to 1} D_n(x:q) := D_n(x)$.

Let χ be the Dirichlet character with a conductor $d \in \mathbb{N}$, *cf.* [16,17,23,25]. The generalized q-Daehee polynomials of both kinds attached to χ are defined by the following p-adic q-integrals (*cf.* [20]):

$$D_{n,\chi,q}(x)=\int_X \chi(\omega)(x+\omega)_n\, d\mu_q(\omega) \text{ and } D_{n,\chi,q}(x)=\int_X \chi(\omega)(-x-\omega)_n\, d\mu_q(\omega) \ \ (n\geq 0).$$

For $n\in\mathbb{N}$, let T_p be the p-adic locally constant space defined by

$$T_p=\bigcup_{0<n}\mathbb{C}_{p^n}=\lim_{n\to\infty}\mathbb{C}_{p^n},$$

where $\mathbb{C}_{p^n}=\left\{\xi \middle| \xi^{p^n}=1\right\}$ is the cyclic group of order p^n. For $t,q\in\mathbb{C}_p$ in conjunction with $|q|_p<p^{-\frac{1}{1-p}}$ and $|t|_p<p^{-\frac{1}{1-p}}$, the twisted q-Daehee polynomials of both kinds are defined by the following q-Volkenborn integrals (*cf.* [17]):

$$D_{n,q}(x|\xi)=\xi^n\int_X (x+\omega)_n\, d\mu_q(\omega) \text{ and } D_{n,q}(x|\xi)=\xi^n\int_X (-x-\omega)_n\, d\mu_q(\omega) \ \ (n\geq 0).$$

The Stirling numbers of the second kind, denoted by $S_2(u,m)$, is the number of ways to partition a set of u objects into m non-empty subsets and have also the following generating function

$$\frac{\left(e^z-1\right)^u}{u!}=\sum_{m=u}^{\infty}S_2(u,m)\frac{z^m}{m!}. \tag{1.9}$$

For further details on the applications of the Stirling numbers of both kinds, take a look at the references [2, 4, 10 - 12, 14, 17, 20, 22, 23, 25 - 27] and see also the references cited therein.

The contents of the present paper are as follows: we firstly give an extension of the bosonic p-adic q-integrals on X involving the parameters α and β to define two types of the generalized twisted q-Daehee polynomials with weight (α,β). Then, using properties of the mentioned integral, we examine several new relationships and interesting identities for these polynomials related to the generalized twisted q-Bernoulli polynomials with weight (α,β), Apostol type Stirling numbers of the second kind, and Stirling numbers of both kinds. We finally get some connections formula including the p-adic Euler constant, p-adic gamma function, and the generalized twisted q-Daehee polynomials with weight (α,β) by utilizing the Mahler series expansion of the p-adic gamma function.

2. Main Results

In this part, let $t,q\in\mathbb{C}_p$ with $|q|_p<p^{-\frac{1}{1-p}}$ and $|t|_p<p^{-\frac{1}{1-p}}$.

As an extension of the bosonic p-adic q-integral, the weighted p-adic q-integral is given by Araci et al. [2] as follows:

$$I_q^{(\alpha,\beta)}(f)=\int_X q^{-\beta x}f(x)\,d\mu_{q^\alpha}(x)=\lim_{N\to\infty}\frac{1}{\left[dp^N\right]_{q^\alpha}}\sum_{r=0}^{dp^N-1}f(r)q^{(\alpha-\beta)r}, \tag{2.1}$$

which satisfies the following formulas, for $f_n(x)=f(x+n)$:

$$q^{(\alpha-\beta)}I_q^{(\alpha,\beta)}(f_1)-I_q^{(\alpha,\beta)}(f)=(q-1)\frac{[\alpha]_q}{\alpha}(\alpha-\beta)f(0)+\frac{[\alpha]_q}{\alpha}\frac{q-1}{\log q}f'(0)$$

and

$$q^{(\alpha-\beta)n}I_q^{(\alpha,\beta)}(f_n)-I_q^{(\alpha,\beta)}(f)=\frac{[\alpha]_q}{\alpha}\left((q-1)\sum_{u=0}^{n-1}q^{(\alpha-\beta)u}f(u)+\frac{q-1}{\log q}\sum_{u=0}^{n-1}q^{(\alpha-\beta)u}f'(u)\right). \qquad (2.2)$$

Araci et al. [2] considered and investigated the q-Daehee polynomials with the weight (α,β) of both kinds given below

$$D_{n,q}^{(\alpha,\beta)}(x)=\int_X q^{-\beta\omega}(x+\omega)_n\,d\mu_{q^\alpha}(\omega) \text{ and } \overset{(\alpha,\beta)}{D_{n,q}}(x)=\int_X q^{-\beta y}(-x-y)_n\,d\mu_{q^\alpha}(y)\,(n\geq 0).$$

If we setting $f(x)=\xi^n\chi(x)(x)_n$ in (2.1), we then have

$$I_q^{(\alpha,\beta)}(\xi^n\chi(x)(x)_n)=\xi^n\int_X\chi(x)q^{-\beta x}(x)_n\,d\mu_{q^\alpha}(x). \qquad (2.3)$$

It follows from Eq. (2.3) that

$$\sum_{n=0}^{\infty}\xi^n\int_X\chi(x)q^{-\beta x}(x)_n\,d\mu_{q^\alpha}(x)\frac{t^n}{n!}=\int_X\chi(x)q^{-\beta x}\left(\sum_{n=0}^{\infty}\binom{x}{n}(\xi t)^n\right)d\mu_{q^\alpha}(x)=\int_X\chi(x)q^{-\beta x}(1+\xi t)^x\,d\mu_{q^\alpha}(x). \qquad (2.4)$$

By (2.2), using the property $\chi(x+d)=\chi(x)$, we have

$$\begin{aligned}
q^{(\alpha-\beta)d}I_q^{(\alpha,\beta)}(f_d)-I_q^{(\alpha,\beta)}(f)&=q^{(\alpha-\beta)d}\int_X\chi(x)q^{-\beta(x+d)}(1+\xi t)^{x+d}\,d\mu_{q^\alpha}(x)\\
&-\int_X\chi(x)q^{-\beta x}(1+\xi t)^x\,d\mu_{q^\alpha}(x)\\
&=\frac{[\alpha]_q}{\alpha}(q-1)\left(\sum_{a=0}^{d-1}q^{(\alpha-\beta)a}\chi(a)(1+\xi t)^a+\frac{\log(1+\xi t)}{\log q}\sum_{a=0}^{d-1}q^{(\alpha-\beta)a}\chi(a)(1+\xi t)^a\right)\\
&=\frac{[\alpha]_q}{\alpha}(q-1)\left(1+\frac{\log(1+\xi t)}{\log q}\right)\sum_{a=0}^{d-1}q^{(\alpha-\beta)a}\chi(a)(1+\xi t)^a. \qquad (2.5)
\end{aligned}$$

We now can state the following Definition 1.

Definition 1 *We define generalized twisted q-Daehee polynomials $D_{n,\chi,q}^{(\alpha,\beta)}(x|\xi)$ with weight (α,β) by the following generating function to be*

$$\sum_{n=0}^{\infty}D_{n,\chi,q}^{(\alpha,\beta)}(x|\xi)\frac{t^n}{n!}=\frac{[\alpha]_q}{\alpha}\frac{1-q+\frac{1-q}{\log q}\log(1+\xi t)}{1-q^{(\alpha-2\beta)d}(1+\xi t)^d}\sum_{a=0}^{d-1}q^{(\alpha-\beta)a}\chi(a)(1+\xi t)^{a+x}. \qquad (2.6)$$

By (2.5) and (2.6), we readily deduce that

$$\sum_{n=0}^{\infty}D_{n,\chi,q}^{(\alpha,\beta)}(x|\xi)\frac{t^n}{n!}=\int_X\chi(y)q^{-\beta y}(1+\xi t)^{x+y}\,d\mu_{q^\alpha}(y) \qquad (2.7)$$

For $n \in \mathbb{N}_0$ and Eq. (2.7), we get

$$D_{n,\chi,q}^{(\alpha,\beta)}(x|\xi) = \xi^n \int_X \chi(y) q^{-\beta y} (x+y)_n \, d\mu_{q^\alpha}(y). \tag{2.8}$$

When $x=0$ in (2.8), we get $D_{n,\chi,q}^{(\alpha,\beta)}(0|\xi) := D_{n,\chi,q,\xi}^{(\alpha,\beta)}$ that we call the generalized twisted q-Daehee numbers with weight (α,β) possessing the following generating function

$$\sum_{n=0}^{\infty} D_{n,\chi,q,\xi}^{(\alpha,\beta)} \frac{t^n}{n!} = \int_X \chi(y) q^{-\beta y} (1+\xi t)^y \, d\mu_{q^\alpha}(y).$$

As a result of (1.6) and (2.8), we give the following theorem.

Theorem 1 *For $n \in \mathbb{N}_0$, the following correlation holds:*

$$D_{n,\chi,q}^{(\alpha,\beta)}(x|\xi) = \sum_{k=0}^{n} S_1(n,k) B_{k,\chi,q}^{(\alpha,\beta)}(x|\xi),$$

where $B_{k,\chi,q}^{(\alpha,\beta)}(x|\xi)$ are the generalized twisted q-Bernoulli polynomials with weight (α,β) defined by

$$B_{k,\chi,q}^{(\alpha,\beta)}(x|\xi) = \xi^n \int_X \chi(y) q^{-\beta y} (x+y)^k \, d\mu_{q^\alpha}(y).$$

Proof. Thanks to (1.6), we compute that

$$\begin{aligned}
D_{n,\chi,q}^{(\alpha,\beta)}(x|\xi) &= \xi^n \int_X \chi(y) q^{-\beta y} (x+y)_n \, d\mu_{q^\alpha}(y) \\
&= \xi^n \int_X \chi(y) q^{-\beta y} \left(\sum_{k=0}^{n} S_1(n,k)(x+y)^k \right) d\mu_{q^\alpha}(y) \\
&= \sum_{k=0}^{n} S_1(n,k) \xi^n \int_X \chi(y) q^{-\beta y} (x+y)^k \, d\mu_{q^\alpha}(y) = \sum_{k=0}^{n} S_1(n,k)(-1)^k B_{k,q}^{(\alpha,\beta)}(x),
\end{aligned}$$

hence, we investigate the asserted result.

In the special case $x=0$, we then attain that

$$D_{n,\chi,q,\xi}^{(\alpha,\beta)} = \sum_{k=0}^{n} S_1(n,k) B_{k,\chi,q,\xi}^{(\alpha,\beta)},$$

where $B_{k,\chi,q}^{(\alpha,\beta)}(0|\xi) := B_{k,\chi,q,\xi}^{(\alpha,\beta)}$ are the generalized twisted q-Bernoulli numbers with weight (α,β).
Since

$$\begin{aligned}
q^{(\alpha-\beta)d} I_q^{(\alpha,\beta)}(f_n) - I_q^{(\alpha,\beta)}(f) &= q^{(\alpha-\beta)d} \int_X \chi(x) q^{-\beta(x+d)} e^{(x+d)t\xi} d\mu_{q^\alpha}(x) - \int_X \chi(x) q^{-\beta x} e^{xt\xi} d\mu_{q^\alpha}(x) \\
&= \frac{[\alpha]_q}{\alpha}(q-1)\left(\sum_{a=0}^{d-1} q^{(\alpha-\beta)a} \chi(a) e^{at\xi} + t\xi \frac{\log(1+\xi t)}{\log q} \sum_{a=0}^{d-1} q^{(\alpha-\beta)a} \chi(a) e^{at\xi} \right),
\end{aligned}$$

the polynomials $B_{n,\chi,q}^{(\alpha,\beta)}(x|\xi)$ satisfy the following equality:

$$\sum_{n=0}^{\infty} B_{n,\chi,q}^{(\alpha,\beta)}(x|\xi)\frac{t^n}{n!} = \frac{1-q+t\xi\frac{1-q}{\log q}}{1-q^{(\alpha-2\beta)d}e^{dt\xi}}\frac{[\alpha]_q}{\alpha}\sum_{a=0}^{d-1} q^{(\alpha-\beta)a}\chi(a)e^{at\xi}.$$

There is a relationship among $D_{n,\chi,q}^{(\alpha,\beta)}(x|\xi)$, $B_{n,\chi,q}^{(\alpha,\beta)}(x|\xi)$ and $S_2(n,k:\omega)$ which we discuss below.

Theorem 2 *Let* $n \in \mathbb{N}_0$. *We then have*

$$B_{n,\chi,q}^{(\alpha,\beta)}(x|\xi) = \xi^n \sum_{k=0}^{n} S_2\left(n,k:\frac{1}{\xi}\right) D_{k,\chi,q}^{(\alpha,\beta)}(x|\xi), \tag{2.9}$$

where the Apostol type Stirling numbers of the second kind are defined by

$$\sum_{k=n}^{\infty} S_2(n,k:\omega)\frac{t^k}{k!} = \frac{(\omega e^t - 1)^n}{n!}, \; cf. \; [27].$$

Proof. Substituting t by $\frac{e^{t\xi}}{\xi} - 1$ in the generating function (2.6), we acquire

$$\frac{[\alpha]_q}{\alpha}\frac{1-q+\frac{1-q}{\log q}t\xi}{1-q^{(\alpha-2\beta)d}e^{t\xi d}}\sum_{a=0}^{d-1} q^{(\alpha-\beta)a}\chi(a)e^{t\xi a}e^{\xi tx} = \sum_{n=0}^{\infty} D_{n,\chi,q}^{(\alpha,\beta)}(x|\xi)\frac{\left(\frac{e^{t\xi}}{\xi}-1\right)^n}{n!}$$

$$= \sum_{n=0}^{\infty} D_{n,\chi,q}^{(\alpha,\beta)}(x|\xi)\sum_{k=n}^{\infty} S_2\left(n,k:\frac{1}{\xi}\right)\frac{\xi^k t^k}{k!} = \sum_{n=0}^{\infty}\left(\xi^n \sum_{k=0}^{n} D_{k,\chi,q}^{(\alpha,\beta)}(x|\xi) S_2\left(n,k:\frac{1}{\xi}\right)\right)\frac{t^n}{n!}$$

and by (2.8), we get

$$\sum_{n=0}^{\infty} B_{n,\chi,q}^{(\alpha,\beta)}(x|\xi)\frac{t^n}{n!} = \frac{[\alpha]_q}{\alpha}\frac{1-q+t\xi\frac{1-q}{\log q}}{1-q^{(\alpha-2\beta)d}e^{dt\xi}}\sum_{a=0}^{d-1} q^{(\alpha-\beta)a}\chi(a)e^{at\xi}e^{x\xi t}, \tag{2.10}$$

which implies the desired result (2.9).

Definition 2 *The generalized twisted* q*-Daehee numbers of the second kind with weight* (α,β) *are defined by the following integral formula:*

$$D_{n,\chi,q,\xi}^{(\alpha,\beta)} = \xi^n \int_X \chi(y) q^{-\beta y}(-y)_n \, d\mu_{q^\alpha}(y) \quad (n \in \mathbb{N}_0). \tag{2.11}$$

In terms of (1.7) and (2.11), we get the following result.

Theorem 3 *The following relationship holds for* $n \geq 0$:

$$D_{n,\chi,q,\xi}^{(\alpha,\beta)} = \sum_{k=0}^{n} \xi^{n-k} S_1(n,k)(-1)^k B_{k,\chi,q,\xi}^{(\alpha,\beta)}.$$

Proof. Using (1.6), we calculate that

$$D_{n,\chi,q,\xi}^{(\alpha,\beta)}=\xi^n\int_X\chi(y)q^{-\beta y}(-y)_n\,d\mu_{q^\alpha}(y)=\xi^n\int_X\chi(y)q^{-\beta y}\left(\sum_{k=0}^n S_1(n,k)(-y)^k\right)d\mu_{q^\alpha}(y)$$

$$=\sum_{k=0}^n S_1(n,k)(-1)^k\,\xi^{n-k}\int_X\xi^k\chi(y)q^{-\beta y}y^k d\mu_{q^\alpha}(y)=\sum_{k=0}^n\xi^{n-k}S_1(n,k)(-1)^k\,B_{k,\chi,q,\xi}^{(\alpha,\beta)},$$

which gives the assertion in the theorem.

To compute the generating function of the numbers $D_{n,\chi,q,\xi}^{(\alpha,\beta)}$, we have

$$\sum_{n=0}^\infty D_{n,\chi,q,\xi}^{(\alpha,\beta)}\frac{t^n}{n!}=\sum_{n=0}^\infty\left(\xi^n\int_X\chi(y)q^{-\beta y}(-y)_n\,d\mu_{q^\alpha}(y)\right)\frac{t^n}{n!}$$

$$=\int_X\chi(y)q^{-\beta y}\left(\sum_{n=0}^\infty\binom{-y}{n}(\xi t)^n\right)d\mu_{q^\alpha}(y)=\int_X\chi(y)q^{-\beta y}(1+\xi t)^{-y}\,d\mu_{q^\alpha}(y).$$

Moreover, in view of (2.11), we investigate

$$q^{(\alpha-\beta)d}I_q^{(\alpha,\beta)}(f_n)-I_q^{(\alpha,\beta)}(f)=q^{(\alpha-\beta)d}\int_X\chi(x)q^{-\beta x-\beta d}(1+\xi t)^{-x-d}\,d\mu_{q^\alpha}(x)-\int_X\chi(x)q^{\beta x}(1+\xi t)^{-x}\,d\mu_{q^\alpha}(x)$$

$$=\frac{[\alpha]_q}{\alpha}(q-1)\left(\sum_{a=0}^{d-1}q^{(\alpha-\beta)a}\chi(a)(1+\xi t)^{-a}+\frac{\log(1+\xi t)^{-1}}{\log q}\sum_{a=0}^{d-1}q^{(\alpha-\beta)a}\chi(a)(1+\xi t)^{-a}\right),$$

which yields the following formula

$$\int_X\chi(y)q^{-\beta y}(1+\xi t)^{-y}\,d\mu_{q^\alpha}(y)=\frac{q-1-\frac{q-1}{\log q}\log(1+\xi t)}{q^{(\alpha-2\beta)d}(1+\xi t)^{-d}-1}\frac{[\alpha]_q}{\alpha}\sum_{a=0}^{d-1}q^{(\alpha-\beta)a}\chi(a)(1+\xi t)^{-a}.$$

Thus let us rewrite Definition 2 as

$$\sum_{n=0}^\infty D_{n,\chi,q,\xi}^{(\alpha,\beta)}\frac{t^n}{n!}=\frac{q-1-\frac{q-1}{\log q}\log(1+\xi t)}{q^{(\alpha-2\beta)d}(1+\xi t)^{-d}-1}\frac{[\alpha]_q}{\alpha}\sum_{a=0}^{d-1}q^{(\alpha-\beta)a}\chi(a)(1+\xi t)^{-a}.$$

We here consider the generalized weighted q-Daehee polynomials of the second kind with weight (α,β) as follows:

$$\sum_{n=0}^\infty D_{n,\chi,q}^{(\alpha,\beta)}(x|\xi)\frac{t^n}{n!}=\int_X\chi(y)q^{-\beta y}(1+\xi t)^{-(x+y)}\,d\mu_{q^\alpha}(y).\qquad(2.12)$$

From (2.12), we acquire

$$\sum_{n=0}^\infty D_{n,\chi,q}^{(\alpha,\beta)}(x|\xi)\frac{t^n}{n!}=\int_X\chi(y)q^{-\beta y}\left(\sum_{n=0}^\infty\binom{-x-y}{n}\xi^n t^n\right)d\mu_{q^\alpha}(y)$$

$$=\sum_{n=0}^\infty\left(\xi^n\int_X\chi(y)q^{-\beta y}(-x-y)_n\,d\mu_{q^\alpha}(y)\right)\frac{t^n}{n!},$$

thus, we get

$$D_{n,\chi,q}^{(\alpha,\beta)}\left(x|\xi\right)=\xi^n\int_X\chi\left(y\right)q^{-\beta y}\left(-x-y\right)_n d\mu_{q^\alpha}\left(y\right). \tag{2.13}$$

Then, we arrive at the following relationship.

Theorem 4 *The following correlation*

$$D_{n,\chi,q}^{(\alpha,\beta)}\left(x|\xi\right)=\sum_{l=0}^{n}\left(-1\right)^l S_1\left(n,l\right)B_{l,\chi,q}^{(\alpha,\beta)}\left(x|\xi\right) \tag{2.14}$$

holds for $n\in\mathbb{N}_0$.

Proof. Owing to the following computations:

$$\begin{aligned}D_{n,\chi,q}^{(\alpha,\beta)}\left(x|\xi\right)&=\xi^n\int_X\chi\left(y\right)q^{-\beta y}\left(-x-y\right)_n d\mu_{q^\alpha}\left(y\right)\\&=\xi^n\int_X\left(\sum_{k=0}^{n}S_1\left(n,k\right)\left(-x-y\right)^k\right)\chi\left(y\right)q^{-\beta y}d\mu_{q^\alpha}\left(y\right)\\&=\sum_{k=0}^{n}S_1\left(n,k\right)\left(-1\right)^k\xi^{n-k}\int_X\xi^k q^{-\beta y}\left(x+y\right)^k d\mu_{q^\alpha}\left(y\right)\\&=\sum_{k=0}^{n}\xi^{n-k}S_1\left(n,k\right)\left(-1\right)^k B_{k,\chi,q}^{(\alpha,\beta)}\left(x|\xi\right),\end{aligned}$$

we get the claimed result (2.14).

We now provide the following theorem.

Theorem 5 *The following correlation*

$$B_{n,\chi,q}^{(\alpha,\beta)}\left(x|-\xi\right)=\sum_{k=0}^{n}D_{k,\chi,q}^{(\alpha,\beta)}\left(x|\xi\right)S_2\left(n,k:\frac{1}{\xi}\right)$$

is valid for $n\geq 0$.

Proof. By changing t by $\frac{e^{t\xi}}{\xi}-1$ in (2.12), we derive

$$\begin{aligned}\frac{q-1+\frac{q-1}{\log q}\left(-\xi t\right)}{q^{(\alpha-2\beta)d}\left(1+\xi t\right)e^{(-t\xi)d}-1}\frac{[\alpha]_q}{\alpha}\sum_{a=0}^{d-1}q^{(\alpha-\beta)a}\chi\left(a\right)e^{(-\xi t)(a+x)}&=\sum_{n=0}^{\infty}D_{n,\chi,q}^{(\alpha,\beta)}\left(x|\xi\right)\frac{\left(\frac{e^{t\xi}}{\xi}-1\right)^n}{n!}\\&=\sum_{n=0}^{\infty}D_{n,\chi,q}^{(\alpha,\beta)}\left(x|\xi\right)\sum_{k=n}^{\infty}S_2\left(k,n:\frac{1}{\xi}\right)\frac{\xi^k t^k}{l!}\\&=\sum_{n=0}^{\infty}\left(\xi^n\sum_{k=0}^{n}D_{k,\chi,q}^{(\alpha,\beta)}\left(x|\xi\right)S_2\left(n,k:\frac{1}{\xi}\right)\right)\frac{t^n}{n!}\end{aligned}$$

and using (2.10), we deduce

$$B_{n,\chi,q}^{(\alpha,\beta)}\left(x\middle|-\xi\right)=\sum_{k=0}^{n}D_{k,\chi,q}^{(\alpha,\beta)}\left(x\middle|\xi\right)S_2\left(n,k:\frac{1}{\xi}\right).$$

The following correlations are valid for the polynomials $D_{n,\chi,q}^{(\alpha,\beta)}\left(x\middle|\xi\right)$ and $\mathsf{D}_{n,\chi,q}^{(\alpha,\beta)}\left(x\middle|\xi\right)$.

Theorem 6 *The following identities*

$$D_{n,\chi,q}^{(\alpha,\beta)}\left(x\middle|\xi\right)=\left(-\xi\right)^{n}\sum_{k=1}^{n}\binom{n}{k}\frac{\mathsf{D}_{k,\chi,q}^{(\alpha,\beta)}\left(x\middle|\xi\right)}{\xi^{k}}$$

and

$$\mathsf{D}_{n,\chi,q}^{(\alpha,\beta)}\left(x\middle|\xi\right)=\left(-\xi\right)^{n}\sum_{k=1}^{n}\binom{n}{k}\frac{D_{k,\chi,q}^{(\alpha,\beta)}\left(x\middle|\xi\right)}{\xi^{k}}$$

hold for $n\in\mathbb{N}_0$.

Proof. By means of the following binomial formulae (*cf.* $[\,2,4,10\text{-}12,14,17,20,22,23,25\text{-}27\,]$)

$$\left(-1\right)^{n}\binom{m}{n}=\binom{-m+n-1}{n}\text{ and }\binom{-m+n-1}{n}=\sum_{k=1}^{n}\binom{n-1}{n-k}\binom{-m}{k},$$

from (2.13), we derive

$$\begin{aligned}\frac{D_{n,\chi,q}^{(\alpha,\beta)}\left(x\middle|\xi\right)}{n!}\left(-1\right)^{n}&=\left(-\xi\right)^{n}\int_{X}\chi\left(y\right)q^{-\beta y}\binom{x+y}{n}d\mu_{q^{\alpha}}\left(y\right)\\&=\xi^{n}\int_{X}\binom{-x-y+n-1}{n}q^{-\beta y}\chi\left(y\right)d\mu_{q^{\alpha}}\left(y\right)\\&=\sum_{k=1}^{n}\binom{n-1}{n-k}\xi^{n-k}\int_{X}\binom{-x-y}{k}\xi^{k}q^{-\beta y}\chi\left(y\right)d\mu_{q^{\alpha}}\left(y\right)\\&=\sum_{k=1}^{n}\frac{\xi^{n-k}}{k!}\binom{n-1}{k-1}\mathsf{D}_{k,\chi,q}^{(\alpha,\beta)}\left(x\middle|\xi\right)\end{aligned}$$

and

$$\begin{aligned}\frac{\mathsf{D}_{n,\chi,q}^{(\alpha,\beta)}\left(x\middle|\xi\right)}{n!}\left(-1\right)^{n}&=\left(-\xi\right)^{n}\int_{X}\binom{-x-y}{n}\chi\left(y\right)q^{-\beta y}d\mu_{q^{\alpha}}\left(y\right)\\&=\xi^{n}\int_{X}\binom{x+y+n-1}{n}q^{-\beta y}\chi\left(y\right)d\mu_{q^{\alpha}}\left(y\right)\\&=\sum_{k=1}^{n}\xi^{n-k}\binom{n-1}{n-k}\int_{X}\binom{x+y}{k}\xi^{k}q^{-\beta y}\chi\left(y\right)d\mu_{q^{\alpha}}\left(y\right)\\&=\sum_{k=1}^{n}\frac{\xi^{n-k}}{k!}\binom{n-1}{k-1}D_{k,\chi,q}^{(\alpha,\beta)}\left(x\middle|\xi\right).\end{aligned}$$

Now, we provide some applications for the polynomials $D_{n,\chi,q}^{(\alpha,\beta)}\left(x\middle|\xi\right)$ and $\mathsf{D}_{n,\chi,q}^{(\alpha,\beta)}\left(x\middle|\xi\right)$ associated with the

Mahler expansion of the p-adic gamma function.

The p-adic gamma function is defined by the following limit expression:

$$\Gamma_p(x) = \lim_{n\to x}(-1)^n \prod_{\substack{h<n \\ (p,h)=1}} h \qquad (x \in \mathbb{Z}_p), \tag{3.1}$$

where n approaches x through positive integers. The mentioned function, as well as its several extensions, have been developed by many physicists and mathematicians, *cf.* [5-8,15,16,19,24]; see also the related references cited in each of these earlier works.

The p-adic Euler constant γ_p is given by the following formula (*cf.* [5,6,15,16,24]):

$$\gamma_p := -\frac{\Gamma'_p(1)}{\Gamma_p(0)} = -\Gamma'_p(0) = \Gamma'_p(1). \tag{3.2}$$

For $x \in \mathbb{Z}_p$, the notation $\binom{x}{n}$ is given by $\binom{x}{n} = \frac{x(x-1)\cdots(x-n+1)}{n!}$ $(n \in \mathbb{N})$ with the initial condition $\binom{x}{0} = 1$. The functions $x \to \binom{x}{n}$ form an orthonormal base of the space $C(\mathbb{Z}_p \to \mathbb{C}_p)$ by the Euclidean norm $\|\cdot\|_\infty$. The aforegoing orthonormal base fulfills the following equality:

$$\binom{x}{n}' = \sum_{j=0}^{n-1} \frac{(-1)^{n-j-1}}{n-j}\binom{x}{j} \quad (\text{see [6-8,18]}). \tag{3.3}$$

Mahler [18] investigated a generalization for continuous maps of a p-adic variable by making use of the special polynomials as a binomial coefficient polynomial in 1958, which means that for any $f \in C(\mathbb{Z}_p \to \mathbb{C}_p)$, there exist unique elements $a_0, a_1, a_2, \ldots$ of $\mathbb{C}_p$ such that

$$f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n} \quad (x \in \mathbb{Z}_p). \tag{3.4}$$

The base $\left\{\binom{*}{m} : m \in \mathbb{N}\right\}$ is named Mahler base of the space $C(\mathbb{Z}_p \to \mathbb{C}_p)$, and the corresponding components $\{a_m : m \in \mathbb{N}\}$ in the function $f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n}$ are termed the Mahler coefficients of $f \in C(\mathbb{Z}_p \to \mathbb{C}_p)$.

The Mahler series expansion of the p-adic gamma function Γ_p and its Mahler coefficients are provided in Robert's book [24] as given below.

Proposition 1 *For $x \in \mathbb{Z}_p$, let $\Gamma_p(x+1) = \sum_{n=0}^{\infty} a_n \binom{x}{n}$ be Mahler series of Γ_p. Then its coefficients satisfy the following identity:*

$$\sum_{n\geq 0}(-1)^{n+1} a_n \frac{x^n}{n!} = \frac{1-x^p}{1-x}\exp\left(x+\frac{x^p}{p}\right). \tag{3.5}$$

We start by giving the following integral result.

Theorem 7 *We have*

$$\int_X \chi(y) q^{-\beta y}\Gamma_p(x+y+1)\,d\mu_{q^\alpha}(y) = \sum_{n=0}^{\infty}\frac{a_n}{\xi^n}\frac{D_{n,\chi,q}^{(\alpha,\beta)}(x|\xi)}{n!}, \tag{3.6}$$

where a_n is given by Proposition 1.

Proof. For $x, y \in \mathbb{Z}$, by the relation $\binom{x+y}{n} = \frac{(x+y)_n}{n!}$ and Proposition 1, we get

$$\xi^n\int_X \chi(y) q^{-\beta x}\Gamma_p(x+y+1)\,d\mu_{q^\alpha}(x) = \xi^n\int_X \chi(y) q^{-\beta x}\sum_{n=0}^{\infty} a_n \frac{(x+y)_n}{n!}\,d\mu_{q^\alpha}(x)$$
$$= \sum_{n=0}^{\infty} a_n \frac{\xi^n}{n!}\int_X \chi(y) q^{-\beta y}(x+y)_n\,d\mu_{q^\alpha}(y),$$

which is the desired result (3.6) via Eq. (2.8).

We now examine a special circumstance of Theorem 7.

Corollary 1 *As a special case of Theorem 7, the following relation holds true:*

$$\int_X \chi(x) q^{-\beta x}\Gamma_p(x+1)\,d\mu_{q^\alpha}(x) = \sum_{n=0}^{\infty}\frac{D_{n,\chi,q,\xi}^{(\alpha,\beta)}}{\xi^n}\frac{a_n}{n!}, \tag{3.7}$$

where a_n is provided by Proposition 1.

We here state the following theorem.

Theorem 8 *Let* $x, y \in \mathbb{Z}$. *We have*

$$\int_X \chi(x) q^{-\beta x}\Gamma_p(-x-y+1)\,d\mu_{q^\alpha}(x) = \sum_{n=0}^{\infty}\frac{a_n}{\xi^n}\frac{D_{n,\chi,q}^{(\alpha,\beta)}(y|\xi)}{n!},$$

where a_n is given by Proposition 1.

Proof. For $x, y \in \mathbb{Z}$, by the relation $\binom{-x-y}{n} = \frac{(-x-y)_n}{n!}$ and Proposition 1, we get

$$\int_X \chi(x) q^{-\beta x}\Gamma_p(-x-y+1)\,d\mu_{q^\alpha}(x) = \int_X q^{-\beta x}\chi(x)\sum_{n=0}^{\infty}\frac{\xi^n a_n}{n!\,\xi^n}(-x-y)_n\,d\mu_{q^\alpha}(x)$$
$$= \sum_{n=0}^{\infty}\frac{a_n}{\xi^n}\frac{1}{n!}\left(\xi^n\int_X \chi(x) q^{-\beta x}(-x-y)_n\,d\mu_{q^\alpha}(x)\right),$$

which is the desired result with Eq. (2.13).

A swift result of Theorem 8 is given below.

Corollary 2 *Setting* $y=0$ *in Theorem 8 gives the following relation:*

$$\int_X q^{-\beta x}\Gamma_p(-x+1)\chi(x)d\mu_{q^\alpha}(x)=\sum_{n=0}^{\infty}\frac{a_n}{\xi^n}\frac{D_{n,\chi,q,\xi}^{(\alpha,\beta)}}{n!},$$

where a_n is given by Proposition 1.

We give the following theorem.

Theorem 9 *For* $x,y\in\mathbb{Z}$, *we have*

$$\int_X \chi(y)q^{-\beta y}\Gamma_p'(x+y+1)d\mu_{q^\alpha}(y)=\sum_{n=0}^{\infty}\sum_{u=0}^{n-1}\frac{a_n}{\xi^n}\frac{(-1)^{n-u-1}}{n-u}\frac{D_{u,\chi,q}^{(\alpha,\beta)}(x|\xi)}{u!},$$

where a_n is given by Proposition 1.
Proof. In view of Proposition 1, we obtain

$$\xi^n\int_X \chi(y)\Gamma_p'(x+y+1)q^{-\beta y}d\mu_{q^\alpha}(y)=\xi^n\int_X\chi(y)\sum_{n=0}^{\infty}a_n\binom{x+y}{n}q^{-\beta y'}d\mu_{q^\alpha}(y)$$
$$=\sum_{n=0}^{\infty}a_n\xi^n\int_X\chi(y)\binom{x+y}{n}'q^{-\beta y}d\mu_{q^\alpha}(y)$$

and using (3.3), we derive

$$\xi^n\int_X\chi(y)q^{-\beta y}\Gamma_p'(x+y+1)d\mu_{q^\alpha}(x)=\sum_{n=0}^{\infty}\sum_{u=0}^{n-1}(-1)^{n-u-1}\frac{a_n}{n-u}\xi^n\int_X\chi(y)q^{-\beta y}\binom{x+y}{u}d\mu_{q^\alpha}(y)$$
$$=\sum_{n=0}^{\infty}\sum_{u=0}^{n-1}a_n\frac{(-1)^{n-u-1}}{n-u}\frac{D_{u,\chi,q}^{(\alpha,\beta)}(x|\xi)}{u!}.$$

We now provide the following theorem.

Theorem 10 *For* $x,y\in\mathbb{Z}$, *we attain*

$$\int_X\chi(y)\Gamma_p'(-x-y+1)q^{-\beta y}d\mu_{q^\alpha}(y)=\sum_{n=0}^{\infty}\sum_{u=0}^{n-1}\frac{(-1)^{n-u-1}}{\xi^n}\frac{a_n}{u!(n-u)}D_{u,\chi,q}^{(\alpha,\beta)}(x|\xi),$$

where a_n is stated in Proposition 1.
Proof. The proof method of this theorem is similar to that of Theorem 10. So, we omit the proof.

The immediate results of Theorems 9 and 10 are given as follows.

Corollary 3 *For* $x\in\mathbb{Z}$, *the following integral formulas holds:*

$$\int_X \chi(y)\Gamma'_p(y+1)q^{-\beta y}d\mu_{q^\alpha}(y)=\sum_{n=0}^{\infty}\sum_{u=0}^{n-1}\frac{(-1)^{n-u-1}}{\xi^n}\frac{a_n}{u!(n-u)}D_{u,\chi,q,\xi}^{(\alpha,\beta)}$$

and

$$\int_X \chi(y)\Gamma'_p(-y+1)q^{-\beta y}d\mu_{q^\alpha}(y)=\sum_{n=0}^{\infty}\sum_{u=0}^{n-1}\frac{(-1)^{n-u-1}}{\xi^n}\frac{a_n}{u!(n-u)}D_{u,\chi,q,\xi}^{(\alpha,\beta)}.$$

We now provide an interesting representation of the p-adic Euler constant utilizing the generalized twisted q-Daehee polynomials attached (α,β).

Theorem 11 *We have*

$$\gamma_p=\frac{\alpha}{q^{2\alpha-\beta}-q^{\alpha-\beta}}\left[\sum_{n=0}^{\infty}\sum_{u=0}^{n-1}\frac{a_n}{\xi^n}\frac{(-1)^{n-u-1}}{n-u}\frac{q^{(\alpha-\beta)d}D_{u,\chi,q}^{(\alpha,\beta)}(d-1|\xi)-D_{u,\chi,q}^{(\alpha,\beta)}(-1|\xi)}{u!}\right.$$
$$\left.-\frac{q^\alpha-1}{\alpha}\sum_{r=2}^{n-1}q^{(\alpha-\beta)r}\chi(r)\left(\Gamma'_p(r)+\frac{\Gamma_p^{(2)}(r)}{\log q^\alpha}\right)-\frac{q^\alpha-1}{\log q^\alpha}q^{(\alpha-\beta)2}\chi(2)\right].$$

Proof. Taking $f(x)=\chi(x)\Gamma'_p(x)$ in Eq. (2.2) and using $\chi(x+d)=\chi(x)$ yield the following result

$$\Psi=q^{(\alpha-\beta)d}\int_X q^{-\beta x}\Gamma'_p(x+d)\chi(x+d)d\mu_{q^\alpha}(x)-\int_X\chi(x)q^{-\beta x}\Gamma'_p(x)d\mu_{q^\alpha}(x) \tag{3.8}$$
$$=\frac{q^\alpha-1}{\alpha}\sum_{r=0}^{n-1}q^{(\alpha-\beta)r}\chi(r)\Gamma'_p(r)+\frac{q^\alpha-1}{\log q^\alpha}\sum_{r=0}^{n-1}q^{(\alpha-\beta)r}\chi(r)\Gamma_p^{(2)}(r),$$

which $\Gamma_p^{(2)}(r)$ is the second derivative of the *p*-adic gamma function at r. With some basic computations and using Theorem 9, we acquire

$$\Psi=q^{(\alpha-\beta)d}\int_X q^{-\beta x}\chi(x)\Gamma'_p(x+d)d\mu_{q^\alpha}(x)-\int_X\chi(x)q^{-\beta x}\Gamma'_p(x)d\mu_{q^\alpha}(x)$$
$$=q^{(\alpha-\beta)d}\sum_{n=0}^{\infty}\sum_{u=0}^{n-1}\frac{a_n}{\xi^n}\frac{(-1)^{n-u-1}}{n-u}\frac{D_{u,\chi,q}^{(\alpha,\beta)}(d-1|\xi)}{u!}-\sum_{n=0}^{\infty}\sum_{u=0}^{n-1}\frac{a_n}{\xi^n}\frac{(-1)^{n-u-1}}{n-u}\frac{D_{u,\chi,q}^{(\alpha,\beta)}(-1|\xi)}{u!}$$
$$=\sum_{n=0}^{\infty}\sum_{u=0}^{n-1}\frac{a_n}{\xi^n}\frac{(-1)^{n-u-1}}{n-u}\frac{q^{(\alpha-\beta)d}D_{u,\chi,q}^{(\alpha,\beta)}(d-1|\xi)-D_{u,\chi,q}^{(\alpha,\beta)}(-1|\xi)}{u!}$$

and

$$\Psi=\frac{q^\alpha-1}{\alpha}q^{\alpha-\beta}\Gamma'_p(1)+\frac{q^\alpha-1}{\alpha}\sum_{r=2}^{n-1}q^{(\alpha-\beta)r}\chi(r)\Gamma'_p(r)+\frac{q^\alpha-1}{\log q^\alpha}\sum_{r=1}^{n-1}q^{(\alpha-\beta)r}\chi(r)\Gamma_p^{(2)}(r)$$
$$\Psi=\frac{q^{2\alpha-\beta}-q^{\alpha-\beta}}{\alpha}\gamma_p+\frac{q^\alpha-1}{\alpha}\sum_{r=2}^{n-1}q^{(\alpha-\beta)r}\chi(r)\left(\Gamma'_p(r)+\frac{\Gamma_p^{(2)}(r)}{\log q^\alpha}\right)+\frac{q^\alpha-1}{\log q^\alpha}q^{(\alpha-\beta)2}\chi(2),$$

where $\Gamma'_p(1)=\gamma_p$, $\chi(1)=1$ and $\chi(0)=0$. Then, the proof is completed.

3. Conclusion

In the present paper, new generalizations of the q-Daehee polynomials, named as the generalized twisted q-Daehee polynomials with weight (α,β) and generalized twisted q-Daehee polynomials of the second kind with weight (α,β), have been considered. For the aforementioned polynomials, several novel and interesting formulas and relations associated with the generalized twisted q-Bernoulli polynomials with weight (α,β), Apostol type Stirling numbers of the second kind, and Stirling numbers of both kinds have been investigated. Moreover, two relationships for the two types of generalized twisted q-Daehee polynomials with weight (α,β) have been acquired. Then, many connections with the p-adic gamma function with its derivative via Mahler expansion for the aforesaid polynomials have been analyzed. Furthermore, a new explicit expression for p-adic Euler constant in view of the generalized twisted q-Daehee polynomials with weight (α,β) has been proved. Some results derived in the second section are generalizations of the results obtained in [2], [10], [11], [12], [17], [22] and [23]. Several results attained in the third section are generalizations of the results given and [6].

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Extension of Leap Condition in Approximate Stochastic Simulation Algorithms of Biological Networks

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Abstract

The stochastic simulation of the biological systems is the realization of the actual biological process by using some Monte Carlo approaches. The direct method, first reaction method and the next reaction method are three major methods in this area. Although these algorithms are successful to generate the systems exactly, they are computationally demanding for large systems. So, the approximate stochastic simulation algorithms (SSA) are the alternative approaches to generate the complex biological systems with a loss in accuracy by gaining from computational demand. The main idea of these methods is to provide a generation of the system via the leap-condition. Basically, this condition implies that the movement in the state of the system should be very slight for the selected time step in the sense that the hazard functions do not change considerably during the change in time from time t to t+τ . By this way, we can compute how many times each reaction can be realized in each small time interval so that we can move along the system's history axis from one time step to the next, instead of moving along from one reaction to the next. Hereby, in this study, we extend the underlying leap condition by deriving a realistic confidence interval for the selection of the number of reactions in every step of the system. For this purpose, we specifically deal with the poisson τ-leap and the Langevin methods as these are two fundamental approximate SSA in the literature, as well as the approximate Gillespie algorithm as the extension of them. We derive the maximum likelihood estimators and the moment estimators of the simulation parameters and construct confidence interval estimators at a given significance level for these three algorithms. From the derivations, we observe theoretically more precise intervals for the plausible values of parameters.

Keywords: Approximate stochastic simulation algorithms, leap condition, biological networks
Acknowledgement: The authors thank to the Scientific Research Project of Middle East Technical University (Project no: BAP-01-09-2017-002) for their support.

1. Introduction

In the biological systems, the stochastic simulation algorithms enable to realize the biological process by using Monte Carlo approaches. Accordingly, the chemical reactions occurring in the biological systems can be simulated numerically in the time evolution. At this point, discreteness and stochasticity may play important roles to describe the system realistically [6]. There are three main algoritmhs for stoshastically simulating the systems. These are the direct method (Gillespie Algorithm), the first reaction method and the next reaction method. Although they are successful in capturing the natural activation of the biological systems, they are computationally not efficient for large systems. Hence, the approximate stochastic simulation algorithm (SSA) are preferred in place of exact SSAs to decrease the computationally demand by losing accuracies of the simulated system. Basically, these approximate SSAs are based on the leap condition. This condition simply implies that the time step τ should be chosen in such a way that there is no significant change in the propensity function during the time change from t to $t+\tau$, i.e. $[t, t+\tau)$ [4]. In mathematically [5],

$$|h_j(Y+\bar{\lambda})(Y,\tau) - h_j(Y)| \leq \varepsilon h_0(Y),$$

where $h_0(Y) = \sum_{j=1}^{r} h_j(Y)$ is the sum of all hazard functions $h_j(Y)$ and ε denotes the error control parameter which presents

$$\bar{\lambda}(Y,\tau) = \sum_{j=1}^{r} [h_j(Y)\tau]v_j = \tau\xi(Y).$$

The above expression shows the expected net change in the state for the given time interval when the system has r members of reactions. In this equation, v_j represents the stoichiometric coefficients of the reaction j which corresponds to the jth row of the net effect matrix V.

In general, estimators provide to find $\hat{Y}$ the plausible value of an unknown population parameter via the observations or the given knowledge about the population. Hence, there are different types of estimators in the literature. Among alternates, in this study, we particularly focus on the maximum likelihood estimators(MLE) and maximum moment estimator(MME) as two well-known parametric estimation approaches in order to inter our model parameters k, the number of simultaneous reactions in the simulations and τ the time interval to realize k numbers of reactions. Simply, the likelihood function can be written as follows,

$$L_n(\theta) = L_n(\theta, y) = f_n(y, \theta),$$

where the observed data set is denoted by $y = (y_1, y_2, \ldots, y_n)$, associated with a vector $\theta = [\theta_1, \theta_2, \ldots, \theta_k]^{\mathsf{T}}$ of parameters that index the probability distribution within a parametric family$\{f(\cdot\,;\theta) \mid \theta \in \Theta\}$. thus, this is called the parameter space and $f_n(y;\theta)$ is the product of univariate density functions. Therefore, the aim of MLE is to find the values of the model parameters that maximize the likelihood function over the parameter space, i.e,

$$\hat{\theta} = \underset{\theta\in\Theta}{\arg\max}\, \widehat{L}_n(\theta\,; y).$$

In addition to MLE, on the other hand, MME can be described in the following way. Suppose that the problem is to estimate k unknown parameters $\theta_1, \theta_2, \ldots, \theta_k$ characterizing the distribution $f_W(w;\theta)$ of the random variable W. Also, assume that the first k moments of the true distribution, i.e, the "population moments", can be expressed as the functions of θ s via

$$\mu_1 \equiv \mathrm{E}[W] = g_1(\theta_1, \theta_2, \dots, \theta_k),$$
$$\mu_2 \equiv \mathrm{E}[W^2] = g_2(\theta_1, \theta_2, \dots, \theta_k),$$
$$\vdots$$
$$\mu_k \equiv \mathrm{E}[W^k] = g_k(\theta_1, \theta_2, \dots, \theta_k).$$

Accordingly, if a sample of size n is drawn, resulting in the values $w_1, \dots, w_n,\ j = 1, \dots, k$, the estimated mean of these values μ_j can be found by

$\widehat{\mu}_j = \frac{1}{n}\sum_{i=1}^{n} w_i^j$ as the j-th sample moment. On conclusion, the method of moments estimator for $\theta_1, \theta_2, \dots, \theta_k$ denoted by $\widehat{\theta}_1, \widehat{\theta}_2, \dots, \widehat{\theta}_k$ can be defined as the solution to the equations:

$$\widehat{\mu}_1 = g_1(\widehat{\theta}_1, \widehat{\theta}_2, \dots, \widehat{\theta}_k),$$
$$\widehat{\mu}_2 = g_2(\widehat{\theta}_1, \widehat{\theta}_2, \dots, \widehat{\theta}_k),$$
$$\vdots$$
$$\widehat{\mu}_k = g_k(\widehat{\theta}_1, \widehat{\theta}_2, \dots, \widehat{\theta}_k).$$

Additionally, a confidence interval gives an estimated range of values which is likely to include an unknown population parameter. This estimated range is calculated from a given set of sample data.

Hereby, in this study, we aim to construct the confidence intervals for the population parameters k and τ int he two well-knowns approximate SSAs, namely, the poisson tau-leap and the langevin tau-leap methods, with one of the recent approach, called the approximate Gillespie method, by using MLE and MME approaches. All these approaches are based on the leap condition. In the current literature, the k and τ in these three simulation approaches have been used via a conservative one-sided confidence interval without controlling the significance level α. Hence, this study suggests realistic and accurate confidence intervals for both parameters by controlling α and by using the MLE and MME of the modal parameters so that narrower and more accurate confidence intervals can be obtained theoretically. Thereby, in the organization of the paper, we present the selected three major approximate SSAs in Section 2. We present our confidence intervals inserted to the leap condition of the underlying three approximate SSAs in Section 3. Finally, we conclude our findings in Section 4.

2. Approximate Stochastic Simulation Algorithms

Although the SSA is exact and it gives more accurate result, it is slow and computationally costly. By the approximate SSA [6], it is possible to gain from computational demand. Because if we control how many times each reaction can be realized in each subinterval, i.e, the leap, it is possible to proceed with one step to the next step rather than the reaction to the next in the history axis of the system. The smaller the intervals are chosen, the more accurate solution we get. Nonetheless, choosing a larger time interval provides approximation solution by gaining the computational efficiency. In this part, we present the Poisson τ-leap, Langevin method and the approximation Gillespie algorithm before we describe our extensions in their derivations.

2.1. Poisson τ-leap Method

In this method, under the leap condition, firstly, for each reaction channel R_j, a random value k_j from a Poisson random value $Poi(h_j(Y)\tau)$ is generated in the time interval [t, t+τ], where $Y(t) = Y$ is a state vector. Then, an admissible τ is found by putting boundary for the difference between

$$|h_j(Y + \lambda(Y)) - h_j(Y)|,$$

where $\lambda(Y) = \sum_{j=1}^{r} k_j v_j$ shows the net change in the state of the system in [t,t+τ]. Since $k_j \sim Poi(h_j(Y)\tau)$, $E(k_j) = h_j(Y)\tau$ and

$$\bar{\lambda}(Y,\tau) = \sum_{j=1}^{r} [h_j(Y),\tau] v_j = \tau\xi(Y), \tag{1}$$

which is the expected net change in the state for the given time interval. Here v_j shows the stoichiometric coefficients of the reaction j corresponding to the jth row of the net effect matrix V as stated before hand and $h_j(Y)$ is the hazard function of the jth reaction that is found by the product of the rate constant c_j and distinct molecular reactant combination of underlying reaction. Then , $\xi(Y) = \sum_{j=1}^{r} h_j v_{ij}$ can be denoted as the mean or expected state change in a unit of time by an n-dimensional vector where each ith component, $\xi_j(Y)$, corresponds to the expected change of the ith species in an unit of time. Then, we get the inequality

$$|h_j(Y + \bar{\lambda})(Y,\tau) - h_j(Y)| \leq \varepsilon h_0(Y) \tag{2}$$

by using $\bar{\lambda}(Y,\tau)$ in Equation (1). This means that there is a limitation for the expected changes in a hazard functions in the time τ, by a fraction ε, error control parameter, lying ($0 < \varepsilon < 1$), of the sum of all hazard functions $h_0(Y) = \sum_{j=1}^{r} h_j(Y)$. In fact, this inequality (2) is a leap condition. Accordingly, by using the first order Taylor expansion and applying Equation (2), we get

$$h_j(Y + \bar{\lambda})(Y,\tau) - h_j(Y) \approx \bar{\lambda})(Y,\tau) h_j(Y) = \sum_{i=1}^{n} \tau\xi_i(Y)\frac{\partial h_j(Y)}{\partial Y_i}. \tag{3}$$

Then, letting $b_{ji} = \frac{\partial h_j(Y)}{\partial Y_i}$ $(i = 1,...,n; j = 1,...,r)$, the following inequality can be obtained.

$$\tau|\sum_{i=1}^{n} \xi_i(Y) b_{ji}| \leq \varepsilon h_0(Y).$$

Therefore, the largest value of τ satisfying the leap condition for the given Y and the preselected ε is calculated by

$$\tau = min\{\frac{\varepsilon h_0(Y)}{|\sum_{i=1}^{n} \xi_i(Y) b_{ji}(Y)|}\}. \tag{4}$$

From Equation (4), it can be still preferable to use the exact time step of SSA when the obtained value of τ is a favorable choice of the leap size. Since the SSA gives $\tau = \frac{1}{h_0(Y)}$, the obtained τ in Equation (4) would not be preferred if $\tau \leq \frac{1}{h_0(Y)}$. However, in terms of the computational cost, the time interval in the Poisson τ-leap method is more demanding than the time of SSA. Notably, the differences are small.

Finally, updating the current state in the Poisson τ-leap method is done by replacing t by $t := t+\tau$ and Y that is necessary to determine the largest value of τ and is compatible with the leap condition.
On the other hand, although it can successfully simulate the systems in majority of cases, it also is possible to generate negative molecular populations. To avoid the negative populations, some alternative approaches have been suggested and the Binomial τ-leap is one of the well-known alternative approaches of Poisson τ-leap to solve the negativity problems.

2.2. Langevin τ-leap Method

Basically, in this method, the number of times of execution of the jth reaction k_j can be found

$$k_j \sim Poi(h_j(Y)\tau) \approx N(h_j(Y)\tau, h_j(Y)\tau)$$

when $j = 1,...,r$ under the condition that it is possible to leap down the history axis of the system by the time step τ with a very large number of reaction and satisfying the leap condition. Since there is no much change that affects the hazard functions $h_j(Y)$'s $(j = 1,...,r)$, the Poisson distribution can be written as a normal distribution with the same mean and variance. Thus, the steps of the Langevin τ-leap method can be shortly described as below:
Firstly, the τ value satisfying leap condition is chosen and $\tau \gg max\{\frac{1}{h_j(Y)}\}$. Secondly, from normal distribution with mean zero and unit variance, $l_j \sim N(0,1)$, a sample value k_j is chosen and put $k_j = h_j(Y)\tau + (h_j(Y)\tau)^{1/2}k_j$ for each $j, (j = 1,...,r)$. Then, the net change in the state $\lambda(Y)$ is calculated by $\lambda(Y) = \sum_{j=1}^{r} k_j v_j$, where v_j is the n-dimensional net effect vector of the jth reaction in n-species in the system. Finally, replacing $Y := Y + \lambda(Y)$ and $t := t+\tau$ the system is updated.

2.3. Approximate Gillespie Algorithm

It is an alternative approximation method of the Gillespie algorithm. In this method, the k number of reactions is executed rather than a single reaction at a time. Here, the k reactions are developed by the Gamma distribution with a parameter $\sum_{j=1}^{r} h_j(Y)$ as each reaction occurs in an exponential time step t. It is also denoted by $\tau \sim \Gamma(k, h_0(Y))$, where τ indicates the time interval of k reactions in the total hazard, $h_0(Y)$, $h_0(Y) := \sum_{j=1}^{r} h_j(Y)$. In the system, the update is performed by replacing t by $t := t+\tau$ and by changing the current state Y by $Y := Y + \lambda(Y)$, where the net change in the state is found via $\lambda(Y) = \sum_{j=1}^{r} k_j v_j$. In this expression, v_j is the net effect of the jth reaction by showing the jth row of the net effect matrix V, as previously.
The total number of reactions during the interval τ is chosen with an assumption that the required time for every reaction is equivalent to that of Gillespie. To choose a suitable k giving a good approximation, there are two proposals, namely, fixed k and varying k approaches.

Fixed k:

Firstly, a fixed k is chosen for the overall simulation and then, by the two-sided Kolmogorov-Smirnov test, the accuracy of the results is controlled by the given k.
Varying k:

A k satisfying the leap condition is described in each time step when the change in hazard function $\Delta h_j(Y),(j=1,...,r)$ is approximated by the first order Taylor expansion in the time interval $[t,t+\tau]$ in a such way that the following equality can be obtained.

$$\Delta h_j(Y) = h_j(Y+\lambda(\bar{Y},\tau)) - h_j(Y) \approx \lambda(\bar{Y},\tau)h_j(Y) = \sum_{i=1}^{n} \lambda(\bar{Y},\tau)\frac{\partial h_{ij}(Y)}{\partial Y_i} \tag{5}$$

in which the expected change in the state by considering k simultaneous reaction is

$$\lambda(\bar{Y},\tau) = Y(t+\tau) - Y(t) = \sum_{j=1}^{r} k_j v_j$$

and k_j is the number of times of the jth reaction, vj is the net effect of the jth reaction by showing the jth row of the net effect matrix V. With a gamma distribution, we have $\tau \sim \Gamma(k,h_0(Y))$ where $k = <\tau> .h_0(Y)$. In this expression, $<\tau>$ shows τ on average.
Substituting Equation (5), we can obtain

$$\Delta h_j(Y) \approx \sum_{j=1}^{r} f_{jj'}(Y)\tau h_0(Y),$$

where the total change in hazard of the reaction j' is described in terms of $f_{jj'}$ via

$$f_{jj'} = \sum_{i=1} \frac{\partial h_j(Y)}{\partial Y_i} V_{ij}$$

for the execution of the reaction j'.
In order to compute the confidence interval, the following expression is written as

$$\Delta h_j(Y) \approx E(\Delta h_j(Y)) \pm \sqrt{Var(\Delta h_j(Y))}. \tag{6}$$

Then, the statistics for $\Delta h_j(Y)$ can be shown by

$$E(\Delta h_j(Y)) \approx \sum_{j=1}^{r} f_{jj'}(Y)E(\tau)h_0(Y) = \sum_{j=1}^{r} f_{jj'}(Y)\frac{k}{h_0(Y)}h_0(Y) = k\sum_{j=1}^{r} f_{jj'}(Y) \tag{7}$$

and

$$Var(\Delta h_j(Y)) \approx \sum_{j=1}^{r} f_{jj'}^2(Y)Var(\tau)h_0(Y) = \sum_{j=1}^{r} f_{jj'}^2(Y)\frac{k}{h_0(Y)}h_0(Y) = k\sum_{j=1}^{r} f_{jj'}^2(Y). \tag{8}$$

By substituting Equation (7) and (8) into the required leap condition, the below expression can be found.

$$|k|\sum_{j'=1}^{r} f_{jj'}(Y) \leq \varepsilon h_0(Y)$$

and

$$\sqrt{(}k\sum_{j'=1}^{r} f_{jj'}^2(Y)) \leq \varepsilon h_0(Y)$$

Finally, the optimal k is computed from

$$k = \min_{j\in[1,r]} \lfloor \frac{\varepsilon h_0(Y)}{\sum_{j'=1}^{r} f_{jj'}(Y)}, \frac{\varepsilon^2 h_0^2(Y)}{\sum_{j'=1}^{r} f_{jj'}^2(Y)} \rfloor \tag{9}$$

3. Confidence Intervals for Leap Condition

As described previously, we derive MLE and MME methods to infer the model parameters of approximate SSA, which are k and τ vie the confidence intervals. Hence, while $\tau \sim Poi(k)$, the MLE of k is found as τ. Then, by inserting it into $\Delta h_j(Y)$, we get

$$\Delta h_j(Y) = \sum_{j=1} f_{jj'}(Y)\tau$$

We know that with the value of $z = 1$, similar to Equation (7),

$$\Delta h_j(Y) \approx E(\Delta h_j(Y)) \pm \sqrt{Var(\Delta h_j(Y))}$$

Since $\tau \sim Poi(k)$, the mean of the value τ is $E(\tau) = k$ and the variance of the value τ is found as $Var(\tau) = k$. Thus, we can obtain the following equalities for the approximate values of $E(\Delta h_j(Y))$ and $Var(\Delta h_j(Y))$, respectively.

$$E(\Delta h_j(Y)) \approx \sum_{j=1}^{r} f_{jj'}(Y)E(\tau) = \sum_{j=1}^{r} f_{jj'}(Y)k = k\sum_{j=1}^{r} f_{jj'}(Y) \tag{10}$$

$$Var(\Delta h_j(Y)) \approx \sum_{j=1}^{r} f_{jj'}^2(Y)Var(\tau) = \sum_{j=1}^{r} f_{jj'}^2(Y)k = k\sum_{j=1}^{r} f_{jj'}^2(Y) \tag{11}$$

After substituting Equation (10) and (11) into the required leap condition, a suitable k can be derived as

$$k = \min_{j\in[1,r]} \lfloor \frac{\varepsilon h_0(Y)}{\sum_{j'=1}^{r} f_{jj'}(Y)}, \frac{\varepsilon h_0(Y)}{\sum_{j'=1}^{r} f_{jj'}^2(Y)} \rfloor \tag{12}$$

It can be observed that the value of $\frac{\varepsilon h_0(Y)}{\sum_{j'=1}^{r} f_{jj'}^2(Y)}$ is smaller than the value of $\frac{\varepsilon^2 h_0^2(Y)}{\sum_{j'=1}^{r} f_{jj'}^2(Y)}$ in Equation (9) .

Secondly, by applying MME, the estimator of the value k is obtained as $k = \frac{\tau}{n}$. Accordingly, the mean of $\Delta h_j(Y)$, $E(\Delta h_j(Y))$ and the variance of $\Delta h_j(Y)$, $Var(\Delta h_j(Y))$ are calculated approximately as the following way.

$$\tag{13}$$

$$Var(\Delta h_j(Y)) \approx \sum_{j=1}^{r} f_{jj'}^2(Y)\frac{E(\tau)}{n} = \sum_{j=1}^{r} f_{jj'}^2(Y)\frac{k}{n} = \frac{k}{n}\sum_{j=1}^{r} f_{jj'}^2(Y) \tag{14}$$

Similar to the process of using MLE, after substituting equation (13) and (14) into the required leap condition, a suitable k value can be obtained from

$$k = \min_{j\in[1,r]} \lfloor \frac{\varepsilon h_0(Y)n}{|\sum_{j'=1}^{r} f_{jj'}(Y)|}, \frac{\varepsilon h_0(Y)n}{|\sum_{j'=1}^{r} f_{jj'}^2(Y)|} \rfloor \tag{15}$$

In order to find k, we also construct the confidence interval.

$k = \tau \pm z_{\frac{\alpha}{2}} \sqrt{\frac{\tau}{n}}$, in this inequality $z_{\frac{\alpha}{2}}$ denotes the tabulated normal value for the significance level α. Then, substituting this expression into $\Delta h_j(Y)$,we get that

$$\Delta h_j(Y) = \sum_{j'=1}^{r} (\tau \pm z_{\frac{\alpha}{2}} \sqrt{\frac{\tau}{n}}).$$

For this statement, $E(\Delta h_j(Y))$ and $Var(\Delta h_j(Y))$ are calculated by using the assumption of $E(\sqrt{\tau}) = \sqrt{k}$ via

$$\begin{aligned} E(\Delta h_j(Y)) &\approx \sum_{j=1}^{r} f_{jj'}(Y) E(\tau \pm z_{\frac{\alpha}{2}} \sqrt{\frac{\tau}{n}}) \\ &= \sum_{j=1}^{r} f_{jj'}(Y)[E(\tau) \pm \frac{z_{\frac{\alpha}{2}}}{\tau} E(\sqrt{\tau})] \\ &= k \pm z_{\frac{\alpha}{2}} \frac{\sqrt{k}}{\sqrt{n}} \sum_{j=1}^{r} f_{jj'}(Y) \end{aligned} \tag{16}$$

$$\begin{aligned} Var(\Delta h_j(Y)) &\approx \sum_{j=1}^{r} f_{jj'}^2(Y) E(\tau \pm z_{\frac{\alpha}{2}} \sqrt{\frac{\tau}{n}}) \\ &= \sum_{j=1}^{r} f_{jj'}^2(Y)[Var(\tau) \pm \frac{z_{\frac{\alpha}{2}}}{\tau} Var(\sqrt{\tau})] \\ &= k \pm z_{\frac{\alpha}{2}} \frac{\sqrt{k}}{\sqrt{n}} \sum_{j=1}^{r} f_{jj'}^2(Y) \end{aligned} \tag{17}$$

Then, by inserting the Equation (16) into the required leap condition, the following inequality is obtained.

$$\left| [(\sqrt{k} + \pm \frac{z_{\frac{\alpha}{2}}}{2\sqrt{n}})^2 + \frac{z_{\frac{\alpha}{2}}}{2\sqrt{n}}] \sum_{j=1}^{r} f_{jj'} \right| \leq \varepsilon h_0(Y)$$

Then ,we can get this inequality for the value of k

$$k \leq (\sqrt{\frac{\varepsilon h_0(Y)}{\sum_{j=1}^{r} f_{jj'}/Y} \pm \frac{z_{\frac{\alpha}{2}}}{2\sqrt{n}})^2} \mp \frac{z_{\frac{\alpha}{2}}}{2\sqrt{n}})^2$$

Applying the same process for Equation (17), a suitable k can be found by

$$k = \min_{j\in[1,r]} \lfloor (\sqrt{\frac{\varepsilon h_0(Y)}{\sum_{j=1}^{r} f_{jj'}(Y)} \pm \frac{z_{\frac{\alpha}{2}}}{2\sqrt{n}})^2} \mp \frac{z_{\frac{\alpha}{2}}}{2\sqrt{n}})^2, (\sqrt{\frac{\varepsilon h_0(Y)}{\sum_{j=1}^{r} f_{jj'}^2(Y)} \pm \frac{z_{\frac{\alpha}{2}}}{2\sqrt{n}})^2} \mp \frac{z_{\frac{\alpha}{2}}}{2\sqrt{n}})^2 \rfloor \tag{18}$$

Also, the confidence interval for the value of k estimated from MME is obtained as $k = \frac{\tau}{n} \pm 2\sqrt{\frac{\tau}{n^2}}$ while taking $z = 1$. If we perform the MLE approach, the suitable k can be derived as

$$k = \min_{j \in [1,r]} \lfloor |(\frac{\varepsilon h_0(Y)n}{\sum_{j=1}^{r} f_{jj'}(Y)} \pm 1)^2 \mp 1|, |(\frac{\varepsilon h_0(Y)n}{\sum_{j=1}^{r} f_{jj'}^2(Y)} \pm 1)^2 \mp 1| \rfloor \tag{19}$$

In addition to these confidence interval for the value of k, there are different approaches for the confidence intervals for the mean of the Poisson distribution [7]. The first approach is based on the normal approximation.

Thus from Equation (18), the lower and the upper bound of the value of k, denoted by k_l and k_u, can be written by following expression.

$$\begin{aligned} k_l &= \tau - \frac{1}{2} + \frac{1}{2} z_{1-\alpha/2}^2 + z_{1-\alpha/2}\sqrt{\tau - \frac{1}{2} + \frac{1}{4} z_{1-\alpha/2}^2}, \\ k_u &= \tau + \frac{1}{2} + \frac{1}{2} z_{1-\alpha/2}^2 + z_{1-\alpha/2}\sqrt{\tau + \frac{1}{2} + \frac{1}{4} z_{1-\alpha/2}^2}. \end{aligned} \tag{20}$$

Hereby, similar to previous approaches, by inserting Equation (20) into the necessary places, $E(\Delta h_j(Y))$ and $Var(\Delta h_j(Y))$ are calculated by the properties of $E(\tau) = k$ and $Var(\tau) = k$. Thus,

$$E(\Delta h_j(Y)) \approx \sum_{j'=1}^{r} f_{jj'}(k \pm \frac{1}{2} + \frac{1}{2} z_{1-\alpha/2}^2 + z_{1-\alpha/2} E(\sqrt{\tau \pm \frac{1}{2} + \frac{1}{4} z_{1-\alpha/2}^2})). \tag{21}$$

$$Var(\Delta h_j(Y)) \approx \sum_{j'=1}^{r} f_{jj'}^2(k \pm \frac{1}{2} + \frac{1}{2} z_{1-\alpha/2}^2 + z_{1-\alpha/2} Var(\sqrt{\tau \pm \frac{1}{2} + \frac{1}{4} z_{1-\alpha/2}^2})). \tag{22}$$

In order to simplify the above expression, we can define

$$A(k) = E(\sqrt{\tau \pm \frac{1}{2} + \frac{1}{4} z_{1-\alpha/2}^2}) = Var(\sqrt{\tau \pm \frac{1}{2} + \frac{1}{4} z_{1-\alpha/2}^2}).$$

since under the poisson distribution, the expected value is the same as the value of the variance. Then, by substituting Equation (21) and (22) into the required leap condition, we get the following inequalities

$$\begin{aligned} k + z_{1-\alpha/2} A(k) &\leq \frac{\varepsilon h_0(Y)}{|\sum_{j'=1}^{r} f_{jj'}(Y)|} \pm \frac{1}{2} - \frac{1}{2} z_{1-\alpha/2}^2, \\ k + z_{1-\alpha/2} A(k) &\leq \frac{\varepsilon h_0(Y)}{|\sum_{j'=1}^{r} f_{jj'}^2(Y)|} \pm \frac{1}{2} - \frac{1}{2} z_{1-\alpha/2}^2. \end{aligned} \tag{23}$$

Let we choose α as $z_{\alpha/2} = 1$ and as $-z_{\alpha/2} = z_{1-\alpha/2}$, we can write the following inequality for the suitable value of k via

$$k \leq \min_{j \in [1,r]} \lfloor \frac{\varepsilon h_0(Y)}{|\sum_{j'=1}^{r} f_{jj'}(Y)|} - \frac{1}{4}, \frac{\varepsilon h_0(Y)}{|\sum_{j'=1}^{r} f_{jj'}^2(Y)|} - \frac{1}{4} \rfloor. \tag{24}$$

Moreover, the confidence interval via the lower and upper bound of k with a normal approximation can be written more simply as [7].

$$k_l, k_u = \tau \pm z_{1-\alpha/2}\sqrt{\tau}.$$

Repeating the same procedure for this value k's, and choosing α such that $z_{\alpha/2} = 1$, the value of k can obtained as

$$k \leq \min_{j\in[1,r]} \lfloor (\sqrt{\frac{\varepsilon h_0(Y)}{|\sum_{j'=1}^{r} f_{jj'}(Y)|} + \frac{1}{4}} \pm \frac{1}{2})^2, (\sqrt{\frac{\varepsilon h_0(Y)}{|\sum_{j'=1}^{r} f_{jj'}^2(Y)|} + \frac{1}{4}} \pm \frac{1}{2})^2 \rfloor. \tag{25}$$

Lastly, an improved normal approximation for the confidence interval gives us the following k with k_l and k_u [7].

$$k_l, k_u = \tau + \frac{2z_{\alpha/2}^2 + 1}{6} \pm (\frac{1}{2} + \sqrt{2z_{\alpha/2}^2(\tau \pm \frac{1}{2} + \frac{2z_{\alpha/2}^2 + 1}{18}}))).$$

Finally, by implemeting a similar process as before and choosing α such that $z_{\alpha/2} = 1$, an appropriate value of k can be presented as

$$k + B(k) \leq \min_{j\in[1,r]} \lfloor \frac{\varepsilon h_0(Y)}{|\sum_{j'=1}^{r} f_{jj'}(Y)|} - \frac{1}{4}, \frac{\varepsilon h_0(Y)}{|\sum_{j'=1}^{r} f_{jj'}^2(Y)|} - \frac{1}{4} \rfloor, \tag{26}$$

where $B(k) = E(\sqrt{(}\tau \pm \frac{1}{2} + \frac{z_{\alpha/2}^2 + 2}{18})).$

4. Conclusion

SSAs are important to show the simulation of the reactions in the biological systems. While they give the exact solution, they are not computationally efficient. Thus, the approximate SSAs can be preferable in order to gain from computational demand. These approximate methods are simply based on the leap condition. This condition implies that the time step τ should be selected so that the propensity function do not change during the time interval $[t, t+\tau]$. In addition to these, it is possible to compute how many times each reaction can be realized in each small time interval so that we can move along the system's history axis from one time step to the next, instead of moving along from one reaction to the next.
In this work, we have used the poisson τ-leap [4] and the Langevin methods [4] as these are two fundamental approximate SSA in the literature, as well as the approximate Gillespie algorithm [5] as the extension of them. By deriving MLE and MME for the parameter k, we have constructed the confidence interval by using this value. Thus, we have extented into the leap condition to have a suitable value of k. We have gotten new confidence intervals for the parameter k which have narrower range.

5. Future Work

As the future study, we consider to derive other types of estimators such as Bayesian and maximum a posterior approaches so that we can obtain theoretically narrower confidence interval for the optimal k in the underlying approximation methods.

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Fault-Tolerant Metric Dimension of 4-th Power of Paths

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Abstract

For a simple connected graph $G = (V(G), E(G))$, a subset $R$of $V(G)$is said to be a resolving set of G if everypair of vertices of G are resolved by some vertices in R i.e., every pair of vertices of G are identified uniquelyby some vertex elements in R. A resolving set of G containing the minimum number of vertices is the metricbasis and the minimum cardinality of the metric basis is called the metric dimension of G. A resolving set$F$ for the graph G is said to be fault tolerant if for each$u \in F$, $F\backslash\{u\}$is also a resolving set for G andthe minimum cardinality of the fault-tolerant resolving set said to be the fault-tolerant metric dimension. In this article, we determine the exact value of fault-tolerant metric dimension of P_n^4 for $n \geq 26$.

Keywords: Resolving set, Fault-tolerant resolving set, Fault-tolerant metric dimension.

1. Introduction

For a simple, undirected, finite and a connected graph G whose set of vertices are denoted by $V(G)$ and the set of edges by $E(G)$, we represent the distance between any two arbitrary vertices say u, v by $d(u,v)$. The distance $d(u,v)$ is considered as the length of the shortest path from the vertex u to the vertex$v$. A vertex u is said to resolve a set of vertices S if $d(u,v) \neq d(u,w)$ for every pair of vertices v and w in$S$. A set $W = \{w_1, w_2, \ldots, w_m\}$ of vertices is said to be the resolving set of G if $code_W(u) \neq code_W(v)$ for every pair of distinct vertices $u$and v of G, where$code_W(x) = \big(d(x,w_1), d(x,w_2), \ldots d(x,w_m)\big)$represent the code of the vertex $x \in V(G)$. The metric basis for a graph G is the resolving set of G containing the minimum number of vertices. The minimum cardinalityof a resolving set W is called the metric dimension of G and denoted by $\beta(G)$.

The concept of metric dimensions was first instigated by Slater [1], Harry and Melter [2]. The metric basis$\beta(G)$ is the minimum cardinality of the resolving set. Elements in the basis were considered as sensors in an application given in [5]. If one of the censors does not work properly, we will not have enough information to deal with the intruder (fire, thief etc). In order to over-come this kind of problems, Hernando et al. introduced concept of fault-tolerant metric dimension in [7]. This concept is defined as follows : A resolving set F of a graphs G is fault- tolerant if $F\backslash\{v\}$ is also a resolving set, for every vertex $v \in F$. The fault-tolerant metric dimension of$G$, denoted by by $\beta'(G)$is the minimum cardinality of a fault-tolerant resolving set. A fault-tolerant resolving set of order by $\beta'(G)$ is called a fault-tolerant metric basis. The t-th power of a connected graph G, denoted by G^t, is the graph on the same vertex set as G and two vertices $u$and $v$are adjacent in G^t if $d_G(u,v) \leq t$in G.

The problem of determining the fault-tolerant metric dimension is NP hard problem and results are known only for some classes of graphs. Hernendo et al. [7] characterized all fault tolerant resolving sets forany tree T . Also in article they have shown the relation$\beta'(G) \leq \beta(G)\left(1 + 2 \cdot 5^{\beta(G)-1}\right)$for every graphs$G$. For a cycle$C_n$, the fault-tolerant metric dimension has been determined by Javaid et al. in [11] as $\beta'(C_n) = 3$. Basak at el. [12] determine the fault-tolerant metric dimension ofC_n^3. In this paper, we studythe fault-tolerant metric dimension of 4-th power of pathsP_n^4 and determine the exact value of fault-tolerant metricdimension of $P_n^4$for$n \geq 26$.

Henceforth we denote the vertex set $V(P_n)$ by $\{v_0, v_1, \dots, v_{n-1}\}$ and hence degree of v_l for $l \in \{1, 2, \dots, n-2\}$, whereas both the vertices v_0 andv_{n-1} has degree one. The following proposition is true P_n^4.

Proposition 1.1. *The distance between two vertices v_i and v_j in P_n^4 is given by $d(v_i, v_j) = \left\lceil \frac{|i-j|}{4} \right\rceil$and thediameter of P_n^4.*

The following lemma gives a basic property of a fault-tolerant resolving set for an arbitrary graph.

Lemma 1.*[11]* A set F of vertices is a fault-tolerant resolving set of G if and only if every pair of vertices in G is resolved by at least two vertices of F.

In next section we show that the value of $\beta'(P_n^4) = 8$ for all integer$n \geq 26$. Our method involves thedetermination of lower bound of $\beta'(P_n^4)$ as 8 and then searching an optimal fault-tolerant resolving set of size 8.

2. Fault-tolerant metric dimension of P_n^4

This section deals with the determination of the fault-tolerant metric dimension of P_n^4 for $n \geq 26$.Firstwe give a lower bound for $\beta'(P_n^4)$ and then determine fault tolerant metric bases. We need following results to determine a lower bound for $\beta'(P_n^4)$.

Lemma 2. *Let v_j resolves two consecutive vertices v_a and v_{a+1}. Then*
(a) $j \equiv a \ (mod\ 4)$with $j \leq a$
(b) $j \equiv a + 1 \ (mod\ 4)$ with $j \geq a + 1$.

Proof :
(a) Let $v_j \in V(P_n^4)$ with $j \leq a$. Then the distances of two vertices v_a and v_{a+1} from v_j are $\left\lceil \frac{a-j}{4} \right\rceil$ and $\left\lceil \frac{a-j+1}{4} \right\rceil$, respectively. Let $a - j = 4q + r$ where $0 \leq r \leq 3$. Then $\left\lceil \frac{a-j}{4} \right\rceil \neq \left\lceil \frac{a-j+1}{4} \right\rceil$ implies$\left\lceil \frac{4q+r}{4} \right\rceil \neq \left\lceil \frac{4q+r+1}{4} \right\rceil$ and this is true only when $r = 0$. Therefore $j \equiv a \ (mod\ 4)$ with$j \leq a$.

(b) Let $v_j \in V(P_n^4)$ with $j \geq a+1$ i.e., v_j be a right side vertex of v_{a+1}. Then the distances of two verticesv_a and v_{a+1} from v_j are $\left\lceil\frac{j-a}{4}\right\rceil$ and $\left\lceil\frac{j-a-1}{4}\right\rceil$, respectively. Let $j-a-1=4q+r$, where $0 \leq r \leq 3$. Then$\left\lceil\frac{j-a}{4}\right\rceil \neq \left\lceil\frac{j-a-1}{4}\right\rceil$ implies $\left\lceil\frac{4q+r+1}{4}\right\rceil \neq \left\lceil\frac{4q+r}{4}\right\rceil$ and this is true only when $r=0$. Therefore $j \equiv a+1 \ (mod\ 4)$ with $j \geq a+1$.

Lemma 3. *Let F be a fault-tolerant resolving set of P_n^4 such that $F \cap \{v_i, v_{i+1}, v_{i+2}\} = \emptyset$ for some $i \in \{1,2,\dots,n-4\}$. Then $|F| \geq 8$.*

Proof : Denote $R_{a,a+1}$ be the set of all vertices which resolve the vertices $v_a$and v_{a+1}. Let $i \equiv r \ (mod\ 4)$, for some $r \in \{0,1,2,3\}$. Then Using Lemma 2,

$$R_{i-1,i} = \{v_j : j \leq i-1, j \equiv r \ (mod\ 4)\} \cup \{v_j : j \geq i, j \equiv r \ (mod\ 4)\};$$
$$R_{i,i+1} = \{v_j : j \leq i, j \equiv r \ (mod\ 4)\} \cup \{v_j : j \geq i+1, j \equiv r+1 \ (mod\ 4)\};$$
$$R_{i+1,i+2} = \{v_j : j \leq i+1, j \equiv r+1 \ (mod\ 4)\} \cup \{v_j : j \geq i+2, j \equiv r+2 \ (mod\ 4)\};$$
$$R_{i+2,i+3} = \{v_j : j \leq i+2, j \equiv r+2 \ (mod\ 4)\} \cup \{v_j : j \geq i+3, j \equiv r+3 \ (mod\ 4)\}.$$

Since $F \cap \{v_i, v_{i+1}, v_{i+2}\} = \emptyset$, $(F \cap R_{a,a+1}) \cap (F \cap R_{b,b+1}) = \emptyset$ for distinct $a, b \in \{i-1, i, i+1, i+2\}$. Again from Lemma 1, $|F \cap R_{a,a+1}| \geq 2$ for every a. Now we may write $|F| \geq \sum_{t=i-1}^{i+2} |F \cap R_{t,t+1}| \geq 8$ as $(F \cap R_{a,a+1}) \cap (F \cap R_{b,b+1}) = \emptyset$. Hence we obtain the result.

Theorem 2.1. *For a path $P_n$with $n \geq 26, \beta(P_n^4) \geq 8$.*

Proof : Let F be a fault-tolerant resolving set of P_n^4. If $F \cap \{v_i, v_{i+1}, v_{i+2}\} = \emptyset$ for some $i \epsilon \{1,2,\dots,n-4\}$, then using Lemma 3, we have $|F| \geq 8$. Now assume that $F \cap \{v_i, v_{i+1}, v_{i+2}\} = \emptyset$ for every $i \in \{1,2,\dots,n-4\}$. Since $n \geq 26$, we have $F \cap \{v_i, v_{i+1}, v_{i+2}\} \neq \emptyset$ for $i \in \{1,4,7,10,13,16,19,21\}$ and hence $|F| \geq 8$ as$\{v_a, v_{a+1}, v_{a+2}\} \cap \{v_b, v_{b+1}, v_{b+2}\} = \emptyset$ for $b = a+3t$ where $t \geq 1$. Therefore, in this case we also have$|F| \geq 8$.Thus we get that any fault-tolerant resolving set Fhas size at least 8 and hence $\beta(P_n^3) \geq 8$, provided $n \geq 26$.

Theorem 2.2. *For an n-vertex path $P_n$with $n \geq 26$, $\beta(P_n^3) = 8$.*

Proof : Using the result of Theorem 2.1, we have $\beta(P_n^3) \geq 8$. Thus to prove this theorem we have show that there exists a fault-tolerant resolving set $F$of $P_n^3$with cardinality 8. Our claim

F= $\{v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ is a fault-tolerant resolving set.

Using Lemma 1, we say that our claim is true if every pair of vertices of P_n^4 is resolved by at least two vertices in F. Let $v_a, v_b \in V(P_n^4)$ be arbitrary two vertices. We show that there exist two vertices u and v in F both of which resolve the vertices $v_a$and v_b. Without loss of generality, we may assume $a < b$. We consider the following three cases depending on the cardinality of $F \cap \{v_a, v_b\}$.

Case-1 : The cardinality of $F \cap \{v_a, v_b\}$ is two.Here both the vertices $v_a$and $v_b$are in F. Let $D = \{v_a, v_b\}$. Then $D \subset F$. Now the $code_D(v_a) = (0, x)$ and $code_D(v_b) = (x, 0)$, where $x = d(v_a, v_b) \neq 0$. Hence $code_D(v_a)$ and $code_D(v_b)$ are differ in two places and hence $v_a$and $v_b$are resolved by two vertices of$D$and hence by two vertices of F as $D \subset F$. So in this case proof is done.

Case-2 : The cardinality of $F \cap \{v_a, v_b\}$is one. Here first we show that $v_b \notin F$. We proof this by contrary. If possible, let $v_b \in F$ Since $a < b$and Fcontains consecutive vertices of P_n^3containing the starting vertex v_0. Hence $v_a$must be in F. But we assume here that $|F \cap \{v_a, v_b\}| = 1$. Therefore, $v_b$cannot be an element of $F$and hence $v_a$must be in Fdue to the fact that $|F \cap \{v_a, v_b\}| = 1$. Since $v_a \in F$ the pair of v_a and v_b must be resolved by v_a as $code_{\{v_a\}}(v_a) = (0)$ and $code_{\{v_a\}}(v_b) = (d(v_a, v_b)) \neq (0)$. Now wehave to find an another vertex $u$in $F\backslash\{v_a\}$that resolve the vertices $v_a$and v_b. Observe that $d(v_a, u) = 1 \; or \; 2$ for every $u \in F\backslash\{v_a\}$. Here the vertex $v_a$is an arbitrary but fixed element of F. For a fixed v_a, definetwo sets $S_a^1 = \{u \in F: d(v_a, u) = 1\}$ and $S_a^2 = \{u \in F: d(v_a, u) = 2\}$. Then $S_a^1 \cup S_a^2 = F\backslash\{v_a\}$. Note that$v_a \notin S_a^1 \cup S_a^2$and S_a^1 and $S_a^2$are disjoint sets. Also it is noted that bothS_a^1 and $S_a^2$are non-empty provided $v_a \notin \{v_3, v_4\}$. For $v_a \notin \{v_3, v_4\}$, S_a^2 is empty. Now we consider following two sub cases according as v_a is in$\{v_3, v_4\}$ or not.

Subcase-(2a) :$\boldsymbol{v_a \in \{v_3, v_4\}}$. In this case S_a^2 is empty and hence $S_a^1 = F\backslash\{v_a\}$ i.e., $d(v_a, u) = 1$ for all $u \in F\backslash\{v_a\}$. Since $v_b \notin F$, $b \geq 8$ and hence $d(u, v_b) \geq 2$ for $u = v_0$. Therefore, in this case there exists a vertex $u = v_0$ that resolve the vertices $v_a$and v_b.

Subcase-(2b) :$\boldsymbol{v_a \notin \{v_3, v_4\}}$ In this case both S_a^1 and $S_a^2$are non-empty. If $d(v_a, u) \geq 2$, then we show that $v_a$and $v_b$are resolved by v_3. Note that $v_3 \in S_a^1$ i.e., $d(v_3, v_a) = 1$. Now $d(v_a, v_b) \geq 2$ implies that $b - a \geq 8$ and hence $d(v_3, v_b) = \left\lceil \frac{b-3}{4} \right\rceil \geq \left\lceil \frac{a+5}{4} \right\rceil \geq 2$. Therefore, v_3 resolve v_a and v_b provided $d(v_a, v_b) \geq 2$. Now we assume $d(v_a, v_b) = 1$. Then $v_a \in \{v_5, v_6, v_7\}$. Let $l$be the largest integer in $\{0,1,2,3\}$such that $d(v_l, v_a) = 1$.Then $a = l + 4$ and hence $b - l \geq 5$. Therefore, $d(v_l, v_b) = \left\lceil \frac{b-l}{4} \right\rceil \geq \left\lceil \frac{5}{4} \right\rceil \geq 2$. Hence in this case $v_a$and $v_b$are resolve by v_l, where $l$is the largest integer in $\{0,1,2,3\}$such that$d(v_l, v_a) = 1$.

***Case-2 F* $\boldsymbol{\cap \{v_a, v_b\}}$ *is empty*.** In this case $a, b \in \{8,9, \ldots, n-1\}$. To prove the result, here we have to identify two vertices $u, v \in F$that resolved $v_a$and v_b. Let a$\equiv l \; (mod\, 4)$. Then both v_l and v_{l+4} are in F. Now we calculate the following distances.

$$d(v_l, v_a) = \left\lceil \frac{a-l}{4} \right\rceil = \frac{a-l}{4}$$

$$d(v_{l+4}, v_a) = \left\lceil \frac{a-l-4}{4} \right\rceil = \frac{a-l-4}{4}$$

$$d(v_l, v_b) = \left\lceil \frac{b-l}{4} \right\rceil \geq \frac{b-l}{4} + 1$$

$$d(v_{l+4}, v_b) = \left\lceil \frac{b-l-4}{4} \right\rceil \geq \frac{b-l-4}{4} + 1.$$

From above we can say that $v_a$and $v_a$are resolved by both the vertices $v_l$and v_{l+4} when $a \equiv l \ (mod\ 4)$. On account of all cases above, we say that $F \cap \{v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ forms a fault-tolerant resolvingset for P_n^4. Hence the proof is complete.

Remark 2.1. By similar argument as used in Theorem 2.2, one may show that any eight consecutive vertices forms a fault-tolerant resolving set for P_n^4.

3. Concluding remark

In this article we have determined the fault-tolerant metric dimension for P_n^4 for all $n \geq 26$. We also have shown the existence of at least n fault-tolerant metric bases for the same graph. The readers may try to find all fault-tolerant metric bases for r-th power of paths or in particular, for square of paths. This article gives a solution to the problem of placement of optimal numbers of sensors in a network when it is structured as square of paths. By giving more than one fault-tolerant metric bases for P_n^4, we present an alternative placement of censors in the network when one solution is not suitable for an organization that are planning to place the sensors. If a sensor fails which can be catastrophic the fault-tolerant system is able to use reversion to fall back to a safer mode. The advantages of using such a system are that it reduces redundancy, there is no slow down of the given system and no assumptions are made for the distribution of fault.

Acknowledgement

The author is thankful to the Science and Engineering Research Board (DST), India for its financial support (Grant No. CRG/2019/006909).

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Global existence of the wave equation with polynomial source and damping terms.

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Abstract

In this work, we consider the wave equation with polynomial source and damping terms. By the stable set method, we prove that under some conditions on the parameters in the system the global solution exists.

Keywords : Wave equation, Source term, Damping term, Global solution.

1. Introduction

In this paper, we consider the following system

$$\begin{cases} u_{tt} - \Delta u = |u|^{p-2}u & in \;\; (0,T)\times\Omega, \\ u = 0 & on \;\; (0,T)\times\Gamma_0, \\ \partial_\nu u = -|u_t|^{m-2}u_t & on \;\; (0,T)\times\Gamma_1, \\ u(0) = u_0, u_t(0) = u_1 & in \;\; \Omega. \end{cases} \tag{1}$$

Where $T > 0$ and Ω is a bounded domain of $\mathbb{R}^n (n \in \mathbb{N}^*)$ with smooth boundary Γ. $\{\Gamma_0, \Gamma_1\}$ is a partition of Γ such that $\Gamma = \Gamma_0 \cup \Gamma_1$ and $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$. Also, we have the following conditions on m and p :

$$\begin{cases} 2 \le p & if \;\; n = 1, 2, \\ 2 \le p \le 2\frac{n-1}{n-2} & if \;\; n \ge 3, \end{cases} \tag{2}$$

and

$$\begin{cases} 2 \le m & if \; n = 1, 2, \\ 2 \le m \le \frac{2n}{n-2} & if \; n \ge 3. \end{cases} \tag{3}$$

In the literature, the problem of global existence of the wave equation was studied by several authors : in the case of internal damping term, Haraux and Zuazua [2] in 1988, proved the global existence for arbitrary initial data. In 1994, Georgiev and Todorova [4] considered the wave equation with internal source and damping terms. They proved that when the damping term dominated the source term then the global solution exists for arbitrary initial data. Ikehata [3] in 1995, considered the same equation.

He used the stable set method, introduced by Sattinger [6] in 1968, to show that when the source term dominated the damping term the global solution exists for small enough initial energy. In the case of the boundary damping term, Park and Ha [5] in 2008, obtained the global existence result. For other results in this direction, we can see [7] and [8].

Our objective is to study the global existence of solution of the system (1) using the stable set method.

This paper contains three sections, in addition to the introduction. In section 2, we give some preliminary results needed to prove our main result. Section 3 contains the proof of the global existence result.

2. Preliminary results

In this section, we start with presenting the local existence result.

Theorem 1 [1] *If $u_0 \in H^1_{\Gamma_0}(\Omega)$ and $u_1 \in L^2(\Omega)$ then there exists a unique weak maximal solution u of the problem (1) in $(0,T)$. Moreover, the following alternative holds :*

$$T = +\infty,$$

or

$$T < +\infty \quad and \quad \lim_{t\to T}(\|u_t\|_2 + \|\nabla u\|_2) = +\infty.$$

Now, we consider the energy functional E associated with our system defined by

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p}\|u\|_p^p \quad for\ all\ t \in (0,T).$$

We have the following derivative energy identity, which shows that the above energy is a decreasing function.

Theorem 2 [1] *Let $u_0 \in H^1_{\Gamma_0}(\Omega)$ and $u_1 \in L^2(\Omega)$. We have*

$$E(t) - E(s) = -\int_s^t \int_{\Gamma_1} |u_t|^{m-1} u_t \, d\Gamma d\tau \qquad for\ all\ 0 \le s \le t \le T.$$

We introduce the following functionals associated to the local solution u of (1) defined for all $t \in (0,T)$ by

$$J(t) = J(u(t)) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p}\|u\|_p^p$$

and

$$K(t) = K(u(t)) = \|\nabla u\|_2^2 - \|u\|_p^p.$$

We consider the following set

$$\mathbb{H} = \{w \in H^1_{\Gamma_0}(\Omega) / K(w) > 0\}. \tag{4}$$

Let C_* be the best constant such that

$$\|u\|_p \le C_* \|\nabla u\|_2 \quad for\ all\ u \in H^1_{\Gamma_0}(\Omega). \tag{5}$$

We have the following property of the set $\mathbb{H}$.

Lemma 1 *If $u_0 \in \mathbb{H}$ and $u_1 \in L^2(\Omega)$ with*

$$\beta = C_*^p(\frac{2p}{p-2}E(0))^{\frac{p-2}{2}} < 1. \tag{6}$$

Then

$$u(t) \in \mathbb{H} \quad for\ all\ t \in (0,T).$$

Proof 1 *Since*

$$u_0 \in \mathbb{H},$$

then

$$K(u_0) > 0.$$

This implies that there exists $T^* \leq T$ such that

$$K(t) \geq 0 \quad for\ all\ t \in [0,T^*]. \tag{7}$$

We have

$$\begin{aligned} J(t) &= \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p}\|u\|_p^p \\ &= \frac{p-2}{2p}\|\nabla u\|_2^2 + \frac{1}{p}(\|\nabla u\|_2^2 - \|u\|_p^p) \\ &= \frac{p-2}{2p}\|\nabla u\|_2^2 + \frac{1}{p}K(t). \end{aligned}$$

By (7), we find

$$J(t) \geq \frac{p-2}{2p}\|\nabla u\|_2^2 \qquad for\ all\ t \in [0,T^*].$$

Hence

$$\|\nabla u\|_2^2 \leq \frac{2p}{p-2}J(t).$$

Since

$$J(t) = E(t) - \frac{1}{2}\|u_t\|_2^2 \leq E(t),$$

then

$$\|\nabla u\|_2^2 \leq \frac{2p}{p-2}E(t).$$

Since E is a decreasing function then

$$\|\nabla u\|_2^2 \leq \frac{2p}{p-2}E(0) \;\; for\ all\ t \in [0,T^*]. \tag{8}$$

By (5), we find

$$\|u\|_p^p \leq C_*^p\|\nabla u\|_2^p = \frac{1}{p}C_*^p\|\nabla u\|_2^{p-2}\|\nabla u\|_2^2.$$

(8) gives

$$\|u\|_p^p \leq C_*^p(\frac{2p}{p-2}E(0))^{\frac{p-2}{2}}\|\nabla u\|_2^2.$$

So

$$\|u\|_p^p \leq C_*^p(\frac{2p}{p-2}E(0))^{\frac{p-2}{2}}\|\nabla u\|_2^2 = \beta\|\nabla u\|_2^2.$$

We then use (6) to find

$$\|u\|_p^p < \|\nabla u\|_2^2 \quad for\ all\ t \in [0, T^*].$$

So

$$K(u(t)) = \|\nabla u\|_2^2 - \|u\|_p^p > 0 \quad for\ all\ t \in [0, T^*].$$

This leads to

$$u(t) \in \mathbb{H} \quad for\ all\ t \in [0, T^*].$$

By noting that

$$C_*^p\left(\frac{2p}{p-2}E(T^*)\right)^{\frac{p-2}{2}} < 1,$$

we can repeat the proceedings above to extend T^* to T. So, we get the desired result.

3. Main result

Now, we prove the main result of this work.

Theorem 3 *If $u_0 \in \mathbb{H}$ and $u_1 \in L^2(\Omega)$ with $\beta < 1$, then the local solution u of (1) is global.*

Proof 2 *From the definition of E and K, we get for all $t \in (0, T)$*

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p}\|u\|_p^p$$

$$= \frac{1}{2}\|u_t\|_2^2 + \frac{p-2}{2p}\|\nabla u\|_2^2 + \frac{1}{p}K(t).$$

Since

$$K(t) \geq 0 \quad for\ all\ t \in (0, T),$$

thus

$$E(t) \geq \frac{1}{2}\|u_t\|_2^2 + \frac{p-2}{2p}\|\nabla u\|_2^2 \quad for\ all\ t \in (0, T).$$

This implies that there exists $C > 0$ such that

$$\|u_t\|_2^2 + \|\nabla u\|_2^2 \leq CE(t) \quad for\ all\ t \in (0, T).$$

Since E is a decreasing function, then

$$\|u_t\|_2^2 + \|\nabla u\|_2^2 \leq CE(0) \quad for\ all\ t \in (0, T).$$

By the alternative statement, we find the desired result.

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Inextensible Flow of Non-Null Curve with Type-3 Bishop Frame in Lorentz 3-space

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Abstract

In this work, we present type-3 Bishop frame for a non-null curve with respect to normal vector field of a non-null curve in Lorentz 3-space. Then, we get the inextensible flow equation for a timelike curve with respect to type-3 Bishop frame and type-3 Bishop frame curvatures.

Key words: Inextensible flow, Non-null curve, Type-3 Bishop frame.

1. Introduction

Curves are one of the important structures of differential geometry. The many fundamental results in the theory of the curves in Euclidean and non-Euclidean were studied by many researchers. The flow of a curve is called to be inextensible if the arc-length of a curve is preserved. Inextensible curve flows have growing importance in many applications such as engineering, computer vision, structural mechanics and computer animation. The terms "inextensible" and "extensible" mostly come up in physics. There are inextensible and extensible collisions in physics. In extensible collision, both the kinetic energy and momentum are conserved. In inextensible collision, the kinetic energy is not conserved in the collision; however, the momentum is conserved, [7]. In [6], studied the evolution of space curves in Euclidean space by considering Frenet equations that describe the evolution for the Frenet vectors according to arc-length and time parameters. He got nonlinear partial differential equations for the Frenet apparatus (κ and τ) of curve. Then in [4], Doliwa and Santini presented a relation between the motion of inextensible curves and solitonic systems in a space of constant curvature. Also, in [8-9], the authors investigated motion of plane and space curves and gave time evolution equation of the moving frame and the curvatures of the evolving curve. Recently, Bishop frame and inextensible flow of curves have studied by many researchers in Euclidean space and non-Euclidean spaces ([3], [5], [12], [13]). Bishop frame which is known alternative or parallel frame of the curve, was firslty presented by Bishop [2]. In [11], the authors described type-3 Bishop frame and gave time evolution equation of space curves with respect to such frame.

The paper is organized as follows. In section 2, we recall some general notions and notations needed throughout the paper and repeat some of the definitions mentioned in the introduction more formally. Then we obtain type-3 Bishop frame for a non-null curve in Lorentz 3-space. In section 3, we get the inextensible flow equation for a timelike curve with respect to type-3 Bishop frame and type-3 Bishop frame curvatures.

2. Type-3 Bishop Frame for Non-Null Curves

The Lorentz 3-space $\mathbb{R}_1^3$ is the real vector space endowed with a symmetric, bilinear and non-degenerate metric

$$< x, y >= -x_1y_1 + x_2y_2 + x_3y_3,$$

where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ are arbitrary vectors in $\mathbb{R}_1^3$. Also, let $= (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ be two vectors in $\mathbb{R}_1^3$, then the Lorentzian cross product $x \times_L y$ is defined by the formula

$$x \times_L y = (x_3y_2 - x_2y_3, -x_1y_3 + x_3y_1, -x_1y_2 - x_2y_1).$$

An arbitrary vector $x \in \mathbb{R}_1^3$ is called spacelike (resp. timelike, lightlike) if it satisfies $< x, x >> 0$ or $x = 0$ (resp. $< x, x >< 0$, $< x, x >= 0$ and $x \neq 0$). An arbitrary curve α in $\mathbb{R}_1^3$ is said spacelike (resp. timelike, lightlike) if its velocity vector is spacelike (resp. timelike, lightlike). Any spacelike or timelike curve is called non-null curve in $\mathbb{R}_1^3$, [10].

Let α(s) be non-null curve and $\{T, N, B\}$ is the moving Frenet frame along the curve α(s) in $\mathbb{R}_1^3$. The Frenet derivative equations are

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\varepsilon_1\varepsilon_2\kappa & 0 & \tau \\ 0 & \varepsilon_1\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}, \tag{1}$$

where κ and τ are curvature and torsion of α, respectively, and $\varepsilon_1 = \mp 1 =< T, T >$, $\varepsilon_2 = \mp 1 =< N, N >$ and $< B, B >= -\varepsilon_1\varepsilon_2$, [1].

Firstly, we present type-3 Bishop frame for a timelike curve with respect to normal vector field of a timelike curve in Lorentz 3-space.

Let T, N and B denotes the unit tangent, the unit normal, the unit binormal vectors on the timelike curve, respectively. If we rotate Frenet frame $\{T, N, B\}$ of the timelike curve by angel θ about the spacelike normal, we get a new frame denoted by $\{B_1, B_2, B_3\}$, then the relation between two frames is written as

$$\begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} cosh\theta(s) & 0 & sinh\theta(s) \\ 0 & 1 & 0 \\ sinh\theta(s) & 0 & cosh\theta(s) \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}, \tag{2}$$

or

$$\begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} cosh\theta(s) & 0 & -sinh\theta(s) \\ 0 & 1 & 0 \\ -sinh\theta(s) & 0 & cosh\theta(s) \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}. \tag{3}$$

Assume that,

$$\begin{gathered} \mathrm{F}=\begin{pmatrix} T \\ N \\ B \end{pmatrix},\ R_1 = \begin{pmatrix} cosh\theta(s) & 0 & sinh\theta(s) \\ 0 & 1 & 0 \\ sinh\theta(s) & 0 & cosh\theta(s) \end{pmatrix},\ R_2 = \begin{pmatrix} cosh\theta(s) & 0 & -sinh\theta(s) \\ 0 & 1 & 0 \\ sinh\theta(s) & 0 & -cosh\theta(s) \end{pmatrix} \\ b = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix},\ K = \begin{pmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}, \end{gathered} \tag{4}$$

then (1), (2) and (3) can be given as

$$\begin{aligned} \frac{\partial F}{\partial s} &= KF, \\ b &= R_1 F, \\ F &= R_2 b. \end{aligned} \tag{5}$$

If we use (5), we can give the rate of change of type-3 Bishop frame as

$$\frac{\partial b}{\partial s} = (\frac{\partial R_1}{\partial s} R_2 + R_1 K R_2) b \tag{6}$$

or by using (4) we obtain

$$\frac{d}{ds}\begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} 0 & \kappa cosh\theta(s) - \tau sinh\theta(s) & \frac{d\theta(s)}{ds} \\ \kappa cosh\theta(s) - \tau sinh\theta(s) & 0 & -\kappa sinh\theta(s) + \tau cosh\theta(s) \\ \frac{d\theta(s)}{ds} & \kappa sinh\theta(s) - \tau cosh\theta(s) & 0 \end{pmatrix}\begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}. \tag{7}$$

Thus we can give the equation (7) as

$$\frac{d}{ds}\begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} 0 & \kappa_1 & 0 \\ \kappa_1 & 0 & \kappa_2 \\ 0 & -\kappa_2 & 0 \end{pmatrix}\begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}, \tag{8}$$

where type-3 Bishop curvatures are

$$\begin{aligned} \kappa_1 &= \kappa cosh\theta(s) - \tau sinh\theta(s), \\ \kappa_2 &= -\kappa sinh\theta(s) + \tau cosh\theta(s), \end{aligned} \tag{9}$$

and

$$\theta(s) = const..$$

Definition. The frame $\{B_1, B_2, B_3\}$ is properly oriented, so we can call the set $\{B_1, B_2, B_3, \kappa_1, \kappa_2\}$ as type-3 Bishop invariants of the timelike curve α(s), [11].

Now, we can give the evolution of an orthonormal type-3 Bishop frame $\{B_1, B_2, B_3\}$ by its angular velocity ω as

$$\begin{aligned} \frac{dB_1}{ds} &= \omega \times B_1, \\ \frac{dB_2}{ds} &= \omega \times B_2, \\ \frac{dB_3}{ds} &= \omega \times B_3. \end{aligned} \tag{10}$$

Also, we can write ω with respect to type-3 Bishop frame $\{B_1, B_2, B_3\}$ as

$$\omega = \omega_1 B_1 + \omega_2 B_2 + \omega_3 B_3. \tag{11}$$

Then equation (10) becomes

$$\begin{aligned} \frac{dB_1}{ds} &= \omega_2 B_3 - \omega_3 B_2, \\ \frac{dB_2}{ds} &= -\omega_1 B_3 - \omega_3 B_1, \\ \frac{dB_3}{ds} &= \omega_1 B_2 + \omega_2 B_1. \end{aligned} \tag{12}$$

If we compare equations (12) with equations (8), we obtain

$$\omega_1 = -\kappa_2\,,\ \omega_2 = 0,\ \omega_3 = -\kappa_1. \tag{13}$$

So, the Darboux vector for to type-3 Bishop frame is given by

$$\omega = -\kappa_2\, B_1 - \kappa_1 B_3. \tag{14}$$

Now, we present type-3 Bishop frame for a spacelike curve with timelike normal in Lorentz 3-space. Let T, N and B denotes the unit tangent, the unit normal, the unit binormal vectors on the spacelike curve with timelike normal, respectively. If we rotate Frenet frame $\{T, N, B\}$ of the spacelike curve by angel θ about the timelike normal, we get a new frame denoted by $\{B_1, B_2, B_3\}$, then the relation between two frames is written as

$$\begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} cos\theta(s) & 0 & sin\theta(s) \\ 0 & 1 & 0 \\ sin\theta(s) & 0 & cos\theta(s) \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}, \tag{15}$$

or

$$\begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} cos\theta(s) & 0 & -sin\theta(s) \\ 0 & 1 & 0 \\ sin\theta(s) & 0 & cos\theta(s) \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}. \tag{16}$$

Assume that,

$$\text{F}=\begin{pmatrix} T \\ N \\ B \end{pmatrix},\ R_1 = \begin{pmatrix} cos\theta(s) & 0 & sin\theta(s) \\ 0 & 1 & 0 \\ sin\theta(s) & 0 & cos\theta(s) \end{pmatrix},\ R_2 = \begin{pmatrix} cos\theta(s) & 0 & -sin\theta(s) \\ 0 & 1 & 0 \\ sin\theta(s) & 0 & cos\theta(s) \end{pmatrix} \tag{17}$$

$$b = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}, \;\; K = \begin{pmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & \tau & 0 \end{pmatrix},$$

then (1), (15) and (16) can be given as

$$\begin{aligned} \frac{\partial F}{\partial s} &= KF, \\ b &= R_1 F, \\ F &= R_2 b. \end{aligned} \tag{18}$$

If we use (18), we can give the rate of change of type-3 Bishop frame as

$$\frac{\partial b}{\partial s} = (\frac{\partial R_1}{\partial s} R_2 + R_1 K R_2) b \tag{19}$$

or by using (17) we obtain

$$\frac{d}{ds}\begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} 0 & \kappa cos\theta(s) + \tau sin\theta(s) & \frac{d\theta(s)}{ds} \\ \kappa cos\theta(s) + \tau sin\theta(s) & 0 & -\kappa sin\theta(s) + \tau cos\theta(s) \\ \frac{d\theta(s)}{ds} & -\kappa sin\theta(s) + \tau cos\theta(s) & 0 \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}. \tag{20}$$

Thus we can give the equation (20) as

$$\frac{d}{ds}\begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} 0 & \kappa_1 & 0) \\ \kappa_1 & 0 & \kappa_2 \\ 0 & \kappa_2 & 0 \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}, \tag{21}$$

where type-3 Bishop curvatures are

$$\begin{aligned} \kappa_1 &= \kappa cos\theta(s) + \tau sin\theta(s), \\ \kappa_2 &= -\kappa sin\theta(s) + \tau cos\theta(s), \end{aligned} \tag{22}$$

and

$$\theta(s) = const..$$

3. Inextensible flow of a timelike curve with respect to type-3 Bishop frame

In this section we get inextensible flow of a timelike curve using type-3 Bishop frame. Also, we get time evolution equation for type-3 Bishop frame and type-3 Bishop curvatures.

Inextensible flow of a timelike curve with respect to type-3 Bishop frame is given as

$$\frac{dC(s)}{ds} = \Lambda_1 B_1 + \Lambda_2 B_2 + \Lambda_3 B_3 \tag{23}$$

where Λ_1, Λ_2 and Λ_3 are functions of arc-lenght parameter s and time parameter s, correspond to the tangent, normal and binormal projections of the velocity. If the timelike curve parametrized by time parameter t, then the general temporal evolution in which the triad $\{B_1, B_2, B_3\}$ remains orthonormal adopts the form [1],

$$\frac{d}{dt}\begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} 0 & \gamma_1 & 0 \\ \gamma_1 & 0 & \gamma_2 \\ 0 & -\gamma_2 & 0 \end{pmatrix}\begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}, \tag{24}$$

where γ_1 and γ_2 are geometric parameters which are in general functions of s and t. These describe the evolution in t of the type-3 Bishop frame $\{B_1, B_2, B_3\}$ on the timelike curve [11]. If we use compatibility condition in (8) and (24), we have

$$\frac{\partial \kappa_1}{\partial t} = \frac{\partial \gamma_1}{\partial s}, \ \frac{\partial \kappa_2}{\partial t} = \frac{\partial \gamma_2}{\partial s}$$

and (25)

$$\gamma_2 \kappa_1 - \gamma_1 \kappa_2 = 0.$$

Also, if we use compatibility condition in (23), we obtain

$$\begin{aligned} \frac{\partial \Lambda_1}{\partial s} &= -\Lambda_2 \kappa_1, \\ \frac{\partial \Lambda_2}{\partial s} &= \gamma_1 + \Lambda_3 \kappa_2 - \Lambda_1 \kappa_1, \\ \frac{\partial \Lambda_3}{\partial s} &= -\Lambda_2 \kappa_2. \end{aligned} \tag{26}$$

For a given functions $\{\Lambda_1, \Lambda_2, \Lambda_3\}$, equations (25) and (26) form a set of 6 nonlinear first order partial differential equations which governing the evolution of the type-3 Bishop curvatures of the evolving timelike curve in the Lorentz 3-space, [11].

From (25), we get

$$\frac{\partial \kappa_1}{\partial t} = \frac{\partial \gamma_1}{\partial s}, \tag{27}$$

$$\frac{\partial \kappa_2}{\partial t} = \gamma_1 \frac{\partial}{\partial s}\left(\frac{\kappa_2}{\kappa_1}\right) + \frac{\kappa_2}{\kappa_1}\frac{\partial \gamma_1}{\partial s}.$$

With the aid of (26), γ_1 and its derivative with respect to s, are given by

$$\gamma_1 = \frac{\partial \Lambda_2}{\partial s} + \Lambda_1\kappa_1 - \Lambda_3\kappa_2, \tag{28}$$

$$\frac{\partial \gamma_1}{\partial s} = \frac{\partial^2 \Lambda_2}{\partial s^2} + \frac{\partial \Lambda_1}{\partial s}\kappa_1 + \Lambda_1\frac{\partial \kappa_1}{\partial s} - \frac{\partial \Lambda_3}{\partial s}\kappa_2 - \Lambda_3\frac{\partial \kappa_2}{\partial s}. \tag{29}$$

Using first and third equations in the (26) and substituting in (29), we get

$$\frac{\partial \gamma_1}{\partial s} = \frac{\partial^2 \Lambda_2}{\partial s^2} + \Lambda_2\kappa_1^2 + \Lambda_1\frac{\partial \kappa_1}{\partial s} - \Lambda_2\kappa_2^2 - \Lambda_3\frac{\partial \kappa_2}{\partial s}. \tag{30}$$

If we use (28) and (30) into (27), the time evolution equation of type-3 Bishop curvatures κ_1 and κ_2 are given by

$$\frac{\partial \kappa_1}{\partial t} = \frac{\partial^2 \Lambda_2}{\partial s^2} + \Lambda_2\kappa_1^2 + \Lambda_1\frac{\partial \kappa_1}{\partial s} - \Lambda_2\kappa_2^2 - \Lambda_3\frac{\partial \kappa_2}{\partial s},$$

$$\frac{\partial \kappa_2}{\partial t} = \left(\frac{\partial \Lambda_2}{\partial s} + \Lambda_1\kappa_1 - \Lambda_3\kappa_2\right)\frac{\partial}{\partial s}\left(\frac{\kappa_2}{\kappa_1}\right) + \frac{\kappa_2}{\kappa_1}\left(\frac{\partial^2 \Lambda_2}{\partial s^2} + \Lambda_2\kappa_1^2 + \Lambda_1\frac{\partial \kappa_1}{\partial s} - \Lambda_2\kappa_2^2 - \Lambda_3\frac{\partial \kappa_2}{\partial s}\right).$$

Lastly, we can give the time evolution of type-3 Bishop frame $\{B_1, B_2, B_3\}$ on the timelike curve as

$$\begin{aligned}
\frac{\partial B_1}{\partial t} &= \left(\frac{\partial \Lambda_2}{\partial s} + \Lambda_1\kappa_1 - \Lambda_3\kappa_2\right)B_2, \\
\frac{\partial B_2}{\partial t} &= \left(\frac{\partial \Lambda_2}{\partial s} + \Lambda_1\kappa_1 - \Lambda_3\kappa_2\right)B_1 + \frac{\kappa_2}{\kappa_1}\left(\frac{\partial \Lambda_2}{\partial s} + \Lambda_1\kappa_1 - \Lambda_3\kappa_2\right)B_2, \\
\frac{\partial B_3}{\partial t} &= -\frac{\kappa_2}{\kappa_1}\left(\frac{\partial \Lambda_2}{\partial s} + \Lambda_1\kappa_1 - \Lambda_3\kappa_2\right)B_2.
\end{aligned}$$

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Integration of TAM model in Students Acceptance of Google Classroom context Using Partial Least Squares -Structural Equation Model.

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Abstract

The development of information technology has an integral role to the promotion of new alternatives in relation to improve teaching and learning for universities.This study has the main aim to analyze through the structure of the Technology Acceptance Model (TAM), how the model factors can predict the behavioral intention towards Google Classroom. The data were collected through an online questionnaire for 194 students from Alexander Moisiu University, who used the Google Classroom platform. The data evaluations are realized using the Partial Least Square-Structural Equation Model (PLS-SEM) method. The results show that Perceived Usefulness (PU) and Perceived Ease of Use (PEOU) significantly influence behavioral intention towards Google Classroom and both of these constructs influence in the actual usage of Google Classroom.

Keywords: Technology Acceptance Model, Partial Least Square-Structural Equation Model, Google Classroom, factors.

1. Introduction

The development of information technology has an integral role in promoting new alternatives related to the improvement of teaching and learning for universities and consequently induce the need for new approaches through the implementation of e-learning tools. We mention the Google Classroom platform as Learning Management System (LMS) which was introduced in 2014, as part of the G Suite for Education, that features Gmail, Calendar, Contacts, Docs, Sheets, Slides, Drive, and Hangouts (1). Google Classroom is the best used in the context that students and instructors can discuss course content outside of the constraints of time and distance (Falvo & Johnson (2)). Being assisted by the platform of Google Classroom, students have access for learning materials, interact with the content, instructor, and other learners referring Ally (10). Several theories / models of information system (IS) are developed to explain the factors of the acceptance of a new technology. One such model is the Technology Acceptance Model (TAM). This model shows that an individual's behavioral intention (BI) to use a technology is determined by two factors that are perceived usefulness (PU) and Perceived ease of use (PEOU) respectively. Perceived usefulness is defined here as "the degree to which a person believes that using a particular system would enhance his or her job performance. Perceived ease of use, refers to "the degree to which a person believes that using a particular system would be free of effort" based on Davis (3). In the study of Rana A. Saeed Al-Maroof (4) implements the TAM model integrated with the PLS-SEM approach, treats the connections of factors that are : perceived usefulness (PU), perceived ease of use (PEOU) respectively. From the empirical data analysis is showed that PU and PEOU has a positive effect with Behavioral Intention (BI) to use Google Classroom, while BI significantly affects the actual usage (AU) of Google Classrooms. So, in tis study are proposed to test the following 4 hypotheses:

H1:Perceived ease of use (PEOU) positively impact the perceived usefulness (PU) of Google Classroom.
H2:Perceived ease of use positively (PEOU) impact the behavioral intention (BI) to use Google Classroom.
H3:Perceived usefulness (PU) positively impact the behavioral intention (BI) to use Google Classroom.
H4:Behavioral intention (BI) to use Google classroom impact the actual use (UB) of Google Classroom

2. Methods

The target populations per kete studim jane 194 studente te Universitetit Aleksander Mosiu te cilet per shkak te pandemise Covid-19 kane perdorur Google Classroom as learning tool. The target populations for this study are 194 students of Alexander Mosiu University who have used Google Classroom as learning tool, due to the Covid-19 pandemic. All the variables were measured using a five-point nominal scale ranging from 1 (strongly disagree) to 5 (strongly agree). In this study is used the Partial Least Square-Structural Equation Model (PLS-SEM) for statistical evaluation, that includes (i) the analysis of the measurement model and (ii) the structural model (5). Firstly we determine if all latent variables are reflective and then evaluate the measurement model which includes reliability and validity (6). In terms of model reliability is evaluated composite reliability, and its values must be greater than 0.7 based on Hair et al (7). Validity includes the assessment of measure convergent validity and discriminant validity. Convergent validity is developed from the average variance extracted (AVE), and outer loadings of the indicators, and their values must be at least 0.5 and 0.7 respectively (Hew et al (8)). In order to evaluate the discriminant validity are used the Fornell-Larcker criterion and heterotrait-monotrait ratio (HTMT) The Fornell–Larcker criterion (Fornell & Larcker (9)) claims that a latent variable shares more variance with its assigned indicators than with any other latent variable. HTMT values must be below the limit value 0.85 to confirm the discriminant validity. After estabilshing all the indicators of the measurement model, we evaluate the structural model. The latter includes the evaluation of path coefficients in the structural models, and coefficient of determination (R^2).

3. Results

Assessment of Measurement Model

Table 1 shows the outer loadings, composite reliability and variance extracted (AVE). As we see all loading values are greater than 0.7 and range from 0.813 to 0.936. Composite reliability values are all greater than the limit 0.7 so satisfying the acceptable minimum values of 0.7. AVE values range from 0.72 to 0.859 that are greater than suggested threshold value i.e. greater than 0.50.

Table 1. Validity and reliability of measurement model

Construct and indicators	Loadings	CR	AVE
Perceived usefulness (PU)		0.924	0.859
PU1	0.932		
PU2	0.921		
Perceived ease of use (PEOU)		0.885	0.72
PEOU 1	0.813		
PEOU 2	0.893		
PEOU3	0.839		
Behavioral Intention (BI)		0.943	0.847
BI 1	0.909		
BI 2	0.936		
BI 3	0.916		
Use Behavior (UB)		0.905	0.826
UB 1	0.902		
UB 2	0.915		

We use the Fornell-Larcker criterion and the heterotrait-monotrait (HTMT) ratio to evaluate the discriminant validity. Table 2 shows the HTMT values in parentheses. We see that all the constructs are showing higher loading on their own respective constructs than other constructs, so the Fornell – Larcker criterion condition is satisfy. Also all HTMT values are less than 1, so since both conditions are satisfied we say that discriminant validity is achieved in this study.

Table 2. Discriminant validity

	BI	PEOU	PU	UB
BI	0.921			
PEOU	0.596(0.666)	0.849		
PU	0.704(0.790)	0.689 (0.826)	0.927	
UB	0.574(0.669)	0.561(0.704)	0.676(0.832)	0.909

Assessment Structural model

The assessment of structural model includes the evaluation of path coefficients, and coefficient of determination (R^2). Figure 1 indicates the path coefficients and p-values for each connection of the constructs.

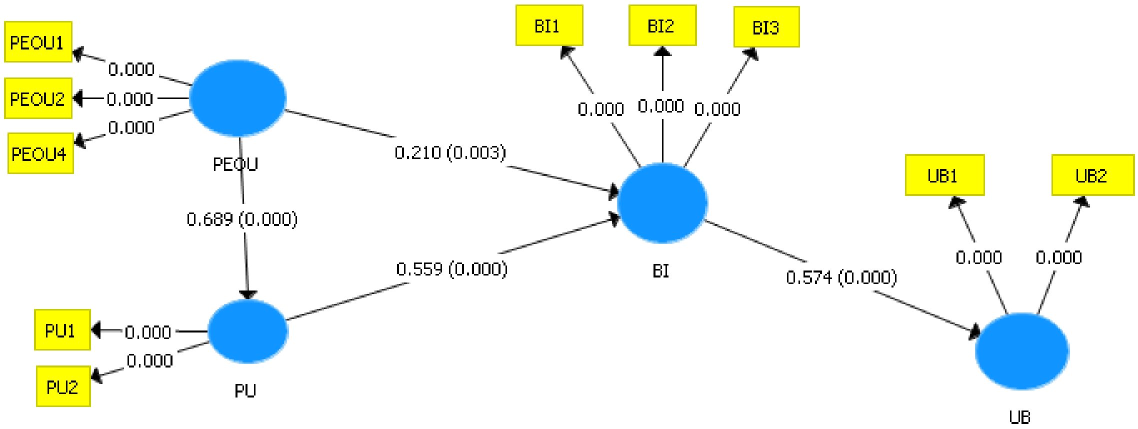

Figure 1. Path coefficients of structural model

The results in Table 3 show the path coefficients, p-values, and significance of each path of the structural model. It is shown that PU (b = 0.559, p = 0.00) and PEOU (b = 0.21, p = 0.002) significantly impacts Behavioral intention (BI) to use Google Classroom. As we see BI has a significant positive relationship with UB, as was hypothesized. Additionally PEOU has positive influence on PU. Finally all path coefficients estimates of the connections of model's construct are all significant at a 5% significance level.

Table 3. Estimation of structural model

Hypothesis	Path coefficients	T-Statistics	P-Values
BI→UB	0.574	10.83	0.000
PEOU →BI	0.21	3.165	0.002
PU→BI	0.559	9.233	0.000
PEOU →PU	0.689	17.114	0.000

Table 4 shows the coefficients of determination of the model constructs. The amount of variance in the endogenous constructs BI explained by all of the exogenous constructs linked to it is 0.519, while for the other variables PU and UB it is (0.475, 0.330) respectively.

Table 4. Coefficient of determination (R^2)

Construct	R Square	R Square Adjusted
BI	0.519	0.514
PU	0.475	0.472
UB	0.330	0.326

4. Conclusions

The projection of the TAM model in this study showed the antecedents of behavioral intention (BI) to use Google Classroom. The TAM model shows that behavioral intention (BI) to use Google Classroom depends on perceived ease of use PEOU and perceived usefulness PU. Using PLS-SEM metode we found that PEOU and PU have positive impact on the BI, further more BI has a significant positive relationship with UB. The Model explains only 51.9% of intentions to use Google Classroom. The findings of this study confirm that acceptance of Google Classroom from students is influenced by perceived ease of use (PEOU) and the perceived usefulness (PU). Perceived ease of use toward Google Classroom has a positive impact on perceived usefulness toward Google Classroom dhe additionally both affect on the actual use of Google Classroom.

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Iterated Binomial Transforms of the Balancing and Lucas-Balancing Polynomials

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Abstract

In this study, we apply r times the binomial transforms to the balancing and Lucas-balancing polynomials. Also, the Binet formulas, summations, generating functions of these transforms are found using recurrence relations. Finally, we obtain the Catalan and Cassini formulas for these transforms.

Keywords: balancing polynomials, Lucas-balancing polynomials, binomial transforms.
2010 AMS: 11B65, 11B83.

1. Introduction

The study of number sequences has been a source of attraction to the mathematicians since ancient times. Since then many of them are focusing their interest on the study of the fascinating triangular numbers. In 1999, Behera and Panda [BeheraPanda] introduced the notion of balancing numbers $\{B_n\}_{n\in\mathbb{N}}$ as solutions to a certain Diophantine equation. Then, the recurence relation of this number is $B_{n+1}=6B_n-B_{n-1}$ for $n\geq 1$, where $B_0=0, B_1=1$. A study on the Lucas-balancing numbers $C_n=\sqrt{8B_n^2+1}$ was published in 2009 by Panda [Panda]. The recurrence relation of this number is $C_{n+1}=6C_n-C_{n-1}$ for $n\geq 1$, where $C_0=1, C_1=3$. In addition, the author in [Frontczak c] studied the binomial sums of the balancing and Lucas-balancing numbers.

Generalizations of balancing numbers can be obtained in various ways (see [Frontczak a, Frontczak b, Ozkoc, PatelIrmakRay, Ray a, Ray b]). A natural extension is to consider for $x\in\mathbb{C}$ sequences of balancing and Lucas-balancing polynomials $\{B_n(x)\}_{n\in\mathbb{N}}$ and $\{C_n(x)\}_{n\in\mathbb{N}}$, respectively.

The authors defined the balancing $\{B_n(x)\}_{n\in\mathbb{N}}$ and Lucas-balancing $\{C_n(x)\}_{n\in\mathbb{N}}$ polynomials as in the forms

$$B_{n+1}(x) = 6xB_n(x) - B_{n-1}(x), \text{ where } B_0(x) = 0, B_1(x) = 1$$

and

$$C_{n+1}(x) = 6xC_n(x) - C_{n-1}(x), \text{ where } C_0(x) = 1, C_1(x) = 3x$$

respectively, in [Ray a, Frontczak b, PatelIrmakRay].

In addition, some matrix based transforms can be introduced for a given sequence. Binomial transform is one of these transforms and there is also other ones such as rising and falling binomial transforms (see [BhadouriaJhalaSingh, Chen, FalconPlaza, Prodinger]). Furthermore, there is the interesting study on the watermarking and the binomial transform. For example, in [MandalGhosalZizka], the binomial transform based fragile image watermarking technique has been proposed for color image authentication.

Given an integer sequence $X = \{x_0, x_1, x_2, \ldots\}$, the binomial transform B of the sequence X, $B(X) = \{b_n\}$, is given by

$$b_n = \sum_{i=0}^{n} \binom{n}{i} x_i.$$

Also, in [Boyadzhiev], author studied the following properties of the binomial transform

$$\sum_{i=0}^{n} \binom{n}{i} i x_i = n(b_n - b_{n-1})$$

and

$$\sum_{i=1}^{n} \binom{n}{i} \frac{x_i}{i} = \sum_{j=1}^{n} \frac{b_j}{j}.$$

In [Yilmaz], the author applied the binomial transforms to the balancing $B_n(x)$ and Lucas-balancing $C_n(x)$ polynomials:

Proposition 1.1. *For* $n>0$,

i) Recurrence relation of sequence $\{b_n(x)\}$ is $b_{n+1}(x)=(6x+2)(b_n(x)-b_{n-1}(x))$, $b_0(x)=0$, $b_1(x)=1$ with initial conditions,

ii) Recurrence relation of sequence $\{c_n(x)\}$ is $c_{n+1}(x)=(6x+2)(c_n(x)-c_{n-1}(x))$, $c_0(x)=1$, $c_1(x)=3x+1$ with initial conditions.

Also, Falcon [Falcon] studied the iterated application of the some binomial transforms to the k-Fibonacci sequence. Yilmaz and Taskara [YilmazTaskara] investigated the iterated application of the some binomial transforms to the k-Lucas sequence.

The goal of this paper is to apply iteratly the binomial transforms to the balancing $B_n(x)$ and Lucas-balancing $C_n(x)$ polynomials. Furthermore, the binet formulas, summations, generating functions of these transforms are found by recurrence relations.

2. Main Results

In this section, we will mainly focus on iterated binomial transforms of the balancing and Lucas-balancing polynomials to get some important results. In fact, we will also present the recurrence relations, binet formulas, summations, generating functions of these transforms.

The iterated binomial transforms of the balancing $B_n(x)$ and Lucas-balancing $C_n(x)$ polynomials are demonstrated by $B^{(r)}(x)=\{b_n^{(r)}(x)\}$ and $C^{(r)}(x)=\{c_n^{(r)}(x)\}$, where $b_n^{(r)}(x)$ and $c_n^{(r)}(x)$ are obtained by applying "r" times the binomial transform to the balancing and Lucas-balancing polynomials. It are obvious that $b_0^{(r)}(x)=0$, $b_1^{(r)}(x)=1$ and $c_0^{(r)}(x)=1$, $c_1^{(r)}(x)=3x+r$.

The following lemma will be key of the proof of the next theorems.

Lemma 2.1. *For* $n\geq 0$ *and* $r\geq 1$, *the following equalities are hold:*

$i)\ b_{n+1}^{(r)}(x)=\sum_{i=0}^{n}\binom{n}{i}\left(b_i^{(r-1)}(x)+b_{i+1}^{(r-1)}(x)\right),$

$ii)\ c_{n+1}^{(r)}(x)=\sum_{i=0}^{n}\binom{n}{i}\left(c_i^{(r-1)}(x)+c_{i+1}^{(r-1)}(x)\right).$

Proof. Firstly, in here we will just prove $i)$, since $ii)$ can be thought in the same manner with *i).*

i) By using definition of binomial transform and the well known binomial equality $\binom{n+1}{i}=\binom{n}{i}+\binom{n}{i-1}$,

we obtain

$$\begin{aligned} b_{n+1}^{(r)}(x) &= \sum_{i=0}^{n+1}\binom{n+1}{i}b_i^{(r-1)}(x) \\ &= \sum_{i=0}^{n+1}\binom{n}{i}b_i^{(r-1)}(x)+\sum_{i=1}^{n+1}\binom{n}{i-1}b_i^{(r-1)}(x) \\ &= \sum_{i=0}^{n}\binom{n}{i}b_i^{(r-1)}(x)+\sum_{i=0}^{n}\binom{n}{i}b_{i+1}^{(r-1)}(x), \end{aligned}$$

which is desired result.

□

Theorem 2.2. *For* $n\geq 0$ *and* $r\geq 1$, *the recurrence relations of sequences* $\left\{b_n^{(r)}(x)\right\}$ *and* $\left\{c_n^{(r)}(x)\right\}$ *are*

$$b_{n+1}^{(r)}(x)=(6x+2r)b_n^{(r)}(x)-\left(6xr+r^2+1\right)b_{n-1}^{(r)}(x), \quad (2.1)$$

$$c_{n+1}^{(r)}(x)=(6x+2r)c_n^{(r)}(x)-\left(6xr+r^2+1\right)c_{n-1}^{(r)}(x), \quad (2.2)$$

with initial conditions $b_0^{(r)}(x)=0,\ b_1^{(r)}(x)=1$ *and* $c_0^{(r)}(x)=1,\ c_1^{(r)}(x)=3x+r.$

Proof. Similarly the proof of the Lemma 2.1, only the first case, the equation (2.1) will be proved. We will omit the equation (2.2) since the proofs will not be different.

The proof will be done by induction steps on r and n.

First of all, for $r=1$, from the $i)$ condition of Proposition 1.1, it is true

$$b_{n+1}(x)=(6x+2)\left(b_n(x)-b_{n-1}(x)\right).$$

Let us consider definition of iterated binomial transform, then we have $b_2^{(r)}(x)=6x+2r.$ The initial conditions are $b_0^{(r)}(x)=0,\ b_1^{(r)}(x)=1.$ Hence, for $n=1$, the equation (2.1) is true, that is

$$b_2^{(r)}(x)=(6x+2r)b_1^{(r)}(x)-\left(6xr+r^2+1\right)b_0^{(r)}(x).$$

Actually, by assuming the equation in (2.1) holds for all $(r-1,n)$ and $(r,n-1)$, that is,

$$b_{n+1}^{(r-1)}(x)=(6x+2r-2)b_n^{(r-1)}(x)-\left(6xr-6x+r^2-2r+2\right)b_{n-1}^{(r-1)}(x),$$

and

$$b_n^{(r)}(x)=(6x+2r)b_{n-1}^{(r)}(x)-\left(6xr+r^2+1\right)b_{n-2}^{(r)}(x).$$

Then, we need to show that it is true for (r,n). That is,

$$b_{n+1}^{(r)}(x)=(6x+2r)b_n^{(r)}(x)-\left(6xr+r^2+1\right)b_{n-1}^{(r)}(x).$$

From Lemma 2.1-$i)$ and assumption, we have

$$\begin{aligned}
b_{n+1}^{(r)}(x)&=\sum_{i=0}^{n}\binom{n}{i}b_i^{(r-1)}(x)+\sum_{i=0}^{n}\binom{n}{i}b_{i+1}^{(r-1)}(x)\\
&=\sum_{i=1}^{n}\binom{n}{i}\left(b_i^{(r-1)}(x)+b_{i+1}^{(r-1)}(x)\right)+b_0^{(r-1)}(x)+b_1^{(r-1)}(x)\\
&=\sum_{i=1}^{n}\binom{n}{i}\left(b_i^{(r-1)}(x)+(6x+2r-2)b_i^{(r-1)}(x)-\left(6xr-6x+r^2-2r+2\right)b_{i-1}^{(r-1)}(x)\right)+b_0^{(r-1)}(x)+b_1^{(r-1)}(x)\\
&=(6x+2r-1)\sum_{i=0}^{n}\binom{n}{i}b_i^{(r-1)}(x)-\left(6xr-6x+r^2-2r+2\right)\sum_{i=1}^{n}\binom{n}{i}b_{i-1}^{(r-1)}(x)+b_0^{(r-1)}(x)+b_1^{(r-1)}(x)\\
&=(6x+2r-1)b_n^{(r)}(x)-\left(6xr-6x+r^2-2r+2\right)\sum_{i=1}^{n}\binom{n}{i}b_{i-1}^{(r-1)}(x)+1.
\end{aligned}$$

Then, we write

$$b_{n+1}^{(r)}(x)-(6x+2r-1)b_n^{(r)}(x)=-(6xr-6x+r^2-2r+2)\sum_{i=1}^{n}\binom{n}{i}b_{i-1}^{(r-1)}(x)+1. \tag{2.3}$$

Now, if n is replaced by $n-1$ in equation (2.3), we get

$$b_n^{(r)}(x)-(6x+2r-1)b_{n-1}^{(r)}(x)=-(6xr-6x+r^2-2r+2)\sum_{i=1}^{n-1}\binom{n-1}{i}b_{i-1}^{(r-1)}(x)+1.$$

Afterward, we obtain by algebraic operation

$$\begin{aligned}
b_n^{(r)}(x)&=(6x+2r-1)b_{n-1}^{(r)}(x)-(6xr-6x+r^2-2r+2)\sum_{i=1}^{n}\left(\binom{n}{i}-\binom{n-1}{i-1}\right)b_{i-1}^{(r-1)}(x)+1\\
&=(6x+2r-1)b_{n-1}^{(r)}(x)-(6xr-6x+r^2-2r+2)\sum_{i=1}^{n}\binom{n}{i}b_{i-1}^{(r-1)}(x)\\
&\quad+(6xr-6x+r^2-2r+2)\sum_{i=1}^{n}\binom{n-1}{i-1}b_{i-1}^{(r-1)}(x)+1\\
&=(6x+2r-1)b_{n-1}^{(r)}(x)-(6xr-6x+r^2-2r+2)\sum_{i=1}^{n}\binom{n}{i}b_{i-1}^{(r-1)}(x)\\
&\quad+(6xr-6x+r^2-2r+2)\sum_{i=0}^{n-1}\binom{n-1}{i}b_{i}^{(r-1)}(x)+1\\
&=(6x+2r-1)b_{n-1}^{(r)}(x)-(6xr-6x+r^2-2r+2)\sum_{i=1}^{n}\binom{n}{i}b_{i-1}^{(r-1)}(x)\\
&\quad+(6xr-6x+r^2-2r+2)b_{n-1}^{(r)}(x)+1\\
&=(6xr+r^2+1)b_{n-1}^{(r)}(x)-(6xr-6x+r^2-2r+2)\sum_{i=1}^{n}\binom{n}{i}b_{i-1}^{(r-1)}(x)+1.
\end{aligned}$$

That is, we can write

$$-(6xr-6x+r^2-2r+2)\sum_{i=1}^{n}\binom{n}{i}b_{i-1}^{(r-1)}(x)+1=b_n^{(r)}(x)-(6xr+r^2+1)b_{n-1}^{(r)}(x). \tag{2.4}$$

Then, we substitute equation (2.4) in equation (2.3)

$$b_{n+1}^{(r)}(x)-(6x+2r-1)b_n^{(r)}(x)=b_n^{(r)}(x)-(6xr+r^2+1)b_{n-1}^{(r)}(x)$$

which is completed the proof of this theorem.

□

The characteristic equation of sequences $\left\{b_n^{(r)}(x)\right\}$ and $\left\{c_n^{(r)}(x)\right\}$ in (2.1) and (2.2) is $\lambda^2-(6x+2r)\lambda+\left(6xr+r^2+1\right)=0.$ Let be λ_1 and λ_2 the roots of this equation. Then, Binet formulas of sequences $\left\{b_n^{(r)}(x)\right\}$ and $\left\{c_n^{(r)}(x)\right\}$ can be expressed as

$$b_n^{(r)}(x)=\frac{\lambda_1^n-\lambda_2^n}{\lambda_1-\lambda_2}, \tag{2.5}$$

and

$$c_n^{(r)}(x)=\frac{\lambda_1^n+\lambda_2^n}{2}, \tag{2.6}$$

where $\lambda_{1,2}=3x+r\pm\sqrt{9x^2-1}$.

Now, we give the sums of iterated binomial transforms for the balancing and Lucas-balancing polynomials.

Theorem 2.3. *Sums of sequences* $\left\{b_n^{(r)}(x)\right\}$ *and* $\left\{c_n^{(r)}(x)\right\}$ *are*

$$i)\sum_{i=0}^{n-1} b_i^{(r)}(x)=\frac{\left(6xr+r^2+1\right)b_{n-1}^{(r)}(x)-b_n^{(r)}(x)+1}{2-6x-2r+6xr+r^2}$$

$$ii)\sum_{i=0}^{n-1} c_i^{(r)}(x)=\frac{\left(6xr+r^2+1\right)c_{n-1}^{(r)}(x)-c_n^{(r)}(x)-3x-r+1}{2-6x-2r+6xr+r^2}.$$

Proof. We omit the balancing case since the proof be quite similar. By considering equation (2.6), we have

$$\sum_{i=0}^{n-1} c_i^{(r)}(x)=\sum_{i=0}^{n-1}\left(\frac{\lambda_1^i+\lambda_2^i}{2}\right).$$

Then we obtain

$$\sum_{i=0}^{n-1} c_i^{(r)}(x)=\frac{1}{2}\left[\frac{\lambda_1^n-1}{\lambda_1-1}+\frac{\lambda_2^n-1}{\lambda_2-1}\right].$$

Afterward, by taking account equations $\lambda_1\cdot\lambda_2=6xr+r^2+1$ and $\lambda_1+\lambda_2=6x+2r,$ we conclude

$$\sum_{i=0}^{n-1} c_i^{(r)}(x) = \frac{\left(6xr + r^2 + 1\right)c_{n-1}^{(r)}(x) - c_n^{(r)}(x) - 3x - r + 1}{2 - 6x - 2r + 6xr + r^2}.$$

□

Theorem 2.4. *The generating functions of the iterated binomial transforms for* $\{B_n(x)\}$ *and* $\{C_n(x)\}$ *are*

i) $\sum_{i=0}^{\infty} b_i^{(r)}(x) t^i = \dfrac{t}{1-(6x+2r)t+\left(6xr+r^2+1\right)t^2},$

ii) $\sum_{i=0}^{\infty} c_i^{(r)}(x) t^i = \dfrac{1-3xt-rt}{1-(6x+2r)t+\left(6xr+r^2+1\right)t^2}.$

Proof. *i)* Assume that $b(x,r,t) = \sum_{i=0}^{\infty} b_i^{(r)}(x) t^i$ is the generating function of the iterated binomial transform for $\{B_n(x)\}$. From Theorem 2.2, we obtain

$$\begin{aligned}
b(x,r,t) &= b_0^{(r)}(x) + b_1^{(r)}(x)t + \sum_{i=2}^{\infty}\left((6x+2r)b_{i-1}^{(r)}(x) - \left(6xr+r^2+1\right)b_{i-2}^{(r)}(x)\right)t^i \\
&= b_0^{(r)}(x) + b_1^{(r)}(x)t + (6x+2r)t\sum_{i=2}^{\infty} b_{i-1}^{(r)}(x)t^{i-1} - \left(6xr+r^2+1\right)t^2\sum_{i=2}^{\infty} b_{i-2}^{(r)}(x)t^{i-2} \\
&= b_0^{(r)}(x) + b_1^{(r)}(x)t + (6x+2r)t\sum_{i=1}^{\infty} b_i^{(r)}(x)t^i - \left(6xr+r^2+1\right)t^2\sum_{i=0}^{\infty} b_i^{(r)}(x)t^i \\
&= b_0^{(r)}(x) + b_1^{(r)}(x)t + (6x+2r)t\sum_{i=0}^{\infty} b_i^{(r)}(x)t^i - (6x+2r)tb_0^{(r)}(x) - \left(6xr+r^2+1\right)t^2\sum_{i=0}^{\infty} b_i^{(r)}(x)t^i \\
&= t + (6x+2r)tb(x,r,t) - \left(6xr+r^2+1\right)t^2 b(x,r,t)
\end{aligned}$$

Now rearrangement the equation implies that

$$b(x,r,t) = \frac{t}{1-(6x+2r)t+\left(6xr+r^2+1\right)t^2},$$

which equal to the $\sum_{i=0}^{\infty} b_i^{(r)}(x) t^i$ in theorem. Hence the result.

ii) The proof of generating function of the iterated binomial transform for $\{C_n(x)\}$ can see by taking account proof of *i)* . □

Using the Binet formulas, we give the Catalan formulas of these transforms, as the next proposition shows.

Proposition 2.5. For integers k, n with $k < n$, we have

$$i)\, b_{n-k}^{(r)}(x)\, b_{n+k}^{(r)}(x) - \left(b_n^{(r)}(x)\right)^2 = -\left(6xr + r^2 + 1\right)^{n-k}\left(b_k^{(r)}(x)\right)^2,$$

$$ii)\, c_{n-k}^{(r)}(x)\, c_{n+k}^{(r)}(x) - \left(c_n^{(r)}(x)\right)^2 = \left(6xr + r^2 + 1\right)^{n-k}\left(9x^2 - 1\right)\left(b_k^{(r)}(x)\right)^2.$$

If we replace $k = 1$ in Proposition 2.5; we give the Cassini formulas of these transforms.

Corollary 2.6. For integers n, we have

$$i)\, b_{n-1}^{(r)}(x)\, b_{n+1}^{(r)}(x) - \left(b_n^{(r)}(x)\right)^2 = -\left(6xr + r^2 + 1\right)^{n-1},$$

$$ii)\, c_{n-1}^{(r)}(x)\, c_{n+1}^{(r)}(x) - \left(c_n^{(r)}(x)\right)^2 = \left(6xr + r^2 + 1\right)^{n-1}\left(9x^2 - 1\right).$$

Conclusion

In this paper, we define the iterated binomial transforms for balancing and Lucas-balancing polynomials and present some properties of these transforms. By the results in Sections 2 of this paper, we have a great opportunity to compare and obtain some new properties over these transforms. Thus, we extend some recent result in the literature. By taking into account these transforms and its properties, it also can be obtained properties of the iterated binomial transforms of balancing and Lucas-balancing numbers. That is,

- *If we replace $x = 1$ in $b_n^{(r)}(x)$; we obtain the iterated binomial transforms for balancing numbers.*
- *If we replace $x = 1$ in $c_n^{(r)}(x)$; we obtain the iterated binomial transforms for Lucas-balancing numbers.*
- *If we replace $x = k$ in $b_n^{(r)}(x)$; we obtain the iterated binomial transforms for k-balancing numbers.*

- *If we replace* $x = k$ *in* $c_n^{(r)}(x)$*; we obtain the iterated binomial transforms for Lucas k-balancing numbers.*

- *If we replace* $r = 1$ *in* $b_n^{(r)}(x)$*; we obtain the binomial transforms for balancing polynomials.*

- *If we replace* $r = 1$ *in* $c_n^{(r)}(x)$ *; we obtain the binomial transforms for Lucas-balancing polynomials.*

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Lacunary Statistical Convergence of Sequences in Neutrosophic Normed Spaces

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Abstract

In this study, we examine lacunary statistical convergence in neutrosophic normed space. We give the lacunary statistically Cauchy sequence in neutrosophic normed space and present the lacunary statistically completeness in connection with a neutrosophic normed space.

Keywords: Lacunary statistical convergence, neutrosophic normed space.

1. Introduction

Fuzzy Sets (FSs) put forward by Zadeh [28] has influenced deeply all the scientific fields since the publication of the paper. It is seen that this concept, which is very important for real-life situations, had not enough solution to some problems in time. New quests for such problems have been coming up. Atanassov [1] examined Intuitionistic fuzzy sets (IFSs). Neutrosophic set (NS) is a new version of the idea of the classical set which is originated by Smarandache [24]. The first world publication related to the concept of neutrosophy was published in 1998 and involved in the literature [22]. Examples of other generalizations are FS [28] interval-valued FS [26], IFS [1], interval-valued IFS [2], the sets dialetheist, paradoxist, paraconsistent and tautological [23], Pythagorean fuzzy sets [27].

Using the concepts Probabilistic metric space and fuzzy, fuzzy metric space (FMS) is introduced in [14]. Kaleva and Seikkala [9] have defined the FMS as a distance between two points to be a non-negative fuzzy number. In [7] some basic properties of FMS studied and the Baire Category Theorem for FMS proved. Further, some properties such as separability, countability are given and Uniform Limit Theorem is proved in [8]. Afterward, FMS has used in the applied sciences such as fixed point theory, image and signal processing, medical imaging, decision-making et al. After defined of the intuitionistic fuzzy set (IFS), it was used in all areas where FS theory was studied. Park [21] defined IF metric space (IFMS), which is a generalization of FMSs. Park used George and Veeramani's [7] thought of applying t-norm and t-conorm to FMS meanwhile defining IFMS and studying its basic features.

Bera and Mahapatra examined the neutrosophic soft linear spaces (NSLSs) [3]. Later, neutrosophic soft normed linear spaces (NSNLS) has been given by Bera and Mahapatra [4]. In [4], neutrosophic norm, Cauchy sequence in NSNLS, convexity of NSNLS, metric in NSNLS were defined. In future studies on this subject, it is also possible to study with the idea of ''Probabilistic metric space'' using neutrosophic probability [25]. Kirişci and Şimşek [31] originated the neutrosophic normed space and the statistical convergence in neutrosophic normed space.

Lacunary statistical convergence was given by Fridy and Orhan [29]. Çakan and Altay [30] presented multidimensional analogues of the results presented by Fridy and Orhan [29]. Mursaleen and Mohiuddien [18] studied lacunary statistical convergence in intuitionistic fuzzy normed space.

Triangular norms (t-norms) (TN) were initiated by Menger [16]. In the problem of computing the distance between two elements in space, Menger offered using probability distributions instead of using numbers for distance. TNs are used to generalize with the probability distribution of triangle inequality in metric space conditions. Triangular conorms (t-conorms) (TC) know as dual operations of TNs. TNs and TCs are very significant for fuzzy operations (intersections and unions).

Definition 1.1. Give an operation $\circ: [0,1] \times [0,1] \to [0,1]$. If the operation $\circ$ is satisfying the following conditions, then it is called that the operation $\circ$ is continuous TN: For $s, t, u, v \in [0,1]$,

(I) $s \circ 1 = s$,
(II) If $s \leq u$ and $t \leq v$, then $s \circ t \leq u \circ v$,
(III) $\circ$ is continuous,
(IV) $\circ$ is commutative and associative.

Definition 1.2. Give an operation $*: [0,1] \times [0,1] \to [0,1]$. If the operation $*$ is satisfying the following conditions, then it is called that the operation $*$ is continuous TC:

(I) $s * 0 = s$,
(II) If $s \leq u$ and $t \leq v$, then $s * t \leq u * v$,
(III) $*$ is continuous,
(IV) $*$ is commutative and associative.

Definition 1.3. Take F be an arbitrary set, $\mathcal{N} = \{\langle u, \mathcal{G}(u), \mathcal{B}(u), \mathcal{Y}(u)\rangle : u \in F\}$ be a NS such that $\mathcal{N} = F \times \mathbb{R}^+ \to [0,1]$. Let $\circ$ and $*$ show the continuous TN and continuous TC, respectively. If the following conditions are hold, then the four tuple $V = (F, \mathcal{N}, \circ, *)$ is called NNS: For all $u, v \in F$ and $\lambda, \mu > 0$ and for each $\sigma \neq 0$,

(I) $0 \leq \mathcal{G}(u,\lambda) \leq 1$, $0 \leq \mathcal{B}(u,\lambda) \leq 1$, $0 \leq \mathcal{Y}(u,\lambda) \leq 1$ $\forall \lambda \in \mathbb{R}^+$,
(II) $\mathcal{G}(u,\lambda) + \mathcal{B}(u,\lambda) + \mathcal{Y}(u,\lambda) \leq 3$, for $\lambda \in \mathbb{R}^+$
(III) $\mathcal{G}(u,\lambda) = 1$ for $\lambda > 0$ iff $u = 0$
(IV) $\mathcal{G}(\sigma u,\lambda) = \mathcal{G}\left(u, \frac{\lambda}{|\sigma|}\right)$,
(V) $\mathcal{G}(u,\mu) \circ \mathcal{G}(v,\lambda) \leq \mathcal{G}(u+v, \lambda+\mu)$,
(VI) $\mathcal{G}(u,.)$ is continuous non-decreasing function,
(VII) $\lim_{\lambda\to\infty} \mathcal{G}(u,\lambda) = 1$,
(VIII) $\mathcal{B}(u,\lambda) = 0$ for $\lambda > 0$ iff $u = 0$,
(IX) $\mathcal{B}(\sigma u,\lambda) = \mathcal{B}\left(u, \frac{\lambda}{|\sigma|}\right)$
(X) $\mathcal{B}(u,\mu) * \mathcal{B}(v,\lambda) \leq \mathcal{B}(u+v, \lambda+\mu)$,
(XI) $\mathcal{B}(u,.)$ is continuous non-decreasing function,

(XII) $\lim_{\lambda\to\infty}\mathcal{B}(u,\lambda)=0$,
(XIII) $\mathcal{Y}(u,\lambda)=0$ for $\lambda>0$ iff $u=0$,
(XIV) $\mathcal{Y}(\sigma u,\lambda)=\mathcal{Y}\left(u,\frac{\lambda}{|\sigma|}\right)$,
(XV) $\mathcal{Y}(u,\mu)*\mathcal{Y}(v,\lambda)\leq\mathcal{Y}(u+v,\lambda+\mu)$,
(XVI) $\mathcal{Y}(u,.)$ is continuous non-decreasing function,
(XVII) $\lim_{\lambda\to\infty}\mathcal{Y}(u,\lambda)=0$,
(XVIII) If $\lambda\leq 0$, then $\mathcal{G}(u,\lambda)=0$, $\mathcal{B}(u,\lambda)=1$ and $\mathcal{Y}(u,\lambda)=1$.

Definition 1.4. Let V be a NNS, then the sequence (a_n) in V, $0<\varepsilon<1$ and $\lambda>0$. Then, the sequence (a_n) is converges to a iff there exists $N\in\mathbb{N}$ such that $\mathcal{G}(a_n-a,\lambda)>1-\varepsilon$, $\mathcal{B}(a_n-a,\lambda)<\varepsilon$ and $\mathcal{Y}(a_n-a,\lambda)<\varepsilon$. That is $\lim_{n\to\infty}\mathcal{G}(a_n-a,\lambda)=1$, $\lim_{n\to\infty}\mathcal{B}(a_n-a,\lambda)=0$ and $\lim_{n\to\infty}\mathcal{Y}(a_n-a,\lambda)=0$ as $\lambda>0$. In that case, the sequence (a_n) is called a convergent sequence in V. The convergent in NNS is denoted by $\mathcal{N}-\lim a_n=a$.

2. Main Results

Definition 2.1. Take a NNS V. A sequence (x_m) is said to be lacunary statistically convergent to $\xi\in F$ with regards to NN (LSC-NN), if for each $\lambda>0$ and $\varepsilon>0$,

$$\delta_\theta(\{m\in\mathbb{N}:\mathcal{G}(x_m-\xi,\lambda)\leq 1-\varepsilon \text{ or } \mathcal{B}(x_m-\xi,\lambda)\geq\varepsilon,\mathcal{Y}(x_m-\xi,\lambda)\geq\varepsilon\})=0,$$

or equivalently

$$\delta_\theta(\{m\in\mathbb{N}:\mathcal{G}(x_m-\xi,\lambda)>1-\varepsilon \text{ and } \mathcal{B}(x_m-\xi,\lambda)<\varepsilon,\mathcal{Y}(x_m-\xi,\lambda)<\varepsilon\})=1.$$

Therefore, we get

$$\lim_{r\to\infty}\frac{1}{h_r}|\{m\in I_r:\mathcal{G}(x_m-\xi,\lambda)\leq 1-\varepsilon \text{ or } \mathcal{B}(x_m-\xi,\lambda)\geq\varepsilon,\mathcal{Y}(x_m-\xi,\lambda)\geq\varepsilon\}|=0.$$

Therefore, we write $S_{\theta(\mathcal{N})}-\lim x_m=\xi$ or $x_m\to\xi\left(S_{\theta(\mathcal{N})}\right)$. The set of LSC-NN will be denoted by $S_{\theta(\mathcal{N})}$.

Example 2.1. Let $(F,\|.\|)$ be a NS. For all $a,b\in[0,1]$, take the TN $a\circ b=ab$ and the TC $a*b=\min\{a+b,1\}$. For all $x\in F$ and every $\lambda>0$, we consider $\mathcal{G}(x,\lambda)=\frac{\lambda}{\lambda+\|x\|}$, $\mathcal{B}(x,\lambda)=\frac{\|x\|}{\lambda+\|x\|}$ and $\mathcal{Y}(x,\lambda)=\frac{\|x\|}{\lambda}$. Then V is a NNS. We define a sequence (x_m) by

$$x_m=\begin{cases} m, & \text{for } k_r-[\![\sqrt{h_r}]\!]+1\leq m\leq k_r, (r\in\mathbb{N}) \\ 0, & \text{otherwise.}\end{cases}$$

Consider

$$A_r(\varepsilon,\lambda):=\{m\in\mathbb{N}:\mathcal{G}(x_m,\lambda)\leq 1-\varepsilon \text{ or } \mathcal{B}(x_m,\lambda)\geq\varepsilon,\mathcal{Y}(x_m,\lambda)\geq\varepsilon\}$$

for every $\varepsilon>0$ and for any $\lambda>0$. Then, we get

$$A_r(\varepsilon,\lambda)=\left\{m\in\mathbb{N}:\frac{\lambda}{\lambda+\|x_m\|}\le 1-\varepsilon \text{ or } \frac{\|x_m\|}{\lambda+\|x_m\|}\ge\varepsilon,\frac{\|x_m\|}{\lambda}\ge\varepsilon\right\}$$
$$=\left\{m\in\mathbb{N}:\quad \|x_m\|\ge\frac{\lambda\varepsilon}{1-\varepsilon} \text{ or } \|x_m\|\ge\lambda\varepsilon\right\}$$
$$=\{m\in\mathbb{N}:x_m=m\}=\left\{m\in\mathbb{N}:k_r-\left[\!\left[\sqrt{h_r}\right]\!\right]+1\le m\le k_r,(r\in\mathbb{N})\right\}.$$

Then we get,

$$\frac{1}{h_r}|A_r(\varepsilon,\lambda)|=\frac{1}{h_r}\left|\left\{m\in\mathbb{N}:k_r-\left[\!\left[\sqrt{h_r}\right]\!\right]+1\le m\le k_r\,,(r\in\mathbb{N})\right\}\right|\le\frac{\left[\!\left[\sqrt{h_r}\right]\!\right]}{h_r},$$

which implies that $\lim_{r\to\infty}\frac{1}{h_r}|A_r(\varepsilon,\lambda)|=0$. Hence

$$\delta_\theta\big(A_r(\varepsilon,\lambda)\big)=\lim_{r\to\infty}\frac{\left[\!\left[\sqrt{h_r}\right]\!\right]}{h_r}=0$$

implies that $x_m\to 0\big(S_{\theta(\mathcal{N})}\big)$. On the other hand $x_m\nrightarrow 0(\mathcal{N})$, since

$$\mathcal{G}(x_m,\lambda)=\frac{\lambda}{\lambda+\|x_m\|}=\begin{cases}\dfrac{\lambda}{\lambda+m}, for\ k_r-\left[\!\left[\sqrt{h_r}\right]\!\right]+1\le m\le k_r\,,(r\in\mathbb{N})\\ 1, \qquad otherwise;\end{cases}\le 1$$

and

$$\mathcal{B}(x_m,\lambda)=\frac{\|x_m\|}{\lambda+\|x_m\|}=\begin{cases}\dfrac{m}{\lambda+m}, for\ k_r-\left[\!\left[\sqrt{h_r}\right]\!\right]+1\le m\le k_r\,,(r\in\mathbb{N})\\ 0, \qquad otherwise;\end{cases}\ge 0,$$
$$\mathcal{Y}(x_m,\lambda)=\frac{\|x_m\|}{\lambda}=\begin{cases}\dfrac{m}{\lambda}, for\ k_r-\left[\!\left[\sqrt{h_r}\right]\!\right]+1\le m\le k_r\,,(r\in\mathbb{N})\\ 0, \qquad otherwise;\end{cases}\ge 0.$$

This completes the proof.

Lemma 2.1. Take a NNS V. The following statements are equivalent, for every $\varepsilon>0$ and $\lambda>0$:

1. $S_{\theta(\mathcal{N})}-\lim x_m=\xi$,
2. $\lim_{r\to\infty}\frac{1}{h_r}|\{m\in I_r:\mathcal{G}(x_m-\xi,\lambda)\le 1-\varepsilon\}|=\lim_{r\to\infty}\frac{1}{h_r}|\{m\in I_r:\mathcal{B}(x_m-\xi,\lambda)\ge\varepsilon\}|=\lim_{r\to\infty}\frac{1}{h_r}|\{m\in I_r:\mathcal{Y}(x_m-\xi,\lambda)\ge\varepsilon\}|=0$,
3. $\lim_{r\to\infty}\frac{1}{h_r}|\{m\in I_r:\mathcal{G}(x_m-\xi,\lambda)>1-\varepsilon \text{ and } \mathcal{B}(x_m-\xi,\lambda)<\varepsilon,\mathcal{Y}(x_m-\xi,\lambda)<\varepsilon\}|=1$,
4. $\lim_{r\to\infty}\frac{1}{h_r}|\{m\in I_r:\mathcal{G}(x_m-\xi,\lambda)>1-\varepsilon\}|=\lim_{r\to\infty}\frac{1}{h_r}|\{m\in I_r:\mathcal{B}(x_m-\xi,\lambda)<\varepsilon\}|=\lim_{r\to\infty}\frac{1}{h_r}|\{m\in I_r:\mathcal{Y}(x_m-\xi,\lambda)<\varepsilon\}|=1$,
5. $S_{\theta(\mathcal{N})}-\lim\mathcal{G}(x_m-\xi,\lambda)=1$, and $S_{\theta(\mathcal{N})}-\lim\mathcal{B}(x_m-\xi,\lambda)=0$, $S_{\theta(\mathcal{N})}-\lim\mathcal{Y}(x_m-\xi,\lambda)=0$.

Theorem 2.1. Take a NNS V. If (x_m) is LSC-NN, then $S_{\theta(\mathcal{N})}-\lim x_m=\xi$ is unique.

Proof. Assume that $S_{\theta(\mathcal{N})}-\lim x_m=\xi_1$ and $S_{\theta(\mathcal{N})}-\lim x_m=\xi_2$ for $\xi_1\neq\xi_2$. Choose $\varepsilon>0$. Then, for a given $\mu>0$, $(1-\varepsilon)\circ(1-\varepsilon)>1-\mu$ and $\varepsilon*\varepsilon<\mu$. For any $\lambda>0$, let's write the following sets:

$$K_{\mathcal{G}_1}(\varepsilon,\lambda):=\left\{m\in\mathbb{N}:\mathcal{G}\left(x_m-\xi_1,\frac{\lambda}{2}\right)\leq 1-\varepsilon\right\},$$
$$K_{\mathcal{B}_1}(\varepsilon,\lambda):=\left\{m\in\mathbb{N}:\mathcal{B}\left(x_m-\xi_1,\frac{\lambda}{2}\right)\geq \varepsilon\right\},$$
$$K_{\mathcal{Y}_1}(\varepsilon,\lambda):=\left\{m\in\mathbb{N}:\mathcal{Y}\left(x_m-\xi_1,\frac{\lambda}{2}\right)\geq \varepsilon\right\}$$

and

$$K_{\mathcal{G}_2}(\varepsilon,\lambda):=\left\{m\in\mathbb{N}:\mathcal{G}\left(x_m-\xi_2,\frac{\lambda}{2}\right)\leq 1-\varepsilon\right\},$$
$$K_{\mathcal{B}_2}(\varepsilon,\lambda):=\left\{m\in\mathbb{N}:\mathcal{B}\left(x_m-\xi_2,\frac{\lambda}{2}\right)\geq \varepsilon\right\},$$
$$K_{\mathcal{Y}_2}(\varepsilon,\lambda):=\left\{m\in\mathbb{N}:\mathcal{Y}\left(x_m-\xi_2,\frac{\lambda}{2}\right)\geq \varepsilon\right\}.$$

We know that $S_{\theta(\mathcal{N})}-\lim x_m=\xi_1$. Then, using the Lemma 2.1, for all $\lambda>0$,

$$\delta_\theta\left(K_{\mathcal{G}_1}(\mu,\lambda)\right)=\delta_\theta\left(K_{\mathcal{B}_1}(\mu,\lambda)\right)=\delta_\theta\left(K_{\mathcal{Y}_1}(\mu,\lambda)\right)=0.$$

Further, since we get $S_{\theta(\mathcal{N})}-\lim x_m=\xi_2$, using the Lemma 2.1, for all $\lambda>0$,

$$\delta_\theta\left(K_{\mathcal{G}_2}(\mu,\lambda)\right)=\delta_\theta\left(K_{\mathcal{B}_2}(\mu,\lambda)\right)=\delta_\theta\left(K_{\mathcal{Y}_2}(\mu,\lambda)\right)=0.$$

Let

$$K_{\mathcal{N}}\big((\mu,\lambda)\big):=\{K_{\mathcal{G}_1}(\mu,\lambda)\cup K_{\mathcal{G}_2}(\mu,\lambda)\}\cap\{K_{\mathcal{B}_1}(\mu,\lambda)\cup K_{\mathcal{B}_2}(\mu,\lambda)\}\cap\{K_{\mathcal{Y}_1}(\mu,\lambda)\cup K_{\mathcal{Y}_2}(\mu,\lambda)\}.$$

Then, observe that $\delta_\theta\big(K_{\mathcal{N}}(\mu,\lambda)\big)=0$ which implies $\delta_\theta\big(\mathbb{N}\setminus K_{\mathcal{N}}(\mu,\lambda)\big)=1$. Then, we have three possible situations, when take $k\in\mathbb{N}/K_{\mathcal{N}}(\mu,\lambda)$:

1. $k\in\mathbb{N}\setminus\left(K_{\mathcal{G}_1}(\mu,\lambda)\cup K_{\mathcal{G}_2}(\mu,\lambda)\right)$,
2. $k\in\mathbb{N}\setminus\left(K_{\mathcal{B}_1}(\mu,\lambda)\cup K_{\mathcal{B}_2}(\mu,\lambda)\right)$,
3. $k\in\mathbb{N}\setminus\left(K_{\mathcal{Y}_1}(\mu,\lambda)\cup K_{\mathcal{Y}_2}(\mu,\lambda)\right)$.

Firstly, consider (1). Then we have

$$\mathcal{G}(\xi_1-\xi_2,\lambda)\geq\mathcal{G}\left(x_m-\xi_1,\frac{\lambda}{2}\right)\circ\mathcal{G}\left(x_m-\xi_2,\frac{\lambda}{2}\right)>(1-\varepsilon)\circ(1-\varepsilon)>1-\mu$$

and so $\mathcal{G}(\xi_1-\xi_2,\lambda)>1-\mu$. For all $\lambda>0$, we obtain $\mathcal{G}(\xi_1-\xi_2,\lambda)=1$, where $\mu>0$ is arbitrary. That is, $\xi_1=\xi_2$ is obtained.

For the situation (2), if we take $k\in\mathbb{N}\setminus\left(K_{\mathcal{B}_1}(\mu,\lambda)\cup K_{\mathcal{B}_2}(\mu,\lambda)\right)$, then we can write

$$\mathcal{B}(\xi_1-\xi_2,\lambda)\leq\mathcal{B}\left(x_m-\xi_1,\frac{\lambda}{2}\right)*\mathcal{B}\left(x_m-\xi_2,\frac{\lambda}{2}\right)<\varepsilon*\varepsilon.$$

Using $\varepsilon * \varepsilon < \mu$, we can see that $\mathcal{B}(\xi_1 - \xi_2, \lambda) < \mu$. For all $\lambda > 0$, we have $\mathcal{B}(\xi_1 - \xi_2, \lambda) = 0$, where $\mu > 0$ is arbitrary. Thus $\xi_1 = \xi_2$. Again, for the situation (3), if we take $k \in \mathbb{N} \setminus \left(K_{\mathcal{Y}_1}(\mu, \lambda) \cup K_{\mathcal{Y}_2}(\mu, \lambda)\right)$, then we write

$$\mathcal{Y}(\xi_1 - \xi_2, \lambda) \leq \mathcal{Y}\left(x_m - \xi_1, \frac{\lambda}{2}\right) * \mathcal{Y}\left(x_m - \xi_1, \frac{\lambda}{2}\right) < \varepsilon * \varepsilon.$$

Using $\varepsilon * \varepsilon < \mu$, we can see that $\mathcal{Y}(\xi_1 - \xi_2, \lambda) < \mu$. For all $\lambda > 0$, we get $\mathcal{Y}(\xi_1 - \xi_2, \lambda) = 0$, where $\mu > 0$ is arbitrary. Thus $\xi_1 = \xi_2$. This step completes the proof.

Theorem 2.2. If $\mathcal{N} - \lim x_m = \xi$ for NNS V, then $S_{\theta(\mathcal{N})} - \lim x_m = \xi$.

Proof. If $\mathcal{N} - \lim x_m = \xi$, then, for every $\varepsilon > 0$ and $\lambda > 0$, there exist a number $N \in \mathbb{N}$ such that $\mathcal{G}(x_m - \xi, \lambda) > 1 - \varepsilon$ and $\mathcal{B}(x_m - \xi, \lambda) < \varepsilon, \mathcal{Y}(x_m - \xi, \lambda) < \varepsilon$ for all $k \geq N$. Therefore, the set

$$\{m \in \mathbb{N}: \mathcal{G}(x_m - \xi, \lambda) \leq 1 - \varepsilon \text{ or } \mathcal{B}(x_m - \xi, \lambda) \geq \varepsilon, \mathcal{Y}(x_m - \xi, \lambda) \geq \varepsilon\}$$

has finite number of terms. Since every finite subset of $\mathbb{N}$ has density zero and so

$$\delta_\theta(\{m \in \mathbb{N}: \mathcal{G}(x_m - \xi, \lambda) \leq 1 - \varepsilon \text{ or } \mathcal{B}(x_m - \xi, \lambda) \geq \varepsilon, \mathcal{Y}(x_m - \xi, \lambda) \geq \varepsilon\}) = 0$$

that is $S_{\theta(\mathcal{N})} - \lim x_m = \xi$.

Theorem 2.3. Let V be an NNS. $S_{\theta(\mathcal{N})} - \lim x_m = \xi$ iff there exists a increasing index sequence $J = \{j_1, j_2, \dots\} \subseteq \mathbb{N}$, while $\delta_\theta(J) = 1, \mathcal{N} - \lim_{n \to \infty} x_{j_n} = \xi$.

Proof. Assume that $S_{\theta(\mathcal{N})} - \lim x_m = \xi$. For any $\lambda > 0$ and $\mu = 1,2, \dots$,

$$P_{\mathcal{N}}(\mu, \lambda) = \left\{m \in \mathbb{N}: \mathcal{G}(x_m - \xi, \lambda) > 1 - \frac{1}{\mu} \text{ and } \mathcal{B}(x_m - \xi, \lambda) < \frac{1}{\mu}, \mathcal{Y}(x_m - \xi, \lambda) < \frac{1}{\mu}\right\}$$

and

$$R_{\mathcal{N}}(\mu, \lambda) = \left\{m \in \mathbb{N}: \mathcal{G}(x_m - \xi, \lambda) \leq 1 - \frac{1}{\mu} \text{ or } \mathcal{B}(x_m - \xi, \lambda) \geq \frac{1}{\mu}, \mathcal{Y}(x_m - \xi, \lambda) \geq \frac{1}{\mu}\right\}.$$

Then, $\delta_\theta\left(R_{\mathcal{N}}(\mu, \lambda)\right) = 0$, since $S_{\theta(\mathcal{N})} - \lim x_m = \xi$. Further, for $\lambda > 0$ and $\mu = 1,2, \dots$, $P_{\mathcal{N}}(\mu, \lambda) \supseteq P_{\mathcal{N}}(\mu + 1, \lambda)$ and so,

$$\delta_\theta\left(P_{\mathcal{N}}(\mu, \lambda)\right) = 1. \tag{1}$$

for $\lambda > 0$ and $\mu = 1,2, \dots$.

Now, we will demonstrate that for $m \in P_{\mathcal{N}}(\mu, \lambda)$, $\mathcal{N} - \lim x_m = \xi$. Suppose that $\mathcal{N} - \lim x_m \neq \xi$, for some $m \in P_{\mathcal{N}}(\mu, \lambda)$. Then, there is $\rho > 0$ and a positive integer N such that $\mathcal{G}(x_m - \xi, \lambda) \leq 1 - \rho$ or $\mathcal{B}(x_m - \xi, \lambda) \geq \rho, \mathcal{Y}(x_m - \xi, \lambda) \geq \rho$, for all $m \geq N$. Let $\mathcal{G}(x_m - \xi, \lambda) > 1 - \rho$ or $\mathcal{B}(x_m - \xi, \lambda) < \rho, \mathcal{Y}(x_m - \xi, \lambda) < \rho$, for all $m < N$. Hence

$$\lim_{r \to \infty} \frac{1}{h_r} |\{m \in I_r: \mathcal{G}(x_m - \xi, \lambda) > 1 - \rho \text{ and } \mathcal{B}(x_m - \xi, \lambda) < \rho, \mathcal{Y}(x_m - \xi, \lambda) < \rho\}| = 0.$$

Since $\rho = \frac{1}{\mu}$, we obtain $\delta_\theta\left(P_{\mathcal{N}}(\mu, \lambda)\right) = 0$, which contradicts (1). That's why, $\mathcal{N} - \lim x_m = \xi$.

Suppose that there exists a subset $J = \{j_1, j_2, \ldots\} \subseteq \mathbb{N}$, such that $\delta_\theta(J) = 1$, If $\mathcal{N} - \lim_{n\to\infty} x_{j_n} = \xi$, i.e. $N \in \mathbb{N}$ such that $\mathcal{G}(x_m - \xi, \lambda) > 1 - \mu$ and $\mathcal{B}(x_m - \xi, \lambda) < \mu, \mathcal{Y}(x_m - \xi, \lambda) < \mu$ for every $\mu > 0$ and $\lambda > 0$. In that case,

$$\begin{aligned} R_{\mathcal{N}}(\mu, \lambda) := \{m \in \mathbb{N}: \mathcal{G}(x_m - \xi, \lambda) \leq 1 - \mu \text{ or } \mathcal{B}(x_m - \xi, \lambda) \geq \mu, \mathcal{Y}(x_m - \xi, \lambda) \geq \mu\} \\ \subseteq N - \{j_{N+1}, j_{N+2}, \ldots\}. \end{aligned}$$

Therefore, we have $\delta_\theta(R_{\mathcal{N}}(\mu, \lambda)) \leq 1 - 1 = 0$. Hence, $S_{\theta(\mathcal{N})} - \lim x_m = \xi$.

Definition 2.2. A sequence (x_m) is called to be lacunary statistically Cauchy with regards to NN $\mathcal{N}$(LSCa-NN) in NNS V, if there exists $N = N(\varepsilon)$, for every $\varepsilon > 0$ and $\lambda > 0$ such that

$$KC_\varepsilon := \{m \in \mathbb{N}: \mathcal{G}(x_m - x_N, \lambda) \leq 1 - \varepsilon \text{ or } \mathcal{B}(x_m - x_N, \lambda) \geq \varepsilon, \mathcal{Y}(x_m - x_N, \lambda) \geq \varepsilon\}$$

has lacunary density zero. That is, $\delta_\theta(KC_\varepsilon) = 0$.

Theorem 2.4. If a sequence (x_m) is LSC-NN in NNS V, then it is LSCa-NN.

Proof. Let (x_m) be LSC-NN. We get $(1 - \varepsilon) \circ (1 - \varepsilon) > 1 - \mu$ and $\varepsilon * \varepsilon < \mu$, for a given $\varepsilon > 0$, choose $\mu > 0$. Then, we get

$$\delta_\theta(A(\varepsilon, \lambda)) = \delta_\theta\left(\left\{m \in \mathbb{N}: \mathcal{G}\left(x_m - \xi, \frac{\lambda}{2}\right) \leq 1 - \varepsilon \text{ or } \mathcal{B}\left(x_m - \xi, \frac{\lambda}{2}\right) \geq \varepsilon, \mathcal{Y}\left(x_m - \xi, \frac{\lambda}{2}\right) \geq \varepsilon\right\}\right) \quad (2)$$

and so,

$$\delta_\theta(A^c(\varepsilon, \lambda)) = \delta_\theta\left(\left\{m \in \mathbb{N}: \mathcal{G}\left(x_m - \xi, \frac{\lambda}{2}\right) > 1 - \varepsilon \text{ and } \mathcal{B}\left(x_m - \xi, \frac{\lambda}{2}\right) < \varepsilon, \mathcal{Y}\left(x_m - \xi, \frac{\lambda}{2}\right) < \varepsilon\right\}\right) = 1,$$

for $\lambda > 0$. Let $p \in A^c(\varepsilon, \lambda)$. Then,

$$\mathcal{G}(x_p - \xi, \lambda) > 1 - \varepsilon \text{ and } \mathcal{B}(x_p - \xi, \lambda) < \varepsilon, \mathcal{Y}(x_p - \xi, \lambda) < \varepsilon.$$

Let

$$B(\varepsilon, \lambda) = \{m \in \mathbb{N}: \mathcal{G}(x_m - x_p, \lambda) \leq 1 - \mu \text{ or } \mathcal{B}(x_m - x_p, \lambda) \geq \mu, \mathcal{Y}(x_m - x_p, \lambda) \geq \mu\}.$$

We claim that $B(\varepsilon, \lambda) \subset A(\varepsilon, \lambda)$. Let $q \in B(\varepsilon, \lambda) \setminus A(\varepsilon, \lambda)$. Then,

$$\mathcal{G}(x_q - x_p, \lambda) \leq 1 - \mu \text{ and } \mathcal{G}(x_q - \xi, \lambda) > 1 - \mu,$$

in particular $\mathcal{G}\left(x_p - \xi, \frac{\lambda}{2}\right) > 1 - \varepsilon$. Then,

$$1 - \mu \geq \mathcal{G}(x_q - x_p, \lambda) \geq \mathcal{G}\left(x_q - \xi, \frac{\lambda}{2}\right) \circ \mathcal{G}\left(x_p - \xi, \frac{\lambda}{2}\right) > (1 - \varepsilon) \circ (1 - \varepsilon) > 1 - \mu,$$

which is not possible. Moreover,

$$\mathcal{B}(x_q - x_p, \lambda) \geq \mu \text{ and } \mathcal{B}(x_q - \xi, \lambda) < \mu,$$

in particular $\mathcal{B}\left(x_p - \xi, \frac{\lambda}{2}\right) < \varepsilon$. Then,

$$\mu < \mathcal{B}(x_q - x_p, \lambda) \leq \mathcal{B}\left(x_q - \xi, \frac{\lambda}{2}\right) * \mathcal{B}\left(x_p - \xi, \frac{\lambda}{2}\right) < \varepsilon * \varepsilon < \mu,$$

which is not possible. Similarly,

$$\mathcal{Y}(x_q - x_p, \lambda) \geq \mu \text{ and } \mathcal{Y}(x_q - \xi, \lambda) < \mu,$$

in particular $\mathcal{Y}\left(x_p - \xi, \frac{\lambda}{2}\right) < \varepsilon$. Then,

$$\mu < \mathcal{Y}(x_q - x_p, \lambda) < \mathcal{Y}\left(x_q - \xi, \frac{\lambda}{2}\right) * \mathcal{Y}\left(x_p - \xi, \frac{\lambda}{2}\right) < \varepsilon * \varepsilon < \mu,$$

which is not possible. In that case, $B(\varepsilon, \lambda) \subset A(\varepsilon, \lambda)$. Then, by (2), $\delta_\theta(A(\varepsilon, \lambda)) = 0$ and (x_m) is LSCa-NN.

Theorem 2.5. The NNS V is called lacunary statistically (LSC-NN) complete, if every LSCa-NN is LSC-NN.

Proof. Let (x_m) be LSCa-NN but not LSC-NN. Choose $\mu > 0$. We get $(1 - \varepsilon) \circ (1 - \varepsilon) > 1 - \mu$ and $\varepsilon * \varepsilon < \mu$, for a given $\varepsilon > 0$, choose $\mu > 0$. Since (x_m) is not LSC-NN,

$$\mathcal{G}(x_m - x_N, \lambda) \geq \mathcal{G}\left(x_m - \xi, \frac{\lambda}{2}\right) \circ \mathcal{G}\left(x_N - \xi, \frac{\lambda}{2}\right) > (1 - \varepsilon) \circ (1 - \varepsilon) > 1 - \mu,$$

$$\mathcal{B}(x_m - x_N, \lambda) < \mathcal{B}\left(x_m - \xi, \frac{\lambda}{2}\right) * \mathcal{B}\left(x_N - \xi, \frac{\lambda}{2}\right) < \varepsilon * \varepsilon < \mu,$$

$$\mathcal{Y}(x_m - x_N, \lambda) < \mathcal{Y}\left(x_m - \xi, \frac{\lambda}{2}\right) * \mathcal{Y}\left(x_N - \xi, \frac{\lambda}{2}\right) < \varepsilon * \varepsilon < \mu.$$

For,

$$T(\varepsilon, \lambda) = \left\{m \in \mathbb{N}: \mathcal{B}_{x_m - x_N}(\varepsilon) \leq 1 - \mu\right\},$$

$\delta_\theta(T^c(\varepsilon, \lambda)) = 0$ and $\delta_\theta(T(\varepsilon, \lambda)) = 1$, which is contradiction, since (x_m) was LSCa-NN. So that (x_m) must be LSC-NN. Hence every NNS is (LSC-NN) complete.

From Theorems 2.3, 2.4, 2.5, we have:

Theorem 2.6. Let V be an NNS. Then, for any sequence $(x_m) \in F$, following conditions are equivalent:

1. (x_m) is LSC-N.
2. (x_m) is LSCa-NN.
3. NNS V is (LSC-NN) complete.
4. There exists an increasing index sequence $J = (j_n)$ of natural numbers such that $\delta_\theta(J) = 1$ and sequence (x_{j_n}) is LSCa-NN.

4. Conclusion

In this study, lacunary statistical convergence in neutrosophic normed space is introduced and some fundamental properties are examined. Further, lacunary statistical Cauchy sequence and lacunary statistically completeness for neutrosophic norm are defined.

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Laplace Transform Collocation Method for Pseudo-hyperbolic Partial Differential Equation

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Abstract

In this paper, Laplace transform collocation method is applied to Pseudo-hyperbolic partial differential equation. First, definition of this method is given for approximation solution. Second, the exact solution is obtained for this equation. Error analysis is calculated by comparing the exact and approximate solution for this problem. This method was found to be convenient and effective on a sample problem.

1. Introduction

Partial differential equations (PDEs) with both constant and variable coefficients arise in many brances of science and engineering, for example, eletromagnetic, thermodynamics, hydrodynamics, elektrodynamics, elasticity, fluid dynamics, wave propogation, and materials science [1-3]. Due to its tremendous scope and applications in several disciplines, a considerable attention has been given to exact and numerical solutions of the partial differential equations. Finite difference methods in particular became very popular and a large number of schemes has been published very recently. Consequently it becomes important to understand how they compare in terms of accuracy, stability and computing times In [4-5], FDTM and MFDTM to solve third-order dispersive partial differential equations were used. Laplace transform collocation method was used for telegraph partial differential equation [7].

In this paper, we consider initial boundary value problem for pseudo-hyperbolic partial differential equation

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} = \lambda \frac{\partial^3 u(t,x)}{\partial t \partial x^2} + \frac{\partial^2 u(t,x)}{\partial x^2} + f(t,x), \ \ 0 < x < L, \ \ 0 < t < T, \\ u(0,x) = r_1(x), \ \ u_t(0,x) = r_2(x), \ 0 \le x \le L, \\ u(t,0) = u(t,L) = 0, \ 0 \le t \le T, \ \ \lambda > 0, \end{cases} \tag{1.1}$$

where λ is known constant coefficient, f , r_1, r_2 are known continuous functions in their respective domains, and the function u is unknown. For this problem, basic definitions are given. For the exact solution of this problem are investigated by Homotopy Analysis Method(HAM). A new form of trial function from the basic equation is obtained with the use of Laplace transform technique for telegraph partial equation [7].

2. Laplace Transform Collocation Method (LTCM)

To put emphasis on the essential mathematical details of the Laplace transform collocation method, we consider (1.1) pseudo-hyperbolic partial differential equation similiar the method [7].

Taking the Laplace transform of (1.1), we have

$$s^2 u(s,x) - su(0,x) - u_t(0,x) = \lambda \mathcal{L}\left\{\frac{\partial^3 u(t,x)}{\partial t \partial x^2}\right\} + \mathcal{L}\left\{\frac{\partial^2 u(t,x)}{\partial x^2}\right\} + \mathcal{L}\{f(t,x)\} \tag{2.1}$$

After simple algebraic simplification, we get

$$U(s,x) = \frac{1}{s^2}\{sr_1(x) + r_2(x)\} + \lambda \mathcal{L}\left\{\frac{\partial^3 u(t,x)}{\partial t \partial x^2}\right\} + \mathcal{L}\left\{\frac{\partial^2 u(t,x)}{\partial x^2}\right\} + \mathcal{L}\{f(t,x)\} \tag{2.2}$$

The function $u(t,x)$and its derivative in the formula (2.2) are there after replaced with a trial function of the form

$$u = u_0 + \sum_{n=1}^{k} c_n u_n \tag{2.3}$$

where c_n are constants to be determined which satisfy the given conditions in the formula (1.1). Thus, we have the following equation

$$\begin{cases} U(s,x) = \frac{1}{s^2}\left\{\mathrm{s}\left(u_0(0,x) + \sum_{n=1}^{k} c_n u_n(0,x)\right)\right. \\ +\frac{\partial}{\partial t}\left(u_0(0,x) + \sum_{n=1}^{k} c_n u_n(0,x)\right) + \lambda \frac{\partial}{\partial t \partial x^2}\left(u_0(0,x) + \sum_{n=1}^{k} c_n u_n(0,x)\right) \\ +\mathcal{L}\left\{\frac{\partial}{\partial x^2}\left(u_0(t,x) + \sum_{n=1}^{k} c_n u_n(t,x)\right)\right\} + \mathcal{L}\{f(t,x)\} \end{cases} \tag{2.4}$$

Taking the inverse Laplace transform of (2.4), we obtain

$$\begin{aligned} u_{new}(t,x) = \ & \mathcal{L}^{-1}\left[\frac{1}{s^2}\left\{\mathrm{s}\left(u_0(0,x) + \sum_{n=1}^{k} c_n u_n(0,x)\right) + \frac{\partial}{\partial t}\left(u_0(0,x) + \sum_{n=1}^{k} c_n u_n(0,x)\right) + \right.\right. \\ & \lambda \frac{\partial}{\partial t \partial x^2}\left(u_0(0,x) + \sum_{n=1}^{k} c_n u_n(0,x)\right) + \\ & \mathcal{L}\left\{\frac{\partial}{\partial x^2}\left(u_0(t,x) + \right.\right. \\ & \left.\left.\sum_{n=1}^{k} c_n u_n(t,x)\right)\right\} + \mathcal{L}\mathcal{L}\{f(t,x)\}\}]. \end{aligned} \tag{2.5}$$

Substituting (2.5) into (1.1), we have the following new collocating at points $x = x_j$

$$\frac{\partial^2 u_{new}(t,x)}{\partial t^2} - \lambda \frac{\partial^3 u_{new}(t,x)}{\partial t \partial x^2} - \frac{\partial^2 u_{new}(t,x)}{\partial x^2} = f(t,x), \tag{2.6}$$

where

$x_j = \frac{L-0}{n+1} j,\ j = 1,2,\dots,n.$

Now, we define the residual function

$$R_n(t,x) = L[u_n(t,x)] - f(t,x), \tag{2.7}$$

where $u_n(t,x)$ denotes the approximate solution, $u(t,x)$ denotes the exact solution and

$$L[u_n(t,x)] = \frac{\partial^3 u_n(t,x)}{\partial t^3} + \lambda \frac{\partial^2 u_n(t,x)}{\partial t^2} + \frac{\partial^\alpha u_n(t,x)}{\partial t^\alpha} - \frac{\partial^2 u_n(t,x)}{\partial x^2}. \tag{2.8}$$

It then follows that

$$\frac{\partial^3 u_n(t,x)}{\partial t^3} + \lambda \frac{\partial^2 u_n(t,x)}{\partial t^2} + \frac{\partial^\alpha u_n(t,x)}{\partial t^\alpha} - \frac{\partial^2 u_n(t,x)}{\partial x^2} = f + R_n \tag{2.9}$$

subject to given initial conditions in the equation (1.1).

Now since L is a linear operator, we get fort he error function $e_n(t,x) = u(t,x) - u_n(t,x)$

$$\frac{\partial^2 e_n(t,x)}{\partial t^2} - \lambda \frac{\partial^3 e_n(t,x)}{\partial t \partial x^2} - \frac{\partial^2 e_n(t,x)}{\partial x^2} = -R_n(t,x) \tag{2.10}$$

with the homogeneous conditions

$$e_n(0,x) = (e_{n)}{}_t(0,x) = (e_{n)}{}_{tt}(0,x) = 0, \tag{2.11}$$

$$e_n(t,0) = e_n(t,L) = 0. \tag{2.12}$$

By solving (2.10) subject to the homogeneous conditons (2.11) and (2.12), we get the error function $e_n(t,x)$. This allows us to compute $u(t,x) = u_n(t,x) + e_n(t,x)$ even for problems without known exact solutions.

3. Numerical Examples

In this section, we implement the new method on some examples to test its efficiency and applicability.

Example 1. W**e** consider the case in which $\lambda = 2$, and (1.1) becomes

$$\frac{\partial^2 u(t,x)}{\partial t^2} - 2\frac{\partial^3 u(t,x)}{\partial t \partial x^2} - \frac{\partial^2 u(t,x)}{\partial x^2} = (t^3 + 3t^2 + 6t)sinx, \tag{3.1}$$

with the following initial conditions

$$\begin{cases} u(0,x) = u_t(0,x) = 0, 0 \leq x \leq \pi, \\ u(t,0) = u(t,1) = 0, \ 0 \leq t \leq 1. \end{cases} \tag{3.2}$$

Using the The exact solution is given by $u(t,x) = t^3 sinx$. We assume that the trial function of the following form:

$$u(t,x) = c_1 x^2(x-1)t^3 + c_2 x(x-1)^2 t^3. \tag{3.3}$$

Taking the Laplace transform of (3.1) and using the formula (2.4), (3.2), (3.3), we obtain

$$U(s,x) = \frac{1}{s^2}[\mathcal{L}\left\{\frac{\partial^2 u(t,x)}{\partial x^2}\right\} + 2\mathcal{L}\left\{\frac{\partial^3 u(t,x)}{\partial t \, \partial x^2}\right\} + \mathcal{L}\{(t^3 + 3t^2 + 6t)sinx\}].$$

Then,

$$\begin{cases} U(s,x) = (\frac{6}{s^6} + \frac{6}{s^5})\{c_1(6x-2) + c_2(6x-4)\} + \left\{\frac{6}{s^6} + \frac{6}{s^5} + \frac{6}{s^4}\right\} sinx. \end{cases} \tag{3.4}$$

Taking the inverse Laplace transform of (3.4), we get the following new trial solution:

$$\begin{cases} u_{new}(t,x) = \left[\left(\frac{t^5}{20} + \frac{t^4}{4}\right)(c_1 + c_2)\right] x \\ -2\left(\frac{t^5}{20} + \frac{t^4}{4}\right)\ (c_1 + 2c_2) + \left[(t^3 + \frac{t^4}{4} + \frac{t^5}{20})sinx\right]. \end{cases} \tag{3.5}$$

Substituting (3.5) into (3.1), we have the following residual formula:

$$R(t,x,c_1,c_2) = \frac{\partial^2 u(t,x)}{\partial t^2} - 2\frac{\partial^3 u(t,x)}{\partial t \partial x^2} - \frac{\partial^2 u(t,x)}{\partial x^2} - (t^3 + 3t^2 + 6t)sinx$$

$$= \left(A(c_1 + c_2)x + B(c_1 + 2c_2)\right) + Ksinx \tag{3.6}$$

where
$A = 6t^3 + 3t^2$,
$B = -2t^3 - 6t^2$,
$K = \frac{t^5}{20} + 3\frac{t^4}{4} + 3t^3 + 6t^2$.

Collocating (3.6) at equally spaced points $x = \frac{\pi}{10}, x = \frac{\pi}{100}$ and $t = 0.01$ and equating to zero, we obtain $c_1 = 3.850930392770440$ and $c_2 = -1.925747071251289$.

Comprasion between the approximate with the method LTCM and the exact approximate for Example is given in Table.

Error Analysis Table

$x =$	t	Maxerror(u(exact)-u(LTCM))
pi/10	0.1	0.0058
pi/100	0.01	6.0753×10^{-6}
pi/1000	0.001	6.0530×10^{-9}
0.0001	0.0001	6.0502×10^{-12}

4. Conclusion

In this work, we adopted a combination of Laplace transform collocation method to develop a numerical method for pseudo-hyperbolic partial differential equation. Numerical example was considered to demonstrate the accuracy and efficiency of the method. The exact solution is compared with the approximate solution. Obtained results are given in result Table.

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Local Existence of Solutions for a p-Laplacian Type Equation with Delay Term and Logarithmic Nonlinearity

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Abstract

In this work, we deal with a p-Laplacian type equation with delay term and logarithmic nonlinearity. Under suitable conditions, we consider the local existence of solutions by using the Faedo-Galerkin method.

Keywords: Delay, Local existence, Logarithmic nonlinearity.

1. Introduction

We consider the following problem

$$\begin{cases} |u_t|^\rho u_{tt} - \Delta u - \Delta_p u + \int_0^t g(t-s)\Delta u(s)ds \\ -\Delta u_{tt} + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) \\ = ku\ln u, & in\ \Omega \times (0,\infty), \\ u_t(x,t-\tau) = f_0(x,t-\tau), & x \in \Omega,\ t \in (0,\tau), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & x \in \Omega, \\ u(x,t) = 0, & x \in \partial\Omega,\ t \geq 0, \end{cases} \tag{1}$$

where Ω is a bounded domain in R^n, $n \geq 1$, with a smooth boundary $\partial\Omega$, $\rho > 0$, k, μ_1 are positive constants, μ_2 is a real number, $\tau > 0$ represents the time delay, the term $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$ is called p -Laplacian, the kernel g is satisfying some conditions to be specified later. $u_{0,}u_{1,}\ f_0$ are the initial data in a suitable function space.

Some other authors studied the related problems (see [Wu 2013, Messaoudi, Guesmia and Al-Gharabli 2020]).

2. Preliminaries

In this part, we give some material needed for the proof of our results. We use the standard Lebesgue space $L^p(\Omega)$ and Sobolev space $H_0^1(\Omega)$ with their usual scalar products and norms $\|\cdot\|_p$ and $\|\cdot\|_{H_0^1(\Omega)}$.

Lemma 1. [Adams and Fournier 2003, Pişkin 2017] Let $2 \leq q \leq \frac{2n}{n-2}$, the inequality

$$\|u\|_q \le c_s \|\nabla u\|_2 \ \ for\ u \in H_0^1(\Omega), \quad (2)$$

holds with some positive constants c_s.

Lemma 2. [Kirane and Belkacem 2011] For any $g \in C^1(R)$ and $\phi \in H^1(0,T)$, we have

$$-2\int_0^t \int_\Omega g(t-s)\phi\phi_t dxds = \frac{d}{dt}\left((go\phi)(t) - \int_0^t g(s)ds\|\phi\|^2\right) + g(t)\|\phi\|^2 - (g'o\phi)(t), \quad (3)$$

where

$$(go\phi)(t) = \int_0^t g(t-s)\int_\Omega |\phi(s)-\phi(t)|^2 dxds.$$

Lemma 3. [Gross 1975, Chen, Luo and Liu 2015] (Logarithmic Sobolev Inequality) Let u be any function in $H_0^1(\Omega)$ and $a > 0$ be any number. Then,

$$\int_\Omega u^2 \ln|u|dx \le \frac{1}{2}\|u\|^2 \ln\|u\|^2 + \frac{a^2}{2\pi}\|\nabla u\|^2 - (1+\ln a)\|u\|^2. \quad (4)$$

Lemma 4. [Cazenave and Haraux 1980] (Logarithmic Gronwall Inequality) Let $C > 0, \gamma \in L^1(0,T;R^+)$ and suppose that the function $w: [0,T] \to [1,\infty)$ satisfies

$$w(t) \le \left(1 + \int_0^t \gamma(s)w(s)\ln\big(w(s)\big)ds\right),\ \forall t \in [0,T]. \quad (5)$$

Then,

$$w(t) \le C\exp(C)\int_0^t \gamma(s)ds,\ \forall t \in [0,T]. \quad (6)$$

Lemma 5. [Messaoudi, Guesmia and Al-Gharabli 2020] Let $\epsilon_0 \in (0,1)$. Then, there exists $d_{\epsilon_0} > 0$ such that

$$s|\ln s| \le s^2 + d_{\epsilon_0}s^{1-\epsilon_0}, \forall s > 0. \quad (7)$$

Now we introduce, similar to the work of Nicaise and Pignotti (2006), the new variable

$$z(x,\kappa,t) = u_t(x, t-\tau\kappa)\ x \in \Omega, \kappa(0,1),$$

thus, we have

$$\tau z_t(x,\kappa,t) + z_\kappa(x,\kappa,t) = 0\ in\ \Omega \times (0,1) \times (0,\infty).$$

Then, problem (1) takes the form

$$\begin{cases} |u_t|^\rho u_{tt} - \Delta u - \Delta_p u + \int_0^t g(t-s)\Delta u(s)ds & \\ -\Delta u_{tt} + \mu_1 u_t(x,t) + \mu_2 z(x,1,t) & \\ = kulnu, & in\ \Omega \times (0,\infty), \\ \tau z_t(x,\kappa,t) + z_\kappa(x,\kappa,t) = 0, & x \in \Omega,\ \kappa \in (0,1), t > 0, \\ z(x,0,t) = u_t(x,t), & x \in \Omega,\ t > 0, \\ z(x,\kappa,0) = f_0(x,-\tau\kappa), & x \in \Omega, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & x \in \Omega, \\ u(x,t) = 0, & x \in \partial\Omega, t \geq 0. \end{cases} \tag{8}$$

Next, we define the energy associated to the solution of system (8) by

$$E(t) = \frac{1}{\rho+2}\|u_t\|_{\rho+2}^{\rho+2} + \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u\|^2$$
$$+\frac{1}{2}(go\nabla u)(t) + \frac{1}{2}\|\nabla u_t\|^2 - \frac{k}{2}\int_\Omega |u|^2 ln\,|u|dx$$
$$+\frac{k}{4}\|u\|^2 + \frac{\zeta}{2}\int_\Omega \int_0^1 z^2(x,\kappa,t)d\kappa dx. \tag{9}$$

We can get the following Lemma similar to Pişkin and Yüksekkaya (2020).

Lemma 6. $\mathrm{E}(t)$ is a nonincreasing function, such that

$$\begin{aligned} E'(t) &\leq -c_1\|u_t\|^2 - c_2\int_\Omega z^2(x,1,s)\,dx + \frac{1}{2}(g'o\nabla u)(t) - \frac{1}{2}g(t)\|\nabla u\|^2 \\ &\leq \frac{1}{2}(g'o\nabla u)(t) - \frac{1}{2}g(t)\|\nabla u\|^2 \leq 0,\ \forall t \geq 0. \end{aligned} \tag{10}$$

3. Local existence

In this part, we state and prove the local existence result for problem (8) by using the Faedo-Galerkin method.

Theorem 7. Suppose that $\mu_2 < \mu_1$. Suppose further that $u_0, u_1 \in H_0^1$ and $f_0 \in L^2(\Omega \times (0,1))$. Then there exists a unique solution (u,z) of (8) satisfying

$$u, u_t \in C([0,T); H_0^1(\Omega)),$$
$$z \in C\left([0,T); L^2(\Omega \times (0,1))\right),$$

for $T > 0$.

Proof. Let us consider

$$u^k(t) = \sum_{j=1}^{k} c^{jk}(t) w^j(x),$$

and

$$z^k(t) = \sum_{j=1}^{k} r^{jk}(t) \varphi^j(x,\kappa),$$

where $(u^k, z^k)$are the solutions of the following approximate problem corresponding to (8)

$$\int_\Omega \left|u_t^k\right|^\rho u_{tt}^k w^j dx + \int_\Omega \nabla u^k \nabla w^j dx - \int_0^t g(t-\tau) \int_\Omega \nabla u^k(\tau) \nabla w^j dx d\tau + \int_\Omega \nabla u_{tt}^k \nabla w^j dx + \int_\Omega |\nabla u^k|^{p-2} |\nabla u^k| \nabla w^j dx + \int_\Omega \left(\mu_1 u_t^k(x,t) + \mu_2 z^k(x,1,t)\right) w^j dx = k \int_\Omega (u^k ln|u^k|) w^j dx. \quad (11)$$

$$u^k(0) = u_0^k \to u_0 \ in\ H_0^1(\Omega),\ u_t^k(0) = u_1^k \to u_1 \ in\ H_0^1(\Omega), \quad (12)$$

and

$$\int_\Omega \left(\tau z_t^k(x,\kappa,t) + z_\kappa^k(x,\kappa,t)\right) \varphi^j dx = 0, \quad (13)$$

$$z^k(0) = z_0^k \to f_0 \ in\ L^2\big(\Omega \times (0,1)\big), \quad (14)$$

where $i = 1,2,\cdots,k$.

First estimate: Since the sequences u_0^k, u_1^k and z_0^k converge, such that

$$E^k(t) - E^k(0) \le -c_1 \int_0^t \left\|u_t^k\right\|^2 ds - c_2 \int_0^t \int_\Omega |z^k(x,1,s)|^2 dx ds. \quad (15)$$

From the estimates for each term in (15), we obtain

$$\left\|u_t^k\right\|_{\rho+2}^{\rho+2} + \|\nabla u^k\|^2 + \|u^k\|^2 + \left\|\nabla u_t^k\right\|^2 + \|\nabla u^k\|_p^p$$

$$+(go\nabla u^k)(t) + \int_\Omega \int_0^1 |z^k(x,\kappa,t)|\, d\kappa dx$$

$$+\int_0^t \|u_t^k\|^2 ds + \int_0^t \int_\Omega |z^k(x,1,s)|^2 dxds$$
$$\leq c(1 + \|u^k\|^2 ln\|u^k\|^2). \qquad (16)$$

From (16) and applying the Logarithmic Gronwall inequality, we have the first estimate:

$$\|u_t^k\|_{\rho+2}^{\rho+2} + \|\nabla u^k\|^2 + \|u^k\|^2 + \|\nabla u_t^k\|^2 + \|\nabla u^k\|_p^p + (go\nabla u^k)(t) + \int_\Omega \int_0^1 |z^k(x,\kappa,t)|\, d\kappa dx + \int_0^t \|u_t^k\|^2 ds + \int_0^t \int_\Omega |z^k(x,1,s)|^2 dxds \leq c\left(1 + Ce^{CT} ln(Ce^{CT})\right) = A_1. \qquad (17)$$

Second estimate: Replacing w^j by $-\Delta w^j$ in (11), multiplying by c_t^{jk} and summing over j from 1 to k, it follows that

$$\int_\Omega |u_t^k|^\rho u_{tt}^k \left(-\Delta u_t^k\right) dx + \int_\Omega \Delta u^k \Delta u_t^k dx + \langle \Delta_p u^k, \Delta u_t^k \rangle - \int_0^t g(t-s) \int_\Omega \Delta u^k \Delta u_t^k dxds + \int_\Omega \Delta u_{tt}^k \Delta u_t^k dx - \mu_1 \int_\Omega \Delta u_t^k \cdot u_t^k dx - \mu_2 \int_\Omega \Delta u_t^k \cdot z^k(x,1,t) dx = -k \int_\Omega \Delta u_t^k u^k ln|u^k| dx. \qquad (18)$$

From the estimates for each term in (18) and by integrating over (0,t), then using Gronwall lemma, we obtain

$$\|\Delta u^k\|^2 + (go\Delta u^k) + \int_0^1 \tau \|\nabla z^k(x,\kappa,t)\|^2 d\rho + \|\nabla u_t^k\|^2 + \|\nabla \Delta u^k\|^2 + \int_0^t \|\Delta u_t^k\|^2 ds + \int_0^t \|\nabla u_{tt}^k\|^2 ds \leq M_2. \qquad (19)$$

4. Conclusion

In recent years, there has been published much work concerning the wave equation with constant delay or time-varying delay. However, to the best of our knowledge, there was no local existence result with a p-laplacian type equation with delay term and logarithmic nonlinearity. We have been proved that local existence of solutions by using Faedo-Galerkin method for problem (1) under the suitable conditions.

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Mathematical description and approximation for the water wave equation

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Abstract

We deal with approximation made to justify the linearization of the equations. Because of the oscillatory motions that they generate, surface gravity waves can bereasonably well described by a linear analysis. This is mathematically justified byrestricting the attention to small wave amplitudes and weakaccompanying motions. In shallow water, the damping of water waves is highly influenced by the bottom friction.

Describing water wave breaking we linearize the problem through the linear parameter. It is interesting to mention that the mathematical model can be derived by alternative proofs. We linearize the wave breaking equation and come to conclusions through the wave absorption experiment on the coast. For instance, it can be derived by imposing the dynamic equilibrium of a string that is vibrating in a two-dimensional plane. Mathematical descriptionand approximation leads us to get conclusions.

Key words: linearization, water waves equation, wave breaking,mathematical description**,** approximation

1. Introduction

We deal with a mathematical model to describe the behavior of shallow water waves, like for instance amplitude and period, depending on the given external forcing and the geometry of the sea bottom, which dictate the initial and boundary conditions. The usual method of obtaining the linear model is to linearize the discrete form of the nonlinear model. This is known as the discrete method and the discrete linear model formed in this way is called the tangent linear model (TLM) (see [5], [6])

From fluid mechanics (see [7]), we know that the motion of fluid is completely described by the velocity vector that generally varies in space and time. We take the motion of fluid in the form

$$\mathbf{v}(x,z,t) = u(x,z,t)\mathbf{i}_x + w(x,z,t)\mathbf{i}_z \qquad (1)$$

where z is the vertical upward-positive coordinate, z = 0 is the mean sea level, x is the longitudinal coordinate along the direction of propagation of the wave, $\boldsymbol{i}_x$ and $\boldsymbol{i}_w$ are unit vectors along the x and z axis, respectively, and finally u and w are the x and z components of the velocity vector, respectively.

Studding the numerical methods in atmospheric and oceanic modeling ([10]), we introduce and study it for small amplitude waves at shallow water wave, one can assume that the values at the sea level of velocity and the potential are given by the corresponding values at the mean sea level obtaining:

$$w(x,0,t) = \frac{\partial \eta(x,t)}{\partial t} \quad \text{and} \quad \frac{\partial \Phi(x,0,t)}{\partial t} + g\eta(x,t) = 0 \tag{2}$$

The resulting equations set the basis for the "**linear wave theory**". Essentially, the linear formulation has been obtained by introducing approximations finalized to give the surface boundary condition at z = 0 instead of z = η. In fact, η is still unknown and therefore in the original formulation the boundary condition depends on the solution, therefore making the equation non-linear (see [11]).

The linear formulation is particularly interesting because it allows application of the superposition principle, which states that the configuration induced by two waves can be obtained by summing the individual configuration originated by each wave taken alone. This would not be possible with the original non-linear formulation, as the configuration induced by each wave would depend on η and therefore on the combined effect of the two waves.

2. Main theory

2.1 Linearization

Mathematically we study small wave amplitudes and weak accompanying motions. In the momentum equations (2), all terms linear in the velocity and pressure are then assumed to be small, while the nonlinear terms are assumed to be even smaller and therefore negligible.

Under the assumption that the velocity, length and time scales, U, L and T, meet the following criterion:

$$U^2 / L \ll U / T \quad \text{or} \quad U \ll L / T \tag{3}$$

The ratio L/T, representing influence over a characteristic time of the phenomenon, provides a scale for the wave speed. Hence a corollary of the approximation made to justify the linearization of the equations is that we restrict our attention to wave motions in which the velocity of the fluid is much smaller than the speed of propagation of wave crests.

By assuming that water is incompressible, the continuity equation applies because of conservation of mass. If we refer to an elementary control volume, we can write

$$\frac{\partial u(x,z,t)}{\partial x} + \frac{\partial w(x,z,t)}{\partial z} = 0 \tag{4}$$

We now assume that the motion of the fluid (the flow) is irrational, it means that the cross gradient of the velocity or shear is zero. This assumption is more realistic for fluids. It follows that the velocity can be expressed in terms of the velocity potential **Φ,** namely

$$u = \frac{\partial \Phi(x,z,t)}{\partial x}$$

$$w = \frac{\partial \Phi(x,z,t)}{\partial Z} \tag{5}$$

By substituting eq. (5) in the continuity equation (4) one can obtain the Laplace equation, which is a second-order partial differential equation.

$$\frac{\partial^2 \Phi(x,z,t)}{\partial x^2} + \frac{\partial^2 \Phi(x,z,t)}{\partial z^2} = 0 \tag{6}$$

2.2 Solutions of the linear wave equations

We are interested in solutions that correspond to a regular wave, namely, waves with constant amplitude, period, and direction. Therefore, we assume that the solution for the potential has the form

$$\Phi(x,z,t) = A(z)\sin(\omega t - kx + \varphi_0) \tag{7}$$

where k is the wavenumber, ω is frequency and φ_0 is the phase of the wave and A is an amplitude which depends on z. Dependence of amplitude on the vertical coordinate is a necessary condition given that the motion of fluid particles depends on their position along the vertical direction. For instance, in the proximity of the sea bottom the vertical component of the velocity is null and therefore the potential has a different gradient with respect to other locations.

The solution given by (5) is called "*traveling wave*". The wave number is related to the wavelength λ by the relationship $k = 2\pi / \lambda$. By plugging this solution into eq. (3), (4), we obtain

$$\eta(x,t) = a\sin(\omega t - kx) \tag{8}$$

$$\omega^2 = gk\tanh(hk) \tag{9}$$

Eq. (7) is called the "*dispersion relation*", which states that *frequency* and *wavenumber* $k = 2\pi / \lambda$ have related each other. Eq. (7), (8) and (9) are the core of the **linear wave theory**. It is a reasonable starting point for shallow water waves, which are greatly influenced by viscosity, surface tension or turbulence.

2.3 The turbulent mixing under shallow water waves breaking

Mathematically, we choose to take the density of the water equal to ρ_0 everywhere and that of the air as zero.

The energy carried by surface waves, however, must eventually be dissipated somewhere and will affect the water contents there. For example, wave energy can be converted into turbulent mixing under wave breaking. After linearization, the equations are:

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \qquad \frac{\partial w}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} - g$$

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{10}$$

This set of equations (10) forms a system for the flow variables u(x, y, z, t), v(x, y, z, t), w(x, y, z, t) and p(x, y, z, t). An accessory variable is the surface elevation α (x, y, t) (Figure 1).

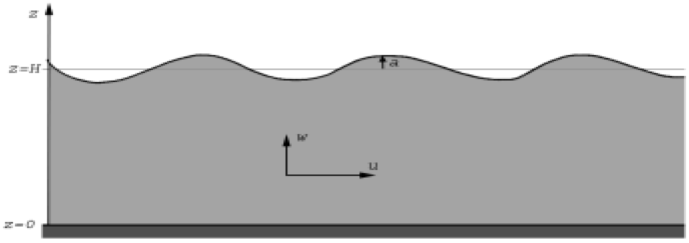

Figure 1: surface wave as the surface elevation

Along the bottom, which we take as horizontal (z = 0), the boundary condition is that there is no vertical velocity (w = 0). After linearization (|α| ≪ H), the boundary conditions become:

$$z = H \; : \quad w = \frac{\partial a}{\partial t} \quad \text{and} \quad p = p_a + \rho_0 g a$$

$$z = 0 \; : \quad w = 0 \tag{11}$$

3. Solution of the problem

The linear equations permit us to seek a periodic solution of sinusoidal shape. Taking λ as the wavelength (the distance from one crest to the next crest at a given time) and T as the period (the interval of time during which a crest travels one wavelength), we introduce for convenience the *wavenumber* $k = 2\pi / \lambda$ and the angular frequency $\omega = 2\pi / T$.
Then, we seek a wave solution of the form:

$$\begin{aligned} u(x,z,t) &= U(z)\ \cos(kx - \omega t) \\ v(x,z,t) &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} w(x,z,t) &= W(z)\ \sin(kx - \omega t) \\ p(x,z,t) &= p_a + \rho_0 g(H - z) + P(z)\ \cos(kx - \omega t) \\ a(x,t) &= A\ \cos(kx - \omega t). \end{aligned} \tag{12}$$

With this notation, the vertical displacement of the surface varies between −A in a trough to +A at a crest. The choice of w in quadrature with the other variables is to ensure a match of trigonometric functions after substitution in the equations.

An interesting aspect of wave propagation is that the energy carried by waves does not always travel in the same direction or at the same speed as crests and troughs. In a wave of a single wavelength, the energy density is uniformly distributed because of the pattern repetition every wavelength. But, in a wave consisting of multiple components, different wavelengths dominate at different locations, and the energy distribution is non uniform.
In a multi-component wave, called a wave group, the trigonometric function sin(kx − ωt) is no longer applicable and needs to be generalized to sin α, where the phase α(x, t) is a more complicated function of space and time. Nonetheless, one can define a local wavenumber k and a local frequency ω as

$$k = \frac{\partial \alpha}{\partial x}, \quad \omega = -\frac{\partial \alpha}{\partial t}, \tag{13}$$

to which correspond a local wavelength λ = 2π/k and a local period T = 2π/ω. The uniqueness of the function α in Equations (13) implies a relation between wavenumber and frequency:

$$\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0 \tag{14}$$

Since the dispersion relation prescribes a relation between frequency and wave number, it follows that ω = ω(k) and ∂ω/∂x = (dω/dk) (∂k/∂x).

4. THE SHALLOW-WATER CONTINUITY EQUATION

4.1 The linearization of the shallow water equations

The figure below shows the interface between two fluids of different, constant densities. The dashed line shows the position of the interface if the fluids are undisturbed (see [1]). The solid line shows the interface displaced. The depth of the lower fluid is H.

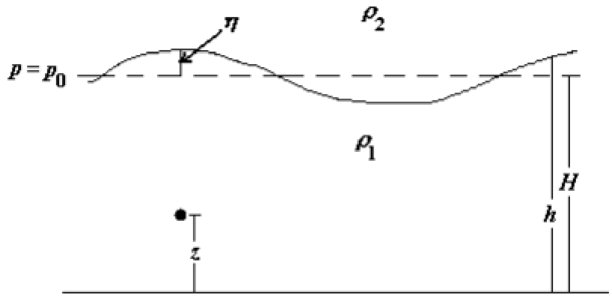

Figure : the interface between two fluids of different, constant densities

If we assume that the upper fluid is very deep compared to the displacement of the interface (see [8]), then we can assume that the pressure at the level of the undisturbed interface (the dashed line in the figure) remains constant at a value of p_0. If the lower fluid is in hydrostatic balance, then the pressure at any point in the lower fluid is proportional to the weight of the fluid above it. Therefore, at the point shown in the diagram the pressure will be

$$p = p_0 + \rho_1 g(H - z) + \rho_1 g\eta - \rho_2 g\eta$$

and the horizontal pressure gradient force will be

$$-\frac{1}{\rho_1}\frac{\partial p}{\partial x} = -\left(\frac{\rho_1 - \rho_2}{\rho_1}\right) g \frac{\partial \eta}{\partial x}.$$

The continuity equation in the lower fluid is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad (15)$$

To linearize the shallow-water equations we use the following perturbation forms for the dependent variables

$$\begin{aligned} u &= \bar{u} + u' \\ v &= \bar{v} + v' \\ h &= H(x,y) + \eta - z_0. \end{aligned} \qquad (16)$$

Note that H is the mean undisturbed height of the fluid surface, which can depend on x, and y. The term z_0 is the elevation of the bottom topography, and for a flat bottom would be zero.
We will assume a flat bottom ($z_0 = 0$), and assume that the base state is in geostrophic balance so that

$$\bar{u} = -\frac{g}{f}\frac{\partial H}{\partial y} \quad \text{and} \quad \bar{v} = \frac{g}{f}\frac{\partial H}{\partial x}, \qquad (17)$$

Putting (16) into equation (15) and using the equations (17), we get

$$\begin{aligned} &\frac{\partial u'}{\partial t} + \bar{u}\frac{\partial u'}{\partial x} + \bar{v}\frac{\partial u'}{dy} = -g'\frac{\partial \eta}{\partial x} + fv' \\ &\frac{\partial v'}{\partial t} + \bar{u}\frac{\partial v'}{\partial x} + \bar{v}\frac{\partial v'}{dy} = -g'\frac{\partial \eta}{\partial y} - fu' \\ &\frac{\partial \eta}{\partial t} + \bar{u}\left(\frac{\partial H}{\partial x} + \frac{\partial \eta}{\partial x}\right) + \bar{v}\left(\frac{\partial H}{\partial y} + \frac{\partial \eta}{dy}\right) + u'\frac{\partial H}{\partial x} + v'\frac{\partial H}{\partial y} = -H\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right). \end{aligned} \qquad (18)$$

Equations (18) are the linearized shallow-water equations.
The effect of linearization errors is another subject that we can study in another section in the next future using some base knowledge (see [9]).

5. Conclusion

In conclusion, we analyzed the linearize the shallow-water equations. We use the perturbation forms for the dependent variables.
The great advantage of our technique is the possibility to measure a two-dimensional field η (x, y, z) with good temporal resolution.
In this article we have observed the importance of the semi-continuous approach. This fact gives the possibility to avoid some of the problems that occur when linearizing complex schemes. For example, it has been shown how problems may arise in the direct linearization of interpolation with semi-Lagrange advection schemes or the direct linearization of iterative solution procedures.
We set out the continuous equations for the nonlinear model and show their linearization for use in the PFM. We discussed how we should measure the accuracy of the linear models. A weakness in applying present methods derives a new method designed to test such a model. The two linear models will be then compared numerically in another section.
We obtain, for the linearization of the momentum equation using the fact that products of perturbations are neglected to give the linear equations.
In conclusion, the linear equation can be solved by using the same direct solver as is used in the nonlinear model. An examination of the resulting time discretization for the tangent linear model shows that it is a second-order approximation to the continuous linear equations.

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Miscellaneous Properties of the Gamma Distribution Polynomials

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Abstract

The main aim of this paper is to investigate multifarious properties and relations for the gamma distribution. The approach to reach this purpose will be introducing a special polynomial including gamma distribution. Several formulas covering addition formula, derivative property, integral representation, and explicit formula are derived utilizing the series manipulation method. Furthermore, two correlations including Bernoulli and Euler polynomials for gamma distribution polynomials are provided by utilizing their generating functions.

Keywords: Special polynomials, gamma distribution, Cauchy product, generating function, Bernoulli polynomials, Euler polynomials.

1. Introduction

The gamma distribution is frequently used to model waiting times. For instance, in life testing, the waiting time until death is a random variable that is frequently modeled with a gamma distribution. Also, the importance of the gamma distribution is mainly due to its connections to the normal and exponential distributions (see [2-5,7,8]). In the literature, Kang et al. [4] considered special polynomials involving exponential distribution, which is related to life testing and then they derived diverse formulas and identities such as the symmetric property, recurrence formula, and summation formula. Jambunathan [3] provided certain significant properties of the gamma distribution to show certain useful applications to sampling problems. Khodabin et al. [5] introduced the generalized gamma distribution that is flexible in the statistical literature, which covers exponential, gamma, and Weibull distributions and also gave several important properties for this distribution. Mead et al. [7] defined modified generalized gamma distribution to investigate greater flexibility in modeling data from a practical viewpoint and they derived multifarious identities and properties of this distribution, including explicit expressions for the moments, quantiles, mode, moment generating function, mean deviation, mean residual lifetime and expression of the entropies. Rahman et al. [8] considered k-gamma and k-beta distributions and moments generating function for the mentioned distributions in terms of a new parameter $k>0$ and developed a lot of properties of these distributions.
Before going to handle the gamma distribution, we need to mention the gamma function.
The gamma function, which is firstly considered by Leonard Euler, is an extension of the factorial

function to real (and complex) numbers. This function denoted by $\Gamma(r)$ is defined by the following improper integral for $r > 0$:

$$\Gamma(r) = \int_0^\infty x^{r-1} e^{-x} dx, \tag{1.1}$$

which holds the following properties (*cf.* [2,3,5,7,8] and also see the references cited therein)

$$\int_0^\infty x^{r-1} e^{-\omega x} dx = \frac{\Gamma(r)}{\omega^r} \quad \text{for } \omega > 0, \tag{1.2}$$

$$\Gamma(r+1) = r\Gamma(r), \tag{1.3}$$

$$\Gamma(n+1) = n! \text{ for } n = 1,2,3,\ldots. \tag{1.4}$$

Dividing both sides of (1.1) by $\Gamma(r)$, it becomes

$$1 = \int_0^\infty \frac{1}{\Gamma(r)} x^{r-1} e^{-x} dx = \int_0^\infty \frac{\omega^r}{\Gamma(r)} t^{r-1} e^{-\omega t} dt, \tag{1.5}$$

where there is a change of variable $x = \omega t$. Thus, the probability density function of a gamma distribution (or called two-parameter family of continuous probability distributions) with parameters $r > 0$ and $\omega > 0$ denoted by $f(t:\omega, r)$, is defined as follows

$$f(t:\omega, r) = \begin{cases} \dfrac{\omega^r}{\Gamma(r)} t^{r-1} e^{-\omega t} & \text{if } t > 0 \\ 0 & \text{if } t \leq 0, \end{cases} \tag{1.6}$$

where X is a continuous random variable (see [2,3,5,7,8]).
Some properties of the gamma distribution are stated below.

Proposition 1 *Let* $\omega > 0$ *and* $r > 0$. *The gamma distribution possesses the following properties (cf. [2,3,5,7,8]):*

$$(i)\,(\text{Mean})\, E(X) = \frac{r}{\omega}$$

$$(ii)\,(\text{Varience})\, V(X) = \frac{r}{\omega^2}$$

$$(iii)\,(\text{Mode})\, M(X) = \frac{r-1}{\omega} \text{ for } r \geq 1$$

$$(iv)\,(\text{Moments})\, E(X^n) = \frac{r}{\omega} \text{ for } n = 1,2,\ldots,$$

The following sections are planned as follows: the second section includes a definition of a special polynomial including gamma distribution which we call gamma distribution polynomials and provides various formulas and identities including addition formula, derivative property, integral representation, and explicit formula. The third section gives two relationships covering the Bernoulli and Euler polynomials for gamma distribution polynomials by using their generating functions. The last section of this paper examines the results attained in this paper.

2. Main Results

In this part, we define special polynomials for gamma distribution polynomials which we call gamma distribution polynomials. We then investigate and research several relations and identities for these polynomials.
Kang and Lee [4] defined special polynomials $\mathfrak{E}(x:\omega)$ including exponential distribution given by the following generating function

$$\sum_{n=0}^{\infty}\mathfrak{E}(x:\omega)\frac{t^n}{n!}=\frac{\omega}{e^{\omega t}}e^{tx}. \tag{2.1}$$

and investigated diverse properties covering recurrence formula, symmetric properties, and summation formula. When $x=0$ in (2.1), we get

$$\sum_{n=0}^{\infty}\mathfrak{E}(0:\omega)\frac{t^n}{n!}=\frac{\omega}{e^{\omega t}}, \tag{2.2}$$

which is the probability density function of exponential distribution (*cf.* $[2,4]$) in conjunction with the parameter $\omega>0$.
Motivated by the above, we now give our main definition.

Definition 1 *Let* $\omega>0$ *and* $r>0$. *We introduce a special polynomial* $\mathfrak{G}_n(x:\omega,r)$ *covering gamma distribution by the following exponential generating function:*

$$\sum_{n=0}^{\infty}\mathfrak{G}_n(x:\omega,r)\frac{t^n}{n!}=\frac{\omega^r}{\Gamma(r)}\frac{t^{r-1}}{e^{\omega t}}e^{tx}, \tag{2.3}$$

which we call the gamma distribution polynomials.

Upon setting $x=0$ in Definition 1, we obtain the following Taylor series expansion about $t=0$:

$$\sum_{n=0}^{\infty}\mathfrak{G}_n(\omega,r)\frac{t^n}{n!}=\frac{\omega^r}{\Gamma(r)}\frac{t^{r-1}}{e^{\omega t}}=f(t:\omega,r), \tag{2.4}$$

where $\mathfrak{G}_n(0:\omega,r):=\mathfrak{G}_n(\omega,r)$ we call the gamma distribution numbers.

If we choose $\omega=\frac{1}{2}$ and $r=\frac{v}{2}$, we then get a new special polynomial for the χ^2-distribution (or chi-squared distribution) (*cf.* [2]), which we call χ^2-distribution polynomials as follows:

$$\sum_{n=0}^{\infty}\mathfrak{K}_n(x:v)\frac{t^n}{n!}=\frac{2^{-\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)}\frac{t^{\frac{v}{2}-1}}{e^{\frac{t}{2}}}e^{tx}. \tag{2.5}$$

Taking $x=0$, the χ^2-distribution polynomials $\mathfrak{K}_n(x:v)$ reduce to the corresponding numbers $\mathfrak{K}_n(0:v):=\mathfrak{K}_n(v)$, which is also new, given by

$$\sum_{n=0}^{\infty}\mathfrak{K}_n(v)\frac{t^n}{n!}=\frac{2^{-\frac{v}{2}}}{\Gamma\left(\frac{v}{2}\right)}\frac{t^{\frac{v}{2}-1}}{e^{\frac{t}{2}}}, \tag{2.6}$$

which is the probability density function of chi-squared distribution with the parameter $v>0$, (*cf.* [2]).

Remark 1 *Substituting* $r=1$ *in (2.3), the gamma distribution polynomials reduce to the exponential distribution polynomials* $\mathfrak{E}(x:\omega)$ *in (2.1), cf.* $[2,4]$.

We now investigate some properties of the gamma distribution polynomials $\mathfrak{G}_n(x:\omega,r)$ and the gamma distribution numbers $\mathfrak{G}_n(\omega,r)$. We firstly give the following relation.

Theorem 1 *The following correlation*

$$\mathfrak{G}_n(x:\omega,r)=\sum_{k=0}^{n}\binom{n}{k}x^k\mathfrak{G}_{n-k}(\omega,r) \tag{2.7}$$

holds for $\omega>0$ and $r>0$.

Proof. By Definition 1, using the Cauchy product rule for the power series, we readily see that

$$\begin{aligned}\sum_{n=0}^{\infty}\mathfrak{G}_n(x:\omega,r)\frac{t^n}{n!}&=\frac{\omega^r}{\Gamma(r)}\frac{t^{r-1}}{e^{\omega t}}e^{tx}\\&=\sum_{n=0}^{\infty}\mathfrak{G}_n(\omega,r)\frac{t^n}{n!}\sum_{n=0}^{\infty}x^n\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}x^k\mathfrak{G}_{n-k}(\omega,r)\frac{t^n}{n!}\end{aligned}$$

which provides the asserted result (2.7).

We state the following theorem.

Theorem 2 *(Derivative Property) The following formula*

$$\frac{d}{dx}\mathfrak{G}_n(x:\omega,r)=n\mathfrak{G}_{n-1}(\omega,r) \tag{2.8}$$

is valid for $\omega>0$ and $r>0$.

Proof. By Definition 1, we acquire

$$\sum_{n=0}^{\infty}\frac{d}{dx}\mathfrak{G}_n(x:\omega,r)\frac{t^n}{n!}=\frac{d}{dx}\left(\frac{\omega^r}{\Gamma(r)}\frac{t^{r-1}}{e^{\omega t}}e^{tx}\right)=\frac{\omega^r}{\Gamma(r)}\frac{t^{r-1}}{e^{\omega t}}te^{tx}=\sum_{n=0}^{\infty}\mathfrak{G}_n(\omega,r)\frac{t^{n+1}}{n!},$$

which means the desired result (2.8).

We give the addition formula for the gamma distribution polynomials.

Theorem 3 *(Addition Formula) The following equality*

$$\mathfrak{G}_n\left(x+y:\omega,r\right)=\sum_{k=0}^{n}\binom{n}{k}y^{n-k}\mathfrak{G}_k\left(x:\omega,r\right) \tag{2.9}$$

holds for $\omega>0,\ r>0$ and x,y being real numbers.

Proof. By Definition 1, using the Cauchy product rule for the power series, we readily see that

$$\sum_{n=0}^{\infty}\mathfrak{G}_n\left(x+y:\omega,r\right)\frac{t^n}{n!}=\frac{\omega^r}{\Gamma(r)}\frac{t^{r-1}}{e^{\omega t}}e^{t(x+y)}=\frac{\omega^r}{\Gamma(r)}\frac{t^{r-1}}{e^{\omega t}}e^{tx}e^{yt}$$

$$=\sum_{n=0}^{\infty}\mathfrak{G}_n\left(x:\omega,r\right)\frac{t^n}{n!}\sum_{n=0}^{\infty}y^n\frac{t^n}{n!}=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}y^k\mathfrak{G}_{n-k}\left(x:\omega,r\right)\frac{t^n}{n!}$$

which implies the claimed result (2.9).

We now give the following theorem.

Theorem 4 *(Integral Representation) The following integral formula*

$$\int_{\mu}^{\xi}\mathfrak{G}_n\left(x+y:\omega,r\right)dy=\frac{\mathfrak{G}_{n+1}\left(x+\xi:\omega,r\right)-\mathfrak{G}_{n+1}\left(x+\mu:\omega,r\right)}{n+1} \tag{2.10}$$

holds for $\omega>0$ and $r>0$.

Proof. In view of derivative property (2.8), using the Fundamental Theorem of Calculus, we easily get

$$\int_{\mu}^{\xi}\mathfrak{G}_n\left(x+y:\omega,r\right)dy=\int_{\mu}^{\xi}\left(\frac{d}{dy}\frac{\mathfrak{G}_{n+1}\left(x+y:\omega,r\right)}{n+1}\right)dy$$

$$=\left.\frac{\mathfrak{G}_{n+1}\left(x+y:\omega,r\right)}{n+1}\right|_{\mu}^{\xi}=\frac{\mathfrak{G}_{n+1}\left(x+\xi:\omega,r\right)-\mathfrak{G}_{n+1}\left(x+\mu:\omega,r\right)}{n+1}$$

which gives the asserted result (2.10).

Some special cases of the integral representation are stated below.

Corollary 1 *For $\omega>0$ and $r>0$, we have*

$$\int_{0}^{1}\mathfrak{G}_n\left(x+y:\omega,r\right)dy=\frac{\mathfrak{G}_{n+1}\left(x+1:\omega,r\right)-\mathfrak{G}_{n+1}\left(x:\omega,r\right)}{n+1},$$

$$\int_{0}^{1}\mathfrak{G}_n\left(y:\omega,r\right)dy=\frac{\mathfrak{G}_{n+1}\left(1:\omega,r\right)-\mathfrak{G}_{n+1}\left(\omega,r\right)}{n+1},$$

$$\int_{\mu}^{\xi}\mathfrak{E}_n\left(x+y:\omega\right)dy=\frac{\mathfrak{E}_{n+1}\left(x+\xi:\omega\right)-\mathfrak{E}_{n+1}\left(x+\mu:\omega\right)}{n+1}.$$

We state the following theorem.

Theorem 5 *The following relations*

$$\mathfrak{G}_n(x:\omega,r)=(-1)^{n+1}\mathfrak{G}_n(-x:-\omega,r) \tag{2.11}$$

and

$$\mathfrak{G}_n(x:\omega,r)=2^r\mathfrak{G}_n\left(x-\frac{\omega}{2}:\frac{\omega}{2},r\right) \tag{2.12}$$

hold for x being a real number, $\omega>0$ and $r>0$.
Proof. Via Definition 1, we obtain

$$\sum_{n=0}^{\infty}\mathfrak{G}_n(x:\omega,r)\frac{t^n}{n!}=\frac{\omega^r}{\Gamma(r)}\frac{t^{r-1}}{e^{\omega t}}e^{tx}=-\frac{(-\omega)^r}{\Gamma(r)}\frac{(-t)^{r-1}}{e^{(-\omega)(-t)}}e^{(-t)(-x)}=-\sum_{n=0}^{\infty}(-1)^n\mathfrak{G}_n(-x:-\omega,r)\frac{t^n}{n!}$$

and

$$\sum_{n=0}^{\infty}\mathfrak{G}_n(x:\omega,r)\frac{t^n}{n!}=\frac{\omega^r}{\Gamma(r)}\frac{t^{r-1}}{e^{\left(\frac{\omega}{2}\right)t}}e^{t\left(x-\frac{\omega}{2}\right)}=2^r\frac{\left(\frac{\omega}{2}\right)^r}{\Gamma(r)}\frac{t^{r-1}}{e^{\left(\frac{\omega}{2}\right)t}}e^{t\left(x-\frac{\omega}{2}\right)}=2^r\sum_{n=0}^{\infty}\mathfrak{G}_n\left(x-\frac{\omega}{2}:\frac{\omega}{2},r\right)\frac{t^n}{n!}$$

which proves the asserted results (2.11) and (2.12).

We give the following theorem.

Theorem 6 *The following relation*

$$x^n=\frac{n!}{(n+r-1)!}\sum_{k=0}^{n+r-1}\binom{n+r-1}{k}\Gamma(r)\mathfrak{G}_{n+r-1-k}(x:\omega,r)\omega^{k-r} \tag{2.13}$$

is valid for x being a real number, $\omega>0$ and $r>0$.
Proof. In terms of Definition 1, we derive

$$e^{tx}=\sum_{n=0}^{\infty}\mathfrak{G}_n(x:\omega,r)\frac{t^n}{n!}e^{\omega t}\frac{\Gamma(r)}{\omega^r}t^{r-1}=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\Gamma(r)\mathfrak{G}_{n-k}(x:\omega,r)\omega^{k-r}\frac{t^{n-r+1}}{n!}$$

which means the desired result (2.13).

We here provide the following theorem.

Theorem 7 *The following explicit formula for the gamma distribution polynomials*

$$\sum_{k=0}^{n+r-1}\binom{n+r-1}{k}\mathfrak{G}_{n+r-1-k}(x:\omega,r)(\omega-x)^k=\begin{cases}\dfrac{\omega^r}{\Gamma(r)} & \text{if } n=1-r\\ 0 & \text{if } n>1-r\end{cases} \tag{2.14}$$

is valid for x being a real number, $\omega>0$ and r being a positive integer.
Proof. From Definition 1, we attain

$$\begin{aligned}\frac{\omega^r}{\Gamma(r)} &= \sum_{n=0}^{\infty}\mathfrak{G}_n(x:\omega,r)\frac{t^n}{n!}e^{t(\omega-x)}t^{1-r}\\ &= \sum_{n=0}^{\infty}\mathfrak{G}_n(x:\omega,r)\frac{t^n}{n!}\sum_{n=0}^{\infty}(\omega-x)^n\frac{t^{n+1-r}}{n!}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\mathfrak{G}_{n-k}(x:\omega,r)(\omega-x)^k\frac{t^{n-r+1}}{n!}\end{aligned}$$

which means the desired result (2.14).

An immediate result of Theorem 7 is given as follows.

Corollary 2 *We have*

$$\sum_{k=0}^{n+r-1}\binom{n+r-1}{k}\mathfrak{G}_{n+r-1-k}(\omega,r)(\omega-x)^k = \{.\frac{\omega^r}{\Gamma(r)} \text{ if } n=1-r, 0 \quad \text{ if } n>1-r. \tag{2.15}$$

The identities (2.14) and (2.15) provide us to compute the gamma distribution polynomials and numbers. The first few polynomials and numbers are calculated as follows:

$\mathfrak{G}_0(x:\omega,r)=\frac{\omega^r}{\Gamma(r)}$,	$\mathfrak{G}_0(\omega,r)=\frac{\omega^r}{\Gamma(r)}$,
$\mathfrak{G}_1(x:\omega,r)=-\frac{\omega^r}{\Gamma(r)}(\omega-x)$,	$\mathfrak{G}_1(\omega,r)=-\frac{\omega^{r+1}}{\Gamma(r)}$,
$\mathfrak{G}_2(x:\omega,r)=\frac{\omega^r}{\Gamma(r)}(\omega-x)^2$,	$\mathfrak{G}_2(\omega,r)=\frac{\omega^{r+2}}{\Gamma(r)}$,
$\mathfrak{G}_3(x:\omega,r)=-\frac{\omega^r}{\Gamma(r)}(\omega-x)^3$,	$\mathfrak{G}_3(\omega,r)=-\frac{\omega^{r+3}}{\Gamma(r)}$,
$\mathfrak{G}_4(x:\omega,r)=\frac{\omega^r}{\Gamma(r)}(\omega-x)^4$,	$\mathfrak{G}_4(\omega,r)=\frac{\omega^{r+4}}{\Gamma(r)}$,

(2.16)

Here, we provide symmetric property for the gamma distribution polynomials.

Theorem 8 *The gamma distribution polynomials satisfy the following summation formula for x being a real number, $\omega>0$ and $r>0$:*

$$\sum_{k=0}^{n}\binom{n}{k}\mathfrak{G}_{n-k}\left(x\frac{b}{a}:\omega\frac{b}{a},r\right)\mathfrak{G}_k\left(y\frac{a}{b}:\omega\frac{a}{b},r\right)\left(\frac{a}{b}\right)^{n-2k} \tag{2.17}$$

$$=\sum_{k=0}^{n}\binom{n}{k}\mathfrak{G}_{n-k}\left(x\frac{a}{b}:\omega\frac{a}{b},r\right)\mathfrak{G}_k\left(y\frac{b}{a}:\omega\frac{b}{a},r\right)\left(\frac{b}{a}\right)^{n-2k}.$$

Proof. By means of Definition 1, we consider

$$U=\frac{\omega^{2r}}{\Gamma(r)\Gamma(r)}\frac{t^{2r-2}}{e^{2\omega t}}e^{t(x+y)}.$$

Thus, we get

$$U=\frac{\omega^r}{\Gamma(r)}\frac{t^{r-1}}{e^{\omega t}}e^{tx}\frac{\omega^r}{\Gamma(r)}\frac{t^{r-1}}{e^{\omega t}}e^{ty}$$
$$=\frac{\left(\omega\frac{b}{a}\right)^r}{\Gamma(r)}\frac{\left(t\frac{a}{b}\right)^{r-1}}{e^{\left(\omega\frac{b}{a}\right)\left(t\frac{a}{b}\right)}}e^{\left(t\frac{a}{b}\right)\left(x\frac{b}{a}\right)}\frac{\left(\omega\frac{a}{b}\right)^r}{\Gamma(r)}\frac{\left(t\frac{b}{a}\right)^{r-1}}{e^{\left(\omega\frac{a}{b}\right)\left(t\frac{b}{a}\right)}}e^{\left(t\frac{b}{a}\right)\left(y\frac{a}{b}\right)}$$
$$=\sum_{n=0}^{\infty}\mathfrak{G}_n\left(x\frac{b}{a}:\omega\frac{b}{a},r\right)\left(\frac{a}{b}\right)^n\frac{t^n}{n!}\sum_{n=0}^{\infty}\mathfrak{G}_n\left(y\frac{a}{b}:\omega\frac{a}{b},r\right)\left(\frac{b}{a}\right)^n\frac{t^n}{n!}$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\mathfrak{G}_{n-k}\left(x\frac{b}{a}:\omega\frac{b}{a},r\right)\mathfrak{G}_k\left(y\frac{a}{b}:\omega\frac{a}{b},r\right)\left(\frac{a}{b}\right)^{n-2k}\frac{t^n}{n!}$$

and similarly

$$U=\frac{\left(\omega\frac{a}{b}\right)^r}{\Gamma(r)}\frac{\left(t\frac{b}{a}\right)^{r-1}}{e^{\left(\omega\frac{a}{b}\right)\left(t\frac{b}{a}\right)}}e^{\left(t\frac{b}{a}\right)\left(x\frac{a}{b}\right)}\frac{\left(\omega\frac{b}{a}\right)^r}{\Gamma(r)}\frac{\left(t\frac{a}{b}\right)^{r-1}}{e^{\left(\omega\frac{b}{a}\right)\left(t\frac{a}{b}\right)}}e^{\left(t\frac{a}{b}\right)\left(y\frac{b}{a}\right)}$$
$$=\sum_{n=0}^{\infty}\mathfrak{G}_n\left(x\frac{a}{b}:\omega\frac{a}{b},r\right)\left(\frac{b}{a}\right)^n\frac{t^n}{n!}\sum_{n=0}^{\infty}\mathfrak{G}_n\left(y\frac{b}{a}:\omega\frac{b}{a},r\right)\left(\frac{a}{b}\right)^n\frac{t^n}{n!}$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\mathfrak{G}_{n-k}\left(x\frac{a}{b}:\omega\frac{a}{b},r\right)\mathfrak{G}_k\left(y\frac{b}{a}:\omega\frac{b}{a},r\right)\left(\frac{b}{a}\right)^{n-2k}\frac{t^n}{n!},$$

which implies the claimed result (2.17).

A specific case of Theorem 8 is given below.

Corollary 3 *We have*

$$\sum_{k=0}^{n}\binom{n}{k}\mathfrak{G}_{n-k}\left(x\frac{1}{a}:\omega\frac{1}{a},r\right)\mathfrak{G}_k\left(ya:\omega a,r\right)a^{n-2k}=\sum_{k=0}^{n}\binom{n}{k}\mathfrak{G}_{n-k}\left(xa:\omega a,r\right)\mathfrak{G}_k\left(y\frac{1}{a}:\omega\frac{1}{a},r\right)a^{2k-n}.$$

Here is a summation formula for the polynomials $\mathfrak{G}_n\left(x:\omega,r\right)$.

Theorem 9 *The following summation formula*

$$\mathfrak{G}_n\left(x+y:\omega,r\right)=\sum_{k=0}^{n}\sum_{u=0}^{k}\binom{n}{k}\binom{k}{u}(-1)^{u+1}\mathfrak{G}_u\left(-\omega,r\right)x^{k-u}y^{n-k}\tag{2.18}$$

holds for x,y being real numbers, ω,r being positive real numbers.

Proof. From Definition 1, by using the Cauchy product for power series, we observe that

$$\sum_{n=0}^{\infty}\mathfrak{G}_n\left(x+y:-\omega,r\right)\frac{(-t)^n}{n!}=(-1)\frac{\omega^r}{\Gamma(r)}\frac{t^{r-1}}{e^{\omega t}}e^{-t(x+y)}=-\frac{\omega^r}{\Gamma(r)}\frac{t^{r-1}}{e^{\omega t}}e^{-tx}e^{-ty}$$

$$=\sum_{n=0}^{\infty}\left(\sum_{u=0}^{n}\binom{n}{u}(-1)^{n-u+1}\mathfrak{G}_u(\omega,r)x^{n-u}\right)\frac{t^n}{n!}\sum_{n=0}^{\infty}(-y)^n\frac{t^n}{n!}$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{u=0}^{k}\binom{n}{k}\binom{k}{u}(-1)^{n+u+1}\mathfrak{G}_u(\omega,r)x^{k-u}y^{n-k}\frac{t^n}{n!},$$

which means the desired result (2.18).

Hence, Theorem 3 and Theorem 9 yield the following results given as Corollary 4, Corollary 5, Corollary 6, and Corollary 7.

Corollary 4 *We have*

$$\sum_{k=0}^{n}\binom{n}{k}y^{n-k}\mathfrak{G}_k(x:\omega,r)=\sum_{k=0}^{n}\sum_{u=0}^{k}\binom{n}{k}\binom{k}{u}(-1)^{u+1}\mathfrak{G}_u(-\omega,r)x^{k-u}y^{n-k}. \tag{2.19}$$

Corollary 5 *We have*

$$\mathfrak{G}_n(x:\omega,r)=\sum_{u=0}^{n}\binom{n}{u}(-1)^{u+1}\mathfrak{G}_u(-\omega,r)x^{n-u}. \tag{2.20}$$

Corollary 6 *We have*

$$\sum_{k=0}^{n}\binom{n}{k}\mathfrak{G}_k(x:\omega,r)=\sum_{k=0}^{n}\sum_{u=0}^{k}\binom{n}{k}\binom{k}{u}(-1)^{u+1}\mathfrak{G}_u(-\omega,r)x^{k-u}. \tag{2.21}$$

Corollary 7 *We have*

$$\sum_{k=0}^{n}\binom{n}{k}\mathfrak{G}_k(x:\omega,r)=\sum_{k=0}^{n}\binom{n}{k}(-1)^{k+1}\mathfrak{G}_k(-\omega,r). \tag{2.22}$$

3 Correlations with the Bernoulli and Euler polynomials

In this section, two correlations including Bernoulli and Euler polynomials for gamma distribution polynomials are provided using their generating functions.
The classical Bernoulli and Euler polynomials are defined utilizing the following generating functions

$$\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}=\frac{t}{e^t-1}e^{xt}\quad(|t|<2\pi) \tag{3.1}$$

and

$$\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}=\frac{2}{e^t+1}e^{xt}\quad(|t|<\pi). \tag{3.2}$$

See $[1,6]$ for details about the aforesaid polynomials. The Bernoulli numbers B_n and Euler numbers E_n are obtained by the special cases of the corresponding polynomials $x=0$, namely

$$B_n(0):=B_n \text{ and } E_n(0):=E_n. \tag{3.3}$$

The Bernoulli polynomials and Euler polynomials are the most famous and important special polynomials in the theory of umbral calculus and they appear everywhere in mathematics, mostly in analytic number

theory, elementary number theory, numerical analysis, *cf.* $[1,6]$ and also see each of the earlier related references cited therein. Cheon [1] acquired a simple correlation for the Bernoulli polynomials and the Euler polynomials by utilizing their exponential generating functions. Kim et al. [6] studied some special polynomials connected to Euler and Bernoulli polynomials and provide several correlations for these polynomials. Then, they presented the zeros of these polynomials by using the computer.
Now, we state the following theorem.

Theorem 10 *The following relation*

$$\mathfrak{G}_n\left(x:\omega,r\right)=\frac{1}{n+1}\sum_{k=0}^{n+1}\binom{n+1}{k}\left(\sum_{u=0}^{k}\binom{k}{u}\mathfrak{G}_u\left(\omega,r\right)-\mathfrak{G}_k\left(\omega,r\right)\right)B_{n+1-k}\left(x\right) \tag{3.4}$$

is valid for x being a real number and ω, r being positive real numbers.
Proof. By Definition 1 and equality (3.1), we see that

$$\begin{aligned}\sum_{n=0}^{\infty}\mathfrak{G}_n\left(x:\omega,r\right)\frac{t^n}{n!}&=\frac{\omega^r}{\Gamma\left(r\right)}\frac{t^{r-1}}{e^{\omega t}}\frac{e^t-1}{t}\frac{t}{e^t-1}e^{tx}\\&=\sum_{n=0}^{\infty}\mathfrak{G}_n\left(\omega,r\right)\frac{t^n}{n!}\left(\sum_{n=0}^{\infty}\frac{t^n}{n!}-1\right)\sum_{n=0}^{\infty}B_n\left(x\right)\frac{t^{n-1}}{n!}\\&=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\left(\sum_{u=0}^{k}\binom{k}{u}\mathfrak{G}_u\left(\omega,r\right)-\mathfrak{G}_k\left(\omega,r\right)\right)B_{n-k}\left(x\right)\frac{t^{n-1}}{n!},\end{aligned}$$

which provides the asserted result (3.4).

The immediate result of the Theorem 10 is given as Corollary 8.
Corollary 8 *We have*

$$\mathfrak{G}_n\left(\omega,r\right)=\frac{1}{n+1}\sum_{k=0}^{n+1}\binom{n+1}{k}\left(\sum_{u=0}^{k}\binom{k}{u}\mathfrak{G}_u\left(\omega,r\right)-\mathfrak{G}_k\left(\omega,r\right)\right)B_{n+1-k}.$$

Now, we state the following correlation with the Euler polynomials.
Theorem 11 *The following relation*

$$\mathfrak{G}_n\left(x:\omega,r\right)=\sum_{k=0}^{n}\binom{n}{k}\left(\sum_{u=0}^{k}\binom{k}{u}\mathfrak{G}_u\left(\omega,r\right)+\mathfrak{G}_k\left(\omega,r\right)\right)E_{n-k}\left(x\right) \tag{3.5}$$

holds for ω, r being positive real numbers and x being a real number.
Proof. From Definition 1 and equality (3.2), we derive

$$\begin{aligned}\sum_{n=0}^{\infty}\mathfrak{G}_n\left(x:\omega,r\right)\frac{t^n}{n!}&=\frac{\omega^r}{\Gamma\left(r\right)}\frac{t^{r-1}}{e^{\omega t}}\frac{e^t+1}{2}\frac{2}{e^t+1}e^{tx}\\&=2\sum_{n=0}^{\infty}\mathfrak{G}_n\left(\omega,r\right)\frac{t^n}{n!}\left(\sum_{n=0}^{\infty}\frac{t^n}{n!}+1\right)\sum_{n=0}^{\infty}E_n\left(x\right)\frac{t^n}{n!}\end{aligned}$$

$$=\sum_{n=0}^{\infty}\left(\sum_{u=0}^{n}\binom{n}{u}\mathfrak{G}_u(\omega,r)+\mathfrak{G}_n(\omega,r)\right)\frac{t^n}{n!}\sum_{n=0}^{\infty}E_n(x)\frac{t^{n-1}}{n!}$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\left(\sum_{u=0}^{k}\binom{k}{u}\mathfrak{G}_u(\omega,r)+\mathfrak{G}_k(\omega,r)\right)E_{n-k}(x)\frac{t^n}{n!}$$

which means the desired result (3.5).
The immediate consequence of the Theorem 11 is stated in Corollary 9 as follows.
Corollary 9 *We have*

$$\mathfrak{G}_n(\omega,r)=\sum_{k=0}^{n}\binom{n}{k}\left(\sum_{u=0}^{k}\binom{k}{u}\mathfrak{G}_u(\omega,r)+\mathfrak{G}_k(\omega,r)\right)E_{n-k}.$$

3. Conclusion

In this paper, we have firstly aimed to investigate some properties and relations for the gamma distribution and introduced a special polynomial including gamma distribution. We then have derived various formulas and equalities such as explicit formula, derivative property, addition formula, and integral representation via the series manipulation method. Moreover, we have proved two relations for the mentioned polynomials related to the classical Bernoulli and usual Euler polynomials by using their generating functions.

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NEW EXACT SOLUTIONS FOR CONFORMABLE FRACTIONAL EQUATION VIA IBSEFM

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Abstract

The aim of this paper to obtain new exact solutions of conformable fractional a Boussinesq-like equation using the method of Improved Bernoulli Sub-Equation Function Method (IBSEFM). This solutions of the equation arises in several problems in coastal and ocean engineering. Compared with other methods, this method is very effective, simple, and easy to calculate, and it is a powerful mathematical tool for obtaining exact travelling wave solutions of other nonlinear conformable fractional partial differential equations.

Keywords: Conformable fractional derivative, Improved Bernoulli Sub-Equation Function Method, Exact Solution.

1. Introduction

Fractional order differential equations are the generalized form of an integer order differential equations. In recent years, the fractional differential equations have become a useful tool for describing nonlinear phenomena of science and engineering models. The fractional differential equation is also widely applied in several physical and engineering fields models such as physics, mathematical biology, fluid mechanics, plasma physics, optic, quantum field theory, etc. [1-3]. In many studies, various effective analytical approaches have been constructed and accomplished for searching more general and new exact solutions of nonlinear partial differential equations such as Kudryashov method [4], The first integral method [5] and IBSEFM [6,7].

In this paper, we obtain the exact solutions of conformable fractional Boussinesq-like equation by using IBSEFM. We consider

$$u_{tt}^{2\alpha} - \left(6u^2 u_x + u_{xtt}^{2\alpha}\right)_x - u_{xx} = 0, \tag{1}$$

where u_t^{α} is the fractional derivative in conformable sense of order $0 < \alpha \le 1$.This equation is an interesting model to describe ocean and coastal sciences [8].

The rest of this paper is organized as follows: In Section 2, we will describe the method for solving conformable fractional differential equations In Section 3, we apply this method to establish exact solution for the conformable fractional Boussinesq-like equation. Some conclusions are presented at the end of the paper.

2. Description of the Method

In this part, let us give the fundamental properties of the IBSEFM. We present the six fundamental steps of the IBSEFM below the following:

Step 1: Let us take consider the following conformable time fractional partial differential equation (CFPDE) of the style

$$P\left(v, D_t^{(\mu)}v, D_x^{(\mu)}v, D_{xt}^{(2\mu)}v, D_{xxt}^{(3\mu)}v, \ldots\right)=0, \tag{2}$$

where $D_t^{(\mu)}$ is the conformable derivative operator, $v(x,t)$ is an unknown function, P is a polynomial in v and its partial derivatives contain fractional derivatives. The aim is to convert conformable fractional partial differential equation with a suitable fractional transformation into the ordinary differential equation (ODE). The wave transformation as:

$$\begin{aligned} &v(x,t)=V(\xi),\\ &\xi=\left(x-ct^{\alpha}\alpha^{-1}\right), \end{aligned} \tag{3}$$

where c is an arbitrary constant and not zero. Using the properties of conformable derivative, it enables us to convert (3) into an ODE in the form

$$N\left(V, V', V'', \ldots\right)=0. \tag{4}$$

Step 2: If we integrate (4) term to term once or more, we acquire integration constant(s) which may be determined then.

Step 3: We hypothesize that the solution of (4) may be presented below:

$$V(\xi)=\frac{\sum_{i=0}^{n} a_i F^i(\xi)}{\sum_{i=0}^{m} b_i F^i(\xi)}, \tag{5}$$

where $a_0, a_1, \ldots, a_n$ and $b_0, b_1, \ldots, b_m$ are coefficients which will be determined later. $m \neq 0, n \neq 0$ are chosen arbitrary constants to balance principle and considering the form of Bernoulli differential equation as follows:

$$F'(\xi)=\sigma F(\xi)+dF^{M}(\xi), \tag{6}$$

where $d \neq 0, \sigma \neq 0, M \in R/\{0,1,2\}$ and $F(\xi)$ is polynomial.

Step 4: The positive integer m, n, M (are not equal to zero) which is found by balance principle that is both nonlinear term and the highest order derivative term of (4).
Substituting (5) and (6) in (3) it gives us an equation of polynomial $\Theta(F)$ of F as following;

$$\Theta(F(\xi)) = \rho_s F(\xi)^s + ... + \rho_1 F(\xi) + \rho_0 = 0,$$

where ρ_i, $\left(i = \overline{0, j}\right)$ are coefficients and will be determined later.
Step 5: The coefficients of $\Theta(F(\xi))$ which will give us a system of algebraic equations, whole be zero.

$$\rho_i = 0,\ i = \overline{0, j}.$$

Step 6: When we solve (4), we get the following two cases with respect to σ and d:

a) For $d \neq \sigma,\ \varepsilon \in R$,

$$F(\xi) = \left[\frac{-de^{\sigma(\varepsilon-1)} + \varepsilon\sigma}{\sigma e^{\sigma(\varepsilon-1)\xi}} \right]^{\frac{1}{1-\varepsilon}}, \tag{7}$$

b) For $d = \sigma,\ \varepsilon \in R$,

$$F(\xi) = \left[\frac{(\varepsilon-1) + (\varepsilon+1)\tanh\left(\sigma(1-\varepsilon)\right)\frac{\varepsilon}{2}}{1 - \tanh\left(\sigma(1-\varepsilon)\frac{\varepsilon}{2}\right)} \right]^{\frac{1}{1-\varepsilon}}. \tag{8}$$

Using a complete discrimination system for polynomial of $F(\xi)$, we obtain the analytical solutions of (4) via software programme and categorize the exact solutions of (4). To achieve better results, we can plot two and three dimensional figures of analytical solutions by considering proper values of parameters.

3. Implementation of the IBSFM

Consider the nonlinear conformable fractional a Boussinesq-like:

$$u_{tt}^{2\alpha} - \left(6u^2 u_x + u_{xtt}^{2\alpha}\right)_x - u_{xx} = 0.$$

In the following, we apply the first integral method to find the solitary wave solutions for conformable fractional a Boussinesq-like. Firstly, we let

$$u(x,t) = U(\eta),\ \eta = px - q\left(t^\alpha \alpha^{-1}\right), \tag{9}$$

where p, q are arbitrary constants (are not zero) to be determined later. Using the (9) into (7), we obtain ordinary differential equation:

$$\left(p^2-q^2\right)U(\eta)+2p^2U^3(\eta)+p^2q^2U''(\eta)=0. \tag{10}$$

According the method, when we apply the balance for the terms $U^3(\eta)$ and $U''(\eta)$, we obtain the relationship for m, n and M as below:

$$M+m=n+1.$$

For this equation, we take one case, assuming that $M=n=3$ and $m=1$, then we get as follows:

$$U(\eta)=\frac{\sum_{i=0}^{3} a_i F^i(\eta)}{\sum_{i=0}^{1} b_i F^i(\eta)}=\frac{a_0+a_1F(\eta)+a_2F^2+a_3F^3(\eta)}{b_0+b_1F(\eta)}=\frac{\psi}{\varphi}, \tag{11}$$

$$U'(\eta)=\frac{\psi'(\eta)\varphi(\eta)-\psi(\eta)\varphi'(\eta)}{\varphi^2(\eta)},$$

$$U''(\eta)=\frac{\psi'(\eta)\varphi(\eta)-\psi(\eta)\varphi'(\eta)}{\varphi^2(\eta)}-\frac{[\psi(\eta)\varphi'(\eta)]'\varphi^2(\eta)-2\psi(\eta)[\varphi(\eta)]^2\varphi(\eta)}{\varphi^4(\eta)}, \tag{12}$$

Substituting (11) and (12) in (10), we get a system of algebraic system. When we equal to zero all the same power of F and solving the algebraic system of equations from coefficients of polynomial of F, it yields the following coefficients:

$$Const: -2p^2a_0^3+q^2a_0b_0^2-p^2a_0b_0^2=0,$$

$$F: -6p^2a_0^2a_1+q^2a_1b_0^2-p^2a_1b_0^2-p^2q^2\sigma^2a_1{b_0}^2+2q^2\,a_0b_0b_1-2p^2a_0b_0b_1-+p^2q^2\sigma^2a_0b_0b_1=0,$$

$$\begin{aligned}F^2: &-6p^2a_0a_1^2-p^26{a_0}^2a_2+q^2a_2{b_0}^2-p^2a_2{b_0}^2-4p^2q^2\sigma^2a_2{b_0}^2+2q^2a_1b_0b_1\\&-2p^2a_1b_0b_1+p^2q^2\sigma^2a_1b_0b_1+qa_0{b_1}^2-c^2\sigma^2a_0{b_1}^2+k^2\sigma^2a_0{b_1}^2+p^2q^2\sigma^2a_0b_1^2=0,\end{aligned}$$

•
•
•

$F^7 : -6p^2a_2^2a_3 - 6p^2a_1a_3^2 - 15q^2d^2p^2a_3b_0^2 - 9q^2d^2p^2a_2b_0b_1 - 12q^2dp^2\sigma a_3b_1^2 = 0,$

$F^8 : -6p^2a_2a_3^2 - 21q^2d^2p^2a_3b_0b_1 - 3q^2d^2p^2a_2b_1^2 = 0,$

$F^9 : -2p^2a_3^3 - 8q^2d^2p^2a_3b_1^2 = 0.$

We now solve the above system of equations and substitute in each case the obtained result of the coefficients to get the new solution(s) $u(x,t)$. By solving the above system with the aid of software, the coefficients are obtained as:

Case1: For $\sigma \neq d$,

$$a_0 = \frac{\mathrm{i}\sqrt{-q^2+p^2}b_0}{\sqrt{2}p}; a_1 = \frac{\mathrm{i}\sqrt{-q^2+p^2}b_1}{\sqrt{2}p}; a_2 = -2\mathrm{i}dqb_0; a_3 = -2\mathrm{i}qdb_1; \sigma = -\frac{\sqrt{-q^2+p^2}}{\sqrt{2}pq}, \tag{13}$$

where $p \neq 0,\ q \neq 0.$

Putting (13) along with (7) in (11), we acquire exponential solution to Boussinesq-like equation as follow:

$$u_1(x,t) = -\frac{2\mathrm{i}\sqrt{2}qd\mathrm{e}^{\frac{\sqrt{2}\sqrt{-q^2+p^2}\left(px-\frac{qt^\alpha}{\alpha}\right)}{pq}}\varepsilon\sqrt{-q^2+p^2}t^{-1+\alpha}}{p\left(\frac{\sqrt{2}qdk}{\sqrt{-q^2+p^2}}+\varepsilon\mathrm{e}^{\frac{\sqrt{2}\sqrt{-q^2+p^2}\left(px-\frac{qt^\alpha}{\alpha}\right)}{pq}}\right)^2},$$

where p,q are constants and are not zero. Thus, we can plot two dimensional (2D), three dimensional (3D) and contour plot for imaginary part of the equation as follow:

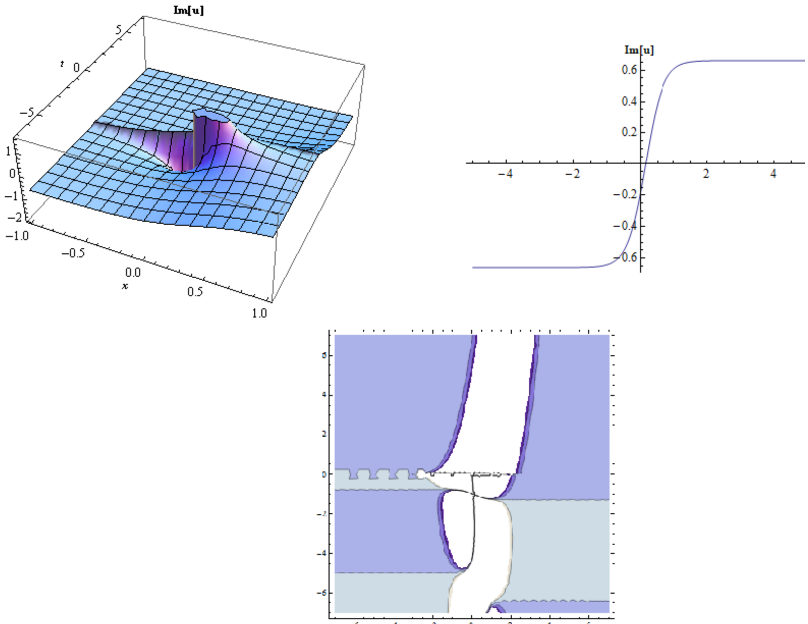

Figure 1: The 3D and 2D Surfaces of the exact solution of $u_1(x,t)$. By considering the values $\alpha = 0.4;\ p = 0.87;\ q = 0.31;\ d = 0.5;\ \varepsilon = 0.5;\ -1 < x < 1,\ -7 < t < 7$ for 3D surface $-5 < x < 5,\ -5 < t < 5$ for 2D surface and contourplot.

4. Conclusion

In this paper, we have succesfully applied the IBSEFM to Eq. (1) and we obtained original new solution. With the aid of mathematical software, a variety of exact solutions consisting of exponential for the conformable fractional partial derivative equations are obtained. And two and three dimensional figures of this solution are plotted according to the suitable values of these parameters. Furthermore, the work shows that the method is effective and can be used for many other conformable fractional partial derivative equations in mathematical physics. The performance of the IBSEFM is reliable. Thus, we deduce that the proposed method can be extended to solve many systems of nonlinear fractional partial differential equations.

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New Fixed Point Theorems on Vector Metric Spaces with w-distance

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Abstract

In this paper, we first define a new concept on vector metric spaces by considering the concept of Q-function which includes the concept of w-distance, and hence we obtain a new contractive mapping. Then, we obtain some fixed point theorems for such mappings on vector metric spaces. Thus, we extend some results existing in the literature.

1. Introduction

The fixed point theory has important place in many disciplines, since it has vital tools for solving some problems in these areas. Therefore, Banach contraction principle [6] which is fundamental result in fixed point theory has attracted interest of authors [4,5,14]. In this sense, Cevik et al. [8] introduced a notion of vector metric space and proved Banach's result on these spaces. Then, many authors have generalized and improved the results in the literature [7,9,13]. Now, before the definition of vector metric spaces, we recall some basic and fundamental concepts about Riesz space and related properties. A real vector space E is said to be an ordered vector space if it is equipped with an partial order which is compatible operations of vector spaces. An ordered vector space $(E, \leq)$ is said to be a Riesz space if the supremum $x \vee y$ and infimum $x \wedge y$ for all $x, y \in E$ are in E. Let $(E, \leq)$ be a Riesz space. We denote positive cone of E by E_+. The notation $a_n \downarrow a$ means that the sequence $\{a_n\}$ is nonincreasing such that $\inf\{a_n : n \in \mathbb{N}\} = a$. If there exists $\{a_n\}$ in E satisfying $a_n \downarrow 0$ such that $|b_n - b| \leq a_n$ for all $n \in \mathbb{N}$, then the sequence $\{b_n\}$ is called order convergence to $b \in E$ where the modul of $a \in E$ is defined by $|a| = a \vee (-a)$. If there exists $\{a_n\}$ in E satisfying $a_n \downarrow 0$ such that $|b_n - b_{n+p}| \leq a_n$ for all $n, p \in \mathbb{N}$, then the sequence $\{b_n\}$ is called order-Cauchy sequence. Also, a Riesz space is called order-Cauchy complete if every order-Cauchy sequence order converges to any point in this space. Moreover, if every subset of E with upper bound (lower bound) has a supremum (infimum) in E, then the Riesz space is called Dedekind complete. A Riesz space E is said to be Archimeden, if $\frac{1}{n}a \downarrow 0$ for all $a \in E_+$. Note that, every Dedekind complete Riesz space is an Archimedean. For more detail see [2].

Definition 1. Let X be a nonempty set and $(E, \leq)$ be a Riesz space. Then, then the mapping $d: X \times X \to E$ is said to be vector metric if it is satisfied for all $x, y, z \in X$,

$m1)$ $d(x, y) = 0$ if and only if $x = y$,

$m2)$ $d(x, y) \leq d(x, z) + d(y, z)$.

Also, (X, d, E) is called vector metric space.

Definition 2. Let (X, d, E) be a vector metric space and $\{x_n\}$ be a sequence in X. Then,

i) It is said to be $\{x_n\}$ E-converges to $x \in X$, denoted by $x_n \xrightarrow{d,E} x$, if there exists $\{a_n\}$ in E satisfying $a_n \downarrow 0$ such that $d(x_n, x) \leq a_n$ for all $n \in \mathbb{N}$.

ii) The sequence $\{x_n\}$ is said to be E-Cauchy if there exists $\{a_n\}$ in E satisfying $a_n \downarrow 0$ such that $d(x_n, x_{n+p}) \leq a_n$ for all $n, p \in \mathbb{N}$.

iii) The vector metric space (X, d, E) is said to be E-complete if every E-Cauchy sequence in X E-converges to a point x in X.

On the other hand, recently, there is a tendency to improve the results obtained in fixed point theory by using some functions such as w-distance. In this sense, Al-Homidian [1] introduced a new concept so called Q-function including the notion of w-distance defined by Kada et al. [11]. Then, many authors studied to generalize this concept in various ways [3,10,12].

In this paper, we first define a new concept on vector metric spaces by considering the concept of Q-function, and hence we obtain a new contractive mapping. Then, we obtain some fixed point theorems for such mappings on vector metric spaces. Thus, we extend many results existing in the literature.

2. Main Results

We begin this section with the definition of our new concept on vector metric spaces.

Definition 3. Let (X, d, E) be a vector metric space. Then, the mapping $q: X \times X \to E$ is called Q_v-function, if it is satisfied for all $x, y, z \in X$,

$Q_1)$ $q(x, y) \leq q(x, z) + q(z, y)$,

Q_2) If $x \in X, M \in E_+$ and the sequence $\{y_n\}$ satisfying $y_n \xrightarrow{d,E} y$ for some $y \in X$ satisfy $q(x, y_n) \leq M$ for all $n \in \mathbb{N}$, then $q(x, y) \leq M$,

Q_3) For each $b \in E_+$, there exists $a \in E_+$ such that

$$q(x, y) \leq a \text{ and } q(x, z) \leq a \Rightarrow d(y, z) \leq b.$$

Note that every vector metric is a Q_v-function.

Theorem 1. Let (X, d, E) be an E-complete vector metric space where $(E, \leq)$ is an Archimedean Riesz space and $q: X \times X \to E$ be a Q_v-function on X. Assume that $T: X \to X$ is a mapping. If there exists $\alpha \in [0,1)$ such that

$$q(Tx, Ty) \leq \alpha q(x, y) \tag{2.1}$$

for all $x, y \in X$, then the mapping T has a fixed point in X.

Proof. Let $x_0 \in X$ be arbitrary point. Define a sequence $\{x_n\}$ with the initial point x_0 such that $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. From (2.1), we have

$$\begin{aligned} q(x_n, x_{n+1}) &= q(Tx_{n-1}, Tx_n) \\ &\leq \alpha q(x_{n-1}, x_n) \\ &= \alpha q(Tx_{n-2}, Tx_{n-1}) \\ &\leq \alpha^2 q(x_{n-2}, x_{n-1}) \\ &\;\;\vdots \\ &\leq \alpha^n q(x_0, x_1) \end{aligned}$$

for all $n \in \mathbb{N}$. Then, for all $n, p \in \mathbb{N}$, we get

$$\begin{aligned} q(x_n, x_{n+p}) &\leq q(x_n, x_{n+1}) + q(x_{n+1}, x_{n+2}) + \cdots + q(x_{n+p-1}, x_{n+p}) \\ &\leq \alpha^n q(x_0, x_1) + \alpha^{n+1} q(x_0, x_1) + \cdots + \alpha^{n+p-1} q(x_0, x_1) \\ &\leq (1 + \alpha + \cdots + \alpha^{p-1}) \alpha^n q(x_0, x_1) \\ &\leq \frac{\alpha^n}{1-\alpha} q(x_0, x_1). \end{aligned}$$

which implies that $q(x_n, x_{n+p}) \leq b_n$ where $b_n = \frac{\alpha^n}{1-\alpha} q(x_0, x_1)$ for all $n \in \mathbb{N}$. Since E is an Archimedean Riesz space, we obtain $b_n \downarrow 0$. Now, we want to show that $\{x_n\}$ is a Cauchy sequence. For this, assume that b is any point in E_+. Then, there exists $a_n \in E_+$ satisfying

$$q(x_n, x_{n+p}) \leq a_n \text{ and } q(x_n, x_{n+r}) \leq a_n \Rightarrow d(x_{n+p}, x_{n+r}) \leq \frac{1}{n} b$$

for all $n \in \mathbb{N}$. Then, there exist $c_n = a_n \vee b_n$ in E for all $n \in \mathbb{N}$ such that $q(x_n, x_{n+p}) \leq a_n \leq c_n$ and $q(x_n, x_{n+r}) \leq a_n \leq c_n$ implies that $d(x_{n+p}, x_{n+r}) \leq \frac{1}{n} b$. Since E is Archimedean, $\frac{1}{n} b \downarrow 0$ and so, $\{x_n\}$ is a Cauchy sequence. Since (X,d,E) is an E-complete vector metric space, there exists a point $x \in X$ such that $x_n \xrightarrow{d,E} x$. On the other hand, we know that $q(x_n, x_{n+p}) \leq b_n \downarrow 0$ for all $n \in \mathbb{N}$. Let fix $n \in \mathbb{N}$. Then, from the property (Q_2), we have

$$q(x_n, x) \leq b_n \downarrow 0 \tag{2.2}$$

for all $n \in \mathbb{N}$. Now, from (2.1), we say that

$$\begin{aligned} q(x_{n+1}, Tx) &= q(Tx_n, Tx) \\ &\leq \alpha q(x_n, x) \\ &\leq \alpha b_n \downarrow 0 \end{aligned} \tag{2.3}$$

for all $n \in \mathbb{N}$. Using the inequalities (2.2) and (2.3), from the property (Q_3), we have $d(x, Tx) = 0$ and so, $x = Tx$. This complete the proof.

If we take $q = d$ in Theorem 1, we obtain the following result which is main result of [8].

Corollary 1. Let (X, d, E) be an E-complete vector metric space where $(E, \leq)$ is an Archimedean Riesz space and $T: X \to X$ be a mapping. If there exists $\alpha \in [0,1)$ such that

$$d(Tx, Ty) \leq \alpha d(x, y)$$

for all $x, y \in X$, then the mapping T has a fixed point in X.

3. Conclusion

In the present paper, we aim to extend the some results in fixed point theory. For this, we first define a new concept so called Q_v-function considering Q-function. Thus, we obtain a new contractive mapping on vector metric spaces. Then, we present a fixed point theorem for such mappings.

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New results on the asymptotic stability, boundedness and square integrability of third order neutral differential equations

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Abstract

Neutral differential equations have many applications. In the qualitative analysis of such systems, the stability and asymptotic behavior of solutions play an important role. In this work by construting a Lyapunov functional, we obtain some sufficient conditions which guarantee the stability, boundedness and square integrability of solutions for some nonlinear neutral delaydifferential equations of third order.

1. Introduction

In this paper we investigate the asymptotic behavior of solutions of third order neutral delay differential equations of the form

$$\left(x(t)+\beta(t)x(t-r)\right)''' + \Psi(x'(t))x''(t) + g(x'(t)) + f(x(t-r)) = p(t), \qquad (1)$$

where throughout the paper we always assume that $\Psi(x'(t)), g(x'(t))), f(x), \beta(t)$ and $p(t, x(t), x(t-r), x'(t), x(t-r), x''(t))$, are continuous in their respective arguments. It is also supposed that the derivatives $f'(x), g'(y)$, and $\beta'(t)$ are continuous for all x, y, t with $f(0) = g(0) = 0$ and $0 \leq \beta(t)$, for all $t \geq T \geq t_0 + r$, where $r > 0$.

In recent years, many books and papers dealt with the delay neutral differential equation and obtained many good results, for example[1-13].
Neutral differential equations have many applications. For example, these equations arise in the study of two or more simple oscillatory systems with some interconnections between them and in modelling physical problems such as vibration of masses attached to an elastic bar. In the qualitative analysis of such systems, the stability and asymptotic behavior of solutions play an important role. There is the permanent interest in obtaining new sufficient conditions for the stability and Boundedness of the solutions of third order neutral differential equations. Many efforts were done to deduce sufficient conditions for the stability, asymptotic behavior and Boundedness of solutions of differential equations of the type (1).
In many references the authors dealt with the problems by considering Lyapunov functions or functionals and obtained the criteria for the stability and boundedness. The asymptotic stability of (1) when $\beta(t) = 0$ has been discussed by many authors. Here, we refer to the papers Omeike [4,5], Remili, and Beldjerd [9] , Tunç¸[22, 23].

For the sake of convenience we introduce the following notation
$X(t) = x(t) + \beta(t)x(t - r)$. (2)

2. Main Results

In this section, we also make use of the following conditions. Suppose that there are positive constants $d_0, d_1, d, M, \delta, \alpha, \rho, \Psi_0$, and Ψ_1 such that the following conditions which will be used on the functions that appeared in equation (1) are satisfied:

i) $\Psi_0 < \Psi(y) < \Psi_1$.

ii) $\frac{f(x)}{x} \geq M > 0$ $(x \neq 0)$, and $|f'(x)| \leq \delta$ for all x.

iii) $d^2 < d_0 \leq \frac{g(y)}{y} \leq d_1$.

iv) $\frac{\delta}{2} < d < \Psi_0$.

v) $\int_T^t |\beta'(s)| ds < \rho$, $-\alpha \leq \beta'(t) \leq 0$,

2.1 Stability

For the case $p(t) \equiv 0$, The following result is introduced

Theorem 2.1. *If in addition to the hypotheses (i)-(v), suppose that there are positive constants* ε_1, ε_2, *such that following is also satisfied*

$H_1)$ $-\varepsilon_1 + \frac{3\alpha}{2} < 0.$

$H_2)$ $-dd_0 + \delta + c(\frac{d_1}{2} + \delta) + \varepsilon_1 + (1 + c)^2 = -\eta_1 < 0.$

$H_3)$ $-\varepsilon_2 + \frac{\alpha}{2}(d + 1) < -\eta_2 < 0.$

$H_4)$ $-B_0 + \frac{c\,(B_1 + B_0)}{2} + (1 + c)^2 + \frac{cd_1}{2} + \varepsilon_2 + \frac{\alpha}{2} < -\eta_3 < 0,$

where $B_0 = \Psi_0 - d$, $B_1 = \Psi_1 - d$ *and* $c = \beta(T)$.

Then every solution of (1) is uniformly asymptotically stable, provided that

$$r < \min\{\frac{2\eta_1}{\delta\,(1 + c + 2d)}, \frac{2\eta_2}{c\delta}, \frac{2\eta_3}{\delta}\}.$$

Proof. In the sequel we will assume the following notations

$$Y(t) = y(t) + \beta(t)y(t - r)$$
$$Z(t) = z(t) + \beta(t)z(t - r).$$

Consider the equivalent system to (1)

$$\begin{cases} x' = y, \\ y' = z, \\ Z' = -\Psi(y)z - g(y) - f(x) + \int_{t-r}^{t} f'(x(s))y(s)ds. \end{cases} \quad (3)$$

The proof of this theorem depends on properties of the continuously differentiable function $W = W(t, x_t, y_t, z_t)$ defined by

$$W = Ve^{-\frac{1}{\omega}\int_T^t |\beta'(s)d|}, \; for\, all t \geq T \geq t_0 + r, \quad (4)$$

where

$$\begin{aligned} V &= dF(x) + f(x)Y + Y^2 + G(y) + \frac{1}{2}Z^2 + dyZ + d\int_0^y \Psi(u)udu \\ &+ \lambda \int_{-r}^{0}\int_{t+s}^{t} y^2(\xi)d\xi ds + E(t), \end{aligned} \quad (5)$$

$$F(x) = \int_0^x f(u)du, \quad G(y) = \int_0^y g(u)du,$$

and

$$E(t) = \mu_1 \int_{t-r}^{t} y^2(s)ds + \mu_2 \int_{t-r}^{t} z^2(s)ds.$$

λ, μ_1 and μ_2 are positives constants which will be specified later in the proof. Since $E(t) \leq 0$, and

$$G(y) = \int_0^y \frac{g(u)}{u} udu \geq \frac{d_0}{2}y^2,$$

it follows from (iv) that

$$\begin{aligned} d\int_0^y \Psi(u)udu &= d\int_0^y [(\Psi(u) - d) + d]udu \\ &\geq \frac{1}{2}(d(1-d)y^2 + \frac{1}{2}d^2y^2 \\ &= \frac{1}{2}dy^2. \end{aligned}$$

and

$$\frac{1}{2}Z^2 + dyZ + \frac{d_0}{2}y^2 = \frac{1}{2}(Z + dy)^2 + (\frac{d_0 - d^2}{2})y^2.$$

Hence, it can be easily seen that there exists k_0 such that

$$d\int_0^y \Psi(u)udu + \frac{1}{2}Z^2 + dyZ + \frac{d_0}{2}y^2 \geq k_0(y^2 + Z^2).$$

On the other hand, by (iv) we have

$$d\,F(x) + f(x)Y + Y^2 = d\int^x f(u)du + (Y + \frac{1}{2}f(x))^2 - \frac{1}{4}f^2(x)$$

$$\geq (d - \frac{\delta}{2})F(x).$$

Condition (ii) implies that

$$F(x) = \int_0^x f(u)du = \int_0^x \frac{f(u)}{u}udu \geq \frac{1}{2}Mx^2.$$

Hence, there exists a positive constant K_0, small enough such that

$$V \geq K_0(x^2 + y^2 + Z^2). \qquad (6)$$

Because

$$\frac{d}{dt}E(t) = \mu_1(y^2(t) - y^2(t-r)) + \mu_2(z^2(t) - z^2(t-r)),$$

the time derivative of functional V along the system (3) leads that

$$V'_{(3)} = U_1 + U_2 + U_3 + Z\int_{t-r}^t f'(x(s)y(s)ds + dy\int_{t-r}^t f'(x(s)y(s)ds - \lambda\int_{t+s}^t y^2(\xi)d\xi,$$

where

$$\begin{aligned} U_1 &= (d - \Psi(y) + \mu_2)z^2(t) - dy\,g(y) + (f'(x) + \mu_1 + \lambda r)\,y^2(t) \\ &\quad + (2\beta(t)\beta'(t) - \mu_1)\,y^2(t-r) + (\beta(t)\beta'(t) - \mu_2)\,z^2(t-r), \\ U_2 &= [\beta(t)f'(x) + 2\beta'(t)]\,y(t)y(t-r) + [\beta'(t) - \beta(t)\Psi(y) + d\beta]\,z(t)z(t-r) \\ &\quad + 2y(t)z(t) + (2\beta(t) + d\beta'(t))\,y(t)z(t-r) - \beta(t)z(t-r)g(y) \\ &\quad + 2\beta(t)y(t-r)z(t) + 2\beta^2(t)y(t-r)z(t-r), \\ U_3 &= \beta'(t)y(t-r)f(x). \end{aligned}$$

On the other hand, from the assumptions (ii) of theorem 2.1 we have $|f\,(x)| < \delta\,|x|$; and by using $2uv \leq u^2 + v^2$ we obtain,

$$\begin{aligned} U_1 &\leq (d - \Psi(y) + \mu_2)\,z^2(t) + (-dd_0 + \delta + \mu_1 + \lambda r)\,y^2(t) \\ &\quad + (2\beta(t)\beta'(t) - \mu_1)\,y^2(t-r) + (\beta(t)\beta'(t) - \mu_2)\,z^2(t-r). \end{aligned}$$

We have also,

$$
\begin{aligned}
U_2 \;\le\; & \left[\frac{\beta(t)\delta}{2}+\left|\beta'(t)\right|+1+\beta(t)\left(1+\frac{d_1}{2}\right)+\frac{d\left|\beta'(t)\right|}{2}\right]y^2(t)\\
&+\left[\frac{1}{2}\left(\left|\beta'(t)\right|+\beta(t)\left[\Psi(y)-d\right]\right)+1+\beta(t)\right]z^2(t)\\
&+\left[\frac{\beta(t)\delta}{2}+\left|\beta'(t)\right|+\beta(t)+\beta^2(t)\right]y^2(t-r)\\
&+\left[\frac{\left|\beta'(t)\right|}{2}+\frac{\beta(t)}{2}\left[\Psi(y)-d\right]+\beta(t)\left(1+\frac{d_1}{2}\right)+\frac{d\left|\beta'(t)\right|}{2}+\beta^2(t)\right]z^2(t-r),
\end{aligned}
$$

and

$$U_3 \le \frac{\left|\beta'(t)\right|}{2}\left(y^2(t-r)+\delta^2x^2\right).$$

In the same way using (ii) we get

$$
\begin{aligned}
z\int_{t-r}^{t} f'(x(s)y(s)ds \;&\le\; \frac{\delta r}{2}z^2+\frac{\delta}{2}\int_{t-r}^{t} y^2(s)ds,\\
\beta(t)z(t-r)\int_{t-r}^{t} f'(x(s)y(s)ds \;&\le\; \frac{\beta_1\delta r}{2}z^2(t-r)+\frac{\delta\beta_1}{2}\int_{t-r}^{t} y^2(s)ds,\\
dy\int_{t-r}^{t} f'(x(s)y(s)ds \;&\le\; \frac{\delta r}{2}dy^2+\frac{\delta d}{2}\int_{t-r}^{t} y^2(s)ds.
\end{aligned}
$$

So, after rearrangement and putting $\lambda = \frac{\delta}{2}(1+c+d)$, by the fact that $\beta'(t) \le 0$, $|\beta'(t)| \le \alpha$ and $\beta(t) \le c$ we obtain from conditions $(H1)$-$(H4)$ that

$$V'_{(3)} \le -K_1\left(y^2(t)+z^2(t)\right)+\left|\beta'(t)\right|\left(1+\frac{d}{2}\right)y^2(t)+\frac{\left|\beta'(t)\right|}{2}\delta^2x^2(t),$$

where

$$K_1=\min\{-\eta_3+\frac{\delta r}{2},-\eta_1+\left(\frac{d\delta}{2}+\lambda\right)r\}.$$

Using (v) we obtain $e^{-\frac{\rho}{\omega}} < e^{-\frac{1}{\omega}\int_T^t|\beta'(s)|ds} < 1$, for all $t \ge T$: Choosing $\omega = \frac{K_0}{K_2}$ where

$$K_2=\max\{\left(1+\frac{d}{2}\right),\frac{\delta^2}{2}\}.$$

We conclude that

$$W'_{(3)} \le -L_1\left[y^2(t)+z^2(t)\right],$$

so that

$L_1 = K_1 e^{-\frac{\varrho}{\omega}}$.
From (3), $W3(\|X\|) = L1\,(y^2(t) + z^2(t))$ is positive definite function. The above discussion guarantees that the null solution of equation (1) is asymptotically stable and completes the proof of Theorem 2.1 .

2.2 Boundedness

Our main theorem in this section is stated with respect to $p(t) \neq 0$ as follows :

Theorem 3.1. *Assume that all the conditions of Theorem 2.1 are satisfied and there exist positive constants p_1 and p_2 such that :*

$I_1)$ $|\dot{p}(t)| \leq < \dot{p}_1$,

$I_2)$ $\left|\int_0^t p(s)ds\right| < p_2$,

Then there exists a positive constant D such that any solution $x(t)$ of (1) and their derivatives $x'(t)$, and $x''(t)$ satisfies

$$|x(t)| \leq D,\ |x'(t)| \leq D,\ |x''(t)| \leq D. \tag{7}$$

Proof. For the case $p(t) \neq 0$; equation (1) is equivalent to the system

$$\begin{cases} x' = y, \\ y' = z, \\ Z' = p(t) - \Psi(y)z - g(y) - f(x) + \int_{t-r}^{t} f'(x(s))y(s)ds. \end{cases} \tag{8}$$

On differentiating (5) along the system (8), using condition ($I1$)we obtain

$$V'_{(8)} \leq -L_1\left(y^2 + z^2\right) + \frac{L_2}{K_0} q(t)V + L_3 p(t), \tag{9}$$

where

$L_3 = 2L_2$ with $L_2 = \max\{1, d\}$.
Integrating both sides (9) from $t1$ to t, we easily obtain

$$V(t) - V(t_1) \leq L_3 \int_{t_1}^{t} p(s)ds + \frac{L_2}{K_0} \int_{t_1}^{t} V(s)p(s)ds.$$

By using Gronwall inequality it follows

$$V(t) \leq p_3 \exp\left(\frac{L2}{K_0} \int_{t_1}^{t} p(s)ds\right) \leq p_4, \tag{10}$$

with

$p_4 = p_3 \exp\left(\frac{L_2}{K_0} q_2\right)$

and

$p_3 = V(t_1) + L_3 p_2.$

This result implies that there exist constant D such that

$$|x(t)| \leq D,\ |y(t)| \leq D, |Z(t)| \leq D.$$

From (8) we have

$$\begin{aligned} |x'(t)| &= |y(t)| < D, \\ .|x''(t)| &= |z(t)| < D. \end{aligned}$$

This completes the proof of Theorem 3.1.

2.3 Square integrability

Our next result concerns the square integrability of solutions of equation (1).

Theorem 4.1. *In addition to the assumptions of Theorem 3.1, If we assume that*

I_3) $M \geq 1 + \frac{\alpha}{2}$

Then all the solutions of (1) and their derivatives are elements of $L^2[0, +\infty)$.

Proof. Define H(t) as

$$H(t) = W(t) + \varepsilon \int_{t_1}^{t} (y^2(s) + z^2(s))ds. \qquad (11)$$

where $\varepsilon > 0$ is a constant to be specified later. By differentiating H(t) and using (9) we obtain

$$H'(t) \leq (\varepsilon - L_1)\,(y^2(s) + z^2(s)) + \left(\frac{L_2}{K_0} V + L_3\right) p(t).$$

If we Choose $\varepsilon - L1 < 0$, then from (10) we get

$$H'(t) \leq L_4 p(t), \qquad (12)$$

where

$$L_4 = \frac{L_2}{K_0} p_4 + L_3.$$

Integrating (11) from $t1$ to t, $t \geq t1, t1 \geq t0 + r$ and using condition ($I2$) of Theorem 3.1 we get

$$H(t) - H(t_1) = \int_{t_1}^{t} H'(s)ds \leq L_4 p_2.$$

We can conclude by (11) that

$$\int_{t_1}^{t} (z^2(s) + y^2(s))ds < \frac{L_4 p_2 + p_3 - L_3 p_2}{\varepsilon},$$

which imply the existence of positive constants σ1 and σ2 such that

$$\int_{t_1}^{t} x'^2(s)ds \leq \sigma_1, \qquad (13)$$

and

$$\int_{t_1}^{t} x''^2(s)ds \leq \sigma_2; \tag{14}$$

Next multiplying (1) by $x(t - r)$, and integrating from t_1 to t we have

$$\int_{t_1}^{t} f(x(s-r))x(s-r)ds = A_1(t) + A_2(t) + A_3(t) + A_4(t),$$

where

$$\begin{aligned} A_1(t) &= -\int_{t_1}^{t}(x'''(s) + \beta(s)x'''(t-r))x(s-r)ds \\ A_2(t) &= -\int_{t_1}^{t}\Psi(x'(s))x''(s)x(s-r)ds, \\ A_3(t) &= -\int_{t_1}^{t}g(x'(s))x(s-r)ds, \\ A_4(t) &= \int_{t_1}^{t}p(t,x,x(s-r),x'(t),x'(s-r),x'')x(s-r)ds. \end{aligned}$$

Integrating by parts and using (13) and (14), we obtain

$$\begin{aligned} A_1(t) &= -\left(\int_{t_1}^{t}(x'''(s)x(s-r)ds + \int_{t_1}^{t}\beta(s)x'''(s-r)x(s-r)ds\right) \\ &\leq R_1 + \frac{\alpha}{2}\int_{t_1}^{t}x^2(s-r)ds \end{aligned}$$

where

$R_1 = |c_1 + c_2| + D^2(1+c) + \frac{1}{2}\left(1 + c + \alpha\right)\sigma_2 + \frac{1+c}{2}\sigma_1.$

In the same way, after using (i)-(iii), (13) and (14) one arrives at

$$\begin{aligned} A_2(t) &\leq \int_{t_1}^{t}|\Psi(x'(s))x''(s)x(s-r)|\,ds \leq \frac{\Psi_1^2}{2}\int_{t_1}^{t}x''^2(s)ds + \frac{1}{2}\int_{t_1}^{t}x^2(s-r)ds \\ &\leq R_2 + \frac{1}{2}\int_{t_1}^{t}x^2(s-r)ds, \\ A_3(t) &\leq \int_{t_1}^{t}\left|g'(x'(s))x(s-r)\right|ds \leq \int_{t_1}^{t}g'^2(x'(s))ds + \int_{t_1}^{t}x^2(s-r)ds \\ &\leq R_3 + \frac{1}{2}\int_{t_1}^{t}x^2(s-r)ds, \end{aligned}$$

$$
\begin{aligned}
A_4(t) &\leq \int_{t_1}^{t} \left|p(s,x,y,x(s-r),x'(s-r),x''(s))x(s-r)\right| ds \\
&\leq D\int_{t_1}^{t} \left|q(s)\right| ds \leq R_4,
\end{aligned}
$$

where

$$R_2 = \frac{\Psi_1^2}{2}\sigma_2,\ R_3 = \frac{d_1^2}{2}\sigma_1,\ R_4 = Dq_2.$$

In the other hand from condition (ii) we have

$$\int_{t_1}^{t} x(s-r)f(x(s-r))ds \geq M\int_{t_1}^{t} x^2(s-r)ds.$$

Hence, by condition $I3$ of Theorem 4.1 we obtain

$$(M - \frac{\alpha}{2} - 1)\int_{t_1}^{t} x^2(s-r)ds \leq R_1 + R_2 + R_3 + R_4,$$

from which it follows that

$$\int_{t_1}^{t} x^2(s-r)ds < \infty$$

hence

$$\int_{t_0}^{+\infty} x^2(s)ds < \infty.$$

which is our desired conclusion.

3. Conclusion

In this paper asymptotic property of solutions of a class of nonlinear neutral delay differential equations are studied. Sufficient conditions are obtained for asymptotic stability, boundedness and square integrability of solutions

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On Fixed Point Theorems of Some Multivalued Mappings

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Abstract

We aim to present new fixed point theorems for multivalued mapping on complete metric spaces with δ-distance following by Jleli and Samet. We introduce the notion of generalized multivalued new contractive in the context of complete metric spaces. Finally, we give an illustrative example, which shows the importance of δ-distance on the contraction condition.

Keyword(s): fixed point, multivalued mapping, metric space.

1. Introduction

Banach [1] introduced a famous fundamental fixed point theorem, also known as the Banach contraction principle. The Banach contraction principle is the simplest and one of the most adaptable elementary results in fixed point theory. Throughout the years, several extensions and generalizations of this principle have appeared in the literature. Recently, Nadler [2], introduced the notion of multi-valued contraction mapping and proved well known Banach contraction principle. Many authors followed Nadler's idea and gave their contributions in that sense, see for example [3, 4]. Denote by $P(Y)$ the family of all nonempty subsets of Y, $CB(Y)$ the family of all nonempty, closed and bounded subsets of Y and $K(Y)$ the family of all nonempty compact subsets of Y. It is well known that, $H: CB(Y) \times CB(Y) \to \mathbb{R}$ is defined by, for every $K, L \in CB(Y)$,

$$H(K,L) = \max\left\{\sup_{k\in K} d(k,L), \sup_{l\in L} d(l,K)\right\}$$

is a metric on $CB(Y)$, which is called the Pompeiu–Hausdorff metric induced by d, where $D(k,L) = \inf\{d(k,l): l \in L\}$ and $D(K,L) = \sup\{D(k,L): k \in K\}$.

Lemma 1.1. [2] Let K and L be nonempty closed and bounded subsets of a metric space. Therefore, for any $k \in K$,

$$D(k,L) \leq H(K,L).$$

Lemma 1.2. [2] Let K and L be nonempty closed and bounded subsets of a metric space and $h > 1$. Subsequently, for all $k \in K$, there exists $l \in L$ such that

$$d(k,l) \leq hH(K,L).$$

Later in [5, 6, 7, 8, 9, 10, 11, 12] different fixed point were introduced for multi-valued contractive mappings using δ-distance. Let $(Y, d$ be a metric spaces and $K, L \in B(Y)$. The δ-distance is denoted by $\delta: B(Y) \times B(Y) \to [0, \infty)$ and defined by $\delta(K, L) = \sup\{d(k, l): k \in K, l \in L\}$. Moreover, $\delta(K, K) = \sup\{d(k, l): k, l \in K\} = diamK$ denotes the diameter of the set K. If $K = \{k\}$ we have $\delta(K, L) = \delta(k, L)$ and if $L = \{l\}$, thus $\delta(K, L) = d(k, l)$. It is easy to prove that, $\delta(K, L) = \delta(L, K) \geq 0$, $\delta(K, L) \leq \delta(K, E) + \delta(E, L)$ and $\delta(K, L) = 0$ implies that $K = L = \{k\}$. If (K_n) is a sequence in $B(Y)$, we say that (K_n) converges to $K \subseteq Y$ have $K_n \to K$ if and only if $k \in K$ implies that $k_n \to k$ for some sequence k_n with $k_n \in K_n$ for $n \in \mathbb{N}$ and for any $\epsilon > 0$, $\exists p \in \mathbb{N}$ such that $K_n \subseteq K_\epsilon$ for $n > p$, where $K_\epsilon = \{y \in Y: d(y, k) < \epsilon \quad for \quad some \quad k \in K\}$.

2. Preliminaries

Berinde [13] introduced the concept of a multi-valued weakly Picard operator.

Definition 2.1. [13] Let (Y, d) be a complete metric space and $T: Y \to P(Y)$ be a multi-valued operator. A mapping $T: Y \to P(Y)$ is said to be a multivalued weakly Picard (MWP) operator if for every $u \in Y$ and any $v \in Tu$, there exists a sequence (u_n) in Y such that

i. $u_0 = u, \quad u_1 = v$;
ii. $u_{n+1} \in Tu_n$;
iii. the sequence u_n is convergent and its limit is a fixed point of T .

Berinde [14, 15, 16] defined almost contraction (or (δ, L)-weak contraction) mappings in a metric space. In the same paper, Berinde [13] introduced the concepts of multi-valued almost contraction (the original name was multi-valued (δ, L)-weak contraction) and proved the following nice fixed point theorem:

Theorem 2.2. [13] Let (Y, d) be a complete metric spaces, $T: Y \to CB(Y)$ be a multi-valued almost contraction, which is, there exist two constants $\delta \in (0,1)$ and $L \geq 0$, such that

$$H(Tu, Tv) \leq \delta d(u, v) + LD(v, Tu)$$

for all $u, v \in Y$. Subsequently, T is a multi-valued almost contraction operator.

Thereafter, Dass and Gupta [17] introduced the concepts of rational contractive and submited the following nice fixed point theorem.

Theorem 2.3. Let (Y, d) be a complete metric space. A mapping $T: Y \to Y$ is said to be rational contractive, that is, a mapping for which there exists two constant $\alpha, \beta > 0$ and $\alpha + \beta < 1$ such that

$$d(Tu, Tv) \leq \alpha d(u, v) + \beta d(v, Tv) \frac{1 + d(u, Tu)}{1 + d(u, v)},$$

for all $(u, v) \in Y$. Then T has a unique fixed point in Y.

Lately, Jleli and Samet [18] introduced a new type of contraction mappings. Jleli and Samet called it as θ-contractivity. They introduced the family of all functions, $\theta: (0, \infty) \to (1, \infty)$ supplying the following particulars by Θ:

- (Θ_1) θ is nondecreasing;
- (Θ_2) For each sequence $\{s_n\} \subset (0, \infty)$, $\lim_{n\to\infty}\theta(s_n) = 1$ if and only if $\lim_{n\to\infty} s_n = 0^+$;
- (Θ_3) There exists $m \in (0,1)$ and $z \in (0, \infty]$ such that $\lim_{s\to 0^+} \frac{\theta(s)-1}{s^m} = z$.

After all Ahmet et al. [19] used the following weaker condition in place of the condition (Θ_3):
(Θ_3') Θ is continuous on $(0, \infty)$.
Let Ω be the family of all functions θ satisfying $(\Theta_1) - (\Theta_3')$.

Recently, Durmaz and Altun [20] introduced some fixed point theorems for multi-valued almost θ-contractive mappings in metric spaces.

The aim of this study, following by Mınak, Altun, [12], Dass, Gupta [17] Jleli and Samet [18], to prove some new multi-valued fixed point theorems with δ-distance.

3. Main Results

Definition 3.1. Let (Y, d) be a metric space and $T: Y \to B(Y)$ be a mapping and $\theta \in \Omega$. Therefore we say that T a rational type multi-valued θ_δ-contractive. Assume that there exists $\theta \in \Omega$ and $k \in (0,1)$ such that

$$\theta(\delta(Tu, Tv)) \leq [\theta(A(u, v))]^k, \tag{1}$$

for all $(u, v) \in Y$, with $\min\{\delta(Tu, Tv), d(u, v)\} > 0$ where

$$A(u, v) = \max\{d(u, v), D(u, Tu), D(v, Tv), \frac{D(v,Tv)[1+D(u,Tu)]}{1+d(u,v)}\}. \tag{2}$$

Theorem 3.2. Let (Y, d) be a complete metric space and $T: Y \to B(Y)$ be a rational type multi-valued θ_δ-contractive mapping. Assume that Tu is closed for all $u \in Y$, then T has a fixed point.

Corollary 3.3. Let's show that known contractive mapping of Theorem 3.2. That is, if $T: Y \to B(Y)$ is a contractive mapping, such as, there exists $k \in (0,1)$ such that

$$\delta(Tu, Tv) \leq kA(u, v) \tag{3}$$

for all $(u, v) \in Y$, with $\min\{\delta(Tu, Tv), d(u, v)\} > 0$ where

$$A(u, v) = \max\{d(u, v), D(u, Tu), D(v, Tv), \frac{D(v,Tv)[1+D(u,Tu)]}{1+d(u,v)}\}. \tag{4}$$

Thus we obtain

$$e^{\delta(Tu,Tv)}(\delta(Tu,Tv)) \leq [e^{\max\{d(u,v),D(u,Tu),D(v,Tv),\frac{D(v,Tv)[1+D(u,Tu)]}{1+d(u,v)}\}}]^k, \tag{5}$$

Obviously, $\theta \in \Omega$ be a function defined as $\theta(p) = e^p$. Corollary (3.3) is a generalization of the rational type multi-valued contractive mapping.

5. Acknowledgement

The authors wish to thank the referees for their careful reading of the manuscript and valuable suggestions.

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On Fourth Fundamental Form of the Translation Hypersurface

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Abstract

We examine the fourth fundamental form of the translation hypersurface in the four dimensional Euclidean space. We also discuss I, II, III and IV fundamental forms of a translation hypersurface.

1. Introduction

A *translation surface* is a surface that is generated by translations: for two space curves α, β with a common point P, the curve α is shifted such that point P is moving on β. Then the curve α generates a translation surface $\mathbf{x}(u, v) = \alpha(u) + \beta(v)$.

Translation surface examples: The *elliptic paraboloid* $z = x^2 + y^2$ can be generated by parabolas α: $(x, 0, x^2)$ and β: $(0, y, y^2)$. *Right circular cylinder*: α is a circle (or another cross section) and β is a line. The *hyperbolic paraboloid* $z = x^2 - y^2$ can be generated by parabolas α: $(x, 0, x^2)$ and β: $(0, y, -y^2)$.

A *translation hypersurface* (TH) in $\mathbb{E}^4$ is a hypersurface that is generated by translations: for three space curves α, β, γ with a common point P, the curve α is shifted such that point P is moving on β and γ, respectively. So, the curve α generates a translation hypersurface in $\mathbb{E}^4$.

Parametrization of the translation hypersurface is given by

$$\mathbf{x}(u, v, w) = \big(u, 0, 0, f(u)\big) + \big(0, v, 0, g(v)\big) + \big(0, 0, w, h(w)\big) = \alpha(u) + \beta(v) + \gamma(w),$$

where $f(u), g(v), h(w)$ are differentiable functions for all $u, v, w \in I \subset \mathbb{R}$. More clear form of it as follows

$$\mathbf{x}(u, v, w) = \big(u, v, w, f(u) + g(v) + h(w)\big). \tag{1.1}$$

Arslan et al [1] studied translation surfaces in 4-dimensional Euclidean space. Chen, Sun, and Tang [2] introduced translation hypersurfaces with constant mean curvature (CMC) in $(n+1)$-dimensional spaces. Dillen, Verstraelen, and Zafindratafa [3] worked a generalization of the translation surfaces of Scherk. Inoguchi, Lopez, and Munteanu [4] studied minimal translation surfaces in the Heisenberg group Nil_3. Lima, Santos, and Sousa [5] gave translation hypersurfaces with constant scalar curvature into the Euclidean space. Lima, Santos, and Sousa [6] considered generalized translation hypersurfaces in Euclidean space. Liu [7] introduced translation surfaces with CMC in 3-dimensional spaces. Lopez [8] studied minimal translation surfaces in hyperbolic space. Lopez and Moruz [9] obtained translation and homothetical surfaces with constant curvature in Euclidean space. Lopez and Munteanu [10] worked minimal translation surfaces in Sol_3. Moruz and Munteanu [11] considered hypersurfaces in the Euclidean space $\mathbb{E}^4$ defined as the sum of a curve and a surface whose mean curvature vanishes. Munteanu, Palmas, and Ruiz-Hernandez [12] studied minimal translation hypersurfaces in Euclidean space. Scherk [13] gave his classical minimal translation surface. Seo [14] worked translation hypersurfaces with constant curvature in space forms. Verstraelen, Walrave, and Yaprak [15] studied on the minimal translation surfaces in $\mathbb{E}^n$ for arbitrary dimension n. Yang and Fu [16] worked affine translation surfaces in affine space. Yoon [17] focused the Gauss map of translation surfaces in Minkowski 3-space.

In this paper, we study the fourth fundamental form of the translation hypersurface in the four dimensional Euclidean space $\mathbb{E}^4$. We give fundamental notions of four dimensional Euclidean geometry. In addition, we obtain fundamental forms I, II, III, and IV of translation hypersurface.

2. Preliminaries

In $\mathbb{E}^{n+1}$, to find the i-th curvature formulas $\mathfrak{C}_i$, where $i = 0,1,\dots n$, we use characteristic polynomial of shape operator **S**:

$$P_{\mathbf{S}}(\lambda) = 0 = \det(\mathbf{S} - \lambda I_n) = \sum_{k=0}^{n} (-1)^k s_k \lambda^{n-k}, \tag{2.1}$$

where I_n denotes the identity matrix of order n. Then, we get curvature formulas $\binom{n}{i} \mathfrak{C}_i = s_i$. Here, $\binom{n}{0} \mathfrak{C}_0 = s_0 = 1$ by definition. k-th fundamental form of hypersurface M^n is defined by

$$\mathrm{I}(\mathbf{S}^{k-1}(X), Y) = \langle \mathbf{S}^{k-1}(X), Y \rangle.$$

So, we get

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} \mathfrak{C}_i \, \mathrm{I}(\mathbf{S}^{k-1}(X), Y) = 0. \tag{2.2}$$

In the rest of this paper, we shall identify a vector $(\mathrm{a, b, c, d})$ with its transpose $(\mathrm{a, b, c, d})^t$.

Let $\mathbf{M} = \mathbf{M}(u, v, w)$ be an isometric immersion of a hypersurface M^3 in $\mathbb{E}^4$. Inner product of vectors $\vec{x} = (x_1, x_2, x_3, x_4)$ and $\vec{y} = (y_1, y_2, y_3, y_4)$ in $\mathbb{E}^4$ is given by as follows:

$$\langle \vec{x}, \vec{y} \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4.$$

Vector product $\vec{x} \times \vec{y} \times \vec{z}$ of $\vec{x} = (x_1, x_2, x_3, x_4)$, $\vec{y} = (y_1, y_2, y_3, y_4)$, $\vec{z} = (z_1, z_2, z_3, z_4)$ in $\mathbb{E}^4$ is defined by as follows:

$$\vec{x} \times \vec{y} \times \vec{z} = \det \begin{pmatrix} e_1 \, e_2 \, e_3 \, e_4 \\ x_1 x_2 x_3 x_4 \\ y_1 y_2 y_3 y_4 \\ z_1 \, z_2 \, z_3 \, z_4 \end{pmatrix}.$$

The Gauss map of a hypersurface **M** is given by

$$e = \frac{\mathbf{M}_u \times \mathbf{M}_v \times \mathbf{M}_w}{\|\mathbf{M}_u \times \mathbf{M}_v \times \mathbf{M}_w\|},$$

where $\mathbf{M}_u = d\mathbf{M}/du$. For a hypersurface **M** in $\mathbb{E}^4$, we have

$$\det \mathrm{I} = \det \begin{pmatrix} E & F & A \\ F & G & B \\ A & B & C \end{pmatrix} = (EG - F^2)C - EB^2 + 2FAB - GA^2,$$

$$\det \mathrm{II} = \det \begin{pmatrix} L & M & P \\ M & N & T \\ P & T & V \end{pmatrix} = (LN - M^2)V - LT^2 + 2MPT - NP^2,$$

$$\det \mathrm{III} = \det \begin{pmatrix} X & Y & O \\ Y & Z & R \\ O & R & S \end{pmatrix} = (XZ - Y^2)S - XR^2 + 2YOR - ZO^2.$$

Here, the coefficients are given by

$$E = \langle \mathbf{M}_u, \mathbf{M}_u \rangle,\ \ F = \langle \mathbf{M}_u, \mathbf{M}_v \rangle,\ \ G = \langle \mathbf{M}_v, \mathbf{M}_v \rangle,\ \ A = \langle \mathbf{M}_u, \mathbf{M}_w \rangle,\ \ B = \langle \mathbf{M}_v, \mathbf{M}_w \rangle,\ \ C = \langle \mathbf{M}_w, \mathbf{M}_w \rangle,$$
$$L = \langle \mathbf{M}_{uu}, e \rangle,\ \ M = \langle \mathbf{M}_{uv}, e \rangle,\ \ N = \langle \mathbf{M}_{vv}, e \rangle,\ \ P = \langle \mathbf{M}_{uw}, e \rangle,\ \ T = \langle \mathbf{M}_{vw}, e \rangle,\ \ V = \langle \mathbf{M}_{ww}, e \rangle,$$
$$X = \langle e_u, e_u \rangle,\ \ Y = \langle e_u, e_v \rangle,\ \ Z = \langle e_v, e_v \rangle,\ \ O = \langle e_u, e_w \rangle,\ \ R = \langle e_v, e_w \rangle,\ \ S = \langle e_w, e_w \rangle,$$

and e is the Gauss map (i.e. the unit normal vector field).

3. The Fourth Fundamental Form

Next, we will obtain the fourth fundamental form matrix for a hypersurface $\mathbf{M}(u, v, w)$ in $\mathbb{E}^4$. Using characteristic polynomial $P_{\mathbf{S}}(\lambda) = a\lambda^3 + b\lambda^2 + c\lambda + d = 0$, we obtain curvature formulas: $\mathfrak{C}_0 = 1$ (by definition),

$$\mathfrak{C}_1 = -\frac{b}{\binom{3}{1} a},\ \ \mathfrak{C}_2 = \frac{c}{\binom{3}{2} a},\ \ \mathfrak{C}_3 = -\frac{d}{\binom{3}{3} a}.$$

Theorem 3.1. For *any hypersurface* M^3 *in* $\mathbb{E}^4$, *the fourth fundamental form is related by*

$$\mathfrak{C}_0 \mathrm{IV} - 3\mathfrak{C}_1 \mathrm{III} + 3\mathfrak{C}_2 \mathrm{II} - \mathfrak{C}_3 \mathrm{I} = 0. \tag{3.1}$$

Proof. Taking $n = 3$ in (2.2), then some computing, we get the fourth fundamental form matrix as follows

$$\mathrm{IV} = \begin{pmatrix} \zeta & \eta & \delta \\ \eta & \phi & \sigma \\ \delta & \sigma & \xi \end{pmatrix}, \tag{3.2}$$

where

$$\zeta = -\frac{\left\{\begin{array}{c} CL^2N - CLM^2 - GLP^2 + B^2LX + A^2NX + GL^2V + F^2VX + NP^2E + M^2VE \\ -CNXE - GVXE - CGLX + 2(BTXE - BL^2T - MPTE + ABMX - ALNP \\ +BLMP + ALMT + CFMX + AGPX - BFPX - AFTX - FLMV + FLPT) \end{array}\right\}}{\det \mathrm{I}},$$

$$\eta = \frac{\left\{\begin{array}{c} CM^3 - FNP^2 - GMP^2 - FLT^2 - B^2LY - A^2NY + FM^2V - F^2VY + MT^2E \\ +CNYE - MNVE + GVYE - CLMN + CGLY + FLNV - GLMV + 2(AFPY \\ -BTYE + ABMY + ANMP - BLMT - CFMY - AGPY + BFPY - TM^2A - BM^2P) \end{array}\right\}}{\det \mathrm{I}},$$

$$\delta = \frac{\left\{\begin{array}{c} GP^3 - B^2LO - A^2NO + ANP^2 + CM^2P - ALT^2 - AVM^2 - F^2OV + PT^2E \\ +CNOE + GOVE - NPVE + CGLO - CLNP + ALNV - GLPV + 2(ABMO \\ -BOTE - CFMO - AGOP + BFOP + AFOT - BLPT + FMPV - BMP^2 - FP^2T) \end{array}\right\}}{\det \mathrm{I}},$$

$$\phi = -\frac{\left\{\begin{array}{c} CLN^2 - CM^2N - GLT^2 + B^2LZ + A^2NZ + GM^2V + F^2VZ - NT^2E + N^2VE \\ -CNZE - GVZE - CGLZ + 2(-AN^2P + BTZE - ABMZ + BMNP + ANMT \\ -BLNT + CFMZ + AGPZ - BFPZ - AFTZ + FMNV + FNPT - GMPT) \end{array}\right\}}{\det \mathrm{I}},$$

$$\sigma=\frac{\left\{\begin{array}{c}ET^3-BNP^2-B^2LR-A^2NR+BLT^2+CM^2T-BM^2V+GP^2T-F^2RV\\+CNRE+GRVE-NTVE+CGLR-CLNT+BLNV-GLTV+2(ABMR\\-BRTE-CFMR-AGPR+BFPR+AFRT+ANPT+FMTV-AT^2M-FT^2P)\end{array}\right\}}{\det\mathrm{I}},$$

$$\xi=-\frac{\left\{\begin{array}{c}CNP^2-B^2LS-A^2NS+CLT^2+GLV^2-GP^2V+F^2SV+NV^2E-T^2VE\\-CNSE-GSVE-CGLS+2(-FMV^2+BSTE-ABMS+CFMS+AGPS\\-BFPS-AFST-CMPT-ANPV+BMPV+ATMV-BLTV-FPTV)\end{array}\right\}}{\det\mathrm{I}}.$$

4. Fundamental Forms of TH

Using the first differentials of (1.1) with respect to u, v, w, we have the Gauss map of the translation hypersurface (1.1):

$$e=\frac{1}{(\det\mathrm{I})^{1/2}}(f',g',h',-1), \tag{4.1}$$

where

$$\det\mathrm{I}=f'^2+g'^2+h'^2+1,$$

and $f'=df/du,\ g'=dg/dv,\ h'=dh/dw$. Using the first differentials of (4.1) with respect to u, v, w, we get the first quantities as follows:

$$\mathrm{I}=\begin{pmatrix}1+f'^2 & f'g' & f'h'\\ f'g' & 1+g'^2 & g'h'\\ f'h' & g'h' & 1+h'^2\end{pmatrix}.$$

Computing the second differentials of (4.1) with respect to u, v, w, we have the second quantities as follows:

$$\mathrm{II}=\begin{pmatrix}-f''/(\det\mathrm{I})^{1/2} & 0 & 0\\ 0 & -g''/(\det\mathrm{I})^{1/2} & 0\\ 0 & 0 & -h''/(\det\mathrm{I})^{1/2}\end{pmatrix}.$$

Taking the first differentials of (4.1) with respect to u, v, w, we have the third quantities as follows:

$$\mathrm{III}=\begin{pmatrix}f'^2(g'^2+h'^2+1)/(\det\mathrm{I})^2 & -f'g'f''g''/(\det\mathrm{I})^2 & -f'h'f''h''/(\det\mathrm{I})^2\\ -f'g'f''g''/(\det\mathrm{I})^2 & g'^2(f'^2+h'^2+1)/(\det\mathrm{I})^2 & -g'h'g''h''/(\det\mathrm{I})^2\\ -f'h'f''h''/(\det\mathrm{I})^2 & -g'h'g''h''/(\det\mathrm{I})^2 & h'^2(f'^2+g'^2+1)/(\det\mathrm{I})^2\end{pmatrix}.$$

Finally, using (3.2), we obtain the fourth quantities of (1.1), i.e. symmetric matrix, as follows

$$\mathrm{IV}=\begin{pmatrix}-2f''^2\Gamma/(\det\mathrm{I})^{11/2} & f'g'f''g''\Theta/(\det\mathrm{I})^{11/2} & f'h'f''h''\Omega/(\det\mathrm{I})^{11/2}\\ f'g'f''g''\Theta/(\det\mathrm{I})^{11/2} & -2g''^2\Upsilon/(\det\mathrm{I})^{11/2} & g'h'g''h''\Lambda/(\det\mathrm{I})^{11/2}\\ f'h'f''h''\Omega/(\det\mathrm{I})^{11/2} & g'h'g''h''\Lambda/(\det\mathrm{I})^{11/2} & -2h''^2\Phi/(\det\mathrm{I})^{11/2}\end{pmatrix}.$$

Here, the functions $\Gamma, \Theta, \Omega, \Upsilon, \Lambda, \Phi$ are given by as follows, respectively,

$$\begin{aligned}\Gamma={}&(1/2)(\det\mathrm{I})^2[f'^2g'^2g''+(g'^2+h'^2+1)^2f'']-\{(\det\mathrm{I})^{1/2}(g'^2+h'^2+1)(f'^2f''+g'^2g'')\\&+[(\det\mathrm{I})^{3/2}(f'^2+g'^2)g''+(1/2)(\det\mathrm{I})^2f'^2]\}h'^2h''\\&-(1/2)[(f'^2+2g'^2+1)g''-(g'^2+1)f'']f'^2h'^2h''^2,\end{aligned}$$

$$
\begin{aligned}
\Theta = {} & -2(\det\mathrm{I})^{1/2}(f'^2f'' + g'^2g'')h'^2h'' + (f'^2f'' + g'^2g'')h'^2h''^2 \\
& -(\det\mathrm{I})^2[h'^2h'' - (g'^2 + h'^2 + 1)f'' - (f'^2 + h'^2 + 1)g''], \\
\Omega = {} & -2(\det\mathrm{I})^{1/2}(f'^2f'' + g'^2g'')h'^2h'' + g'^2h'^2g''^2h'' + [(f'^2h'^2f''h'' - (\det\mathrm{I})^2g'^2)]g'' \\
& +(\det\mathrm{I})^2[(g'^2 + h'^2 + 1)f'' + (f'^2 + g'^2 + 1)h''], \\
\Upsilon = {} & -(\det\mathrm{I})^{1/2}(f'^2 + h'^2 + 1)(f'^2f'' + g'^2g'')h'^2h'' \\
& -\{[f'^2(1/2)(g'^2 + 1)f''] - (1/2)(f'^2 + 1)g''\}g'^2h'^2h''^2, \\
& +\left[(\det\mathrm{I})^{3/2}(f'^2 + g'^2)f'' + (1/2)(\det\mathrm{I})^2g'^2\right]h'^2h'' \\
& +(1/2)(\det\mathrm{I})^2[f'^2g'^2f'' + (f'^2 + h'^2 + 1)^2g''], \\
\Lambda = {} & -2(\det\mathrm{I})^{1/2}(f'^2f'' + g'^2g'')h'^2h'' - [(\det\mathrm{I})^2f'^2 - g'^2h'^2f''h''] \\
& +(\det\mathrm{I})^2[(f'^2 + h'^2 + 1)g'' + (f'^2 + g'^2 + 1)h''], \\
\Phi = {} & -(\det\mathrm{I})^{1/2}(f'^2 + g'^2 + 1)h'^2(f'^2f'' + g'^2g'')h'' + \{(1/2)(\det\mathrm{I})^2f'^2 - (1/2)g'^2(h'^2 + 1)g''^2 \\
& +(1/2)[(\det\mathrm{I})^2g'^2h'^2g'' + (f'^2 + 1)g'^2h'^2g''^2h'' + (\det\mathrm{I})^2(f'^2 + g'^2 + 1)h''] \\
& +\left[(\det\mathrm{I})^{3/2}(f'^2 + g'^2) - f'^2g'^2h''\right]g''\}h'^2f'' \\
& -(1/2)[(h'^2 + 1)g'' - (g'^2 + 1)h'']f'^2h'^2f''^2
\end{aligned}
$$

Corollary 4.1. *The curves* $\alpha(u), \beta(v)$ *and* $\gamma(w)$ *on translation hypersurface* (1.1) *have unit speed if and only if the fourth fundamental form matrix of* (1.1) *is* $\mathrm{IV} = (\mathrm{O})_{3\times3}$. *That is, the hypersurface* (1.1) *is a* IV*-minimal translation hypersurface.*

5. Conclusion

Translation hypersurfaces have been worked by a number of authors, recently. We have extended some well-known results of the translation hypersurfaces with the help of the fourth fundamental form.

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On Non-null Rational Bézier Curves on 2-dimensional de Sitter Space S_1^2

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Abstract

We constitute an orthonormal frame field of an arbitrary speed non-null quadratic rational Bézier curve at the end points on 2-dimensional de Sitter space S_1^2 known as one of the hyperquadrics in Minkowski 3-space. We get the formulas of geodesic curvature for a non-null quadratic rational Bézier curve that allows a curve to be characterized on the surface.

Key words: Darboux frame field, non-null Rational Bézier curve, 2-dimensional de Sitter space.

1. Introduction

Bézier curves are significant geometrical representations of curves that is used in many areas such as computer graphics, computer aided design, etc. Although Bézier curves can represent a wide variety of curves, the conic sections cannot be represented in Bézier form. In order to be able to include conic sections in Bézier form, rational quadratic Bézier curves are defined. A rational Bézier curve is defined by n control points b_0 through b_n and corresponding scalar weights ω_i, $0 \leq i \leq n$, where n is called its order. Bézier curves are called linear, quadratic, cubic or higher-grade curves according to n. b_0 and b_n are called end points of the curve. Recently, orthonormal frame fields and curvatures of rational Bézier curves are investigated by different spaces or surfaces (see for example [6, 7, 8]). Especially, Darboux frame and geodesic curvature of a quadratic rational Bézier curve at end points on two-dimensional sphere is studied in [8]. The frame field for quadratic rational Bézier curves on 2- dimensional sphere can be generalized hyperquadrics in Minkowski 3- space. Therefore, we construct the Darboux frame of a non-null quadratic rational Bézier curve on 2-dimensional de Sitter space S_1^2. Then we obtain the formula of geodesic curvature of a non-null quadratic rational Bézier curve.

2. Preliminaries

Let $\mathbb{R}_1^3$ be the Minkowski 3-space endowed with a symmetric, bilinear and non-degenerate metric

$$< x, y >= -x_1y_1 + x_2y_2 + x_3y_3,$$

where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ are tangent vectors in $\mathbb{R}_1^3$. The Lorentzian vector product of x and y is defined by

$$x \times_L y = (x_3y_2 - x_2y_3, -x_1y_3 + x_3y_1, -x_1y_2 - x_2y_1).$$

A tangent vector x in $\mathbb{R}_1^3$ is called spacelike (resp. timelike, lightlike) if it satisfies $< x, x >> 0$ or $x = 0$ (resp. $< x, x >< 0$, $< x, x >= 0$ and $x \neq 0$). A curve α in $\mathbb{R}_1^3$ is said spacelike (resp. timelike, lightlike) if its velocity vector is spacelike (resp. timelike, lightlike). Any spacelike or timelike curve is called non-null curve in $\mathbb{R}_1^3$. A surface in $\mathbb{R}_1^3$ is a non-degenerate (or degenerate) if induced metric on its tangent plane is non-degenerate (or degenerate). A non-degenerate surface is named in terms of the induced metric. If the induced metric is positive definite, a non-degenerate surface is called spacelike, if the induced metric is indefinite, a non-degenerate surface is called timelike (see [2,4]). Let S denote a non-degenerate surface with non-null normal vector field in $\mathbb{R}_1^3$. We can assign an orthonormal frame to any point of a smooth regular non-null curve since we investigate the geometry of the curve. Generally, we choose the Darboux frame field if the smooth curve lies on a surface. This frame field is a reasonable way to set up on orthonormal trihedron to measure the change of a non-null curve on a non-degenerate surface.

Now, we assign a Darboux frame field at $\alpha(t)$ by using the unit normal vector field $U(t)$ of the surface S. Then T=$\frac{\alpha'(t)}{\|\alpha'(t)\|}$ is tangent vector and $\varepsilon_1 W(t) = U(t) \times_L T(t)$ is known the binormal vector field of α, where $\varepsilon_1 = \mp 1 =< W, W >$. Thus $\{\mathrm{T}, \mathrm{W}, \mathrm{U}\}$ is the Darboux frame field on S along α with the following derivative equations

$$\begin{pmatrix} T' \\ W' \\ U' \end{pmatrix} = \begin{pmatrix} 0 & \varepsilon_1 v\kappa_g & \varepsilon_2 v\kappa_n \\ -\varepsilon_0 v\kappa_g & 0 & \varepsilon_2 v\tau_g \\ -\varepsilon_0 v\kappa_n & -\varepsilon_1 v\tau_g & 0 \end{pmatrix} \begin{pmatrix} T \\ W \\ U \end{pmatrix}, \quad (1)$$

where $v = \|\alpha'\|$, κ_g, κ_n and τ_g are the speed, geodesic curvature, the normal curvature and the geodesic torsion of α, respectively, and $\varepsilon_0 = \mp 1 = < T, T >$ and $\varepsilon_2 = \mp 1 = < U, U >$. We can obviously see from (1) that the geodesic curvature of α at a point $\alpha(\mathrm{t})$ can be computed by the following equation by

$$\kappa_g = \varepsilon_1 \varepsilon_2 \frac{\det(\alpha', \alpha'', U)}{\|\alpha'\|^3}. \tag{2}$$

If a non-null curve on S has zero geodesic curvature, then it is called a geodesic on the surface.

Definition 1: A rational Bézier curve of degree n with control points $b_0, b_1, \ldots, b_n$ and corresponding scalar weights ω_i ,$0 \leq i \leq n,$ is defined to be

$$\mathrm{P(t)} = \frac{\sum_{i=0}^{n} \omega_i b_i B_{i,n}(t)}{\sum_{i=0}^{n} \omega_i B_{i,n}(t)}, \qquad t \in [0,1]$$

where

$$B_{i,n}(t) = \begin{cases} \dfrac{n!}{(n-i)!\, i!}, & if\ 0 \leq i \leq n \\ 0, & otherwise \end{cases}$$

are called the Bernstein polynomials, with the understanding that if $\omega_i = 0$, then $\omega_i b_i$ is to be replaced by b_i. It is assumed that all the weights are non-zero. Rational Bézier curves are called quadratic for $n = 2$ [1,3]. If $\omega_0 = \omega_1 = \cdots = \omega_n$, then the curve becomes an integral Bézier curve.

3. Darboux Frame of a Non-null Quadratic Rational Bézier Curves on 2-dimensional de Sitter Space S_1^2

A quadratic rational Bézier curve is defined by

$$\mathrm{P(t)} = \frac{\sum_{i=0}^{2} \omega_i b_i B_{i,2}(t)}{\sum_{i=0}^{2} \omega_i B_{i,2}(t)}, \tag{3}$$

where ω_i and b_i, $0 \leq i \leq 2$, are called weights and control points, respectively.

The first order derivative of the quadratic rational Bézier curve is given by the following equation:

$$P'(t) = \frac{\left(\sum_{i=0}^{2} \omega_i b_i B_{i,2}(t)\right)' - \left(\sum_{i=0}^{2} \omega_i B_{i,2}(t)\right)' B(t)}{\sum_{i=0}^{2} \omega_i B_{i,2}(t)} \tag{4}$$

By the help of the derivative of Bernstein polynomials and the derivative of Bézier curves [3], we get the following equation:

$$P'(t) = 2\frac{\omega_0\omega_1(b_1-b_0)(1-t)^2+\omega_1\omega_2(b_2-b_1)t^2+\omega_0\omega_2(b_2-b_0)t(1-t)}{\sum_{i=0}^{2}\omega_i B_{i,2}(t)^2}. \qquad (5)$$

Let $S_1^2(r)=\{x\in\mathbb{R}_1^3 \mid <x,x>=r^2\}$ be a 2-dimensional pseudo-sphere which is also known as 2-dimensional de Sitter space $S_1^2(r)$. Let $P\colon I\subset R\longrightarrow S_1^2(r)$ be a non-null quadratic rational Bézier curve. The position vector of P is expressed by

$$P(t)=r\,\mathrm{U}(P(t)), \qquad (6)$$

then we have $P'=\mathrm{r}\mathrm{U}'$. In this situation, (1) and (2) implies that

$$\kappa_n=-\varepsilon_0\frac{1}{r},\ \ \tau_g=0 \qquad (7)$$

for every $t\in R$ $(see,$ [5]). So, the derivative formulas of the Darboux frame along P is given by

$$T(t)=\frac{P'(t)}{\|P'(t)\|},\ \ \varepsilon_1 W(t)=\frac{P(t)\times_L P'(t)}{\|P(t)\times_L P'(t)\|},\ U(t)=\frac{1}{r}\ P(t) \qquad (8)$$

with formulas

$$\begin{pmatrix}T'\\W'\\U'\end{pmatrix}=\begin{pmatrix}0 & \varepsilon_1 v\kappa_g & -\frac{\varepsilon_0\varepsilon_2 v}{r}\\ -\varepsilon_0 v\kappa_g & 0 & 0\\ \frac{v}{r} & 0 & 0\end{pmatrix}\begin{pmatrix}T\\W\\U\end{pmatrix}. \qquad (9)$$

The following theorem give some results for the Darboux frame and the geodesic curvature for a quadratic rational Bézier curve at the point $P(0)=b_0$.

Theorem 3. The Darboux frame {T,W,U} and geodesic curvature κ_g of the non-null quadratic rational Bézier curve P(t) defined by (3) at t=0 on the 2-dimensional de Sitter space. $S_1^2(r)$ are given by

$$T|_{t=0}=\frac{\Delta^1 b_0}{\|\Delta^1 b_0\|}, \qquad (10)$$

$$W|_{t=0}=\varepsilon_1\frac{b_0\times_L b_1}{\|b_0\times_L b_1\|}, \qquad (11)$$

$$U|_{t=0}=\frac{b_0}{r} \qquad (12)$$

and

$$\kappa_g\big|_{t=0}=\varepsilon_1\varepsilon_2\frac{4\omega_1\omega_2<b_1\times_L b_2,b_0>}{r{\omega_0}^3\|\left(\frac{2\omega_1}{\omega_0}(b_1-b_0)\right)\|^3}, \qquad (13)$$

where $\Delta^1 b_0 = b_1 - b_0 \neq 0$.

Proof. Taking into consideration (8) for t=0, we get

$$T|_{t=0} = \frac{\omega_0\omega_1(b_1 - b_0)}{\|\omega_0\omega_1(b_1 - b_0)\|} = \frac{(b_1 - b_0)}{\|(b_1 - b_0)\|} = \frac{\Delta^1 b_0}{\|\Delta^1 b_0\|},$$

$$W|_{t=0} = \varepsilon_1 \frac{\omega_0 b_0 \times_L \Delta^1 \omega_0 b_0}{\|\omega_0 b_0 \times_L \Delta^1 \omega_0 b_0\|} = \varepsilon_1 \frac{\omega_0 b_0 \times_L (\omega_1 b_1 - \omega_0 b_0)}{\|\omega_0 b_0 \times_L (\omega_1 b_1 - \omega_0 b_0)\|} = \varepsilon_1 \frac{\omega_0 b_0 \times_L \omega_1 b_1}{\|\omega_0 b_0 \times_L \omega_1 b_1\|}$$

and

$$U|_{t=0} = \frac{\omega_0 b_0}{r\omega_0}.$$

From (2) and (6), we have

$$\kappa_g\big|_{t=0} = \varepsilon_1\varepsilon_2 \frac{< P'(0) \times_L P''(0), P(0) >}{\|P'(0)\|^3}\,. \tag{14}$$

We get from (3) and (5)

$$P(0) = b_0, \tag{15}$$

$$P'(0) = 2\frac{\omega_0\omega_1(b_1 - b_0)}{{\omega_0}^2} = \frac{\omega_1 \Delta^1 b_0}{\omega_0}\,, \tag{16}$$

Taking derivative of (5), we obtain

$$P''(0) = \frac{\big(2\omega_0\omega_2(b_2 - b_0) - 4\omega_0\omega_1(b_1 - b_0)\big)\omega_0 - 2\omega_0\omega_1(b_1 - b_0)(2\omega_0(-2\omega_0 + 2\omega_1))}{{\omega_0}^4}$$

$$= \frac{(-4\omega_1 + 8\omega_0\omega_1 - 8{\omega_1}^2)\Delta^1 b_0 + 2\omega_2(b_2 - b_0)}{{\omega_0}^4}. \tag{17}$$

Cross product of (16) and (17) can be calculated as follows

$$P'(0) \times_L P''(0) = \frac{4\omega_1\omega_2((b_1 - b_0) \times (b_2 - b_1))}{{\omega_0}^3}. \tag{18}$$

Substituting (17), (18) and (4) into (14), we obtain (13).

Corollary 4. Any non-null quadratic rational Bézier curve P(t) defined by (3) at t=0 on the 2-dimensional de Sitter space $S_1^2(r)$ lies on a geodesic arc of $S_1^2(r)$ when

$$< b_1 \times_L b_2, b_0 >= 0\,.$$

The following theorem give some results for the Darboux frame and the geodesic curvature for a non-null quadratic rational Bézier curve P(t) at the point P(1)= b_2. The proof of the next theorem closely mirrors proof of Theorem 3 and is omitted.

Theorem 5. The Darboux frame {T,W,U} and geodesic curvature κ_g of the quadratic non-null rational Bézier curve P(t) defined by (3) at t=1 on the 2-dimensional de Sitter space $S_1^2(r)$ are given by

$$T|_{t=1} = \frac{\Delta^1 b_1}{\|\Delta^1 b_1\|},$$
$$W|_{t=1} = \varepsilon_1 \frac{b_1 \times_L b_2}{\|b_1 \times_L b_2\|},$$
$$U|_{t=1} = \frac{b_2}{r}$$

and

$$\kappa_g\Big|_{t=1} = \varepsilon_1 \varepsilon_2 \frac{4\omega_1\omega_0 < b_1 \times_L b_0, b_2 >}{r{\omega_2}^2 \parallel \left(\frac{2\omega_1}{\omega_2}(b_2 - b_1)\right) \parallel^3},$$

where $\Delta^1 b_1 = b_2 - b_1 \neq 0$.

Corollary 6. Any non-null quadratic rational Bézier curve P(t) defined by (3) at t=1 on the 2-dimensional de Sitter space $S_1^2(r)$ lies on a geodesic arc of $S_1^2(r)$ when

$$< b_1 \times_L b_0, b_2 >= 0.$$

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On Oscillation of Nonlinear Impulsive Differential Equations System with Piecewise Constant Mixed Arguments

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Abstract

A nonlinear impulsive differential equations system with piecewise constant mixed arguments is considered. Also, the existence and uniqueness of the solutions are proved. Moreover, sufficient conditions for the oscillation of the solutions are obtained. Finally, two examples are given.

1. Introduction

The applications of various neural networks in many areas, such as signal processing, image processing, pattern recognition, fault diagnosis, associative memory, and combinatorial optimization have developed rapidly ([5], [6], [8], [10], [13]). In addition, piecewise constant systems proposed by Busenberg and Cooke [2] exist in widely expanded areas such as biomedicine, chemistry, mechanical engineering, physics, etc. Such systems are described as a combination of continuous and discrete (hybrid dynamical systems) and so combine properties of both differential and difference equations. Moreover, cellular neural networks with piecewise constant argument have been investigated by some authors ([4], [7], [11], [12]).

On the other hand, in the real world, many evolutionary processes are characterized by sudden changes at certain times. These changes are said to be impulsive phenomena that take place in many areas, such as physics, chemistry, population dynamics, optimal control, etc.

Most neural networks can be classified into two categories, either continuous or discrete. However, they exhibit characteristics both continuous and discrete styles in many real world systems and natural processes. So, the impulsive differential equations with piecewise constant argument which are named by Wiener [12] have been important. Recently, impulsive cellular neural networks models with piecewise constant argument have been studied in the papers ([1], [3], [9]).

Abbas and Xia [1] discussed the existence and uniqueness of almost automorphic solutions of the following impulsive model of neural network with piecewise constant argument. These kinds of solutions are more general than periodic and almost periodic solutions. They gave several sufficient conditions for the exponential and global attractivity of the solution.

$$\frac{dx_i(t)}{dt} = -a_i(t)x_i(t) + \sum_{j=1}^{m} b_{ij}(t)\, f_j\left(x_j(t)\right) + \sum_{j=1}^{m} c_{ij}(t)\, g_j\left(x_j\left(2\left[\frac{t+1}{2}\right]\right)\right) + I_i(t),$$

$$\Delta x_i|_{t=\tau_k} = \mathrm{J}_k\left(x_i(\tau_k{}^-)\right), i = 1, 2, \cdots, n, k \in \mathbb{N},$$

$$\Delta\left(x_i(\tau_k)\right) = x_i(\tau_k) - x_i(\tau_k{}^-), \qquad x_i(\tau_k{}^-) = \lim_{\mathrm{h}\to 0^-} x_i(\tau_k + \mathrm{h}),$$

where $\tau_k = 2k-1, \tau_k{}^- = 2k-1^-$.

Chiu [3] introduced the following impulsive cellular neural network models with piecewise alternately advanced and retarded arguments. Some sufficient conditions were established for the existence and global exponential stability of a unique periodic solution.

$$\frac{dx_i(t)}{dt} = -a_i(t)x_i(t) + \left\{\sum_{j=1}^{m} b_{ij}(t)\, f_j\left(x_j(t)\right) + c_{ij}(t) g_j\left(x_j\left(2\left[\frac{t+1}{2}\right]\right)\right)\right\} + d_i(t), t \neq 2k-1,$$

$$\Delta x_i|_{t=2k-1} = \mathrm{J}_k\left(x_i(2k-1^-)\right), i = 1,2,\cdots, n, k \in \mathbb{N}.$$

Karakoc et al. [9] proved the existence and uniqueness of a solution of the following nonlinear impulsive differential equation system with piecewise constant argument and obtained sufficient conditions for the oscillation of the solution.

$$x'(\mathrm{t}) = -a(t)x(t) - x([t-1])f\left(y([t])\right) + h_1(x\left([t]\right)),$$
$$y'(\mathrm{t}) = -b(t)y(t) - y([t-1])g\left(x([t])\right) + h_2(y\left([t]\right)), \qquad t \neq n \in \mathbb{Z}^+, t > 0,$$
$$\Delta x(n) = x(n^+) - x(n^-) = \mathrm{c}_n x(n),$$
$$\Delta y(n) = y(n^+) - y(n^-) = \mathrm{d}_n y(n), \qquad t = n \in \mathbb{Z}^+.$$

In this paper, we consider the following nonlinear impulsive differential equations system with piecewise constant mixed arguments:

$$\begin{aligned} x'(\mathrm{t}) &= -a(t)x(t) - f_1(x([t-1]))g_1\left(y([t])\right) + h_1(x\left([t+1]\right)), \\ y'(\mathrm{t}) &= -b(t)y(t) - f_2(y([t-1]))g_2\left(x([t])\right) + h_2(y\left([t+1]\right)), \qquad t \neq n \in \mathbb{Z}^+, t > 0 \end{aligned} \tag{1}$$

with the impulse conditions

$$\begin{aligned} \Delta x(n) &= \mathrm{c}_n x(n), \\ \Delta y(n) &= \mathrm{d}_n y(n), \qquad t = n \in \mathbb{Z}^+ \end{aligned} \tag{2}$$

and the initial conditions

$$x(-1) = x_{-1}, x(0) = x_0, y(-1) = y_{-1}, y(0) = \mathrm{y}_0, \tag{3}$$

where $a, b\colon (0,\infty) \to \mathbb{R}$ are continuous functions, $f_i, g_i, h_i \epsilon C(\mathbb{R},\mathbb{R})$, $i = 1,2$, c_n and d_n are sequences of real numbers such that $\mathrm{c}_n \neq 1$ and $\mathrm{d}_n \neq 1$ for all $n \geq 1$. $\Delta u(n) = u(n^+) - u(n^-)$, $u(n^+) = \lim_{t\to n^+} u(t)$, $u(n^-) = \lim_{t\to n^-} u(t)$, $[.]$ denotes the greatest integer functions, and x_{-1}, x_0, y_{-1}, y_0 are given real numbers.

2. Main Results

Definition 1 A pair of functions $(x(t), y(t))$ is said to be a solution of (1)–(2) if it satisfies the following conditions:
(i) $x\colon \mathbb{R}^+ \cup \{-1\} \to \mathbb{R}$ and $y\colon \mathbb{R}^+ \cup \{-1\} \to \mathbb{R}$ are continuous with a possible exception at the points $[t] \in [0, \infty)$,
(ii) $x(t)$ and $y(t)$ are right continuous and have left-hand limits at the points $[t] \in [0, \infty)$,
(iii) $x(t)$ and $y(t)$ are differentiable and satisfy (1) for any $t \epsilon \mathbb{R}^+$ with a possible exception at the points $[t] \in [0, \infty)$, where one-sided derivatives exist,
(iv) $(x(n), y(n))$ satisfies (2) for $n \in \mathbb{Z}^+$.

Theorem 1 If $c_n \neq 1$ and $d_n \neq 1$ for all $n \geq 1$, then the initial value problem (1)–(3) has a unique solution $(x(t), y(t))$ on $[0, \infty) \cup \{-1\}$, which can be formulated on the interval $n \leq t < n+1$, $n \in \mathbb{N} = \{0, 1, 2, \dots\}$ as

$$\begin{aligned} x(t) &= e^{-\int_n^t a(s)ds} \left\{ x(n) + [-f_1(x(n-1)g_1(y(n)) + h_1(x(n+1))] \int_n^t e^{\int_n^u a(s)ds} du \right\}, \\ y(t) &= e^{-\int_n^t b(s)ds} \left\{ y(n) + [-f_2(y(n-1)g_2(x(n)) + h_2(y(n+1))] \int_n^t e^{\int_n^u b(s)ds} du \right\}, \end{aligned} \tag{4}$$

where $(x(n), y(n))$ is the unique solution of the difference equations system

$$\begin{aligned} x(n+1) &= \frac{1}{1-c_{n+1}} e^{-\int_n^{n+1} a(s)ds} \left\{ x(n) + [-f_1(x(n-1))g_1(y(n)) + h_1(x(n+1))] \int_n^{n+1} e^{\int_n^u a(s)ds} du \right\}, \\ y(n+1) &= \frac{1}{1-d_{n+1}} e^{-\int_n^{n+1} b(s)ds} \left\{ y(n) + [-f_2(y(n-1))g_2(x(n)) + h_2(y(n+1))] \int_n^{n+1} e^{\int_n^u b(s)ds} du \right\}, \end{aligned} \tag{5}$$

for $n \geq 0$ with initial conditions (3).

Proof Let $(x_n(t), y_n(t)) \equiv (x(t), y(t))$ be a solution of (1)-(2) on the interval $n \leq t < n+1$. So, system (1) can be rewritten in the following form

$$\begin{aligned} x'(\mathrm{t}) + a(t)x(t) &= -f_1(x(n-1))g_1(y(n)) + h_1(x(n+1)), \\ y'(\mathrm{t}) + b(t)y(t) &= -f_2(y(n-1))g_2(x(n)) + h_2(y(n+1)). \end{aligned} \tag{6}$$

From (6), for $n \leq t < n+1$, we get

$$\begin{aligned} x_n(t) &= e^{-\int_n^t a(s)ds} \left\{ x(n) + [-f_1(x(n-1))g_1(y(n)) + h_1(x(n+1))] \int_n^t e^{\int_n^u a(s)ds} du \right\}, \\ y_n(t) &= e^{-\int_n^t b(s)ds} \left\{ y(n) + [-f_2(y(n-1))g_2(x(n)) + h_2(y(n+1))] \int_n^t e^{\int_n^u b(s)ds} du \right\}. \end{aligned} \tag{7}$$

On the other hand, for $n-1 \leq t < n$, we have

$$\begin{aligned} x_{n-1}(t) &= e^{-\int_{n-1}^{t} a(s)ds}\left\{x(n-1) + \left[-f_1(x(n-2))g_1\big(y(n-1)\big) + h_1(x(n))\right]\int_{n-1}^{t} e^{\int_{n-1}^{u} a(s)ds}du\right\}, \\ y_{n-1}(t) &= e^{-\int_{n-1}^{t} b(s)ds}\left\{y(n-1) + \left[-f_2(y(n-2))g_2\big(x(n-1)\big) + h_2(y(n))\right]\int_{n-1}^{t} e^{\int_{n-1}^{u} b(s)ds}du\right\}. \end{aligned} \tag{8}$$

Using impulse conditions (2), from (7) and (8), we obtain difference equations system (5). Considering initial conditions (3), the solution of system (5) is obtained uniquely. Thus, the solution of (1)-(3) is obtained as (4).

Definition 2 A function $x(t)$ (or $y(t)$) defined on $[0,\infty)$ is called oscillatory if there exist two real-valued sequences $\{t_n\}_{n\geq 0}, \{t_n'\}_{n\geq 0} \subset [0,\infty)$ such that $t_n \to +\infty$, $t_n' \to +\infty$ as $n \to +\infty$ and $x(t_n) \leq 0 \leq x(t_n')$ (or $y(t_n) \leq 0 \leq y(t_n')$) for $n \geq N$, where N is sufficiently large. Otherwise, $x(t)$ (or $y(t)$) is called nonoscillatory.

Definition 3 A solution $x(t)$ and $y(t)$ of the first and second equations of system (1), respectively are said to be oscillatory if $x(t)$ and $y(t)$ have arbitrarily large zeros. Otherwise, it is said to be nonoscillatory.

Definition 4 The solution of system (1)-(2) is called oscillatory if $x(t)$ and $y(t)$ are oscillatory.

Definition 5 A solution $x(n)$ (or $y(n)$) is said to oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is said to be nonoscillatory.

Theorem 2 Let $\big(x(n), y(n)\big)$, $n \geq -1$ be a nonoscillatory solution of system (5) with the initial conditions (3), $uf_i(u) > 0$, $i = 1,2$ for $u \neq 0$ and $f_i(u) \neq 0$, $i = 1,2$ for all $u \in \mathbb{R}$. Then the solution $(x(t), y(t))$ of the problem (1)-(3) is nonoscillatory if and only if there exist a $N \in \mathbb{N}$ such that

$$\left.\begin{aligned} \frac{x(n)}{f_1(x(n-1))} &> \left[g_1\big(y(n)\big) + \frac{h_1(x(n+1))}{f_1(x(n-1))}\right]\int_{n}^{n+1} e^{\int_{n}^{u} a(s)ds}du, \\ \frac{y(n)}{f_2(y(n-1))} &> \left[g_2\big(x(n)\big) + \frac{h_2(y(n+1))}{f_2(y(n-1))}\right]\int_{n}^{n+1} e^{\int_{n}^{u} b(s)ds}du. \end{aligned}\right. \quad n \leq t < n+1, n > N. \tag{9}$$

Proof Without loss of generality, we may suppose that $x(n), y(n) > 0, x(n-1), y(n-1) > 0$ for $n > N$. If $x(t)$ and $y(t)$ are nonoscillatory, then $x(t), y(t) > 0$, $t > T \geq N$. So, from (4), the condition (9) is obtained easily.

Now let us assume that the condition (9) is true. It is enough to show that $x(t)$ is nonoscillatory. This can show for $y(t)$, similarly. For contradiction, let $x(t)$ be oscillatory. Therefore, there exist sequences $\{t_k\}_{k\geq 0}, \{t_k'\}_{k\geq 0}$ such that $t_k \to +\infty$, $t_k' \to +\infty$ as $k \to +\infty$ and $x(t_k) \leq 0 \leq x(t_k')$. Let $n_k = [t_k]$. It is clear that $n_k \to +\infty$ as $k \to +\infty$. So, from (4), we get

$$x(t_k) = e^{-\int_{n_k}^{t_k} a(s)ds}\left\{x(n_k) + \left[-f_1(x(n_k-1)g_1\big(y(n_k)\big) + h_1(x(n_k+1))\right]\int_{n_k}^{t_k} e^{\int_{n_k}^{u} a(s)ds}du\right\}.$$

Since $x(n_k)>0$, $x(n_k-1)>0, f_1(x(n_k-1))>0$ and $x(t_k)\le 0$, we obtain

$$x(n_k)\le\left[f_1(x(n_k-1))g_1\big(y(n_k)\big)+h_1(x(n_k+1))\right]\int_{n_k}^{t_k}e^{\int_{n_k}^{u}a(s)ds}du,\ n_k\le t_k<n_k+1.$$

Hence

$$\frac{x(n_k)}{f_1(x(n_k-1))}\le\left[g_1\big(y(n_k)\big)+\frac{h_1(x(n_k+1))}{f_1(x(n_k-1))}\right]\int_{n_k}^{t_k}e^{\int_{n_k}^{u}a(s)ds}du,\ n_k\le t_k<n_k+1$$

which is a contradiction to (9).

If $x(n),y(n)<0,x(n-1),y(n-1)<0$ for $n>N$, then the proof is done by similar method.

Theorem 3 Suppose that $c_n<1$, $d_n<1$ for $n\in\mathbb{Z}^+$, and there exist $K_i>0$, $i=1,2$ such that $g_i(u)\ge K_i$ for all $u\in\mathbb{R}$, $\frac{f_i(u)}{u}\ge 1$, $uh_i(u)<0$, $i=1,2$ for $u\neq 0$ and

$$\begin{aligned}&\lim_{n\to\infty}\sup(1-c_n)\int_{n}^{n+1}e^{\int_{n-1}^{u}a(s)ds}\,du>\frac{1}{K_1},\\&\lim_{n\to\infty}\sup(1-d_n)\int_{n}^{n+1}e^{\int_{n-1}^{u}b(s)ds}\,du>\frac{1}{K_2},\end{aligned}\qquad(10)$$

then all solutions of system (5) are oscillatory.

Proof We prove that the existence of eventually positive (or negative) solutions leads to a contradiction. Let $(x(n),y(n))$ be a solution of system (5). Assume that $x(n)>0,x(n-1)>0,x(n-2)>0$ for $n>N$, where N is sufficiently large. From the first equation of system (5), we find

$$(1-c_n)x(n)e^{\int_{n-1}^{n}a(s)ds}=x(n-1)+\left[-f_1(x(n-2))g_1\big(y(n-1)\big)+h_1(x(n))\right]\int_{n-1}^{n}e^{\int_{n-1}^{u}a(s)ds}du.$$

Since $x(n-2)>0$, $f_1(x(n-2)>0, g_1\big(y(n-1)\big)>0$, $h_1\big(x(n)\big)<0$, we have

$$(1-c_n)x(n)e^{\int_{n-1}^{n}a(s)ds}\le f_1(x(n-1)).\qquad(11)$$

Multiplying both sides of this inequality (11) by $-g_1\big(y(n)\big)\int_n^{n+1}e^{\int_n^u a(s)ds}du$ and adding $x(n)+h_1\big(x(n+1)\big)\int_n^{n+1}e^{\int_n^u a(s)ds}du$, from (5), we obtain

$$\begin{aligned}&x(n)-(1-c_n)x(n)e^{\int_{n-1}^{n}a(s)ds}g_1\big(y(n)\big)\int_n^{n+1}e^{\int_n^u a(s)ds}du+h_1\big(x(n+1)\big)\int_n^{n+1}e^{\int_n^u a(s)ds}du\\&\ge x(n)-f_1(x(n-1))g_1\big(y(n)\big)\int_n^{n+1}e^{\int_n^u a(s)ds}du+h_1\big(x(n+1)\big)\int_n^{n+1}e^{\int_n^u a(s)ds}du\\&=(1-c_{n+1})x(n+1)e^{\int_n^{n+1}a(s)ds}>0.\end{aligned}\qquad(12)$$

Since $x(n)>0,x(n+1)>0,1-c_{n+1}>0$ and $h_1\big(x(n+1)\big)<0$, from (12), we get

$$1>(1-c_n)e^{\int_{n-1}^{n}a(s)ds}g_1\big(y(n)\big)\int_n^{n+1}e^{\int_n^u a(s)ds}du.$$

So, from $g_1(y(n))\ge K_1$, we have

$$\frac{1}{K_1} \geq \lim_{n\to\infty} \sup(1-c_n) \int_n^{n+1} e^{\int_{n-1}^u a(s)ds}\, du,$$

which contradicts (10). $x(n) < 0, x(n-1) < 0, x(n-2) < 0$ for $n > N$, then we obtain the same contradiction. So, the component $x(n)$ of the solution $(x(n), y(n))$ is oscillatory. Similarly, we can show that the component $y(n)$ is oscillatory under condition (10).

Corollary 1 Under the hypotheses of Theorem 3, all solutions of (1)-(2) are oscillatory.

Theorem 4 Assume that there exist constants $K_i, M_i, N_i > 0, i = 1,2$ such that $\frac{f_i(u)}{u} \geq M_i, uh_i(u) < 0, i = 1,2$ for $u \neq 0$ and $g_i(u) \geq K_i, i = 1,2$ for all $u \in \mathbb{R}$, $1 - c_n \geq N_1$, $1 - d_n \geq N_2, n = 0,1,2,\cdots$. Suppose that the following conditions are satisfied:

$$\begin{aligned} \frac{1}{4K_1M_1N_1} &< \lim_{n\to\infty} inf e^{\int_n^{n+1} a(s)ds} \lim_{n\to\infty} inf \int_n^{n+1} e^{\int_n^u a(s)ds}\, du < \infty, \\ \frac{1}{4K_2M_2N_2} &< \lim_{n\to\infty} inf e^{\int_n^{n+1} b(s)ds} \lim_{n\to\infty} inf \int_n^{n+1} e^{\int_n^u b(s)ds}\, du < \infty. \end{aligned} \tag{13}$$

Then all solutions of (5) are oscillatory.

Proof Let $(x(n), y(n))$ be a solution of system (5). We need to show that under first inequality of condition (13), $x(n)$ is oscillatory. Assume that $x(n), x(n-1) > 0$ for $n > N$, where N is sufficiently large. From system (5), we have

$$\begin{aligned} &(1-c_{n+1})x(n+1)e^{\int_n^{n+1} a(s)ds} \\ &\qquad = x(n) + \left[-f_1(x(n-1))g_1\big(y(n)\big) + h_1(x(n+1))\right] \int_n^{n+1} e^{\int_n^u a(s)ds} du. \end{aligned} \tag{14}$$

Let $w_n = \frac{x(n)}{x(n-1)}$. Since $w_n > 0$, we consider two cases:

Case 1: Let $\lim_{n\to\infty} inf w_n = \infty$. Then dividing the equation (14) by $x(n)$, we have

$$(1-c_{n+1})w_{n+1}e^{\int_n^{n+1} a(s)ds} + \mathrm{M_1K_1}\frac{1}{w_n}\int_n^{n+1} e^{\int_n^u a(s)ds} du \leq 1. \tag{15}$$

Taking the inferior limit on both sides of inequality (15), we get

$$\lim_{n\to\infty} inf(1-c_{n+1}) \lim_{n\to\infty} inf w_{n+1} \lim_{n\to\infty} inf e^{\int_n^{n+1} \boldsymbol{a(s)ds}} + \mathrm{M_1K_1} \lim_{n\to\infty} inf \frac{1}{w_n} \lim_{n\to\infty} inf \int_n^{n+1} e^{\int_n^u a(s)ds} du \leq 1$$

which is a contradiction to $\lim_{n\to\infty} inf w_n = \infty$. So, we consider the second case.

Case 2: Let $0 \leq \lim_{n\to\infty} inf w_n < \infty$. Dividing the equation (14) by $x(n-1)$, we have

$$\begin{aligned} \frac{x(n)}{x(n-1)} &= (1-c_{n+1})\frac{x(n+1)}{x(n-1)} e^{\int_n^{n+1} a(s)ds} \\ &\quad + \left[\frac{f_1\big(x(n-1)\big)}{x(n-1)} g_1\big(y(n)\big) - \frac{h_1(x(n+1))}{x(n-1)}\right] \int_n^{n+1} e^{\int_n^u a(s)ds} du \end{aligned}$$

which yields

$$w_n \geq (1-c_{n+1})w_n w_{n+1} e^{\int_n^{n+1} a(s)ds} + K_1 M_1 \int_n^{n+1} e^{\int_n^u a(s)ds} du. \qquad (16)$$

Let $\lim_{n\to\infty} inf w_n = w$, $\lim_{n\to\infty} inf e^{\int_n^{n+1} a(s)ds} = A$, $\lim_{n\to\infty} inf \int_n^{n+1} e^{\int_n^u a(s)ds} du = B$. Taking the inferior limit on both sides of inequality (16), we have

$$w \geq \lim_{n\to\infty} inf(1-c_{n+1})w^2 A + K_1 M_1 B. \qquad (17)$$

Now, there are two subcases.

(i) $\lim_{n\to\infty} inf(1-c_{n+1}) = \infty$, then we obtain a contradiction from (17).

(ii) If $0 < N_1 \leq \lim_{n\to\infty} inf(1-c_{n+1}) < \infty$, then from (17) we have

$$AN_1 w^2 - w + K_1 M_1 B \leq 0$$

or

$$AN_1\left[\left(w - \frac{1}{2AN_1}\right)^2 + \frac{4K_1M_1N_1AB - 1}{4A^2{N_1}^2}\right] \leq 0.$$

Since $A > 0$ and $N_1 > 0$, we have

$$\frac{4K_1M_1N_1AB - 1}{4A^2{N_1}^2} \leq 0,$$

which contradicts condition (13).

In the case of $x(n) < 0, x(n-1) < 0$ for sufficiently large $n > N$, the proof is similar, and we obtain the same contradiction.

On the other hand, if we assume that $y(n)$ is a nonoscillatory sequence, then we have a contradiction to second condition in (13). Hence, $\big(x(n), y(n)\big)$ is an oscillatory solution of system (5).

Corollary 2 Under the hypotheses of Theorem 4, all solutions of (1)-(2) are oscillatory.

3. Examples

Example 1 We consider the following system

$$\begin{aligned} x'(\mathrm{t}) &= -\tfrac{1}{t+1}x(t) - 2x([t-1])e^{x([t-1])}\big(e^{y([t])}+3\big) - x([t+1]),\\ y'(\mathrm{t}) &= -\tfrac{1}{t+1}y(t) - 3y([t-1])e^{y([t-1])}\big(e^{x([t])}+2\big) - y([t+1]),\ t \neq n, t > 0 \end{aligned} \qquad (18)$$

and the impulse conditions

$$\begin{aligned} \Delta x(n) &= \tfrac{1}{3^n}x(n),\\ \Delta y(n) &= \tfrac{1}{4^n}y(n),\ t = n. \end{aligned} \qquad (19)$$

It is clear that it is satisfied all hypotheses of Theorem 3. Then all solutions (18)-(19) are oscillatory. The solution $(x_n(t), y_n(t))$ of system (18)-(19) with initial conditions $x(-1) = 0, x(0) = 0.5, y(-1) = 0, y(0) = -0.5$ is as shown in Figure 1 obtained by using Mathematica.

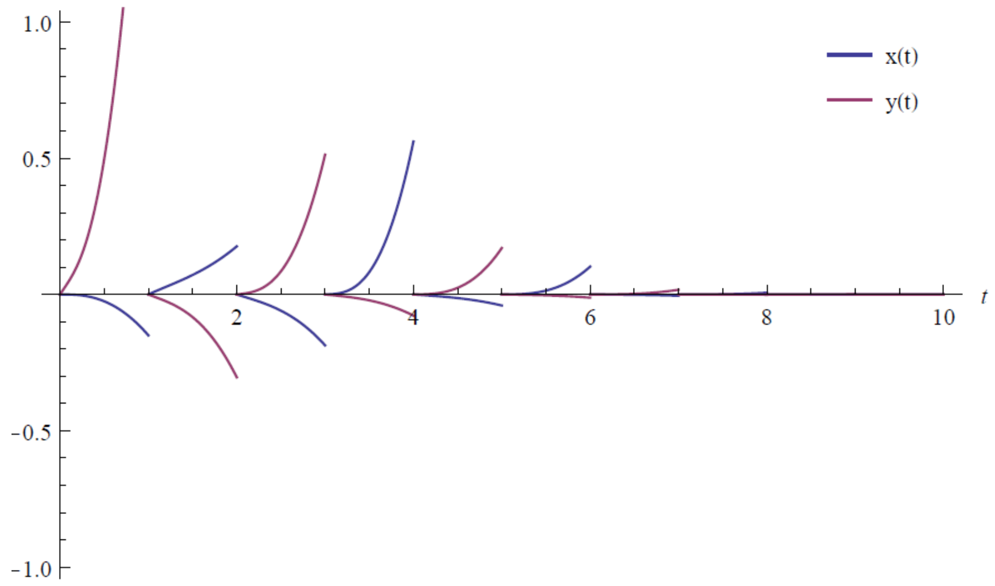

Figure 1: Oscillatory solutions of system (18)-(19) with the initial conditions $x(-1) = 0, x(0) = 0.5, y(-1) = 0, y(0) = -0.5$.

Example 2 As a special case of system (1)-(2), let us take the following system

$$\begin{aligned} x'(\mathrm{t}) &= -(ln2)x(t) - x([t-1])(x^2([t-1]) + 2)(y^2([t-1]) + 1) - x^3([t+1]), \\ y'(\mathrm{t}) &= -(ln3)y(t) - y([t-1])(y^2([t-1]) + 3)(x^4([t-1]) + 1) - y([t+1]), \qquad t \neq n, t > 0 \end{aligned} \tag{20}$$

and the impulse conditions

$$\begin{aligned} \Delta x(n) &= \frac{(-1)^n}{2} x(n), \\ \Delta y(n) &= \frac{(-1)^n}{3} y(n), \ t = n. \end{aligned} \tag{21}$$

All hypotheses of Theorem 4 are satisfied for (20)-(21). Then all solutions (20)-(21) are oscillatory. The solution $(x_n(t), y_n(t))$ of system (18)-(19) with initial conditions $x(-1) = 0.2, x(0) = 0.1, y(-1) = 0.2, y(0) = -0.1$ is as shown in Figure 2 obtained by using Mathematica.

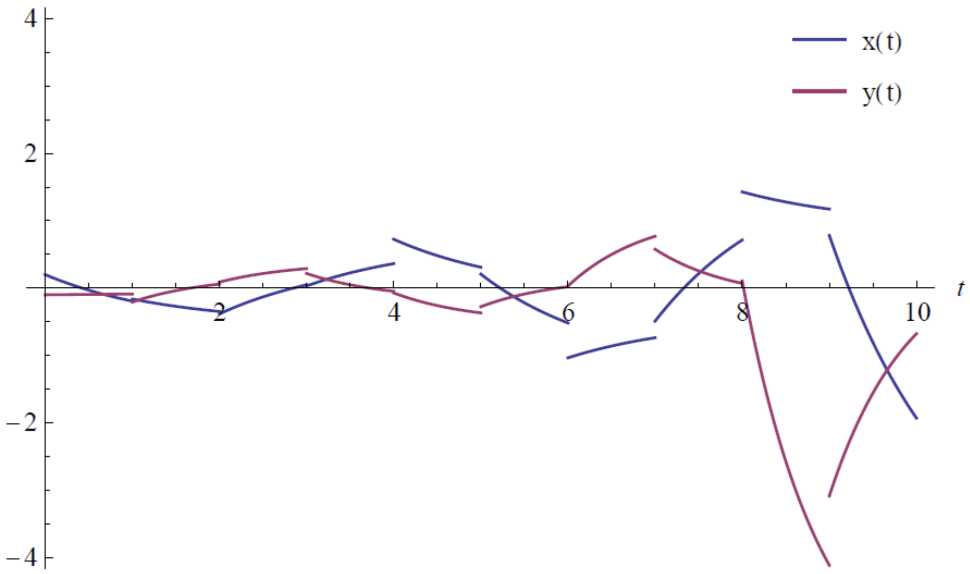

Figure 2: Oscillatory solutions of system (20)-(21) with the initial conditions $x(-1) = 0.2, x(0) = 0.1, y(-1) = 0.2, y(0) = -0.1$.

4. Conclusion

We have studied the existence and uniqueness of the solutions of a nonlinear impulsive differential equations system with piecewise constant mixed arguments and obtained sufficient conditions for oscillation of the solutions. Additionally, we give two examples to support our main results. Also, Mathematica has been used to draw two figures. Figure 1 in first example exemplifies oscillatory solutions of system (18)-(19) with the initial conditions $x(-1) = 0, x(0) = 0.5, y(-1) = 0, y(0) = -0.5$ since the conditions of Theorem 3 are satisfied. Figure 2 shows oscillatory solutions of system (20)-(21) with the initial conditions $x(-1) = 0.2, x(0) = 0.1, y(-1) = 0.2, y(0) = -0.1$ from Theorem 4. Moreover, system (1)-(3) is a generalization of system in [9].

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Optimal Control of Higher Order Differential Inequality

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Abstract

This paper discusses optimal control problems for differential inclusions described by the nonlinear differential inequality with higher order. To this end, sufficient optimality conditions for such problems are based on the method of discretization of the continuous problem and equivalence theorems. Optimality conditions for the Mayer problem are proved in the form of Euler - Lagrange inclusions and transversality conditions. Besides, a numerical illustration is provided to demonstrate the theoretical result.

Key words: Differential inequality, Euler-Lagrange inclusion, higher-order, transversality.

1. Introduction

One of the intensively developable fields of optimal processes in mathematical theory covers the theory of extremal problems involving inequality constraints and related results. In literature, several mathematical problems can be formulated as variational inequality problems that are an important unifying methodology for the analysis of equilibrium problems [1], [2].

In the recent literature, the existence of solutions and other qualitative properties has been extensively analyzed for second-order differential inequalities and/or inclusions. The existence result of solutions to a nonlinear second-order ordinary differential equation is presented in the paper [13], using the suggested method based on differential inequalities. Some stability results for generalized vector quasi-variational inequality problems are obtained from the paper [4]. The solution set is shown to be closed. The upper semi-continuity property of the solution set is also defined for perturbed generalized vector quasi-variational inequality problems. Based on the properties of differential inclusion solutions with unbounded measurable-pseudo-Lipschitz right-hand side, the necessary conditions of optimality for the Mayer extremal problem are obtained in the paper [12]. Their effective resolution requires the inclusion in the mathematical model of the particular peculiarities of each problem.

We analyze a particular form of an optimization problem with higher-order differential inequalities in the present paper. Since by the existence of higher-order derivatives, the problem is much more complicated and the difficulty with higher-order differential inequalities is rather than determining higher-order adjoint inclusions of the Euler-Lagrange form and the conditions of transversality. The paper [3] presents the necessary and sufficient conditions of optimality for the Bolza problem with higher-order polyhedral discrete and differential inclusions. The paper [10] discusses the Mayer problem with differential inclusions of higher-order evolution and functional constraints of optimal control theory. Furthermore, in the Mahmudov survey papers for differential inclusions, many of the necessary and sufficient conditions for optimization inevitably require the formation of new forms of equivalence [5]-[9]. By passing formally to the limit, the derivation of sufficient conditions is completed, as the discrete steps tend to be zero. Using the proposed methods for ordinary differential Mordukhovich [11] inclusions, of course, it can be seen that sufficient conditions are also necessary to achieve optimum results.

We recall some notions and main results of the convex analysis from the books [5] and [11]. As usual, $\mathbb{R}^n$ is an n-dimensional Euclidean space, $\langle x,v\rangle$ is the inner product of elements $x,v\in\mathbb{R}^n$ and (x,v) is a pair of x,v. A convex cone $K_A(z_0)$ is called the cone of tangent directions at a point $z_0=(x^0,v_1^0,...,v_s^0)\in A$ if from $\overline{z}=(\overline{x},\overline{v}_1,...,\overline{v}_s)\in K_A(z_0)$ it follows that $\overline{z}$ is a tangent vector to the set A, i.e., there exists a function $\varphi(\lambda)\in\mathbb{R}^{(s+1)n}$ satisfying $z_0+\lambda\overline{z}+\varphi(\lambda)\in A$ for sufficiently small $\lambda>0$, where $\lambda^{-1}\varphi(\lambda)\to 0$, as $\lambda\downarrow 0$. Evidently, for a convex set A at a point $(x^0,v_1^0,...,v_s^0)\in A$ setting $\varphi(\lambda)\equiv 0$ we have $K_A(z_0)=\left\{(\overline{x},\overline{v}_1,...,\overline{v}_s):\overline{x}=\lambda(x-x^0),\overline{v}_i=\lambda(v_i-v_i^0),i=1,2,...s,\lambda>0\right\},\forall\,(x_1,v_1,...,v_s)\in A$.

As usually, $K_A^*(x^0,v_1^0,...,v_s^0)$ is the dual cone to a cone of tangent vectors $K_A(x^0,v_1^0,...,v_s^0)$. A function W is called a proper function if it does not assume the value $-\infty$ and is not identically equal to $+\infty$.

Subdifferentiation theory is a basic method in the study of extremum problems. It is known that convex functions are not differentiable in general. However, these functions have several useful differential properties, one of which is the universal existence of directional derivatives. Also, the definition of subgradient can be defined for convex function, and the subdifferential conception is the set of subgradients. It should be noted that the subdifferential notion can be defined for different functional classes in different ways. Clarke (see, for example, [2] and references therein), who did pioneering work in the field of non-smooth analysis, spread well beyond the domain of convexity, introduced the first definition of subdifferential for general non-convex functions. For his generalized gradients of local Lipschitzian functions described in Banach spaces, Clarke created a comprehensive subdifferential calculus.

The Mordukhovich subdifferential is the main class of generalized differentials and, in pure and applied analysis, plays a vital role. Mordukhovich recently published two volumes of the book [11] which, from theoretical and applied points of view, provide important questions in modern variational analysis of these subdifferentials.

In this paper the problem (PC) given by s-th order differential inequality constraint is studied:

$$\text{minimize } \psi\left(x(1), x'(1), ..., x^{(s-1)}(1)\right) \quad (1)$$

$$\text{(PC)} \qquad W\left(x(t), x'(t), ..., x^{(s)}(t)\right) \le 0, \text{ a.e. } t \in [0,1], \quad (2)$$

$$x^{(k)}(0) = \alpha_k, k = 0,1,..., s-1. \quad (3)$$

Here $\psi : (\mathbb{R}^n)^s \to \mathbb{R}$ and $W : (\mathbb{R}^n)^{s+1} \to \mathbb{R}$ are continuous convex functions, $\alpha_k, k = 0,1,..., s-1$ are fixed vectors.

It is necessary to find the solution $\tilde{x}(t)$ of Cauchy problem (PC) satisfying (2) almost everywhere (a.e.) on [0,1] and the initial conditions (3) which minimize the Mayer functional $\psi\left(x(1), x'(1), ..., x^{(s-1)}(1)\right)$.

A feasible trajectory $x(t), t \in [0,1]$ is an absolutely continuous function together with the higher-order derivatives until s-1 and $x^{(s)}(t) \in L_1^n\left([0,1]\right)$. Clearly such a class of functions is Banach space, endowed with different equivalent norms.

2. Construction of Discrete-Approximation Problem

We use difference derivatives in this section to approximate the problem (PC) and formulate the necessary and sufficient conditions for it. Let us choose step δ on the t axis and the grid function $x(t)$ on [0,1]. We introduce the following s-th order difference operators

$$\Delta^s x(t) = \frac{1}{\delta^s}\left(\sum_{j=0}^{s}(-1)^j C_s^j x(t+(s-j)\delta)\right), t = 0, \delta, ..., 1-\delta$$

where $C_s^j = \begin{pmatrix} s \\ j \end{pmatrix} = \dfrac{s!}{j!(s-j)!}$ is a binomial coefficient.

Now, concerning the problem (PC), we associate the following s-th order discrete approximation problem (PDA):

$$
(PDA) \qquad \begin{aligned} &\text{minimize } \psi\left(x(1-(s-1)\delta), \Delta x(1-(s-1)\delta), ..., \Delta^{s-1} x(1-(s-1)\delta)\right), \\ &W\left(x(t), \Delta x(t), ..., \Delta^s x(t)\right) \le 0, t = 0, \delta, ..., 1-s\delta, \\ &\Delta^k x(0) = \alpha_k, k = 0, 1, ..., s-1. \end{aligned} \qquad (4)
$$

It is noted that the transition to a discrete-approximation problem requires a following special equivalence Theorem 2.1 linking the main results of discrete and discrete-approximation problems. The Chain rule of subdifferentiation of the composition of convex functions is used for the construction of the transversality condition for the Mayer problem in its general form, taking into account a Pascal triangle with binomial coefficients. As a result, the secret to our progress is the approximation and formulation of the following equivalence Theorem 2.1.

Theorem 2.1. Suppose $\psi_0 : (\mathbb{R}^n)^s \to \mathbb{R}$ is a function defined by the relationship $\psi_0\left(x, w_1, ..., w_{s-1}\right) \equiv \psi\left(x, \xi_1, ..., \xi_{s-1}\right)$ where $\xi_r = \dfrac{1}{\delta^r}\left(\sum_{j=0}^{r-1} (-1)^j \binom{r}{j} w_{r-j} + (-1)^r x\right)$, $r = 0, 1, ..., s-1,$ $(x^0, w_1^0, ..., w_{s-1}^0) \in \operatorname{dom}\psi_0$, $(x^0, \xi_1^0, ..., \xi_{s-1}^0) \in \operatorname{dom}\psi$. The following subdifferential inclusions are equivalent:

(i) $\left(\overline{x}^*, \overline{w}_1^*, ..., \overline{w}_{s-1}^*\right) \in \partial\psi_0\left(x^0, w_1^0, ..., w_{s-1}^0\right)$,

(ii) $\left(\overline{x}^* + \sum_{j=1}^{s-1} \overline{w}_j^*,\ w_1^*, w_2^*, ..., w_{s-1}^*\right) \in \partial\psi(x^0, \xi_1^0, ..., \xi_{s-1}^0)$.

where $w_r^* = \delta^r \left(\sum_{j=0}^{s-(r+1)} \binom{j+r}{j} \overline{w}_{j+r}^* \right), r = 1, ..., s-1.$

Theorem 2.2. For optimality of the trajectory $\left\{\tilde{x}(t)\right\}_{t=0}^{1}$ in the problem (PDA), it is necessary and sufficient that there is an adjoint trajectory of vectors $\left\{x^*(t)\right\}_{t=0}^{1}$ simultaneously not all equal to zero satisfying the approximate Euler-Lagrange inclusions

$$
\left((-1)^{s+1}\Delta^s x^*(t) + \sum_{j=1}^{s-1}(-1)^j \Delta^j u^{s-j*}(t),\ -u^{s-1*}(t+\delta),\ -u^{s-2*}(t+2\delta), ..., -u^{1*}(t+(s-1)\delta), x^*(t+s\delta)\right)
$$

$$
\in \mu(t)\partial W\left(\tilde{x}(t), \Delta\tilde{x}(t), ..., \Delta^s \tilde{x}(t)\right)
$$

$$\mu(t)W\left(\tilde{x}(t),\Delta\tilde{x}(t),...,\Delta^{s}\tilde{x}(t)\right)=0,\mu(t)\geq 0,t=s,...,1-s\delta,$$

and transversality conditions

$$(\theta_1,\theta_2,...,\theta_s)\in\partial\psi\left(\tilde{x}(1-(s-1)\delta),\Delta\tilde{x}(1-(s-1)\delta),...,\Delta^{s-1}\tilde{x}(1-(s-1)\delta)\right)$$

where

$$\theta_k=(-1)^{k+s+1}\Delta^{s-k}x^*(1-(s-k)\delta)-\sum_{j=1}^{s-k}(-1)^j\Delta^{j-1}u^{s-k-j+1*}(1-(s-k)\delta),$$

$$\eta^{s-k*}(t)=\sum_{j=0}^{k-1}(-1)^j\begin{pmatrix}s-k+j\\ s-k\end{pmatrix}\delta^{k-j}u^{k-j*}(t+j\delta)-(-1)^k\begin{pmatrix}s\\ s-k\end{pmatrix}x^*(t+k\delta),$$

$$k=1,...,s-1,\text{ and }\theta_s=-x^*(1)$$

3.Main Results

The results obtained in the previous section are useful for determining the conditions of optimality for the continuous problem (PC). At least, we would like to emphasize that the transformation to the limit under the conditions of Theorem 2.2. as $\delta\to 0$ is the secret to our success. Sufficient conditions of optimality for the continuous problem (PC) can be justified by the fact that these conditions are also necessary conditions of optimality by using the functional analysis method in convex problems. Then the so-called higher-order adjoint Euler-Lagrange differential inclusion and transversality conditions at the point $t=0$ consist of the following, respectively:

(a) $$\left((-1)^{s+1}\frac{d^s x*(t)}{dt^s}+\sum_{j=1}^{s-1}(-1)^j\frac{d^j u^{s-j*}(t)}{dt^j},\ -u^{s-1*}(t),-u^{s-2*}(t),\ldots,-u^{1*}(t),\ x^*(t)\right)$$

$$\in\mu(t)\partial W\left(\tilde{x}(t),\tilde{x}'(t),...,\tilde{x}^{(s)}(t)\right)$$

(b) $\mu(t)W\left(\tilde{x}(t),\tilde{x}'(t),...,\tilde{x}^{(s)}(t)\right)=0,\ \mu(t)\geq 0,t\in[0,1]$,

and transversality conditions

(c) $(\gamma_1,\gamma_2,...,\gamma_s)\in\partial\psi\left(\tilde{x}(1),\tilde{x}'(1),...,\tilde{x}^{(s-1)}(1)\right)$

where

$$\gamma_k = (-1)^{k+s+1} \frac{d^{s-k} x*(1)}{dt^{s-k}} - \sum_{j=1}^{s-k} (-1)^j \frac{d^{j-1} u^{s-k-j+1*}(1)}{dt^{j-1}}, k = 1, ..., s-1,$$

$$\gamma_s = -x^*(1).$$

Theorem 3.1. Let ψ and W be convex continuous functions. Then for optimality of the trajectory $\tilde{x}(t)$ in the problem (PC) with higher order differential constraint, it is sufficient that there are absolutely continuous functions $\{x^*(t), u^*(t)\}$ and functions $\mu(t) \ge 0, t \in [0,1]$, which satisfy a.e. the higher-order adjoint Euler-Lagrange differential inclusions (a) and the conditions (b) and transversality conditions (c).

Example 3.1. This numerical example is to show the feasibility and efficiency of the theoretic results obtained:

$$\text{minimize } x(1) - x'(1) + x''(1),$$

$$\begin{aligned} & x(t) - x'(t) - x''(t) + x'''(t) \le 0, t \in [0,1] \\ & x(0) = 0,\ x'(0) = 1, x''(0) = -2. \end{aligned} \tag{5}$$

According to problem (PC) $W\left(x(t), x'(t), x''(t), x'''(t)\right) = x(t) - x'(t) - x''(t) + x'''(t)$, and $\psi(x(1), x'(1), x''(1)) = x(1) - x'(1) + x''(1)$, where $W : \mathbb{R}^4 \to \mathbb{R}^1$ is a linear function, i.e., $W\left(x, v_1, v_2, v_3\right) = x - v_1 - v_2 + v_3$. By conditions (a)-(c) we show that the optimal trajectory is $\tilde{x}(t) = (1-t)e^t - e^{-t}$ and the minimal value of the problem (5) is $\tilde{x}(1) - \tilde{x}'(1) + \tilde{x}''(1) = -e - 3e^{-1}$. First, it can be easily seen that the subdifferential of W is the gradient vector, $\{1, -1, -1, 1\}$, that is, $\partial_{(x,u_1,u_2,u_3)} W\left(x, u_1, u_2, u_3\right) = \{1, -1, -1, 1\}$. By analogy $\partial\psi(x, u_1, u_2) = \{1, -1, 1\}$. Then by the sufficient condition (a) of Theorem 3.1., we obtain

$$\left(\frac{d^3 x^*(t)}{dt^3} + \frac{d^2 u^{1*}(t)}{dt^2} - \frac{du^{2*}(t)}{dt}, -u^{2*}(t), -u^{1*}(t), x^*(t) \right) \in \mu(t) \partial W\left(\tilde{x}(t), \tilde{x}'(t), \tilde{x}''(t), \tilde{x}'''(t)\right).$$

Therefore, we have

$$\frac{d^3 x^*(t)}{dt^3} + \frac{d^2 u^{1*}(t)}{dt^2} - \frac{du^{2*}(t)}{dt} = \mu(t);\ u^{2*}(t) = u^{1*}(t) = x^*(t) = \mu(t),$$

$$-\frac{d^2 x^*(1)}{dt^2} - \frac{du^{1*}(1)}{dt} + u^{2*}(1) = 1, \quad \frac{dx^*(1)}{dt} + u^{1*}(1) = -1 \quad , \quad -x^*(1) = 1,$$

$$\mu(t)\left[\tilde{x}(t)-\tilde{x}'(t)-\tilde{x}''(t)+\tilde{x}'''(t)\right]=0,\ \ \mu(t)\geq 0. \tag{6}$$

In view of first relations of (6) we have an initial value problem given by third-order linear equation with constant coefficients

$$\frac{d^3x^*(t)}{dt^3}+\frac{d^2x^*(t)}{dt^2}-\frac{dx^*(t)}{dt}-x^*(t)=0,$$

$$-\frac{d^2x^*(1)}{dt^2}-\frac{dx^*(1)}{dt}+x^*(1)=1,\ \frac{dx^*(1)}{dt}+x^*(1)=-1,\ -x^*(1)=1. \tag{7}$$

Then, we find that the characteristic equation of differential equation (7) is $r^3+r^2-r-1=0$ therefore, the roots are $r_1=1, r_2=r_3=-1$ and the general solution of third-order linear homogeneous differential equation with constant coefficients is $x^*(t)=C_1e^t+(C_2+tC_3)e^{-t}$. If we impose the initial conditions, we obtain $C_1=-\dfrac{3}{4e}, C_2=\dfrac{-3e}{4}, C_3=\dfrac{e}{2}$. Then the solution of the given initial value problem $x^*(t)=\dfrac{(2t-3)e^{1-t}-3e^{t-1}}{4}$. Then since $x^*(t)=\mu(t)$ we have

$$\mu(t)=\frac{(2t-3)e^{1-t}-3e^{t-1}}{4} \tag{8}$$

The function $\mu(t)$ in (8) is nonzero over the entire time interval [0,1]. Hence, by the second condition of (6) we have $\tilde{x}(t)-\tilde{x}'(t)-\tilde{x}''(t)+\tilde{x}'''(t)=0$ and the solution to the initial value problem

$$\begin{gathered}\tilde{x}(t)-\tilde{x}'(t)-\tilde{x}''(t)+\tilde{x}'''(t)=0\\ x(0)=0,\ x'(0)=1, x''(0)=-2\end{gathered}$$

is $\tilde{x}(t)=(1-t)e^t-e^{-t}$ and the minimal value of the problem (5) is $\tilde{x}(1)-\tilde{x}'(1)+\tilde{x}''(1)=-e-3e^{-1}$.

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4th INTERNATIONAL CONFERENCE ON MATHEMATICS
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P-adic Apostol-Bernoulli Measures

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Abstract

The family p-adic Apostol-Bernoulli measures parametrized by the complement of an open ball in the algebraic closure of p-adic fields is introduced. It is proved that these measures are the higher order versions of the p-adic measure which interpolates the p-adic polylogarithms. The decomposition of Bernoulli measures as a linear combination of Apostol-Bernoulli measures is also given.

1. Introduction

Bernoulli polynomials have important applications in number theory and related fields. Among them a significant apperance of Bernoulli polynomials occurs in p-adic analysis via Bernoulli measures relating them to (p-adic) L-functions defined in [7]. Let $\mu_{B,k,\alpha}$ denote the k-th Bernoulli measure for $k \in \mathbb{Z}_{\geq 1}$ and $\alpha \in \mathbb{Z}_p^*$. Then the p-adic zeta function is defined by

$$\zeta_p(1-k) := \frac{1}{k(\alpha^{-k}-1)} \int\limits_{\mathbb{Z}_p^*} 1 d\mu_{B,k,\alpha} = (1-p^{k-1})(-B_k/k)$$

and extended to $\mathbb{Z}_p$ by p-adic interpolation. The reader is referred to [2], [3] and [6] for construction and various aspects of p-adic zeta function or more generally of

p-adic L-functions.

The p-adic polylogarithms can be defined in a similar way as we outline below. The details can be found in [4] and [5]. Let μ_z be the p-adic measure defined as $\mu_z(a+(p^N)) = z^a/(1-z^{p^N})$ for some z with $|z-1|_p \geq 1$. Then the p-adic polylogartihms are defined as

$$\mathrm{Li}^{(p)}_{1-k}(z) = \int\limits_{\mathbb{Z}_p^*} x^{k-1} d\mu_z.$$

for any $k \in \mathbb{Z}$ and z with $|z-1|_p \geq 1$. In terms of the Apostol-Bernoulli numbers denoted by $\beta_k(z)$, we can rewrite it as

$$\mathrm{Li}^{(p)}_{1-k}(z) = \int\limits_{\mathbb{Z}_p^*} x^{k-1} d\mu_z = -\frac{\beta_k(z) - p^{k-1}\beta_k(z^p)}{k}$$

where again $|z-1|_p \geq 1$.

In this paper we construct a family of p-adic measures $\mu_{k,z}$ for $k \in \mathbb{Z}_{\geq 1}$ using Apostol-Bernoulli polynomials where the measure μ_z appears as $\mu_z = -\mu_{1,z}$ in this family and

$$\int\limits_{\mathbb{Z}_p^*} x^{k-1} d\mu_{1,z} = (1/k) \int\limits_{\mathbb{Z}_p^*} 1 d\mu_{k,z}.$$

We will also see the relation between this faimly of measures and the Bernoulli measures.

We set the following notation. Throughout the paper $\mathbb{C}_p$ will denote the completion of a fixed algebraic closure of $\mathbb{Q}_p$. For $z \in \mathbb{C}$, $|z|$ denotes the Euclidean norm and for $\alpha \in \mathbb{C}_p$, $|\alpha|_p$ denotes the normalized norm on $\mathbb{C}_p$ as $|p|_p = 1/p$. For the basic facts about p-adic distributions/measures and integration the reader is referred to [1] and [3].

2. Distribution Property of Apostol - Bernoulli Polynomials

We can define the Apostol-Bernoulli polynomials $\beta_k(x,z)$ by the generating function

$$\frac{te^{xt}}{ze^t - 1} = \sum_{k=0}^{\infty} \frac{\beta_k(x,z)t^k}{k!}. \tag{1}$$

where $z \neq 1$. The k-th Apostol-Bernoulli number is defined as $\beta_k(z) := \beta_k(0,z)$ for $z \neq 1$.

Lemma 1. *Apostol-Bernoulli numbers/polynomials satisfy the following identities.*

- $\beta_k(x,z)$ *is a polynomial in* x *over* $\mathbb{Z}[z, 1/(z-1)]$.
- $\beta_k(z)$ *is defined everywhere except* $z = 1$ *where it has a pole.*
- $\beta_0 = 0$ *and* $\beta_1(x,z) = 1/(z-1)$.
- $\beta_k(x,z) = \sum_{i=0}^{k} \binom{k}{i} \beta_i(z) x^{k-i}$.
- *For* $k \geq 2$, $\beta_k(z) = z \sum_{i=0}^{k} \binom{k}{i} \beta_i(z)$.
- $\beta_k(a+b,z) = \sum_{i=0}^{k} \binom{k}{i} \beta_i(a,z) b^{k-i}$.
- $z\beta_k(x+1,z) - \beta_k(x,z) = kx^{k-1}$
- *For any* $k \geq 0$ *and* $z \neq 0, 1$, $(-1)^k \beta_k(x, 1/z) = z\beta_k(1-x,z)$.

Proof. These follow by manipulating (1) or we may refer to [8]. □

Theorem 1. *Let $\beta_k(x, z)$ be the k-th Apostol-Bernoulli polynomial. Then the map defined as*

$$\mu_{k,z}(a + (p^N)) = p^{N(k-1)} z^a \beta_k(a/p^N, z^{p^N})$$

extends to a p-adic distribution on $\mathbb{Z}_p$.

Proof. First we see that

$$\sum_{k=0}^{\infty}\sum_{b=0}^{M-1} \beta_k((a+b)/M, z^M) z^b \frac{t^k}{k!} = \sum_{b=0}^{M-1} \frac{tz^b e^{t(a+b)/M}}{z^M e^t - 1} = \frac{te^{ta/M}}{z^M e^t - 1} \sum_{b=0}^{M-1} (ze^{t/M})^b.$$

Summing the last geometric progression and then equating the coefficients of t^k we find

$$\beta_k(a, z) = M^{k-1} \sum_{b=0}^{M-1} z^b \beta_k((a+b)/M, z^M).$$

Now we choose $M = p^N$ to obtain that

$$\mu_{k,z}(a + (p^N)) = \sum_{b=0}^{p-1} \mu_{k,z}(a + bp^N + (p^{N+1}))$$

$\square$

Definition. *Let $z \in \mathbb{C}_p$ with $z \neq 1$ and $k \in \mathbb{Z}_{\geq 1}$. We define the k-th Apostol-Bernoulli distribution on $\mathbb{Z}_p$ as*

$$\mu_{k,z}(a + (p^N)) = p^{N(k-1)} z^a \beta_k(a/p^N, z^{p^N})$$

on the subsets of the form $a + (p^N)$ and extend it linearly.

Example 1. Since $\beta_1(a,z) = 1/(z-1)$ we have

$$\mu_{1,z}(a + (p^N)) = z^a/(z^{p^N} - 1)$$

which is the distribution μ_z of Koblitz given in the Introduction up to a minus sign.

Also $\beta_2(a,z) = 2\beta_1(z)a + \beta_2(z) = 2a/(z-1) - 2z/(z-1)^2$, and so

$$\mu_{2,z}(a + (p^N)) = z^a \left(\frac{a}{z^{p^N} - 1} + \frac{-p^N z^{p^N}}{(z^{p^N} - 1)^2} \right)$$

3. Construction and Some Properties of Apostol-Bernoulli Measures

First we show that $\mu_{k,z}$ is bounded if $|z-1|_p \geq 1$ and so is indeed a p-adic measure. Note that the boundedness of $\mu_{1,z}$ is already known [4].

Proposition 1. *If $|z-1|_p \geq 1$ then the k-th Apsotol-Bernoulli distribution*

$$\mu_{k,z}(a + (p^N)) = p^{N(k-1)} z^a \beta_k(a/p^N, z^{p^N})$$

is bounded and so extends to a p-adic measure on $\mathbb{Z}_p$.

Proof. Note that for $k \geq 2$, we have

$$z\frac{d\beta(z)}{dz} = \frac{k}{k+1}\beta_{k+1}(z)$$

Write

$$\beta_k(z) = \frac{\gamma_k(z)}{(z-1)^{s_k}}$$

where $\gamma_k(z) \in \mathbb{Z}$ is a polynomial of degree at most s_k (This follows by induction on

k noting that $\beta_2(z) = -2z/(z-1)^2)$. Also $\beta_k(z)/k$ has integer coefficients. So since $|z-1|_p \geq 1$, then we have that $|\beta_k(z)|_p \leq 1$. Then the result follows by considering

$$\beta_n(x,z) = \sum_{k=0}^{n} \binom{n}{k} \beta_k(z) x^{n-k}$$

term by term. □

Corollary 1. *Let X be any compact open subset of $\mathbb{Z}_p$. Let $k \in \mathbb{Z}_{\geq 1}$. Then*

$$\int_X 1 d\mu_{k,z} = k \int_X x^{k-1} d\mu_{1,z}$$

In particular

$$\int_{\mathbb{Z}_p^*} 1 d\mu_{k,z} = \beta_k(z) - p^{k-1}\beta_k(z^p).$$

Proof. The proof follows by definition of p-adic integration and Lemma 1. □

Finally we give the decomposition of Bernoulli measures as a sum of Apostol-Bernoulli measures.

Theorem 2. *Let $\alpha \in \mathbb{Z}_p^*$, and $\mu_{B,k,\alpha}$ be the k-th Bernoulli measure defined as $\mu_{B,k,\alpha}(a + (p^N)) = \mu_{B,k}\left(a + (p^N)\right) - \alpha^{-k}\mu_{B,k}(\{a\alpha\}_N + (p^N))$. Let $c \in \mathbb{Z}_{\geq 1}$ with $(p,c) = 1$ and let μ_c denote the c-th roots of unity. Then*

$$c^{-k} \sum_{z \in \mu_c - \{1\}} \mu_{k,z}(a + (p^N)) = -\mu_{B,k,c^{-1}}(a + (p^N)).$$

where $\{a\alpha\}_N$ denotes the residue class of $a\alpha$ mod (p^N).

Proof. We set $\beta_k(x,1) = B_k(x)$ in the sequel. Now

$$\begin{aligned}\sum_{k=0}^{\infty}\sum_{\zeta\in\mu_c}\zeta^a\beta_k(a/p^N,\zeta^{p^N})\frac{t^k}{k!} &= \sum_{\zeta\in\mu_c}\frac{te^{at/p^N}\zeta^a}{\zeta^{p^N}e^t-1} = -te^{at/p^N}\sum_{\zeta\in\mu_c}\frac{\zeta^a}{1-\zeta^{p^N}e^t}\\ &= -te^{at/p^N}\sum_{\zeta\in\mu_c}\zeta^a\sum_{l=0}^{\infty}\left(\zeta^{p^N}e^t\right)^l = -t\sum_{l=0}^{\infty}e^{t(a+lp^N)/p^N}\sum_{\zeta\in\mu_c}\zeta^{a+lp^N}\\ &= -ct\sum_{l=0,c|(a+lp^N)}^{\infty}e^{t(a+lp^N)/p^N}\end{aligned}$$

Now we manipulate the sum $\sum_{l=0,c|(a+lp^N)}^{\infty}e^{t(a+lp^N)/p^N}$. We have

$$c \mid (a+lp^N),\ l=0,1,2...\iff l=a_N+sc,\ s=0,1,2,...$$

where a_N is the unique integer with $0\le a_N\le c-1$ and $p^Na_N\equiv -a \pmod c$. Then we have

$$\sum_{l=0,c|(a+lp^N)}^{\infty}e^{t(a+lp^N)/p^N} = e^{t(a+p^Na_N)/p^N}\sum_{s=0}^{\infty}e^{(ct)s}$$

which implies that

$$\begin{aligned}\sum_{k=0}^{\infty}\sum_{\zeta\in\mu_c}\zeta^a\beta_k(a/p^N,\zeta^{p^N})\frac{t^k}{k!} &= -cte^{t(a+p^Na_N)/p^N}\sum_{s=0}^{\infty}e^{(ct)s}\\ &= \frac{ct}{e^{ct}-1}e^{ct(a+p^Na_N)/(cp^N)}\\ &= \sum_{k=0}^{\infty}B_k\left(\frac{a+p^Na_N}{cp^N}\right)\frac{c^kt^k}{k!}.\end{aligned}$$

Taking out the coefficient of $t^k/k!$ and multiplying it by $p^{N(k-1)}$ we obtain

$$\begin{aligned}\sum_{z\in\mu_c-\{1\}} \mu_{k,z}(a+(p^N)) &= c^k\mu_{B,k}\left(\frac{a+p^Na_N}{c}+(p^N)\right)-\mu_{B,k}(a+(p^N))\\ &= c^k\mu_{B,k}\left(\frac{a}{c}+(p^N)\right)-\mu_{B,k}(a+(p^N)).\end{aligned}$$

The last equality follows from the fact that $p^Na_N/c \in (p^N)$. Multiplying both sides by c^{-k} we obtain the desired equality. $\square$

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Periodic Solutions and Stability of a Fifth Order Difference Equation

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Abstract

In this paper, we study the periodic solutions of difference equation $x_{n+1} = x_{n-2}x_{n-4} - 1, n = 0,1, \ldots$ where the initial conditions are real numbers. We also investigate the stability of related difference equation.

1. Introduction and Preliminaries

Difference equations or their systems have a huge interest among the researchers. This interest related to applications of these equations or systems. There are many applications of difference equations in many fields of science.

In this paper, we study the dynamics of solutions of the following difference equation

$$x_{n+1} = x_{n-2}x_{n-4} - 1, n = 0,1, \ldots \qquad (1)$$

where the initial conditions are real numbers. We also investigate the stability of related difference equation.

Eq.(1) be a member of the class of equations of the form

$$x_{n+1} = x_{n-l}x_{n-k} - 1, n = 0,1, \ldots \qquad (2)$$

with special choices of l and k, where $k, l \in \mathbb{N}_0$ and $l < k$. Many authors studied different forms of Eq.(2). See following table.

Equations	Papers
$x_{n+1} = x_n x_{n-1} - 1$	[7, 10, 20]
$x_{n+1} = x_n x_{n-2} - 1$	[9]
$x_{n+1} = x_{n-1} x_{n-2} - 1$	[8]
$x_{n+1} = x_n x_{n-3} - 1$	[6]
$x_{n+1} = x_{n-1} x_{n-3} - 1$	[15, 16]
$x_{n+1} = x_{n-2} x_{n-3} - 1$	[17]
$x_{n+1} = x_{n-1} x_{n-4} - 1$	[18]
$x_{n+1} = x_{n-3} x_{n-4} - 1$	[19]

Additionally, in literature, there are many papers and books related to difference equations (see [1]-[20]).

Now we present some important definitions and theorems.

Definition 1. Let I be some interval of real numbers and let $f: I^{k+1} \to I$ be a continuously differentiable function. A difference equation of order (k+1) is an equation of the form

$$x_{n+1} = f(x_n, x_{n-1}, \cdots, x_{n-k}), \qquad n = 0,1, \cdots. \qquad (3)$$

A solution of Eq.(3) is a sequence $\{x_n\}_{n=-k}^{\infty}$ that satisfies Eq.(3) for all $n \geq -k$.

As a special case of Eq.(3), for every set of initial conditions $x_{-2}, x_{-1}, x_0 \in I$, the third order difference equation

$$x_{n+1} = f(x_n, x_{n-1}, x_{n-2}), \qquad n = 0,1,\cdots. \qquad (4)$$

has a unique solution $\{x_n\}_{n=-2}^{\infty}$.

Definition 2. A solution of Eq.(3) that is constant for all $n \geq -k$i s called an equilibrium solution of Eq.(3). If

$$x_n = \bar{x}, for\ all\ n \geq -k$$

is an equilibrium solution of Eq.(3), then $\bar{x}$ is called an equilibrium point, or simply an equilibrium of Eq.(3).

So a point $\bar{x} \in I$ is called an equilibrium point of Eq.(3) if

$$\bar{x} = f(\bar{x}, \bar{x}, \cdots, \bar{x}),$$

that is,

$$x_n = \bar{x}, for\ n \geq -k$$

is a solution of Eq.(3).

Definition 3. Suppose that the function f is continuously differentiable in some open neighborhood of an equilibrium point $\bar{x}$. Let

$$q_i = \frac{\partial f}{\partial u_i}(\bar{x}, \bar{x}, \cdots, \bar{x}), for\ i = 0,1,2,\cdots,k$$

denote the partial derivative of $f(u_0, u_1, \cdots, u_k)$ with respect to u_i evaluated at the equilibrium point $\bar{x}$ of Eq.(3).

The equation

$$z_{n+1} = q_0 z_n + q_1 z_{n-1} + \cdots, q_k z_{n-k}, k = 0,1,\cdots. \qquad (5)$$

is called the linearized equation of Eq.(3) about the equilibrium point $\bar{x}$.

Definition 4. The equation

$$\lambda^{k+1} - q_0 \lambda^k + q_1 \lambda^{k-1} + \cdots, q_k = 0 \qquad (6)$$

is called the characteristic equation of Eq.(5) about $\bar{x}$.

Definition 5. Let $\bar{x}$ an equilibrium point of Eq.(3).

(a) An equilibrium point $\bar{x}$ of Eq.(3) is called locally stable if, for every $\varepsilon > 0$; there exists $\delta > 0$ such that if $\{x_n\}_{n=-k}^{\infty}$ is a solution of Eq.(3) with

$$|x_{-k} - \bar{x}| + |x_{1-k} - \bar{x}| + \cdots + |x_0 - \bar{x}| < \delta$$

then

$$|x_n - \bar{x}| < \varepsilon, for\ all\ n \geq -k.$$

(b) An equilibrium point $\bar{x}$ of Eq.(3) is called locally asymptotically stable if, it is locally stable, and if in addition there exists $\gamma > 0$ such that if $\{x_n\}_{n=-k}^{\infty}$ is a solution of Eq.(3) with

$$|x_{-k} - \bar{x}| + |x_{1-k} - \bar{x}| + \cdots + |x_0 - \bar{x}| < \delta$$

then we have

$$\lim_{n\to\infty} x_n = \bar{x}.$$

(c) An equilibrium point $\bar{x}$ of Eq.(3) is called a global attractor if, for every solution $\{x_n\}_{n=-k}^{\infty}$ of Eq.(3), we have

$$\lim_{n\to\infty} x_n = \bar{x}.$$

(d) An equilibrium point $\bar{x}$ of Eq.(3) is called globally asymptotically stable if it is locally stable, and a global attractor.

(e) An equilibrium point $\bar{x}$ of Eq.(3) is called unstable if it is not locally stable.

Theorem 1 (The Linearized Stability Theorem). Assume that the function F is a continuously differentiable function defined on some open neighborhood of an equilibrium point $\bar{x}$. Then the following statements are true:

(a) When all the roots of Eq.(6) have absolute value less than one, then the equilibrium point $\bar{x}$ of Eq.(3) is locally asymptotically stable. Moreover, in this here the equilibrium point $\bar{x}$ of Eq.(3) is called sink.

(b) If at least one root of Eq.(6) has absolute value greater than one, then the equilibrium point $\bar{x}$ of Eq.(3) is unstable.

(i) The equilibrium point $\bar{x}$ of Eq.(3) is called hyperbolic if no root of Eq.(6) has absolute value equal to one.

(ii) If there exists a root of Eq.(6) with absolute value equal to one, then the equilibrium $\bar{x}$ is called nonhyperbolic.

(iii) An equilibrium point $\bar{x}$ of Eq.(3) is called a saddle point if it is hyperbolic and if there exists a root of Eq.(6) with absolute value less than one and another root of Eq.(6) with absolute value greater than one.

(iv) An equilibrium point $\bar{x}$ of Eq.(3) is called a repeller if all roots of Eq.(6) have absolute value greater than one.

2. Stability Analysis of Eq.(1)

In this here, we firstly find out the equilibrium points of Eq.(1). Then we examine the characteristic equation of Eq.(1). Moreover we investigate the stability of Eq.(1).

Lemma 1 There are two equilibrium points of Eq.(1) such that

$$\bar{x}_{1,2} = \frac{1 \pm \sqrt{5}}{2}. \tag{7}$$

Proof. Let $x_n = \bar{x}$ for all $n \geq -4$. Therefore, we get from Eq.(1)

$$\bar{x} = \bar{x}\bar{x} - 1.$$

Hence, we obtain that,

$$\bar{x}_1 = \frac{1 + \sqrt{5}}{2},$$

and

$$\bar{x}_2 = \frac{1 - \sqrt{5}}{2}. \square$$

Lemma 2 Assume that $\bar{x}$ is an equilibrium point of Eq.(1). Hence the linearized equation of Eq.(1) is

$$z_{n+1} - \bar{x}z_{n-2} - \bar{x}z_{n-4}. \tag{8}$$

Proof. Let I be some interval of real numbers and let $f: I^5 \to I$ be a continuously differentiable function such that f is defined by

$$f(x_n, x_{n-1}, x_{n-2}, x_{n-3}, x_{n-4}) = x_{n-2}x_{n-4} - 1.$$

Thus, we obtain the followings,

$$q_0 = \frac{\partial f}{\partial x_n}(\bar{x},\bar{x},\bar{x},\bar{x},\bar{x}) = 0,$$
$$q_1 = \frac{\partial f}{\partial x_{n-1}}(\bar{x},\bar{x},\bar{x},\bar{x},\bar{x}) = 0,$$
$$q_2 = \frac{\partial f}{\partial x_{n-2}}(\bar{x},\bar{x},\bar{x},\bar{x},\bar{x}) = \bar{x},$$
$$q_3 = \frac{\partial f}{\partial x_{n-3}}(\bar{x},\bar{x},\bar{x},\bar{x},\bar{x}) = 0,$$
$$q_4 = \frac{\partial f}{\partial x_{n-4}}(\bar{x},\bar{x},\bar{x},\bar{x},\bar{x}) = \bar{x}.$$

Therefore, the linearized equation associated with Eq.(1) about the equilibrium point $\bar{x}$ is

$$z_{n+1} = q_0 z_n + q_1 z_{n-1} + q_2 z_{n-2} + q_3 z_{n-3} + q_4 z_{n-4}$$

then

$$z_{n+1} - \bar{x}z_{n-2} - \bar{x}z_{n-4}.$$

Lemma 3 The characteristic equation of Eq.(1) about its equilibrium point $\bar{x}$ is

$$\lambda^5 - \bar{x}\lambda^2 - \bar{x} = 0. \qquad (9)$$

Now, we examine the stability of equilibrium points of Eq.(1).

Theorem 4 The positive equilibrium $\bar{x}_1 = \frac{1+\sqrt{5}}{2}$ of Eq.(1) is unstable.

Proof. We take (9) with $\bar{x}_1 = \frac{1+\sqrt{5}}{2}$. Hence, we obtain five roots of (9):

$$\lambda_1 \approx 1.35669,$$
$$\lambda_{2,3} \approx -0.817695 \pm 0.913305\mathrm{i},$$
$$\lambda_{4,5} \approx 0.139352 \pm 0.879896\mathrm{i}.$$

Hence we have,

$$|\lambda_1| > |\lambda_{2,3}| > 1 > |\lambda_{4,5}|.$$

Consequently, the equilibrium $\bar{x}_1 = \frac{1+\sqrt{5}}{2}$ of Eq.(1) is unstable.

Theorem 5 The negative equilibrium $\bar{x}_2 = \frac{1-\sqrt{5}}{2}$ of Eq.(1) is unstable.

Proof. We take (9) with $\bar{x}_2 = \frac{1-\sqrt{5}}{2}$. Hence, we obtain five roots of (9):

$$\lambda_1 \approx -1.054797,$$
$$\lambda_{2,3} \approx -0.163078 \pm 0.772937\mathrm{i},$$
$$\lambda_{4,5} \approx 0.690472 \pm 0.679856\mathrm{i}.$$

Hence we have,

$$|\lambda_1| > 1 > |\lambda_{2,3}| > |\lambda_{4,5}|.$$

Consequently, the equilibrium $\bar{x}_2 = \frac{1-\sqrt{5}}{2}$ of Eq.(1) is unstable.

3. Periodic Solutions of Eq.(1)

Now, we study the periodic solutions of Eq.(1) with period two and three.

Theorem 6 There are two periodic solutions of Eq.(1).

Proof. Let a, b be real numbers, $a \neq b$. Suppose that $\{x_{2n}\}_{n=-2}^{\infty} = a$ and $\{x_{2n-1}\}_{n=-1}^{\infty} = b$. Hence we obtain from Eq.(1)

$$b = a^2 - 1, \qquad (10)$$
$$a = b^2 - 1. \qquad (11)$$

From (10) and (11), we have four cases such that

i. $a = b = \bar{x}_1$,
ii. $a = b = \bar{x}_2$,
iii. $a = 0, b = -1$,

iv. $a = -1, b = 0$.

Since cases i and ii are trivial solutions, they are not periodic solutions. Other cases iii and iv are periodic solutions with period two.

Theorem 7 There are no three periodic solutions of Eq.(1).

Proof. Let a, b, c be real numbers such that at least two are different from each other. Assume that $\{x_{3n}\}_{n=-1}^{\infty} = a$, $\{x_{3n-1}\}_{n=-1}^{\infty} = b$ and $\{x_{3n-2}\}_{n=0}^{\infty} = c$. Therefore we get from Eq.(1) that

$$a = ac - 1, \qquad (12)$$
$$b = ba - 1, \qquad (13)$$
$$c = cb - 1. \qquad (14)$$

From (12) - (14), we have two cases that

$$a = b = c = \bar{x}_1, \qquad (15)$$
$$a = b = c = \bar{x}_2. \qquad (16)$$

So, (15) and (16) are not three periodic solutions, because these are trivial solutions of Eq.(1). The proof complete.

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Quantum Batteries Driven via Feedback Algorithms

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Abstract

Feedback algorithms can be efficiently applied to control the basic characteristics of quantum batteries (QBs): their ergotropy, their charging power, their storage capacity and others. We discuss pros and cons of the proposed algorithms for bosonic types of quantum batteries. We compare alternative algorithms, including the standard speed gradient and target attractor feedback and our original speed anti-gradient feedback. With their application we study the control efficiency for the ergotropy based on the example of bosonic QB.

1. Introduction

Quantum Battery (QB) is a quantum device for the efficient storage of energy and its transfer to consumption centers. QB can have different physical realizations (Dicke QB, spin QB, harmonic oscillator QB) and vary with their basic characteristics, such as the ergotropy, the charging power, the storage capacity and others [1].

The optimization of different working properties for QBs demands the application of control methods driving the basic characteristics of the quantum battery itself and its charger. Feedback algorithms can be efficiently applied to control the QB dynamics. They are based on different approaches: Kolesnikov's target attractor and Fradkov's speed gradient, and they could be efficiently applied to different configurations of the QB-charger system.

Here we compare few alternative algorithms for the control over the egrotropy of bosonic quantum battery. We make a short review of gradient target attractor methods, their pros and cons, and describe in details our original approach in the form of target repeller feedback.

2. Mathematical model for bosonic quantum battery

Let's consider a model quantum system consisting of two parts: the charger A and the battery B with the corresponding Hamiltonians H_A and H_B. Both Hamiltonian terms have a zero ground-state energy. Apart from that there is the Hamiltonian component H_1 coupling the charger A and the battery B together [2]:

$$H(t) = H_A + H_B + u(t) H_1 \ , \tag{1}$$

where $u(t)$ is a time-dependent coupling parameter playing a role of control.

In the case of ***bosonic harmonic oscillator*** *QB* the battery *B* is composed by *N* non-mutually interacting elements marked with the index *k*. In the terms of creation-annihilation operators the Hamiltonians (1) is given by:

$$H_A = \omega_0 a^+ a;\quad H_B = \omega_0 \sum_k b_k^+ b_k;\quad H_1 = g\sum_k \left(ab_k^+ + a^+ b_k\right), \tag{2}$$

with the positive constants ω_0 and g.

For simplicity we discuss here a single-qubit based quantum battery in the form of quantum oscillator in a Markovian bath, such that its density operator ρ is described with the Lindblad-type operator [3]:

$$\frac{d\rho}{dt} = -i[H_0 + u(t)\hat{Q}, \rho] + \hat{L}[\rho] , \tag{3}$$

with

$$H_0 = \left(\omega_0 + \frac{1}{2}\right) b^+ b ;\quad \hat{Q} = \frac{b^+ + b}{\sqrt{2\omega_0}} ;\quad \hat{P} = i\sqrt{\frac{\omega_0}{2}}(b^+ - b) , \tag{4}$$

and

$$\hat{L}[\rho] = \gamma \cdot (n(t)+1)\left(2b\rho b^+ - \rho b^+ b - b^+ b\rho\right) + \gamma \cdot n(t)\left(2b^+\rho b - bb^+\rho - \rho bb^+\right) \tag{5}$$

The model (3)-(5) covers also a decay due to the interaction of QB with the environment, and for that reason it possesses one extra control parameter $n(t)$. The positive constant γ defines the rate of this decay.

The system (3)-(5) can be re-written in the form of quasi-classical representation [4]:

$$\frac{dE}{dt} = 2\gamma \cdot (\omega_0 n(t) - E) - u(t)P ;\quad \frac{dQ}{dt} = P - \gamma \cdot Q ;\quad \frac{dP}{dt} = -\omega_0^2 Q - \gamma \cdot P - u(t) , \tag{6}$$

where we used the notation:

$$E(t) = Tr(H_0\rho);\; Q(t) = Tr(\hat{Q}\rho);\; P(t) = Tr(\hat{P}\rho). \tag{7}$$

Thus, our finalized form for the mathematical model involves three ODEs for the real functions $E(t)$, $P(t)$ and $Q(t)$, and two control parameters: $u(t)$ and $n(t)$.

The energy storage of quantum battery depends on the time-independent reference Hamiltonian H with the finite Hilbert space of the battery system. The difference between the useful energy exacted from QB in the state ρ and its energetically lowest accessible state σ_ρ defines its ***ergotropy*** [5]:

$$W_\rho = \mathrm{Tr}(\rho H) - \mathrm{Tr}(\sigma_\rho H). \tag{8}$$

In our model (6) the ergotropy could be defined by (8) as:

$$W(t) = E(t) - E_0, \tag{9}$$

where E_0 is the energy of the lowest accessible passive battery state.

3. Feedback control algorithms

There are few alternative approaches to perform an efficient feedback control over the ergotropy (9). Let's study the two most popular of them.

One is based on the family of gradient algorithms, for example, on Fradkov's speed gradient [6]. In this algorithm we need to define a positive goal function to drive the dynamical system toward its minimization. Fradkov's approach creates in the system a sort of 'target friction' which provides the maximum decay of the dynamical trajectories in the neighborhood of the control goal. As soon as the goal is achieved, Fradkov's control becomes off.

Alternative form is represented with the Kolesnikov's 'synergetic' control [7]. Here the goal function is served to design a dynamical target attractor in the system locking the trajectories in the phase space in its neighborhood. Kolesnikov's algorithm forms in the phase space trajectories which are converging exponentially fast to the target attractor. The existence of the attactor demands the permanent pumping of the energy to the dynamical system.

Very recently we proposed a modification of Kolesnikov's control based on designing a target repeller in the system [8]. Here we develop the similar idea via the speed anti-gradient approach based on Fradkov's feedback. To do that, let's define the non-negative function of the control goal as:

$$G = \frac{W^2}{2} = \frac{1}{2}(E - E_0)^2. \tag{10}$$

This goal (10) should drive our system far away from the lowest accessible energy E_0 of QB to increase its ergotropy W.

Then, following Fradkov's approach [6], we define the control signals:

$$\begin{aligned} n_{\mathrm{SG}} &= \Gamma_n \frac{\partial}{\partial n}\frac{dG}{dt} = 2\gamma\omega_0\Gamma_n(E - E_0)\ ; \\ u_{\mathrm{SG}} &= \Gamma_u \frac{\partial}{\partial u}\frac{dG}{dt} = -\Gamma_u(E - E_0)P; \\ \Gamma_n, \Gamma_u &= \mathrm{const} > 0. \end{aligned} \tag{11}$$

By (11) Eqs (6) becomes:

$$\frac{dW}{dt} = \left[\Gamma_u P^2 - 2\gamma + 4\gamma^2\omega_0^2\Gamma_n\right]\cdot W - 2\gamma E_0;$$
$$\frac{dQ}{dt} = P - \gamma \cdot Q; \tag{12}$$
$$\frac{dP}{dt} = -\omega_0^2 Q - \gamma \cdot P + \Gamma_u (E - E_0) P.$$

The control system (12) provides the most possible maximization of the ergotropy. Particularly, we can study the achievability of the control goal (10) as $\gamma \to 0$. For $\Gamma_u >> \omega_0$ we get:

$$P(t) \cong \sqrt{\frac{c_1 e^{c_1 t}}{\Gamma_u (c_2 - e^{c_1 t})}}\ ;\ W(t) \cong \frac{c_1 c_2}{2\Gamma_u (c_2 - e^{c_1 t})}\ , \tag{13}$$

with the constants based on the initial conditions:

$$c_1 = \Gamma_u [2W(0) - P^2(0)]\ ;\ c_2 = \frac{2W(0)}{P^2(0)}\ . \tag{14}$$

If the upper limit for the control signal magnitude, then:

$$|\,u\,| = |\,\Gamma_u WP\,| \le u_{\max}\ . \tag{15}$$

Thus, the ergotropy *W*(*t*) cannot become infinite for any physically reasonable initial conditions *W*(0) and *P*(0).

4. Conclusion

All algorithms mentioned here (Kolesnikov's and Fradkov's versions of feedback and our new proposed speed anti-gradient) are robust, they are stable under the perturbation of the initial conditions and the relatively small external noise. They also can be easily extended for a multi-qubit model.

The choice of the particular form of the feedback depends on the control conditions. In general, the gradient-based algorithms are less energy-consuming, and they could be easily computed in the real time regime. From another side, they are less accurate in the achievement of the goal to compare with target attractor / repeller feedback. Thus, the basic criteria for the choice could be: the computational time cost and the cost of the energy that we need to pump into the system to support the control dynamics.

Additionally we point out that the repeller approach which seems to be more natural in the frame of Kolesnikov's control paradigm, could be also realized for Fradkov's algorithmic approach. Two alternative repeller-based algorithms posses the same pros and cons as their attractor-based analogs.

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Ranking the dimensions and attributes of SERVQUAL model for hotel satisfactory customers in Albania : A fuzzy AHP method.

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Abstract

The quality of service SERVQUAL model is mainly used as a multi-dimensional research instrument for customer satisfaction, and has a direct impact on hotels and their business development in Albania. Keeping tourist satisfied is more important for hotel industry to survive in the competitive market. Customer Satisfaction includes five dimensions named criteria as are: Tangibles, Reliability, Responsiveness, Empathy and Assurance. Each of them has their attributes in total 22 items, named as sub-criteria. The aim of this paper is to find the most important criteria, and their sub-criteria. The survey was developed for 200 tourists from 10 hotels of Tirana in Albania during the year 2018. The Fuzzy AHP (FAHP) method was applied by using a fuzzy conversion of the Saaty scale into Triangular Fuzzy Numbers (TFN) numbers. FAHP is an extension of AHP and shows how the decision maker thinks about using the information to estimate uncertainty in producing decisions under some subjective criteria and their sub-criteria. Due to the data, study results showed that the most important criteria was Responsiveness, the second Empathy, then Reliability, Assurance and the last one was Tangibles. Also the sub-criteria have been ranked related to criteria with FAHP method. These study results help Albania to be more sensitized to tourism in the future.

Keywords: Customer Satisfaction, Fuzzy AHP, Fuzzy Weights, TFN numbers.

1. Introduction

The quality of service has been used to measure customer satisfaction. İt consists of five dimensions or attributes: tangibles, reliability, responsiveness, empathy and assurance. These attributes for customer satisfaction are important challenges for managers and even for researchers to optimize better the success of hotel business and tourism. In this way the reason for evaluating customer satisfaction is to understand what customers require to have and in the other side what managers are able to perform, even something differently. For managers is important to attrack new customer, but also to retain the previous customer. The model of the service quality was developed firstly by A. Parasuraman,Valerie A. Zeithaml, and Len Berry (1),(2). They proposed a scale for measuring SERVQUAL, with ten dimensions which reduced to five. The definitions for these five dimensions were proposed by them (2). Tangibles are defined as the appearance of physical facilities, equipment, personel and communication materials, Reliability represents the ability to perform the promised service dependably and accurately, Responsiveness are the

willingness to help customers and to provide prompt service, Assurance shows the knowledge and courtesy of employees and their ability to convey trust and confidence, Empathy establishes the provision of caring, individualized attention to customer. For all these dimensions they developed a questionnaire with 22 attributes in total. Respectively were 4 to capture tangible, 5 items to capture reliability, 4 items for Responsiveness, 4 items for Assurance and 5 items for Empathy. Also R. Ramsaran-Fowdar (3) developed a service quality questionnaire for the hotel industry in Mauritius.
The model applied by Dell (4) for customer service in the computer industry was "beginning with the customer and ending with the customer" helped him to provide personalized service to its customers. Dell's strategies: "build-to-order"; "direct relationship" and "single point of accountability" helped to orient the entire firm to serve the customer better, Kandampully (5). Efficient and personalized service offered by Southwest employes have been useful to win the heart of its customers Bunz and Maes (6) Rhoades (7). According to the literature review measuring customer satisfaction has many different ways. Yu-Cheng Lee (8) studyed customer satisfaction in different levels of performance by some methods including the decision making problem. Laika Satish (9) treats the types of analyses which can be done for customer satisfaction, with behavioral and preferences of customers variable, that are predictive and descriptive analyse. Dimitris Mourtzis (10) attend the customer feedback in product service system, by using OEM website for creating different groups of customers and different weights for each target group. According to Ali Ahani (11) used a multi-criteria decision making and machine learning to identify the important factors for hotel selections based on previous travelers reviews on Trip Advisor. Robert Kosova et al (24) measured the customer satisfaction with the qualitative analyses for hotel's industry in Durres, Albania. Godolja M. (12) measured hotel guest satisfaction in Tirana, using means and factor analysis. There are many algorithms related to criteria rankings like AHP method. The AHP is used to determine relative priorities on absolute scales from both discrete and continuous cases of the paired comparisons in hierarchic structures developed by Saaty et al (13), (14). The importance measurement has been developed to represent the relative importance of the criteria, known as Saaty Scale (15). Saaty introduced AHP to solve real life problems in decision making, but to deal with uncertainty in complex problems and multi criteria decision making (MCDM), AHP is combined with fuzzy logic set (16) as Fuzzy AHP (FAHP). Mehdi Fasanghari (17) studied the fuzzy evaluation of e-commerce customer satisfaction, with triangular fuzzy numbers. Chih-Hsuan Wang (18) incorporate the customer satisfaction into the decision making process of product konfiguration, by a fuzzy Kano perspective. Medjoudj Rabah (19) evaluate the customer satisfaction and profitability analysis using multi criteria decision making problems. There are a few studies about costumer satisfaction for hotel's industry using the fuzzy AHP model. In this paper the Fuzzy AHP method will convert the Saaty scale of AHP from 1-9, into triangular fuzzy numbers, thus constructing the fuzzy decision matrix and applying the Fuzzy AHP. İn our study we have prepared the full survey based on Godolja (12), (25), and Ramsaran-Fowdar (3). In fact FAHP offers better performance than the AHP method. FAHP also uses hierarchical structures, decomposition and comparison matrices, decreases inconsistency and produces more important eigen vector, according to the

Saaty scale implemented in membership functions used as Fuzzy function. Fuzzy AHP is performed by calculating the value of the Triangular Fuzzy Number (TFN) scale, calculating the value of Fuzzy Priority Synthesis, calculating the minimum value, normalizing and calculating priority weights. In the figure 1 are shown the FAHP steps.

Proposed framework

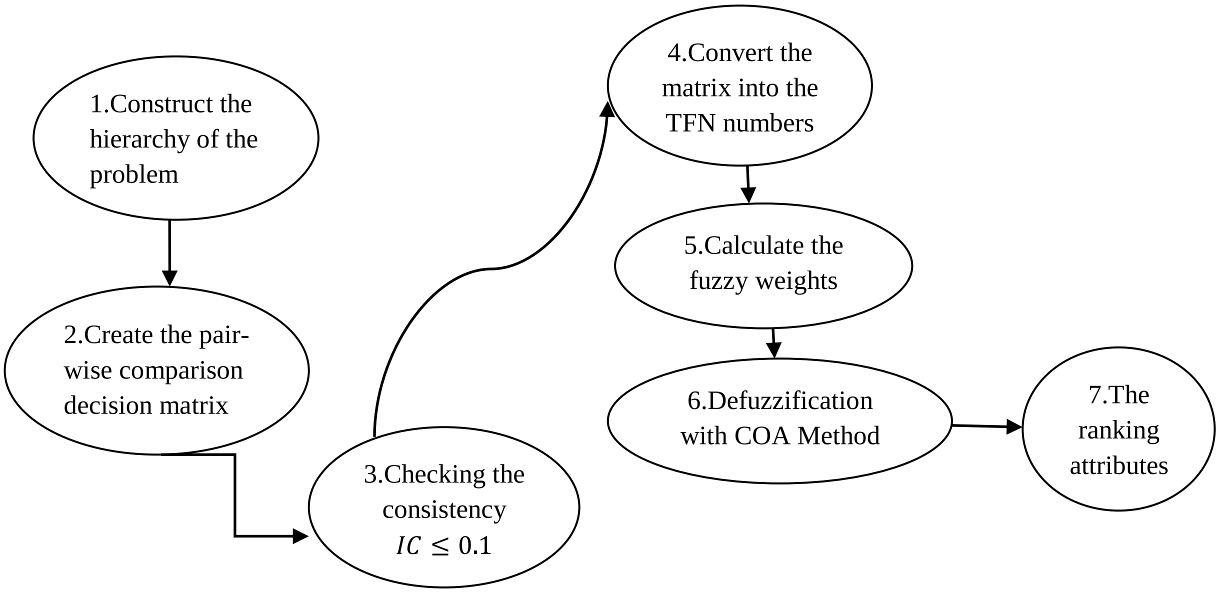

Figure 1. Steps for FAHP

2. Materials and Methods

Data collections

The data in this paper are collected from 200 customers in 10 different hotels in Tirana, by the online survey developed based in Ramsaran-Fowdar (3), Godolja (25). The survey used the 1-7 likert scale to measure expectation and perception of customers, where 1 represent the unsatisfied customer and 7 for very satisfied customer. The hotels were located in different geographical, and central zones of Tirana. Customer satisfaction consists of five main criteria that are: Tangibles, Reliability, Responsiveness, Empathy and Assurance.

Each of them have respectively 4, 5, 4, 4, 5 sub-criteria from A_1 to A_{22}. The full questionnaire is in appendix A1. For all these data firstly we construct the pair-wise comparison matrices with Saaty scale. The methods that are based on the use of pair-wise comparisons represent an important group in Multi Criteria Decision Making Models (MCDM). One of them is the Fuzzy AHP method, mostly applied in decision making problems. The aim of this study is to rank the criteria and their sub-criteria using the Fuzzy AHP. İn figure 2 is shown the hierarchy structure of the survey used to evaluate the customer satisfaction for our data.

Hierarchy structure

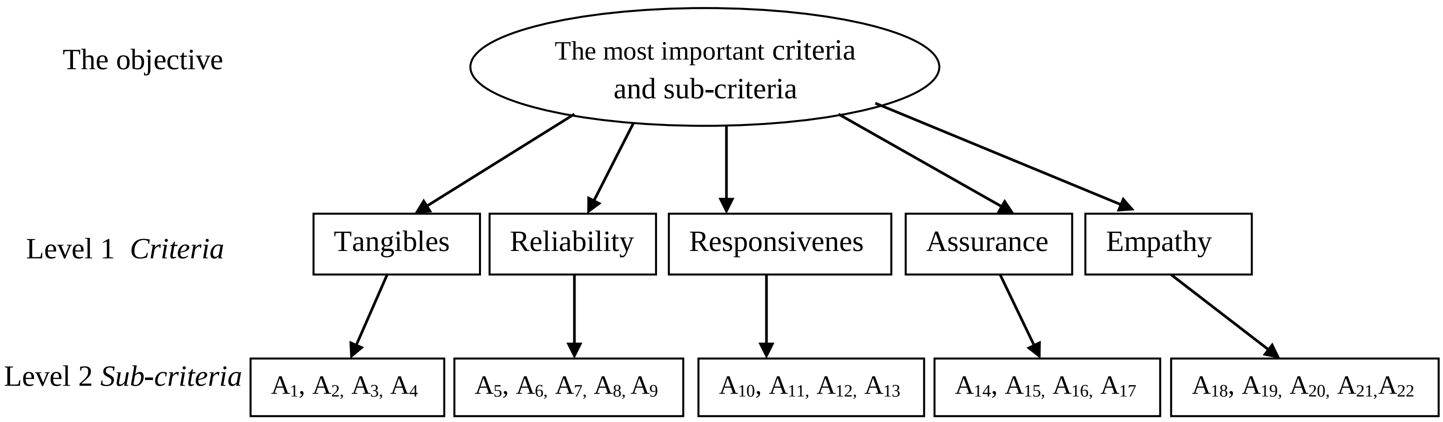

Figure 2. Hierarchy structure of the questionnaire with 22 items

Fuzzy AHP method

Fuzzy AHP is an improvement of AHP method for solving complicated decision problems with criteria and subcriteria. Zadeh (16) introduced the fuzzy sets, and fuzzy membership functions with elements [0,1]. There are different fuzzy membership functions such as: triangular, trapezoidal, s-shape, sigmoid, and Gaussian. In this study we use triangular fuzzy membership function and show how it is adapted to the data (20). A fuzzy number $\breve{\alpha} = (l, m, u)$ is called a triangular fuzzy number (TFN) with lower, medium and upper bounds respectively,if its membership function $\mu_{\breve{\alpha}}(x): R \to [0,1]$ is as follows:

$$\mu_{\breve{\alpha}}(x) = \begin{cases} \frac{x-l}{m-l} & l \le x \le m \\ \frac{u-x}{u-m} & m \le x \le u \\ 0 & x \ne [l, u] \end{cases} \quad (1)$$

The operational laws with Fuzzy numbers based on Gao (21), Nagoor (22):

a) $\breve{\alpha}_1 \oplus \breve{\alpha}_2 = (l_1, m_1, u_1) \oplus (l_2, m_2, u_2) = (l_1 + l_2, m_1 + m_2, u_1 + u_2)$
b) $\breve{\alpha}_1 \otimes \breve{\alpha}_2 = (l_1, m_1, u_1) \otimes (l_2, m_2, u_2) = (l_1 l_2, m_1 m_2, u_1 u_2)$
c) $\breve{\alpha}_1 \oslash \breve{\alpha}_2 = (l_1, m_1, u_1) \oslash (l_2, m_2, u_2) = (l_1/l_2, m_1/m_2, u_1/u_2)$
d) $\breve{\alpha}^{-1} = (l, m, u)^{\ -1} = (\frac{1}{u}, \frac{1}{m}, \frac{1}{l})$

Firstly we construct the decision matrix with Saaty scale, decided from the decision makers experts. The most important thing is that this matrix has to be consistent with index CI less or equal to 0.1

($CI = \frac{\lambda_{max}-n}{n-1} \leq 0.1$). Also is necessary that the consistence ratio $CR = CI/RI \leq 0.1$, where RI is random index of the matrix n-order showed in table 1.

Table 1. Random Index for matrix of n-order, simple AHP (Saaty 1980)

n	1	2	3	4	5	6	7	8	9	10
RI	0.00	0.00	0.58	0.9	1.12	1.24	1.32	1.41	1.45	1.49

In this study all crisp numbers of the Saaty decision matrix are converted into triangular fuzzy numbers as in matrix $\check{A}$ by applying the values of table 2, that indicates the Saaty scale for relative importance.

Table 2. The relative importance values with triangular fuzzy numbers.

Relative (Saaty) importance value	Importance	Fuzzy triangular scale
1	Equal	(1,1,1)
3	Moderate	(2,3,4)
5	Strong	(4,5,6)
7	Very strong	(6,7,8)
9	Extremely strong	(9,9,9)
2	Intermediate values	(1,2,3)
4	Intermediate values	(3,4,5)
6	Intermediate values	(5,6,7)
8	Intermediate values	(7,8,9)

The decision matrix $\check{A} = (\widetilde{a_{ij}})$, $\widetilde{a_{ij}} = (l_{ij}, m_{ij}, u_{ij})$, where $\widetilde{a_{ij}}^{-1} = (\frac{1}{u_{ij}}, \frac{1}{m_{ij}}, \frac{1}{l_{ij}})$ (2)

For each of the criteria is calculated the fuzzy geometric mean value:

$$\check{r}_i = \left(\prod_{i=1}^{n} \widetilde{a_{ij}}\right)^{1/n} \qquad (3)$$

After step (3) becames the defuzzification of fuzzy numbers $\breve{\omega}_i$ with the method "Center of Area" named as COA for the fuzzy weights. According to Voskoglou (23) the coordinates of the Center of Area for the triangular formed with fuzzy numbers are G($\frac{l+m+n}{3}, \frac{1}{3}$). Point G is the intersection of the medians of the triangle formed by fuzzy numbers.

$$\breve{\omega}_i = \check{r}_i \otimes (\check{r}_1 \oplus \check{r}_2 \oplus \ldots \oplus \check{r}_n)^{-1} \qquad (4)$$

The last is the average M_i and the normalized weightsN_i for all the criteria:

$$M_i = \frac{\breve{\omega}_1 \oplus \breve{\omega}_2 \oplus \ldots \oplus \breve{\omega}_n}{n} \quad (5) \qquad\qquad N_i = \frac{M_i}{M_1 \oplus M_2 \oplus \ldots \oplus M_n} \quad (6)$$

3. Results

The decision matrix constructed for the criteria level with AHP method by Saaty scale crisp numbers is in table 3.

Table 3. The decision matrix for Criteria level with Saaty crisp numbers.

Criteria	Tangibles	Reliability	Responsiveness	Empathy	Assurance
Tangibles	1	1/2	1/9	1/5	1/2
Reliability	2	1	1/7	1/4	4
Responsiveness	9	7	1	3	5
Empathy	5	4	1/3	1	2
Assurance	2	1/4	1/5	1/2	1

The consistency index $CI = 0.1$, $RI = 0.089$ so the matrix is consistent. The fuzzy matrix related with the TFN numbers according to equation (2) is in table 4.

Table 4. The decision matrix for Criteria level with Saaty triangular fuzzy numbers.

Criteria	Tangibles	Reliability	Responsiveness	Empathy	Assurance
Tangibles	(1,1,1)	(0.33, 0.5,1)	(0.11,0.11,0.11)	(0.16,0.2,0.25)	(0.33, 0.5,1)
Reliability	(1,2,3)	(1,1,1)	(0.125,0.14,0.16)	(0.2,0.25,0.33)	(3,4,5)
Responsiveness	(9,9,9)	(6,7,8)	(1,1,1)	(2,3,4)	(4,5,6)
Empathy	(4,5,6)	(3,4,5)	(0.25,0.33,0.5)	(1,1,1)	(1,2,3)
Assurance	(1,2,3)	(0.2,0.25,0.33)	(0.16,0.2,0.25)	(0.33, 0.5,1)	(1,1,1)

In table 5 are shown the values of equations (2) till to (6) for FAHP method.

Table 5. The ranked criteria

Criteria	$\breve{r}_i$	$\breve{\omega}_i$	COA	N_i	Rank
Tangibles	(0.28,0.35, 0.48)	(0.03, 0.05, 0.08)	0.053	0.052	5
Reliability	(0.59,0.77, 0.95)	(0.065, 0.1, 0.16)	0.1	0.098	3
Responsiveness	(3.36,3.93 ,4.44)	(0.36, 0.55, 0.75)	0.55	0.54	1
Empathy	(1.24,1.67, 2.14)	(0.13,0.23, 0.36)	0.24	0.23	2
Assurance	(0.4, 0.55, 0.75)	(0.044,0.077,0.12)	0.08	0.078	4

The results show that for Albanian's customer satisfaction the most important criteria is Responsiveness, the second is Empathy, then Reliability, Assurance and the last Tangibles.

For the next level sub-criteria, in the same way are calculated the FAHP equations from (2) to (6) in order to evaluate their importance in accordance of the criteria they apart. The following tables 6, 7, 8, 9, 10 indicate the sub-criteria level evaluations and their rankings.

Table 6. The sub-criteria A_1, A_2, A_3, A_4 ranked with the relative importance for Tangibles criteria.

Sub-Criteria	$\breve{r}_i$	$\breve{\omega}_i$	COA	N_i	Rank
A_1	(0.47, 0.63, 0.83)	(0.07, 0.11, 0.19)	0.123	0.12	3
A_2	(2.45, 3.02, 3.55)	(0.36, 0.54, 0.81)	0.57	0.55	1
A_3	(0.3, 0.36, 0.48)	(0.045,0.064, 0.11)	0.073	0.07	4
A_4	(1.15, 1.41, 1.69)	(0.17, 0.25, 0.38)	0.266	0.26	2

The most preferred sub-criteria for Tangibles criteria is A_2 "The hotel has communication facilities", the second is A_4, then A_1 and the last is A_3.

Table 7. The sub-criteria A_5, A_6, A_7, A_8 ranked with the relative importance for Reliability criteria.

Sub-Criteria	$\breve{r}_i$	$\breve{\omega}_i$	COA	N_i	Rank
A_5	(0.32, 0.41, 0.6)	(0.041, 0.065, 0.144)	0.083	0.075	5
A_6	(1.37, 2.17, 2.29)	(0.17, 0.34, 0.55)	0.353	0.32	1
A_7	(1.21, 1.72, 2.35)	(0.157, 0.27, 0.56)	0.329	0.3	2
A_8	(0.8, 1.14, 1.72)	(0.1, 0.18, 0.41)	0.23	0.2	3
A_9	(0.4, 0.55, 0.78)	(0.052, 0.088, 0.187)	0.109	0.099	4

The most preferred sub-criteria for Reliability criteria is A_6 "When you have a problem, the hotel has dependability to solve it", the second is A_7, then A_8, A_9 and the last is A_5.

Table 8. The sub-criteria A_{10}, A_{11}, A_{12}, A_{13} ranked with the relative importance for Responsiveness criteria.

Sub-Criteria	$\breve{r}_i$	$\breve{\omega}_i$	COA	N_i	Rank
A_{10}	(0.33, 0.42, 0.59)	(0.049,0.075, 0.13)	0.084	0.08	4
A_{11}	(0.61, 0.89, 1.16)	(0.091, 0.16, 0.25)	0.167	0.16	2
A_{12}	(0.57, 0.74, 1)	(0.085, 0.13, 0.22)	0.145	0.14	3
A_{13}	(2.99, 3.49, 3.98)	(0.45, 0.62, 0.87)	0.65	0.62	1

The most preferred sub-criteria for Responsiveness criteria is A_{13}"The hotel employees usually respond to your requests", the second is A_{11}, then A_{12} and the last is A_{10}.

Referring the results in table 9 the most preferred sub-criteria for Assurance criteria is A_{16} "The hotel employees are consistently courteous towards you", the second is A_{14}, then A_{15} and the last is A_{17}.

Table 9. The sub-criteria A_{14}, A_{15}, A_{16}, A_{17} ranked with the relative importance for Assurance criteria.

Sub-Criteria	$\check{r}_i$	$\breve{\omega}_i$	COA	N_i	Rank
A_{14}	(1.27, 1.65, 2.21)	(0.19, 0.297, 0.53)	0.339	0.32	2
A_{15}	(0.74, 0.89, 1.15)	(0.11, 0.16, 0.27)	0.18	0.17	3
A_{16}	(1.86, 2.54, 3.13)	(0.28, 0.45, 0.75)	0.49	0.46	1
A_{17}	(0.23, 0.26, 0.29)	(0.034,0.046,0.069)	0.049	0.046	4

Table 10. The sub-criteria A_{18}, A_{19}, A_{20}, A_{21}, A_{22} ranked with the relative importance for Empathy criteria.

Sub-Criteria	$\check{r}_i$	$\breve{\omega}_i$	COA	N_i	Rank
A_{18}	(0.32, 0.41, 0.6)	(0.038, 0.069, 0.144)	0.083	0.073	5
A_{19}	(1.38, 2.04, 2.85)	(0.16, 0.34, 0.68)	0.39	0.34	1
A_{20}	(1.21, 1.72, 2.35)	(0.145, 0.29, 0.56)	0.33	0.29	2
A_{21}	(0.79, 1.15, 1.72)	(0.094, 0.19, 0.41)	0.23	0.2	3
A_{22}	(0.4, 0.55, 0.78)	(0.048, 0.093, 0.187)	0.1	0.088	4

The most preferred sub-criteria for Empathy criteria is A_{19} "The hotel has operating hours convenient to all its customers", the second is A_{20}, then A_{21}, A_{22} and the last is A_{18}.

4. Conclusions

This paper considers five dimensions of SERVQUAL model:Tangibles, Reliability, Responsiveness, Assurance and Empathy, that have their impact on customer satisfaction for hotel industries in Albania. We developed a survey for SERVQUAL model and named the five dimensions as criteria and their attributes as sub-criteria from A_1 to A_{22}. As study purpose was the evaluation of the most important criteria, and their most important sub-criteria using Fuzzy AHP method. FAHP is an extension of AHP when the fuzziness of the decision maker is considered. The FAHP method for the uncertain or the deviation of the decision maker proved to be a convenient method. The study results showed that the most important criteria was Responsiveness, the second Empathy, then Reliability, Assurance and the last one was Tangibles. As for the sub-criteria the most preferred of Tangible criteria was attribute A_2 "The hotel has communication facilities", for Reliability the most preferred sub-criteria was A_6 "When you have a problem, the hotel has dependability to solve it", according to Responsiveness the most preferred sub-criteria was A_{13}"The hotel employees usually respond to your requests", for Assurance the most preferred sub-criteria was A_{16} "The hotel employees are consistently courteous towards you", and for Empathy the most preferred sub-criteria was A_{19} "The hotel has operating hours convenient to all its customers". According to customer satisfaction related with the hotels of Albania is very important the Responsiveness criteria first of all, and the last remain Tangibles. The explanation for this fact is subjective but we think that Albania needs to be more sensitized to tourism in future. Additionally in further studies we will include more hotels from all over Albania, not only tourists but guests etc, and another Fuzzy AHP method may be applied via α-cuts with a bigger data set from all of the customers.

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Appendix
(Table of the SERVQUAL attributes)

Tangibles (Look sharp)	A_1: The hotel has a good location. A_2: The hotel has communication facilities. A_3: The hotel's employees are well dressed. A_4: Materials associated with the service are visually appealing at the hotel.
Reliability (Just do it)	A_5: When the hotel promises to do something by a certain time, it does so A_6: When you have a problem, the hotel has dependability to solve it. A_7: The hotel has a correct performance of the service at the first time. A_8: The hotel provides its services at the time it promises to do so. A_9: The hotel insists on maintenance for error-free records.
Responsiveness (Do it now)	A_{10}.The hotel employees keep you informed about service perform. A_{11}. The hotel employees provide you prompt service. A_{12}. The hotel employees are always willing to help you. A_{13}. The hotel employees usually respond to your requests.
Assurance (Know what you're doing)	A_{14}: The ability of hotel's staff instills confidence in customers. A_{15}: You feel safe in your transaction with the hotel. A_{16}: The hotel employees are consistently courteous towards you. A_{17}: The hotel has knowledgeable staff to answer your questions.
Empathy (Care about customers as much as the service)	A_{18}: The hotel gives you special attention. A_{19}: The hotel has operating hours convenient to all its customers. A_{20}: The hotel has employees who give you individual attention. A_{21}: The hotel has your best interests at heart. A_{22}: Employees of the hotel understand your special needs.

Relaxed Elastic Lie Quadratics

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Abstract

We introduce the problem of the relaxed elastic Lie quadratic in the Lie algebra of a Lie group endowed with bi-invariant Riemannian metric. Then we characterize relaxed elastic Lie quadratics by the Euler-Lagrange equations with two boundary conditions in Lie algebra $\mathbf{g}$.

Key words: Relaxed elastic Lie quadratics, Lie groups, Lie algebra.

1. Introduction

A relaxed elastic line as defined by Manning in [4] and characterized in [6] is a variational problem which minimizes the total squared curvature $\int_\gamma \kappa^2 ds$ among curves with fixed initial point, direction, and length ℓ, where s, $0 \leq s \leq \ell$, is arc length. So far this variational problem and its generalizations have been analyzed from various perspectives. Especially in [10], authors consider this problem in a Riemannian manifold by using a different method from the previous works, which is known Langer and Singer's approach (see, [3]). The authors derive a differential equation with two boundary conditions for the relaxed elastic line in a Riemannian manifold with constant sectional curvature. Inspired by the characterization of relaxed elastic line in a Riemannian manifold in [10] and studying elastic curve in Lie group in [9], we consider the problem in a Lie group equipped with bi-invariant Riemannian metric. The Lie algebra $\mathbf{g}$ of a Lie group G is defined in the identity element e of G. A curve γ in G corresponds a Lie reduction V in the Lie algebra $\mathbf{g}$ of G. We call a Lie reduction of the critical point of the relaxed elastic energy functional as relaxed elastic Lie quadratic. We consider that this characterization a more geometric and algebraic approach to questions concerning a relaxed elastic problem.

2. Preliminaries

We consider that M is a Riemannian manifold with Riemannian metric <,>, Levi-Civita connection ∇ and Riemannian curvature tensor R. The relaxed elastic line problem aims to minimize the energy

$$F(\gamma) = \frac{1}{2}\int_0^{\ell} \kappa^2 ds \tag{2.1}$$

among smooth curves in M with fixed initial point, direction, and length ℓ. Critical points of the functional (2.1) are characterized by the Euler-Lagrange equation

$$\nabla^3_{\frac{d}{ds}}\dot{\gamma}(s) + R\left(\nabla_{\frac{d}{ds}}\dot{\gamma}(s), \dot{\gamma}(s)\right)\dot{\gamma}(s) - \nabla_{\frac{d}{ds}}\big(\Lambda(s)\dot{\gamma}(s)\big) = 0, \tag{2.2}$$

where

$$\Lambda(s) = -\frac{3}{2}<\nabla_{\frac{d}{ds}}\dot{\gamma}(s), \nabla_{\frac{d}{ds}}\dot{\gamma}(s)> \tag{2.3}$$

with two boundary conditions

$$\kappa(\ell) = 0,\ \kappa_s(\ell) = 0 \tag{2.4}$$

where κ is the geodesic curvature (see, [10]). In the following proposition, we reconsider Eq. (2.2) as solutions of an unconstrained differential equation with initial conditions of a particular form.

Proposition 1 (see, [9]). Any C^∞ curve $\gamma\colon [0,\ell] \longrightarrow M$ is a relaxed elastic line if and only if the Euler Lagrange equation

$$\nabla^3_{\frac{d}{ds}}\dot{\gamma}(s) + R\left(\nabla_{\frac{d}{ds}}\dot{\gamma}(s), \dot{\gamma}(s)\right)\dot{\gamma}(s) + \nabla_{\frac{d}{ds}}\left(\frac{3}{2}\left\|\nabla_{\frac{d}{ds}}\dot{\gamma}(s)\right\|\dot{\gamma}(s)\right) = 0$$

with boundary conditions

$$\left\|\nabla_{\frac{d}{ds}}\dot{\gamma}(\ell)\right\| = 0, \qquad \frac{d}{ds}\left\|\nabla_{\frac{d}{ds}}\dot{\gamma}(s)\right\|\Big|_{s=\ell} = 0$$

for all $s \in [0,\ell]$ and following equalities are satisfied

$$1=\|\dot{\gamma}(s_0)\|$$

$$0=<\nabla_{\frac{d}{ds}}\dot{\gamma}(s)\Big|_{s=s_0}, \nabla_{\frac{d}{ds}}\dot{\gamma}(s_0)>$$

and

$$0=<\nabla^2_{\frac{d}{ds}}\dot{\gamma}(s)\Big|_{s=s_0}, \dot{\gamma}(s_0)>+\left\|\nabla_{\frac{d}{ds}}\dot{\gamma}(s)\Big|_{s=s_0}\right\|^2$$

for some $s_0 \in [0, \ell]$.

Let G be a Lie group endowed with bi-invariant Riemannian metric <,>. If e denotes the identity element of G, then $\mathbf{g}$ =T_eG is the Lie algebra of G. Bi-invariance of the Riemannian metric <,> for all $X, Y, Z \in \mathbf{g}$ is equivalent to the following condition

$$< [X, Y], Z >=< [Z, X], Y >$$

where [,] is the Lie bracket operator. Let $\|.\|$ be the norm corresponding to the restriction of the Riemannian metric <,> to $\mathbf{g}$. For the Lie group G, we have

$$\nabla_X Y = \frac{1}{2}[X, Y]$$

and

$$R(X, Y)Z = \frac{1}{4}[Z, [X, Y]],$$

where ∇ denotes the Levi-Civita connection of G and R is the Riemannian curvature tensor (see, [2, 5]). Given any C^∞ lifting $u: I \to TG$ of a curve $\gamma: I \to G$, $U: I \to \mathbf{g}$ defined by

$$U(t) = \left(dL_{\gamma(t)^{-1}}\right)_{\gamma(t)} u(\mathrm{t})$$

is known the Lie reduction of u, where L_x is the left multiplication by $x \in G$ and $(dL_x)_y$ is the derivative of L_x at $y \in \mathbf{g}$ [7]. Especially the Lie reduction of $\gamma'(t): I \to TG$ (or the curve $\gamma: I \to G$) corresponds to a differentiable curve $V: I \to \mathbf{g}$ defined by

$$V(t) = \left(dL_{\gamma(t)^{-1}}\right)_{\gamma(t)} \gamma'(t). \tag{2.5}$$

(2.5) is also equivalent to the first order differential equation

$$\gamma'(t) = \left(dL_{\gamma(t)}\right)_e V(t),$$

where $\gamma'(t) = \frac{d\gamma(t)}{dt}$. For C^∞ liftings u_0, u_1 and u_2 of γ, we have the following equalities:

$$\left(dL_{\gamma(t)^{-1}}\right)_{\gamma(t)} \nabla_{\frac{d}{dt}} u_0 = \nabla_{\frac{d}{dt}} U_0 + \frac{1}{2}[V, U_0]$$

and

$$\left(dL_{\gamma(t)^{-1}}\right)_{\gamma(t)} R(u_0, u_1)u_2 = \frac{1}{4}[U_2, [U_0, U_1]],$$

where U_0, U_1 and U_2 are Lie reductions of u_0, u_1 and u_2, respectively (see, [7]).

Lemma 1(see, [8, 9]). Let $\gamma: I \to G$ be a differentiable curve. Suppose that the Lie reduction of γ is given by $V: I \to \mathbf{g}$. Then we have for all $s \in I$ in the following equations:

i) $\left(dL_{\gamma(s)^{-1}}\right)_{\gamma(s)} \nabla_{\frac{d}{ds}} \dot{\gamma}(s) = \dot{V}(s)$

ii) $\left(dL_{\gamma(s)^{-1}}\right)_{\gamma(s)}\nabla^2_{\frac{d}{ds}}\dot{\gamma}(s)=\ddot{V}(s)+\frac{1}{2}[V(s),\dot{V}(s)]$

iii) $\left(dL_{\gamma(s)^{-1}}\right)_{\gamma(s)}\nabla^3_{\frac{d}{ds}}\dot{\gamma}(s)=\frac{d^3V(s)}{ds^3}+\left[V(s),\ddot{V}(s)\right]+\frac{1}{4}\left[V(s),\left[V(s),\dot{V}(s)\right]\right]$

iv) $\left(dL_{\gamma(s)^{-1}}\right)_{\gamma(s)}\mathrm{R}\left(\nabla_{\frac{d}{ds}}\dot{\gamma}(s),\dot{\gamma}(s)\right)\dot{\gamma}(s)=-\frac{1}{4}\left[V(s),\left[V(s),\dot{V}(s)\right]\right].$

3. Relaxed Elastic Lie Quadraics

Theorem 1. Any differentiable curve $\gamma: I\to G$ in the Lie group G is called a relaxed elastic line if and only if the curve $V: I\to\mathbf{g}$., which is the Lie reduction of γ, in the Lie algebra g satisfies

$$\ddot{V}(s)=\left[\dot{V}(s),V(s)\right]-\left(<V(s),C>+\left\|\dot{V}(s)\right\|^2\right)V(s)+C, \tag{3.1}$$

$$\|V(s)\|^2=1 \tag{3.2}$$

and

$$\left\|\dot{V}(\ell)\right\|=0,\ \frac{d}{ds}\left\|\dot{V}(s)\right\|\Big|_{s=\ell}=0 \tag{3.3}$$

for some constant $C\in\mathbf{g}$ and all $s\in I$.

Proof. Assume that $\gamma: I\to G$ is a relaxed elastic line in G. Then γ satisfies the Euler-Lagrange equation (2.2). Applying $\left(dL_{\gamma(s)^{-1}}\right)_{\gamma(s)}$ to (2.2) and using Lemma 1, we obtain

$$\frac{d^3V(s)}{ds^3}+\left[V(s),\ddot{V}(s)\right]-\frac{d}{ds}\left(\Lambda(s)\dot{V}(s)\right)=0.$$

Integrating once, we have

$$\frac{d^2V(s)}{ds^2}=\left[\dot{V}(s),V(s)\right]+\Lambda(s)\dot{V}(s)+C \tag{3.4}$$

where $C\in\mathbf{g}$ is a integration constant. On the other hand, we have

$$1=\|\dot{\gamma}(s)\|^2=\left\|\left(dL_{\gamma(s)^{-1}}\right)_{\gamma(s)}\dot{\gamma}(s)\right\|^2=\|V(s)\|^2. \tag{3.5}$$

The first and second derivative of (3.5) are found as follows

$$<\dot{V}(s),V(s)>=0 \tag{3.6}$$

and

$$<\ddot{V}(s),V(s)>+\left\|\dot{V}(s)\right\|^2=0. \tag{3.7}$$

Taking inner product of (3.4) with V(s) and using (3.7), we have

$$\Lambda(s)=-\left\|\dot{V}(s)\right\|^2-<V(s),C>. \tag{3.8}$$

From (3.4) and (3.8), we obtain (3.1). Finally, taking into consider the equations

$$\kappa(s) = \left\|\nabla_{\frac{d}{ds}}\dot{\gamma}(s)\right\| = \left\|\left(dL_{\gamma(s)^{-1}}\right)_{\gamma(s)}\nabla_{\frac{d}{ds}}\dot{\gamma}(s)\right\| = \|\dot{V}(s)\|$$

and (2.4), we obtain the conditions (3.3).

Conversely, let $V: I \to \mathbf{g}$ corresponds the Lie reduction of a curve $\gamma: I \to G$. Suppose that $\|V(s)\|^2 = 1$ and Eq. (3.1) with conditions (3.3) is satisfied. From Lemma 1, we have

$$\|V(s)\|^2 = \left\|\left(dL_{\gamma(s)^{-1}}\right)_{\gamma(s)}\dot{\gamma}(s)\right\|^2 = \|\dot{\gamma}(s)\|^2 = 1.$$

Choosing $-\left(< V(s), C > + \left\|\dot{V}(s)\right\|^2\right) = \Lambda(s)$ and taking derivative of (3.1), we obtain

$$\frac{d^3V(s)}{ds^3} = \left[\ddot{V}(s), V(s)\right] + \frac{d}{ds}\left(\Lambda(s)\dot{V}(s)\right)$$

If we add $\pm\frac{1}{4}\left[V(s), \left[V(s), \dot{V}(s)\right]\right]$ and using Lemma 1, we have for all $s \in I$

$$\nabla^3_{\frac{d}{ds}}\dot{\gamma}(s) + R\left(\nabla_{\frac{d}{ds}}\dot{\gamma}(s), \dot{\gamma}(s)\right)\dot{\gamma}(s) - \nabla_{\frac{d}{ds}}\left(\Lambda(s)\dot{\gamma}(s)\right) = 0$$

since $\left(dL_{\gamma(s)^{-1}}\right)_{\gamma(s)}$ is an isomorphism. On the other hand, by using the equations

$$\left\|\dot{V}(s)\right\| = \left\|\left(dL_{\gamma(s)^{-1}}\right)_{\gamma(s)}\nabla_{\frac{d}{ds}}\dot{\gamma}(s)\right\| = \left\|\nabla_{\frac{d}{ds}}\dot{\gamma}(s)\right\| = \kappa(s)$$

and (3.3), we have (2.4). Therefore, γ is a relaxed elastic line in G.

By similarly with the Lie quadratics of [7-9], we can give the following the definition:

Definition 1. Any curve $V: I \to \mathbf{g}$ satisfying differential equation (3.1) with conditions (3.2) and (3.3) for some $C \in \mathbf{g}$ and all $s \in I$ is called a relaxed elastic Lie quadratic with constant C. Also, V defined by (2.5) is called a relaxed elastic Lie quadratic associated with γ, if γ is a relaxed elastic line.

Corollary 1. Let $V: I \to \mathbf{g}$ be a relaxed elastic Lie quadratic. If we define $Q: I \to \mathbf{g}$ by

$$Q(s) = \ddot{V}(s) + \left(< V(s), C > + \left\|\dot{V}(s)\right\|^2\right)V(s) \tag{3.9}$$

then we have for all $s \in I$

$$\dot{Q}(s) = [Q(s), V(s)] \tag{3.10}$$

and $\|Q(s)\|$ is a constant as follows

$$\sqrt{\left\|\ddot{V}(\ell)\right\|^2 + (2 < V(\ell), \ddot{V}(\ell) > + < V(\ell), C >) < V(\ell), C >}.$$

Proof. Writing (3.9) in (3.1), we obtain

$$Q(s) = \left[\dot{V}(s), V(s)\right] + C. \tag{3.11}$$

Taking derivative of (3.11), we calculate

$$\dot{Q}(s) = \left[\ddot{V}(s), V(s)\right] \tag{3.12}$$

Combining (3.9) and (3.12), we find

$$\dot{Q}(s) = \left[Q(s) - \left(< V(s), C > + \left\|\dot{V}(s)\right\|^2\right) V(s), V(s)\right] = [Q(s), V(s)].$$

Finally, we calculate in the following result

$$\frac{d}{ds}\|Q(s)\|^2 = 2 < \dot{Q}(s), Q(s) >= 2 < [Q(s), V(s)], Q(s) >= 0.$$

This implies that $\|Q(s)\|$ is a constant. Taking into consideration (3. 9) with boundary conditions (3.3), we especially have for $s = \ell$

$$\|Q(\ell)\|^2 = \left\|\ddot{V}(\ell)\right\|^2 + (2 < V(\ell), \ddot{V}(\ell) > + < V(\ell), C >) < V(\ell), C >.$$

Differential equations of the form (3.10) are called Lax equations. The Lax equation (3.10) is crucial to solution of (2.5) for a relaxed elastic line γ in term of its relaxed elastic Lie quadratic V. In [9], authors prove that the differential equation that gives the elastic curve can expand the whole real axis by Picard's theorem and Lax equations. Then by the Theorem 3.1 in [9] and the Proposition 1, all relaxed elastic lines in G extend uniquely to R when G is compact.

4. Relaxed elastic lines in SO(3)

We suppose G=SO(3) which is the group of rotations of Euclidean 3-space. Then the Lie algebra of G is $\mathbf{g}=\mathbf{so}(3)$ which is the set off all skew symmetric real 3×3 matrices. $\mathbf{so}(3)$ is a Lie algebra with the Lie bracket $[A, B] = AB - BA$, $A, B \in \mathbf{so}(3)$ and E^3 is a Lie algebra with the Lie bracket the cross product ×. The map $B: E^3 \to \mathbf{so}(3)$ defined by

$$B(v)w = v \times w$$

is a Lie algebra isomorphism.

Let $\gamma: I \to$SO(3) be a relaxed elastic line and $\tilde{V}: I \to \mathbf{so}(3)$ the associated relaxed elastic Lie quadratic with the constant $\tilde{C}$. Define the inverse function as follows:

$$V = B^{-1}(\tilde{V}): R \to E^3$$

and $C = B^{-1}(\tilde{C})$ for convenience. Since B is a Lie algebra isomorphism and isometry, V satisfies for all $s \in I$

$$\|V(s)\|^2 = \left\|B^{-1}(\tilde{V}(s))\right\|^2 = \left\|\tilde{V}(s)\right\|^2 = 1, \tag{4.1}$$

$$\left\|\dot{V}(\ell)\right\| = 0, \ \frac{d}{ds}\left\|\dot{V}(s)\right\|\Big|_{s=\ell} = 0 \tag{4.2}$$

and from (3.1), we get

$$\ddot{V}(s)=\dot{V}(s)\times \mathrm{V}(s)-\left(< V(s), C > +\left\|\dot{V}(s)\right\|^{2}\right)V(s)+C. \tag{4.3}$$

This implies V is a relaxed elastic Lie quadratic with constant C in the Lie algebra $(E^3,\times)$. We study with V rather than $\tilde{V}$, solving (4.3) with (4.2) and (4.1). So, we can say that for any $A \in \mathrm{SO}(3)$ and $s_0 \in R$, $s \rightarrow A(V(s))$ is a relaxed elastic Lie quadratic in E^3 with constant A(C) and $s \rightarrow V(s-s_0)$ is a relaxed elastic Lie quadratic in $E^3$with constant C by local uniqueness in Picard theorem.

If V is a relaxed elastic Lie quadratic in E^3 with constant $C=0$, then we call that V is null relaxed elastic Lie quadratic. Then Eq. (4.3) reduces to

$$\ddot{V}(s)=\dot{V}(s)\times \mathrm{V}(s)-\left\|\dot{V}(s)\right\|^{2}V(s).$$

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Rough Statistical Convergence of Double Sequences of Fuzzy Numbers

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Abstract

In this study, we examine the concept of rough statistical convergence of double sequences in the fuzzy setting, which generalizes rough convergence of sequences of fuzzy numbers. We define the set of rough statistical limit points of a double sequence of fuzzy numbers and prove some results associated with these notions.

Keywords: Rough convergence, fuzzy number, statistical convergence.

1. Introduction

The notion of fuzzy set was examined by L.A. Zadeh in 1965. The potential of the notion of fuzzy set was realized by different scientific groups and many researchers were interested in further investigation and its applications.

It has been studied in different branches of science, where mathematics has applications. Authors interested in sequence spaces have also applied this concept and defined different classes of sequences of fuzzy real numbers and studied their different properties. The concept of the convergence of a sequence of fuzzy numbers was introduced by Matloka [14], who proved some basic theorems. Later on, several mathematicians, such as Nanda [15], Savaş [20], Tripathy and Debnath [22] and many others have generalized the concept.

The classical analysis is often based on fine behavior, valid for all points of some subsets, even if some distance tends to zero. Since many objects of the material universe and many objects represented by the digital computers cannot satisfy such requirements, the so-called rough analysis was developed as an approach to a rough world. The idea of rough convergence was first introduced by Phu [17], in finite dimensional normed linear spaces. In [17], Phu proved that the set $LIM^r x$ is bounded, closed and convex. Also he investigated the relationship between rough convergence and other types of convergence and the dependence of $LIM^r x$ with respect to the roughness degree r. Later on, Aytar [3] extended this concept and introduced the rough statistical convergence. Recently, in [8], the authors examined the rough convergence of fuzzy numbers based on α-level sets. Also, in [26] rough statistical convergence of sequences of fuzzy numbers was examined. In this paper, we introduce the rough statistical convergence in fuzzy setting and the r-statistical limit set of a double sequence of fuzzy numbers. We note that our results are analogue to those of Phu's [17] and Aytar's [3]. The actual origin of most of our results and proof techniques is in those papers. We actually present those results in generalized form. This increases the interest in finding applications of these concepts.

Throughout the paper r denotes a non-negative real number. The sequence (x_n) in a metric space (X, d) is said to be r-convergent to a point $x_* \in X$, denoted as $x_n \xrightarrow{r} x_*$ if, given $\varepsilon > 0$, there exists a natural number n_0 such that $d(x_n, x_*) < r + \varepsilon$, $\forall n \geq n_0$, and the r - *limit set* of (x_n) is defined as $LIM^r x_n \coloneqq \left\{x_* \in X: x_n \xrightarrow{r} x_*\right\}$. This is the rough convergence with r as roughness degree. For *r = 0*, we get the usual convergence in a metric space. A sequence (x_n) is said to be r-convergent if $LIM^r x_n \neq \emptyset$.

We first recall some basic notions in the theory of fuzzy numbers. We denote by D, the set of all closed and bounded intervals on the real line $\mathbb{R}$, i.e. $D = \{A \subset \mathbb{R}: A = [A_l, A_u]\}$. For $A, B \in D$, we have $A \leq B$, iff $A_l \leq B_l$, $A_u \leq B_u$ and $d = max\{|A_l - B_l|, |A_u - B_u|\}$. Then (D, d) forms a complete metric space.

Definition 1.1. A fuzzy number is a function X from $\mathbb{R}$ to [0; 1], which
satisfies the following conditions:
(i) X is normal.
(ii) X is fuzzy convex.
(iii) X is upper semi-continuous.
(iv) The closure of the set $\{x \in \mathbb{R}: X(x) > 0\}$ is compact.

Properties (i)-(iv) imply that, for each $\alpha \in [0,1]$, the α-level set $X^\alpha = \{x \in \mathbb{R}: X(x) > \alpha\} = [X_l^\alpha, X_u^\alpha]$ is a non-empty compact convex subset of $\mathbb{R}$. The 0-level set is the class of the strong 0-cut, i.e. $cl\{x \in \mathbb{R}: X(x) > \alpha\}$. Let $L(\mathbb{R})$ denote the set of all fuzzy numbers. Define a map on $L(\mathbb{R})$, by $\bar{d}(X, Y) = sup_{\alpha \in [0,1]} d(X^\alpha, Y^\alpha)$. Then $\left(L(\mathbb{R}), \bar{d}\right)$ forms a complete metric space (see [19]).

Definition 1.2. A subset A of $\mathbb{N}$ is said to have density $\delta(A)$ if $\delta(A) = lim_{n\to\infty} \frac{1}{n}|A|$ where $|A|$ denotes the number of elements in A.

Definition 1.3. ([16]). A fuzzy number sequence $X = (X_n)$ is said to be statistically convergent to the fuzzy number X_0, if, for every $\varepsilon > 0$,

$$\delta\left(\left\{n \in \mathbb{N}: \bar{d}(X_n, X_0)\right\}\right) = 0,$$

and X_0 is called the statistical limit of X, written as $st - limX_n = X_0$.

Definition 1.4 ([2]). If $\left(X_{k(j)}\right)$ is a subsequence of $X = (X_n)$ and $K = \{k(j) \in \mathbb{N}: j \in \mathbb{N}\}$, then we abbreviate $\left(X_{k(j)}\right)$ by $(X)_K$, which, in the case $\delta(K) = 0$, is called a subsequence of density zero or a thin subsequence. On the other hand, $(X)_K$ is a nonthin subsequence of X, if $\delta(K) \neq 0$.

Definition 1.5 ([2]). The fuzzy number v is called statistical limit point of sequence of fuzzy number $X = (X_n)$, provided that there is a nonthin subsequence of X that converges to v. Let Λ_X denote the set of statistical limit points of the sequence X.

Definition 1.6 ([2]). The fuzzy number μ is called statistical cluster point of sequence of fuzzy number $X = (X_n)$, if $\delta\left(\left\{n \in \mathbb{N}: \bar{d}(X_n, \mu) < \varepsilon\right\}\right) > 0$ for every $\varepsilon > 0$. Let Γ_X denote the set of statistical cluster points of X.

2. Main Results

Definition 2.1. Let (X_{kl}) be a double sequence of fuzzy numbers in the metric space $\left(L(\mathbb{R}),\bar{d}\right)$ and r be a non negative real number. (X_{kl}) is said to be r-statistically convergent to X_* if, for all $\varepsilon > 0$,

$$\delta\left(\left\{(k,l) \in \mathbb{N} \times \mathbb{N}: \bar{d}(X_{kl}, X_*) \geq r + \varepsilon\right\}\right) = 0.$$

This is the r-statistical convergence with r as roughness degree. For $r = 0$ we get the statistical convergence of double sequences. In this case we denote it by $X_{kl} \xrightarrow{r-S_2} X_*$.

Theorem 2.1. If (X_{kl}) and (Y_{kl}) are two double sequences in $\left(L(\mathbb{R}),\bar{d}\right)$ such that $Y_{kl} \xrightarrow{r_1-S_2} X_*$ and $\delta\left(\left\{(k,l) \in \mathbb{N} \times \mathbb{N}: \bar{d}(X_{kl}, Y_{kl}) \leq r_2\right\}\right) = 1$, then (X_{kl}) is $(r_1 + r_2)$-statistically convergent to X_*, for $r_1 \geq 0, r_2 \geq 0$.

Proof. Since $Y_{kl} \xrightarrow{r_1-S_2} X_*$, we have, for all $\varepsilon > 0$,

$$\delta\left(\left\{(k,l) \in \mathbb{N} \times \mathbb{N}: \bar{d}(Y_{kl}, X_*) \geq r_1 + \varepsilon\right\}\right) = 0,$$

i.e. $\delta(A) = 1$, where $A = \left\{(k,l) \in \mathbb{N} \times \mathbb{N}: \bar{d}(Y_{kl}, X_*) < r_1 + \varepsilon\right\}$.

Let $B = \left\{(k,l) \in \mathbb{N} \times \mathbb{N}: \bar{d}(X_{kl}, Y_{kl}) \leq r_2\right\}$. It is given that $\delta(B) = 1$. Then $\delta(A \cap B) = 1$. So, for all $(k,l) \in A \cap B$, we get

$$\bar{d}(X_{kl}, X_*) \leq \bar{d}(X_{kl}, Y_{kl}) + \bar{d}(Y_{kl}, X_*) < r_1 + r_2 + \varepsilon,$$

for $k \geq k_0, l \geq l_0$. Hence, the proof is complete.

In particular, for $r_1 = 0$ and $r_2 = r > 0$, we get an approximate sequence (X_{kl}) of a statistically convergent sequence $Y_{kl} \to X_*$ with

$$\delta\left(\left\{(k,l) \in \mathbb{N} \times \mathbb{N}: \bar{d}(X_{kl}, Y_{kl}) \leq r\right\}\right) = 1,$$

which is r-statistically convergent to X_*.

Theorem 2.2. If (X_{kl}) and (Y_{kl}) are two double sequences in $\left(L(\mathbb{R}),\bar{d}\right)$ such that $X_{kl} \xrightarrow{r_1-S_2} X_*$ and $Y_{kl} \xrightarrow{r_2-S_2} Y_*$, then $X_{kl} + Y_{kl} \xrightarrow{(r_1+r_2)-S_2} X_* + Y_*$, for $r_1 \geq 0, r_2 \geq 0$.

Proof. Let $X_{kl} \xrightarrow{r_1-S_2} X_*$ and $Y_{kl} \xrightarrow{r_2-S_2} Y_*$. Then, we have $\delta(A_1) = 1$ and $\delta(A_2) = 1$, where $A_1 = \left\{(k,l) \in \mathbb{N} \times \mathbb{N}: \bar{d}(X_{kl}, X_*) < r_1 + \frac{\varepsilon}{2}\right\}$ and $A_2 = \left\{(k,l) \in \mathbb{N} \times \mathbb{N}: \bar{d}(Y_{kl}, Y_*) < r_2 + \frac{\varepsilon}{2}\right\}$. Therefore $\delta(A_1 \cap A_2) = 1$. So, for all such $(k,l) \in A_1 \cap A_2$, we get $\bar{d}(X_{kl} + Y_{kl}, X_* + Y_*) \leq \bar{d}(X_{kl}, X_*) + \bar{d}(Y_{kl}, Y_*) < (r_1 + r_2) + \varepsilon$ for $k \geq k_0, l \geq l_0$. Hence the proof follows.

In view of the existing techniques, we state the following results without proof.

Theorem 2.3. For any $c \in \mathbb{R}$, if $X_{kl} \xrightarrow{r-S_2} X_*$, then $cX_{kl} \xrightarrow{|c|r-S_2} cX_*$.

It is known that the limit of a statistically convergent sequence of fuzzy numbers has an unique limit point. But this property is not maintained in the case of rough statistical convergence with roughness degree $r > 0$. So, in the case of rough statistical convergence we get an *r*-statistical limit set. We discuss some basic properties of the *r*-statistical limit set of a sequence of fuzzy numbers.

Definition 2.2. Let X_* be an *r*-statistical limit point of (X_{mn}), which is not necessarily unique. Consider the *r*-statistical limit set of (X_{mn}), defined by $st_2 - LIM^r X_{mn} = \left\{X_* \in L(\mathbb{R}): X_{mn} \xrightarrow{r-S_2} X_*\right\}$, i.e. $st_2 - LIM^r X_{mn} = \{X_* \in L(\mathbb{R}): [X_*^\alpha] \subseteq [st_2 limsupX_{lmn}^\alpha - r, st_2 liminf X_{umn}^\alpha + r]\}$ where $[X_*^\alpha] = [X_{lmn}^\alpha, X_{umn}^\alpha]$.

If $st_2 - LIM^r X_{mn} = \emptyset$, for any sequence (X_{mn}) of fuzzy numbers, then (X_{mn}) is not *r*-statistically convergent, for any $r \geq 0$.

Proposition 2.1. The set $st_2 - LIM^r X_{kl}$ of an arbitrary sequence of fuzzy numbers (X_{kl}) of $(L(\mathbb{R}), \bar{d})$ is a closed set.

Proof. If $st_2 - LIM^r X_{kl} = \emptyset$, then the hypothesis is true. Assume that $st_2 - LIM^r X_{kl} \neq \emptyset$. Let (Y_{kl}) be a sequence in $st_2 - LIM^r X_{kl}$ which converges to *Y*. We show that $Y \in LIM^r X_{kl}$. Since (Y_{kl}) converges to Y, we have $\bar{d}(Y_{kl}, Y) < \frac{\varepsilon}{2}$, for all $k \geq k_0, l \geq l_0$. In particular, $\bar{d}\left(Y_{k_0 l_0}, Y\right) < \frac{\varepsilon}{2}$ and, by the definition of $st_2 - LIM^r X_{kl}$, we have $\delta\left(\left\{(k,l) \in \mathbb{N} \times \mathbb{N}: \bar{d}\left(X_{kl}, Y_{k_0 l_0}\right) \geq r + \frac{\varepsilon}{2}\right\}\right) = 0$. Now, for all $(k,l) \in \left\{(k,l) \in \mathbb{N} \times \mathbb{N}: \bar{d}\left(X_{kl}, Y_{k_0 l_0}\right) < r + \frac{\varepsilon}{2}\right\}$, we have $\bar{d}(X_{kl}, Y) \leq \bar{d}\left(X_{kl}, Y_{k_0 l_0}\right) + \bar{d}\left(Y_{k_0 l_0}, Y\right) < r + \varepsilon$. So $\left\{(k,l) \in \mathbb{N} \times \mathbb{N}: \bar{d}(X_{kl}, Y) < r + \varepsilon\right\} \supseteq \left\{(k,l) \in \mathbb{N} \times \mathbb{N}: \bar{d}\left(X_{kl}, Y_{k_0 l_0}\right) < r + \frac{\varepsilon}{2}\right\}$, i.e. $\delta\left(\left\{(k,l) \in \mathbb{N} \times \mathbb{N}: \bar{d}(X_{kl}, Y) \geq r + \varepsilon\right\}\right) = 0$, which completes the proof.

Proposition 2.2. For any fuzzy number sequence (X_{kl}) in $(L(\mathbb{R}), \bar{d})$, we have $diam(st_2 - LIM^r X_{kl}) \leq 2r$.

Proof. If possible, let $diam(st_2 - LIM^r X_{kl}) > 2r$. Then, there exist $Y, Z \in st_2 - LIM^r X_{kl}$ such that $\bar{d}(Y, Z) = d_1 > 2r$. Let $\varepsilon = \frac{d_1 - 2r}{2}$. Since $Y, Z \in st_2 - LIM^r X_{kl}$, we have $\delta(K_1) = 1$ and $\delta(K_2) = 1$, where $K_1 = \left\{(k,l) \in \mathbb{N} \times \mathbb{N}: \bar{d}(X_{kl}, Y) \geq r + \varepsilon\right\}$ and $K_2 = \left\{(k,l) \in \mathbb{N} \times \mathbb{N}: \bar{d}(X_{kl}, Z) \geq r + \varepsilon\right\}$. Hence, for all $(k,l) \in (K_1^c \cap K_2^c)$, we have $\bar{d}(Y, Z) \leq \bar{d}(X_{kl}, Y) + \bar{d}(X_{kl}, Z) < 2(r + \varepsilon) = d_1$, for all $k \geq k_0, l \geq l_0$, which is contradiction. Hence $diam(st_2 - LIM^r X_{kl}) \leq 2r$.

Proposition 2.3. Let (X_{kl}) be statistically convergent to X_*. Then $st_2 - LIM^r X_{kl} = \overline{B_r}(X_*)$.

Proof. Since $X_{kl} \xrightarrow{st_2} X_*$, we have $\delta\left(\left\{(k,l) \in \mathbb{N} \times \mathbb{N}: \bar{d}(X_{kl}, X_*) \geq \varepsilon\right\}\right) = 0$. Let $Y \in \overline{B_r}(X_*) = \left\{Y \in L(\mathbb{R}): \bar{d}(Y, X_*) \leq r\right\}$. Now, for all $(k,l) \in \mathbb{N} \times \mathbb{N}$ with $\bar{d}(X_{kl}, X_*) < \varepsilon$, we have $\bar{d}(X_{kl}, Y) \leq \bar{d}(X_{kl}, X_*) + \bar{d}(Y, X_*) < r + \varepsilon$.

Since $\delta\left(\left\{(k,l) \in \mathbb{N} \times \mathbb{N}: \bar{d}(X_{kl}, X_*) < \varepsilon\right\}\right) = 1$ and $\left\{(k,l) \in \mathbb{N} \times \mathbb{N}: \bar{d}(X_{kl}, Y) < r + \varepsilon\right\} \supseteq \left\{(k,l) \in \mathbb{N} \times \mathbb{N}: \bar{d}(X_{kl}, X_*) < \varepsilon\right\}$, we get $Y \in st_2 - LIM^r X_{kl}$. Consequently, we can write $st_2 - LIM^r X_{kl} = \overline{B_r}(X_*)$.

Proposition 2.4. For all $r \geq 0$, a statistically bounded sequence (X_{kl}) of fuzzy numbers always contains a subsequence $\left(X_{k_n l_m}\right)$ with $st_2 - LIM^{(X_{k_n l_m}),r} X_{k_n l_m} \neq \emptyset$.

Proof. It is known that every statistically bounded sequence has a statistically convergent subsequence, so (X_{kl}) contains a statistically convergent subsequence $\left(X_{k_n l_m}\right)$. Let X_* be the limit point of $\left(X_{k_n l_m}\right)$. Then, $st_2 - LIM^r X_{k_n l_m} = \overline{B_r}(X_*)$ and, for *r>0*, $st_2 - LIM^{(X_{k_n l_m}),r} X_{k_n l_m} = \left\{X_{k_n l_m} : \bar{d}\left(X_{k_n l_m}, X_*\right) \leq r\right\} \neq \emptyset$.

Proposition 2.5. If $(X_{kl}{}')$ is a non-thin subsequence of (X_{kl}), then $st_2 - LIM^r X_{kl} \subseteq st_2 - LIM^r X_{kl}{}'$.

Proof. The proof is obvious.

Proposition 2.6. For an arbitrary $C \in \Gamma_X$ of a sequence (X_{kl}), we have $\bar{d}(X_*, \mathrm{C}) \leq r$, for all $X_* \in st_2 - LIM^r X_{kl}$.

Proof. Since $C \in \Gamma_X$, $\delta(A) \neq 0$, where $A = \left\{(k,l) \in \mathbb{N} \times \mathbb{N} : \bar{d}(X_{kl}, C) < \frac{\varepsilon}{2}\right\}$. Let $X_* \in st_2 - LIM^r X_{kl}$. Then, for all $\varepsilon > 0$, $\delta(B) = 1$, where $B = \left\{(k,l) \in \mathbb{N} \times \mathbb{N} : \bar{d}(X_{kl}, X_*) < r + \frac{\varepsilon}{2}\right\}$. Therefore, for all $(k,l) \in A \cap B$, we get $\bar{d}(X_*, C) \leq \bar{d}(X_{kl}, X_*) + \bar{d}(X_{kl}, C) < r + \varepsilon$. Hence the proof follows.

Proposition 2.7. (a) If $C \in \Gamma_X$, then $st_2 - LIM^r X_{kl} \subseteq B_r(C)$.

(b) $st_2 - LIM^r X_{kl} = \cap_{C \in \Gamma_X} \overline{B_r}(C) = \{Y_* \in L(\mathbb{R}) : \Gamma_X \subseteq B_r(Y_*)\}$.

Proof. (a) Let $X_* \in LIM^r X_{kl}$ and $C \in \Gamma_X$. Then $\bar{d}(X_*, C) \leq r$, i.e. $st_2 - LIM^r X_{kl} \subseteq B_r(C)$.

(b) From the above theorem, we get

$$st_2 - LIM^r X_{kl} \subseteq \cap B_r(C). \qquad (1)$$

Let $Y \in \cap_{C \in \Gamma_X} B_r(C)$. Then $\bar{d}(Y, C) \leq r$, for all $C \in \Gamma_X$, which is equivalent to $\Gamma_X \subseteq \overline{B_r}(Y)$, i.e.

$$\cap_{C \in \Gamma_X} B_r(C) \subseteq \{Y_* \in L(\mathbb{R}) : \Gamma_X \subseteq \overline{B_r}(Y_*)\}. \qquad (2)$$

Now, let $Y \notin st_2 - LIM^r X_{kl}$. Then there exists an $\varepsilon > 0$ such that $\delta\left(\left\{(k,l) \in \mathbb{N} \times \mathbb{N} : \bar{d}(X_{kl}, Y) \geq \mathrm{r} + \varepsilon\right\}\right) \neq 0$, which implies the existence of a statistical cluster point *C* of the double sequence *X* that satisfies $\bar{d}(Y, C) > r + \frac{\varepsilon}{2}$, i.e. Γ_X is not a subset of $\overline{B_r}(Y)$ and $Y \notin \{Y_* \in L(\mathbb{R}) : \Gamma_X \subseteq \overline{B_r}(Y_*)\}$.

Therefore,

$$\{Y_* \in L(\mathbb{R}) : \Gamma_X \subseteq \overline{B_r}(Y_*)\} \subseteq st_2 - LIM^r X_{kl}. \qquad (3)$$

The proof follows, by following (1), (2), (3).

Proposition 2.8. Let (X_{kl}) be a statistically bounded sequence of fuzzy numbers. If $r = diam(\Gamma_X)$, then $\Gamma_X \subseteq st_2 - LIM^r X_{kl}$.

Proof. Take $C \notin st_2 - LIM^r X_{kl}$. Then there exists $\varepsilon > 0$ such that $\delta\left(\left\{(k,l) \in \mathbb{N} \times \mathbb{N}: \bar{d}(X_{kl}, C) \geq \mathrm{r} + \varepsilon\right\}\right) \neq 0$. Since the sequence is statistically bounded and in view of the above inequality, there exists another statistical cluster point C' such that $\bar{d}(C', C) > r + \frac{\varepsilon}{2}$. Hence, we get $diam(\Gamma_X) > r + \frac{\varepsilon}{2}$, which is a contradiction. Hence the result follows.

4. Conclusion

The rough convergence have been recently studied by several authors. In view of the recent applications of statistical convergence in the theory of convergence of sequences of fuzzy numbers, it seems very natural to extend the interesting concept of rough statistical convergence of double sequences of fuzzy numbers which we mainly do here and investigate some properties of this new type convergence. So, we have extended some well-known results.

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Γ-semigroups are concrete

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Abstract

Me dhwnien e njw monoidi *M* me grupin e njwsh Γ, mund të përcaktojmë një Γ-gjysmëgrup të qartw (*M*, Γ) ku pwr çdo γ ∈ Γ dhe s, t ∈ M, *γ*-shumwzimi i *s* me *t* wshtw thjwshtw produkti sγt ∈ M. E quajmw (*M*, Γ) Γ-gjysmwgrupi njwsi i *M*. Qëllimi i këtij punimi është të provojë se çdo Γ-gjysmëgrup pwrfshihet në Γ-gjysmëgrupin njësi të një monoidi të caktuar *M* grupi njësi i të cilit ka bashkwsi themelor Γ. Ky fakt tregon natyrwn konkrete tw Γ-gjysmëgrupeve dhe marrëdhënia e tyre tw afwrt me monoidet.

Keywords: monoid, Γ –semigroup, semigroup, group of units.

1. Introduction

If S and Γ are two non empty sets, then every map $\cdot : S \times \Gamma \times S \to S$ will be called a Γ- multiplication in S. The result of this multiplication for $a, b \in S$ and $\gamma \in \Gamma$ is denoted by $a\gamma b$. According to Sen and Saha [6], a Γ-semigroup S is an ordered pair (S, Γ) equipped with a Γ-multiplication · on S which satisfies the following property

$$\forall (a, b, c, \alpha, \beta) \in S^3 \times \Gamma^2, (a\alpha b)\beta c = a\alpha(b\beta c).$$

We give below only one example of Γ-semigroups. Most of the known examples are as artificial as this one and it would be appreciated to find other examples that would become useful in other branches of mathematics and motivate the study of Γ-semigroups. Such a possible brunch of mathematics is the algebraic theory of semigroups.

2. Examples

Let S be the set of $m \times n$ matrices with entries in any field, and Γ the set of $m \times n$ matrices with entries in the same field. For every triple $A \in S$, $G \in \Gamma$ and $B \in S$, we have a $m \times n$ matrix associated with it which is the usual product $A \cdot G \cdot B$. This assignment gives in fact a Γ-semigroup

(S, Γ).
For a Γ-semigroup (S,Γ) and some particular $\gamma \in \Gamma$, one considers the semigroup (S, o_γ) where o_γ is defined by setting $x o_\gamma y = x\gamma y$. It turns out that (S, o_γ) is a semigroup which is sometimes called the laminator of γ. It is well known that if one laminator is a group then so is every laminator.

3. Main Results

The enveloping monoid of a Γ-semigroup

Given a Γ-semigroup S we will define a monoid $\Omega_{\gamma_0}(S,\Gamma)$ where γ_0 is a fixed element of Γ. The construction is very similar to that of [1] except that here we have a unit element. The definition of $\Omega_{\gamma_0}(S,\Gamma)$ uses the fact that we can always define a multiplication $\bullet$ on any non empty set Γ in such a way that $(\Gamma, \bullet)$ becomes a group. This in fact is equivalent to the axiom of choice (see [2]). We write the unit element of $(\Gamma, \bullet)$ by γ_0. To define $\Omega_{\gamma_0}(S,\Gamma)$, we first let $(F, \cdot)$ be the free semigroup on S. Its elements are finite strings $(x_1, \ldots, x_n)$ where each $x_i \in S$ and the product $\cdot$ is the concatenation of words. Now we define $\Omega_{\gamma_0}(S,\Gamma)$ as the quotient semigroup of the free product $F * \Gamma$ of two semigroups $(F, \cdot)$ with $(\Gamma, \bullet)$ by the congruence generated from the set of relations

$$((x, y), x\gamma_0 y), ((x, \gamma, y), x\gamma y), (\gamma_0 x, x), (x\gamma_0, x),$$

for all $x, y \in S, \gamma \in \Gamma$ and with $\gamma_0 \in \Gamma$ being the unit element of $(\Gamma, \bullet)$. We can also regard the group $(\Gamma, \bullet)$ as given by a presentation with generators the elements of Γ, and relations arising from the multiplication table of the group. So a presentation of $\Omega_{\gamma_0}(S,\Gamma)$ has now as a generating set $S \cup \Gamma$, and relations those mentioned above together with those arising from the multiplication table of $(\Gamma, \bullet)$. It is obvious that $\Omega_{\gamma_0}(S,\Gamma)$ becomes thus a monoid with unit element the class of γ_0 modulo the defining relations.

Lemma 2.1 *Every element of $\Omega_{\gamma_0}(S,\Gamma)$ can be represented by an irreducible word which has the form $(\alpha, x, \beta), (\alpha, x), (x, \beta), \gamma$ or x where $x \in S$ and $\alpha, \beta \in \Gamma \setminus \{\gamma_0\}$ and $\gamma \in \Gamma$.*

Proof. First we have to prove that the reduction system arising from the given presen- tation is Noetherian and confluent, and therefore any element of $\Omega_{\gamma_0}(S,\Gamma)$ is given by a unique irreducible word with letters from $S \cup \Gamma$. Secondly, we have to prove that the irreducible words have one of these five forms. So if ω is a word of the form $\omega = (u, x, \gamma, y, v)$ for $\gamma \in \Gamma, x, y \in S$ and u,v possibly empty words, then ω reduces to $\omega' = (u, x\gamma y, v)$. And if $\omega = (u, x, y, v)$, then it reduces to $\omega' = (u, x\gamma_0 y, v)$. In this way we obtain a reduction system which is length reducing and therefore it is Noetherian. To prove that this system is confluent, from Newman's lemma, it is sufficient to prove that it is locally confluent. For this we need to see only the overlapping pairs.

1. $(x, y, z) \to (x\gamma_0 y, z)$ and $(x, y, z) \to (x, y\gamma_0 z)$ which both resolve to $(x\gamma_0 y\gamma_0 z)$

2. $(x,\gamma,y,z) \to (x\gamma y,z)$ and $(x,\gamma,y,z) \to (x,\gamma,y\gamma_0 z)$ which both resolve to $(x\gamma y\gamma_0 z)$
3. $(x,y,\gamma,z) \to (x\gamma_0 y,\gamma,z)$ and $(x,y,\gamma,z) \to (x,y\gamma z)$ which both resolve to$(x\gamma_0 y\gamma z)$
4. $(x,\gamma,y,\gamma',z) \to (x\gamma y,\gamma',z)$ and $(x,\gamma,y,\gamma',z) \to (x,\gamma,y\gamma' z)$ which both resolve to $(x\gamma y\gamma' z)$
5. $(x,\gamma,y,\gamma_0) \to (x\gamma y,\gamma_0,)$ and $(x,\gamma,y,\gamma_0,) \to (x\gamma\, y)$ which both resolve to $(x\gamma y)$
6. $(\gamma_0,x,\gamma,y) \to (x,\gamma,y)$ and $(\gamma_0,x,\gamma,y) \to (\gamma_0,x\gamma\, y)$ which both resolve to $(x\gamma y)$
7. $(\gamma_1,\gamma_2,\gamma_3) \to (\gamma_1 \bullet \gamma_2,\gamma_3)$ and $(\gamma_1,\gamma_2 \bullet \gamma_3)$ which both resolve to $(\gamma_1 \bullet \gamma_2 \bullet \gamma_3)$

To complete the proof we need to show that the irreducible word representing the element of Ω_{γ_0} has one of the five forms stated. If the word which has neither a prefix nor a suffix made entirely of letters from Γ, then it reduces to an element of S by performing the appropriate reductions. If the word has the form (α,ω,α'), (α,ω), or (ω,α'), where ω is a word which has neither a prefix nor a suffix made entirely of letters from Γ, and α,α' have only letters from $\Gamma \setminus \{\gamma_0\}$, then it reduces to an element of one of the first three forms. When either α or α' is γ_0, then it is omitted from the word.∎

We note that there is a map

$$\iota : S \to \Omega_{\gamma_0}(S,\Gamma) \;\; st\; x \mapsto cls(x)$$

where $cls(x)$ is the class of $x \in F$ modulo the defining relations of $\Omega_{\gamma_0}(S,\Gamma)$. This map is injective from lemma 2.1 and this allows us to regard S as a subset of $\Omega_{\gamma_0}(S,\Gamma)$. Furthermore the image $\iota(x\gamma y)$ of the product $x\gamma y$ of any two elements $x,y \in S$ by means of an arbitrary operation $\gamma \in \Gamma$, belongs to $\Omega_{\gamma_0}(S,\Gamma)$. This motivates us to call $\Omega_{\gamma_0}(S,\Gamma)$ the *enveloping monoid of* (S,Γ) *relative to the group* $(\Gamma,\bullet)$. We finally note here that there is a monomorphism of monoids

$$j : \Gamma \to \Omega_{\gamma_0}(S,\Gamma)$$

defined by setting

$$\gamma \to cls(\gamma).$$

The mapping j is clearly a homomorphism, and lemma 2.1 insures that j is injective. So we may now regard the group $(\Gamma,\bullet)$ as a submonoid of $\Omega_{\gamma_0}(S,\Gamma)$ via j.

Γ-semigroups are concrete

Consider first the following general situation. Let $(M,\cdot)$ be a monoid with unit 1 and group of units $(\Gamma,\cdot)$. We can form a Γ-semigroup by setting

$$\cdot : M \times \Gamma \times M \to M \;\; st \;\; x \cdot \gamma \cdot y = x\gamma y,$$

where the product on the right hand side is the product of x, γ and y in $(M,\cdot)$. It is straightforward that we have a Γ-semigroup (M, Γ). We call (M, Γ) *the* Γ-*semigroup of units of* M . The result of the "product" of two elements $x, y \in M$ by some $\gamma \in \Gamma$ is $x\gamma y$ which is an element of M having as a factor an invertible element. To put it in another way, the laminator S_γ gives all the possible ways the unit γ is enveloped in M.

We apply this general situation in the special case when $(M, \cdot)$ is the enveloping monoid $\Omega_{\gamma_0}(S,\Gamma)$ of an arbitrary Γ-semigroup S. More specifically, if S is any Γ- semigroup, $(\Gamma,\bullet)$ is a group on Γ with unit $1 \in \Gamma$, then we have already defined $\Omega_1(S,\Gamma)$ the enveloping monoid of (S, Γ) relative to the group $(\Gamma, \bullet)$. The following holds true.

Lemma 3.1 *The group of units of* $\Omega_1(S,\Gamma)$ *is exactly the group* $(\Gamma, \bullet)$ *regarded as a subgroup of* $\Omega_1(S,\Gamma)$ *via j.*

Proof. It is folklore that the group of units of some monoid is its $\mathcal{H}$-class of 1. So if $w \in \Omega_1(S,\Gamma)$ is a unit, then the quasi-ideal $(w)_q = \Omega_1(S,\Gamma) \cdot w \cap w \cdot \Omega_1(S,\Gamma)$ of $\Omega_1(S,\Gamma)$ generated by w will contain 1, hence there are $R, L \in \Omega_1(S,\Gamma)$ such that $1 = wR = Lw$. It is obvious why w cannot contain as a factor an element of S. It remains that w is just an element of Γ. ■

As we mentioned above, for the monoid $\Omega_1(S,\Gamma)$ and its group of units Γ (ref. lemma 3.1), we may consider the Γ-semigroup of units $(\Omega_1(S,\Gamma),\Gamma)$ of $\Omega_1(S,\Gamma)$ relative to $(\Gamma,\bullet)$. The following relates (S, Γ) with $(\Omega_1(S,\Gamma),\Gamma)$.

Theorem 3.1 (S, Γ) *embeds into* $(\Omega_1(S,\Gamma), \Gamma)$.

Proof. Define

$$\psi : (S,\Gamma) \to (\Omega_1(S,\Gamma),\Gamma) \text{ by } \psi(s) = \iota(s) = cls(s).$$

This mapping is injective since ι is such. Let us prove that ψ is a homomorphism of Γ-semigroups. Indeed, if $s, t \in S$ and $\gamma \in \Gamma$, then

$$\psi(s\gamma t) = cls(s\gamma t) = cls(s)cls(\gamma)cls(t) = \psi(s)j(\gamma)\psi(t) = \psi(s)\gamma\psi(t),$$

since $j(\gamma)$ is identified with γ. ■

Denote by Γ-**Sgrp** the category of Γ-semigroups and Γ-semigroup morphisms. Also we consider the category **Mon** of monoids and monoid morphisms. We give without proof the following.

Proposition 3.1 *Assume we have given to* Γ *the structure of some group* $(\Gamma,\bullet)$. *There is a covariant functor* $F : \Gamma\text{-}\boldsymbol{Sgrp} \to \boldsymbol{Mon}$ *which maps each* (S,Γ) *to* $(\Omega_1(S,\Gamma),\Gamma)$ *and each* Γ-*semigroup morphism* $f: S \to S'$ *to the induced morphism* $F(f) : (\Omega_1(S,\Gamma),\Gamma) \to (\Omega_1(S',\Gamma),\Gamma)$.

Problem 3.1 Is it possible to define an adjoint (right, or left) $F : \boldsymbol{Mon} \to \Gamma - \boldsymbol{Sgrp}$ of F ?

This question is legitimate since as we saw earlier, we can always assign to each monoid M a Γ-

semigroup provided that M has group of units Γ.

4. Conclusion

We relate Γ-semigroups with semigroups by constructing for any Γ-semigroup a monoid with group of units a group with underlying set Γ and than define a Γ-semigroup structure on this monoid. We prove that the latter Γ-semigroup includes the former one. This shows that we can always realize Γ-semigroups as those arising from some monoid and its group of units.

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R^3_1 Semi-Riemannian Manifoldların Birasyonel Kobordizm İnvaryantları Üzerine

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Abstract

Duggal ve Bejancu 1996 da yayınladıkları kitapta bir semi-Riemannian manifoldda lightlike (null) alt uzayın varlığını gösterdiler ve alt manifoldların geometrisi için ihtiyaç duyulan önemli bir boşluğu doldurdular. Semi-Riemannian manifoldlar için unireglelik, kodaira boyutu gibi birasyonel invaryantların yanında maximum lineer bağımsız lightlike vektörlerin sayıları olan k(U) değerlerinin de bir birasyonel invaryant olduğu vurgıulanarak R^3_1 Semi-Riemannian Uzayda 2-Cob Üreteç olan eğik boru kobordizm örneği verilmiş, bunun kodaira boyutu ve k(U) invaryantı ifade edilmiştir.

1. Giriş

Uzun zamandır simplektik geometride birasyonel denkliğin uygun bir kavramının ne olduğu gerçekten açık değildi. Simplektik geometride blow-up/blow-down gibi basit birasyonel operasyonlar biliniyordu[1,2]. Fakat esnek simplektik kategoride genel birasyonel fonksiyon kavramının açık bir genellemesi yoktu. Bu durum son zamanlarda zayıf faktörizasyon teoreminin geliştirilmesiyle büyük ölçüde değişmiştir [3] ki bu teoreme göre projektif manifoldlar arasındaki herhangi bir birasyonel fonksiyon blow-up ve blow-down ların (yukarı ve aşağı etkilerin) bir dizisi halinde ayrıştırılabilir.[4]

Birasyonel geometrinin en temel kavramı unireglelikt. Cebiro-geometrik bir şekilde bunun anlamı bir manifoldun cebirsel eğrilerle kaplanabilmesi demektir. Dikkat etmek gerekir ki, cebirsel geometrideki tanımı basit bir şekilde taklit ederek bu kavramı tanımlamak anlamsızdır [5] ve her noktadan geçen sabit sınıflarda bir simplektik kürenin olması da gerekir. Aksi takdirde her basit bağlantılı manifold uniregle olmalıdır. [4]

Diğer yandan Kollar- Ruan'ın teoremiyle [6,7] bir uniregle projektif manifold, bir nokta arakesitiyle, sıfırdan farklı bir cins sıfır GW-invaryanta sahiptir. Bu nedenle, eğer bir nokta kısıtlaması dahil ederek sıfırdan farklı bir tür sıfır GW-invaryant varsa (M,w) simplektik manifolda uniregledir denir.[4]

Duggal ve Bejancu 1996 da yayınladıkları kitapta (Duggal, Krishan L., Bejancu,A., 1996) bir semi-Riemannian manifoldda lightlike (null) alt uzayın varlığını gösterdiler [8] ve alt manifoldların geometrisi için ihtiyaç, duyulan önemli bir boşluğu doldurdular. Bu kitabin yayınlanmasından sonra hedef, lightlike geometrideki yeni geometrik sonuçların ispatı ve lightlike geometrinin fizikteki uygulamaları oldu. Böylece geometrinin önemli bir boşluğu dolduruldu ve yeni bir çalışma alanı ortaya çıktı..

1942 yılında Moskova Üniversitesinde Lev Pontjagin, Charles Ehresmann sayesinde bir hücre alt bölümünü kullanarak Grassmann manifoldlarının homolojisini çalışmaya başlamıştır. Bu onun yeni önemli bir karakteristik sınıf oluşturmasına olanak sağlamıştır.[11]

1946 yılında Shiing-Shen Chern kompleks vektör demetleri için bir sınıf tanımlamıştır. [12] Chern göstermiştir ki kohomoloji yapısına sahip olan kompleks Grassmann manifoldlarını anlamak, reel Grassmann manifoldlarını anlamaktan daha kolaydır[11]. Chern temel makalesinde Hermitian manifoldları için karakteristik sınıflarının bazı inşaalarını vermiştir. Makale Obstriction teori, Schubert değişkenleri diferansiyel formlar, transgresyonlar vb. arasındaki ilişki için temel oluşturur [11].

2. Temel Tanımlar

Tanım 1: M bir C^{∞} manifold olsun. $p \in M$ noktasındaki tanjant uzay T_pM olmak üzere
$g_p: T_pM \times T_pM \longrightarrow R$
$\left(X_p, Y_p\right) \longrightarrow g_p\left(X_p, Y_p\right)$
biçiminde tanımlı sabit indeksli, simetrik, bilineer, non-degenere $(0,2)$ tensörüne M üzerinde bir metrik tensör denir [15].
Tanım 2: M bir C^{∞} manifold olsun. M bir g metrik tensörü ile donatılmışsa, M ye bir semi-Riemannian manifold denir. [15]
Tanım 3: Bir M Semi-Riemannian manifoldu üzerinde tanımlı g metrik tensörünün indeksine semi-Riemannian manifoldun indeksi denir ve $indM$ ile gösterilir.
Eğer indeks ν ise $0 \leq \nu \leq boyM$ dir. Özel olarak, $\nu = 0$ ise $\forall p \in M$ için g_p, T_pM üzerinde pozitif tanımlı bir iç çarpım olduğundan, M bir Riemannian manifold olur. $\nu = 1$ ve $n \geq 2$ olması durumunda ise, M ye bir Lorentz manifoldu denir [15].
Tanım 4: V sonlu boyutlu reel vektör uzayı, V üzerindeki simetrik bilineer form

$$\gamma: V \times V \to R$$

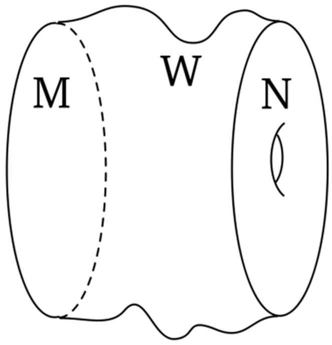
R-bilineer fonksiyonu olsun. V üzerinde tanımlı γ simetrik bilineer formu

- **(i)** $\nu \neq 0$ iken $\gamma(\nu, \nu) > 0$ ise γ pozitif tanımlıdır.
- **(ii)** $\nu \neq 0$ iken $\gamma(\nu, \nu) < 0$ ise γ negatif tanımlıdır.
- **(iii)** $\forall \omega \in V$ iken $\gamma(\nu, \omega) = 0$ şartı sadece $\nu = 0$ için sağlanıyorsa γ ye non-degeneredir denir [15].

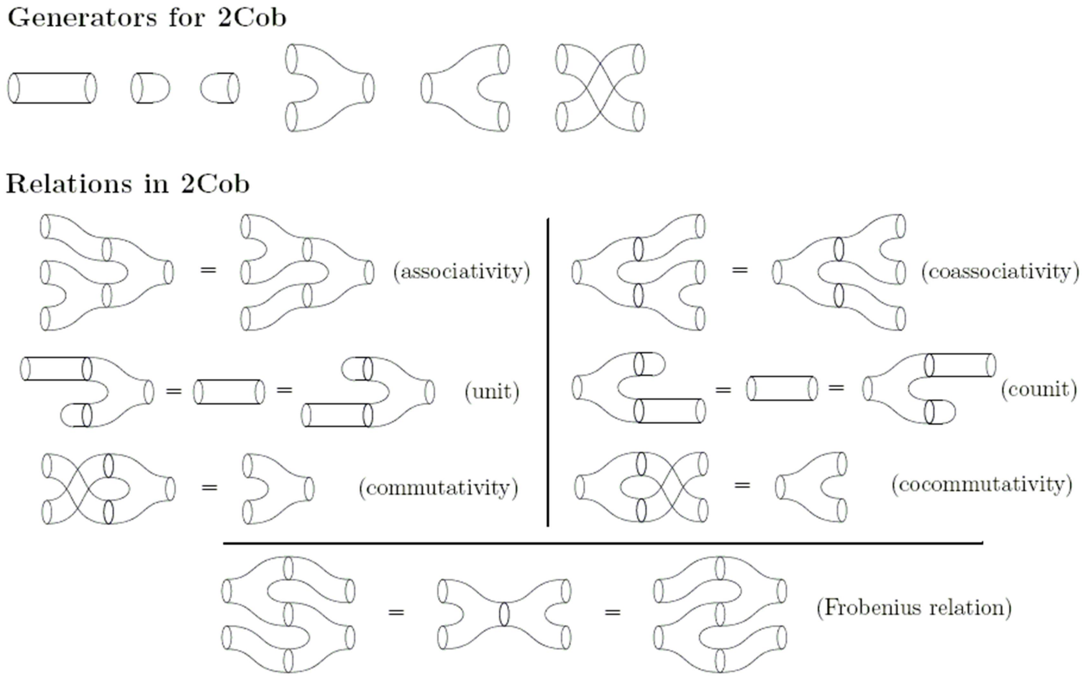
Tanım 5: M bir Semi-Riemannian manifold olsun. $X_p \in T_pM$ olmak üzere,

- **i)** $g_p\left(X_p, X_p\right) > 0$ veya $X_p = 0$ ise X_p vektörüne spacelike,
- **ii)** $g_p\left(X_p, X_p\right) < 0$ ise X_p vektörüne timelike,
- **iii)** $g_p\left(X_p, X_p\right) = 0$, $X_p \neq 0$ ise X_p vektörüne lightlike (null) denir. [15]

Tanım 6: M ve N, n boyutlu iki kompakt manifoldlar olmak üzere, eğer n+1 boyutlu bir W kompakt manifoldu, sınırı M ve N nin ayrık birleşimi olarak yazılabilecek şekilde bulunabilirse M ile N ye kobordant, W ye de M ile N arasında bir kobordizm denir. [20] (bkz. Şekil 1)



Şekil 1: Kobordizm (W; M, N).



Şekil 2: 2 boyutlu kobordizmler için üreteçler

Tanım 7: Bir Semi-Rieamannian manifoldda maximum lineer bağımsız lightlike (null) vektörlerinin sayısına k(U) indexi denir. [21]

Önerme 1:

$$k(U) = 0 \Leftrightarrow \begin{cases} U - spacelike, & boyU = 1 \\ U - spacelike, & boyU > 1 \\ U - timelike, & boyU = 1 \end{cases}$$

[21].

3. Main Results

Cebirsel geometride, Kodaira boyutu κ (X), projektif bir X yapısının kanonik modelinin boyutunu ölçer. Bir cisim üzerinde tanımlı n boyutlu bir X düzgün cebirsel yapısının kanonik demeti, X in kotanjant demetinin n. nci dış kuvveti olan Bir d tamsayısı için K_X in d. nci tensor kuvveti yine bir doğru demetidir.

$$K_X = \bigwedge^{n} \Omega_X^1$$

n-formlarının doğru demetidir. $d \geq 0$ için $H^0(X,K_X{}^d)$ global bölümlerin vector uzayı, X düzgün projektif yapısının birasyonel invaryantı olmasından dolayı olağanüstü özelliğe sahiptir. Yani, bu vektör uzayı, daha düşük boyutlu alt kümelerin dışında X'e izomorfik olan herhangi bir düzgün projektif yapı için karşılık gelen uzay ile kanonik olarak tanımlanır. $d \geq 0$ için X in d.nci P_d çoğul genusu (plurigenus), $K_X{}^d$ nin global bölümlerinin vector uzayının boyutu olarak tanımlanır:

$P_d = dim\, H^0(X,K_X{}^d)$

Buna gore X in kodaira boyutu κ (X),

$$\kappa(X) = \begin{cases} -\infty, & eğer \text{ her } d > 0 \text{ için } P_d = 0 \ \ ise \\ P_d/_{d^k} \ yı\ sınırlı\ yapan\ k\ ların\ minimumu, & diğer\ durumlarda \end{cases}$$

biçiminde tanımlanır.

Böylece n boyutlu bir X projektif yapısının kodaira boyutu ya $-\infty$ dur ya da 0 ile n arasında bir tamsayı değeridir. [25]

Cebirsel geometride K -cismi üzerinde tanımlı bir cebirsel yapıya "ruled" denir, eğer o, bir projektif doğru ile K üzerindeki bazı yapıların çarpımına birasyonel ise. Bir cebirsel yapı "uniruled" dir eğer, o bir rasyonel eğriler ailesi ile kaplanırsa. (Daha kesin bir ifadeyle, X uniruleddir eğer, vardır bir γ ve $\gamma \times P_1 \rightarrow X$ dominant rasyonel map öyle ki, Y ye projeksiyon boyunca etki etmez.)[25].

Karakteristiği 0 (sıfır) olan bir cisim üzerindeki her uniruled yapı, -∞ kodaira boyutuna sahiptir. Tersi, en fazla 3 boyutta bilinen bir varsayımdır: karakteristiği sıfır olan bir cisim üzerindeki cebirsel yapının kodaira boyutu -∞ ise uniruled olmalıdır. Bununla ilgili şu ifade tüm boyutlarda bilinir: Boucksom, Dewailly, Pâun, ve Peternell gösterdi ki, karakteristiği sıfır olan bir cisim üzerindeki bir düzgün projektif X yapısı, uniruleddir. ⇔ X in kanonik demeti pseudo-effektive değildir (Yani, reel sayılarda tensörlendirilmiş Neron-Severi grubundaki, efektif bölenlerle elde edilmiş kapalı konveks koni içinde değil) [22,25]. Çok özel bir durumda karakteristiği sıfır olan bir cisim üzerindeki P^n de derecesi d olan bir düzgün hiperyüzey, uniruleddir ⇔ $d \leq n$ ise, (Aslında P^n deki derecesi $d \leq n$ olan bir düzgün hiperyüzey Fano yapısındadır ve bundan dolayı, (unireglelikten daha güçlü olan) rasyonel bağlantılıdır) [25].

Sayılamayan cebirsei kapalı K- cismi üzerindeki X cebirsel yapısı uniruleddir. ⇔ En az bir rasyonel eğri vardır öyle ki X in her k noktasından geçer. Tersine, bir K- sonlu cisminin cebirsel kapanışı üzerinde, cebirsel yapılar vardır ve bunlar uniruled değildir. Fakat bunların her k noktasından geçen bir rasyonel eğri vardır (p -tek alınmak üzere herhangi bir F_p non-super singüler abelyen yüzeyinin Kummer yapısı bu özelliğe sahiptir.)[23,25]

Uniruledlik bir geometrik özelliktir. Cisim genişlemeleri altında değişmezdir. Oysa ki ruled yapı böyle değildir. Pozitif karakteristikte uniruledlik çok farklı davranır. Özellikle genel tipte uniruled yüzeyler (hatta unirational) vardır. $p \geq 5$ olan p asal sayıları için $\bar{F}_p$ üzerindeki P^3 de $x^{p+1} + y^{p+1} + z^{p+1} + w^{p+1} = 0$ yüzeyi bunun bir örneğidir [24]. Böylece uniruledlik, pozitif karakteristikte kodira boyutununun $-\infty$ olmasını ima etmez. [25]

Kısaca özetlersek;

1 boyutlu eğriler için
Düzgün projektif eğriler, g=0,1,… gibi herhangi bir doğal sayı olan genuslarına göre ayrık olarak sınıflandırılır. Buradaki ayrk olarak sınıflandırma şu anlamdadır: bir genus verildiğinde o genuslu eğrilerin indirgenemez bir moduli uzayı vardır.
Bir X eğrisinin Kodaira bayutu :
$K=-\infty$: genus 0 (projective line P^1): K_X efektif değil, her $d>0$ için $P_d=0$
$K=0$: genus 1(eliptik eğriler): K_X aşikar demet, her $d \geq 0$ için $P_d=1$
$K=1$: genus $g \geq 2$ (genel tipten eğri): K_X geniştir (ample), her $d \geq 2$ için $P_d=(2d-1)(g-1)$
2,3 ya da daha büyük boyutların sınıflandırmaları için bakınız.[25]

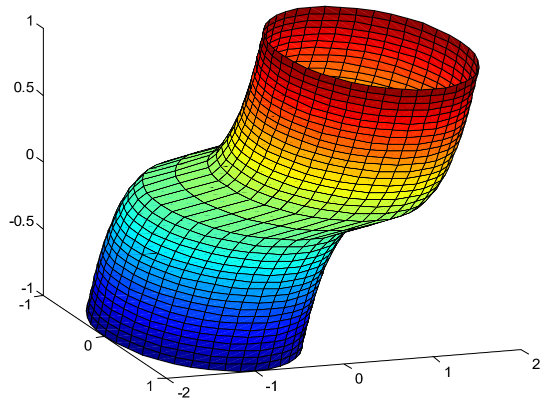
Teorem : Bir Semi-Rieamannian manifoldda maximum lineer bağımsız lightlike (null) vektörlerinin sayısı olan k(U) indexi bir birasyonel invaryanttır.

3. Example

Şimdi bir örnek olarak yine R_1^3 uzayında eğik boru şeklindeki bir kobordizm göz önüne alalım. Bu yüzeyin parametrik denklemi $u \in R$, $v \in [0, 2\pi]$ olmak üzere:

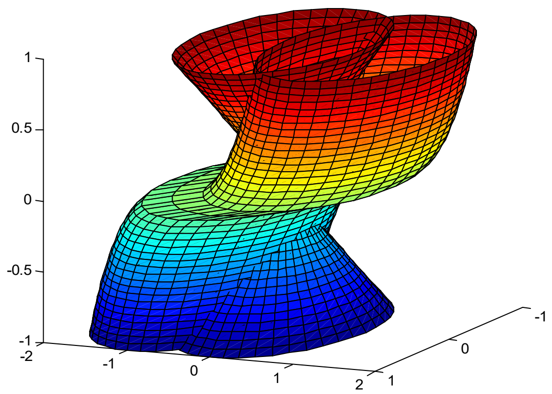
$$X(u,v) = \sin v, \quad Y(u,v) = \sqrt[3]{u} - \cos v, \quad Z(u,v) = u$$

Olur. Şimdi bu yüzeyin üzerindeki lightlike yada ışıksal vektörleri elde edecek olursak bunlar bir hiperbol çifti oluşturacaklardır. Bu çift yukarıdaki kobordizm ile $X^2 + Y^2 - Z^2 = 0$ denklemiyle verilen uzay-zaman konisinin arakesitiyle bulunur.



Şekil 3: Eğik boru şeklindeki kobordizm örneği

Bu eğrileri elde etmek için koninin parametrik denklemi yukarıdaki gibi, $u \in R$, $v \in [0,2\pi]$ olmak üzere parametrik denklem
$X(u,v)=u\cos(v)$; $Y(u,v)=u\sin(v)$; $Z(u,v)=u$;
biçimindedir. O halde arakesit grafiği aşağıdaki gibi olur:



Şekil 4: Eğik boru kobordizmi üzerindeki ışıksal vektörler

Bu hiperbol çifti bir boyutlu manifold olduğundan boyutu 1 dir. O halde kobordizm üzerindeki lightlike yada ışıksal vektörlerin boyutu k(U)=1 olur. Bu da yine (Ören,2007) e göre birasyonel kobordizm invaryantıdır. Benzer şekilde, bu yüzeyler her biri karakteristiği sıfır olan complex cisim C üzerinde olduğundan örnekteki semi-Rieamannian manifoldun bir diğer birasyonel kobordizm invaryantı , kodaira boyutu $-\infty$ dur. Eğer yukarıdaki eğik boru şeklindeki kobordizm yüzeyi tamamen space-like ya da tamamen timelike olsaydı bu durumlarda da yine k(u) invaryantı k(u)=0, kodaira boyutu yine $-\infty$ olurdu

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Structures of Timelike Canal Surfaces using Quasi Frame

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Abstract

In geometry, a canal surface in three dimensional space is a surface formed as the envelope of a family of spheres whose centers lie on a space curve, its directrix. If the radius of the generating spheres are constant the canal surface is called pipe surface. In this study, we introduce three kinds of canal surfaces with respect to quasi frame {t,nq,bq} in Minkowski 3-space. Giving characterizations of timelike canal surfaces by using quasi frame in Minkowski 3 space, some properties of these surfaces are examined by computing first and second fundamental forms. We, also give the some geometric properties of pipe surfaces with the help of Gaussian and mean curvatures of canal surfaces in Minkowski 3-space.

1. Introduction

In geometry, a canal surface in three dimensional space is a surface formed as the envelope of a family of spheres whose centers lie on a space curve, its directrix shown in Fig 1. If the radius of the generating spheres are constant the canal surface is called tube surface shown in Fig 2.

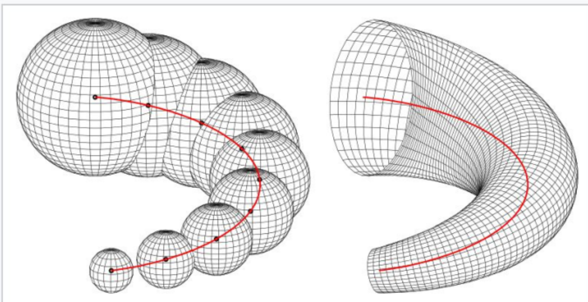

Figure 1: Canal Surface

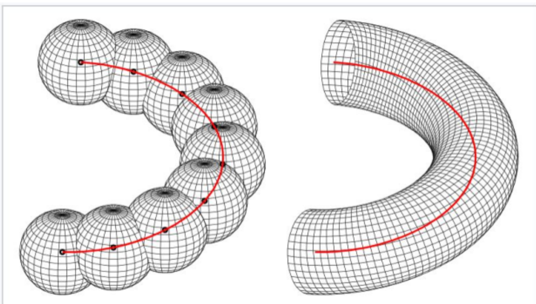

Figure 2: Tube Surface

Surfaces such as canal surfaces, pipe surfaces, ruled surfaces, Bonnet surfaces, evolution surfaces,... are one of the most charming subjects in differential geometry since the application of surfaces to physics and engineering is countless. Among these surfaces, the canal surfaces have been studied by many researchers in various spaces in [1],[3], [9]-[11], [16]-[18]. The tube (pipe) surfaces have been worked by [4], [6]-[8], [13], [14].

In this study, we introduce three kinds of canal surfaces with respect to quasi frame {t,nq,bq} in Minkowski 3-space. Giving characterizations of timelike canal surfaces by using quasi frame in Minkowski 3–space, some properties of these surfaces are examined by computing first and second fundamental forms. We, also give the some geometric properties of pipe surfaces with the help of Gaussian and mean curvatures of canal surfaces in Minkowski 3-space.

2 Preliminaries

In three dimensional Minkowski space $\mathbb{R}_1^3$, the dot and cross products of two vectors $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta = (\beta_1, \beta_2, \beta_3)$ are defined as

$$<\alpha, \beta> = \alpha_1\beta_1 + \alpha_2\beta_2 - \alpha_3\beta_3$$

and

$$\alpha \wedge \beta = (\alpha_3\beta_2 - \alpha_2\beta_3)u_1 + (\alpha_1\beta_3 - \alpha_3\beta_1)u_2 + (\alpha_1\beta_2 - \alpha_2\beta_1)u_3$$

where $u_1 \wedge u_2 = u_3$, $u_2 \wedge u_3 = -u_1$, $u_3 \wedge u_1 = -u_2$, respectively. The norm of the vector α is given by

$$\|\alpha\| = \sqrt{|\langle \alpha, \alpha \rangle|}$$

We say that a Lorentzian vector α is spacelike, lightlike or timelike if $\langle \alpha, \alpha \rangle > 0$ or $\alpha = 0$, $\langle \alpha, \alpha \rangle = 0$ and $\alpha \neq 0$, $\langle \alpha, \alpha \rangle < 0$, respectively [15].

As an alternative to the Frenet frame we use a new adapted frame along a space curve, the quasi frame defined in [5] as

$$\mathbf{t} = \frac{\alpha'}{\|\alpha'\|}, \quad \mathbf{n}_q = \frac{\mathbf{t} \wedge \mathbf{k}}{\|\mathbf{t} \wedge \mathbf{k}\|}, \quad \mathbf{b}_q = \mathbf{t} \wedge \mathbf{n}_q$$

where k is unit projection vector.

Case1: Let $\alpha(s)$ be a spacelike curve with timelike normal vector (n) and spacelike projection vector $\mathbf{k} = (0,1,0)$. It is easy to see that unit tangent vector $\mathbf{t}$ (spacelike), the quasi-normal $\mathbf{n}_q$ (timelike) and the quasi-binormal vector $\mathbf{b}_q$ (spacelike). The variation equations of the q-frame in the following form are given as

$$\begin{bmatrix} \mathbf{t}' \\ \mathbf{n}_q' \\ \mathbf{b}_q' \end{bmatrix} = \begin{bmatrix} 0 & -k_1 & k_2 \\ -k_1 & 0 & k_3 \\ -k_2 & k_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n}_q \\ \mathbf{b}_q \end{bmatrix}.$$

Case2: Let $\alpha(s)$ be a spacelike curve with spacelike normal vector (n) and timelike projection vector $\mathbf{k} = (0,0,1)$. It is easy to see that the unit tangent vector $\mathbf{t}$ (spacelike), the quasi-normal $\mathbf{n}_q$ (spacelike) and the quasi-binormal vector $\mathbf{b}_q$ (timelike). The variation equations of the q-frame in the following form are given as

$$\begin{bmatrix} \mathbf{t}' \\ \mathbf{n}_q' \\ \mathbf{b}_q' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & -k_2 \\ -k_1 & 0 & -k_3 \\ -k_2 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n}_q \\ \mathbf{b}_q \end{bmatrix}.$$

Case3: Let $\alpha(s)$ be a timelike curve with spacelike normal vector (n) and spacelike projection vector $\mathbf{k} = (0,1,0)$. It is easy to see that the unit tangent vector $\mathbf{t}$ (timelike), the quasi-normal $\mathbf{n}_q$ (spacelike) and the quasi-binormal vector $\mathbf{b}_q$ (spacelike). The variation equations of the q-frame in the following form are given as

$$\begin{bmatrix} \mathbf{t}' \\ \mathbf{n}_q' \\ \mathbf{b}_q' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ k_1 & 0 & k_3 \\ k_2 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n}_q \\ \mathbf{b}_q \end{bmatrix}.$$

Let M be a regular surface given with the parametrization $\varphi(s,v)$ in E^3. The tangent space of M at an arbitrary point is spanned by the vectors φ_s and φ_v. The coefficients of the first fundamental form of M are defined in [2], [12] as

$$E = \langle \varphi_s, \varphi_s \rangle, F = \langle \varphi_s, \varphi_v \rangle, G = \langle \varphi_v, \varphi_v \rangle,$$

where $\langle , \rangle$ is the Euclidean inner product. Then the unit normal vector field of M is defined as

$$N = \frac{\varphi_s \wedge \varphi_v}{||\varphi_s \wedge \varphi_v||}.$$

The coefficients of the second fundamental form of M are defined as

$$e = \langle \varphi_{ss}, N \rangle, f = \langle \varphi_{sv}, N \rangle, g = \langle \varphi_{vv}, N \rangle.$$

The Gaussian curvature and the mean curvature of M are given by

$$K = \frac{eg - f^2}{EG - F^2}$$

and

$$H = \frac{Eg + Ge - 2Ff}{2(EG - F^2)},$$

respectively.

3 On The Timelike Directional Canal Surfaces

3.1 Timelike Directional Canal Surfaces around a Spacelike Curve

Case1. Timelike directional canal surfaces, centered at a spacelike curve $\alpha(s)$ with timelike normal vector (n) and spacelike projection vector $\mathbf{k} = (0,1,0)$ and radius $r(s) = r$ can be parametrized as follows

$$\phi(s,v) = \alpha(s) + r\big(\sinh(v)\mathbf{n_q} + \cosh(v)\mathbf{b_q}\big).$$

where $\{\mathbf{t}, \mathbf{n}_q, \mathbf{b}_q, \mathbf{k}\}$ is the q-frame of the spacelike curve $\alpha(s)$. The partial derivatives of $\phi(s,v)$, with respect to s, and v are determined by respectively

$$\phi_s(s,v) = \big(1 - r(k_1 \sinh(v) + k_2 \cosh(v))\big)\mathbf{t} + (r' \sinh(v) + rk_3 \cosh(v))\mathbf{n_q} \\ +(r'\cosh(v) + rk_3\sinh(v))\mathbf{b_q}. \tag{1}$$

$$\phi_v(s,v) = r\cosh(v)\mathbf{n_q} + r\sinh(v)\mathbf{b_q} \tag{2}$$

From the equations (1) and (2), the coefficients of the first fundamental form are given as

$$E = (1 - r(k_1\sinh(v) + k_2\cosh(v)))^2 + r'^2 - r^2k_3^2$$
$$F = -r^2k_3$$
$$G = -r^2$$

The unit normal vector is written as follows

$$N = \pm \frac{1}{\sqrt{r'^2 + W^2}} (r', -\sinh(v)W, -\cosh(v)W)$$

where $W = 1 - r(k_1 \sinh(v) + k_2 \cosh(v))$.

The second partial derivatives of $\phi_s(s,v)$, with respect to s and v, are determined by

$$\begin{aligned}\phi_{ss}(s,v) = &\big(-2r'(k_1 \sinh(v) + k_2 \cosh(v)) - rk_3(k_1 \cosh(v) + k_2 \sinh(v))\big)\mathbf{t} \\ &+\big((r'' + rk_1^2 + rk_3^2) \sinh(v) + (2r'k_3 + rk_1k_2) \cosh(v) - k_1\big)\mathbf{n_q} \\ &+((r'' - rk_2^2 + rk_3^2)\cosh(v) + (2r'k_3 - rk_1k_2)\sinh(v) + k_2)\mathbf{b_q}.\end{aligned}$$

$$\begin{aligned}\phi_{sv}(s,v) = &-r(k_1 \cosh(v) + k_2 \sinh(v))\mathbf{t} + (r' \cosh(v) + rk_3 \sinh(v))\mathbf{n_q} \\ &+(r'\sinh(v) + rk_3\cosh(v))\mathbf{b_q}\end{aligned}$$

$$\phi_{vv}(s,v) = r(\sinh(v)\mathbf{n_q} + \cosh(v)\mathbf{b_q})$$

The coefficients of second fundamental form are given as

$$\begin{aligned}e = \pm \frac{1}{\sqrt{r'^2 + W^2}} &[-2r'^2(k_1 \sinh(v) + k_2 \cosh(v)) - r'rk_3(k_1 \cosh(v) + k_2 \sinh(v)) \\ &-(r'' + rk_3^2 + W(k_1 \sinh(v) + k_2 \cosh(v)))W]\end{aligned}$$

$$f = \pm \frac{1}{\sqrt{r'^2 + W^2}} [-rr'(k_1 \cosh(v) + k_2 \sinh(v) - rk_3W]$$

$$g = \pm \frac{-rW}{\sqrt{r'^2 + W^2}}$$

Gauss and mean curvatures are calculated by

$$K = \frac{rW[(AW + rk_3^2 + r'')W + rr'A'k_3 + 2r'^2A] - r^2(r'A' - k_3W)^2}{-r^2(r'^2 + W^2)^2}$$

and

$$H = \pm \frac{-rW^3 + W^2r^2A + W(r'^2 - rr'') - 2rr'^2A + r^2r'k_3A'}{-2r(r'^2 + W^2)^{\frac{3}{2}}}$$

where $A = k_1 \sinh(v) + k_2 \cosh(v)$, $A' = \frac{dA}{dv}$ respectively.

When the radius of the generating spheres of this surface are constant, we get following theorems found in [7].

Theorem1. The Gauss and mean curvatures of the tube surface around a spacelike curve with timelike normal vector are given as

$$K = -\frac{k_1 \sinh(v) + k_2 \cosh(v)}{r(1 - (k_1 \sinh(v) + k_2 \cosh(v))}$$

and

$$H = \pm \frac{2r(k_1 \sinh(v) + k_2 \cosh(v)) - 1}{r(1 - (k_1 \sinh(v) + k_2 \cosh(v))^2}$$

respectively.

Corollary1. The tube surfaces are flat if and only if

$$r = \frac{k_1}{k_2} = -\coth(v).$$

Case 2. Timelike directional canal surfaces, centered at a spacelike curve $\alpha(s)$ with spacelike normal vector (n) and timelike projection vector $\mathbf{k} = (0,0,1)$ and radius $r(s) = r$ can be parametrized as follows

$$\bar{\phi}(s,v) = \alpha(s) + r\big(\cosh(v)\mathbf{n_q} - \sinh(v)\mathbf{b_q}\big).$$

where $\{\mathbf{t}, \mathbf{n}_q, \mathbf{b}_q, \mathbf{k}\}$ is the q-frame of the spacelike curve $\alpha(s)$. The partial derivatives of $\bar{\phi}(s,v)$, with respect to s, and v are determined by respectively

$$\begin{aligned}\bar{\phi}_s(s,v) = {} & \big(1 - r(k_1\cosh(v) - k_2\sinh(v))\big)\mathbf{t} + (r'\cosh(v) + rk_3\sinh(v))\mathbf{n_q} \\ & -(r'\sinh(v) + rk_3\cosh(v))\mathbf{b_q}. \end{aligned} \tag{3}$$

$$\bar{\phi}_v(s,v) = r(\sinh(v)\mathbf{n_q} - \cosh(v))\mathbf{b_q} \tag{4}$$

From the equations (3) and (4), the coefficients of the first fundamental form are given as

$$E = (1 - r(k_1\cosh(v) - k_2\sinh(v)))^2 + r'^2 - r^2k_3^2$$

$$F = -r^2k_3$$

$$G = -r^2$$

The unit normal vector is written as follows

$$N = \frac{1}{\sqrt{r'^2 + \bar{W}^2}}(r', -\cosh(v)\bar{W}, \sinh(v)\bar{W})$$

where $\bar{W} = 1 - r(k_1\cosh(v) - k_2\sinh(v))$.

The second partial derivatives of $\bar{\phi}(s,v)$, with respect to s and v, are determined by

$$\begin{aligned}\bar{\phi}_{ss}(s,v) = {} & \big(-2r'(k_1\cosh(v) - k_2\sinh(v)) + rk_3(k_1\sinh(v) - k_2\cosh(v))\big)\mathbf{t} \\ & +\big((r'' - rk_1^2 + rk_3^2)\cosh(v) + (2r'k_3 + rk_1k_2)\sinh(v) + k_1\big)\mathbf{n_q} \\ & -((r'' + rk_2^2 + rk_3^2)\sinh(v) + (2r'k_3 - rk_1k_2)\cosh(v) + k_2)\mathbf{b_q}.\end{aligned}$$

$$\begin{aligned}\bar{\phi}_{sv}(s,v) = {} & -r(k_1\sinh(v) - k_2\cosh(v))\mathbf{t} + (r'\sinh(v) + rk_3\cosh(v))\mathbf{n_q} \\ & -(r'\cosh(v) + rk_3\sinh(v))\mathbf{b_q}\end{aligned}$$

$$\bar{\phi}_{vv}(s,v) = r(\cosh(v)\mathbf{n_q} - \sinh(v)\mathbf{b_q})$$

The coefficients of second fundamental form are given as

$$\begin{aligned}e = {} & \frac{1}{\sqrt{r'^2 + \bar{W}^2}}((-2r'^2 - W^2)(k_1\cosh(v) - k_2\sinh(v)) - r'rk_3(k_1\sinh(v)) \\ & -k_2\cosh(v)) - (r'' + rk_3^2)\bar{W})\end{aligned}$$

$$f = \frac{1}{\sqrt{r'^2 + \bar{W}^2}}(-rr'(k_1\sinh(v) - k_2\cosh(v)) - rk_3\bar{W})$$

$$g = \frac{-r\bar{W}}{\sqrt{r'^2 + \bar{W}^2}}$$

Gauss and mean curvatures are calculated by

$$K=\frac{\bar{W}(B(2rr'^2+r\bar{W}^2)-r^2B'r'k_3)+rr''\bar{W})-rr'^2B'^2}{-r^2(r'^2+\bar{W}^2)^2}$$

and

$$H=\frac{\bar{W}(rr''-rr'^2-r\bar{W}^2+r^2B\bar{W})+2r'^2r^2B-r^3r'k_3B'}{-2r^2(r'^2+\bar{W}^2)^{\frac{3}{2}}}$$

where $B=k_1\sinh(v)-k_2\cosh(v),\ B'=\frac{dB}{dv}$ respectively.

When the radius of the generating spheres of this surface are constant, we have following theorems found in [7].

Theorem2. The Gauss and mean curvatures of the tube surface around a spacelike curve with spacelike normal vector are given as

$$K=-\frac{k_1\cosh(v)-k_2\sinh(v)}{r(1-r(k_1\cosh(v)-k_2\sinh(v)))}$$

and

$$H=\pm\frac{1-2r(k_1\cosh(v)-k_2\sinh(v))}{2r(1-r(k_1\cosh(v)-k_2\sinh(v)))}$$

respectively.

Corollary2. The tube surfaces are flat if the following equation satisfies

$$\frac{k_1}{k_2}=\tanh(v).$$

3.2 Timelike Directional Canal Surfaces around a Timelike Curve

Case 3. Timelike directional canal surfaces, centered at a timelike curve $\alpha(s)$ with spacelike normal vector (n) and spacelike projection vector $\mathbf{k}=(0,1,0)$ and radius $r(s)=r$ can be parametrized as follows

$$\bar{\bar{\phi}}(s,v)=\alpha(s)+r\big(\cos(v)\mathbf{n_q}+\sin(v)\mathbf{b_q}\big).$$

where $\{\mathbf{t},\mathbf{n}_q,\mathbf{b}_q,\mathbf{k}\}$ is the q-frame of the timelike curve $\alpha(s)$. The partial derivatives of $\bar{\bar{\phi}}(s,v)$, with respect to s, and v are determined by respectively

$$\bar{\bar{\phi}}_s(s,v)=\big(1+r(k_1\cos(v)+k_2\sin(v))\big)\mathbf{t}+(r'\cos(v)-rk_3\sin(v))\mathbf{n_q}$$
$$+(r'\sin(v)+rk_3\cos(v))\mathbf{b_q}.$$

$$\bar{\bar{\phi}}_v(s,v)=-r\sin(v)\mathbf{n_q}+r\cos(v)\mathbf{b_q}$$

The coefficients of the first fundamental form are given as

$$E=r'^2+r^2k_3^2-(1+r(k_1\cos(v)+k_2\sin(v)))^2$$

$$F=r^2k_3$$

$$G=r^2$$

The unit normal vector is written as follows

$$N=\frac{1}{\sqrt{\bar{\bar{W}}^2-r'^2}}(r',\cos(v)\bar{\bar{W}},\sin(v)\bar{\bar{W}})$$

where $\bar{\bar{W}} = 1 + r(k_1\cos(v) + k_2\sin(v))$.
The second partial derivatives of $\bar{\bar{\phi}}(s,v)$, with respect to s and v, are determined by

$$\begin{aligned}\bar{\bar{\phi}}_{ss}(s,v) = &\big(2r'(k_1\cos(v) + k_2\sin(v)) + rk_3(k_2\cos(v) - k_1\sin(v))\big)\mathbf{t}\\ &+\left((r'' - rk_3^2)\cos(v) - 2r'k_3\sin(v) + k_1\bar{\bar{W}}\right)\mathbf{n_q}\\ &+\big((r'' - rk_3^2)\sin(v) + 2r'k_3\cos(v) + k_2\bar{\bar{W}}\big)\mathbf{b_q}.\end{aligned}$$

$$\begin{aligned}\bar{\bar{\phi}}_{sv}(s,v) = &-r(k_1\sin(v) - k_2\cos(v))\mathbf{t} - (r'\sin(v) + rk_3\cos(v))\mathbf{n_q}\\ &+(r'\cos(v) - rk_3\sin(v))\mathbf{b_q}\end{aligned}$$

$$\bar{\bar{\phi}}_{vv}(s,v) = -r\big(\cos(v)\mathbf{n_q} + \sin(v)\mathbf{b_q}\big).$$

The coefficients of second fundamental form are given as

$$\begin{aligned}e = \frac{1}{\sqrt{\bar{\bar{W}}^2 - r'^2}}&(-2r'^2(k_1\cos(v) + k_2\sin(v)) - r'rk_3(k_2\cos(v) - k_1\sin(v))\bar{\bar{W}}(r''\\ &-rk_3^2 + (k_1\cos(v) + k_2\sin(v))\bar{\bar{W}}))\end{aligned}$$

$$f = \frac{-r}{\sqrt{\bar{\bar{W}}^2 - r'^2}}[r'(k_2\cos(v) - k_1\sin(v)) + k_3\bar{\bar{W}}]$$

$$g = \frac{-r\bar{\bar{W}}}{\sqrt{\bar{\bar{W}}^2 - r'^2}}$$

Gauss and mean curvatures are calculated by

$$K = \frac{\bar{\bar{W}}(C(2r'^2 - \bar{\bar{W}}^2) - rr'k_3C' - r''\bar{\bar{W}}) - rr'^2C'^2}{r(\bar{\bar{W}}^2 - r'^2)^2}$$

and

$$H = \frac{\bar{\bar{W}}(rr'' - r'^2 + \bar{\bar{W}}^2 + rC\bar{\bar{W}}) - 2r'^2rC + r^2r'k_3C'}{2r(\bar{\bar{W}}^2 - r'^2)^{\frac{3}{2}}}$$

where $C = k_1\cos(v) + k_2\sin(v)$, $C' = \frac{dC}{dv}$, respectively.

When the radius of the generating spheres of this surface are constant, we have proved the theorems found in [7].

Theorem3. The Gauss and mean curvatures of the tube surface around a timelike curve are given as

$$K = -\frac{k_1\cos(v) + k_2\sin(v)}{r(1 + r(k_1\cos(v) + k_2\sin(v)))}$$

and

$$H = \frac{1 + 2r(k_1\cos(v) + k_2\sin(v))}{2r(1 + r(k_1\cos(v) + k_2\sin(v)))}$$

respectively.

Corollary3. The tube surfaces are flat if the following equation satisfies

$$\frac{k_1}{k_2} = -\tan(v).$$

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Sum Formulas for Gaussian Generalized Tribonacci Numbers: Closed Form Formulas of the Sums $\sum_{k=0}^{n} GW_k$ and $\sum_{k=1}^{n} GW_{-k}$

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Abstract

In this paper, closed forms of the sum formulas $\sum_{k=0}^{n} GW_k$ and $\sum_{k=1}^{n} GW_{-k}$ of Gaussian generalized Tribonacci numbers are presented. Consequently, some previous results are recovered as particular cases of the present results. As special cases, we give summation formulas of Gaussian Tribonacci, Gaussian Tribonacci-Lucas, Gaussian Padovan, Gaussian Perrin, Gaussian Narayana and some other third order linear recurrance sequences. All the summing formulas of well known recurrence sequences are linear except the cases Gaussian Pell-Padovan and Gaussian Padovan-Perrin.

1. Introduction

In this work, we investigate linear summation formulas of Gaussian generalized Tribonacci numbers. Sequences have been fascinating topic for mathematicians for centuries. The Fibonacci sequence is very well-known example of second order recurrence sequences.

The sequence of Fibonacci numbers $\{F_n\}$ is defined by

$$F_n = F_{n-1} + F_{n-2}, n \geq 2, F_0 = 0, F_1 = 1.$$

The Fibonacci numbers and their generalizations have many interesting properties and applications to almost every field. The generalized Tribonacci sequence $\{W_n(W_0, W_1, W_2; r, s, t)\}_{n\geq 0}$ (or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3}, W_0 = a, W_1 = b, W_2 = c, n \geq 3 \qquad (1.1)$$

where W_0, W_1, W_2 are arbitrary complex numbers and r, s, t are real numbers.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{s}{t}W_{-(n-1)} - \frac{r}{t}W_{-(n-2)} + \frac{1}{t}W_{(-(n-3))}$$

for $n = 1,2,3, \ldots$ when $t \neq 0$. Therefore, recurrence (1.1) holds for all integer n.

If we set $r = s = t = 1$ and $W_0 = 0, W_1 = 1, W_2 = 1$ then $\{W_n\}_{n\geq 0}$ is the well-known Tribonacci sequence and if we set $r = s = t = 1$ and $W_0 = 3, W_1 = 1, W_2 = 3$ then $\{W_n\}_{n\geq 0}$ is the well-known Tribonacci-Lucas sequence.

In fact, the generalized Tribonacci sequence is the generalization of the well-known sequences like Tribonacci, Tribonacci-Lucas, Padovan (Cordonnier), Perrin, Padovan-Perrin, Narayana, third order Jacobsthal and third order Jacobsthal-Lucas.

We now present some background about Gaussian and Gaussian generalized Tribonacci numbers. In literature, there have been so many studies of the sequences of Gaussian numbers. A Gaussian integer

z is a complex number whose real and imaginary parts are both integers, i.e., $= a + ib, a, b \in \mathbb{Z}$. These numbers is denoted by $\mathbb{Z}[i]$. The norm of a Gaussian integer $a + ib, a, b \in \mathbb{Z}$ is its Euclidean norm, that is, $N(a + ib) = \sqrt{a^2 + b^2} = \sqrt{(a + ib)(a - ib)}$. For more information about this kind of integers, see the work of Fraleigh [3].

If we use together sequences of integers defined recursively and Gaussian type integers, we obtain a new sequences of complex numbers such as Gaussian Fibonacci, Gaussian Lucas, Gaussian Pell, Gaussian Pell-Lucas and Gaussian Jacobsthal numbers; Gaussian Padovan and Gaussian Pell-Padovan numbers; Gaussian Tribonacci numbers. The Gaussian generalized Tribonacci sequence $\{GW_n(GW_0, GW_1, GW_2; r, s, t)\}_{n\geq 0}$ (or shortly $\{GW_n\}_{n\geq 0}$) is defined as follows:

$$GW_n = rGW_{n-1} + sGW_{n-2} + tGW_{n-3}, GW_0 = W_0 + W_{-1}i, GW_1 = W_1 + W_0 i, GW_2 = W_2 + W_1 i, \quad n \geq 3 \qquad (1.2)$$

where r, s, t are real numbers.

The sequence $\{GW_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$GW_{-n} = -\frac{s}{t}GW_{-(n-1)} - \frac{r}{t}GW_{-(n-2)} + \frac{1}{t}GW_{-(n-3)}$$

for $n = 1,2,3, \ldots$ when $t \neq 0$. Therefore, recurrence (1.2) holds for all integer n.

Note that for $n \geq 0$

$$GW_n = W_n + iW_{n-1}. \qquad (1.3)$$

and

$$GW_{-n} = W_{-n} + iW_{-n-1}$$

In fact, the Gaussian generalized Tribonacci sequence is the generalization of the well-known sequences like Gaussian Tribonacci, Gaussian Tribonacci-Lucas, Gaussian Padovan (Cordonnier), Gaussian Perrin, Gaussian Padovan-Perrin, Gaussian Narayana, Gaussian third order Jacobsthal and Gaussian third order Jacobsthal-Lucas. In literature, for example, the following names and notations (see Table 1) are used for the special case of r, s, t and initial values.

Table 1 A few special case of Gaussian generalized Tribonacci sequences.

Sequences (Numbers)	Notation
Gaussian Tribonacci	$\{GT_n\} = \{W_n(0,1,1+i; 1,1,1)\}$
Gaussian Tribonacci-Lucas	$\{GK_n\} = \{W_n(3-i, 1+3i, 3+i; 1,1,1)\}$
Gaussian third order Pell	$\{GP_n^{(3)}\} = \{W_n(0,1,2+i; 2,1,1)\}$
Gaussian third order Pell-Lucas	$\{GQ_n^{(3)}\} = \{W_n(3-i, 2+3i, 6+2i; 2,1,1)\}$
Gaussian third order modified Pell	$\{GE_n^{(3)}\} = \{W_n(-i, 1,1+i; 2,1,1)\}$
Gaussian Padovan (Cordonnier)	$\{GP_n\} = \{W_n(1,1+i, 1+i; 0,1,1)\}$
Gaussian Perrin	$\{GE_n\} = \{W_n(3-i, 3i, 2; 0,1,1)\}$
Gaussian Padovan-Perrin	$\{GS_n\} = \{W_n(i, 0,1; 0,1,1)\}$
Gaussian Pell-Padovan	$\{GR_n\} = \{W_n(1-i, 1+i, 1+i; 0,2,1)\}$
Gaussian Pell-Perrin	$\{GC_n\} = \{W_n(3-4i, 3i, 2; 0,2,1)\}$
Gaussian Jacobsthal-Padovan	$\{GQ_n\} = \{W_n(1,1+i, 1+i; 0,1,2)\}$
Gaussian Jacobsthal-Perrin	$\{GD_n\} = \{W_n(3-\frac{1}{2}i, 3i, 2; 0,1,2)\}$
Gaussian Narayana	$\{GN_n\} = \{W_n(0,1,1+i; 1,0,1)\}$

Gaussian third order Jacobsthal	$\{GJ_n^{(3)}\} = \{W_n(0,1,1+i;1,1,2)\}$
Gaussian third order Jacobsthal-Lucas	$\{Gj_n^{(3)}\} = \{W_n(2+i,1+2i,5+i;1,1,2)\}$

In 1963, Horadam [9] introduced the concept of complex Fibonacci number called as the Gaussian Fibonacci number. Pethe [13] defined the complex Tribonacci numbers at Gaussian integers, see also [5].

There are other several studies dedicated to these sequences of Gaussian numbers. We present some works on Gaussian Generalized Fibonacci Numbers in the following Table 2.

Table 2. A few special study of Gaussian Generalized Fibonacci Numbers.

Name of sequence	Papers which deal with Gaussian Numbers
Gaussian Generalized Fibonacci	[1,2,3,5,6,7,8,9,10,11,12,14]
Gaussian Generalized Tribonacci	[15,18,19,20]
Gaussian Generalized Tetranacci	[16]
Gaussian Generalized Pentanacci	[17]

2. Linear Sum formulas of Gaussian Generalized Tribonacci Numbers with Positive Subscripts

The following Theorem presents some linear summing formulas of Gaussian generalized Tribonacci numbers with positive subscripts.

Theorem 2.1 For $n \geq 0$, we have the following formulas:

(a) (Sum of the generalized Tribonacci numbers) If $r+s+t-1 \neq 0$, then

$$\sum_{k=0}^{n} GW_k = \frac{G\Delta_1}{r+s+t-1}$$

where

$$G\Delta_1 = GW_{n+3} + (1-r)GW_{n+2} + (1-r-s)GW_{n+1} - GW_2 + (r-1)GW_1 + (r+s-1)GW_0.$$

(b) If $2s + 2rt + r^2 - s^2 + t^2 - 1 = (r+s+t-1)(r-s+t+1) \neq 0$ then

$$\sum_{k=0}^{n} GW_{2k} = \frac{G\Delta_2}{(r+s+t-1)(r-s+t+1)}$$

where

$$G\Delta_2 = (-s+1)GW_{2n+2} + (t+rs)GW_{2n+1} + (t^2+rt)GW_{2n} + (-1+s)GW_2 + (-t-rs)GW_1 + (-1+r^2-s^2+rt+2s)GW_0.$$

(c) If $2s + 2rt + r^2 - s^2 + t^2 - 1 = (r+s+t-1)(r-s+t+1) \neq 0$ then

$$\sum_{k=0}^{n} GW_{2k+1} = \frac{G\Delta_3}{(r-s+t+1)(r+s+t-1)}$$

where

$G\Delta_3 = (r+t)GW_{2n+2} + (s-s^2+t^2+rt)GW_{2n+1} + (t-st)GW_{2n} + (-r-t)GW_2 + (-1+s+r^2+rt)GW_1 + (-t+st)GW_0.$

Proof. Using the recurrence relation

$$GW_n = rGW_{n-1} + sGW_{n-2} + tGW_{n-3}$$

i.e.

$$tGW_{n-3} = GW_n - rGW_{n-1} - sGW_{n-2}$$

we obtain

$$\begin{aligned}
tGW_0 &= GW_3 - rGW_2 - sGW_1 \\
tGW_1 &= GW_4 - rGW_3 - sGW_2 \\
tGW_2 &= GW_5 - rGW_4 - sGW_3 \\
&\vdots \\
tGW_{n-1} &= GW_{n+2} - rGW_{n+1} - sGW_n \\
tGW_n &= GW_{n+3} - rGW_{n+2} - sGW_{n+1}.
\end{aligned}$$

If we add the equations by side by, we get **(a)**.

(b) and **(c)**: Using the recurrence relation

$$GW_n = rGW_{n-1} + sGW_{n-2} + tGW_{n-3}$$

i.e.

$$rGW_{n-1} = GW_n - sGW_{n-2} - tGW_{n-3}$$

we obtain

$$\begin{aligned}
rGW_3 &= GW_4 - sGW_2 - tGW_1 \\
rGW_5 &= GW_6 - sGW_4 - tGW_3 \\
&\vdots \\
rGW_{2n+1} &= GW_{2n+2} - sGW_{2n} - tGW_{2n-1}. \\
rGW_{2n+3} &= GW_{2n+4} - sGW_{2n+2} - tGW_{2n+1}
\end{aligned}$$

Now, if we add the above equations by side by, we get

$$\begin{aligned}
r(-GW_1 + \textstyle\sum_{k=0}^{n} GW_{2k+1}) &= (GW_{2n+2} - GW_2 - GW_0 + \textstyle\sum_{k=0}^{n} GW_{2k}) \\
&\quad -s(-GW_0 + \textstyle\sum_{k=0}^{n} GW_{2k}) - t(-GW_{2n+1} + \textstyle\sum_{k=0}^{n} GW_{2k+1}). \qquad (2.1)
\end{aligned}$$

Similarly, using the recurrence relation

$$GW_n = rGW_{n-1} + sGW_{n-2} + tGW_{n-3}$$

i.e.

$$rGW_{n-1} = GW_n - sGW_{n-2} - tGW_{n-3}$$

we write the following obvious equations;

$$\begin{aligned}
rGW_2 &= GW_3 - sGW_1 - tGW_0 \\
rGW_4 &= GW_5 - sGW_3 - tGW_2 \\
rGW_6 &= GW_7 - sGW_5 - tGW_4 \\
&\vdots \\
rGW_{2n} &= GW_{2n+1} - sGW_{2n-1} - tGW_{2n-2} \\
rGW_{2n+2} &= GW_{2n+3} - sGW_{2n+1} - tGW_{2n}.
\end{aligned}$$

Now, if we add the above equations by side by, we obtain

$$\begin{aligned}
r(-GW_0 + \textstyle\sum_{k=0}^{n} GW_{2k}) &= (-GW_1 + \textstyle\sum_{k=0}^{n} GW_{2k+1}) - s(-GW_{2n+1} + \textstyle\sum_{k=0}^{n} GW_{2k+1}) \\
&\quad -t(-GW_{2n} + \textstyle\sum_{k=0}^{n} GW_{2k}). \qquad (2.2)
\end{aligned}$$

Then, solving the system (2.1)-(2.2), the required results of (b) and (c) follow.

Taking $r = s = t = 1$ in Theorem 2.1 (a) and (b) (or (c)), we obtain the following Proposition.

Proposition 2.2 If $r = s = t = 1$ then for $n \geq 0$ we have the following formulas:

(a) $\sum_{k=0}^{n} GW_k = \frac{1}{2}(GW_{n+3} - GW_{n+1} - GW_2 + GW_0)$.

(b) $\sum_{k=0}^{n} GW_{2k} = \frac{1}{2}(GW_{2n+1} + GW_{2n} - GW_1 + GW_0)$.

(c) $\sum_{k=0}^{n} GW_{2k+1} = \frac{1}{2}(GW_{2n+2} + GW_{2n+1} - GW_2 + GW_1)$.

From the above Proposition, we have the following Corollary which gives linear sum formulas of Gaussian Tribonacci numbers (take $GW_n = GT_n$ with $GT_0 = 0, GT_1 = 1, GT_2 = 1 + i$).

Corollary 2.3 For $n \geq 0$, Gaussian Tribonacci numbers have the following properties.

(a) $\sum_{k=0}^{n} GT_k = \frac{1}{2}(GT_{n+3} - GT_{n+1} - (1 + i))$.

(b) $\sum_{k=0}^{n} GT_{2k} = \frac{1}{2}(GT_{2n+1} + GT_{2n} - 1)$.

(c) $\sum_{k=0}^{n} GT_{2k+1} = \frac{1}{2}(GT_{2n+2} + GT_{2n+1} - i)$.

Taking $GW_n = GK_n$ with $GK_0 = 3 - i, GK_1 = 1 + 3i, GK_2 = 1$ in the above Proposition, we have the following Corollary which presents linear sum formulas of Gaussian Tribonacci-Lucas numbers.

Corollary 2.4 *For $n \geq 0$, Gaussian Tribonacci-Lucas numbers have the following properties.*

(a) $\sum_{k=0}^{n} GK_k = \frac{1}{2}(GK_{n+3} - GK_{n+1} - 2i)$.

(b) $\sum_{k=0}^{n} GK_{2k} = \frac{1}{2}(GK_{2n+1} + GK_{2n} + (2 - 4i))$.

(c) $\sum_{k=0}^{n} GK_{2k+1} = \frac{1}{2}(GK_{2n+2} + GK_{2n+1} + (-2 + 2i))$.

Taking $r = 2, s = 1, t = 1$ in Theorem 2.1 (a) and (b) (or (c)), we obtain the following Proposition.

Proposition 2.5 If $r = 2, s = 1, t = 1$ then for $n \geq 0$ we have the following formulas:

(a) $\sum_{k=0}^{n} GW_k = \frac{1}{3}(GW_{n+3} - GW_{n+2} - 2GW_{n+1} - GW_2 + GW_1 + 2GW_0)$.

(b) $\sum_{k=0}^{n} GW_{2k} = \frac{1}{3}(GW_{2n+1} + GW_{2n} - GW_1 + 2GW_0)$.

(c) $\sum_{k=0}^{n} GW_{2k+1} = \frac{1}{3}(GW_{2n+2} + GW_{2n+1} - GW_2 + 2GW_1)$.

From the last Proposition, we have the following Corollary which gives linear sum formulas of Gaussian third-order Pell numbers (take $GW_n = GP_n^{(3)}$ with $GP_0^{(3)} = 0, GP_1^{(3)} = 1, GP_2^{(3)} = 2 + i$).

Corollary 2.6 For $n \geq 0$, Gaussian third-order Pell numbers have the following properties:

(a) $\sum_{k=0}^{n} GP_k^{(3)} = \frac{1}{3}(GP_{n+3}^{(3)} - GP_{n+2}^{(3)} - 2GP_{n+1}^{(3)} - (1 + i))$.

(b) $\sum_{k=0}^{n} GP_{2k}^{(3)} = GP_{2n+1}^{(3)} + GP_{2n}^{(3)} - 1$.

(c) $\sum_{k=0}^{n} GP_{2k+1}^{(3)} = \frac{1}{3}(GP_{2n+2}^{(3)} + GP_{2n+1}^{(3)} - i)$.

Taking $GQ_0^{(3)} = 3 - i, GQ_1^{(3)} = 2 + 3i, GQ_2^{(3)} = 6 + 2i$ in the last Proposition, we have the following Corollary which presents linear sum formulas of Gaussian third-order Pell-Lucas numbers.

Corollary 2.7 For $n \geq 0$, Gaussian third-order Pell-Lucas numbers have the following properties:

(a) $\sum_{k=0}^{n} GQ_k^{(3)} = \frac{1}{3}\left(GQ_{n+3}^{(3)} - GQ_{n+2}^{(3)} - 2GQ_{n+1}^{(3)} + (2 - i)\right)$.

(b) $\sum_{k=0}^{n} GQ_{2k}^{(3)} = GQ_{2n+1}^{(3)} + G^{(3)}Q_{2n} + (4 - 5i)$.

(c) $\sum_{k=0}^{n} GQ_{2k+1}^{(3)} = \frac{1}{3}\left(GQ_{2n+2}^{(3)} + GQ_{2n+1}^{(3)}\right) + (-2+4i)$.

From the last Proposition, we have the following Corollary which gives linear sum formulas of Gaussian third-order modified Pell numbers (take $GW_n = GE_n^{(3)}$ with $E_0^{(3)} = -i, GE_1^{(3)} = 1, GE_2^{(3)} = 1+i$).

Corollary 2.8 For $n \geq 0$, Gaussian third-order modified Pell numbers have the following properties:

(a) $\sum_{k=0}^{n} GE_k^{(3)} = \frac{1}{3}(GE_{n+3}^{(3)} - GE_{n+2}^{(3)} - 2GE_{n+1}^{(3)} - 3i)$

(b) $\sum_{k=0}^{n} GE_{2k}^{(3)} = \frac{1}{3}(GE_{2n+1}^{(3)} + GE_{2n}^{(3)} + (-1-2i))$

(c) $\sum_{k=0}^{n} GE_{2k+1}^{(3)} = \frac{1}{3}(GE_{2n+2}^{(3)} + GE_{2n+1}^{(3)} + (1-i))$.

Taking $r=0, s=1, t=1$ in Theorem 2.1 (a) and (b) (or (c)), we obtain the following Proposition.

Proposition 2.9 If $r=0, s=1, t=1$ then for $n \geq 0$ we have the following formulas:

(a) $\sum_{k=0}^{n} GW_k = GW_{n+3} + GW_{n+2} - GW_2 - GW_1$.

(b) $\sum_{k=0}^{n} GW_{2k} = GW_{2n+1} + GW_{2n} - GW_1$.

(c) $\sum_{k=0}^{n} GW_{2k+1} = GW_{2n+2} + GW_{2n+1} - GW_2$.

From the last Proposition, we have the following Corollary which gives linear sum formulas of Gaussian Padovan numbers (take $GW_n = GP_n$ with $GP_0 = 1, GP_1 = 1+i, GP_2 = 1+i$).

Corollary 2.10 For $n \geq 0$, Gaussian Padovan numbers have the following properties.

(a) $\sum_{k=0}^{n} GP_k = GP_{n+3} + GP_{n+2} - 2(1+i)$.

(b) $\sum_{k=0}^{n} GP_{2k} = GP_{2n+1} + GP_{2n} - (1+i)$.

(c) $\sum_{k=0}^{n} GP_{2k+1} = GP_{2n+2} + GP_{2n+1} - (1+i)$.

Taking $GW_n = GE_n$ with $GE_0 = 3-i, GE_1 = 3i, GE_2 = 2$ in the last Proposition, we have the following Corollary which presents linear sum formulas of Gaussian Perrin numbers.

Corollary 2.11 For $n \geq 0$, Gaussian Perrin numbers have the following properties.

(a) $\sum_{k=0}^{n} GE_k = GE_{n+3} + GE_{n+2} - (2+3i)$.

(b) $\sum_{k=0}^{n} GE_{2k} = GE_{2n+1} + GE_{2n} - 3i$.

(c) $\sum_{k=0}^{n} GE_{2k+1} = GE_{2n+2} + GE_{2n+1} - 2$.

Taking $GW_n = GS_n$ with $GS_0 = i, GS_1 = 0, GS_2 = 1$ in the last Proposition, we have the following Corollary which gives linear sum formulas of Gaussian Padovan-Perrin numbers.

Corollary 2.12 For $n \geq 0$, Gaussian Padovan-Perrin numbers have the following properties.

(a) $\sum_{k=0}^{n} GS_k = GS_{n+3} + GS_{n+2} - 1$.

(b) $\sum_{k=0}^{n} GS_{2k} = GS_{2n+1} + GS_{2n}$.

(c) $\sum_{k=0}^{n} GS_{2k+1} = GS_{2n+2} + GS_{2n+1} - 1$.

If $r=0, s=2, t=1$ then $(r-s+t+1)=0$ so we can't use Theorem 2.1 (b) and (c). In other words, the method of the proof Theorem 2.1 (b) and (c) can't be used to find $\sum_{k=0}^{n} GW_{2k}$ and $\sum_{k=0}^{n} GW_{2k+1}$. Therefore we need another method to find them which is given in the following Theorem.

Theorem 2.13 If $r=0, s=2, t=1$ then for $n \geq 0$ we have the following formulas:

(a) $\sum_{k=0}^{n} GW_k = \frac{1}{2}(GW_{n+3} + GW_{n+2} - GW_{n+1} - GW_2 - GW_1 + GW_0)$.
(b) $\sum_{k=0}^{n} GW_{2k} = GW_{2n+1} + (GW_2 - GW_1 - GW_0)n + GW_0 - GW_1$.
(c) $\sum_{k=0}^{n} GW_{2k+1} = \frac{1}{2}\begin{pmatrix} GW_{2n+3} + GW_{2n+2} - GW_{2n+1} + 2n(-GW_2 + GW_1 + GW_0) - GW_2 \\ +GW_1 - GW_0 \end{pmatrix}$.

Proof. Taking $r = 0, s = 2, t = 1$ in Theorem 2.1 (a) we obtain (a).
Using the recurrence relation

$$GW_n = 2GW_{n-2} + GW_{n-3}$$

we obtain

$$\sum_{k=0}^{0} GW_{2k} = GW_0$$
$$\sum_{k=0}^{1} GW_{2k} = GW_0 + GW_2 = GW_3 + GW_2 - 2GW_1$$
$$\sum_{k=0}^{2} GW_{2k} = GW_0 + GW_2 + GW_4 = GW_5 + 2GW_2 - 3GW_1 - GW_0$$
$$\vdots$$
$$\sum_{k=0}^{n} GW_{2k} = GW_{2n+1} + (GW_2 - GW_1 - GW_0)n + GW_0 - GW_1.$$

This result can be also proved by mathematical induction. Note that from (a) we get

$$\sum_{k=0}^{n} GW_{2k+1} = \frac{1}{2}(GW_{2n+3} + GW_{2n+2} + GW_{2n+1} - GW_2 - GW_1 + GW_0) - \sum_{k=0}^{n} GW_{2k}$$

Now, (c) follows from the last equation.
From the above Theorem we have the following Corollary which gives sum formulas of Gaussian Pell-Padovan numbers (take $GW_n = GR_n$ with $GR_0 = 1 - i, GR_1 = 1 + i, GR_2 = 1 + i$).

Corollary 2.14 For $n \geq 0$, Gaussian Pell-Padovan numbers have the following property:
(a) $\sum_{k=0}^{n} R_k = \frac{1}{2}(R_{n+3} + R_{n+2} - R_{n+1} - (1 + 3i))$.
(b) $\sum_{k=0}^{n} R_{2k} = R_{2n+1} + (-1 + i)n - 2i$.
(c) $\sum_{k=0}^{n} R_{2k+1} = \frac{1}{2}(R_{2n+3} + R_{2n+2} - R_{2n+1} + (2 - 2i)n + (-1 + i))$.

Taking $GW_n = GC_n$ with $GC_0 = 3 - 4i, GC_1 = 3i, GC_2 = 2$ in the last Theorem, we have the following Corollary which presents sum formulas of Gaussian Pell-Perrin numbers.

Corollary 2.15 For $n \geq 0$, Gaussian Pell-Perrin numbers have the following property:
(a) $\sum_{k=0}^{n} GC_k = \frac{1}{2}(GC_{n+3} + GC_{n+2} - GC_{n+1} + (1 - 7i))$.
(b) $\sum_{k=0}^{n} GC_{2k} = GC_{2n+1} + (-1 + i)n + (3 - 7i)$.
(c) $\sum_{k=0}^{n} GC_{2k+1} = \frac{1}{2}(GC_{2n+3} + GC_{2n+2} - GC_{2n+1} + (2 - 2i)n + (-5 + 7i))$.

Taking $r = 0, s = 1, t = 2$ in Theorem 2.1 (a) and (b) (or (c)), we obtain the following Proposition.

Proposition 2.16 If $r = 0, s = 1, t = 2$ then for $n \geq 0$ we have the following formulas:
(a) $\sum_{k=0}^{n} GW_k = \frac{1}{2}(GW_{n+3} + GW_{n+2} - GW_2 - GW_1)$.
(b) $\sum_{k=0}^{n} GW_{2k} = \frac{1}{2}(GW_{2n+1} + 2GW_{2n} - GW_1)$.
(c) $\sum_{k=0}^{n} GW_{2k+1} = \frac{1}{2}(GW_{2n+2} + 2GW_{2n+1} - GW_2)$.

Taking $GW_n = GQ_n$ with $GQ_0 = 1, GQ_1 = 1 + i, GQ_2 = 1 + i$ in the last Proposition, we have the following Corollary which presents linear sum formulas of Gaussian Jacobsthal-Padovan numbers.

Corollary 2.17 For $n \geq 0$, Jacobsthal-Padovan numbers have the following properties.

(a) $\sum_{k=0}^{n} GQ_k = \frac{1}{2}(GQ_{n+3} + GQ_{n+2} - 2(1 + i))$.

(b) $\sum_{k=0}^{n} GQ_{2k} = \frac{1}{2}(GQ_{2n+1} + 2GQ_{2n} - (1 + i))$.

(c) $\sum_{k=0}^{n} GQ_{2k+1} = \frac{1}{2}(GQ_{2n+2} + 2GQ_{2n+1} - (1 + i))$.

From the last Proposition, we have the following Corollary which gives linear sum formulas of Jacobsthal-Perrin numbers (take $GW_n = GD_n$ with $GD_0 = 3 - \frac{1}{2}i, GD_1 = 3i, GD_2 = 2$).

Corollary 2.18 For $n \geq 0$, Gaussian Jacobsthal-Perrin numbers have the following properties.

(a) $\sum_{k=0}^{n} GD_k = \frac{1}{2}(GD_{n+3} + GD_{n+2} - (2 + 3i))$.

(b) $\sum_{k=0}^{n} GD_{2k} = \frac{1}{2}(GD_{2n+1} + 2GD_{2n} - 3i)$.

(c) $\sum_{k=0}^{n} GD_{2k+1} = \frac{1}{2}(GD_{2n+2} + 2GD_{2n+1} - 2)$.

Taking $r = 1, s = 0, t = 1$ in Theorem 2.1 (a) and (c), we obtain the following Proposition.

Proposition 2.19 If $r = 1, s = 0, t = 1$ then for $n \geq 0$ we have the following formulas:

(a) $\sum_{k=0}^{n} GW_k = GW_{n+3} - GW_2$.

(b) $\sum_{k=0}^{n} GW_{2k} = \frac{1}{3}(GW_{2n+2} + GW_{2n+1} + 2GW_{2n} - GW_2 - GW_1 + GW_0)$.

(c) $\sum_{k=0}^{n} GW_{2k+1} = \frac{1}{3}(2GW_{2n+2} + 2GW_{2n+1} + GW_{2n} - 2GW_2 + GW_1 - GW_0)$.

From the last Proposition, we have the following Corollary which presents linear sum formulas of Gaussian Narayana numbers (take $GW_n = GN_n$ with $GN_0 = 0, GN_1 = 1, GN_2 = 1 + i$).

Corollary 2.20 For $n \geq 0$, Gaussian Narayana numbers have the following properties.

(a) $\sum_{k=0}^{n} N_k = N_{n+3} - (1 + i)$.

(b) $\sum_{k=0}^{n} N_{2k} = \frac{1}{3}(N_{2n+2} + N_{2n+1} + 2N_{2n} - (2 + i))$.

(c) $\sum_{k=0}^{n} N_{2k+1} = \frac{1}{3}(2N_{2n+2} + 2N_{2n+1} + N_{2n} - (1 + 2i))$.

Taking $r = 1, s = 1, t = 2$ in Theorem 2.1 (a) and (c), we obtain the following Proposition.

Proposition 2.21 If $r = 1, s = 1, t = 2$ then for $n \geq 0$ we have the following formulas.

(a) $\sum_{k=0}^{n} GW_k = \frac{1}{3}(GW_{n+3} - GW_{n+1} - GW_2 + GW_0)$.

(b) $\sum_{k=0}^{n} GW_{2k} = \frac{1}{3}(GW_{2n+1} + 2GW_{2n} - GW_1 + GW_0)$.

(c) $\sum_{k=0}^{n} GW_{2k+1} = \frac{1}{3}(GW_{2n+2} + 2GW_{2n+1} - GW_2 + GW_1)$.

Taking $GW_n = GJ_n^{(3)}$ with $GJ_0^{(3)} = 0, GJ_1^{(3)} = 1, GJ_2^{(3)} = 1 + i$ in the last Proposition, we have the following Corollary which presents linear sum formulas of Gaussian Jacobsthal numbers.

Corollary 2.22 For $n \geq 0$, Jacobsthal numbers have the following properties.

(a) $\sum_{k=0}^{n} GJ_k^{(3)} = \frac{1}{3}(GJ_{n+3}^{(3)} - GJ_{n+1}^{(3)} - (1+i))$.

(b) $\sum_{k=0}^{n} GJ_{2k}^{(3)} = \frac{1}{3}(GJ_{2n+1}^{(3)} + 2GJ_{2n}^{(3)} - 1)$.

(c) $\sum_{k=0}^{n} GJ_{2k+1}^{(3)} = \frac{1}{3}(GJ_{2n+2}^{(3)} + 2GJ_{2n+1}^{(3)} - i)$.

From the last Proposition, we have the following Corollary which gives linear sum formulas of Gaussian Jacobsthal-Lucas numbers (take $GW_n = Gj_n^{(3)}$ with $Gj_0^{(3)} = 2+i, Gj_1^{(3)} = 1+2i, Gj_2^{(3)} = 5+i$).

Corollary 2.23 For $n \geq 0$, Gaussian Jacobsthal-Lucas numbers have the following properties.

(a) $\sum_{k=0}^{n} Gj_k^{(3)} = \frac{1}{3}(Gj_{n+3}^{(3)} - Gj_{n+1}^{(3)} - 3)$.

(b) $\sum_{k=0}^{n} Gj_{2k}^{(3)} = \frac{1}{3}(Gj_{2n+1}^{(3)} + 2Gj_{2n}^{(3)} + (1-i))$.

(c) $\sum_{k=0}^{n} Gj_{2k+1}^{(3)} = \frac{1}{3}(Gj_{2n+2}^{(3)} + 2Gj_{2n+1}^{(3)} + (-4+i))$.

3. Linear Sum formulas of Gaussian Generalized Tribonacci Numbers with Negative Subscripts

The following Theorem presents some linear summing formulas (identities) of Gaussian generalized Tribonacci numbers with negative subscripts.

Theorem 3.1 For $n \geq 1$, we have the following formulas:

(a) If $r+s+t-1 \neq 0$, then

$$\sum_{k=1}^{n} GW_{-k} = \frac{G\Delta_4}{r+s+t-1}$$

where

$$G\Delta_4 = -(rGW_{-n-1} + s(GW_{-n-1} + GW_{-n-2}) + t(GW_{-n-1} + GW_{-n-2} + GW_{-n-3}) - GW_2 \\ +(r-1)GW_1 + (r+s-1)GW_0).$$

(b) If $2s + 2rt + r^2 - s^2 + t^2 - 1 = (r+s+t-1)(r-s+t+1) \neq 0 \wedge r+t \neq 0$ then

$$\sum_{k=1}^{n} GW_{-2k} = \frac{G\Delta_5}{(r+s+t-1)(r-s+t+1)}$$

where

$$G\Delta_5 = -(r+t)GW_{-2n+1} + (r^2+rt+s-1)GW_{-2n} + (st-t)GW_{-2n-1} + (1-s)GW_2 \\ +(t+rs)GW_1 + (1-rt-2s-r^2+s^2)GW_0.$$

(c) If $2s + 2rt + r^2 - s^2 + t^2 - 1 = (r+s+t-1)(r-s+t+1) \neq 0 \wedge r+t \neq 0$ then

$$\sum_{k=1}^{n} GW_{-2k+1} = \frac{G\Delta_6}{(r+s+t-1)(r-s+t+1)}$$

where

$$G\Delta_6 = (s-1)GW_{-2n+1} - (t+rs)GW_{-2n} - (t^2+rt)GW_{-2n-1} + (r+t)GW_2 \\ +(1-r^2-rt-s)GW_1 + (t-st)GW_0.$$

Proof. Using the recurrence relation

$$GW_{-n+3} = rGW_{-n+2} + sGW_{-n+1} + tGW_{-n} \Rightarrow GW_{-n} = -\frac{s}{t}GW_{-(n-1)} - \frac{r}{t}GW_{-(n-2)} + \frac{1}{t}GW_{-(n-3)}$$

i.e.

$$tGW_{-n} = GW_{-n+3} - rGW_{-n+2} - sGW_{-n+1}$$

or

$$GW_{-n} = \frac{1}{t}GW_{-n+3} - \frac{r}{t}GW_{-n+2} - \frac{s}{t}GW_{-n+1}$$

we obtain

$$\begin{aligned}
tGW_{-n} &= GW_{-n+3} - rGW_{-n+2} - sGW_{-n+1}\\
tGW_{-n+1} &= GW_{-n+4} - rGW_{-n+3} - sGW_{-n+2}\\
tGW_{-n+2} &= GW_{-n+5} - rGW_{-n+4} - sGW_{-n+3}\\
&\vdots\\
tGW_{-2} &= GW_1 - r \times GW_0 - s \times GW_{-1}\\
tGW_{-1} &= GW_2 - r \times GW_1 - s \times GW_0.
\end{aligned}$$

If we add the above equations by side by, we get **(a).**

(b) and **(c):** Using the recurrence relation

$$GW_{-n+3} = rGW_{-n+2} + sGW_{-n+1} + tGW_{-n}$$

i.e.

$$sGW_{-n+1} = GW_{-n+3} - rGW_{-n+2} - tGW_{-n}$$

we obtain

$$\begin{aligned}
sGW_{-2n+1} &= GW_{-2n+3} - rGW_{-2n+2} - tGW_{-2n}\\
sGW_{-2n+3} &= GW_{-2n+5} - rGW_{-2n+4} - tGW_{-2n+2}\\
&\vdots\\
sGW_{-3} &= GW_{-1} - rGW_{-2} - tGW_{-4}\\
sGW_{-1} &= GW_1 - rGW_0 - tGW_{-2}.
\end{aligned}$$

If we add the equations by side by, we get

$$\begin{aligned}
s\sum_{k=1}^{n} GW_{-2k+1} = {} & (-GW_{-2n+1} + GW_1 + \sum_{k=1}^{n} GW_{-2k+1})\\
& -r(-GW_{-2n} + GW_0 + \sum_{k=1}^{n} GW_{-2k}) - t(\sum_{k=1}^{n} GW_{-2k}).
\end{aligned} \tag{3.1}$$

Similarly, using the recurrence relation

$$GW_{-n+3} = rGW_{-n+2} + sGW_{-n+1} + tGW_{-n}$$

i.e.

$$sGW_{-n+1} = GW_{-n+3} - rGW_{-n+2} - tGW_{-n}$$

we obtain

$$\begin{aligned}
sGW_{-2n} &= GW_{-2n+2} - rGW_{-2n+1} - tGW_{-2n-1}\\
sGW_{-2n+2} &= GW_{-2n+4} - rGW_{-2n+3} - tGW_{-2n+1}\\
&\vdots\\
sGW_{-6} &= GW_{-4} - rGW_{-5} - tGW_{-7}\\
sGW_{-4} &= GW_{-2} - rGW_{-3} - tGW_{-5}\\
sGW_{-2} &= GW_0 - rGW_{-1} - tGW_{-3}.
\end{aligned}$$

If we add the above equations by side by, we get

$$s\sum_{k=1}^{n} GW_{-2k} = \left(-GW_{-2n} + GW_0 + \sum_{k=1}^{n} GW_{-2k}\right) - r\left(\sum_{k=1}^{n} GW_{-2k+1}\right)$$
$$-t\left(GW_{-2n-1} - GW_{-1} + \sum_{k=1}^{n} GW_{-2k+1}\right).$$

Since

$$GW_{-1} = (-\tfrac{s}{t}GW_0 - \tfrac{r}{t}GW_1 + \tfrac{1}{t}GW_2).$$

It follows that

$$s\textstyle\sum_{k=1}^{n} GW_{-2k} = (-GW_{-2n} + GW_0 + \sum_{k=1}^{n} GW_{-2k}) - r(\sum_{k=1}^{n} GW_{-2k+1})$$
$$-t(GW_{-2n-1} - (-\tfrac{s}{t}GW_0 - \tfrac{r}{t}GW_1 + \tfrac{1}{t}GW_2) + \textstyle\sum_{k=1}^{n} GW_{-2k+1}). \qquad (3.2)$$

Then, solving system (3.1)-(3.2) the required results of **(b)** and **(c)** follow.

Next, we present several sum formulas (identities).
Taking $r = s = t = 1$ in Theorem 3.1 (a) and (b), we obtain the following Proposition.

Proposition 3.2 If $r = s = t = 1$ then for $n \geq 1$ we have the following formulas:

(a) $\sum_{k=1}^{n} GW_{-k} = \frac{1}{2}(-3GW_{-n-1} - 2GW_{-n-2} - GW_{-n-3} + GW_2 - GW_0)$.

(b) $\sum_{k=1}^{n} GW_{-2k} = \frac{1}{2}(-GW_{-2n+1} + GW_{-2n} + GW_1 - GW_0)$.

(c) $\sum_{k=1}^{n} GW_{-2k+1} = \frac{1}{2}(-GW_{-2n} - GW_{-2n-1} + GW_2 - GW_1)$.

From the above Proposition, we have the following Corollary which gives linear sum formulas of Gaussian Tribonacci numbers (take $GW_n = GT_n$ with $GT_0 = 0, GT_1 = 1, GT_2 = 1 + i$).

Corollary 3.3 For $n \geq 1$, Gaussian Tribonacci numbers have the following properties.

(a) $\sum_{k=1}^{n} GT_{-k} = \frac{1}{2}(-3GT_{-n-1} - 2GT_{-n-2} - GT_{-n-3} + (1 + i))$.

(b) $\sum_{k=1}^{n} GT_{-2k} = \frac{1}{2}(-GT_{-2n+1} + GT_{-2n} + 1)$.

(c) $\sum_{k=1}^{n} GT_{-2k+1} = \frac{1}{2}(-GT_{-2n} - GT_{-2n-1} + i)$.

Taking $GW_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3$ in the above Proposition, we have the following Corollary which gives linear sum formulas of Gaussian Tribonacci-Lucas numbers.

Corollary 3.4 For $n \geq 1$, Gaussian Tribonacci-Lucas numbers have the following properties.

(a) $\sum_{k=1}^{n} GK_{-k} = \frac{1}{2}(-3GK_{-n-1} - 2GK_{-n-2} - GK_{-n-3} + 2i)$.

(b) $\sum_{k=1}^{n} GK_{-2k} = \frac{1}{2}(-GK_{-2n+1} + GK_{-2n} + (-2 + 4i))$.

(c) $\sum_{k=1}^{n} GK_{-2k+1} = \frac{1}{2}(-GK_{-2n} - GK_{-2n-1} + (2 - 2i))$.

Taking $r = 2, s = 1, t = 1$ in Theorem 3.1 (a) and (b), we obtain the following Proposition.

Proposition 3.5 If $r = 2, s = 1, t = 1$ then for $n \geq 1$ we have the following formulas:

(a) $\sum_{k=1}^{n} GW_{-k} = \frac{1}{3}(-4GW_{-n-1} - 2GW_{-n-2} - GW_{-n-3} + GW_2 - GW_1 - 2GW_0)$.
(b) $\sum_{k=1}^{n} GW_{-2k} = \frac{1}{3}(-GW_{-2n+1} + 2GW_{-2n} + GW_1 - 2GW_0)$.
(c) $\sum_{k=1}^{n} GW_{-2k+1} = \frac{1}{3}(-GW_{-2n} - GW_{-2n-1} + GW_2 - 2GW_1)$.

From the last Proposition, we have the following Corollary which gives linear sum formulas of Gaussian third-order Pell numbers (take $GW_n = GP_n^{(3)}$ with $GP_0^{(3)} = 0, GP_1^{(3)} = 1, GP_2^{(3)} = 2 + i$).

Corollary 3.6 For $n \geq 1$, Gaussian third-order Pell numbers have the following properties.
(a) $\sum_{k=1}^{n} GP_{-k}^{(3)} = \frac{1}{3}(-4GP_{-n-1}^{(3)} - 2GP_{-n-2}^{(3)} - GP_{-n-3}^{(3)} + (1 + i))$.
(b) $\sum_{k=1}^{n} GP_{-2k}^{(3)} = \frac{1}{3}(-GP_{-2n+1}^{(3)} + 2GP_{-2n}^{(3)} + 1)$.
(c) $\sum_{k=1}^{n} GP_{-2k+1}^{(3)} = \frac{1}{3}(-GP_{-2n}^{(3)} - GP_{-2n-1}^{(3)} + i)$.

Taking $GW_n = GQ_n^{(3)}$ with $GQ_0^{(3)} = 3 - i, GQ_1^{(3)} = 2 + 3i, GQ_2^{(3)} = 6 + 2i$ in the last Proposition, we have the following Corollary which gives linear sum formulas of Gaussian third-order Pell-Lucas numbers.

Corollary 3.7 For $n \geq 1$, Gaussian third-order Pell-Lucas numbers have the following properties.
(a) $\sum_{k=1}^{n} GQ_{-k}^{(3)} = \frac{1}{3}(-4GQ_{-n-1}^{(3)} - 2GQ_{-n-2}^{(3)} - GQ_{-n-3}^{(3)} + (-2 + i))$.
(b) $\sum_{k=1}^{n} GQ_{-2k}^{(3)} = \frac{1}{3}(-GQ_{-2n+1}^{(3)} + 2GQ_{-2n}^{(3)} + (-4 + 5i))$.
(c) $\sum_{k=1}^{n} GQ_{-2k+1}^{(3)} = \frac{1}{3}(-GQ_{-2n}^{(3)} - GQ_{-2n-1}^{(3)} + (2 - 4i))$.

From the last Proposition, we have the following Corollary which presents linear sum formulas of third-order modified Pell numbers (take $GW_n = GE_n^{(3)}$ with $GE_0^{(3)} = -i, GE_1^{(3)} = 1, GE_2^{(3)} = 1 + i$).

Corollary 3.8 For $n \geq 1$, third-order modified Pell numbers have the following properties.
(a) $\sum_{k=1}^{n} GE_{-k}^{(3)} = \frac{1}{3}(-4GE_{-n-1}^{(3)} - 2GE_{-n-2}^{(3)} - GE_{-n-3}^{(3)} + 3i)$.
(b) $\sum_{k=1}^{n} GE_{-2k}^{(3)} = \frac{1}{3}(-GE_{-2n+1}^{(3)} + 2GE_{-2n}^{(3)} + (1 + 2i))$.
(c) $\sum_{k=1}^{n} GE_{-2k+1}^{(3)} = \frac{1}{3}(-GE_{-2n}^{(3)} - GE_{-2n-1}^{(3)} + (-1 + i))$.

Taking $r = 0, s = 1, t = 1$ in Theorem 3.1 (a) and (b), we obtain the following Proposition.

Proposition 3.9 If $r = 0, s = 1, t = 1$ then for $n \geq 1$ we have the following formulas:
(a) $\sum_{k=1}^{n} GW_{-k} = -2GW_{-n-1} - 2GW_{-n-2} - GW_{-n-3} + GW_2 + GW_1$.
(b) $\sum_{k=1}^{n} GW_{-2k} = -GW_{-2n+1} + GW_1$.
(c) $\sum_{k=1}^{n} GW_{-2k+1} = -GW_{-2n} - GW_{-2n-1} + GW_2$.

Taking $GW_n = GP_n$ with $GP_0 = 1, GP_1 = 1 + i, GP_2 = 1 + i$ in the last Proposition, we have the following Corollary which gives linear sum formulas of Gaussian Padovan numbers.

Corollary 3.10 For $n \geq 1$, Gaussian Padovan numbers have the following properties.
(a) $\sum_{k=1}^{n} GP_{-k} = -2GP_{-n-1} - 2GP_{-n-2} - GP_{-n-3} + 2(1 + i)$.

(b) $\sum_{k=1}^{n} GP_{-2k} = -GP_{-2n+1} + (1+i)$.
(c) $\sum_{k=1}^{n} GP_{-2k+1} = -GP_{-2n} - GP_{-2n-1} + (1+i)$.

From the last Proposition, we have the following Corollary which presents linear sum formulas of Gaussian Perrin numbers (take $GW_n = GE_n$ with $GE_0 = 3 - i, GE_1 = 3i, GE_2 = 2$).

Corollary 3.11 For $n \geq 1$, Gaussian Perrin numbers have the following properties.

(a) $\sum_{k=1}^{n} GE_{-k} = -2GE_{-n-1} - 2GE_{-n-2} - GE_{-n-3} + (2+3i)$.
(b) $\sum_{k=1}^{n} GE_{-2k} = -GE_{-2n+1} + 3i$.
(c) $\sum_{k=1}^{n} GE_{-2k+1} = -GE_{-2n} - GE_{-2n-1} + 2$.

Taking $GW_n = GS_n$ with $GS_0 = i, GS_1 = 0, GS_2 = 1$ in the last Proposition, we have the following Corollary which gives linear sum formulas of Gaussian Padovan-Perrin numbers.

Corollary 3.12 *For $n \geq 1$, Gaussian Padovan-Perrin numbers have the following properties.*

(a) $\sum_{k=1}^{n} GS_{-k} = -2GS_{-n-1} - 2GS_{-n-2} - GS_{-n-3} + 1$.
(b) $\sum_{k=1}^{n} GS_{-2k} = -GS_{-2n+1}$.
(c) $\sum_{k=1}^{n} GS_{-2k+1} = -GS_{-2n} - GS_{-2n-1} + 1$.

If $r = 0, s = 2, t = 1$ then $(r+s+t-1)(r-s+t+1) = 0$ so we can't use Theorem 3.1 (b) and (c). In other words, the method of the proof Theorem 3.1 (b) and (c) can't be used to find $\sum_{k=1}^{n} GW_{-2k}$ and $\sum_{k=1}^{n} GW_{-2k+1}$. Therefore we need another method to find them which is given in the following Theorem.

Theorem 3.13 If $r = 0, s = 2, t = 1$ then for $n \geq 1$ we have the following formulas:

(a) $\sum_{k=1}^{n} GW_{-k} = \frac{1}{2}(-3GW_{-n-1} - 3GW_{-n-2} - GW_{-n-3} + GW_2 + GW_1 - GW_0)$.
(b) $\sum_{k=1}^{n} GW_{-2k} = -GW_{-2n+1} + GW_{-2n} + (GW_1 - GW_0) + (GW_2 - GW_1 - GW_0)n$.
(c) $\sum_{k=1}^{n} GW_{-2k+1} = \frac{1}{2}(GW_{-2n+1} - 3GW_{-2n} - GW_{-2n-1} + (GW_2 - GW_1 + GW_0) + 2(-GW_2 + GW_1 + GW_0)n)$.

Proof. **(a)** Taking $r = 0, s = 2, t = 1$ in Theorem 3.1 (a) we obtain **(a)**.
(b) and **(c)**: Proof can be done as in the proof of Theorem 2.13. Induction also can be used for the proof.

From the last Theorem, we have the following Corollary which gives sum formula of Gaussian Pell-Padovan numbers (take $GW_n = GR_n$ with $GR_0 = 1 - i, GR_1 = 1 + i, GR_2 = 1 + i$).

Corollary 3.14 For $n \geq 1$, Gaussian Pell-Padovan numbers have the following property:

(a) $\sum_{k=1}^{n} R_{-k} = \frac{1}{2}(-3R_{-n-1} - 3R_{-n-2} - R_{-n-3} + (1+3i))$.
(b) $\sum_{k=1}^{n} R_{-2k} = -R_{-2n+1} + R_{-2n} + 2i + (-1+i)n$.
(c) $\sum_{k=1}^{n} R_{-2k+1} = \frac{1}{2}(R_{-2n+1} - 3R_{-2n} - R_{-2n-1} + (1-i) + (2-2i)n)$.

Taking $GW_n = GC_n$ with $GC_0 = 3 - 4i, GC_1 = 3i, GC_2 = 2$ in the last Theorem, we have the following Corollary which gives sum formulas of Gaussian Pell-Perrin numbers.

Corollary 3.15 For $n \geq 1$, Gaussian Pell-Perrin numbers have the following property:

(a) $\sum_{k=1}^{n} GC_{-k} = \frac{1}{2}(-3GC_{-n-1} - 3GC_{-n-2} - GC_{-n-3} + (-1+7i))$.
(b) $\sum_{k=1}^{n} GC_{-2k} = -GC_{-2n+1} + GC_{-2n} + (-3+7i) + (-1+i)n$.
(c) $\sum_{k=1}^{n} GC_{-2k+1} = \frac{1}{2}(GC_{-2n+1} - 3GC_{-2n} - GC_{-2n-1} + (5-7i) + (2-2i)n)$.

Taking $r = 0, s = 1, t = 2$ in Theorem 3.1 (a) and (b), we obtain the following Proposition.

Proposition 3.16 If $r = 0, s = 1, t = 2$ then for $n \geq 1$ we have the following formulas:
(a) $\sum_{k=1}^{n} GW_{-k} = \frac{1}{2}(-3GW_{-n-1} - 3GW_{-n-2} - 2GW_{-n-3} + GW_2 + GW_1)$.
(b) $\sum_{k=1}^{n} GW_{-2k} = \frac{1}{2}(-GW_{-2n+1} + GW_1)$.
(c) $\sum_{k=1}^{n} GW_{-2k+1} = \frac{1}{2}(-GW_{-2n} - 2GW_{-2n-1} + GW_2)$.

From the last Proposition, we have the following Corollary which gives linear sum formulas of Gaussian Jacobsthal-Padovan numbers (take $GW_n = GQ_n$ with $GQ_0 = 1, GQ_1 = 1+i, GQ_2 = 1+i$).

Corollary 3.17 For $n \geq 1$, Gaussian Jacobsthal-Padovan numbers have the following properties.
(a) $\sum_{k=1}^{n} GQ_{-k} = \frac{1}{2}(-3GQ_{-n-1} - 3GQ_{-n-2} - 2GQ_{-n-3} + 2(1+i))$.
(b) $\sum_{k=1}^{n} GQ_{-2k} = \frac{1}{2}(-GQ_{-2n+1} + (1+i))$.
(c) $\sum_{k=1}^{n} GQ_{-2k+1} = \frac{1}{2}(-GQ_{-2n} - 2GQ_{-2n-1} + (1+i))$.

Taking $GW_n = GD_n$ with $GD_0 = 3 - \frac{1}{2}i, GD_1 = 3i, GD_2 = 2$ in the last Proposition, we have the following Corollary which gives linear sum formulas of Gaussian Jacobsthal-Perrin numbers.

Corollary 3.18 For $n \geq 1$, Gaussian Jacobsthal-Perrin numbers have the following properties.
(a) $\sum_{k=1}^{n} GD_{-k} = \frac{1}{2}(-3GD_{-n-1} - 3GD_{-n-2} - 2GD_{-n-3} + (2+3i))$.
(b) $\sum_{k=1}^{n} GD_{-2k} = \frac{1}{2}(-GD_{-2n+1} + 3i)$.
(c) $\sum_{k=1}^{n} GD_{-2k+1} = \frac{1}{2}(-GD_{-2n} - 2GD_{-2n-1} + 2)$.

Taking $r = 1, s = 0, t = 1$ in Theorem 3.1, we obtain the following Proposition.

Proposition 3.19 If $r = 1, s = 0, t = 1$ then for $n \geq 1$ we have the following formulas:
(a) $\sum_{k=1}^{n} GW_{-k} = -2GW_{-n-1} - GW_{-n-2} - GW_{-n-3} + GW_2$.
(b) $\sum_{k=1}^{n} GW_{-2k} = \frac{1}{3}(-2GW_{-2n+1} + GW_{-2n} - GW_{-2n-1} + GW_2 + GW_1 - GW_0)$.
(c) $\sum_{k=1}^{n} GW_{-2k+1} = \frac{1}{3}(-GW_{-2n+1} - GW_{-2n} - 2GW_{-2n-1} + 2GW_2 - GW_1 + GW_0)$.

From the above Proposition, we have the following Corollary which gives linear sum formulas of Gaussian Narayana numbers (take $GW_n = GN_n$ with $GN_0 = 0, GN_1 = 1, GN_2 = 1+i$).

Corollary 3.20 For $n \geq 1$, Gaussian Narayana numbers have the following properties.
(a) $\sum_{k=1}^{n} N_{-k} = -2N_{-n-1} - N_{-n-2} - N_{-n-3} + (1+i)$.
(b) $\sum_{k=1}^{n} N_{-2k} = \frac{1}{3}(-2N_{-2n+1} + N_{-2n} - N_{-2n-1} + (2+i))$.

(c) $\sum_{k=1}^{n} N_{-2k+1} = \frac{1}{3}(-N_{-2n+1} - N_{-2n} - 2N_{-2n-1} + (1+2i))$.

Taking $r = 1, s = 1, t = 2$ in Theorem 3.1, we obtain the following Proposition.

Proposition 3.21 If $r = 1, s = 1, t = 2$ then for $n \geq 1$ we have the following formulas:

(a) $\sum_{k=1}^{n} GW_{-k} = \frac{1}{3}(-4GW_{-n-1} - 3GW_{-n-2} - 2GW_{-n-3} + GW_2 - GW_0)$.

(b) $\sum_{k=1}^{n} GW_{-2k} = \frac{1}{3}(-GW_{-2n+1} + GW_{-2n} + GW_1 - GW_0)$.

(c) $\sum_{k=1}^{n} GW_{-2k+1} = \frac{1}{3}(-GW_{-2n} - 2GW_{-2n-1} + GW_2 - GW_1)$.

Taking $GW_n = GJ_n$ with $GJ_0 = 0, GJ_1 = 1, GJ_2 = 1 + i$ in the last Proposition, we have the following Corollary which gives linear sum formulas of Gaussian Jacobsthal numbers.

Corollary 3.22 For n $\geq$ 1, Gaussian Jacobsthal numbers have the following properties.

(a) $\sum_{k=1}^{n} GJ_{-k}^{(3)} = \frac{1}{3}(-4GJ_{-n-1}^{(3)} - 3GJ_{-n-2}^{(3)} - 2GJ_{-n-3}^{(3)} + (1+i))$.

(b) $\sum_{k=1}^{n} GJ_{-2k}^{(3)} = \frac{1}{3}(-GJ_{-2n+1}^{(3)} + GJ_{-2n}^{(3)} + 1)$.

(c) $\sum_{k=1}^{n} GJ_{-2k+1}^{(3)} = \frac{1}{3}(-GJ_{-2n}^{(3)} - 2GJ_{-2n-1}^{(3)} + i)$.

From the last Proposition, we have the following Corollary which gives linear sum formulas of Gaussian Jacobsthal-Lucas numbers (take $GW_n = Gj_n^{(3)}$ with $Gj_0^{(3)} = 2 + i, Gj_1^{(3)} = 1 + 2i, Gj_2^{(3)} = 5 + i$).

Corollary 3.23 For $n \geq 1$, Gaussian Jacobsthal-Lucas numbers have the following properties.

(a) $\sum_{k=1}^{n} Gj_{-k}^{(3)} = \frac{1}{3}(-4Gj_{-n-1}^{(3)} - 3Gj_{-n-2}^{(3)} - 2Gj_{-n-3}^{(3)} + 3)$.

(b) $\sum_{k=1}^{n} Gj_{-2k}^{(3)} = \frac{1}{3}(-Gj_{-2n+1}^{(3)} + Gj_{-2n}^{(3)} + (-1+i))$.

(c) $\sum_{k=1}^{n} Gj_{-2k+1}^{(3)} = \frac{1}{3}(-Gj_{-2n}^{(3)} - 2Gj_{-2n-1}^{(3)} + (4-i))$.

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Teaching Mathematics through coding and programming. Programming with students during math lessons.

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Abstract

Mathematics has always been a traditional science. And, that will be always the case in the future. Algebra, Euclidean Geometry, Trigonometry, and all the classic definitions and theorems of mathematics will be always in our books and notebooks. Blackboard and chalk will be permanent elements of teaching, where math professors will write and prove math theorems, draw graphs, and solve problems. However, with the development of Information Technology, we are witnessing that many mathematical issues, problems, and calculations, are solved with the help of familiar programs like Java, C ++, Python, etc. The first examples of computer programming to explore mathematics are found in the 1960s, (Feurzeig, 1969), although the most prominent pedagogical approach was proposed by Seymour Papert, [8]. He was the first to note that young people learn best when they are engaged in the construction of digital and/or physical artifacts that are personally meaningful to them and that can be shared with others (Papert, 1980). The use of software and applications will help students to understand and solve math problems, this applies especially to students who will become high school, math teachers. Starting from EXCEL, students can try to do simple tasks such as multiplication of matrices, solving algebraic equations, and others. Students can also write code and programs in Java, C ++, Python, etc., to handle and solve problems of Discrete Mathematics, Number Theory courses, etc. They can generate Pythagorean triples, verify prime and perfect numbers, generate Ferma and Mersenne numbers, etc. The coding and programming, in addition to being a fascinating process in itself, will become even more fascinating when it comes to finding solutions and results, that will strengthen the knowledge of math concepts, aiding students and professors in working together for a better and more effective classroom.

Keywords. Mathematics, coding, teaching, programming, classroom.

1. Introduction

Information nowadays and, what interests us, scientific information as an important and very useful part of information also grows extremely fast. A big problem today is not with communication and acquaintance with the latest information, but with its evaluation, classification, evaluation of its usefulness, and its use. The school today, and as an important part of it, the mathematics and natural sciences programs, are under great pressure to recognize and evaluate the scientific information that is added every day. The use of information technology, computers, the internet, increasing the speed and amount of information has greatly increased the pressure on our school to find ways and to use this information effectively and to put it at the service of the process of teaching and education to students. How much part of this information and how

can it be used and included in our school curricula, to not only increase the knowledge acquired by students and teachers but also to help the acquisition of programs that are used in our schools, this is a challenge we face every day.
Our pre-university and university school programs will continue with materials and topics that have been known for centuries and that is right. Arithmetic, Algebra, Number Theory, Geometry, Trigonometry, etc., will continue to bring together the same topics, some of which have been around for centuries. Students need to learn basic knowledge of arithmetic, algebra; numbers, integers, properties of numbers, sets, numerical sets, operations with numbers, equations, inequalities, inequations, [6]. They need and will continue to learn fractions, properties, and operations with them, will continue to learn Euclidean geometry, straight lines, properties, angles, properties and relations of angles, triangles, they will continue with polygons, circles, they will learn many theorems and many wonderful proofs that have to do with their properties, [7].
School and university programs can not change so quickly and there is no need to change so quickly. The need and usefulness of classical mathematics will remain the same and necessary for the scientific and math education of pupils and students. The school will continue to successfully address and bring together classical knowledge and knowledge. What worries us most today is the daily practice of assessing pupils and students, by giving up theoretical knowledge, proofs, reasoning. Our school today values almost only information, and in this regard, the use of information technology seems to help. There is nothing wrong with using the well-known math solving applications, which helps solve equations, among other things, but the student should know the formula for calculating the roots of any algebraic equation, and especially the discussion of solutions, the number of roots according to the values of the discriminant. Information Technology tools provide information for everyone, and very easily, and this is very attractive to pupils and students, including especially the ease of obtaining information and solving problems. The lure of finding ready-made solutions and answers is immense, and in fact, this is happening rapidly today. High school students get a lot more information than 20 years ago, but, and this is disturbing, they know much less than 20 years ago, especially when it comes to theoretical knowledge, theorems, proofs and reasoning on formulas and operations, properties of functions, figures, etc. It seems that this "war" for students, for the education of students, a "war" that should not have been at all, is gained by formal memory, photography, plain information instead of true knowledge, theory, cognition, mathematical education, reasoning.

The problem is;

- How can we find out how we can use information technology, software, applications, the web, with the purpose to increase students' interest in science subjects, especially mathematics?
- How can we incorporate elements of information technology, software, applications, to engage more students in the teaching process?
- How can we make them feel better and more valued?
  How can we make them self-motivated to work and learn?
- How can we involve pupils and students to create in the learning process using software and applications?
- How can we make them improve themselves during the process of learning and creating?

2. Main results

There is a big difference in the reaction and attention of students' when they simply hear the new lessons from the teacher or the math professor and see the solution of problems or the proving theorems by him, in comparison to another way, when they participate by creating and becoming part of the new lessons. The use of software, applications, helps students solve math problems, this is more helpful for students who will become high school, math teachers. Software and applications that are associated with theoretical knowledge and explanations about what they do are also more useful to them. Such software and applications help students to read and understand better. There are plenty of such applications for number theory, statistics, discrete mathematics, etc. But, more than knowing and using applications and computer software, creating codes and programming during the process of learning is much more valuable and most useful than the execution of software and applications, [7].

This because;

- The process of coding and creating applications and programs is the same as solving mathematical problems,
- During coding and programming students repeat all the information they need to solve math problems,
- Instead of the "boring" formulas of mathematics and the "boring" process of teaching mathematics to prove, calculate, reason, they have an "interesting" challenge, that of creation,
- Young people respond very well to almost personal challenges, the desire to create is a personal challenge and success is personal.
- Pupils, students do not have the teacher to correct them and advise what to do next, it is the coding and programming process itself that does this and the application that suggests where they made a mistake and what the mistake is, this is a very objective assessment.
- Coding to help solve math problems is a unique learning experience for students. In this type of class, when possible, students engage in solving exercises and problems while learning math as well as programming.
- When a student writes code, the student is practically "learning" the computer how to complete a task and uses the entire normal process of solving the exercise and programming. This puts him in an active, engaged, and positive position.
- When students write in code to solve an exercise or problem, they immediately see if the program they have created is successful or not. Thus they become self-teachers and become self-correctors. This is better than being evaluated in the process by the teacher, whether they are wrong or not. This puts them in a new position and encourages them more to want to succeed because the process and the result are a purely personal challenge or a group challenge and they are interested in success.
  Instead of asking students what they have learned and read, we;
- We ask them to create a working group and start problem-solving projects through coding and programming; we know all the benefits of working groups and projects and creating together, as a very useful and effective process.
- Students will immediately begin to deal with the problems, to understand them as they need to start the codes. This is a much more effective process than reading the same math material for knowledge purposes, or because they may be asked in class.
- The process of writing code is a process towards perfection, better understanding, and finding the best solutions, and most optimally, the challenge is not only the solution but also the most effective solution

that uses less time to be realized. We will see a realization of the "Polya" method of solving the program problem.

- During the process, students will have to repeat the correction process many times to make their code work properly and will do so on their own, without the need for encouragement from teachers, professors.
- They will seek help from the professors when needed, they will also seek out math and computer forums online, and collaborate with other groups.
- Teaching students how to solve a math problem and how to spot errors in their code and improve it follows the same method as solving a math problem, which we know very well from Polya. This is a very educational process.
- While many students lack the desire to solve problems or prove mathematical theorems during a traditional way of teaching, on the blackboard, when asked during math lessons, they will be more likely to ask themselves and seek to solve their own mistakes. them.
- Instead of a "boring" class and math class they already have a challenging and interesting class called "code and math".

3. Examples- Case studies

We have monitored three Bachelor MI (Mathematics and Informatics) classes and one Master MI class, with the purpose to evaluate and promote the use of Information Technology, coding and programming and, to help students have a different approach to learning math. At the same time, we have encouraged students as future high school teachers to have a better evaluation and understanding attitude towards IT knowledge and practices. There are a total of 100 students of the bachelor program and 45 students of the master program to conduct this study.

The University of Durres was founded in 2006, it has about 14,000 students, 5 Faculties, and many bachelor and master programs. Among the faculties, the Faculty of IT has 5 bachelor programs and 3 master programs.

The main programs in the Bachelor are;
Computer Science, CS,
Information Technology, IS,
Information systems, IS,
Mathematics and Informatics, MI,
Informatics and English, IE.

The MI program, which is a program of the Math Department, graduates students who continue their master's degree in MI. The purpose of the MI master program is the education of students and their preparation as future teachers of high school teachers of Mathematics and IT. We believe that our graduate students will succeed as future math teachers, the bachelor study program best meets the needs of math teachers for high schools. Three years of a bachelor program and two years of master program are enough for future math teachers, the problem is that they lack the knowledge and proper training to become successful IT teachers. Their knowledge about MS Office, IT basics, coding, programming is very weak and this problem is becoming more serious with the years to come and go.

We are witnessing that most graduated high school students have a very poor Informatics education.

The problem consists mostly of the Informatics teachers, poor laboratory equipment, as we think that there are very good IT textbooks which contain more than basics of Microsoft Office, introduction to programming, programming languages, such as C ++, Java, and even Java Script.
The solution is the improvement of teaching in IT high school, better laboratories, and more qualified teachers. Most of the solutions start from University; with the education of students to become future teachers.
The MI bachelor program of our Math Department contains some important courses of IT such as;
Introduction to programming, Web programming, Programming languages C++, Java, etc.
Among math courses, some important courses are; Algebra, Math Analyses, Discrete mathematics, Number Theory, Probability and Statistics, Combinatorics, etc.
The best math courses to "get into coding are"

1. Number theory, with topics;

- Find divisions of any integer,
- Verify primes, find prime numbers, the perfect number, abundant and deficient numbers,
- Convert positive integers in any numeric system,
- Application of Euclid Algorithm,
- Solve linear Diophantine equations, find several solutions,
- Generate triple Pythagorean numbers, Generate Fermat numbers,
- Generate Mersenne numbers, verify if they are prime,

2. Probability and Statistics, with topics;

- Calculate probability for any probability distributions,
- Evaluating intervals, testing hypothesis, etc.,
- Statistic data, numerical parameters, create classes, create histograms, etc.,
- Graphics, linear regressions, quadratic, exponential, etc.,

3. Combinatorics; with topics;

- Calculate factorials, formulas,
- Graphs, etc.,

4. Discrete Math, with topics;

- Logic, sets, integers,
- Congruences, etc.

Microsoft Office- Excel logical functions for Algebra.

Microsoft Office- Excel has good use of logical statements. There are some logical functions used in Microsoft- Excel, some of these are: if, count if, and, or, etc. The program verifies the fulfillment of a condition, if the condition is met then the cell receives a value that we give it to ourselves, otherwise, it receives another value.
Students were asked to build a multiple-choice test, for the use of any math topics that they are learning. İt is not difficult to convert a normal test to a multiple-choice test, this can be done working together with students. The column of correct answers as well as the column of students' answers can be "hidden" on another worksheet, and they are to be discovered after students have finished the test. Both columns of answers and the result are made known to the students immediately, after the test, making an easy and very fast and transparent process, table 1.

Table 1. A multiple-choice test, with the result.

	Alternatives					Answers	Answers	Comparison
Questions	A	B	C	D	E	Right	Student	Values
1	A	B	C	D	E	C	C	10
2	A	B	C	D	E	B	B	10
3	A	B	C	D	E	C	C	10
4	A	B	C	D	E	C	C	10
5	A	B	C	D	E	C	C	10
6	A	B	C	D	E	E	A	0
7	A	B	C	D	E	D	B	0
8	A	B	C	D	E	B	B	10
9	A	B	C	D	E	A	D	0
10	A	B	C	D	E	E	E	10
							Result	70%

There are many other topics and problems from Algebra that can be covered by the use of software and applications. Some problems that can be addressed in Excel to learn Algebra and Excel, such as; solving algebraic equations, doing matrix multiplication, verify if a matrix is singular or not, calculating the determinant of matrices, finding the inverse of a nonsingular square matrix, table 2.

Table 2, Matrix, the inverse, and multiplication.

|A|=1			
	1	2	3
1	3	5	0
2	1	2	1
3	0	0	1

A (-1)			
	1	2	2
1	2	-5	5
2	-1	3	-3
3	0	0	1

A*A (-1)			
	1	2	3
1	1	0	0
2	0	1	0
3	0	0	1

A (-1) *A			
	1	2	3
1	1	0	0
2	0	1	0
3	0	0	1

Microsoft Office- Excel and Number Theory.

Perfect, abundant, and deficient numbers:

The concept of perfect number is connected with the concept of the divisors of an integer.
Definition: A positive integer a is a proper divisor of n if a is a divisor of n and a<n.
So the only "improper" divisor of a number is the number itself; all other divisors are proper.
We'll write P(n) for the sum of all the proper divisors of the number n.
Example: P(12)=16, because the proper divisors of 12 are 1, 2,3,4,6, P(12)=1+2+3+4+6=16.
If P(n) > n, then n is called an abundant number.
If P(n) < n, then n is called a deficient number.
If P(n) = n, then n is called a perfect number.
İt is not difficult to find all the divisors of a positive integer using Excel and counting them, but C++, Java is better to work. And, there are many software and applications to do that.

Generation of Pythagorean triples: $c^2 = a^2 + b^2, a, b, c \in N$.
The generation of Pythagorean triples is made by the Euclid formula. We choose a pair of integers m, n with $m > n > 0$. Then, the integers $a = m^2 - n^2, b = 2mn, c = m^2 + n^2$ form a pythagoren triple.
Generation of Ferma, Mersenne numbers; $c = 2^{2^n} + 1,\ c = 2^n - 1$, and verifying if they are prime.
All the programs were written in C++ or Java.

Generate Fibonacci, Lukas sequences:
The Fibonacci sequence starts with two first numbers 1 and 1. The recurrent formula for the Fibonacci sequence is;
$F_1 = 1, F_2 = 1, F_n = F_{n-2} + F_{n-1}, n \geq 3$
The first 15 numbers in the Fibonacci sequence are;
1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610,
Lukas sequence starts with two first numbers 1 and 3. The recurrent formula for the Lukas sequence is;
$L_0 = 1, L_1 = 3, L_n = L_{n-2} + L_{n-1}, n \geq 3$
The first 15 numbers in the Lukas sequence are;
1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843, 1364.

Table 3. Fibonacci, Lukas, with Excel.

NR	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
F	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610
L	1	3	4	7	11	18	29	47	76	123	199	322	521	843	1364

Statistics:
It is very simple and easy for Excel to build graphs, frequency histograms, etc. There are many statistical functions to do such as calculation of probabilities for different probabilistic distributions such as Binomial, Poisson, Normal, etc. Other useful tasks to do with excel are; a complete analysis of descriptive statistics, calculation of various statistical tests, etc.
One task performed by the students is to write a numerical sequence as statistical data and;
count the number of values,
count the number of values, on some condition,
find all the statistical parameters such as min, max, mode, median, mean,
find quartiles, interquartile range, variance,
divide the data into several classes or groups,
find the frequency of each class, the modal class, the average according to the classes, etc.
All these tasks have been very well completed by the students, learning statistics and Excel, together.

4. Conclusions

- We may say, everything can be programmed, coded, especially math problems.
- Theoretical mathematics, definitions in mathematics, theorems, proofs, etc., starting with Euclid, Pythagoras, Thales, are necessary to be learned by students, as part of their mathematical education.
- While proving theorems will remain a permanent task of mathematical education, much can be done by using IT, applications, software, such as those mentioned above.

- For simple things, we can use Office-Excel, especially for statistical studies.
- For more complex problems, such as number theory, discrete mathematics, etc., we can use programming languages, such as C ++, Java, etc.
- The attitude of the students changes completely when it comes to the realization of projects on the topics mentioned above. They are much more very interested, engaged, and cooperative.
- The process will help them not only to learn to program, it will help them to use math formulas, definitions, concepts. İt will help them teach and promote IT for their future pupils.

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The Bounds for the Length Between Dirichlet and the Semi-Periodic Eigenvalues of Hill's Equation with Symmetric Single Well Potential

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Abstract

In this paper, the length between Dirichlet and the semi-periodic eigenvalues is estimated for Hill's equation with continuous and periodic potential. The potential is also symmetric to the midpoint and nonincreasing on the first half of the related interval. This function is known as symmetric single well potential in quantum mechanics.

1. Introduction

We are interested in the following equation:

$$y''(t)+\left[\lambda-q(t)\right]y(t)=0, \tag{1.1}$$

where $t\in[0,a]$, λ is a real parameter, $q(t)$ is a real-valued, continuous and periodic function with period a. This equation is called as Hill's equation or one-dimensional Schrödinger equation. If we take Equation (1.1) with boundary conditions $y(0)=y(a)$ and $y'(0)=y'(a)$, this is periodic problem and we show the eigenvalues of this problem by $\{\lambda_n\}$. If we take Equation (1.1) with boundary conditions $y(0)=-y(a)$ and $y'(0)=-y'(a)$, this is semi-periodic problem or anti-periodic problem and we show the eigenvalues of this problem by $\{\mu_n\}$. Equation (1.1) with boundary conditions $y(0)=y(a)=0$ is known by Dirichlet boundary value problem with eigenvalues $\{\Lambda_n\}$ and Equation (1.1) with boundary conditions $y'(0)=y'(a)=0$ is known by Neumann boundary value problem with eigenvalues $\{\nu_n\}$.

We have the relationship between eigenvalues of Hill's equation proven by [4] as following:

$$-\infty<\lambda_0<\mu_0\le\mu_1<\lambda_1\le\lambda_2<\mu_2\le\mu_3<\cdots$$

and for $n=0,1,2,...$

$$\mu_{2n}\le\Lambda_{2n}\le\mu_{2n+1},\qquad \lambda_{2n+1}\le\Lambda_{2n+1}\le\lambda_{2n+2}, \tag{1.2}$$

$$\mu_{2n} \le \nu_{2n+1} \le \mu_{2n+1}, \qquad \lambda_{2n+1} \le \nu_{2n+2} \le \lambda_{2n+2}.$$

Also, we emphasize that for $\lambda \in (\lambda_{2n}, \mu_{2n}) \cup (\mu_{2n+1}, \lambda_{2n+1})$, all solutions of Hill's equation are bounded in $(-\infty, +\infty)$; and for $\lambda \in (-\infty, \lambda_0) \cup (\mu_{2n}, \mu_{2n+1}) \cup (\lambda_{2n+1}, \lambda_{2n+2})$, all nontrivial solutions of Hill's equation are unbounded in $(-\infty, +\infty)$.

Hydrogen bonding is a fundamental aspect of chemical structure and reactivity. Is a hydrogen bond symmetric (single-well potential) or asymmetric (double-well potential)? It is a key to understanding the structure and properties of water, proteins, DNA, and it is currently of interest for designing systems that exhibit molecular recognition. Hydrogen bonds are usually thought to arise from electrostatic attraction between an O-H (or N-H or H-F) dipole and the electron density on a nearby O (or N or F). Hydrogen bonds may have extra stabilization, often viewed as arising from a covalent character, if the two contributing resonance forms, $O-H\cdots O^-$ and $O^-\cdots H-O$, are of equal or nearly equal energy. This is more likely if the two donor atoms have the same basicity (matched pK_a values) and if the hydrogen is centered between them. Indeed, symmetric hydrogen bonds seem to be stronger than asymmetric ones [9]. Besides, especially in recent years, since quantum mechanic has gained importance, there are a lot of studies on eigenvalues of Hill's equation with symmetric single well potential such as anharmonic oscillator. The eigenvalues of these equations represent excitation energy and eigenfunctions are named as wavefunction in physics. Instability intervals for Hill's equation with symmetric single well potential have been investigated by many authors. Some important studies of them are [1,2,3,6,7,8]. Especially, we refer to [7] includes the bound for $\Lambda_0 - \lambda_0$.

In this study, the length between Dirichlet and the semi-periodic eigenvalues is estimated for Hill's equation with continuous and periodic potential. The potential is also symmetric to the midpoint and nonincreasing on the first half of the related interval. This function is known as symmetric single well potential in quantum mechanics. We remark that a symmetric single well potential on [0,a] means a continuous function $q(t)$ on [0,a] which is symmetric about t=a/2 and non-increasing on [0,a/2], so we have $q(t) = q(a-t)$, mathematically.

2. Main Results

Before giving main result, we note that $q'(t)$ exists since a monotone function on an interval I is differentiable almost everywhere on I [5]. Our analysis is based on the following theorem of [3]:

The periodic and semi-periodic eigenvalues of Equation (1) satisfy, as

$$\begin{aligned}\begin{matrix}\lambda_{2n+1}^{1/2}\\ \lambda_{2n+2}^{1/2}\end{matrix} &= \frac{2(n+1)\pi}{a} \mp \frac{a}{8(n+1)^2\pi^2}\left|\int_0^{a/2} q'(t)\sin\left[\frac{4(n+1)\pi}{a}t\right]dt\right| \\ &-\frac{a^2}{64(n+1)^3\pi^3}\left[aq^2(a)+2a\int_0^{a/2} q(t)q'(t)dt-4\int_0^{a/2} tq(t)q'(t)dt\right] \\ &+o\left(n^{-3}\right)\end{aligned} \tag{2.1}$$

and

$$\begin{aligned}\begin{matrix}\mu_{2n}^{1/2}\\ \mu_{2n+1}^{1/2}\end{matrix} &= \frac{(2n+1)\pi}{a} \mp \frac{a}{2(2n+1)^2\pi^2}\left|\int_0^{a/2} q'(t)\sin\left[\frac{2(2n+1)\pi}{a}t\right]dt\right| \\ &-\frac{a^2}{8(2n+1)^3\pi^3}\left[aq^2(a)+2a\int_0^{a/2} q(t)q'(t)dt-4\int_0^{a/2} tq(t)q'(t)dt\right] \\ &+o\left(n^{-3}\right).\end{aligned} \tag{2.2}$$

The purpose of this study is to prove the following theorems:

Theorem: Let $q(t)$ be a symmetric single well potential on $[0,a]$. Then, the bounds for the length between Dirichlet and the semi-periodic eigenvalues, as $n \to \infty$

$$\begin{aligned}\Lambda_{2n+1}-\mu_{2n+1} \geq &\frac{(4n+3)\pi^2}{a^2} - \frac{1}{2(n+1)\pi}\left|\int_0^{a/2} q'(t)\sin\left[\frac{4(n+1)\pi}{a}t\right]dt\right| \\ &-\frac{1}{(2n+1)\pi}\left|\int_0^{a/2} q'(t)\sin\left[\frac{2(2n+1)\pi}{a}t\right]dt\right| \\ &-\frac{a}{4\pi^2}\left[\frac{1}{4(n+1)^2}-\frac{1}{(2n+1)^2}\right]\left[aq^2(a)+2a\int_0^{a/2} q(t)q'(t)dt-4\int_0^{a/2} tq(t)q'(t)dt\right] \\ &+o\left(n^{-2}\right)\end{aligned}$$

and

$$\begin{aligned}\Lambda_{2n+1}-\mu_{2n+1}\le{}&\frac{(4n+3)\pi^2}{a^2}+\frac{1}{2(n+1)\pi}\left|\int_0^{a/2}q'(t)\sin\left[\frac{4(n+1)\pi}{a}t\right]dt\right|\\&-\frac{1}{(2n+1)\pi}\left|\int_0^{a/2}q'(t)\sin\left[\frac{2(2n+1)\pi}{a}t\right]dt\right|\\&-\frac{a}{4\pi^2}\left[\frac{1}{4(n+1)^2}-\frac{1}{(2n+1)^2}\right]\left[aq^2(a)+2a\int_0^{a/2}q(t)q'(t)dt-4\int_0^{a/2}tq(t)q'(t)dt\right]\\&+o\left(n^{-2}\right).\end{aligned}$$

Proof of Theorem: Firstly, if we take the squares of the eigenvalues in Equation (2.1) and Equation (2.2) asymptotically, we find that

$$\begin{aligned}\begin{matrix}\lambda_{2n+1}\\\lambda_{2n+2}\end{matrix}={}&\frac{4(n+1)^2\pi^2}{a^2}\mp\frac{1}{2(n+1)\pi}\left|\int_0^{a/2}q'(t)\sin\left[\frac{4(n+1)\pi}{a}t\right]dt\right|\\&-\frac{a}{16(n+1)^2\pi^2}\left[aq^2(a)+2a\int_0^{a/2}q(t)q'(t)dt-4\int_0^{a/2}tq(t)q'(t)dt\right]\\&+o\left(n^{-2}\right)\end{aligned}\tag{2.3}$$

and

$$\begin{aligned}\begin{matrix}\mu_{2n}\\\mu_{2n+1}\end{matrix}={}&\frac{(2n+1)^2\pi^2}{a^2}\mp\frac{1}{(2n+1)\pi}\left|\int_0^{a/2}q'(t)\sin\left[\frac{2(2n+1)\pi}{a}t\right]dt\right|\\&-\frac{a}{4(2n+1)^2\pi^2}\left[aq^2(a)+2a\int_0^{a/2}q(t)q'(t)dt-4\int_0^{a/2}tq(t)q'(t)dt\right]\\&+o\left(n^{-2}\right).\end{aligned}\tag{2.4}$$

By Equation (1.2), we obtain the bounds for $\Lambda_{2n+1}-\mu_{2n+1}$ that

$$\lambda_{2n+1}-\mu_{2n+1}\le\Lambda_{2n+1}-\mu_{2n+1}\le\lambda_{2n+2}-\mu_{2n+1}.\tag{2.5}$$

By using Equation (2.3) and Equation (2.4), we calculate the upper bound for $\Lambda_{2n+1}-\mu_{2n+1}$ as following:

$$\begin{aligned}\lambda_{2n+2}-\mu_{2n+1}=&\frac{(4n+3)\pi^2}{a^2}+\frac{1}{2(n+1)\pi}\left|\int_0^{a/2}q'(t)\sin\left[\frac{4(n+1)\pi}{a}t\right]dt\right|\\&-\frac{1}{(2n+1)\pi}\left|\int_0^{a/2}q'(t)\sin\left[\frac{2(2n+1)\pi}{a}t\right]dt\right|\\&-\frac{a}{4\pi^2}\left[\frac{1}{4(n+1)^2}-\frac{1}{(2n+1)^2}\right]\left[aq^2(a)+2a\int_0^{a/2}q(t)q'(t)dt-4\int_0^{a/2}tq(t)q'(t)dt\right]\\&+o\left(n^{-2}\right).\end{aligned}\tag{2.6}$$

Similarly, by using Equation (2.3) and Equation (2.4), we calculate the lower bound for $\Lambda_{2n+1}-\mu_{2n+1}$ as following:

$$\begin{aligned}\lambda_{2n+1}-\mu_{2n+1}=&\frac{(4n+3)\pi^2}{a^2}-\frac{1}{2(n+1)\pi}\left|\int_0^{a/2}q'(t)\sin\left[\frac{4(n+1)\pi}{a}t\right]dt\right|\\&-\frac{1}{(2n+1)\pi}\left|\int_0^{a/2}q'(t)\sin\left[\frac{2(2n+1)\pi}{a}t\right]dt\right|\\&-\frac{a}{4\pi^2}\left[\frac{1}{4(n+1)^2}-\frac{1}{(2n+1)^2}\right]\left[aq^2(a)+2a\int_0^{a/2}q(t)q'(t)dt-4\int_0^{a/2}tq(t)q'(t)dt\right]\\&+o\left(n^{-2}\right).\end{aligned}\tag{2.7}$$

Hence, Equation (2.5), Equation (2.6) and Equation (2.7) prove the theorem.

Example: The following eigenvalue problem is known as anharmonic oscillator

$$-y''(x)+q(x)y(x)=\lambda y(x),\quad x\in[0,\pi)$$

where $q(x)=\frac{1}{4}\left(x-\frac{\pi}{2}\right)^4+\frac{1}{2}\left(x-\frac{\pi}{2}\right)^2$ and extended by periodicity. Since $q(x)$ has mean value zero in the reference theorem from [3], $q(x)$ can taken as follows:

$$q(x)=\frac{1}{4}\left(x-\frac{\pi}{2}\right)^4+\frac{1}{2}\left(x-\frac{\pi}{2}\right)^2-\frac{\pi^4}{320}-\frac{\pi^2}{24}.$$

In this case, by calculating integral terms in our theorem, we get as $n\to\infty$

$$\begin{aligned}\Lambda_{2n+1}-\mu_{2n+1} \geq (4n+3) & \\ & -\frac{1}{128(n+1)^4}\left\{2\left[\left(\pi^2+4\right)n(n+2)+\pi^2\right]+5\right\} \\ & -\frac{1}{16(2n+1)^4}\left\{4\left(\pi^2+4\right)n(n+1)+\pi^2-2\right\} \\ & -\frac{1}{4\pi}\left[\frac{1}{4(n+1)^2}-\frac{1}{(2n+1)^2}\right]\frac{\pi^5}{6451200}\left(112\pi^4+1920\pi^2+8960\right) \\ & +o\left(n^{-2}\right)\end{aligned}$$

and

$$\begin{aligned}\Lambda_{2n+1}-\mu_{2n+1} \leq (4n+3) & \\ & +\frac{1}{128(n+1)^4}\left\{2\left[\left(\pi^2+4\right)n(n+2)+\pi^2\right]+5\right\} \\ & -\frac{1}{16(2n+1)^4}\left\{4\left(\pi^2+4\right)n(n+1)+\pi^2-2\right\} \\ & -\frac{1}{4\pi}\left[\frac{1}{4(n+1)^2}-\frac{1}{(2n+1)^2}\right]\frac{\pi^5}{6451200}\left(112\pi^4+1920\pi^2+8960\right) \\ & +o\left(n^{-2}\right)\end{aligned}$$

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The evaluation of online teaching performance in high education using Google Classroom platform : A Fuzzy AHP method.

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Abstract

The emergence of the Pandemic Covid-19 caused to the education system the transition from classroom learning to online learning almost all over the globe. One of the main difficulties was the learning and using of a platform by lecturer and students. Google classroom was one of the platforms adapted very quickly. Through the unified theory of acceptance and use of technology 2 (UTAUT2) we have evaluated five criteria out of eight in total, relating to performance and use of this new technology for students of mathematics courses at the “Alexander Moisiu” University Durres Albania. The main criteria are considered Performance Expectancy (PE), Social influence (SI), Habit (HT), Behavioral Intention (BI), and Use Behavior (UB). Each of these criteria has their sub-criteria: PE_1, PE_2, PE_3, PE_4, SI_1, SI_2, SI_3, HT_1, HT_2, HT_3, HT_4, BI_1, BI_2, BI_3 UB_1, UB_2. The method used to evaluate the most important criteria and their sub-criteria is Fuzzy AHP with trapezoidal fuzzy numbers. The students answered the survey after completing the course study and passing the exam. According to the study results, the most important criterion was Performance Expectancy followed by Behavioral Intention, Social Influence, then Habit, and the last Use Behavior. Regarding the sub-criteria the most important were PE_4, BI_2, SI_3, HT_2 and UB_1. These results help students and lecturers to orient themselves towards online learning and to improve this teaching method.

Keywords: Google Classroom, Fuzzy AHP, trapezoidal fuzzy numbers, online learning.

1. Introduction

Taking trigger from the situation caused by the Covid-19 virus all around the globe, the high education system had to pass from traditional auditorium learning to online learning. The biggest difficulty was in implementing a digital platform in such a way that lecturers and students could interact during the online learning. A digital platform adapted very quickly was Google Classroom. It’s a free web service developed by Google for teachers/lecturer and students/pupil, and has the primary purpose to streamline the process of sharing files between them. This platform was easy to navigate because many people are familiar with Google layout, and according to students there are a lot of means to access Google Classroom. Android, Smartphone and Personal computer are some of the ways to access online learning. The use of Google Classroom in learning does not need to be complicated in installation. Besides that, Google Classroom can be used free of charge by anyone with an internet network (1). There is an ongoing debate about optimal ways to teach mathematics for students (2). On the one hand are the students who can’t understand the explanations of new the knowledge online, and on the other hand are the students who can understand the new knowledge and even do some exercises themselves.

In addition there are some other online application platforms used in several studies about the unified theory for acceptance and use of technology 2 (UTAUT2). The theory UTAUT2 has eight constructs (also named by us as the criteria) related to the use and the acceptance of a new technology that are: Performance Expectancy, Effort Expectancy, Social Influence, Facilitating Conditions, Hedonic Motivations, Habit, Behavioral Intention, and Use Behavior referring Venkatesh et al (3), (4). We will specify only five criteria. Performance expectancy is defined as the degree to which using a technology will provide benefits to students to perform certain activities. Social Influence is defined as the student's perceive that important others so family or friends, believe they should use a particular technology. Habit is defined as the extent to which people tend to perform behaviors automatically because of learning (5),(6). Behavioral Intention is defined as the degree to which a person has formulated conscious plans to perform or not perform some specified future behavior (7), (8). Use behavior is defined from the subsequent effects of habit and others, but mostly from the effect of behavioral intention (4).
Some of the articles that have studied UTAUT2 theory for online education are: T.H Tseng studied the number of the massive open online courses (MOOCs) to investigate the drivers of teachers' acceptance and use of MOOCs from the perspective of the extended unified theory of acceptance and use of technology (UTAUT2) (9). Ashwin Mehta et al in their study integrate values with technology adoption models and applied the novel conceptual model to the context of digital education (10). Marko Radovan examined the acceptance and use of learning management systems (LMS) among higher-education teachers and the relation between their use of such systems and their teaching approaches in the context of online learning, following the community of inquiry (CoI) framework (11). Silvana Dakduk evaluates the factors involved in the acceptance of Blended Learning through UTAUT2, and showed that hedonic motivation, performance expectancy, and effort expectancy predict the intention to adopt Blended Learning (12). Mateus Martins studied the acceptance of e-books by identifying the effects of the variables Performance Expectancy, Effort Expectancy, Social Influence, Facilitating Conditions, Habit, Price Value, and Hedonic Motivation, moderated by Age, Gender and Experience on the intention of use and actual use of this technology (13). Shuiqing Yang showed an adoption model that reflects the determinants of undergraduate students' mobile learning acceptance in a consumer context. The results indicated that hedonic motivation, performance expectancy, social influence, and price value positively affect students' mobile learning adoption (14).
But there are fewer studies in the field of decision making problems to estimate better the importance of each of these constructs and which of them according to students, is thought to be the most effective for the online learning. One of the most used method in decision making is Analytic Hierarchic Process known as the AHP developed by Thomas Saaty (15), (16). Saaty introduced AHP to solve real life problems in decision making, but to deal with uncertainty in complex problems and multi criteria decision making (MCDM), AHP is combined with fuzzy logic by applying fuzzy set of Zadeh (17), called as the Fuzzy AHP (FAHP). In some articles dealing with the learning management systems (LMS) with Fuzzy AHP are used the triangular fuzzy numbers. Ali Hakan Işık used the Fuzzy AHP (FAHP) to determine the suitable LMS for the e-learning environment (18). Yasemin A. Turker showed that the decision making and selection process is crucial, because LMS functions are different from each other, and they all have various features. Because of this, in her study , it was targeted to help and facilitate the decision-making process of LMS for institutions (19).
The main purpose of this study is to evaluate the performance and the use of Google Classroom into the higher education, during the online learning caused from COVID-19 for the students of "Alexander

Moisiu" University Durres. We apply the UTAUT2 theory in forms of a survey for the students of bachelor studies in the mathematical courses. We have take into consider only five constructs from eight of the UTAUT2: Performance Expectancy (PE), Social influence (SI), Habit (HT), Behavioral Intention (BI), and Use Behavior (UB). In order to estimate which of them are more important, we use Fuzzy AHP to rank these criteria. Each of the criteria has their sub-criteria. The survey is developed for about 200 students, after doing online lectures and done the exams in the auditor of the university. The full survey with the criteria and their sub-criteria is as follows:

Criteria/Sub-criteria

Performance Expectancy (PE)

1. I find Google Classroom useful in this course of math. (PE_1)
2. Using Google Classroom enables me to achieve course related tasks more quickly. (PE_2)
3. Using Google Classroom increases my learning productivity. (PE_3)
4. If I use Google Classroom, I will increase my chances of passing the course. (PE_4)

Social Influence (SI)

1.My friends who are important to me think that I should participate in Google Classroom.(SI_1)
2. My peers who influence my behaviour think that I should use Google Classroom.(SI_2)
3.Other people whose opinions I value prefer that I use Google Classroom. (SI_3)

Habit (HT)

1. Using Google Classroom has become a habit for me. (HT_1)
2. Using Google Classroom has become natural to me. (HT_2)
3. Using Google Classroom is addictive. (HT_3)
4. I feel that I must use Google Classroom. (HT_4)

Behavioural Intention (BI)

1. I intend to continue using Google Classroom in the future. (BI_1)
2. It is worth to recommend the Google Classroom for other students. (BI_2)
3. I plan to continue to use Google Classroom frequently. (BI_3)

Use Behaviour (UB)

1. I use Google Classroom for writing quizzes and submitting assignments behaviour .(UB_1)
2. I use Google Classroom to interact with online materials,peers and instructor. (UB_2)

Additionally another aim for this paper is to find the rank of the sub-criteria related to each of their criteria, therefore from the most important one to the last important applying Fuzzy AHP method.
This method has its complex steps to be applied for every decision making problem with fuzzy numbers. The diagram below shows the steps for applying FAHP with trapezoidal fuzzy numbers (TPFN).

Proposed framework

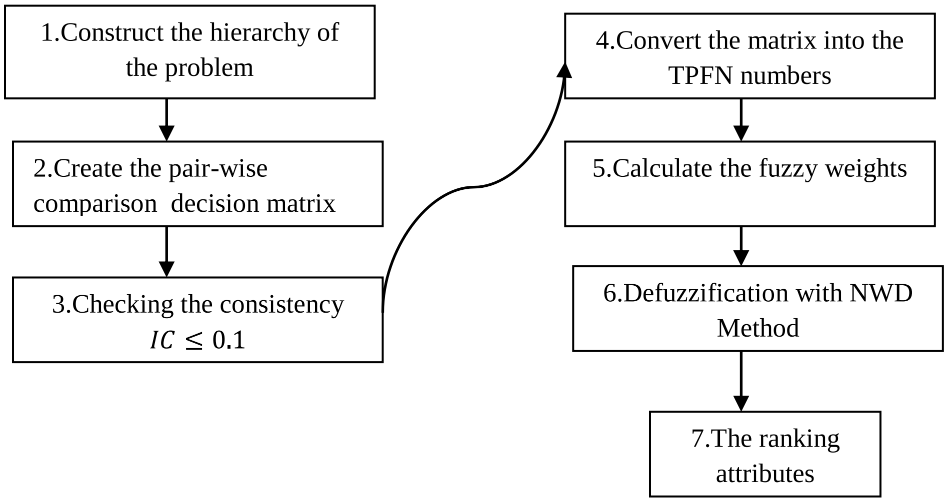

Figure 1. Steps for FAHP with TPFN

2. Materials and Methods

Data collections

The data used for this study were collected from about 200 students of mathematics courses following bachelor degree from Alexander Moisiu University, age from 18-21 years old. For this purpose is developed an online survey based in the model UTAUT2 by Venkatesh et.al (5), (20). By practicing the model with the 5 criteria PE, SI, HT, BI, and UB of UTAUT2 theory, in our survey for these students we give them the options to answer on a 1-5 likert scale. İn order to estimate if acceptance or use of Google Classroom had affected them or not, we choose the response options 1-5 likert scale that are as follows: 1-Strongly disagree, 2-Disagree, 3- Nor agree or disagree, 4-Agree, 5- Strongly Agree. Students were chosen from mathematic courses in various Bachelor degree studies, and completed the online survey after studying their subject and done the exam. In this way they used the Google Classroom platform, and were able to answer immediately.

According to their responses related with our criteria and sub-criteria, we have applied FAHP with its main purpose to rank them. The evaluations for the sub-criteria are considered with respect to their importance related to each of the criteria, so from the 4 sub-criteria PE1, PE2, PE3, PE4 we ranked them compared to their importance related with the criteria Performance Expectancy, and so on for all the others. The reason that we choose trapezoidal fuzzy numbers for AHP is because they are usually adopted to deal with the vagueness of decisions related to the performance of the sub-criteria in respect to each criterion. So a trapezoidal fuzzy number can deal with more general situations described by linguistic variables (21). In figure 2 is shown the hierarchy structure of the survey.

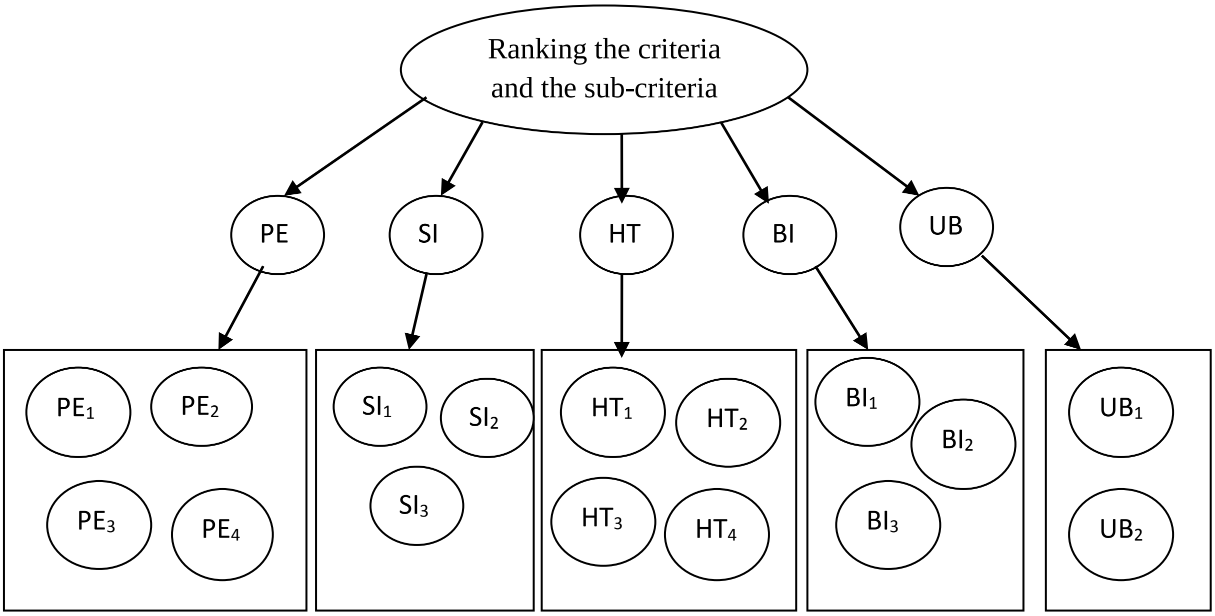

Figure 2. The hierarchy structure of the survey

Method **:** *Fuzzy AHP with trapezoidal fuzzy numbers*

The fuzzy numbers represent in the real line the fuzzy sets (22). Using the trapezoidal fuzzy numbers is so convenient to express the vagueness and uncertainty. İn our study the linguistic variables are used to describe the decision makers judgments by expressing them as trapezoidal fuzzy numbers. Fuzzy set introduced by Zadeh (17) extended the AHP into the fuzzy environment as the Fuzzy AHP (FAHP). İn order to construct a decision matrix the decision maker has to do $\frac{n(n-1)}{2}$ comparisons for a level with n-criteria. Firslty is the construction of the decision matrix based on the Saaty scale for the importance with the crisp numbers 1,...,9. So the first condition to start AHP (also the Fuzzy AHP) is the consistency index of the decision matrix IC to be less than 0.1 ($IC \leq 0.1$). After that all the crisp numbers are converted into trapezoidal fuzzy numbers, and the last applying FAHP. Table 1 shows the order of the decision matrix A and its random index RI.The consistency index $CI = \frac{\lambda_{max}-n}{n-1} \leq 0.1$, even the consistence ratio $CR = CI/RI \leq 0.1$. Table 2 shows the relative importance values of Saaty scale equivalented into trapezoidal fuzzy numbers.

Table 1.Random Index for simple AHP (Saaty 1980).

n	1	2	3	4	5	6	7	8	9	10
RI	0.00	0.00	0.58	0.9	1.12	1.24	1.32	1.41	1.45	1.49

Table 2.The relative importance values for TPFN.

Relative importance value	Importance	Trapezoidal fuzzy numbers scale
1	Equal	(1,1,1,1)
3	Moderate	(2,5/2,7/2,4)
5	Strong	(4,9/2,11/2,6)
7	Very strong	(6,13/2,15/2,8)
9	Extremely strong	(8,17/2,9,9)
2	Intermediate values	(1,3/2,5/2,3)
4	Intermediate values	(3,7/2,9/2,5)
6	Intermediate values	(5,11/2,13/2,7)
8	Intermediate values	(7,15/2,17/2,9)

A trapezoidal fuzzy number is denoted as $\tilde{\alpha} = (l, m, n, u)$, and has a membership function $\mu_{\tilde{\alpha}}(x): R \to [0,1]$.

$$\mu_{\tilde{\alpha}}(x) = \begin{cases} \frac{x-l}{m-l} & l \le x \le m \\ 1 & m \le x \le n \\ \frac{u-x}{u-m} & m \le x \le u \\ 0 & x \ne [l, u] \end{cases} \qquad [1]$$

Some law for operating with two Trapezoidal Fuzzy Numbers $\tilde{\alpha}_1, \tilde{\alpha}_2$ are:

a) $\tilde{\alpha}_1 \oplus \tilde{\alpha}_2 = (l_1, m_1, n_1, u_1) \oplus (l_2, m_2, n_2, u_2) = (l_1 + l_2, m_1 + m_2, n_1 + n_2, u_1 + u_2)$
b) $\tilde{\alpha}_1 \otimes \tilde{\alpha}_2 = (l_1, m_1, n_1, u_1) \otimes (l_2, m_2, n_2, u_2) = (l_1 \cdot l_2, m_1 \cdot m_2, n_1 \cdot n_2, u_1 \cdot u_2)$
c) $\tilde{\alpha}^{-1} = (l, m, n, u)^{-1} = (\frac{1}{u}, \frac{1}{n}, \frac{1}{m}, \frac{1}{l})$ [2]
d) For converting crisp numbers less than 1 into trapezoidal fuzzy numbers it is used the generalized trapezoidal fuzzy number $\tilde{\alpha} = (l, m, n, u: w)$, $0 < w \le 1$, we have w=1 in our case (23).

$$R(\tilde{\alpha}) = \frac{w}{2}[(n + u) - (l + m)]$$

After constructing the decision matrix A it is converted into $\widetilde{A} = \left(\widetilde{\alpha_{ij}}\right), \left(\widetilde{\alpha_{ij}}\right) = \left(l_{ij},\ m_{ij}, n_{ij}, u_{ij}\right)$ [3]
as the fuzzy TpFN matrix .For each of the criteria is calculated the fuzzy geometric mean value:

$$\tilde{r}_i = \left(\prod_{i=1}^{n} \widetilde{\alpha_{ij}}\right)^{1/n} \qquad [4]$$

Next step is to calculate the fuzzy weights as follow: $\widetilde{\omega}_i = \tilde{r}_i \otimes (\tilde{r}_1 \oplus \tilde{r}_2 \oplus \dots \oplus \tilde{r}_n)^{-1}$, then becames the defuzzification with the method of nearest weighted symmetry (NWS) for trapezoidal fuzzy numbers according to Saneifard (23). $a_{ij} = \frac{l+2m+2n+u}{6}$ [5]

The last step is the normalization and the ranking attributes: $N_i = \frac{a_{ij}}{\sum a_{ij}}$ [6]

3. Results

The decision matrix constructed for the criteria level with AHP method by Saaty scale crisp numbers is in table 3.

Table 3.The decision matrix for Criteria level by Saaty crisp numbers.

Criteria	PE	SI	HT	BI	UB
PE	1	6	4	4	4
SI	1/6	1	3	1/2	2
HT	1/4	1/3	1	1/3	2
BI	1/4	2	3	1	2
UB	1/4	1/2	1/2	1/2	1

The consistency index $CI = 0.083$, $RI = 0.074$ so the matrix is consistent. The fuzzy matrix related with the TFN numbers according to equation [2] is in table 4.

Table 4.The decision matrix for Criteria level with Saaty TPFN

	PE	SI	HT	BI	UB
PE	(1,1,1,1)	(5, 5.5, 6.5, 7)	(3,3.5,4.5,5)	(3,3.5,4.5,5)	(3,3.5,4.5,5)
SI	(0.01,0.155,0.175,0.32)	(1,1,1,1)	(2,2.5,3.5,4)	(0.2,0.3,0.7,0.8)	(1,1.5,2.5,3)
HT	(0.1,0.15,0.35,0.4)	(0.13,0.2,0.46,0.53)	(1,1,1,1)	(0.13,0.2,0.46,0.53)	(1,1.5,2.5,3)
BI	(0.1,0.15,0.35,0.4)	(1,1.5 ,2.5, 3)	(2,2.5,3.5,4)	(1,1,1,1)	(1,1.5,2.5,3)
UB	(0.1,0.15,0.35,0.4)	(0.2, 0.3, 0.7, 0.8)	(0.2,0.3,0.7,0.8)	(0.2,0.3,0.7,0.8)	(1,1,1,1)

After applying equations [3] - [6], table 5 shows the ranked criteria.

Table 5.The ranked criteria.

Criteria	$\check{r}_i$	$\breve{\omega}_i$	NWS	N_i	Rank
PE	(2.66, 2.98, 3.58, 3.87)	(0.32, 0.38, 0.64, 0.89)	0.54	0.5	1
SI	(0.33, 0.7, 1.01, 1.25)	(0.04, 0.09, 0.18, 0.28)	0.14	0.13	3
HT	(0.27, 0.38, 0.71, 0.8)	(0.032, 0.05, 0.127, 0.18)	0.094	0.087	4
BI	(0.72, 0.96, 1.5, 1.7)	(0.086, 0.12, 0.27, 0.39)	0.209	0.195	2
UB	(0.24, 0.33, 0.65, 0.72)	(0.03, 0.04, 0.12, 0.16)	0.085	0.079	5

As is shown from the results, the Performance Expectancy (PE) is the most important criteria referring all the others. PE for our students indicates that the use of the technology with Google Classroom provided benefits for them in performing certain activities. The second ranked is (BI) Behavioral Intention that for our students indicates the performance for future behavior of the Google Classroom usage, then are ranked the SI Social Influence, HT Habit and UB Use behavior the last. The use behavior is the subsequent effect of behavioral intention, social influence and habit, so our students intend to use Google Classroom for sharing online materials, peers etc. In table 6 are shown the results for sub-criteria PE1, PE2, PE3, PE4 after applying the equations [2]-[6] in importance relation with Performance Expectancy criteria.

Table 6. The ranked sub-criteria for Performance Expectancy.

Alternative	$\check{r}_i$	$\breve{\omega}_i$	NWS	N_i	Rank
PE_1	(0.12, 0.27, 0.44, 0.54)	(0.0168, 0.04, 0.083, 0.12)	0.063	0.06	4
PE_2	(0.29, 0.46, 0.75, 0.87)	(0.04, 0.07, 0.142, 0.19)	0.109	0.1	3
PE_3	(0.67, 0.86, 1.32, 1.48)	(0.09, 0.13, 0.25, 0.32)	0.195	0.18	2
PE_4	(3.46, 3.72, 4.17, 4.35)	(0.48, 0.56, 0.79,0.95)	0.688	0.65	1

The most important sub-criteria evaluated from them is PE_4 "If I use Google Classroom, I will increase my chances of passing the course". So our student believe that using Google Classroom increrases chanses of passing the course. The last important was PE_1 "I find Google Classroom useful in this course of math." This result depends on the subject of math they were studying for. İn table 7 are shown the results for the sub-criteria SI_1, SI_2, SI_3, referring the relative importance with Social Influence.

Table 7. The ranked sub-criteria for Social Influence.

Alternative	$\check{r}_i$	$\breve{\omega}_i$	NWS	N_i	Rank
SI_1	(0.1, 0.27, 0.38, 0.48)	(0.0185, 0.054, 0.087, 0.13)	0.071	0.07	3
SI_2	(0.46, 0.72, 0.95, 1.12)	(0.085, 0.144, 0.218, 0.3)	0.184	0.18	2
SI_3	(3.17, 3.36, 3.67, 3.78)	(0.58, 0.67, 0.844, 1.02)	0.771	0.75	1

The most important from them is SI_3 "Other people whose opinions I value prefer that I use Google Classroom", and the last is SI_1 "My friends who are important to me think that I should participate in Google Classroom". So for these students is more important the opinion of other people they prefer then the opinion of their friends to participate in Google Classroom.

İn table 8 are shown the results for the sub-criteria HT_1,HT_2,HT_3, HT_4, related to HT Habit criteria. HT_2 is the most important sub-criteria evaluated from the students. HT_2 "Using Google Classroom has become natural to me", as a habit this usage has became natural. The last one is HT1 "Using Google Classroom has become a habit for me". These results show that it is difficult for students to perform Google Classroom as a habit.

Table 8. The ranked sub-criteria for Habit.

Alternative	$\check{r}_i$	$\breve{\omega}_i$	NWS	N_i	Rank
HT1	(0.25, 0.34, 0.64, 0.71)	(0.04, 0.06, 0.18, 0.25)	0.128	0.111	4
HT2	(1.19, 1.54, 2.16, 2.45)	(0.19, 0.27, 0.6, 0.88)	0.468	0.4	1
HT3	(0.4, 0.55, 0.95, 1.06)	(0.064, 0.1, 0.26, 0.38)	0.194	0.17	3
HT4	(0.88, 1.12, 1.67, 1.86)	(0.14, 0.2, 0.46, 0.67)	0.355	0.31	2

In table 9 are ranked the sub-criteria of Behavioral Intention, BI_1, BI_2, BI_3. Ranked as the most important is BI_2 "It is worth to recommend the Google Classroom for other students", and the last one is BI_1 "I intend to continue using Google Classroom in the future."

Table 9. The ranked sub-criteria for Behavioral Intention.

Alternative	$\check{r}_i$	$\breve{\omega}_i$	NDW	N_i	Rank
BI1	(0.27, 0.35, 0.62, 0.68)	(0.056, 0.08, 0.186, 0.238)	0.137	0.131	3
BI2	(2.08, 2.3, 2.72, 2.92)	(0.43, 0.529, 0.816, 1.022)	0.69	0.66	1
BI3	(0.46, 0.6, 0.95, 1.06)	(0.096, 0.138, 0.285, 0.371)	0.218	0.2	2

In table 10 are ranked the two sub-criteria of the Use Behavior criteria. The most important is UB_1 "I use Google Classroom for writing quizzes and submitting assignments behaviour", and the last UB2 "I use Google Classroom to interact with online materials, peers and instructor".

Table 10. The ranked sub-criteria for Use Behavior.

Alternative	$\check{r}_i$	$\breve{\omega}_i$	NDW	N_i	Rank
UB1	(1.41, 1.58, 1.87, 2)	(0.5, 0.616, 0.916, 1.12)	0.78	0.75	1
UB2	(0.36, 0.44, 0.67, 0.73)	(0.129, 0.171, 0.328, 0.4)	0.254	0.245	2

4. Conclusions

This paper used the UTAUT2 theory for evaluations of online teaching performance via Google Classroom platform, for students of bachelor degree in math courses. We have include only five criteria from the UTAUT2 theory that are: Performance Expectancy, Social Influence, Habit, Behavioral Intention and Use Behavior. The online survey was developed for these students after they had studied and pass the exams of their subjects. The main aim is to rank these criteria with fuzzy AHP method via trapezoidal fuzzy numbers. Fuzzy AHP method is the extension of AHP, and is better than the other MCDM methods because is more capable of capturing a human judgment for complex decision problems. According to the respondent results show that the most important criteria was Performance Expectancy, then Behavioral Intention, Social Influence, habit and the last Use Behavior. As for the sub-criteria the most preferred of Performance Expectancy criteria was PE_1 "I find Google Classroom useful in this course of math." This result depends on the subject of math they were studying for. The next criteria was Socila influence and the most important subcriteria was SI_3 "Other people whose opinions I value prefer that I use Google Classroom". For Habit criteria the most important sub-criteria was HT_2 "Using Google Classroom has become natural to me", for Behavioral Intention the most important sub-criteria was BI_2 "It is worth to recommend the Google Classroom for other students", and for Use Behavior criteria the most important sub-criteria was UB_1 "I use Google Classroom for writing quizzes and submitting assignments behaviour". The factor that has mainly influenced these results has been the online learning in the situation of COVID-19, and also the various courses of math in the bachelor degree. The students have tried to accept and use the Google Classroom platform in their way, and all these results show this. İn order to improve the online learning is better for students to familiarize with the platform used by the lecturers. The study helps students and lecturers to interact together for online learning, and guides lecturers in focusing in the most important UTAUT2 criteria evaluated by their students. İn further studies we will use the Fuzzy AHP with Z-number, including some other Albanian Universities, not only students but even lecturers.

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The Lattice of Fuzzy Topologies Generated by Fuzzy Relations

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Abstract

In a recent paper, Mishra and Srivastava have introduced and studied the notion of fuzzy topology generated by fuzzy relation and some basic properties were proved. In this paper, we provide the lattice structure to a family of fuzzy topologies generated by fuzzy relations. More specifically, we analyze the most important properties and characteristics of this lattice.

1. Introduction

In [2], Chang introduced the notion of fuzzy topology on a set X as a familyτ, satisfying the well-know axioms, and he referred to each member of τ as a fuzzy open set. After that, several researches were conducted on the properties and characterizations of the notion of fuzzy topology.

In 2018, Mishra and Srivastava [6] introduced the notion of fuzzy topology generated by a fuzzy relation and studied several related results. The present study is motivated by the above approach. First, we construct the lattice structure to a family of fuzzy topologies generated by fuzzy relations. Moreover, we provide necessary structural characteristics such as distributivity, modularity and complementary of this lattice.

This paper is organized as follows. We briefly recall some basic concepts in Section 2. In Section 3, we provide a lattice structure to the family of fuzzy topologies generated by fuzzy relations, and we discuss some basic properties about distributivity, modularity and complementary of this lattice. Finally, we present some conclusions and we discuss future research in Section 4.

2. Preliminaries

Let X be a universe, then a fuzzy subset $A = \{\langle x, \mu_A(x)\rangle | x \in X\}$ of X defined by Zadeh [8] is characterized by a membership function $\mu_A: X \to [0,1]$, where $\mu_A(x)$ is interpreted as the degree of a membership of the element x in the fuzzy subset A for each $x \in X$.

The notion of fuzzy topology was first introduced by Chang.

Definition 2.1 A fuzzy topology (FT, for short) on a nonempty set X is a family τ of fuzzy sets on X which satisfies the following axioms:

1. $\emptyset, X \in \tau$;
2. $G_1 \cap G_2 \in \tau$ for any $G_1, G_2 \in \tau$;
3. $\bigcup \quad G_i \in \tau$ for any $\{G_i : i \in J\} \subseteq \tau$.

In this case, the pair (X, τ) is called a fuzzy topological space (FTS, for short) and any FS in τ is known as a fuzzy open set (FOS, for short) in X. The complement of a fuzzy open set is called a fuzzy closed set (FCS, for short) in X.

Example 2.2 Let $X = \{x, y, z\}$ and $A, B, C \in FS(X)$ such that

$$A = \{\langle x, 0.3\rangle, \langle y, 0.4\rangle, \langle z, 0.1\rangle\};$$

$$B = \{\langle x, 0.4\rangle, \langle y, 0.5\rangle, \langle z, 0.3\rangle\};$$

$$C = \{\langle x, 0.3\rangle, \langle y, 0.4\rangle, \langle z, 0.3\rangle\}.$$

Then, $\tau = \{\emptyset, X, A, B, C\}$ is a fuzzy topology on X.

Mishra and Srivastava [6] introduced the notion of fuzzy topology generated by fuzzy relation.

Definition 2.3 (Fuzzy topology generated by fuzzy relation) Let X be a nonempty crisp set and $R = \{\langle (x, y), \mu_R(x, y)\rangle | x, y \in X\}$ be a fuzzy relation on X. Then for any $x \in X$, the fuzzy sets $\mathcal{L}_x$ et $\mathcal{R}_x$ are defined by

$$\mu_{\mathcal{L}_x}(y) = \mu_R(y, x), \text{ for any } y \in X,$$

$$\mu_{\mathcal{R}_x}(y) = \mu_R(x, y), \text{ for any } y \in X,$$

they are called the lower and the upper contour, respectively, of x.

We denote by τ_1, the fuzzy topology generated by the set of all lower contours and τ_2, the fuzzy topology generated by the set of all upper contours. Consequently, we denote by τ_R, the fuzzy topology generated by S the set of all lower and upper contours and it's called the fuzzy topology generated by R.

Example 2.4 Let $X = \{x, y\}$ and R be a fuzzy relation on X given by

$\mu_R(.,.)$	x	y
x	0.6	0.8
y	0.3	0.7

Then, $\mathcal{L}_x, \mathcal{L}_y, \mathcal{R}_x$ and $\mathcal{R}_y$ are the fuzzy sets on X given by :

$$\mathcal{L}_x = \{\langle x, 0.6\rangle; \langle y, 0.3\rangle\};$$

$$\mathcal{L}_y = \{\langle x, 0.8\rangle; \langle y, 0.7\rangle\};$$

$$\mathcal{R}_x = \{\langle x, 0.6\rangle; \langle y, 0.8\rangle\};$$

$$\mathcal{R}_y = \{\langle x, 0.3\rangle; \langle y, 0.7\rangle\}.$$

We note that, $\mathcal{L}_x \subset \mathcal{L}_y, \mathcal{L}_x \subset \mathcal{R}_y, \mathcal{R}_y \subset \mathcal{R}_x$ and $\mathcal{R}_y \subset \mathcal{L}_y$. Then the fuzzy topology $\tau_{\mathcal{R}}$ is generated by $S = \{\mathcal{L}_x, \mathcal{L}_y\} \cup \{\mathcal{R}_x, \mathcal{R}_y\}$. Thus, $\tau_{\mathcal{R}} = \{\emptyset, X, \mathcal{L}_x, \mathcal{L}_y, \mathcal{R}_x, \mathcal{R}_y, \mathcal{L}_x \cap \mathcal{R}_y, \mathcal{L}_y \cap \mathcal{R}_x, \mathcal{L}_x \cup \mathcal{R}_y, \mathcal{L}_y \cup R_x\}$, where

$$\mathcal{L}_x \cap \mathcal{R}_y = \{\langle x, 0.6\rangle; \langle y, 0.7\rangle\}, \mathcal{L}_y \cap \mathcal{R}_x = \{\langle x, 0.8\rangle; \langle y, 0.8\rangle\},$$

$$\mathcal{L}_x \cup \mathcal{R}_x = \{\langle x, 0.7\rangle; \langle y, 0.7\rangle\} \text{ and } \mathcal{L}_x \cup \mathcal{R}_x = \{\langle x, 0.6\rangle; \langle y, 0.4\rangle\}.$$

3. The lattice of fuzzy topologies generated by fuzzy relations

In this section, we mainly investigate the lattice of all fuzzy topologies generated by fuzzy relations. First, we introduce the notion of fuzzy inclusion between topologies.

Definition 3.1 Let R_1, R_2 are two fuzzy relations on the set X and τ_{R_1}, τ_{R_2} are the fuzzy topologies generated by R_1 and R_2 respectively. Then, τ_{R_1} is said to be contained in τ_{R_2} (in symbols, $\tau_{R_1} \sqsubseteq \tau_{R_2}$) if $G \in \tau_{R_2}$ for any $G \in \tau_{R_1}$.

In this case, we also say that τ_{R_1} is smaller than τ_{R_2}.

Proposition 3.2 Let τ_{R_1} and τ_{R_2} are the fuzzy topologies on the set X generated by R_1 and R_2 respectively. Then it holds that, $\tau_{R_1} \sqsubseteq \tau_{R_2}$ if and only if $R_1 \subseteq R_2$ (i.e., $\mu_{R_1}(x, y) \leq \mu_{R_2}(x, y)$, for any $(x, y) \in X \times X$).

Now, we introduce the intersection of fuzzy topologies generated by fuzzy relations.

Definition 3.3 Let τ_{R_1} and τ_{R_2} are the fuzzy topologies on the set X generated by R_1 and R_2 respectively. The intersection of τ_{R_1} and τ_{R_2} (in symbols, $\tau_{R_1} \sqcap \tau_{R_2}$) is a fuzzy topology τ_R such that $G \in \tau_R$ if and only if $G \in \tau_{R_1}$ and $G \in \tau_{R_2}$.

Next, we introduce the union of fuzzy topologies generated by fuzzy relations.

Definition 3.4 Let τ_{R_1} and τ_{R_2} are the fuzzy topologies on the set X generated by R_1 and R_2 respectively. The union of τ_{R_1} and τ_{R_2} (in symbols, $\tau_{R_1} \sqcup \tau_{R_2}$) is a fuzzy topology τ_R such that $G \in \tau_R$ if and only if $G \in \tau_{R_1}$ or $G \in \tau_{R_2}$.

The following theorem provides the lattice structure to a family of F-topologies generated by F-relations.

Theorem 3.5 Let X be a finite set and $\mathfrak{L} = \{\tau_{R_i} | R_i \in FR(X^2)\}$ is a family of all fuzzy topologies on X generated by the fuzzy relations R_i. Then $\mathfrak{L}$ is a fuzzy lattice on X.

Proposition 3.6 Let X be a finite set and $\mathfrak{L} = \{\tau_{R_i} | R_i \in FR(X^2)\}$ is the lattice of all fuzzy topologies on X generated by the fuzzy relations R_i. Then $\mathfrak{L}$ is complete.

Corollary 3.7 Let $\mathfrak{L}$ be the complete lattice of all fuzzy topologies generated by fuzzy relations, then $\mathfrak{L}$ is bounded. Indeed, the least element of $\mathfrak{L}$ is $0_{\mathfrak{L}} = \sqcap \tau_{R_i}$ and the greatest element of $\mathfrak{L}$ is $1_{\mathfrak{L}} = \sqcup \tau_{R_i}$.

The following proposition discuss the distributivity property of the lattice of all F-topologies generated by F-relations.

Proposition 3.8 *Let $\mathfrak{L}$ be the lattice of fuzzy topologies generated by the fuzzy relations R_i, then $\mathfrak{L}$ is distributive.*

Corollary 3.9 Since $\mathfrak{L}$ is a distributive lattice, then it holds that $\mathfrak{L}$ is modular.

In 1958, Hartmanis [3] proved that the lattice of all topologies on a finite set is complemented. In the following proposition, we prove that the lattice of fuzzy topologies generated by fuzzy relations is also complemented.

Proposition 3.10 Let $\mathfrak{L}$ be the lattice of all fuzzy topologies generated by the fuzzy relations R_i, then $\mathfrak{L}$ is complemented.

Corollary 3.11 Since $\mathfrak{L}$ is a distributive lattice and complemented with a least element $0_{\mathfrak{L}}$ and a greatest element $1_{\mathfrak{L}}$, then $\mathfrak{L}$ is a boolean algebra denoted by $(\mathfrak{L}, \sqcap, \sqcup, 0_{\mathfrak{L}}, 1_{\mathfrak{L}})$.

In the following theorem, we will show that the lattice of fuzzy relations and the lattice of all fuzzy topologies generated by fuzzy relations are isomorphic.

Theorem 3.12 Let $\varphi\colon \mathrm{FR}(X^2) \to \mathfrak{L}$ be a mapping defined as $\varphi(R) = \tau_R$, for any $R \in \mathrm{FR}(X^2)$, then $(\mathrm{FR}(X^2), \cap, \cup)$ and $\mathfrak{L}$ are isomorphic.

4. Conclusion

In this work, we have studied the lattice structure of fuzzy topologies generated by fuzzy relations and we have investigated its most important properties. In particular, we have focused on the characteristics of this lattice such as distributivity, modularity and complementary. In a future work, we plan to generalize this study to the intuitionistic fuzzy setting.

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The notion of continuous additive units of product systems of Hilbert modules

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Abstract

We consider continuous additive units and roots of a central unital unit in a spatial product system of two-sided Hilbert modules. We prove that the set of all continuous additive units of a central unital unit has a two-sided Hilbert module structure. Also, we show that the set of all roots is a Hilbert two-sided submodule therein.

1. Introduction

We observe a spatial product system of two-sided Hilbert modules over a unital C*-algebra B (it presents a product system that contains a central unital unit). We consider the notion of continuous additive units and continuous roots of a central unital unit. The set of all continuous additive units of a central unital unit ω can be provided with a structure of two-sided Hilbert $B-B$ module wherein the set of all continuous roots of ω is a Hilbert $B-B$ submodule. As an example, we find all continuous additive units and roots in the spatial product system from [1, Example 4.2.4].

Throughout the whole paper, B denotes a unital C*-algebra and 1 denotes its unit. Also, we use $\otimes$ for tensor product, although $\odot$ is in common use.

Definition 1.1. a) A Hilbert B-module F is a right B-module with a map $\langle\, ,\rangle: F \times F \to B$ that satisfies the following properties:

- $\langle x, \lambda y + \mu z\rangle = \lambda\langle x, y\rangle + \mu\langle x, z\rangle$ for $x, y, z \in F$ and $\lambda, \mu \in \mathbb{C}$;
- $\langle x, y\beta\rangle = \langle x, y\rangle\beta$ for $x, y \in F$ and $\beta \in B$;
- $\langle x, y\rangle = \langle y, x\rangle^*$ for $x, y \in F$;
- $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0 \iff x = 0$ for $x \in F$

and F is complete with respect to the norm $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$.

b) A Hilbert $B-B$ module is a Hilbert B-module with a non-degenerate *-representation of B by elements in the C*-algebra $B^a(F)$ of all adjointable mappings on F. The homomorphism $j: B \to B^a(F)$ is contractive. In particular, since B is a unital C*-algebra, the unit of B acts as the unit of $B^a(F)$. Also, for $x, y \in F$ and $\beta \in B$ there holds $\langle x, \beta y\rangle = \langle \beta^* x, y\rangle$ where $\beta y = j(\beta)(y)$.

For basic facts about Hilbert C*-modules we refer the reader to [2].

Definition 1.2. a) A product system over a C*-algebra B is a family $(E_t)_{t\geq 0}$ of Hilbert $B-B$ modules, with $E_0 \cong B$, and a family of isomorphisms

$$\varphi_{t,s}: E_t \otimes E_s \to E_{t+s},$$

where $\otimes$ is the so-called inner tensor product obtained by identifications $ub \otimes v \sim u \otimes bv$, $u \otimes vb \sim (u \otimes v)b$, $bu \otimes v \sim b(u \otimes v)$, $(u \in E_t,\ v \in E_s,\ b \in B)$ and then completing in the inner product $\langle u \otimes v, u' \otimes v'\rangle = \langle v, \langle u, u'\rangle v'\rangle$;
b) A unit on E is a family $u = (u_t), u_t \in E_t$, so that $u_0 = 1$ and $\varphi_{t,s}(u_t \otimes u_s) = u_{t+s}$, which we abbreviate to $u_t \otimes u_s = u_{t+s}$. A unit u is unital if $\langle u_t, u_t\rangle = 1$. It is central if for all $\beta \in B$ and all $t \geq 0$ there holds $\beta u_t = u_t \beta$.

Definition 1.3. The spatial product system is a product system that contains a central unital unit.
We refer the reader to [1], [3], [4].

2. Additive units

Let $\omega = (\omega_t)$ be a central unital unit in a spatial product system $E = (E_t)$ over a unital C*-algebra B.

Definition 2.1. A family $a = (a_t)$, $a_t \in E_t$, is an additive unit of ω if $a_0 = 0$ and

$$a_{s+t} = a_s \otimes \omega_t + \omega_s \otimes a_t, \quad s, t \geq 0.$$

An additive unit a of ω is a root if $\langle a_t, \omega_t\rangle = 0$ for all $t \geq 0$.

Definition 2.2. For $\beta \in B$, let $F_\beta^{a,b}: [0, \infty) \to B$ be the map defined by

$$F_\beta^{a,b}(s) = \langle a_s, \beta b_s\rangle$$

where a, b are additive units of ω in E.
A set of additive units S of ω is continuous if the map $F_\beta^{a,b}$ is continuous for all $a, b \in S$ and all $\beta \in B$.
We say that a is a continuous additive unit of ω if the set $\{a\}$ is continuous, i.e. if the map $F_\beta^{a,a}$ is continuous for all $\beta \in B$. Denote the set of all continuous additive units of ω by A_ω and the set of all continuous roots of ω by R_ω.

Lemma 2.3. (1) If $a \in A_\omega$, then $\langle \omega_s, a_s\rangle = s\langle \omega_1, a_1\rangle,\ s \geq 0$.
(2) If $a, b \in R_\omega$, then $F_\beta^{a,b}(s) = sF_\beta^{a,b}(1),\ s \geq 0,\ \beta \in B$.
(3) If $a \in A_\omega$, then a family (a'_s), where $a'_s = a_s - \langle \omega_s, a_s\rangle\omega_s$, belongs to R_ω.

Proof. (1) Let $H^a: [0, \infty) \to B$ be the map defined by $H^a(s) = \langle \omega_s, a_s\rangle,\ s \geq 0$. For $s, t \geq 0$ we obtain

$$H^a(s+t) = \langle \omega_s \otimes \omega_t, a_s \otimes \omega_t\rangle + \langle \omega_s \otimes \omega_t, \omega_s \otimes a_t\rangle =$$
$$= \langle \omega_t, \langle \omega_s, a_s\rangle \omega_t\rangle + \langle \omega_t, \langle \omega_s, \omega_s\rangle a_t\rangle = \langle \omega_s, a_s\rangle + \langle \omega_t, a_t\rangle = H^a(s) + H^a(t)$$

and

$$\|H^a(s) - H^a(0)\|^2 = \|\langle \omega_s, a_s \rangle\|^2 \leq \|\omega_s\|^2 \|a_s\|^2 = \|F_1^{a,a}(s)\| \to \|F_1^{a,a}(0)\| = 0,\ s \to 0\,.$$
Hence, the map H^a is continuous. Therefore, $H^a(s) = sH^a(1)$, i.e. $\langle \omega_s, a_s \rangle = s\langle \omega_1, a_1 \rangle$.

(2) Let $s, t \geq 0$. Since $a, b \in R_\omega$, we see that
$$F_\beta^{a,b}(s+t) = \langle a_s \otimes \omega_t + \omega_s \otimes a_t, \beta(b_s \otimes \omega_t + \omega_s \otimes b_t) \rangle =$$
$$= \langle \omega_t, \langle a_s, \beta b_s \rangle \omega_t \rangle + \langle a_t, \langle \omega_s, \beta \omega_s \rangle b_t \rangle = F_\beta^{a,b}(s) + F_\beta^{a,b}(t)$$
and
$$\left\|F_\beta^{a,b}(s) - F_\beta^{a,b}(0)\right\|^2 = \|\langle a_s, \beta b_s \rangle\|^2 \leq \|\langle a_s, a_s \rangle\|\ \|\beta\|^2\ \|\langle b_s, b_s \rangle\| \to 0,\ s \to 0.$$
Hence, the map $F_\beta^{a,b}$ is continuous and, therefore, $F_\beta^{a,b}(s) = sF_\beta^{a,b}(1)$.

(3) For $s, t \geq 0$, we obtain that
$$a'_{s+t} = a_s \otimes \omega_t + \omega_s \otimes a_t - (\langle \omega_t, \langle \omega_s, a_s \rangle \omega_t \rangle + \langle \omega_t, \langle \omega_s, \omega_s \rangle a_t \rangle)\omega_s \otimes \omega_t =$$
$$= a_s \otimes \omega_t + \omega_s \otimes a_t - \langle \omega_s, a_s \rangle \omega_s \otimes \omega_t - \langle \omega_t, a_t \rangle \omega_s \otimes \omega_t =$$
$$= (a_s - \langle \omega_s, a_s \rangle \omega_s) \otimes \omega_t + \omega_s \otimes (a_t - \langle \omega_t, a_t \rangle \omega_t) = a'_s \otimes \omega_t + \omega_s \otimes a'_t$$
and
$$\langle a'_s, \omega_s \rangle = 0.$$
Therefore, a' is a root of ω.
Let $\beta \in B$. By (1) it follows that
$$F_\beta^{a',a'}(s) = F_\beta^{a,a}(s) - s^2 \langle a_1, \omega_1 \rangle \beta \langle \omega_1, a_1 \rangle, \qquad s \geq 0.$$
Therefore, the map $F_\beta^{a',a'}$ is continuous and we obtain that $a' \in R_\omega$. □

Remark 2.4. By Lemma 2.3, every $a \in A_\omega$ can be decomposed as $a_s = s\langle \omega_1, a_1 \rangle \omega_s + a'_s,\ a' \in R_\omega$.

Let $a, b \in A_\omega$. By Remark 2.4, $a_s = s\langle \omega_1, a_1 \rangle \omega_s + a'_s$ and $b_s = s\langle \omega_1, b_1 \rangle \omega_s + b'_s$, $s \geq 0,\ a', b' \in R_\omega$.
Let $s \geq 0$ and $\beta \in B$. By Lemma 2.3(2), there holds
$$F_\beta^{a',b'}(s) = sF_\beta^{a,b}(1) - s\langle a_1, \omega_1 \rangle \beta \langle \omega_1, b_1 \rangle.$$
Now, using decomposition of a, b, it follows that the map $F_\beta^{a,b}$ is continuous since
(4) $$F_\beta^{a,b}(s) = sF_\beta^{a,b}(1) + (s^2 - s)\langle a_1, \omega_1 \rangle \beta \langle \omega_1, b_1 \rangle.$$
Remark 2.5. The set A_ω is continuous in the sense of Definition 2.2.

3. Main results

Throughout this section, $\omega = (\omega_t)_{t\geq 0}$ is a central unital unit in a spatial product system $E = (E_t)_{t\geq 0}$ over a unital C*-algebra B.

Theorem 3.1. The set A_ω is a $B-B$ module under the point-wise addition and point-wise scalar multiplication. The set R_ω is a $B-B$ submodule therein.

Proof. Let $a,b \in A_\omega$, $\beta \in B$ and $s,t \geq 0$. Since $(a+b)_{s+t} = (a+b)_s \otimes \omega_t + \omega_s \otimes (a+b)_t$ and $F_\beta^{a+b,a+b}$=$F_\beta^{a,a} + F_\beta^{b,a}$+$F_\beta^{a,b}$+$F_\beta^{b,b}$, it follows that $a+b \in A_\omega$ by Remark 2.5.

Let $\theta \in B$. Since the unit ω is central, we obtain $(a\theta)_{s+t} = (a\theta)_s \otimes \omega_t + \omega_s \otimes (a\theta)_t$ and similarly $(\theta a)_{s+t} = (\theta a)_s \otimes \omega_t + \omega_s \otimes (\theta a)_t$. Also, $F_\beta^{a\theta,a\theta}(s) = \theta^* F_\beta^{a,a}(s)\theta$ and, using Remark 2.4, $F_\beta^{\theta a,\theta a}(s) = s^2\langle a_1,\omega_1\rangle\theta^*\beta\theta\langle\omega_1,a_1\rangle + F_{\theta^*\beta\theta}^{a',a'}(s)$. It follows $a\theta, \theta a \in A_\omega$. The other axioms of $B-B$ module follow directly.

Let $a,b \in R_\omega$. Then $\langle a_s + b_s, \omega_s\rangle = 0, \langle a_s\beta,\omega_s\rangle = \beta^*\langle a_s,\omega_s\rangle = 0$ and $\langle \beta a_s,\omega_s\rangle = \langle a_s, \beta^*\omega_s\rangle = 0$. Hence, $a+b, a\beta, \beta a \in R_\omega$. Therefore, R_ω is a $B-B$ submodule in A_ω. □

Theorem 3.2. The set A_ω is a Hilbert $B-B$ module under the inner product $\langle\ ,\ \rangle : A_\omega \times A_\omega \longrightarrow B$ defined by

(5) $$\langle a,b\rangle = \langle a_1, b_1\rangle, \quad a,b \in A_\omega.$$

The set R_ω is a Hilbert $B-B$ submodule in A_ω.

Proof. At the beginning we notice that A_ω is a $B-B$ module by Theorem 3.1. The map $\langle\ ,\ \rangle$ in (5) is a B-valued inner product in A_ω. Therefore, the map satisfies the following properties:

1) $\langle a, \lambda b + \mu c\rangle = \lambda\langle a,b\rangle + \mu\langle a,c\rangle$ for all $a,b,c \in A_\omega$ and $\lambda,\mu \in \mathbb{C}$;
2) $\langle a, b\beta\rangle = \langle a,b\rangle\beta$ for all $a,b \in A_\omega$ and $\beta \in B$;
3) $\langle a,b\rangle = \langle b,a\rangle^*$ for all $a,b \in A_\omega$;
4) $\langle a,a\rangle \geq 0$ for all $a \in A_\omega$;
5) $\langle a,a\rangle = 0 \Leftrightarrow a = 0$ for all $a \in A_\omega$;
6) $\langle a, \beta b\rangle = \langle \beta^* a, b\rangle$ for all $a,b \in A_\omega$ and $\beta \in B$.

We consider only the property 5) since all the others properties are straightforward calculation.
If $\langle a,a\rangle = 0$, then $a_1 = 0$. By Remark 2.4, $a_s = s\langle\omega_1,a_1\rangle\omega_s + a'_s$, $a' \in R_\omega$, $s \geq 0$, implying $a_s = a'_s$. By Lemma 2.3(2), $\langle a_s,a_s\rangle = s\langle a'_1,a'_1\rangle$. It follows that $a_s = 0$ for all $s \geq 0$, i.e. $a = 0$.

Therefore, A_ω is a pre-Hilbert $B-B$ module. Also, A_ω is complete with respect to the given inner product (5):
Let (a^n) be a Cauchy sequence in A_ω and $s \geq 0$. If $a = b = a^m - a^n$ in (4), it follows that

$$\|a_s^m - a_s^n\|^2 \leq (s^2+2s)\|a_1^m - a_1^n\|^2 = (s^2+2s)\|a^m - a^n\|^2.$$

Therefore, (a_s^n) is a Cauchy sequence in E_s and denote

(6) $$a_s = \lim_{n\to\infty} a_s^n.$$

Let $\varepsilon > 0$ and $s, t \geq 0$. There is $n_0 \in \mathbb{N}$ so that $\|a_s^n - a_s\| \leq \frac{\varepsilon}{3}$, $\|a_t^n - a_t\| \leq \frac{\varepsilon}{3}$, $\|a_{s+t}^n - a_{s+t}\| \leq \frac{\varepsilon}{3}$ for $n > n_0$. Then,

$$\|a_{s+t} - a_s \otimes \omega_t - \omega_s \otimes a_t\| \leq \|a_{s+t} - a_{s+t}^n\| + \|a_{s+t}^n - a_s \otimes \omega_t - \omega_s \otimes a_t\| \leq$$
$$\leq \|a_{s+t} - a_{s+t}^n\| + \|(a_s^n - a_s) \otimes \omega_t\| + \|\omega_s \otimes (a_t^n - a_t)\| \leq \varepsilon,$$

implying that a is an additive unit of ω. Let $\beta \in B$. By (6) and (4),

$$F_\beta^{a,a}(s) = \lim_{n\to\infty} F_\beta^{a^n,a^n}(s) = \lim_{n\to\infty}\left(sF_\beta^{a^n,a^n}(1) + (s^2 - s)\langle a_1^n, \omega_1\rangle\beta\langle\omega_1, a_1^n\rangle\right) =$$
$$= sF_\beta^{a,a}(1) + (s^2 - s)\langle a_1, \omega_1\rangle\beta\langle\omega_1, a_1\rangle.$$

Hence, $a \in A_\omega$. By (5) and (6), $\|a^n - a\| = \|a_1^n - a_1\| \to 0, n \to \infty$. Therefore, A_ω is complete with respect to the inner product (5).

Now, let (a^n) be a sequence in R_ω satisfying $\lim_{n\to\infty} a^n = a$. It only remains to see whether the continuous additive unit a belongs to R_ω. This follows from (6) since $\langle a_s, \omega_s\rangle = \lim_{n\to\infty}\langle a_s^n, \omega_s\rangle = 0$ for all $s \geq 0$. □

4. Example

As an example, we find the form of an arbitrary continuous additive unit of a central unital unit in the spatial product system from [1, Example 4.2.4].

Let $B = K(H) + \mathbb{C}1 \subset B(H)$ be the unitization of compact operators on an infinite-dimensional Hilbert space H. Let E_t be a Hilbert $B - B$ module so that $E_t = B$ as right Hilbert module and with left multiplication $b \cdot x_t = e^{ith}be^{-ith}x_t$ where $h \in B(H)$ is a self-adjoint operator. These Hilbert $B - B$ modules form a product system $E = (E_t)_{t\geq 0}$ with the identification

(7) $$x_s \otimes y_t = e^{ith}x_s e^{-ith}y_t.$$

Notice that E has a unital unit $\omega = (\omega_t)$, $\omega_t = 1 \in B$. Also, if $h = 0$, ω is a central unital unit.

Taking into account Definition 2.1. and Definition 2.2, we easily see that for every $\theta \in B$, $a = (a_t)$ with $a_t = t\omega_t\theta = t\theta$ is a continuous additive unit of ω.

Let $a = (a_t) \in A_\omega$. For $s, t \geq 0$, by (7), there holds

$$a_{s+t} = a_s \otimes \omega_t + \omega_s \otimes a_t = a_s\omega_t + \omega_s a_t = a_s + a_t.$$

Also, $\|a_t\|^2 = \|\langle a_t, a_t\rangle\| = \|F_1^{a,a}(t)\| \to \|F_1^{a,a}(0)\| = 0,\ t \to 0$, so the map $[0,\infty) \ni t \mapsto a_t \in B$ is continuous. Therefore, there is $\theta_a \in B$ so that $a_t = t\theta_a$ for all $t \geq 0$.

Let $a \in R_\omega$. Therefore, there is $\theta_a \in B$ so that $a_t = t\theta_a$ for all $t \geq 0$ and

$$\langle\omega_t, a_t\rangle = t\theta_a = 0$$

for all $t \geq 0$. We conclude that $\theta_a = 0$, i.e. $a_t = 0$. Hence, $R_\omega = \{0\}$.

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The Relationship Between Mathematical Problem-Solving and Planning Ability

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Abstract

Mathematical problem-solving skills depend on many variables and cognitive skills are one of the most important of them. The purpose of this study is to determine the mathematical problem-solving skills in terms of cognitive aspects. This aims to reveal the relationship between the planning ability and mathematical problem-solving skills at the secondary school level. An empirical quantitative research was done for this purpose. For the measurement of problem-solving skills, curriculum-based mathematics achievement test and for assessing the planning ability, a computerized Tower of London test was used. It was applied to a total of 416 students from the fifth, sixth, and seventh grades. As a result of the statistical analysis, a positive significant relationship was found between planning skills and problem-solving skills but there are not any significant differences in terms of gender.

Keywords: Mathematics, Mathematics Achievement, Executive Function, Tower of London Test

1. Introduction

Understanding of math concepts requires the formation of abstract representations of quantitative and qualitative relations between variables (Primi et al., 2010). Therefore, to teach mathematical concepts, problem-solving can be one of the best methods that can be used to teach mathematical concepts to school-age children (Donaldson, 2011).

Problem solving in mathematics refers to invisible, non-routine problems that require high-level understanding and cognitive skills to solve the problem in a meaningful way. (Jagals and Van der Walt, 2016). It can be said that this way of solving problems related to the very hierarchical nature of mathematics; because the understanding of new material depends upon what has been learned before, and so the learning of the new topic becomes intimately tied to the knowledge of the previous topics (Siadat, 2011). Therefore parts of a task need to be solved first and kept in mind to be able to solve the overall task (Lipnevich et al, 2016). For instance, some students consciously pay attention to the given problem by solving it in a hierarchical way, but there are also students who just carelessly answer the problem when facing the test, in this frame it can be stated that each student has a different process of thinking or plan of thinking in solving the problem.

A problem schema can be defined as a cognitive construct that allows problem solvers to recognize a problem as belonging to a specific category that requires particular moves for a solution. This definition includes problem states, problem-solving operators, and their relations. Although there are different problem-solving models such as linear (hierarchical), ideal or recursive by different researchers (Montague et al. 2000; Polya 1962; Verschaffel et al.1999), most of these models were developed on Polya's linear problem-solving model. According to Polya (2004), there are four steps in problem-solving. These are understanding the problem, planning, implementing the plan and controlling the process. Therefore, solving a problem requires a good plan, in which a child takes decisions, selects strategies, and uses them, for a successful score (Kroesbergen et al., 2010).

All the steps in problem-solving need the cognitive process to solve a mathematics problem. Bull et al. (2008) have propounded as a consequence of their research that cognitive skills help children to gain more math proficiency, and children identified as having a limitation in some aspects of cognitive functioning can have mathematical problems presented in a format that reduces the reliance on weaker cognitive skills.

Domain-general cognitive abilities especially executive functions (EF) may provide good explanations for variability in early math learning (Kroesbergen et al., 2007). The EF comprise mental capacities necessary for formulating goals, planning how to achieve them, and carrying out the plans effectively (Lezak, 1982). EF contains different mathematical functions in its different structures such as planning, mental set-shifting, updating and monitoring, and inhibition (Miyake et al., 2000). For instance, inhibition is likely to be especially important at younger ages to suppress less sophisticated strategies, e.g. counting on the first addend, in order to use more sophisticated strategies, such as counting on the larger addend (Cragg and Gilmore, 2013). Shifting skills may help to switch between operations, solution strategies, quantity ranges, and notations (e.g. between verbal digits, written Arabic symbols, and non-symbolic quantity representations) and between the steps of a complex multi-step problem (Bull and Lee, 2014). Planning requires all these skills to think of alternatives, to weigh and make choices, and to evolve a conceptual framework or structure which can serve to direct activity (Lezak, 1982). There is a clear theoretical rationale that executive control might play a central role in the acquisition of early counting and mathematical concepts (Clark et al., 2013).

Cragg et al. (2017) made a theoretical model predicting relationships between EF skills and components of mathematical knowledge (Fig1).

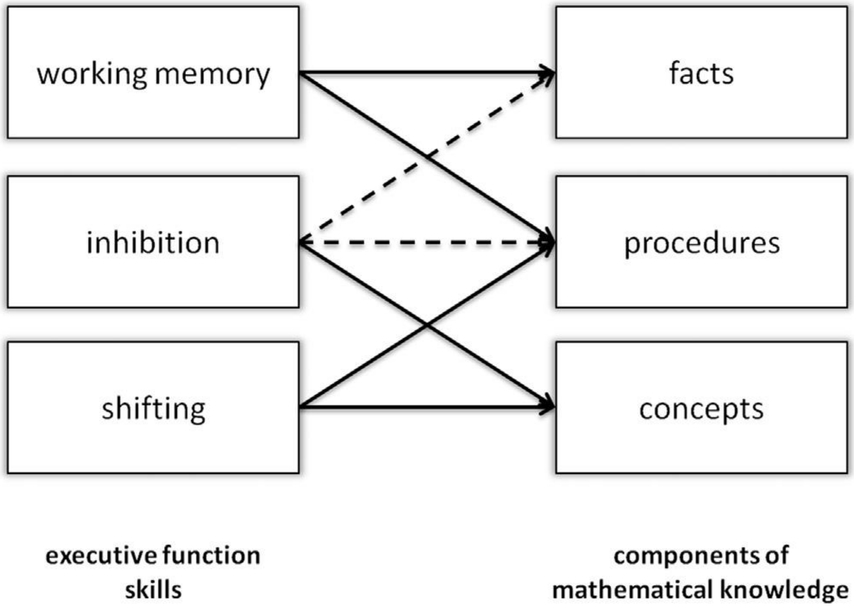

Fig. 1. Dashed lines represent relationships that change over the course of development. Taken from Cragg et al. (2017).

A complex phenomenon like executive function skills specifically planning ability (PA) cannot be measured with imprecise and over-simplified operationalizations. Researchers demonstrate the performance on cognitive tests such as Tower of Hanoi (TOH) or Tower of London (TOL) utilize working memory processes to generate, execute, manipulate, and concatenate these move sequences (Humes et al., 1997).

A number of different paradigms for measuring these skills have been developed, for instance, TOH or TOL illustrate the influence of individual structural problem parameters on planning performance (Borys et al. 1982; Klahr and Robinson 1981; Spitz et al. 1982). When the TOL test was used as a single-time task to assess EF, the performance was shown to develop with age, in line with children's increasing ability to control thoughts and actions as they grow older (Flavell 1971; Siegler 1983).

In light of these studies, for demonstrating the cognitive side of the mathematical problem-solving skills in a structure of planning ability in executive functions; the relationship between PA and math problem-solving skills was investigated in this study.

Materials and Participants

Participants of this study were secondary school students from the fifth, sixth, and seventh grades in Istanbul. Participation rates at each grade level are close to each other (36.5% in 5th grades, 26.4% in 6th grades, and 37.00% in 7th grades). The proportions of girls and boys are also close to each other at each grade level (203 females and 213 males).

To assess the PA, TOL-Freiburg (TOL-F) version and for the mathematical problem-solving skills a curriculum based mathematics achievement test was used. TOL-F hold problems of increasing difficulty. TOL has been widely used as a single-time task to assess executive functioning abilities such as planning and problem-solving as well as implicit learning (Schiff and Vakil, 2015).
Implementation of the TOL-F version is computerized in the form of a realistic three-dimensional representation of a wooden model of the tower configuration shown in Fig.2

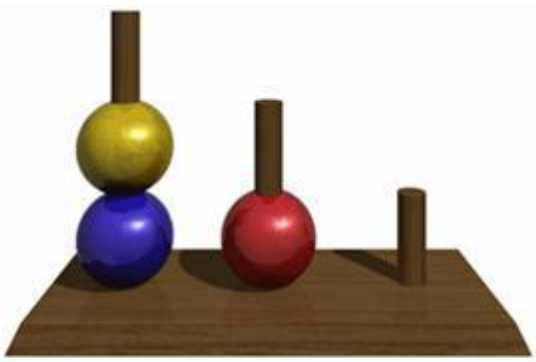

Fig2. Implementation of the "Tower of London" in the Freiburg Version

The aim of the test is to convert a starting state into a defined finishing state in as few moves as possible. For mathematical problem-solving skills, a curriculum mathematics achievement (MA) test was used, a total of 20 questions were asked from the subjects of algebra, geometry, data analysis and number in the secondary school curriculum.

2. Main Results

Students' self-perceived performance in the two tests was measured in order to analyze the relationship between mathematical problem-solving skills and PA.

Depending on gender differences there are not any significant differences in terms of mathematical and cognitive problem-solving. It has been shown in figure 4 and 5 that the participants showed normal distribution in MA ($M = 10.03$, $SD = 4.63$) and TOL-F test ($M= 3.00$, $SD = 1.47$). Mean score differences are related to the number of questions and the differences between standard deviations released that students show a more homogeneous distribution in terms of planning skills than mathematical problem-solving skills.

Relationship Between Planning Ability and Math Performance

All correlational analyses were conducted by the use of the Pearson Product Moment correlation coefficients. statistical analyses revealed that there was a significant positive correlation between PA and mathematics problem-solving skills ($p < .05$).

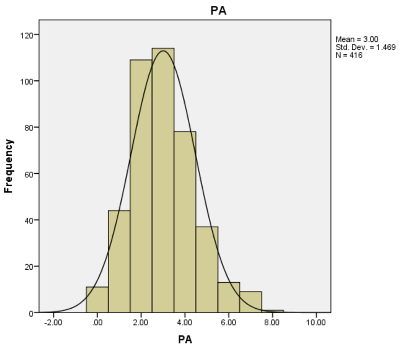

Fig.4: Distribution of the planning ability scores of the participants

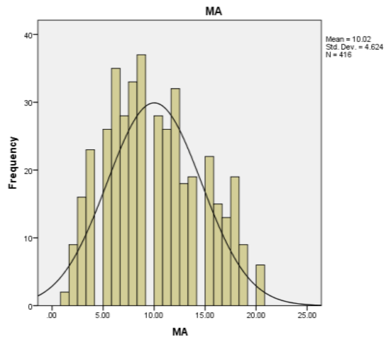

Fig.5: Distribution of the mathematics achievement scores of the participants

Statistical analyses revealed that there was a significant positive correlation between the PA within the scope of executive functions and mathematical problem skills. This correlation is practically significant

$p < .05$, indicating that, when students plan the solution of a problem, they monitor their progress and reflect on their mathematical knowledge. Thus, planning of the problem will construct their correct solution. In both tests, it was determined that students who tried to do a problem without planning and thinking about the solution could not reach the result.

4. Conclusion

The most important thing in problem-solving is understanding, reasoning, and methodology up to producing the right solution (Nursyahidah et al., 2018). In this research, the total number of correct solutions in each test indicates the level of children's planning and problem-solving efforts. In a new problem, planning is the priority for the solution. Therefore, it has been seen in the analyses that competence in mathematical problem-solving is related to PA.

High scores show that children are capable of developing and maintaining mental planning patterns over time. The findings of this quantitive investigation suggest that students who solve mathematical problems reflect their planning skills on problem-solving processes and become aware of their strategy of knowledge.

The reason why operations based on arithmetic in mathematics learning are better understood or performed compared to word problems is that students try to find results by using only the numbers they see, without doing the problem-solving stages, understanding, and planning the problem (Soylu and Soylu, 2006). Therefore, the low rate of word math problems can be attributed to this. This may be due to the inability or deficiencies of students in using their cognitive skills.

The cognitive sides of Mathematics are important in terms of a better understanding of mathematics concepts and more effective use of problem-solving skills from the foundation of mathematics learning. By expanding the scope of this study, the effect of cognitive skills on mathematics education and problem-solving skills can be investigated in terms of other executive functions.

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The Solution of Linear First Order Stiff Differential Equations by Applying Haar Wavelet Collocation Method

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Abstract

Generally, there are many sorts of problems experienced by various disciplines that can be expressed by differential equations. These issues can be interpreted analytically in more straightforward cases; notwithstanding, computational methods are needed for more complex cases. In such manner, the wavelet-based approaches have been utilizing to compute these sorts of equations more dramatically. The Haar Wavelet is one of the relevant procedures that have a place with the wavelet family which is used to solve stiff ordinary differential equations (ODEs). In this paper, The Haar Wavelet method is applied to stiff differential equations to show the accuracy and ability of this method by comparing it to the analytical solutions. In conclusion, the Haar wavelet method gives satisfactory outcomes to stiff differential problems compared to exact solutions.

1. Introduction

Diverse types of engineering and practical problems are explained and interpreted as stiff ordinary differential equations (ODEs) [1]. While some problems can be solved by analytical methods, some cases are not always suitable for having an analytical solution. However, this factor itself is not the only reason to use numerical approaches. In practical use, even finding an analytical solution is not always helpful since the required computation power is exceedingly high [2].

Knowing only the differential equation is not sufficient to find the solutions to the problem, meaning that the need for extra information is evident. Since the proposed solution to the given stiff linear ODE is an initial value problem (IVP), the explanation of extra information can be said as "Initial Conditions". When all the components of y are defined in terms of x, which is specified at a certain value, then the problem can be classified as "Initial Value Problem". Therefore, an initial value problem is in the structure of;

$$y'(x) = f\big(y(x)\big), where\ y(x_0) = y_0 \qquad (1)$$

Various types of other numerical methods have been used to solve these problems previously [3]. In this paper, the Haar Wavelet Method is used to solve such differential equations numerically. The proposed method was first used by Chen and Hsiao (1997) and was applied to the stiff systems of linear ODEs by Lepik (2009) [4,5,6]. The aim of this work is to discuss the quality, efficiency and accuracy of the Haar wavelet method for solving linear stiff ODEs with initial conditions. The resolution and accuracy of the method can be altered by changing only the value of J, which will be explained in the following sections. Typical computation time is taken for the solution of such problems, at J=7,8,9 which yields to very low errors, just takes a few seconds in an average commercial computer with 2 core processors. When the required computation power and low error deviation is taken into consideration, the proposed solution shows the effectiveness of Haar Wavelet Method.

2. Haar Wavelets Collocation Methods

In 1910, a Hungarian mathematician who is Alfred Haar introduced the general form of Haar wavelets which are based on the functions. Haar wavelets are the simplest method in the wavelet families as mathematically and Haar wavelets consist of piecewise constant functions [5].

The following parameters are used in using this method:

The function considered to be in the interval $[A, B]$, where A and B are constant numbers. The quantity M was defined as;

$M = 2^J$, where J defines the maximal level of resolution

$\Delta x = \frac{B-A}{2M}$, where Δx is the length of each equal subinterval [5,6]

Other two parameters are;

$$j = 0,1, \ldots..J\ and\ k = 0,1, \ldots\ldots., m-1$$

j is the dilatation parameter and k is the translation parameter. m, which is introduced in the parameter k, is

$$m = 2^j$$

The wavelet numbers are denoted by the letter i, which is identified as [6,7]

$$i = m + k + 1$$

The following formula represents the i-th Haar wavelet [5].

$$h_i(x) = \begin{cases} 1 \; for \; x \in \left[\xi_1(i), \xi_2(i)\right), \\ -1 \; for \; x \in \left[\xi_2(i), \xi_3(i)\right), \\ 0 \; elsewhere, \end{cases} \tag{2}$$

where

$$\xi_1(i) = A + 2k\mu\Delta x, \qquad \xi_2(i) = A + (2k+1)\mu\Delta x,$$
$$\xi_3(i) = A + 2(k+1)\mu\Delta x, \qquad \mu = \frac{M}{m}$$

With the help of the parameters described above the Haar function can be determined. Haar functions are used to define the Haar matrix which consists of only $1, -1, and\ 0$. To solve the differential equation another operational matrix, which is P, are needed.

$$\int_A^B h_i(x) h_l(x) dx = \begin{cases} (B-A)2^{-j} & for \; l = i, \\ 0 & for \; l \neq i, \end{cases} \tag{3}$$

In the following, analytically calculated integral obtained based on Eqn. 2 [5].

$$p_{\alpha,i}(x) = \begin{cases} 0 & for \; x < \xi_1(i), \\ \frac{1}{\alpha!}[x - \xi_1(i)]^\alpha & for \; x \in [\xi_1(i), \xi_2(i)], \\ \frac{1}{\alpha!}\{[x - \xi_1(i)]^\alpha - 2[x - \xi_2(i)]^\alpha\} & for \; x \in [\xi_2(i), \xi_3(i)], \\ \frac{1}{\alpha!}\{[x - \xi_1(i)]^\alpha - 2[x - \xi_2(i)]^\alpha + [x - \xi_3(i)]^\alpha\} & for \; x > \xi_3(i) \end{cases} \tag{4}$$

Both Haar and operational matrices are constructed in a same way that Lepik (2009) explained in his work [6]. Moreover, to find the Haar function and p function, we need to use collocation points. In other words, Haar matrices and P matrices can be calculated with the help of collocation points [5,6,7].

3. Stiff ODE's

As it is previously introduced, stiff ordinary differential equations in many different fields such as in fluid mechanics, elasticity, electrical networks, chemical reactions, etc. [4]

$$\boldsymbol{y}' = f\,(x, \boldsymbol{y})\,, \quad \boldsymbol{y}\,(\,x_0\,) = \boldsymbol{y_0}\,, \quad a \le x \le b \tag{5}$$

Eqn. (2) is the general form of the first-order linear system of ODEs where $f(x, y)$ can be defined in the interval $[a, b]$. Stiffness of Eqn. (2) can be checked as follows [8];

If the following expressions (a and b) are valid, then a linear system is stiff,

a) Re($\lambda_s(x)$) < 0, s = 1, 2, 3, . . ., m

b) max$|Re(\lambda_s)| \gg$ min$|Re(\lambda_s)|$ s = 1, 2, 3, . . ., m

The ratio of max$|Re(\lambda_s)|$ *and* min$|Re(\lambda_s)|$ is called the stiffness ratio and λ_s, which is expressed in the first and second cases are the eigenvalues of the Jacobian of the system [9].

4. Applications

In this section, several stiff differential problems are computed with the aid of the Haar Wavelet method, which is investigated elaborately in previous sections.

4.1 The first example of the first order stiff differential problem is formulated as follows [10]:

$$y' = 9y \text{ with initial condition } y(0) = e, \qquad t \in [0,1] \tag{6}$$

The exact solution is given as;

$$y(t) = e^{(1-9t)} \tag{7}$$

Fig.1 shows the exact solution and solution of Haar Wavelet. For this solution, the resolution number J is specified as 6. Fig.1 indicated that the solution computed by the Haar Wavelet method converges to the exact solution.

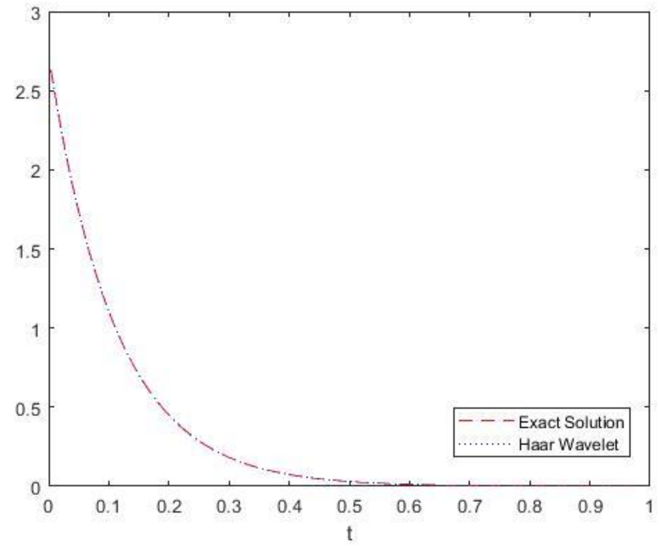

Fig. 1 *Exact solution and the approximate solutions calculated for J=6 and I = 128 by Haar Wavelet for Eqn. 6*

Table 1. Numerical and Analytical Results of Example-1 for J=6

t	Haar Solution	Exact Solution	Absolute Error	Logarithmic Error
0,0039	2.625962823	2.624377565	0.001585258	-2.7999001774
0.1	1.129019714	1.128730726	2.88988E-04	-3.5391205533
0.2	0.452444753	0.452499442	5.46882E-05	-4.2621059785
0.3	0.181313269	0.181403536	9.02670E-05	-4.0444710228
0.4	0.072659703	0.072723278	6.35752E-05	-4.1967125528
0.5	0.029117739	0.029154201	3.64622E-05	-4.4381569194
0.6	0.012519028	0.012539065	2.00380E-05	-4.6981459530
0.7	0.005016891	0.005026815	9.92410E-06	-5.0033087088
0.8	0.002010475	0.002015211	4.73631E-06	-5.3245598520
0.9	8.05680E-04	8.07883E-04	2.20244E-06	-5.6570964184
1	3.46398E-04	3.47464E-04	1.06778E-06	-5.9715182840

Table 2. Numerical and Analytical Results of Example-1 for J=10

t	Haar Solution	Exact Solution	Absolute Error	Logarithmic Error
2.44E-04	2.712322136	2.712315598	6.53793E-06	-5.1845597
0.1	1.106629962	1.106628891	1.07091E-06	-5.9702487
0.2	0.449526249	0.449526466	2.16714E-07	-6.6641141
0.3	0.182602908	0.182603261	3.52772E-07	-6.4525062
0.4	0.074175472	0.074175723	2.50841E-07	-6.6006019
0.5	0.030130959	0.030131104	1.45579E-07	-6.8369017
0.6	0.012293459	0.012293536	7.71328E-08	-7.1127616
0.7	0.004993749	0.004993788	3.85723E-08	-7.4137250
0.8	0.002028520	0.002028540	1.86093E-08	-7.7302691
0.9	8.24009E-04	8.24018E-04	8.75413E-09	-8.0577869
1	3.36196E-04	3.36201E-04	4.05675E-09	-8.3918220

4.2 Following is the second example of the first order stiff differential problem [11]:

$$y' + 100y = (99t + 1)e^{-t} \ \ with\ initial\ condition \quad y(0) = 1, \qquad t \in [0,1] \tag{8}$$

The exact solution is given as;

$$y(t) = e^{-100t} + te^{-t} \tag{9}$$

The solutions that belong to the exact solution and Haar Wavelet method respectively are given in Fig.3. With the assistance of Haar Wavelet method this particular solution computed for J=6. Results of both Haar wavelet and exact solution are matching for certain.

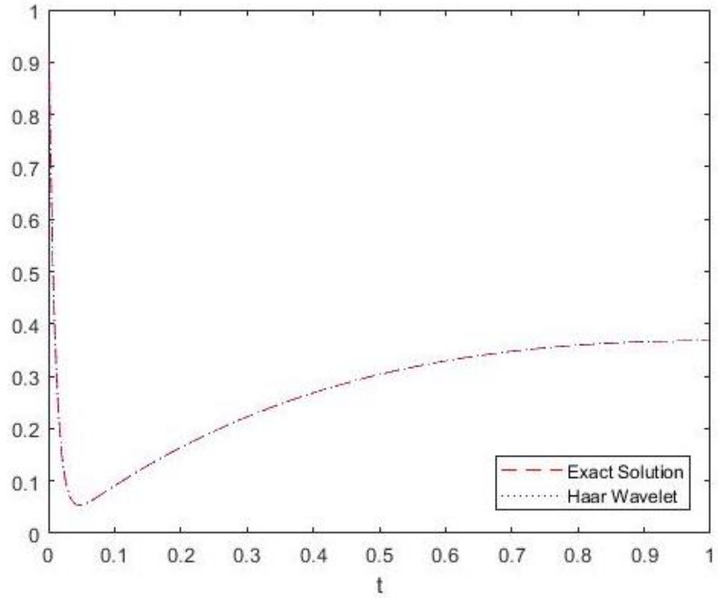

Fig. 2 *Exact solution and the approximate solutions calculated for J=8 and I = 512 by Haar Wavelet for Eqn. 8*

Table 3. Numerical and Analytical Results of Example-2 for J=8

t	**Haar Solution**	**Exact Solution**	**Absolute Error**	**Logarithmic Error**
0.009	0.912006769	0.907936227	0.004070542	-2.390347695
0.1	0.091002275	0.091003423	1.1475117506E-06	-5.940242858
0.2	0.163874058	0.163874051	7.2694935666E-09	-8.138495843
0.3	0.222144164	0.222144158	6.4484494959E-09	-8.1905446971
0.4	0.267892180	0.267892174	5.6225828435E-09	-8.2500641369
0.5	0.303561058	0.303561053	4.8819844834E-09	-8.3114036052
0.6	0.329415481	0.329415477	4.2437159892E-09	-8.3722536889
0.7	0.347638800	0.347638797	3.6825082428E-09	-8.4338562723
0.8	0.359445612	0.359445609	3.1896046396E-09	-8.4962631461
0.9	0.365888797	0.365888794	2.7570746904E-09	-8.5595515663
1	0.367879268	0.367879265	2.3779136998E-09	-8.6238039111

Table 4. Numerical and Analytical Results of Example-2 for J=12

t	**Haar Solution**	**Exact Solution**	**Absolute Error**	**Logarithmic Error**
0	0.993994539	0.993976104	1.843459257E-05	-4.734366456
0.1	0.090558792	0.090558797	4.743521422E-09	-8.323899133
0.2	0.163754148	0.163754147	2.834754653E-11	-10.54748452
0.3	0.222239135	0.222239135	2.510605612E-11	-10.60022150
0.4	0.268113289	0.268113288	2.186090197E-11	-10.66033192
0.5	0.303246818	0.303246818	1.923927683E-11	-10.71581125
0.6	0.329295020	0.329295020	1.706934593E-11	-10.76778311
0.7	0.347611531	0.347611531	1.532352023E-11	-10.81464145
0.8	0.359462074	0.359462074	1.286493134E-11	-10.89059252
0.9	0.365916166	0.365916166	1.099381696E-11	-10.95885149
1	0.367879440	0.367879440	8.837763854E-12	-11.05365760

5. Conclusion

As denoted earlier in the research, the Haar wavelet has a broad variety of use areas such as linear and nonlinear ODE. The investigation shows that the Haar Wavelet, which belongs to the Wavelet family, is applicable to many types of linear stiff initial differential problems. Compared with their exact solution results, the results of examples 4.1 and 4.2 are determined by the Haar wavelet method. Besides, numerical values and comparison graphs demonstrate how the Haar wavelet method is efficient. As a consequence, the outcomes of the examples can be obtained swiftly with the assistance of the construction of the Haar matrices.

6. References

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Timelike Harmonic Evolute Surfaces of Quasi Binormal Surfaces

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Abstract

In this study, we work on timelike harmonic surfaces of quasi binormal surfaces with quasi frame in three dimensional Minkowski space. Using quasi frame, we first construct special type of ruled surface called quasi binormal ruled surface, we then study timelike harmonic evolute surfaces of them. The condition for timelike harmonic evolute surface to be Bonnet surface is given for each cases. Later, some geometric properties are examined and application of results is given.

1. Introduction

The theory of surfaces, in differential geometry, is one of the most important subjects to study since they have too many applications in engineering and physics. Both ruled surfaces and harmonic evolute surfaces, two of these impressive surfaces, have been studied by many geometers.

Harmonic evolutes of surfaces in Euclidean space were studied in [2]. In 2014, After Sipus and Viladimir worked on properties of harmonic evolutes of constant mean curvature surfaces and their relation to parallel surfaces in Minkowski space [11], Protrrka focused on harmonic evolute of both timelike ruled surfaces and helicoidal surfaces in Minkowski space [9], [10]. Also, harmonic evolutes of B-scrolls with constant mean curvature in Lorentz-Minkowski space can be found in [7]. Most recently, in [4], harmonic evolute of quasi normal surfaces have been studied.

In this study, we work on the surfaces associated to a given regular surface. That is, timelike harmonic surfaces of quasi binormal surfaces with quasi frame in three dimensional Minkowski space. Using quasi frame, we first construct special type of ruled surface called quasi binormal ruled surface, we then study timelike harmonic evolute surfaces of them. The condition for timelike harmonic evolute surface to be Bonnet surface is given for each cases. Lastly, an application of results is given.

2. Preliminaries

In three dimensional Minkowski space R_1^3, the dot and cross products of two vectors $v=(v_1,v_2,v_3)$ and $w=(w_1,w_{2,}w_3)$ are defined as

$$\langle v,w\rangle = v_1w_1 + v_2w_2 - v_3w_3$$

and

$$v\times w = v_3w_2 - v_2w_3)u_1 + (v_1w_3 - v_3w_1)u_2 + (v_1w_2 - v_2w_1)u_3$$

where $u_1\times u_2 = u_3,\ u_2\times u_3 = -u_1,\ u_3\times u_1 = -u_2,$ respectively [5],[6]. The norm of the vector v is is given by $\|v\| = \sqrt{|\langle v,v\rangle|}.$

We say that Lorentzian vector v is spacelike, lightlike or timelike if $\langle v,v\rangle > 0$ or $v=0,$ $\langle v,v\rangle = 0$ and $v\neq 0,$ $\langle v,v\rangle < 0,$ respectively [8].

As an alternative to the Frenet frame, we use a new adapted frame along both timelike and spacelike curves, the quasi frame defined in [12], [3]. General formula for derivation of quasi frame is given

$$\begin{bmatrix}\mathbf{t}'\\ \mathbf{n}_q'\\ \mathbf{b}_q'\end{bmatrix} = \begin{bmatrix}0 & \varepsilon_1 k_1 & -\varepsilon_0\varepsilon_1 k_2\\ -\varepsilon_0 k_1 & 0 & -\varepsilon_0\varepsilon_1 k_3\\ -\varepsilon_0 k_2 & -\varepsilon_1 k_3 & 0\end{bmatrix}\begin{bmatrix}\mathbf{t}\\ \mathbf{n}_q\\ \mathbf{b}_q\end{bmatrix}, \tag{1.1}$$

where $\langle \mathbf{t},\mathbf{t}\rangle = \varepsilon_0,$ $\langle \mathbf{n}_q,\mathbf{n}_q\rangle = \varepsilon_1,$ $\langle \mathbf{b}_q,\mathbf{b}_q\rangle = -\varepsilon_0\varepsilon_1$ and k_i be quasi curvatures when $1\leq i\leq 3.$ The vector products are defined as $\mathbf{t}\times\mathbf{n}_q = \varepsilon_0\varepsilon_1\mathbf{b}_q,$ $\mathbf{n}_q\times\mathbf{b}_q = -\varepsilon_0\mathbf{t},$ and $\mathbf{b}_q\times\mathbf{t} = -\varepsilon_1\mathbf{n}_q.$

In equation (1.1) if $\varepsilon_0 = 1$ or $\varepsilon_0 = -1,$ then a curve is spacelike or timelike, respectively. If $\varepsilon_1 = -1$ or $\varepsilon_1 = 1,$ then a spacelike curve is called type I or type II, respectively.

Let M be a regular surface given with the parametrization $\varphi(s,t)$ in $E^3.$ Let N be the standard unit normal vector field on a surface φ defined by

$$N = \frac{\varphi_s\times\varphi_t}{\|\varphi_s\times\varphi_t\|},$$

The tangent space of M at an arbitrary point is spanned by the vectors φ_s and $\varphi_t.$ The coefficients of the first and fundamental form of M are defined in [1], [5] as

$$\begin{aligned} E &= \langle\varphi_s,\varphi_s\rangle, F = \langle\varphi_s,\varphi_t\rangle, G = \langle\varphi_t,\varphi_t\rangle,\\ e &= \langle\varphi_{ss},N\rangle, f = \langle\varphi_{st},N\rangle, g = \langle\varphi_{tt},N\rangle. \end{aligned}$$

The Gaussian and the mean curvatures of M are given by

$$K = \frac{eg - f^2}{EG - F^2}$$

and

$$H = \frac{Eg - 2Ff + Ge}{2(EG - F^2)},$$

respectively.

Theorem 2.1. The surface is minimal if and only if it has vanishing mean curvature [1].

Theorem 2.2. If $E = G,\ F = 0,\ f = c \neq 0\ \ (c = constant)$ are satisfied then the surface is called A-net on a surface [1].

Theorem 2.3. A surface to be a Bonnet surface if and only if surface has an A-net [1].

3. Timelike Harmonic Evolute Surfaces of Binormal Surface

In this part, we work on the timelike harmonic evolute surfaces of binormal surfaces which are not minimal. The parametrization of quasi binormal ruled surfaces is given as

$$\varphi(s,t) = \alpha(s) + t\mathbf{b_q}.$$

Theorem 3.1. *The ruled surface generated by quasi binormal vector is minimal if and only if*

$$t^2 k_3(\varepsilon_1 k_1 k_3 - (k_2)_s) + \varepsilon_0(1 - t\varepsilon_0 k_2)(k_1(1 - t\varepsilon_0 k_2) - t(k_3)_s = 0.$$

Proof. Using definition of quasi binormal surface, we get first and second partial derivatives of φ as

$$\begin{aligned} \varphi_s &= (1 - t\varepsilon_0 k_2)\mathbf{t} - t\varepsilon_1 k_3 \mathbf{n_q}, \\ \varphi_t &= \mathbf{b_q} \end{aligned}$$

and

$$\begin{aligned} \varphi_{ss} &= t\varepsilon_0(\varepsilon_1 k_1 k_3 - (k_2)_s)\mathbf{t} + \varepsilon_1(k_1(1 - t\varepsilon_0 k_2) - t(k_3)_s)\mathbf{n_q} + \varepsilon_0(tk_3^2 - \varepsilon_1 k_2(1 - tk_2))\mathbf{b_q}, \\ \varphi_{ts} &= -\varepsilon_0 k_2 \mathbf{t} - \varepsilon_1 k_3 \mathbf{n_q}, \\ \varphi_{tt} &= 0, \end{aligned}$$

respectively. By the definition of the unit normal vector, we have

$$\varphi_s \times \varphi_t = t\varepsilon_0 \varepsilon_1 k_3 \mathbf{t} + \varepsilon_1(1 - t\varepsilon_0 k_2)\mathbf{n_q}$$

and so

$$N = \frac{t\varepsilon_0 \varepsilon_1 k_3}{\sqrt{t^2 \varepsilon_0 k_3^2 + \varepsilon_1(1 - t\varepsilon_0 k_2)^2}}\mathbf{t} + \frac{\varepsilon_1(1 - t\varepsilon_0 k_2)}{\sqrt{t^2 \varepsilon_0 k_3^2 + \varepsilon_1(1 - t\varepsilon_0 k_2)^2}}\mathbf{n_q}.$$

Therefore, we have the coefficients of the first and second fundamental forms given

$$\begin{aligned} E &= \varepsilon_0(1 - t\varepsilon_0 k_2)^2 + t^2 \varepsilon_1 k_3^2, \\ F &= 0, \\ G &= -\varepsilon_0 \varepsilon_1, \end{aligned}$$

and

$$
\begin{aligned}
e &= \frac{1}{\sqrt{t^2\varepsilon_0 k_3^2+\varepsilon_1(1-t\varepsilon_0 k_2)^2}}(t^2\varepsilon_0\varepsilon_1 k_3(\varepsilon_1 k_1 k_3-(k_2)_s)+\varepsilon_1(1-t\varepsilon_0 k_2)(k_1(1-t\varepsilon_0 k_2)-t(k_3)_s),\\
f &= -\frac{\varepsilon_1 k_3}{\sqrt{t^2\varepsilon_0 k_3^2+\varepsilon_1(1-t\varepsilon_0 k_2)^2}},\\
g &= 0,
\end{aligned}
$$

respectively.

The mean curvature of timelike harmonic evolute surfaces of binormal surfaces is calculated

$$
H=\frac{1}{2(t^2\varepsilon_0 k_3^2+\varepsilon_1(1-t\varepsilon_0 k_2)^2)^{\frac{3}{2}}}(t^2 k_3(\varepsilon_1 k_1 k_3-(k_2)_s)+\varepsilon_0(1-t\varepsilon_0 k_2)(k_1(1-t\varepsilon_0 k_2)-t(k_3)_s).
$$

Using Theorem 2.1. completes the proof.

Theorem 3.2.The Gauss curvature of timelike harmonic evolute surfaces of binormal surfaces is calculated

$$
K=-\frac{\varepsilon_0\varepsilon_1 k_3^2(t^2\varepsilon_1 k_3^2+(1-t\varepsilon_0 k_2)^2)}{t^2\varepsilon_0 k_3^2+\varepsilon_1(1-t\varepsilon_0 k_2)^2}.
$$

Corollary 3.3. The ruled surfaces generated by spacelike type II curve ($\varepsilon_0=\varepsilon_1=1$) is flat only when $k_3=0$.

Theorem 3.4. *The ruled surfaces generated by quasi binormal vector are not a Bonnet surface.*

Proof. Using Definition 2.2. and Theorem 2.3, it is easily shown.

Suppose φ is not minimal. Then, a harmonic evolute surface of the quasi binormal surface is given by

$$
\varphi^h(s,t)=\varphi(s,t)+\frac{1}{H}N.
$$

Theorem 3.5. *A harmonic evolute surface of* φ *is given by*

$$
\varphi^h(s,t)=\alpha(s)+\frac{t\varepsilon_0\varepsilon_1 k_3}{H\sqrt{t^2\varepsilon_0 k_3^2+\varepsilon_1(1-t\varepsilon_0 k_2)^2}}\mathbf{t}+\frac{\varepsilon_1(1-t\varepsilon_0 k_2)}{H\sqrt{t^2\varepsilon_0 k_3^2+\varepsilon_1(1-t\varepsilon_0 k_2)^2}}\mathbf{n_q}+t\mathbf{b_q}.
$$

Theorem 3.6. *A harmonic evolute surface of* φ *is a Bonnet surface if and only if*

$$\varepsilon_0(1-t\varepsilon_0k_2-\varepsilon_0\varepsilon_1k_1(\frac{1-t\varepsilon_0k_2}{H\sqrt{W}})+(\frac{t\varepsilon_0\varepsilon_1k_3}{H\sqrt{W}})_s)^2+\varepsilon_1(t\varepsilon_1k_3-\frac{t\varepsilon_0k_1k_3}{H\sqrt{W}}-\varepsilon_1(\frac{1-t\varepsilon_0k_2}{H\sqrt{W}})_s)^2-\varepsilon_0\varepsilon_1(\frac{\varepsilon_0k_3}{H\sqrt{W}})^2$$

$$-\varepsilon_0((\frac{tk_3}{H\sqrt{W}})_t)^2+\varepsilon_1((\frac{1-t\varepsilon_0k_2}{H\sqrt{W}})_t)^2+\varepsilon_0\varepsilon_1=(1-t\varepsilon_0k_2-\varepsilon_0\varepsilon_1k_1(\frac{1-t\varepsilon_0k_2}{H^{\mathbf{b_q}}\sqrt{W}})+(\frac{t\varepsilon_0\varepsilon_1k_3}{H\sqrt{W}})_s)(\frac{t\varepsilon_1k_3}{H\sqrt{W}})_t$$

$$-(t\varepsilon_1k_3-\frac{t\varepsilon_0k_1k_3}{H\sqrt{W}}-\varepsilon_1(\frac{1-t\varepsilon_0k_2}{H\sqrt{W}})_s)(\frac{1-t\varepsilon_0k_2}{H\sqrt{W}})_t+\frac{\varepsilon_1k_3}{H\sqrt{W}}=0$$

and

$$\begin{aligned}f^h \quad = \quad & \frac{1}{\omega}[(1-t\varepsilon_0k_2-\varepsilon_0\varepsilon_1k_1(\frac{1-t\varepsilon_0k_2}{H\sqrt{W}})+(\frac{t\varepsilon_0\varepsilon_1k_3}{H\sqrt{W}})_s)_t((t\varepsilon_1k_3-\frac{t\varepsilon_0k_1k_3}{H\sqrt{W}}-\varepsilon_1(\frac{1-t\varepsilon_0k_2}{H\sqrt{W}})_s)\\
&-\varepsilon_1\frac{k_3}{H\sqrt{W}}(\frac{1-t\varepsilon_0k_2}{H\sqrt{W}})_t)\qquad -\varepsilon_1((t\varepsilon_1k_{3-}\frac{t\varepsilon_0k_1k_3}{H\sqrt{W}}-\varepsilon_1(\frac{1-t\varepsilon_0k_2}{H\sqrt{W}})_s)_t(\varepsilon_1(1-t\varepsilon_0k_2\\
&-\varepsilon_0\varepsilon_1k_1(\frac{1-t\varepsilon_0k_2}{H\sqrt{W}})+(\frac{t\varepsilon_0\varepsilon_1k_3}{H\sqrt{W}})_s)+(\frac{k_3}{H\sqrt{W}})(\frac{tk_3}{H\sqrt{W}})_t))+(\frac{\varepsilon_1k_3}{H\sqrt{W}})_t((t\varepsilon_1k_3-\frac{t\varepsilon_0k_1k_3}{H\sqrt{W}}\\
&-\varepsilon_1(\frac{1-t\varepsilon_0k_2}{H\sqrt{W}})_s)(\frac{tk_3}{H\sqrt{W}})_t)+(\varepsilon_0(\frac{1-t\varepsilon_0k_2}{H\sqrt{W}})_t(1-t\varepsilon_0k_2-\varepsilon_0\varepsilon_1k_1(\frac{1-t\varepsilon_0k_2}{H\sqrt{W}})+(\frac{t\varepsilon_0\varepsilon_0k_3}{H\sqrt{W}})_s))]=\rho,\end{aligned}$$

where $W=t^2\varepsilon_0k_3^2+\varepsilon_1(1-t\varepsilon_0k_2)^2$ and $\omega=\|\varphi_s\times\varphi_t\|=(t\varepsilon_1k_3)^2+(1-t\varepsilon_0k_2)^2$.

Proof. The first and second partial derivatives of φ_s^h are given

$$\begin{aligned}\phi_s^h(s,t) \quad &= \quad (1-t\varepsilon_0k_2-\varepsilon_0\varepsilon_1k_1(\frac{1-t\varepsilon_0k_2}{H\sqrt{W}})+(\frac{t\varepsilon_0\varepsilon_1k_3}{H\sqrt{W}})_s)\mathbf{t}-(t\varepsilon_1k_3-\frac{t\varepsilon_0k_1k_3}{H\sqrt{W}}-(\frac{1-t\varepsilon_0k_2}{H\sqrt{W}})_s)\mathbf{n_q}-(\frac{\varepsilon_0k_3}{H\sqrt{W}})\mathbf{b_q},\\
\phi_t^h(s,t) \quad &= \quad (\frac{t\varepsilon_0\varepsilon_1k_3}{H\sqrt{W}})_t\mathbf{t}+\varepsilon_1(\frac{1-t\varepsilon_0k_2}{H\sqrt{W}})_t\mathbf{n_q}+\mathbf{b_q},\end{aligned}$$

and

$$\begin{aligned}\phi_{ss}^h(s,t) \quad = \quad & ((1-t\varepsilon_0k_2-\varepsilon_0\varepsilon_1k_1(\frac{1-t\varepsilon_0k_2}{H\sqrt{W}})+(\frac{t\varepsilon_0\varepsilon_1k_3}{H\sqrt{W}})_s)_s-\varepsilon_0((t\varepsilon_1k_3-\frac{t\varepsilon_0k_1k_3}{H\sqrt{W}}-\varepsilon_1(\frac{1-t\varepsilon_0k_2}{H\sqrt{W}})_s)k_1+(\frac{\varepsilon_0k_2k_3}{H\sqrt{W}}))\mathbf{t}\\
&+(t\varepsilon_1k_3-\frac{t\varepsilon_0k_1k_3}{H\sqrt{W}}-\varepsilon_1(\frac{1-t\varepsilon_0k_2}{H\sqrt{W}})_s)_s-\varepsilon_1((\frac{\varepsilon_0k_3^2}{H\sqrt{W}})(1-t\varepsilon_0k_2-\varepsilon_0\varepsilon_1k_1(\frac{1-t\varepsilon_0k_2}{H\sqrt{W}})+(\frac{t\varepsilon_0\varepsilon_1k_3}{H\sqrt{W}})_s)k_1))\mathbf{n_q}\\
&+(\frac{\varepsilon_0k_3}{H\sqrt{W}})_s-\varepsilon_0\varepsilon_1((t\varepsilon_1k_3-\frac{t\varepsilon_0k_1k_3}{H\sqrt{W}}-\varepsilon_1(\frac{1-t\varepsilon_0k_2}{H\sqrt{W}})_s)k_3+(1-t\varepsilon_0k_2-\varepsilon_0\varepsilon_1k_1(\frac{1-t\varepsilon_0k_2}{H\sqrt{W}})+(\frac{t\varepsilon_0\varepsilon_1k_3}{H\sqrt{W}})_sk_2))\mathbf{b_q},\\
\phi_{tt}^h(s,t) \quad = \quad & (\frac{t\varepsilon_0\varepsilon_1k_3}{H\sqrt{W}})_{tt}\mathbf{t}+\varepsilon_1(\frac{1-t\varepsilon_0k_2}{H\sqrt{W}})_{tt}\mathbf{n_q},\\
\phi_{ts}^h(s,t) \quad = \quad & (1-t\varepsilon_0k_2-\varepsilon_0\varepsilon_1k_1(\frac{1-t\varepsilon_0k_2}{H\sqrt{W}})+(\frac{t\varepsilon_0\varepsilon_1k_3}{H\sqrt{W}})_s)_t\mathbf{t}-(t\varepsilon_1k_3-\frac{t\varepsilon_0k_1k_3}{H\sqrt{W}}-\varepsilon_1(\frac{1-t\varepsilon_0k_2}{H\sqrt{W}})_s)_t\mathbf{n_q}-(\frac{\varepsilon_0k_3}{H\sqrt{W}})_t\mathbf{b_q}.\end{aligned}$$

Using first derivative equations of the harmonic evolute surface with respect to s and t, the normal vector of $\varphi^h(s,t)$ is calculated by

$$\begin{aligned} N^h \;=\; & \frac{1}{\omega}[\varepsilon_0((t\varepsilon_1k_3-\frac{t\varepsilon_0k_1k_3}{H\sqrt{W}}-\varepsilon_1(\frac{1-t\varepsilon_0k_2}{H\sqrt{W}})_s)-\varepsilon_1(\frac{k_3}{H\sqrt{W}})(\frac{1-t\varepsilon_0k_2}{H\sqrt{W}})_t)\mathbf{t}+(\varepsilon_1(1-t\varepsilon_0k_2-\varepsilon_0\varepsilon_1k_1(\frac{1-t\varepsilon_0k_2}{H\sqrt{W}}) \\ & +(\frac{t\varepsilon_0\varepsilon_1k_3}{H\sqrt{W}})_s)+(\frac{k_3}{H\sqrt{W}})(\frac{tk_3}{H\sqrt{W}})_t)\mathbf{n_q}+(\varepsilon_0(\frac{1-t\varepsilon_0k_2}{H\sqrt{W}})_t(1-t\varepsilon_0k_2-\varepsilon_0\varepsilon_1k_1(\frac{1-t\varepsilon_0k_2}{H\sqrt{W}})+(\frac{t\varepsilon_0\varepsilon_1k_3}{H\sqrt{W}})_s) \\ & +(t\varepsilon_1k_3-\frac{t\varepsilon_0k_1k_3}{H\sqrt{W}}-\varepsilon_1(\frac{1-t\varepsilon_0k_2}{H\sqrt{W}})_s)(\frac{tk_3}{H\sqrt{W}})_t)\mathbf{b_q}]. \end{aligned}$$

where $\omega = \| \varphi_s^h \wedge \varphi_t^h \|$.

Then, we have the coefficients of the first and second fundamental forms of harmonic evolute surface as

$$\begin{aligned} E^h \;=\; & \varepsilon_0(1-t\varepsilon_0k_2-\varepsilon_0\varepsilon_1k_1(\frac{1-t\varepsilon_0k_2}{H\sqrt{W}})+(\frac{t\varepsilon_0\varepsilon_1k_3}{H\sqrt{W}})_s)^2+\varepsilon_1(t\varepsilon_1k_3-\frac{t\varepsilon_0k_1k_3}{H\sqrt{W}}-\varepsilon_1(\frac{1-t\varepsilon_0k_2}{H\sqrt{W}})_s)^2-\varepsilon_0\varepsilon_1(\frac{k_3}{H\sqrt{W}})^2, \\ F^h \;=\; & (1-t\varepsilon_0k_2-\varepsilon_0\varepsilon_1k_1(\frac{1-t\varepsilon_0k_2}{H\sqrt{W}})+(\frac{t\varepsilon_0\varepsilon_1k_3}{H\sqrt{W}})_s)(\frac{t\varepsilon_1k_3}{H\sqrt{W}})_t-(t\varepsilon_1k_3-\frac{t\varepsilon_0k_1k_3}{H\sqrt{W}}-\varepsilon_1(\frac{1-t\varepsilon_0k_2}{H\sqrt{W}})_s)(\frac{1-t\varepsilon_0k_2}{H\sqrt{W}})_t+(\frac{\varepsilon_1k_3}{H\sqrt{W}}), \\ G^h \;=\; & \varepsilon_0((\frac{tk_3}{H\sqrt{W}})_t)^2+\varepsilon_1((\frac{1-t\varepsilon_0k_2}{H\sqrt{W}})_t)^2-\varepsilon_0\varepsilon_1, \end{aligned}$$

and

$$\begin{aligned} e^h \;=\; & \frac{1}{\omega}[((1-t\varepsilon_0k_2-\varepsilon_0\varepsilon_1k_1(\frac{1-t\varepsilon_0k_2}{H\sqrt{W}})+(\frac{t\varepsilon_0\varepsilon_1k_3}{H\sqrt{W}})_s)_s-\varepsilon_0((t\varepsilon_1k_3-\frac{t\varepsilon_0k_1k_3}{H\sqrt{W}}-\varepsilon_1(\frac{1-t\varepsilon_0k_2}{H\sqrt{W}})_s)k_1 \\ & +(\frac{\varepsilon_0k_2k_3}{H\sqrt{W}}))((t\varepsilon_1k_3-\frac{t\varepsilon_0k_1k_3}{H\sqrt{W}}-\varepsilon_1(\frac{1-t\varepsilon_0k_2}{H\sqrt{W}})_s)-\varepsilon_1(\frac{k_3}{H\sqrt{W}})(\frac{1-t\varepsilon_0k_2}{H\sqrt{W}})_t)+\varepsilon_1((t\varepsilon_1k_3-\frac{t\varepsilon_0k_1k_3}{H\sqrt{W}} \\ & -\varepsilon_1(\frac{1-t\varepsilon_0k_2}{H\sqrt{W}})_s)_s-\varepsilon_1((\frac{\varepsilon_0k_3^2}{H\sqrt{W}})(1-t\varepsilon_0k_2-\varepsilon_0\varepsilon_1k_1(\frac{1-t\varepsilon_0k_2}{H\sqrt{W}})+(\frac{t\varepsilon_0\varepsilon_1k_3}{H\sqrt{W}})_s)k_1))(\varepsilon_1(1-t\varepsilon_0k_2 \\ & -\varepsilon_0\varepsilon_1k_1(\frac{1-t\varepsilon_0k_2}{H\sqrt{W}})+(\frac{t\varepsilon_0\varepsilon_1k_3}{H\sqrt{W}})_s)+(\frac{k_3}{H\sqrt{W}})(\frac{tk_3}{H\sqrt{W}})_t)+((\frac{\varepsilon_0k_3}{H\sqrt{W}})_s-\varepsilon_0\varepsilon_1((t\varepsilon_1k_3-\frac{t\varepsilon_0k_1k_3}{H\sqrt{W}} \\ & -\varepsilon_1(\frac{1-t\varepsilon_0k_2}{H\sqrt{W}})_s)k_3+(1-t\varepsilon_0k_2-\varepsilon_0\varepsilon_1k_1(\frac{1-t\varepsilon_0k_2}{H\sqrt{W}})+(\frac{t\varepsilon_0\varepsilon_1k_3}{H\sqrt{W}})_s)k_2))((t\varepsilon_1k_3-(\frac{t\varepsilon_0k_1k_3}{H\sqrt{W}}) \\ & -\varepsilon_1(\frac{1-t\varepsilon_0k_2}{H\sqrt{W}})_s)(\frac{tk_3}{H\sqrt{W}})_t+(\varepsilon_0(\frac{1-t\varepsilon_0k_2}{H\sqrt{W}})_t(1-t\varepsilon_0k_2-\varepsilon_0\varepsilon_1k_1(\frac{1-t\varepsilon_0k_2}{H\sqrt{W}})+(\frac{t\varepsilon_0\varepsilon_1k_3}{H\sqrt{W}})_s))], \end{aligned}$$

$$
\begin{aligned}
f^{h} \;=\;& \frac{1}{\omega}[(1-t\varepsilon_0 k_2-\varepsilon_0\varepsilon_1 k_1(\frac{1-t\varepsilon_0 k_2}{H\sqrt{W}})+(\frac{t\varepsilon_0\varepsilon_1 k_3}{H\sqrt{W}})_s)_t(t\varepsilon_1 k_3-\frac{t\varepsilon_0 k_1 k_3}{H\sqrt{W}}-\varepsilon_1(\frac{1-t\varepsilon_0 k_2}{H\sqrt{W}})_s)v \\
& -\varepsilon_1(\frac{k_3}{H\sqrt{W}})(\frac{1-t\varepsilon_0 k_2}{H\sqrt{W}})_t-\varepsilon_1((t\varepsilon_1 k_3-\frac{t\varepsilon_0 k_1 k_3}{H\sqrt{W}}-\varepsilon_1(\frac{1-t\varepsilon_0 k_2}{H\sqrt{W}})_s)_t(\varepsilon_1(1-t\varepsilon_0 k_2 \\
& -\varepsilon_0\varepsilon_1 k_1(\frac{1-t\varepsilon_0 k_2}{H\sqrt{W}})+(\frac{t\varepsilon_0\varepsilon_1 k_3}{H\sqrt{W}})_s)+(\frac{k_3}{H\sqrt{W}})(\frac{k_3}{H\sqrt{W}})_t))+(\frac{\varepsilon_1 k_3}{H\sqrt{W}})_t((t\varepsilon_1 k_3-\frac{t\varepsilon_0 k_1 k_3}{H\sqrt{W}} \\
& -\varepsilon_1(\frac{1-t\varepsilon_0 k_2}{H\sqrt{W}})_s)(\frac{tk_3}{H\sqrt{W}})_t)+(\varepsilon_0(\frac{1-t\varepsilon_0 k_2}{H\sqrt{W}})_t(1-t\varepsilon_0 k_2-\varepsilon_0\varepsilon_1 k_1(\frac{1-t\varepsilon_0 k_2}{H\sqrt{W}})+(\frac{t\varepsilon_0\varepsilon_1 k_3}{H\sqrt{W}})_s))], \\
g^{h} \;=\;& \frac{1}{\omega}[(\frac{t\varepsilon_0\varepsilon_1 k_3}{H\sqrt{W}})_{tt}(t\varepsilon_1 k_3-\frac{t\varepsilon_0 k_1 k_3}{H\sqrt{W}}-\varepsilon_1(\frac{1-t\varepsilon_0 k_2}{H\sqrt{W}})_s)-\varepsilon_1(\frac{k_3}{H\sqrt{W}})(\frac{1-t\varepsilon_0 k_2}{H\sqrt{W}})_t) \\
& +(\frac{1-t\varepsilon_0 k_2}{H\sqrt{W}})_{tt}(\frac{k_3}{H\sqrt{W}})(\frac{tk_3}{H\sqrt{W}})_t+(\varepsilon_1(1-t\varepsilon_0 k_2+-\varepsilon_0\varepsilon_1 k_1(\frac{1-t\varepsilon_0 k_2}{H\sqrt{W}})-(\frac{t\varepsilon_0\varepsilon_1 k_3}{H\sqrt{W}})_s))],
\end{aligned}
$$

respectively. Using the definition of a Bonnet surface, we have proved the theorem.

4. Application

Application to timelike helix:

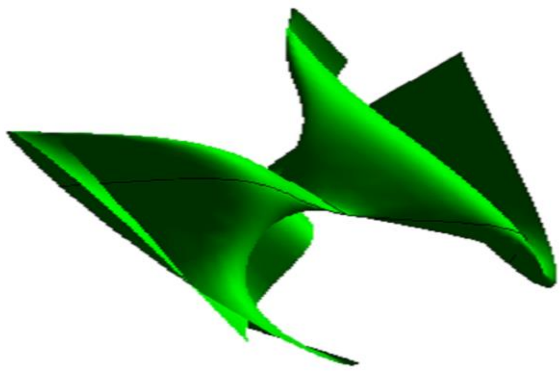

Figure 1. The quasi binormal surface of timelike helix

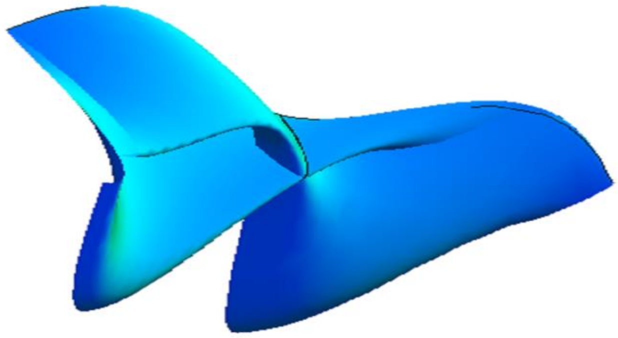

Figure 2. A harmonic evolute surface of $\varphi(s,t)$

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4th INTERNATIONAL CONFERENCE ON MATHEMATICS
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M-Projective Curvature Tensor on a Para-Kenmotsu Manifold with a type of Semi-symmetric Metric Connection

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Abstract

In this paper we study on para-Kenmotsu manifolds with a type of semi-symmetric metric connection under certain conditions of M-projective curvature tensor. Firstly, we obtain the form of M-projective curvature tensor with a type of semi-symmetric metric connection. Later, we examine para-Kenmotsu manifolds under some semi-symmetry conditions on this tensor.

Keywords: para-contact manifold, para-Kenmotsu manifolds, curvature tensors, symmetry conditions, semi-symmetric metric connection

1 INTRODUCTION

An almost para-contact structure is a triple (ϕ, ξ, η, g) such that $\phi^2 = I + \eta \otimes \xi$ for a contact form η, $(1,1)-$tensor field ϕ and chracteristic vector field ξ. Any almost para-contact structure admits a pseudo-Riemannian metric with signature $(n+1; n)$ [1]. There are many subclasess of para-contact manifolds similar to contact manifolds. A para-Kenmotsu manifold is an analogy of a Kenmotsu manifold [2] in para-contact geometry. These type of manifolds were defined by Sinha and Sai Prasad [3] as follow;

Definition 1. *Let (M, ϕ, ξ, η, g) be an almost para-contact metric manifold of dimension $(2m+1)$. M is said to be an almost para-Kenmotsu manifold if $1-$form η is closed and $d\Phi = 2\eta \wedge \Phi$. A normal almost para-Kenmotsu manifold M is called a para-Kenmotsu manifold .*

Para-Kenmotsu manifolds were studied by several authors [4, 5, 6, 7, 8, 9, 10]. In 1930, Hayden [11] introduced a semi-symmetric metric connection on a Riemannian manifold. K. Yano [12] developed this definition. Later, manifolds with special structures have been studied by using geometric properties of this type of connection [13, 14]. Also there many different forms of semi-symmetric metric connections. Semi-symmetric metric connections on para-Kenmotsu manifolds were studied in [3, 6].

In this work, our aim is to study a p-Kenmotsu manifold admitting a type of semi-symmetric metric connection. After obtained some basic results we focus on the M-projective curvature tensor. We compare with flatness conditions of this tensor for the Levi-Civita connection and the semi-symmetric metric connection. Finally, we examine some semi-symmetry conditions.

2 PRELIMINARIES

Definition 2. *Let M be a $(2m+1)-$dimensional smooth manifold. (ϕ, ξ, η) is called an almost para-contact structure on M such that for ϕ is $(1,1)$ a tensor field ξ is a vector field and η is a $1-$ form M we have*

$$\phi^2 X = X - \eta(X)\xi \ \ , \ \phi(\xi) = 0, \quad , \ \eta \circ \phi = 0 \ \ , \ \eta(\xi) = 1 \tag{1}$$

where X is an arbitrary vector fields on M.

The tensor field ϕ induces an almost paracomplex structure on the distribution $D = ker\eta$. There two distributions $D\pm$ corresponding to the eigenvalues ± 1 and they are in same dimension m. The rank of ϕ is $2m$. A manifold with an almost para-contact structure is called an almost para-contact manifold [1].

Let g be a pseudo-Riemann metric on M.

- g is called compatible metric, if $g(\phi X_1, \phi X_2) = -g(X_1, X_2) + \eta(X_1)\eta(X_2)$ for all $X_1, X_2 \in \Gamma(TM)$
- g is called associated metric, if $d\eta(X_1, X_2) = g(\phi X_1, X_2)$ for all $X_1, X_2 \in \Gamma(TM)$.

If g is an associated metric then (M, ϕ, ξ, η, g) is called almost para-contact metric manifold. Also by taking $X_2 = \xi$ we get $\eta(X_1) = g(X_1, \xi)$. The $2-$form $\Phi(X_1, X_2) = d\eta(X_1, X_2)$ is called second fundamental form. For a C^∞ function f on a para-contact metric manifold M, a complex structure J is defined as $M \times \mathbb{R}$ by $J(X_1, f\frac{d}{dt}) = (\phi X_1 + f\xi, \eta(X_1)\xi)$. If J is integrable i.e Nijenhuis tensor vanishes almost para-contact structure (ϕ, ξ, η) is called normal. Like contact manifolds this condition is restricted to $N_\phi + 2d\eta \otimes \xi = 0$.

Theorem 1. *Let (M, ϕ, ξ, η, g) be an almost para-contact metric manifold. M is a para-Kenmotsu manifold if and only if*

$$(\nabla_{X_1}\phi) X_2 = g(\phi X_1, X_2)\xi - \eta(X_2)\phi X_1 \tag{2}$$

for all $X_1, X_2 \in \Gamma(TM)$ [15].

For brevity we use "p-Kenmotsu" for para-Kenmotsu. By using (2) on a p-Kenmotsu manifold we have

$$\nabla_{X_1}\xi = \phi^2 X_1 \tag{3}$$

for all $X_1 \in \Gamma(TM)$.
Let X_1, X_2, X_3 be arbitrary vector fields on M. For the expression of Riemannian curvature

$$R(X_1, X_2)X_3 = [\nabla_{X_1}, \nabla_{X_2}] - \nabla_{[X_1, X_2]}X_3$$

we have following properties [19];

$$R(X_1, X_2)\xi = \eta(X_1)X_2 - \eta(X_2)X_1 \tag{4}$$

$$R(X_1, \xi)\xi = -X_1 + \eta(X_1)\xi \tag{5}$$

$$\eta(R(X_1, X_2)X_3) = g(X_1, X_3)\eta(X_2) - g(X_2, X_3)\eta(X_1) \tag{6}$$

Also for Ricci curvature of M we have

$$Ric(X_1, \xi) = -2n\eta(X_1) \tag{7}$$

3 A Type of Semi-symmetric metric connection on p-Kenmotsu manifold

A semi-symmetric metric connection is a type of connections in Riemannian geometry which has non-zero torsion. There are many different kinds of semi-symmetric metric connections. In this section we use one of them, then we obtain many results. Let M be a $(2m+1)-$dimensional p-Kenmotsu manifold and define $\nabla^\star : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ as follow

$$\nabla^{\star\star}_{X_1} X_2 = \nabla_{X_1} X_2 + \eta(X_2)X_1 - g(X_1, X_2)\xi \tag{8}$$

$\nabla^\star$ is a semi-symmetric connection i.e. the torsion of $\nabla^\star$ is not zero. Also $\nabla^{\star\star}$ is a metric connection that is $(\nabla^{\star\star}_{X_1} g)(X_2, X_3) = g(\nabla^{\star\star}_{X_1} X_2, X_3) + g(X_2, \nabla^{\star\star}_{X_1} X_2)$. A p-Kenmotsu manifold admitting $\nabla^\star\star$ were studied by Sinha and Prasad [3].

By using (3), (4), (7) we have

$$(\nabla^{\star\star}_{X_1}\phi)X_2 = -2(\nabla_{X_1}\phi)X, \tag{9}$$

$$\nabla^{\star\star}_X \xi = 2[X - \eta(X)\xi], \tag{10}$$

$$\nabla^{\star\star}_\xi \xi = 0, \tag{11}$$

$$(\nabla^{\star\star}_{X_1}\eta)X_2 = 2g(\phi X_1, \phi X_2). \tag{12}$$

Thus, from (11) we state:

Corollary 1. *The characteristic vector field of a p-Kenmotsu manifold M with semi-symmetric metric connection $\nabla^{\star\star}$ is a geodesic vector field on M.*

The Riemannian curvature of $\nabla^{\star\star}$ is obtained as

$$\begin{aligned} R^{\star\star}(X_1, X_2)X_3 \quad &= R(X_1, X_2)X_3 + [2\eta(X_1)\eta(X_2) + g(X_2, X_3)]X_1 \\ &-[2\eta(X_1)\eta(X_3) + g(X_1, X_3)]X_2 \\ &+2[\eta(X_1)g(X_2, X_3) - \eta(X_2)g(X_1, X_3)]\xi \end{aligned} \tag{13}$$

By using the expression of $R^{\star\star}$ we get followings;

$$R^{\star\star}(X_1, \xi)\xi = -2[\eta(X_1)X_2 - \eta(X_2)X_1],$$
$$R^{\star\star}(X_1, X_2)\xi = -2[\eta(X_3)X_1 - g(X_1, X_3)\xi],$$
$$\eta(R^{\star\star}(X_1, X_2)X_3) = 2[\eta(X_1)g(X_3, X_2) - \eta(X_2)g(X_3, X_1)].$$

Also we have symmetry properties of $R^{\star\star}$ as:

$$R^{\star\star}(X_1,X_2)X_3 + R^{\star\star}(X_2,X_3)X_1 + R^{\star\star}(X_3,X_1)X_2 = 0,$$
$$R^{\star\star}(X_1,X_2)X_3 + R^{\star\star}(X_2,X_1)X_3 = 0,$$
$$g(R^{\star\star}(X_1,X_2)X_3,X_4) - g(R^{\star\star}(X_3,X_4)X_1,X_2) = 0.$$

for all $X_1, X_2, X_3, X_4 \in \Gamma(TM)$. The Ricci curvature of a p-Kenmotsu manifold with $\nabla^{\star\star}$ is given by

$$Ric^{\star\star}(X_1,X_2) = Ric(X_1,X_2) + (2m+3)g(X_1,X_2) + (4m-3)\eta(X_1)\eta(X_2)$$

and the scalar curvature is obtained as

$$scal^{\star\star}(X_1,X_2) = scal + 6m.$$

Also we have

$$Ric^{\star\star}(\xi,\xi) = 4m. \tag{14}$$

A well known results is that ; a p-Kenmotsu manifold can not be flat [3], because of the fact that $Ric(\xi,\xi) = -2m$. Similarly, from (14) we state;

Theorem 2. *A p-Kenmotsu manifold M with a semi-symmetric metric connection $\nabla^{\star\star}$ can not be flat.*

4 M-projective curvature tensor on p-Kenmotsu manifolds with a type of semi-symmetric metric connection

Curvature tensors play a major role in the Riemannian geometry of manifolds with structure. There are many different kinds of curvature tensors. On of them is $\mathcal{M}-$projective curvature tensor which is defined in [20]. For a p-Kenmotsu manifold M the $\mathcal{M}-$projective curvature tensor is defined as

$$\begin{aligned} \mathcal{W}(X_1,X_2)X_3 &= R(X_1,X_2)X_3 - \frac{1}{4m}[Ric(X_2,X_3)X_1 \\ &\quad - Ric(X_1,X_3)X_2 + g(X_2,X_3)QX_1 - g(X_1,X_3)QX_2] \end{aligned} \tag{15}$$

for $X_1, X_2, X_3 \in \Gamma(TM)$.
This type of tensor on a p-Kenmotsu manifold was studied by Singh and Kishor in [21]. With using curvature form and Ricci curvature of a p-Kenmotsu manifold with $\nabla^{\star\star}$ we get the $\mathcal{M}-$projective curvature tensor of $\nabla^{\star\star}$ as follow;

$$\mathcal{W}^{\star\star}(X_1,X_2)X_3 = \mathcal{W}(X_1,X_2)X_3 + \frac{4m+3}{4m}\mathcal{K}(X_1,X_2)X_3$$

where

$$\mathcal{K}(X_1,X_2)X_3 = [g(\phi X_1,\phi X_2)X_1 - g(\phi X_1,\phi X_3)X_2]$$

for all $X_1, X_2, X_3 \in \Gamma(TM)$. Thus, we get

Corollary 2. *Let M be a $\mathcal{W}$-flat p-Kenmotsu manifold . Then, M is $\mathcal{W}^{\star\star}$-flat if and only if $\mathcal{K}$ vanishes identically.*

The operation of "$\cdot$" acts like as a derivation on curvature tensor and it is defined as follow ;

$$\begin{aligned}(R(X_1,X_2).\mathcal{T})(X_3,X_4)X_5 \quad &= R(X_1,X_2)\mathcal{T}(X_3,X_4)X_5 - \mathcal{T}(R(X_1,X_2)X_3,X_4)X_5 \\ &-\mathcal{T}(X_3,R(X_1,X_2)X_4)X_5 - \mathcal{T}(X_3,X_4)R(X_1,X_2)X_5.\end{aligned}$$

for all X_1,X_2,X_3,X_4,X_5 vector fields on M, where $\mathcal{T}$ is a $(1,3)-$tensor field. A p-Kenmotsu manifold is called locally symmetric or semi-symmetric if $R\cdot R=0$. Also we recall a Riemann manifold as $T-$semi-symmetric if $R\cdot T=0$.

Let $\mathcal{S}$ be a $(0,2)$-type tensor field on a p-Kenmotsu manifold M. Then, we have

$$(\mathcal{W}^{\star\star}(X_1,X_2)\cdot\mathcal{S})(X_3,X_4) = -\mathcal{S}(\mathcal{W}^{\star\star}(X_1,X_2)X_3,X_4) - \mathcal{S}(X_3,\mathcal{W}^{\star\star}(X_1,X_2)X_4). \tag{16}$$

With using the expression $\mathcal{W}^{\star\star}$ we obtain

$$(\mathcal{W}^{\star\star}(X_1,X_2)\cdot\mathcal{S})(X_3,X_4) = (\mathcal{W}(X_1,X_2)\cdot\mathcal{S})(X_3,X_4) + \mathcal{P}(X_1,X_2,X_3,X_4)$$

where

$$\begin{aligned}\mathcal{P}(X_1,X_2,X_3,X_4) \quad &= \quad \frac{4m+3}{4m}[g(\phi X_2,\phi x_3)S(X_1,X_3) - g(\phi X_1,\phi X_3)\mathcal{S}(X_2,X_4) \\ &+g(\phi X_2,\phi X_4)S(X_1,X_3) - g(\phi X_1,\phi X_3)\mathcal{S}(X_2,X_3)]\end{aligned} \tag{17}$$

We can state following theorem from the expression of $\mathcal{W}^{\star\star}\cdot\mathcal{S}$:

Theorem 3. *Let M be p-Kenmotsu manifold satisfying the condition $\mathcal{W}\cdot\mathcal{S}=0$. Then , $\mathcal{W}^{\star\star}\cdot\mathcal{S}=0$ if and only if $\mathcal{P}=0$.*

A p-Kenmotsu manifold is called as Ricci-semi symmetric if $R\cdot Ric=0$. Similar to locally symmetry we recall a p-Kenmotsu manifold by $\mathcal{T}$-Ricci semi-symmetric if $\mathcal{T}\cdot Ric=0$.

Theorem 4. *Let M be an $\mathcal{M}-$projectively Ricci-semi-symmetric p-Kenmotsu manifold. Then, M is an $\mathcal{M}-$projectively Ricci-semi-symmetric p-Kenmotsu manifold with a semi-symmetric metric connection $\nabla^{\star\star}$ if and only if $Ric(X_1,X_2)=\frac{scal}{2m}g(\phi X_1,\phi X_2)$.*

Proof. Let take $\mathcal{S}=Ric$ in (16) and suppose that $\mathcal{P}=0$. For an orthonormal basis $\{E_i,\phi E_i,\xi\}$, $1\leq i\leq 2m+1$ of $\Gamma(TM)$ and by getting sum over i in (17), we obtain

$$Ric(X_1,X_2) = \frac{scal}{2m}g(\phi X_1,\phi X_2).$$

This completes the proof. □

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Using Factor Analysis and Cluster Analysis as tools for studying the service quality. An evidence from the field of hospitality in Albania.

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Abstract

In an environment of strong competition and continuous development, the hotel sector must improve the quality of service in order to increase customer satisfaction to cope the competition and long-term position in the market. The main purpose of this study is to evaluate the dimensions of service quality and the relationship between the latter, and customer satisfaction in the field of hospitality. The data belong to the year 2018 and the questionnaire was developed by 200 tourists from 10 hotels in Tirana. Is used the service quality model (SERVQUAL) that includes the dimensions reliability, empathy, responsiveness, assurance, and tangibility. The Factor Analysis (FA) technique has been applied to determine the factors that are most important in defining customer satisfaction and has been integrated the Cluster Analysis method to realize a division of the data in groups based on the customer satisfaction level. According to the FA results it was found out that from all of the dimensions included in our study, tangibility has the strongest conection with customer satisfaction. The results of this study serve in improving service quality enhancement programs in hotel's sector.

Keywords: Factor Analysis, Cluster analysis, dimension, SERVQUAL.

1. Introduction

Service quality has a great importance in the success of the hotel industry because it promotes customer satisfaction and consequently hotels remain competitive, sustainable and profitable. Service quality is a complex multidimensional concept and difficult to be measured (1) so in this way some indicators are used that summarize its multidimensional nature. Acording Parasuraman (1) SERVQUAL consists of 5 dimensions that are: reliability, empathy, responsiveness, assurance, and tangibility.
Reliability: The ability to perform the service dependably, consistently and accurately. Responsiveness: The willingness to help customers and provide prompt service. Assurance: The knowledge and courtesy of employees and their ability to convey trust and confidence. Tangibles: The physical evidence of service including physical facilities, appearance of personnel, tools, and equipment used to provide the service. Empathy: Caring individualized attention to its customer. A large number of empirical research have studied through the help of the SERVQUAL model the relationship between quality of service and customer satisfaction in the hotel industry (Debasish, Dey (2), Priyo (3), Godolja (4), Rahman (11), Markovic (5)). In the study of Rahman et al (11) is showed that the general structure loading of factorial analysis affirm that all the five factors serve as dimensions of service quality in the hospitality industry in Melaka. On the other hand, it is shown from the analysis of linear multiple regression that the dimensions of empathy,

responsiveness, tangibles are important in explaining the service quality. Marcovic et al (5) explores customers' perceptions of service quality in the Croatian hotel industry and determines through FA technique that structure of service quality perception includes reliability, empathy and competence of staff, accessibility and "tangibles" because these variables are the best that explained customers' expectations of Croatian hotel service quality. İn the study of Masrurul (12) has highlighted the issue of the impact of "service quality on customer satisfaction" and notes that four dimensions influenced service quality, except the assurance that has no impact. The findings in the study of Fah et al (6) are consistent with previous studies because based on reliability and validity test, show that tangibility, reliability and assurance, responsiveness and empathy are dimensions of service quality. Multiple Linear Regression was performed in order to see the relationship between the dimensions tangibility, reliability, assurance, responsiveness, empathy, hotel technology, ecological design and customer satisfaction towards hotels in Langkawi, and it turns out that all dimensions have a significant effect on customer satisfaction.

2. Methods

A total of 200 surveys were distributed among 3-star hotels in Tirana, 10 in total in 2018. The survey was developed based on the SERVQUAL model Parasuraman (1) in the spring period. All variables were measured on a 1–7 likert scale, from "Strongly Disagree" to "Strongly Agree". Factor Analysis and Cluster Analysis was applied through SPSS program. Factor analysis shows that with whether the correlations between a set of observed variables can be explained in terms of a smaller number of unobservable constructs known common factors (B. G Tabachnick, L. Fidell) (7). So the FA technique reduces the number of variables by using a smaller number that we will refer as dimensions. Since it is based on the correlation between variables, the correlation matrix is calculated. There are several methods used to obtain factor solutions, but in our study has been applied Principal Components Analysis (PCA). Since the PCA method has been applied we will use the term principal component for the construct customer satisfaction. The variables: reliability, responsiveness, assurance, tangibility, empathy are called dimensions of the main factor or of the principal component customer satisfaction. As part of the FA procedure is the determination of the number of factors to be extracted. Only eigenvalues of 1 or greater will be considered (8). We need to test if the data matrix has sufficient correlations in order to apply FA. In order to test collinearity we need before two measures that are The Kaiser – Meyer – Olkin (KMO) statistic and Bartlett's test of sphericity. Kaiser – Meyer – Olkin (KMO) statistics are called model suitability measure and values that are less than 0.5 are called unacceptable. The Bartlett criterion controls the hypothesis that the correlation matrix is an identity matrix. In our study, we find also the factor scores that describe the relations between factors and data. Factors scores are used as inputs in cluster analysis and specifically the K-means algorithm has been developed. In this way we obtain the separation of data into 3 groups that represent the level of hotel's customer satisfaction.

3. Results

Table 1 indicates factor loadings concerning the components. Referring to Henry et al. (9) we should take into consider all those dimensions that have a load greater than 0.3. It is seen from table 1 that all loading complete the criterion therefore they are important in explaining the principal component. The principal component is interpreted as "the component of the customer satisfaction level". The principal component has a strong correlation with all variables so we will consider only that.

Table 1. Matrix Component

Dimension	Component
	1
Tangibles	0.859
Reliability	0.793
Responsiveness	0.833
Empathy	0.505
Assurance	0.787

Table 2 below, shows all the components derived from eigenvalue factor analysis, the percentage of variance attributed to each factor and cumulative variance. The eigenvalue shows us the total amount of variance, explained by each component in the PCA.
First factor has an eigenvalue of 2.934. The third column shows the percentage of variance explained by each component. We see that the first component explains 58.6% of the overall variance while the other components have eigenvalue less than 1 therefore we don't take them into consideration.

Table 2. Component's variance

Component	Initial Eigenvalues			Extraction Sums of Squared Loadings		
	Total	% of Variance	Cumulative %	Total	% of Variance	Cumulative %
1	2.934	58.671	58.671	2.934	58.671	58.671
2	0.82	16.408	75.078			
3	0.507	10.131	85.209			
4	0.409	8.179	93.388			
5	0.331	6.612	100			

In Table 3 is shown The Kaiser – Meyer – Olkin (KMO) statistic and Bartlett's test of sphericity. It is shown from table 3 that the value of the statistic KMO turned out to be equal to 0.838, then we say that we are within an acceptable interval; the data are suitable for the FA. The Bartlett criterion controls whether the correlation matrix is an identity matrix.
The level of significance or this criterion is 0.000 so it is less than 0.05 therefore in this case we reject the above hypothesis. So the main purpose in this case of PCA, is to extract the principal component "customer satisfaction" and to be able to use it to create the factor scores of dimensions.

While loading describes the relation between dimensions and the principal component, whereas factor scores represent the relation of the factors to the data.

Table 3. KMO and and Bartlett's Test

Kaiser-Meyer-Olkin Measure of Sampling Adequacy.		0.837674
Bartlett's Test of Sphericity	Approx. Chi-Square	327.2341
	df	10
	Sig.	2.68E-64

For the obtained factor scores we develop the K-means algorithm to realize the separation into groups. Table 4 shows the groups.

Table 4. Final Cluster Centers

	Cluster		
	1	2	3
REGR factor score 1 for analysis 1	-1.04824	0.64212	2.57718
Percentage	41%	56%	3%

The results of table 4 show that value greater than zero for a factor score, exhibits that the characteristic described by the factor (in our case the level of customer satisfaction) is above the average and if a factor score is below zero, then show that the characteristic is below average (Moi et al) (10).
The groups based on the above explanations are named as follows: The first group is the group where the level of customer satisfaction is low, the second group is the group where the level of customer satisfaction is average, the third group is the group where the level of customer satisfaction is high. This separation is made based on the values of factor scores. It is shown that the majority of tourists (56%) taken into consideration in our study, have an average level of satisfaction, regarding the quality of service.

4. Conclusions

Factor analysis shows that with whether the correlations between a set of observed variables can be explained in the context of a smaller number of latent constructs known common factors.
The FA method used in this study confirmed the 5-dimensional structure of the SERVQUAL model. The findings of this study show that from the application of group analysis with the method of K-means, is developed the separation into 3 groups which showed that they are distinct from each other. There is a high percentage of tourists in group 2 (average level) which corresponds to 56% of the total number of tourists surveyed. The findings of this study serve to improve programs in order to increase the quality of service in the hotel sector.

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A Note on the Commutativity of Riemann Differential Equation

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Abstract

In this study, commutativity conditions of Riemann differential equation are considered. Using commutativity conditions of second-order continuous time-varying linear systems [1], it is proved that the system described by a Riemann differential equation has commutative pairs, which depends on the parameters of the equation. An example is considered to find the commutative pair of Riemann differential equation.

1. Introduction

Time-varying linear systems are used for modeling, solving and analyzing many physical and engineering problems. There have been many publications on this subject in the literature (see [2-4] and the references therein). The commutativity of linear time-varying analog systems has gained attraction in the last few decades. This concept was studied by Marshal [5] for the first time for first-order systems in 1977. Then, commutativity of second-order linear time-varying systems was studied by Koksal [1] in 1982.

Although most of the famous second-order time-varying differential equations are subjected from the commutativity point of view in [6, 7], Riemann differential equation [8] is not among them. So, the purpose of this study is to find commutativity conditions of Riemann differential equation

$$y''(x) + \left(\frac{1-\alpha-\alpha'}{x-a} + \frac{1-\beta-\beta'}{x-b} + \frac{1-\gamma-\gamma'}{x-c}\right) y'(x)$$
$$+ \left[\frac{\alpha\alpha'(a-b)(a-c)}{x-a} + \frac{\beta\beta'(b-c)(b-a)}{x-b} + \frac{\gamma\gamma'(c-a)(c-b)}{x-c}\right] \frac{y(x)}{(x-a)(x-b)(x-c)} = 0, \quad (1)$$

such that $\alpha + \alpha' + \beta + \beta' + \gamma + \gamma' = 1$. Here, $\alpha, \alpha', \beta, \beta', \gamma, \gamma', a, b, c$ are constants.

2. Commutativity Conditions of Second-order Systems

We assume that A be linear time-varying second-order differential system described by the following second-order differential equations:

$$a_2(t)\ddot{y}_A(t) + a_1(t)\dot{y}_A(t) + a_0(t)y_A(t) = x_A(t), y_A(t_0) = y_{A0}; t \geq t_0. \quad (2)$$

Here, $x_A(t)$ and $y_A(t)$ are input and output functions, respectively. The initial time is t_0 and the initial value is y_{A0}. For the unique solution of Eq. (2) for $t \geq t_0$, it is sufficient that the excitation and the time-varying coefficients $a_2(t),\ a_1(t),\ a_0(t)$ be piece-wise continuous functions of time with $a_2(t) \neq 0$ for all $t \geq t_0$.

The system modeled by Eq. (2) has commutative pairs if and only if the coefficients of the system satisfy the following equation:

$$-a_2^{0.5}\frac{d}{dt}\left[a_0-\frac{1}{16a_2}(4a_1^2+3\dot{a}_2^2-8a_1\dot{a}_2+8\dot{a}_1a_2-4a_2\ddot{a}_2)\right]k=0, \qquad (3)$$

where k is a constant. In [1], Koksal proved that commutative pairs of (2) are obtained by

$$\begin{bmatrix} b_2 \\ b_1 \\ b_0 \end{bmatrix} = \begin{bmatrix} a_2 & 0 & 0 \\ a_1 & a_2^{0.5} & 0 \\ a_0 & a_2^{-0.5}(2a_1-\dot{a}_2)/4 & 1 \end{bmatrix} \begin{bmatrix} k_2 \\ k_1 \\ k_0 \end{bmatrix}, \qquad (4)$$

where k_2, $k_1 = \text{same k in Eq. (3)}$, k_0 are some constants. Then, any commutative pair B of A is described by

$$b_2(t)\ddot{y}_B(t)+b_1(t)\dot{y}_B(t)+b_0(t)y_B(t)=x_B(t). \qquad (5)$$

3. Commutativity of Riemann Differential Equation

For Eq. (1), $a_0=\frac{1}{(x-a)(x-b)(x-c)}\left[\frac{\alpha\alpha'(a-b)(a-c)}{x-a}+\frac{\beta\beta'(b-c)(b-a)}{x-b}+\frac{\gamma\gamma'(c-a)(c-b)}{x-c}\right]$ and $a_1=\frac{1-\alpha-\alpha'}{x-a}+\frac{1-\beta-\beta'}{x-b}+\frac{1-\gamma-\gamma'}{x-c}$. Because of the fact that $a_2=1, \dot{a}_2=\ddot{a}_2=0$. So, Eq. (3) becomes

$$-a_2^{0.5}\frac{d}{dt}\left[a_0-\frac{a_1^2+2\dot{a}_1}{4}\right]k=0. \qquad (6)$$

Excluding the special case $k=k_1\neq 0$ which corresponds to constant feedforward and constant feedback connections of the system A [9], inside of the parenthesis of Eq. (6) must be equal to a constant. Putting the values of $a_0, a_1, \dot{a}_1=-\frac{1-\alpha-\alpha'}{(x-a)^2}-\frac{1-\beta-\beta'}{(x-b)^2}-\frac{1-\gamma-\gamma'}{(x-c)^2}$ in, we obtain following expression:

$$-\frac{A^2}{4(x-a)^2}-\frac{B^2}{4(x-b)^2}-\frac{C^2}{4(x-c)^2}-\frac{2AB}{4(x-a)(x-b)}-\frac{2AC}{4(x-a)(x-c)}-\frac{2BC}{4(x-b)(x-c)}$$
$$+\frac{2A}{4(x-a)^2}+\frac{2B}{4(x-b)^2}+\frac{2C}{4(x-c)^2}+\frac{D}{(x-a)^2(x-b)(x-c)}+\frac{E}{(x-a)(x-b)^2(x-c)}$$
$$+\frac{F}{(x-a)(x-b)(x-c)^2}=constant$$

where $A=1-\alpha-\alpha', B=1-\beta-\beta', C=1-\gamma-\gamma', D=\alpha\alpha'(a-b)(a-c), E=\beta\beta'(b-c)(b-a)$ and $F=\gamma\gamma'(c-a)(c-b)$.

If we equate the denominators, multiply the whole equation by $4(x-a)^2(x-b)^2(x-c)^2$, equate the coefficients of like powers of x on both sides of the equation we obtain $constant=0$ for the 6th and 5th powers. Further powers lead to:

For x^4,

$$A^2+B^2+C^2+2(AB+AC+BC)=2(A+B+C). \qquad (7)$$

For x^3,

$$A^2(b+c)+B^2(a+c)+C^2(a+b)=2A(b+c)+2B(a+c)+2C(b+a)$$
$$-AB(a+b+2c)-AC(a+2b+c)-BC(2a+b+c).$$

For x^2,

$$4(D+E+F)+(-A^2+2A)[c^2+2(b+c)+b^2]+(-B^2+2B)[a^2+2(a+c)+c^2]$$
$$(-C^2+2C)[a^2+2(a+b)+b^2]=2AB[c^2+2c(a+b)+ab]+2AC[b^2+2b(a+c)+ac]$$
$$+2BC[a^2+2a(b+c)+bc].$$

For x,

$$-4D(b+c)-4E(a+c)-4F(a+b)-(-A^2+2A)(2bc^2+2b^2c)-(-B^2+2B)(2ac^2+2a^2c)$$
$$-(-C^2+2C)(2ab^2+2a^2b)=2AB[-(a+b)c^2-2abc]+2AC[-(a+c)b^2-2abc]$$
$$+2BC[-(b+c)a^2-2abc].$$

For x^0,

$$4(Dbc+Eac+Fab)+2(Ab^2c^2+Ba^2c^2+Ca^2b^2)$$
$$=A^2b^2c^2+B^2a^2c^2+C^2a^2b^2+2(ABabc^2+ACab^2c+BCa^2bc).$$

So, we have five different algebraic equations. For $a=b=c$, all equations are equivalent to each other. By solving Eq. (7), we obtain $A+B=-C$ and $A+B=2-C$ which mean $\alpha+\alpha'+\beta+\beta'+\gamma+\gamma'=3$ and $\alpha+\alpha'+\beta+\beta'+\gamma+\gamma'=1$ respectively. From the definition of Riemann differential equation, $\alpha+\alpha'+\beta+\beta'+\gamma+\gamma'$ must be equal to 1. So, $A+B$ can not be equal to $-C$. Hence choosing$A+B=2-C$, Riemann differential equation has an unconditional commutative pair when $a=b=c$.

4. Example

Let us consider the following Riemann differential equation:

$$y''(x)+\left(\frac{1-\alpha-\alpha'}{x-a}+\frac{1-\beta-\beta'}{x-a}+\frac{1-\gamma-\gamma'}{x-a}\right)y'(x)=0$$

such that $\alpha+\alpha'+\beta+\beta'+\gamma+\gamma'=1$. In this differential equations, $a_2=1, a_1=\frac{1-\alpha-\alpha'}{x-a}+\frac{1-\beta-\beta'}{x-a}+\frac{1-\gamma-\gamma'}{x-a}=\frac{3-[\alpha+\alpha'+\beta+\beta'+\gamma+\gamma']}{x-a}=\frac{2}{x-a}, a_0=0$. Using Eq. (4), we find

$$b_2=a_2k_2=k_2,$$
$$b_1=a_1k_2+k_1=\frac{2}{x-a}k_2+k_1,$$
$$b_0=\frac{a_1k_1}{2}+k_0=\frac{2}{x-a}k_1+k_0.$$

So, commutative pair of system A is described by the following differential equation:

$$k_2\ddot{y}_B(t)+\left[\frac{2}{x-a}k_2+k_1\right]\dot{y}_B(t)+\left[\frac{2}{x-a}k_1+k_0\right]y_B(t)=x_B(t).$$

5. Conclusions

In this contribution, the existence of the commutative pairs of Riemann differential equation other than the constant feedback conjugates is studied. The nonlinear algebraic simultaneous equations that must be satisfied by the parameters of Riemann differential equation for the existence are derived. It is shown that Their always exist commutative pairs for the special case $a=b=c$. The study for the existence of the commutative pairs needs a deeper study for the difficulty of dealing derived nonlinear equations that must be satisfied by the parameters and this is left as a future work.

Acknowledgments: This study was supported by the Scientific and Technological Research Council of Turkey (TUBITAK) under the project no. 115E952.

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An Evaluation Of The Albanian Electricity Market Through Optimization Models

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Abstract

The electrical energy system in Albania is toward the development and approval of the Albanian Market Model (AMM). As a Mediterranean country this is the moment where Albania should take advantage of its geographical position as an asset of the hydropower system and take lead position in the development of the regional market. Hydropower system plays an important role in the domestic produced energy and the country's production capacities are still unused. In this work we will discuss on the benefits of a larger common market which will improve the investment climate and will be more attractive to potential investors. Optimization and time series models are presented to give an overview of the potential production in hydropower's of the largest cascade in Albania (drin cascade). This analysis will help to evaluate the importance and capacity of hydropower as an excellent natural resource with the ability to adapt to demand with appropriate quantitative optimization and enhance competition.

Keywords: electrical energy, market, hydropower, optimization, forecast .

1. Introduction

The electric power system is one of the most important which affects directly the economic development of a country. In Albania, the electricity market works on fixed prices due to the contract between the government and the corporate. Being a country that is new to free market, it is a necessary that the electricity price should be assigned by the economic trade, not by fixed price. Our goal is to model a stable structure of free market that give us the opportunity to integrate with the region countries. For the model to be valuable it is needed a detailed analysis of the most important factors, which are rainfalls, air temperature, water inflow. These items affect directly in electric power production. The relation between unit operators could not be neglected, because they are in series geographical position.

The electrical power system is divided into three main sectors: manufacturing sector, transmission and distribution. The relation between these three sec-tors is presented in the report of the Electrical Power corporation (2016). In Albania the manufacturing sector (KESH) produces energy based on the distribution operator demand (OSHE).

The modelling of the electrical situation in the country is given in various studies using optimization techniques and approximation (2015. a) (Ferrja) Simoni and Dhamo (Gjika) has done the Forecasting of the maximum power in hydropower plant using PSO technique and time series . (2015.b) they have combined evolutionary algorithm PSO and Holt-Winters method to optimize the energy protected in hydro Power Plants and is shown the Efficiency of Time Series Models on Predicting Water Inflow.
Gjika and Ferrja (2019) have studied an Hybrid Models in Forecasting Precipitations and Water Inflow Albania Case Study. They have shown that the most accurate models were Time series models combined with PSO technique. Regarding International Hydropower Association energy demand is expected to increase by 60 per cent in 2020, and there is a clear need for Albania to strengthen its energy security and propose strategies on the electric market.
Many models are given for the energetic situation in different countries. A good review is presented by Weron (2014) related to electricity price forecasting Andrade et al. widely used statistical techniques such as regression model, Markov models, SARIMA models, neural networks and hybrid models. Andrade at al (2017) have presented probabilistic methods for price forecasting for day-ahead. Monteiro et al. (2016) conducted a short-term price forecasting model using neural network.
Janke and Steinke (2019) conducted a study for the German intraday market to forecast the price distribution in the time from three hours to 30 minutes. (2019) Hubicka et al. have done a study based on averaging day-ahead electricity price forecasts.
Albania electric power imports are high because the power generation is based only on hydropower plants. Our country is dependent on import due to the hydrology, this has nothing to do the establishment of the market. We seek to exchange energetic production with region countries to ensure the economical development of the country. A competitive market will help to present the real value for Albanian power sector and reduce the import expenses. Albania can take advantage of its location and unique hydro power assets and should use opportunity to take lead position in development of the regional power market.
It is necessary to fulfill the Public Service Ob-ligation (PSeO) which allows the company to trade excessive electricity in the unregulated market. The geo-graphical position of our country favors us to develop the Albanian electricity exchange in the Balkan region. which can be considered as the last step towards market liberalization. Gradual market liberalization will in-crease the amount of energy that can be sold in the unregulated market. Detailed expertise is needed to operate the energy exchange.

2. Case study

To maximize the amount of energy produced in hydro plant, numerous efforts have been made to improve the supply systems.(2012) Fontana at.al has Losses reduction and Energy Production in Water-Cascade. (2012) Möderl at. al, (2014) Van Vuurenat.al have been implemented different generating schemes in order to analyze the hydro power generation potential.
The Mediterranean geographical position and climatic conditions of Albania makes the power sector heavily dependent on electrical energy produced mainly by hydropower plants (HPP). The electrical power system is divided into three main sectors, manufacturing sector, transmission and distribution.

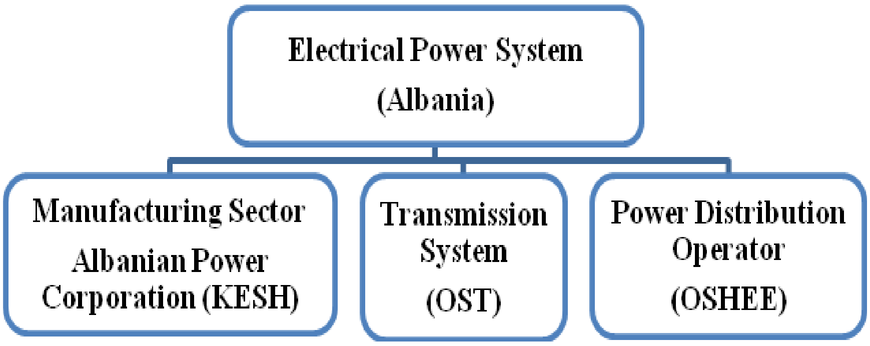

Figure1. Scheme of main sector of electrical power system.

The power distribution operator (OSHEE) makes the demand for energy to manufacturing sector (KESH) which is obligated to full fill the request. KESH is the main public producer of electrical energy in the country. It has into administration the main HPP positioned in Drin Cascade (Fierza HPP, Koman HPP, Vau-Dejes HPP) with an installed capacity of 1,350 MW. The cascade built on the Drin River is the largest in Balkan region both for its installed capacity and the size of hydro-tech works. Having in operation 79% of production capacity in the country, KESH supplies about 70-75% of the demand for electricity.

The produced capacity of thre main hydropower plant of Drin Cascade is given by the table below.

Table 1. Production capacity in drin Cascade divided in thre main HPP.

PRODUCTION **CAPACITIES**

Generation Plants		No. of Units	Installed Capacity	Mean Annual Production
DRIN RIVER CASCADE	Fierza HPP	4	500 MW	1,290 GWh
	Komani HPP	4	600 MW	1,690 GWh
	Vau i Dejes HPP	5	250 MW	870 GWh
	Vlora TPP	1	98 MW	0 GWh
TOTAL		14	1,448 MW	3,850 GWh

Fierza is the upper HPP of the Drin river cascade. For the installed power, the position and volume of the reservoir, Fierza plays a key role in the utilization, regulation and security of the entire cascade.

KESH is not only one of the producers of electricity from important hydropower sources in the region, but is also considered a regionally influential factor in the safety of hydro cycles. Position of HPP in Drin Cascade is shown in figure below.

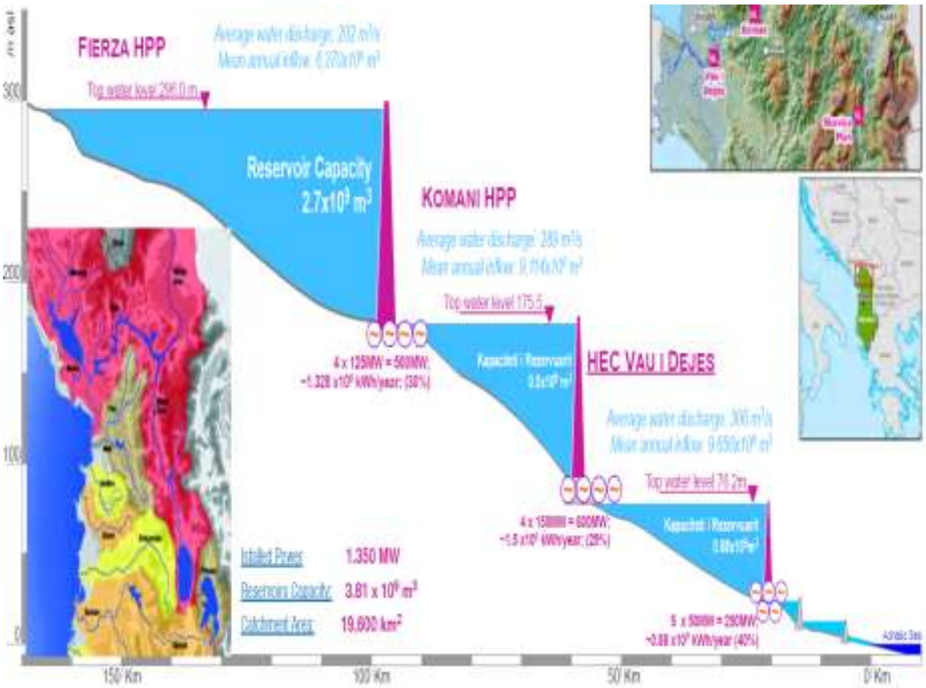

Figure 2. Geographic position of three bazed HPP in Drin Cascade.

Figure 3. A picture of Fierza HPP.

3. Αναλψσινγ ανδ πρεδιχτινγ τηε προδυχτιον ιν υπχομινγ μοντησ

The time series of power generation in the Drin River cascade display monthly seasonality in their history. It is observed that the output in this cascade reaches the minimum values in months of July and August and the maximum values in wet months where the precipitations and waterinflow are higher.

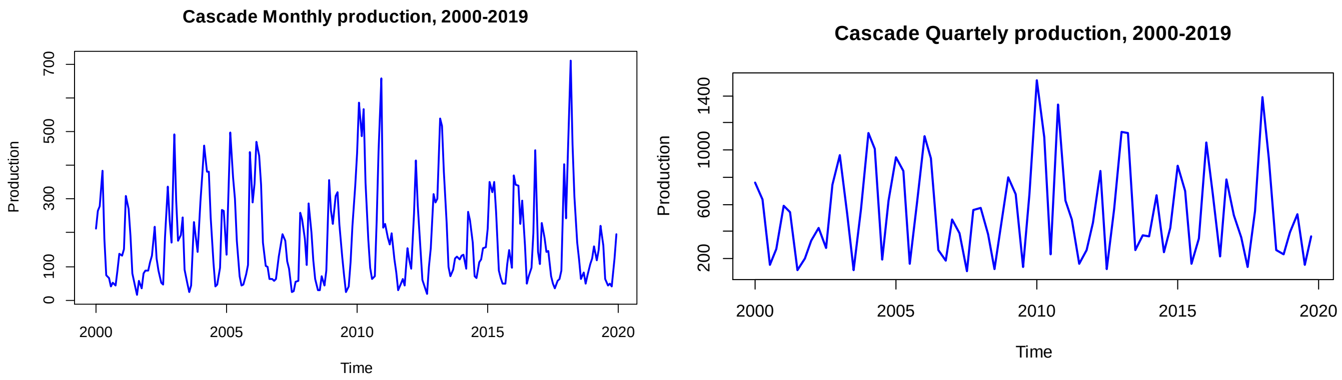

Figure 4. Monthly and quarterly time series of the production in Drin cascade (source: authors)

The seasonality is also shown in monthly production (Fig.a) in quarterly production (Fig.b) and a stability in production is observed in the last years (Fig.c).

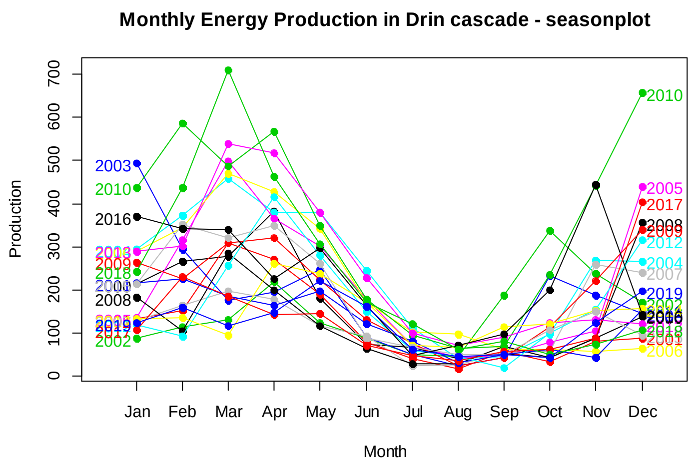

Figure 5. Seasonal Plot of monthly time series production in Drin cascade (Source:author)

During the period from March-April the production is high and also during the period December-January. The level of energy produced in January-February-March-April and also November –December have a considerable range compared to other months where the production remains in low levels due to low precipitation and water inflow.

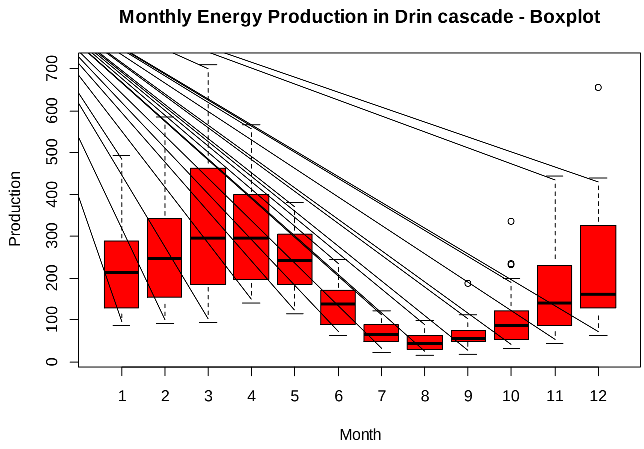

Figure 6. Box-Plot of monthly energy production in Drin cascade (soruce:author)

Several modeling strategies among: hierarchical forecasting, neural network, multistage forecasting, econometric forecasting models may be used to obtain the predictions of electric energy demand in the country. These models can help the Albanian Power Corporation (KESH) to build various scenarious on optimizing the country demand and production capacities on HPP cascade.
Figure 7 show the forecast for energy production in Drin cascade using a Error-Trend-Seasonality model.

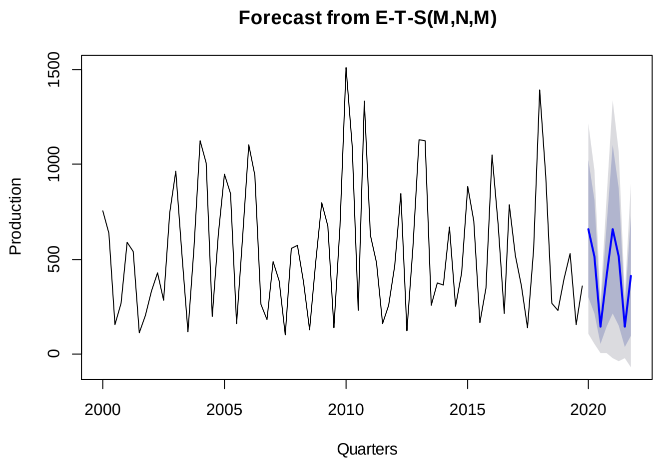

Figure 7. ETS forecast and confidence intervals for monthy energy production during 2020 in Drin cascade (source:authors)

We have used the forecasting packages in R (forecast library). Using an ARIMA, ETS and Neural network models we have simulated the forecast for each month in 2020. The forecast obtained from each model show that the situation of energy produced in the cascade is expected to remain within the average levels. We want to emphases again that the production in based on the demand from the distribution operator OSHEE, and not on the capacity production of the cascade. Which means that the production power of the HPP is higher than the demand.

4. Optimization formulation

In this section we will present profit modeling as an optimization problem. After doing a detailed analysis of the main factors that affect the cascade energy production we can maximize the profit. (2020) Janczura and Michalak propose an optimization scheme for a selling strategy of an electricity producer who in advance decides on the share of electricity sold on the day-ahead market. They modeled maximization of the overall profit, minimization of the sellers risk, or maximization of the median of portfolio values. The energy corporationin ouer country is obligate to fulfill the required demand from (PSeO) then we can trade the amount of excess energy from the maximum output. We have two views of the situation. the generation capacity of the generating system according to the operating rules should be set at the maximum capacity, at the same time the economic portfolio should be optimized. In Drin Cascade where located the most important HPP, Fierza, Koman, Vau i Dejes. These three HPP taken in consideration, cannot work independently but as a compound system. We should make the right coordination of work in order to generate the maximum electric power in each of the HPP units. For this purpose, we set some conditions which are necessary to finish the process.

Our goal is to generate maximum power from the whole system so we can optimize the distance between optimal capacity and PSeO. The objective function is optimizing the economic portfolio which impacts directly on maximizing profit.

$$\max F = f(p_t * E_{free})$$

Where F is economic profit, p_t price stock exchange and E_{free} (free portfolio optimization capacity) this variable represents the energy purchased and sold at a different price. At the end of the operations in market the amount of energy interchanged is required to be zero.

$$\sum_t E^t_{free} = 0$$

$$\text{where} \qquad E_{free} = E_{\max} - E_{PSeO}$$

E_{max} is the maximal capacity and E_{PSeO} is the amount of electricity that we are obliged to afford at a regulated price.

$$E_{free} + E_{PSeO} = \sum_{i=1}^{I} \sum_{t=1}^{T} g\eta_i \rho Q_{i,t}' H_{i.t}$$

$$\text{where} \quad P_{i,t} = g\eta_{i,t} \rho Q_{i,t}' H_{i.t}$$

T is the total number of the computation time interval index; I is the total number of reservoirs in system; i is the index for the number of reservoirs; t is the index for the current period; $P_{i,t}$ is the output of power; η_i is the hydropower generation efficiency of i-th reservoir; $Q_{i.t}'$ is the release through the reservoir turbines of the ith reservoir in the t-th period; $H_{i,t}$, is the difference between reservoir water level and tail-racewater level of ith reservoir in t th period.

The head is defined as the difference between the water levels in the upstream and downstream reservoirs of the hydro power plant, depending on the water storages in the respective reservoirs

$$H_{i.t} = X_{i.t} - X_{i+1,t}, i = 1, 2, 3$$
$$H_{4,t} = \omega, \text{ where } \omega = 23$$

The optimal value of the given objective function is computed subjected to constraints of two kinds ; equality constraints and inequality constraints or simple variable bounds as given below.

Water balance equation. Generation associated discharge varies with time and there are environmental and weather factors which also contribute directly to change in water level.The water balance equation for each reservoir is given:

$$M_{i,t} = M_{i,t-1} + I_{i,t} + (\sum_{m \in M_j} q_{t.m} + s_{t,m}) - q_{i,t} - s_{ki,t}$$

where $M_{i,t}$ is the water storage of reservoir j at end of period k ; $I_{i,t}$ is the natural inflow to reservoir j during the period k; $q_{k,j}$ is the water discharge of plants j ; $s_{k,j}$ is the water spillage by reservoir j; M_j is the set of reservoir upstream to reservoir j; $s_{k,m}$ is the water quantity discharged from the upper reservoir gates; $q_{k.m}$ is the water quantity discharged from the upper reservoir gates

Inequality constraints. Reservoir storage, turbine discharges rates, spillages and power generation limit should be in minimum and maximum bound due to the physical limitation of the dam and turbine.

Reservoir storage bounds: Generally in the hydro power plants the volume of water available is between the minimum draw down level and full reservoir level.

$$M_{i.\min} \le M_{i,t} \le M_{i.\max} \ (hm^3) \quad \text{or} \quad X_{i.\min} \le X_{i,t} \le X_{i,\max} \ (m)$$

$M_{i,t}$ water content of reservoir i at the end of position t; $M_{i,max}$ installed storage capacity of a hydro power unit; $M_{i,min}$ min storage contento of a hydro power unit; $X_{i,t}$ water level of reservoir i at the end of position t; $X_{i,max}$ max water level of reservoir i during position t; $X_{i,min}$ min water level of reservoir i during position t.

Water Discharge Bounds: This is a minimum and maximum bound on the release of the hydro power plant through turbine.

$$Q_{i,\min} \le Q_{i,t} \le Q_{i,\max} \qquad \left(m/s^2\right)$$

$Q_{i,t}$ water discharge of reservoir i during position t; $Q_{i,max}$ max discharge of a hydro power plant; $Q_{i,min}$ min discharge of a hydro power plant.

Power Generation Bounds: Power generated through hydro power plants should be minimum and maximum bounds.

$$P_{i,\min} \le P_{i.t} \le P_{i,\max} \qquad (MW)$$

$P_{i,t}$ generated power of hydro unit i during position t (MW); $P_{i,max}$ max generated power of a hydro plant (MW); $P_{i,min}$ min generated power of a hydro plant (MW).

Spillage: Spillage from the reservoir is allowed only when to be released from reservoir exceeds the maximum discharge limits. Water spilled from reservoir i during time t can be calculated as follows

$$S_{i,t} \ge 0 \qquad S_{i,t} = Q'_{i,t} - Q_{i,t} \quad \text{if} \;\; Q'_{i,t} > Q_{i,t}$$

Initial & end reservoir storage volume: This constraint implies the total quantity of utilized water for short term scheduling should be within limit.

$$X_j^0 = X_j^{begin} \text{ and } X_j^T = X_j^{end}$$

The level of water in the reservoir after we operate must be the same as in the beginning. If we want to take an optimal choice than we should preserve the amount of water available in the reservoir.

5. Conclusions

The power situation in the country is in the development of an open market in order to maximize production capacity. Albania is a country with high potential for generating electricity power through HPP cascades. Being a country with a geographical position, Mediterranean climate and hydropower system across rivers all over the country it has all the potential to generate electricity not only for domestic purposes but also to sell internationally to neighbour countries or beyond.

In this paper we have presented the the current situation and production capability over the years in the largest cascade in Albania that provides the bulk of inner product. It was shown through time series forecasts that the production potential in Drin cascade would not decrease and could be maximized according to the needs of the internal and external market. Optimization models were mentioned for the purpose of constructing in next steps a concrete problem for estimating the electricity price in the country.

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SOME PROPERTIES OF THE PRE-ORDERS OBTAINED BY SOME TOPOLOGICAL CLOSURE OPERATORS

FUNDA KARAÇAL, ÜMIT ERTUĞRUL, AND M. NESIBE KESICIOĞLU

Abstract. In this paper, the pre-order is defined by means of the topological operator $c^*_{U,A}$. Some properties of the pre-orders are investigated under some restrictions on the set A. The relationship between the selection of the set A and the obtained pre-orders is revealed. Moreover, an equivalence relation is defined and some properties of the relation are researched. Also, we define a uninorm on the quetient set of L by the relation $\sim$.

1. Introduction

Closure operators are important maps for many areas of mathematics such as lattices, topological spaces, fuzzy sets, graphs, categories, etc. Also, they are not only theoretically important operators but also have many applications ([3, 14]).

Uninorms are aggregation functions satisfying monotonicity, associativity, commutativity properties and with the neutral element e. They were first defined on unit real interval ([15]). Later, it was proven that there always exists a uninorm on any bounded lattice with a given neutral element ([10]). Following this study, uninorms on bounded lattices have been a hot topic for researchers and uninorms have been studied from many different aspects such as construction methods, ordering obtained by uninorms, etc ([5, 12, 13]).

When we look at the literature in terms of closure operator and uninorm, it is seen that closure operators were used in constructing uninorms ([13]) and obtaining a partial order from t-closure operator obtaining through triangular norms which is a special uninorm ([8]).

Inspiring by the idea of partially order in [8], we focus on the pre-orders obtained from closure operators in this study. The remainder of this paper is organized as follows. After recalling some necessary notions on lattices and uninorms on a bounded lattice in Section 2, we give the definitions of the pre-orders in Section 3. Also, some properties of the pre-orders are investigated under some restrictions on the set A or the uninorm U. The relationship between the selection of the set A and the obtained pre-orders is proved. Moreover, we define the equivalence relation via $c^*_{U,A}$, and we investigate on some properties of the equivalence relation. Also, we define a uninorm on the quetient set of L by the relation $\sim$.

2. Notations, definitions and a review of previous results

In this section, we recall some basic notions and results.

2020 *Mathematics Subject Classification.* 03E72, 03B52, 08A72, 54A10, 06B30, 06F30.
Key words and phrases. Bounded lattice, preorder, closure operator.

A bounded lattice $(L, \leqslant)$ has a top and a bottom element, denoted by 1 and 0, respectively, i.e., there exist two elements $1, 0 \in L$ such that $0 \leqslant x \leqslant 1$, for all $x \in L$.

Definition 2.1. [10] Let $(L, \leq, 0, 1)$ be a bounded lattice. An operation $U : L^2 \to L$ is called a uninorm on L, if it is commutative, associative, increasing with respect to the both variables and has a neutral element $e \in L$.

Recall that a uninorm U possessing a neutral element $e = 1$ is called a triangular norm. Similarly, a uninorm U with a neutral element $e = 0$ is called a triangular conorm.

Definition 2.2. [7, 12] A uninorm U is called conjunctive (disjunctive) if $U(0, 1) = 0$ $(U(0, 1) = 1)$ respectively.

Definition 2.3. [11] A map $c : \wp(X) \to \wp(X)$ is called a topological closure operator on non-empty set X if it satisfies following conditions

C1.: $c(\emptyset) = \emptyset$,
C2.: $A \subseteq c(A)$ for all $A \subseteq X$,
C3.: $c(c(A)) = c(A)$ for all $A \subseteq X$,
C4.: $c(A \cup B) = c(A) \cup c(B)$ for all $A, B \subseteq X$.

The topology defined below is the topology associated with a closure operator.

Theorem 2.4. [11] *Let c be a topological closure operator on X, Let $\mathfrak{F}$ be the family of all subsets A of X for which $c(A) = A$, and let $\mathfrak{T}$ be the family of complements of members of $\mathfrak{F}$. That is, $\mathfrak{T} = \{L \setminus A : A \in \mathfrak{F}\}$. Then $\mathfrak{T}$ is a topology for X, and $c(A)$ is the τ_c-closure of A for each subset A of X.*

Proposition 2.5. [9] $(L, \leq, 0, 1)$ be a bounded lattice, $A \subseteq L$ such that $e \in A$ and $U(A, A) \subseteq A$ where U is a uninorm on L with the neutral element e. The operation $c^*_{U,A} : \wp(L) \to \wp(L)$ defined by

$c^*_{U,A}(X) = \{y \in L :$ there exists $a \in A$ and $x \in X$ such that $U(a, x) \leq y\}$ is a topological closure operator.

The topology defined by $\mathfrak{T}_{c^*_{U,A}} = \{L \setminus X \mid \quad X \in \mathfrak{F}_{c^*_{U,A}}\}$ is the topology with the closure operator $c^*_{U,A}$, where $\mathfrak{F}_{c^*_{U,A}} = \{X \subseteq L \mid \quad c^*_{U,A}(X) = X\}$.

Proposition 2.6. [9] $(L, \leq, 0, 1)$ be a bounded lattice, $A \subseteq L$ such that $e \in A$ and $U(A, A) \subseteq A$ where U is a uninorm on L with the neutral element e. The operation $c^{**}_{U,A} : \wp(L) \to \wp(L)$ defined by $c^{**}_{U,A}(X) = \{y \in L :$ there exists $a \in A$ and $x \in X$ such that $U(a, x) \geq y\}$ is a topological closure operator.

The topology defined by $\mathfrak{T}_{c^{**}_{U,A}} = \{X' \mid \quad X \in \mathfrak{F}_{c^{**}_{U,A}}\}$ is the topology with the closure operator $c^{**}_{U,A}$, where $\mathfrak{F}_{c^{**}_{U,A}} = \{X \subseteq L \mid \quad c^{**}_{U,A}(X) = X\}$.

3. The Pre-orders Obtained by Some Topological Closure Operators

In this section, the pre-orders obtained from closure operators are defined. We investigate some properties of the pre-orders. Moreover, the equivalence relation via $c^*_{U,A}$ defined and some properties of the equivalence relation researched. Also, a uninorm is defined on the quetient set of L by the relation $\sim$.

Definition 3.1. $(L, \leq, 0, 1)$ be a bounded lattice, $A \subseteq L$ such that $e \in A$ and $U(A, A) \subseteq A$ where U is a uninorm on L with the neutral element e and consider the topological closure operator $c^*_{U,A}$. The pre-order $x \preceq_{c^*_{U,A}} y \Leftrightarrow x \in c^*_{U,A}(\{y\})$ is called a U-preorder for U.

Lemma 3.2. *$(L, \leq, 0, 1)$ be a bounded lattice, $A \subseteq L$ such that $e \in A$ and $U(A, A) \subseteq A$ where U is a uninorm on L with the neutral element e and consider the topological closure operator $c^*_{U,A}$. Then,*
(i) $1 \preceq_{c^*_{U,A}} x$ for all $x \in L$.
(ii) If U is a conjunctive uninorm and $0 \in A$, $x \preceq_{c^*_{U,A}} y$ and $y \preceq_{c^*_{U,A}} x$ for all $x, y \in L$.
(iii) $y \preceq_{c^*_{U,A}} x$ if $x \leq y$ for $x, y \in L$.
(iv) If $0 \in A$, 0 is greater than all elements of L w.r.t the pre-order $\preceq_{c^*_{U,A}}$.
(v) If $0 \in A$ and U is a conjunctive uninorm, 0 is smaller than all elements of L w.r.t. the pre-order $\preceq_{c^*_{U,A}}$.

Proof. (i) Since $U(x, e) = x \leq 1$ for all $x \in L$, $1 \in c^*_{U,A}(\{x\})$. Therefore, $1 \preceq_{c^*_{U,A}} x$ for all $x \in L$.
(ii) Let U is a conjunctive uninorm and $0 \in A$. Therefore, $U(x, 0) = 0 \leq y$ and $U(y, 0) = 0 \leq x$. Then, $x \preceq_{c^*_{U,A}} y$ and $y \preceq_{c^*_{U,A}} x$ for all $x, y \in L$.
(iii) Let $x \leq y$ for $x, y \in L$ and $e \in A$. Since $U(x, e) = x \leq y$, i.e., $y \in c^*_{U,A}(\{x\})$, $y \preceq_{c^*_{U,A}} x$.
(iv) Let $0 \in A$. Since $U(0, 0) = 0 \leq x$ for all $x \in L$, $x \in c^*_{U,A}(\{0\})$, i.e $x \preceq_{c^*_{U,A}} 0$ for all $x \in L$.
(v) Let $0 \in A$ and U be a conjunctive uninorm. Since $U(x, 0) = 0 \leq 0$ for $0 \in A$ and for all $x \in L$, $0 \in c^*_{U,A}(\{x\})$, i.e., $0 \preceq_{c^*_{U,A}} x$ for all $x \in L$. □

Proposition 3.3. Let $(L, \leq, 0, 1)$ be a bounded lattice, $e \in L$, $A = \{e, 1\}$ and U be a uninorm on L with the neutral element $e \in L$ and consider the topological closure operator $c^*_{U,A}$ and $x \in L$. Then, $y \preceq_{c^*_{U_e,A}} x$ for $y \in [x, 1]$.

Proof. (i) Let $x \in L$. Then,

$$\begin{aligned} c^*_{U,A}(\{x\}) &= \{y \in L \ : \text{there exists } a \in A = \{e, 1\} \text{ such that } U(a, x) \leq y\} \\ &= \{y \in L \ : \ U(e, x) \leq y\} \cup \{y \in L \ : \ U(1, x) \leq y\}. \end{aligned}$$

Since $U(e, x) \leq U(1, x)$, it is obtained that

$$\begin{aligned} c^*_{U,A}(\{x\}) &= \{y \in L \ : \ U(e, x) \leq y\} \\ &= \{y \in L \ : \ x \leq y\} = [x, 1]. \end{aligned}$$

Therefore, $y \preceq_{c^*_{U,A}} x$ for $y \in [x, 1]$. □

Proposition 3.4. Let $(L, \leq, 0, 1)$ be a bounded lattice, U is a uninorm on L with the neutral element e. Consider the sets $A_1, A_2 \subseteq L$ such that $e \in A_1, A_2$, $U(A_1, A_1) \subseteq A_1$, $U(A_2, A_2) \subseteq A_2$ and $A_1 \subseteq A_2$. Then, $\preceq_{c^*_{U,A_1}} \subseteq \preceq_{c^*_{U,A_2}}$.

Proof. Let $(x, y) \in \preceq_{c^*_{U,A_1}}$. Then, $x \preceq_{c^*_{U,A_1}} y$. It means that $x \in c^*_{U,A_1}(\{y\})$. It follows from that there exists $a_1 \in A_1$ such that $U(a_1, y) \leq x$. Since $A_1 \subseteq A_2$, $a_1 \in A_2$. Therefore, $x \in c^*_{U,A_2}(\{y\})$. Then, it is clearly obtained that $x \preceq_{c^*_{U,A_2}} y$, i.e., $(x, y) \in \preceq_{c^*_{U,A_2}}$. □

Remark 3.5. The converse of Proposition 3.4 may not be satisfied.
For example: Consider the lattice $((L=\{0,a,b,e,d,1\},\leq,0,1)$ whose lattice diagram is displayed in Fig. 1.

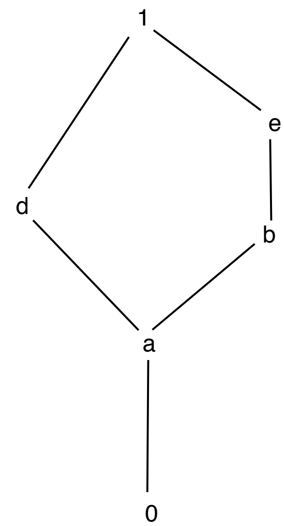

Figure 1. $(L,\leq,0,1)$

Define the function $U:L^2\to L$ as follows (Table 1):

U	0	a	b	e	d	1
0	0	0	0	0	d	1
a	0	0	0	a	d	1
b	0	0	0	b	d	1
e	0	a	b	e	d	1
d	d	d	d	d	1	1
1	1	1	1	1	1	1

Table 1. The uninorm U on L

It is clear that the function U is a uninorm on L with the neutral element e ([5]). Consider the topological closure operators c^*_{U,A_1} and c^*_{U,A_2}, where $A_1=\{0,a,b,e\}$ and $A_2=\{0,a,e\}$, respectively. Then, it is obtained that

$$\begin{aligned}
c^*_{U,A_1}(\{0\})&=L \quad (c^*_{U,A_2}(\{0\})=L),\\
c^*_{U,A_1}(\{a\})&=L \quad (c^*_{U,A_2}(\{a\})=L),\\
c^*_{U,A_1}(\{b\})&=L \quad (c^*_{U,A_2}(\{b\})=L),\\
c^*_{U,A_1}(\{e\})&=L \quad (c^*_{U,A_2}(\{e\})=L),\\
c^*_{U,A_1}(\{d\})&=\{d,1\} \quad (c^*_{U,A_2}(\{d\})=\{d,1\}),\\
c^*_{U,A_1}(\{1\})&=\{1\} \quad (c^*_{U,A_2}(\{1\})=\{1\}).
\end{aligned}$$

Therefore, it is seen that $x\preceq_{c^*_{U,A_2}} y$ when $x\preceq_{c^*_{U,A_1}} y$ for $x,y\in L$, i.e., $\preceq_{c^*_{U,A_1}}\subseteq\preceq_{c^*_{U,A_2}}$. Thus, it is clear that $\preceq_{c^*_{U,A_1}}\subseteq\preceq_{c^*_{U,A_2}}$ does not require that $A_1\subseteq A_2$.

Theorem 3.6. *Let $(L, \leq, 0, 1)$ be a bounded lattice, $A \subseteq L$ such that $e \in A$ and $U(A, A) \subseteq A$ where U is a uninorm on L with the neutral element e and consider the topological closure operator $c^*_{U,A}$. Then, the set $\mathcal{A} = \{\preceq_{c^*_{U,A}} \mid A \subseteq L\}$ is a pre-ordered set according to inclusion relation.*

Proof. Considering Proposition 3.4, it is obtained that $\preceq_{c^*_{U,A_1}} \subseteq \preceq_{c^*_{U,A_2}}$ when $A_1 \subseteq A_2$. Moreover, it is clear that $\preceq_{c^*_{U,A_1}} \subseteq \preceq_{c^*_{U,A_3}}$ when $\preceq_{c^*_{U,A_1}} \subseteq \preceq_{c^*_{U,A_2}}$ and $\preceq_{c^*_{U,A_2}} \subseteq \preceq_{c^*_{U,A_3}}$. □

Proposition 3.7. Let $(L, \leq, 0, 1)$ be a bounded lattice, $A \subseteq L$ such that $e \in A$ and $U(A, A) \subseteq A$ where U is a uninorm on L with the neutral element e and consider the topological closure operator $c^*_{U,A}$. Then, the function $\alpha : L \to \mathfrak{F}_{c^*_{U,A}}$ given by

$$\alpha(x) = c^*_{U,A}(\{x\})$$

is an order-reversing map.

Proof. Let $x \leq y$ and $k \in c^*_{U,A}(\{y\})$ for $x, y, k \in L$. Then, there exists an element $a \in L$ such that $U(a, y) \leq k$. Since $U(a, x) \leq U(a, y) \leq k$, $k \in c^*_{U,A}(\{x\})$. Therefore, it is obtained that $c^*_{U,A}(\{y\}) \subseteq c^*_{U,A}(\{x\})$. □

Proposition 3.8. Let $(L, \leq, 0, 1)$ be a bounded lattice, $A \subseteq L$ such that $e \in A$ and $U(A, A) \subseteq A$ where U is a uninorm on L with the neutral element e and consider the topological closure operator $c^{**}_{U,A}$. Then, the function $\alpha : L \to \mathfrak{F}_{c^{**}_{U,A}}$ given by

$$\beta(x) = c^{**}_{U,A}(\{x\})$$

is an order-preserving map.

Proof. Let $x \leq y$ and $l \in c^{**}_{U,A}(\{x\})$ for $x, y, l \in L$. Then, there exists an element $a \in L$ such that $U(a, x) \geq l$. Since $U(a, y) \geq U(a, x) \geq l$, $l \in c^{**}_{U,A}(\{y\})$. Therefore, it is obtained that $c^{**}_{U,A}(\{x\}) \subseteq c^{**}_{U,A}(\{y\})$. □

Definition 3.9. Let $(L, \leq, 0, 1)$ be a bounded lattice, $A \subseteq L$ such that $e \in A$ and $U(A, A) \subseteq A$ where U is a uninorm on L with the neutral element e and consider the topological closure operator $c^*_{U,A}$. Define a relation $\sim$ on L by $x \sim y$ if and only if $c^*_{U,A}(\{x\}) = c^*_{U,A}(\{y\})$.

Proposition 3.10. The relation $\sim$ given in Definition 3.9 is an equivalence relation.

Proof. The proof is obvious. □

Definition 3.11. Let $(L, \leq, 0, 1)$ be a bounded lattice, $A \subseteq L$ such that $e \in A$ and $U(A, A) \subseteq A$ where U is a uninorm on L with the neutral element e and consider the topological closure operator $c^*_{U,A}$. We denote by $\overline{x}$ the $\sim$ equivalence class linked to x, i.e., $\overline{x} = \{y \in L : y \sim x\}$. The quotient set of L by the relation $\sim$ is denoted as $L/\sim$, i.e.,

$$L/\sim = \{\overline{x} \mid x \in L\}.$$

Proposition 3.12. $(L, \leq, 0, 1)$ be a bounded lattice, $A \subseteq L$ such that $e \in A$ and $U(A, A) \subseteq A$ where U is a uninorm on L with the neutral element e and consider the topological closure operator $c^*_{U,A}$. Consider the equivalence class $\overline{x}$ the $\sim$ equivalence class linked to $x \in L$. Then, $\overline{U(a, x)} = \overline{U(a, y)}$ for all $a \in A$ if $\overline{x} = \overline{y}$.

Proof. Let $\overline{x} = \overline{y}$ and $a \in L$. Then, $x \sim y$, i.e., $x \in c^*_{U,A}(\{y\})$. Therefore, there exists an element $a_1 \in A$ such that $U(a_1, y) \leq x$. Since $U(U(a, y), a_1) = U(a, U(a_1, y)) \leq U(a, x)$, it is obtained that $U(a, x) \in c^*_{U,A}(\{U(a, y)\})$, i.e., $\{U(a, x)\} \subseteq c^*_{U,A}(\{U(a, y)\})$. Thus,

$$c^*_{U,A}(\{U(a, x)\}) \subseteq c^*_{U,A}(\{U(a, y)\}). \tag{3.1}$$

On the other side, it follows from $x \sim y$ that $y \in c^*_{U,A}(\{x\})$. Then, there exists an element $a_2 \in A$ such that $U(a_2, x) \leq y$. Since $U(U(a, x), a_2) = U(a, U(a_2, x)) \leq U(a, y)$, it is obtained that $U(a, y) \in c^*_{U,A}(\{U(a, x)\})$, i.e., $\{U(a, y)\} \subseteq c^*_{U,A}(\{U(a, x)\})$. Thus,

$$c^*_{U,A}(\{U(a, y)\}) \subseteq c^*_{U,A}(\{U(a, x)\}). \tag{3.2}$$

Then, $c^*_{U,A}(\{U(a, x)\}) = c^*_{U,A}(\{U(a, y)\})$ for all $a \in A$ from the (3.1) and (3.2). Therefore,

$$\begin{aligned}
\overline{U(a, x)} &= \{t \in L \ : \ t \sim U(a, x)\} \\
&= \{t \in L \ : \ c^*_{U,A}(\{t\}) = c^*_{U,A}(\{U(a, x)\})\} \\
&= \{t \in L \ : \ c^*_{U,A}(\{t\}) = c^*_{U,A}(\{U(a, y)\})\} \\
&= \{t \in L \ : \ t \sim U(a, y)\} \\
&= \overline{U(a, y)}.
\end{aligned}$$

□

Corollary 3.13. $(L, \leq, 0, 1)$ be a bounded lattice, $A \subseteq L$ such that $e \in A$ and $U(A, A) \subseteq A$ where U is a uninorm on L with the neutral element e and consider the topological closure operator $c^*_{U,A}$. Consider the equivalence class $\overline{x}$ the $\sim$ equivalence class linked to $x \in L$. Then, $\overline{x} = \{x\}$ for all $x \in L$ if $A = \{e\}$.

Proposition 3.14. The equivalence relation given by Definition 3.9 is an U-congruence relation.

Proof. Let $x, y, z, t \in L$, $x \sim y$ and $z \sim t$. Since $x \sim y$, there exists an element $a_1 \in A$ such that $U(x, a_1) \leq y$. Similarly, there exists an element $a_2 \in A$ such that $U(z, a_2) \leq t$ since $z \sim t$. Therefore, $U(U(x, z), U(a_1, a_2)) = U(U(x, a_1), U(z, a_2)) \leq U(y, t)$, $U(y, t) \in c^*_{U,A}(\{U(x, z)\})$. Then,

$$c^*_{U,A}(\{U(y, t)\}) \subseteq c^*_{U,A}(\{U(x, z)\}). \tag{3.3}$$

On the contrary, there exists an element $a_1^* \in A$ such that $U(y, a_1^*) \leq x$ since $x \sim y$. Similarly, there exists an element $a_2^* \in A$ such that $U(t, a_2^*) \leq z$ since $z \sim t$. Therefore, $U(U(y, t), U(a_1^*, a_2^*)) = U(U(y, a_1^*), U(t, a_2^*)) \leq U(x, z)$, i.e., $U(x, z) \in c^*_{U,A}(\{U(y, t)\})$. Then,

$$c^*_{U,A}(\{U(x, z)\}) \subseteq c^*_{U,A}(\{U(y, t)\}). \tag{3.4}$$

Thus, it is obtained that $c^*_{U,A}(\{U(x, z)\}) = c^*_{U,A}(\{U(y, t)\})$ from the (3.3) and (3.4). Therefore, $U(x, z) \sim U(y, t)$. □

Definition 3.15. Consider the quotient set $L/\sim$ defined in Definition 3.11 and define the $\overline{\leq}$ relation on $L/\sim$ by

$$\overline{x} \ \overline{\leq} \ \overline{y} \ :\Longleftrightarrow \ \overline{x} \ \subseteq \ \overline{y}.$$

Proposition 3.16. $(L/\sim, \overline{\leq})$ is a partially ordered set.

Proof. The proof is obvious. □

Proposition 3.17. Consider the partially ordered set $(L/\sim, \overline{\leq})$. Then, the followings are satisfied for $\overline{x}, \overline{y} \in L/\sim$ such that $\overline{x} = c^*_{U,A}(\{x\}), \overline{y} = c^*_{U,A}(\{y\})$.
(i) $\overline{x} \cup \overline{y} \subseteq \overline{x \wedge y}$,
(ii) $\overline{x \vee y} \subseteq \overline{x} \cap \overline{y}$

Proof. (i) It is obvious that $a \wedge b \leq a, b$ for all $a, b \in L$. Taking into account the definition of $c^*_{U,A}$, $x, y \in c^*_{U,A}(\{x \wedge y\}) = \overline{x \wedge y}$ since $U(x \wedge y, e) \leq x, y$ for $e \in A$. It follows from $\{x\}, \{y\} \subseteq c^*_{U,A}(x \wedge y) = \overline{x \wedge y}$ that $\overline{x} = c^*_{U,A}(\{x\}), \overline{y} = c^*_{U,A}(\{y\}) \subseteq c^*_{U,A}(\{x \wedge y\}) = \overline{x \wedge y}$. Therefore, $\overline{x} \cup \overline{y} \subseteq \overline{x \wedge y}$.
(ii) The proof can be done in a similar fashion as the proof of (i). □

Proposition 3.18. Consider the partially ordered set $(L/\sim, \overline{\leq})$ and the uninorm U on L with the neutral element e. The function $\overline{U} : (L/\sim)^2 \to L/\sim$ defined by $\overline{U}(\overline{x}, \overline{y}) = \overline{U(x, y)}$ is a uninorm on $L/\sim$.

Proof. The funciton $\overline{U}$ is a well-defined function since the equivalence relation is a U-congruence relation.
(i) The fact that $\overline{e}$ is a neutral element of $\overline{U}$ is obvious from the equation $\overline{U}(\overline{x}, \overline{e}) = \overline{U(x, e)} = \overline{x}$ for all $\overline{x} \in L/\sim$.
(ii) The fact that the associativity and commutativity of $\overline{U}$ are obvious from the associativity and commutativity of U.
(iii) Let $\overline{x} \overline{\leq} \overline{y}$ and $m \in \overline{U}(\overline{x}, \overline{t}) = \overline{U(x, t)}$ arbitrarily to prove that $\overline{U}(\overline{x}, \overline{t}) \overline{\leq} \overline{U}(\overline{y}, \overline{t})$ (i.e., $\overline{U(x, t)} \subseteq \overline{U(y, t)}$) for all $\overline{t} \in L/\sim$. Then, $m \sim U(x, t)$, i.e.,

$$c^*_{U,A}(\{m\}) = c^*_{U,A}(\{U(x, t)\}). \tag{3.5}$$

$c^*_{U,A}(\{m\}) = c^*_{U,A}(\{U(x, t)\})$. Since $m \in c^*_{U,A}(\{m\})$, $m \in c^*_{U,A}(\{U(x, t)\})$. Then, there exists an element $a_m \in A$ such that

$$U(a_m, U(x, t)) \leq m. \tag{3.6}$$

Since $\overline{x} \overline{\leq} \overline{y}$, $\overline{x} \subseteq \overline{y}$. Then, it follows from $x \in \overline{y}$ that $x \sim y$, i.e., $c^*_{U,A}(\{x\}) = c^*_{U,A}(\{y\})$. Therefore, $x \in c^*_{U,A}(\{y\})$. Then, there exists an element $a_x \in A$ such that

$$U(a_x, y) \leq x. \tag{3.7}$$

It is obtained that

$$\begin{aligned} m &\geq U(a_m, U(x, t)) \\ &\geq U(a_m, U(U(a_x, y), t)) \\ &= U(a_m, U(a_x, U(y, t))) \\ &= U(U(a_m, a_x), U(y, t)) \end{aligned}$$

from (3.6) and (3.7). Then, $m \in c^*_{U,A}(\{U(y, t)\})$. Therefore,

$$c^*_{U,A}(\{m\}) \subseteq c^*_{U,A}(\{U(y, t)\}). \tag{3.8}$$

On the contrary, let $k \in c^*_{U,A}(\{U(y, t)\})$. Then, there exists an element $a^* \in A$ such that

$$U(a^*, U(y, t)) \leq k. \tag{3.9}$$

We have $y \in c^*_{U,A}(\{x\})$ since $c^*_{U,A}(\{x\}) = c^*_{U,A}(\{y\})$. Then, there exists an element $a^{**} \in A$ such that

$$U(a^{**}, x) \leq y. \tag{3.10}$$

It is obtained that

$$\begin{aligned} k &\geq U(a^*, U(y,t)) \\ &\geq U(a^*, U(U(a^{**}, x), t)) \\ &= U(a^*, U(a^{**}, U(x,t))) \\ &= U(U(a^*, a^{**}), U(x,t)) \end{aligned}$$

from (3.9) and (3.10). Then, it follows from $k \in c^*_{U,A}(\{U(x,t)\})$ that $k \in c^*_{U,A}(\{m\})$ from (3.5). Therefore,

$$c^*_{U,A}(\{U(y,t)\}) \subseteq c^*_{U,A}(\{m\}). \tag{3.11}$$

It is obtained that $c^*_{U,A}(\{m\}) = c^*_{U,A}(\{U(y,t)\})$ from (3.8) and (3.11). Therefore, we have

$$\begin{aligned} m &\sim U(y,t) \\ m &\in \overline{U(y,t)}. \end{aligned}$$

Then, $\overline{U(x,t)} \subseteq \overline{U(y,t)}$. Summarizing all the above facts, we have shown the monotonicity of $\overline{U}$.

Therefore, $\overline{U}$ is a uninorm on $L/\sim$.

□

4. Concluding Remarks

After investigating the partial order obtained from the closure operator defined with the help of triangular norms, it is quite natural to investigate the closure operators obtained from uninorms that generalize triangular norms and to research on the orders obtained from these operators. Main idea of this paper to define the pre-orders and to investigate of their properties. Moreover, we also define an equivalence relation via the closure operator and obtain a uninorm on its qutient set. As a follow-up study, obtaining a partial order from the closure operators can be discussed and the properties of the order can be examined in detail.

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The Normalizer of $H_0^5(I)$ in H^5

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Abstract. Let $\lambda = 2cos\frac{\pi}{5}$ and H^5 be the Hecke group assosiated to λ. Let I be an ideal of $\mathbb{Z}[\lambda]$ and $H_0^5(I)$ the principal congruence subgroup of H^5 associated to I. In this paper, the normalizer of the congruence subgroups $H_0^5(I)$ in H^5 are given in the case where $I = (2^\alpha)I', (2, I') = 1$ and I' is a nonsquare ideal.

Keywords. Normalizer, Congruence subgroup, Hecke group.

1. Introduction

The normalizer has acquired its importance in several areas of mathematics. For instance, the genus zero subgroups of the normalizer $\Gamma_0(N)$ in $PSL(2, \mathbb{R})$ have a mysterious correspondence to the conjugacy classes of the monster simple group [2, 3]. Moreover, the normalizer $\Gamma_0(N)$ in $PSL(2, \mathbb{R})$ played an important role in the work on Weierstrass points on the modular curve $X_0(N)$ associated $\Gamma_0(N)$ [9] and on ternary quadratic forms [10]. In [1], Akbaş and Singerman calculated the above mentioned the normalizer of $\Gamma_0(n)$ in $PSL(2, \mathbb{R})$. On the other hand many studies were done by some authors on the normalizer of the Hecke group H^q in $PSL(2, \mathbb{Z}[\lambda_q])$ and $PSL(2, \mathbb{R})$ when q is 4 and 5 in [6, 7, 8]. Furthermore, in [5], it is conjectured that the normalizer of $H_0^5(I)$ in H^5 is $H_0^5((2)^{\alpha'}I')$ where $I = (2)^\alpha I'$ is an ideal of $\mathbb{Z}[\lambda], (2, I') = 1$ and $\alpha' = \alpha - min(2, [\frac{\alpha}{2}])$ by Ivrissimtz in his Ph.D thesis. In [11], this conjecture is proved in the case where I' is a prime ideal. In this study, we give a proof to the conjecture in the case where I' is a nonsquare ideal. So, our result is more strong than the other result.

2. Hecke group H^q

The subgroups of $PSL(2, \mathbb{R})$ generated by the two transformations

$$X : z \to -\frac{1}{z} \text{ and } Z : z \to z + \lambda,\ \lambda \in \mathbb{R}$$

are of special interest when they are discrete. So, it is naturally arising question to find the values of λ for which these groups are discrete. It was shown by Hecke in [4] that the above groups are discrete for

$$\lambda = \lambda_q = 2cos\tfrac{\pi}{q}, q = 3, 4, 5, \ldots \text{ and } \lambda \geq 2.$$

We will call these discrete groups Hecke groups, and we will use the notation H^q for the Hecke group corresponding to

$$\lambda = \lambda_q = 2cos\frac{\pi}{q}.$$

Using the matrix representation in $PSL(2, \mathbb{R})$ of the two transformations X, Z we find that

$$H^q =< X, Z >=< \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & \lambda_q \\ 0 & 1 \end{pmatrix} > .$$

We notice that if $q = 3$ then

$$z = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and in this case corresponding Hecke group is the modular group.

For any particular Hecke group H^q we define $Y = XZ$, that is in terms of matrix representations

$$Y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \lambda_q \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & \lambda_q \end{pmatrix}$$

We notice that X, Y are elliptic elements of order 2 and q respectively, while Z is parabolic. Because X, Z generate H^q and because $XY = Z$ we have that X, Y also generate H^q.

A very basic result concerning the Hecke groups H^q is that they are the free product of the cyclic groups generated by X and Y.

3. The congruence subgroups of H^q

For any ideal I of $\mathbb{Z}[\lambda_q]$ we can define the corresponding principal congruence subgroup of $PSL(2, \mathbb{Z}[\lambda_q)$ in a way similar to the definition of the principal congruence subgroups of modular group. Namely we define

$$H^q(I) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H^q | \ a - 1, b, c, d - 1 \equiv 0 \ modI \right\}$$

and similarly we define the special congruence subgroup corresponding to I by

$$H_1^q(I) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H^q | \ a - 1, c, d - 1 \equiv 0 \ modI \right\},$$

$$H_0^q(I) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H^q | \ c \equiv 0 \ modI \right\}.$$

Obviously the inclusions

$$H^q(I) \leq H_1^q(I) \leq H_0^q \geq H^q$$

hold. It is a well known result that $H^q(I)$ is normal in H^q and we can also prove that $H_1^q(I)$ is normal in $H_0^q(I)$ in the same way we proved the corresponding result for the modular group. That is by defining

$$f : H_0^q(I) \to (\mathbb{Z}[\lambda_q]/I)/\{\pm 1\} \text{ with } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \pm\theta(a)$$

where θ is the canonical map $\mathbb{Z}[\lambda_q] \to \mathbb{Z}[\lambda_q]/I$, and then finding that the kernel of f is $H_1^q(I)$.

4. The normalizer of $H_0^5(I)$ in H^5

Firstly, we will give the following lemma necessary for our result.

Lemma 4.1. [11] *Let the ideal $I = (2) = 2\mathbb{Z}[\lambda]$ and $A \in H^5$. Then*

$$A \in H_0^5 \textit{ if and only if } A \equiv \pm \begin{pmatrix} 1 & r\lambda \\ 0 & 1 \end{pmatrix} \ mod I,$$

where $r = 0, 1$.

Theorem 4.2. *Let $I = (2)^\alpha I'$ be an ideal of $\mathbb{Z}[\lambda]$, where I' is a nonsquare ideal of $\mathbb{Z}[\lambda]$ and $(2, I') = 1$. Then the normalizer of $H_0^5(I)$ in H^5 is $H_0^5((2)^{\alpha'} I')$ where $\alpha' = \alpha - min\{2, [\frac{\alpha}{2}]\}$.*

Proof. Let $X = \begin{pmatrix} x & y \\ z & t \end{pmatrix} \in N_{H^5}(H_0^5(I))$. Because of the definition of the normalizer

$$A = XKX^{-1} \in H_0^5(I),$$

such that $K = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H_0^5(I)$.

From the above multiplication, we have

$$ayt - by^2 - dty \equiv 0 \ mod I.$$

If we take $K = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$, and consider $(-1+\lambda)\lambda = 1$, we have that

$$(a-d)ty \equiv 0 \ mod I.$$

Multiplying the above congruence by $x \in \mathbb{Z}$, and using $|X| = xt - yz = 1$ and $|K| = ad - bc = 1$, it can be seen that

$$(a^2 - 1)y \equiv 0 \ mod I.$$

On the other hand, because $I = (2)^\alpha I' \subseteq (2)^\alpha \cap I' \subset (2) \cap I'$, $\alpha \geq 1$ and $y^2 \equiv 0 \ mod I$, then

$$y^2 \equiv 0 \ mod 2$$

and

$$y^2 \equiv 0 \ mod I'$$

As (2) is the prime ideal,
$$y \equiv 0 \; mod2.$$
Indeed, because I' is a nonsquare ideal, it can be written as a multiplying of the prime ideals. So,
$$I' = P_1 P_2 ... P_n,$$
where P_i is prime ideal and $1 \leq i \leq n$. Thus
$$y^2 \equiv 0 \; modI' \Rightarrow y^2 \in I'.$$
From the above congruence
$$y^2 = p_1 p_2 ... p_n (k + l\lambda),$$
with $(k + l\lambda) \in \mathbb{Z}[\lambda]$. Using p_i is prime, we can write
$$p_i | y^2 \Rightarrow p_i | y$$
and
$$p_1 p_2 ... p_n | y \Rightarrow y \in I'.$$
Now, let us consider the following chain. In here, there is $n \in \mathbb{N}^+$ such that
$$...2^{n+1}\mathbb{Z}[\lambda] \subset 2^n \mathbb{Z}[\lambda] \subset ... \subset 2^2\mathbb{Z}[\lambda] \subset 2\mathbb{Z}[\lambda] \subset \mathbb{Z}[\lambda].$$
Now let us consider $\alpha' \in \mathbb{N}$ where
$$y \in (2)^{\alpha'} \text{ and } y \notin (2)^{\alpha'+1}$$
If we take a square of the first expression
$$y^2 \in (2)^{2\alpha'}.$$
Moreover, as $y^2 \in (2)^{\alpha}, \; \alpha \leq 2\alpha'$.
Now let us consider the conditions of α and α' relative to each other.

i. Let $\alpha < \alpha'$. In this case,
$$y^2 \equiv 0 \; mod 2^{\alpha'} I'.$$
From $\alpha < \alpha'$, $(2)^{\alpha'} \subset (2)^{\alpha}$ and so
$$y^2 \equiv 0 \mod I.$$

ii. Let us see $\alpha' \leq \alpha$. Then we have
$$\frac{\alpha}{2} \leq \alpha' \leq \alpha.$$
From Lemma 4.1 ,
$$a^2 - 1 \equiv 0 \; mod 2^2.$$
Moreover
$$(a^2 - 1)y \in 2^{\alpha'+2}\mathbb{Z}[\lambda],$$
for $y \in 2^{\alpha'}\mathbb{Z}[\lambda]$ and considering $(a^2 - 1)y \equiv 0 \; modI$ we have
$$\alpha \leq \alpha' + 2.$$
In this case, it is enough to find the smallest number $\alpha' \in \mathbb{N}$ for values of α.
$$\alpha' = 1, \text{ for } \alpha = 1,$$

$$\alpha' = 1, \text{ for } \alpha = 2,$$
$$\alpha' = 2, \text{ for } \alpha = 3,$$

So,

$$\alpha' = \begin{cases} 1 = 1 - [\frac{1}{2}], & \alpha = 1 \\ 1 = 2 - [\frac{2}{2}], & \alpha = 2 \\ 2 = 3 - [\frac{3}{2}]. & \alpha = 3 \end{cases}$$

for $\alpha \in \{1, 2, 3\}$.

Now, let $\alpha \geq 4$. From this, there is an element $\beta' \in \mathbb{N}$, such that $\alpha = \beta + 4$. So,

$$\alpha' = \alpha - 2, \text{ for } \alpha \geq 4.$$

Thus, for every $\alpha \in \mathbb{Z}^+$

$$y \equiv 0 mod\ 2^{\alpha'} I'$$

and so

$$N_{H^5}(H_0^5(I)) \leq H_0^5((2)^{\alpha-2} I').$$

As a consequnce, we have

$$N_{H^5}(H_0^5(I)) \leq H_0^5((2)^{\alpha} I'),$$

$\alpha' = \alpha - min\{2, [\frac{\alpha}{2}]\}$.

Conversely, it is clear that

$$H_0^5((2)^{\alpha} I') \leq N_{H^5}(H_0^5(I)).$$

□

Now, let we examine the following example for the special values of some cases in Theorem (4.2).

Example. If we take $I' = (3), \alpha = 3, (2, I') = 1$ and $I = ((2)^{\alpha} 3)$ in Theorem (4.2), then we have

$$N_{H^5}(H_0^5(((2)^3 3))) = H_0^5(((2)^2 3)),$$

$\alpha' = \alpha - min\{2, [\frac{\alpha}{2}]\}$.

Let $X = \begin{pmatrix} x & y \\ 2^2 3z & t \end{pmatrix} \in N_{H^5}(H_0^5(((2)^3 3)))$. In this case, there is an element $K = \begin{pmatrix} a & b \\ 2^3 3c & d \end{pmatrix} \in H_0^5(((2)^3 3))$ such that

$$XKX^{-1} \in H_0^5(((2)^3 3)).$$

So, if we calculate the above multiplaying, we get

$$XKX^{-1} = \begin{pmatrix} * & * \\ 2^2 3azt + 2^3 3t^2 c - 2^4 3^2 z^2 - 12ztd & * \end{pmatrix}.$$

We wonder if

$$2^2 3zt(a - d) \equiv 0\ mod 2^3 3,$$

so,

$$a - d \equiv 0 \ mod 2.$$

Using the determinant of $|K| = ad - 12bc = 1$, we can say $a \equiv d \equiv 1 \ mod 2$ and so $a - d \equiv 0 \ mod 2$.

The other inclusion is clear.

Corollary 4.3. *Let I be a nonsquare ideal of $\mathbb{Z}[\lambda]$. Then*

$$N_{H^5}(H_0^5(I)) = H_0^5(I)$$

Proof. From 4.2, it is easily seen that

$$N_{H^5}(H_0^5(I)) \leq H_0^5(I).$$

Conversely, let $X = \begin{pmatrix} x & y \\ z & t \end{pmatrix} \in H_0^5(I)$. We need to show

$$XKX^{-1} \in H_0^5(I)$$

with $K = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H_0^5(I)$.

This is obvious. Thus,

$$N_{H^5}(H_0^5(I)) = H_0^5(I).$$

The proof is complete.

□

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EQUIVALENCE CLASSES OF IMPLICATIONS ON BOUNDED LATTICES

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Abstract. In this paper, an equivalence relation is defined by implications. An implication on the bounded lattice of the equivalence classes of implications is introduced. Some basic properties of the generated implication are studied. A set is introduced by the linear combination of implications on bounded lattices and the lattice theoretical structure of the mentioned set is investigated.

1. Introduction

Fuzzy implications generalize the classical implication to fuzzy logic. They have the great quantity of applications, viz., approximate reasoning, fuzzy control, fuzzy image processing [1, 5, 6, 17, 18, 27].

In the literature, the convex combination of logical operators, which is one of the remarkable generating method, provides new logical operators from two given ones [7, 19, 20, 21, 23, 24, 25, 26]. The convex combination of fuzzy implications I and J on the unit interval $[0, 1]$ is given as follows (see [1, 20, 22]): for $\lambda \in [0, 1]$,

$$K(x, y) = \lambda I(x, y) + (1 - \lambda)J(x, y).$$

As extensions of the convex combination of triangular norms and fuzzy implications, the notions of convex combination for two implications and two triangular norms on a bounded lattice are introduced in [11, 12]. In this paper, we study an equivalence relation defined by implications on bounded lattices and generate a new implication defined on the equivalence classes. The paper is organized as follows. In Section 2, we shortly recall some basic notions. In Section 3, we define an equivalence relation by implications on bounded lattices and introduce an implication on the bounded lattice of the equivalence classes based on the mentioned equivalence relation. Some basic properties of the new implication are studied. In Section 4, we introduce a set by the linear combination of implications on bounded lattices and investigate the properties of the set in the term of the lattice theoretical structure.

2. Notations, definitions and a review of previous results

In this section, we give some basic definitions, notions and results.

Definition 2.1. [2] A bounded lattice $(L, \leq)$ is a lattice which has the top and bottom elements, which are written as 1 and 0, respectively, that is, there exist two elements $1, 0 \in L$ such that $0 \leq x \leq 1$, for all $x \in L$.

2020 *Mathematics Subject Classification.* 03E72, 03B52, 08A72, 54A10, 06B30, 06F30.

Key words and phrases. Implication; Bounded lattice; Equivalence class; Convex combination; G-convex combination; Linear combination.

Definition 2.2. [13, 14] A binary operation T (S) on a bounded lattice L is called a triangular norm (triangular conorm) if it is commutative, associative, increasing with respect to the both variables and has a neutral element 1 (0).

The followings are the four basic t-norms T_M, T_P, T_{LK} and T_D on the unit interval $[0,1]$ given by respectively:

$$T_M(x,y) = \min(x,y), \qquad T_P(x,y) = xy,$$

$$T_{LK}(x,y) = \max(x+y-1,0), \qquad T_D(x,y) = \begin{cases} 0 & (x,y) \in [0,1)^2, \\ \min(x,y) & \text{otherwise.} \end{cases}$$

The t-norms T_M and T_D are generalized to a bounded lattice L, respectively, as follows.

$$T_\wedge(x,y) = x \wedge y \text{ and } T_D(x,y) = \begin{cases} y & x = 1, \\ x & y = 1, \\ x \wedge y & \text{otherwise.} \end{cases}$$

The followings are the four basic t-conorms S_M, S_P, S_{LK} and S_D on the unit interval $[0,1]$ given by respectively:

$$S_M(x,y) = \max(x,y), \qquad S_P(x,y) = x + y - xy,$$

$$S_{LK}(x,y) = \min(x+y,1), \qquad S_D(x,y) = \begin{cases} 1 & (x,y) \in (0,1]^2, \\ \max(x,y) & \text{otherwise.} \end{cases}$$

The t-conorms S_M and S_D are generalized to a bounded lattice L, respectively, as follows.

$$S_\vee(x,y) = x \vee y \text{ and } S_D(x,y) = \begin{cases} y & x = 0, \\ x & y = 0, \\ x \vee y & \text{otherwise.} \end{cases}$$

Definition 2.3. [1, 9, 10] A function $I : L^2 \to L$ on a bounded lattice $(L, \leq, 0, 1)$ is called an implication if it satisfies the following conditions:

(I1) I is a decreasing operation on the first variable, that is, for every $a, b \in L$ with $a \leq b$, $I(b,y) \leq I(a,y)$ for all $y \in L$.

(I2) I is an increasing operation on the second variable, that is, for every $a, b \in L$ with $a \leq b$, $I(x,a) \leq I(x,b)$ for all $x \in L$.

(I3) $I(0,0) = 1$.

(I4) $I(1,1) = 1$.

(I5) $I(1,0) = 0$.

Denote by $\mathcal{F}$ the set of all implications on a bounded lattice L.

Example 2.4. [1] The followings are well-known implications on the unit interval $[0,1]$.

$$I_{LK}(x,y) = \min(1, 1 - x + y), \qquad I_{RC}(x,y) = 1 - x + xy,$$

$$I_{KD}(x,y) = \max(1 - x, y), \qquad I_{GD}(x,y) = \begin{cases} 1 & x \leq y, \\ y & x > y, \end{cases}$$

$$I_{GG}(x,y) = \begin{cases} 1 & x \leq y, \\ \frac{y}{x} & x > y, \end{cases} \qquad I_{RS}(x,y) = \begin{cases} 1 & x \leq y, \\ 0 & x > y, \end{cases}$$

$$I_{YG}(x,y) = \begin{cases} 1 & x = 0 \text{ and } y = 0, \\ y^x & x > 0 \text{ or } y > 0, \end{cases} \qquad I_{WB}(x,y) = \begin{cases} 1 & x < 1, \\ y & x = 1, \end{cases}$$

$$I_{FD}(x,y) = \begin{cases} 1 & x \leq y, \\ \max(1 - x, y) & x > y. \end{cases}$$

The least and the greatest implications are respectively given by:

$$I_0(x,y) = \begin{cases} 1 & x = 0 \text{ or } y = 1, \\ 0 & x > 0 \text{ and } y < 1, \end{cases} \qquad I_1(x,y) = \begin{cases} 1 & x < 1 \text{ or } y > 0, \\ 0 & x = 1 \text{ and } y = 0. \end{cases}$$

Definition 2.5. [1, 9, 14] Let $(L, \leq, 0, 1)$ be a bounded lattice. A decreasing function $N : L \to L$ is called a negation if $N(0) = 1$ and $N(1) = 0$. A negation N on L is called strong if it is an involution, i.e., $N(N(x)) = x$, for all $x \in L$.

The weakest and strongest negations on L are given by respectively

$$N_{D_1}(x) = \begin{cases} 0 & x \neq 0, \\ 1 & x = 0, \end{cases} \qquad N_{D_2}(x) = \begin{cases} 1 & x \neq 1, \\ 0 & x = 1. \end{cases}$$

The natural negation of an implication I on a bounded lattice is the function $N_I : L \to L$ defined by $N_I(x) = I(x, 0)$, for all $x \in L$.

Let N be a fuzzy negation and I be an implication on a bounded lattice L. The implication $I_N : L^2 \to L$ defined by

$$I_N(x, y) = I(N(y), N(x)), \qquad x, y \in L$$

is called the N-reciprocal of I.

Definition 2.6. [1, 28] An implication I on a bounded lattice L is said to satisfy

(i) the left neutrality property if for all $y \in L$

$$(NP) \qquad I(1, y) = y$$

holds.

(ii) the ordering property if for all $x, y \in L$

$$(OP) \qquad x \leq y \Leftrightarrow I(x, y) = 1$$

holds.

(iii) the identity principle if for all $x \in L$

$$(\text{IP}) \qquad I(x, x) = 1$$

holds.

(iv) the exchange principle if for all $x, y, z \in L$

$$(\text{EP}) \qquad I(x, I(y, z)) = I(y, I(x, z)).$$

Definition 2.7. [1] Let S be a t-conorm and N a negation on a bounded lattice L. We say that the pair (S, N) satisfies the law of excluded middle if for all $x \in L$

$$(\text{LEM}) \qquad S(N(x), x) = 1.$$

Definition 2.8. [1, 20, 22] Let I and J be two fuzzy implications on $[0, 1]$. The operation K defined as follows is called a convex combination of I and J for any $x, y \in [0, 1]$ and $\lambda \in [0, 1]$,

$$K(x, y) = \lambda I(x, y) + (1 - \lambda) J(x, y).$$

Definition 2.9. [8] (i) An operation M on a lattice is called $\vee$-distributive in the second place if for any $a, b_1, b_2 \in L$

$$M(a, b_1 \vee b_2) = M(a, b_1) \vee M(a, b_2).$$

(ii) Let A be an infinite indexing set. An operation M on a complete lattice is called infinitely $\vee$-distributive in the second place if for any $a \in L$ and $\{b_\tau \mid \tau \in A\} \subseteq L$,

$$M(a, \vee_{\tau \in A} b_\tau) = \vee_{\tau \in A} M(a, b_\tau).$$

The (infinitely) $\wedge$-distributivity in the first place of any operation can be defined as similar to Definition 2.9.

(iii) An operation M on a lattice is called (infinitely) $\vee$-distributive if it is (infinitely) $\vee$-distributive in both the first and second places.

(iv) The (infinitely) $\wedge$-distributivity of any operation can be defined similarly.

Definition 2.10. [11] Let L be a bounded lattice, I and J be two implications on L. For $a \in L$, define the operation $K^{I,J}_{a,T,S,N}$ as follows

$$(2.1) K^{I,J}_{a,T,S,N}(x, y) = S(T(a, I(x, y)), T(N(a), J(x, y))) \quad \text{for all} \quad x, y \in L$$

where T, S and N are respectively a t-norm, a t-conorm and a negation on L.

The operation $K^{I,J}_{a,T,S,N}$ defined by formula (2.1) is called the linear combination of the implications I and J by means of T, S and N for $a \in L$.

Definition 2.11. [11] The implication $K^{I,J}_{a,T,S,N}$ is called the generalized convex combination of the implications I and J by means of T, S and N for any $a \in L$ with $S(a, N(a)) = 1$.

For reader convenience, the operation $K^{I,J}_{a,T,S,N}$ given in Definition 2.11 will be called as the g-convex combination of I and J.

3. The implication defined on an equivalence class

In this section, we define an equivalence relation by implications on bounded lattices and introduce an implication on the bounded lattice of the equivalence classes based on the mentioned equivalence relation. Some basic properties of the new implication are studied.

Definition 3.1. Let L be a bounded lattice, I an implication on L. Define a relation $\sim$ as follows: For any $x, y \in L$

$$(3.1) \qquad x \sim y \Leftrightarrow I(1, x) = I(1, y).$$

Proposition 3.2. The relation defined in (3.1) is an equivalence relation. Denote the set of all equivalence classes w.r.t. the relation $\sim$ by $L_{/\sim}$.

Definition 3.3. Let L be a bounded lattice and I an implication on L. Define a relation $\leq$ on $L_{/\sim}$ as follows: For any $\bar{x}, \bar{y} \in L_{/\sim}$

$$(3.2) \qquad \bar{x} \leq \bar{y} \Leftrightarrow I(1, x) \leq I(1, y).$$

Proposition 3.4. $\left(L_{/\sim}, \leq\right)$ is a bounded partially ordered set.

Proof. It is clear that $(L_{/\sim}, \leq)$ is a partially ordered set. For any $\bar{y} \in L_{/\sim}$, since

$$I(1, 0) \leq I(1, y) \text{ and } I(1, y) \leq 1 = I(1, 1)$$

we obtain that $\overline{0} \leq \bar{y}$ and $\bar{y} \leq \overline{1}$. Thus, $\left(L_{/\sim}, \leq\right)$ is a bounded partially ordered set. $\square$

Theorem 3.5. *The operation $\bar{I} : L_{/\sim} \times L_{/\sim} \to L_{/\sim}$ defined by*

$$\bar{I}(\bar{x}, \bar{y}) = \overline{I(I(1,x), I(1,y))}, \text{ for all } \bar{x}, \bar{y} \in L_{/\sim}$$

is an implication on $L_{/\sim}$.

Proof. (I1) Let $\bar{x}_1 \leq \bar{x}_2$. Then, $I(1, x_1) \leq I(1, x_2)$. Since I is decreasing in the first place, we have that for all $y \in L$

$$I(I(1, x_1), I(1,y)) \geq I(I(1, x_2), I(1,y)).$$

Since I is an increasing operation in the second place, we obtain that

$$I(1, I(I(1, x_2), I(1,y))) \leq I(1, I(I(1, x_1), I(1,y))).$$

Then, $\bar{I}(\bar{x}_2, \bar{y}) = \overline{I(1, I(I(1, x_2), I(1,y)))} \leq \overline{I(1, I(I(1, x_1), I(1,y)))} = \bar{I}(\bar{x}_1, \bar{y})$.
(I2) Let $\bar{y}_1 \leq \bar{y}_2$. Then, $I(1, y_1) \leq I(1, y_2)$. Since I is increasing in the second place, we have that

$$I(I(1,x), I(1, y_1)) \leq I(I(1,x), I(1, y_2)),$$

whence

$$I(1, I(I(1,x), I(1, y_1))) \leq I(1, I(I(1,x), I(1, y_2))).$$

Then, $\overline{I(I(1,x), I(1, y_1))} \leq \overline{I(I(1,x), I(1, y_2))}$. Thus, $\bar{I}(\bar{x}, \bar{y}_1) \leq \bar{I}(\bar{x}, \bar{y}_2)$.
(I3) $\bar{I}(\bar{0}, \bar{0}) = \overline{I(I(1,0), I(1,0))} = \overline{I(0,0)} = \bar{1}$.
(I4) $\bar{I}(\bar{1}, \bar{1}) = \overline{I(I(1,1), I(1,1))} = \overline{I(1,1)} = \bar{1}$.
(I5) $\bar{I}(\bar{1}, \bar{0}) = \overline{I(I(1,1), I(1,0))} = \overline{I(1,0)} = \bar{0}$.
Thus, $\bar{I}$ is an implication on $L_{/\sim}$. □

Remark 3.6. If I satisfies (NP), the relation $\sim$ coincides with the equivalent relation. Thus, $\bar{I}$ satisfies (NP).

Proposition 3.7. Let L be a bounded lattice, I an implication on L.
(i) If I satisfies (OP), then $\bar{I}$ satisfies (OP).
(ii) If I satisfies (IP), then $\bar{I}$ satisfies (IP).
(iii) If I satisfies (EP), then $\bar{I}$ satisfies (EP).
(iv) If I satisfies (NP), then $N_{\bar{I}}(\bar{x}) = \overline{N_I}(x)$.

Proof. (i) Let $\bar{x}, \bar{y} \in L_{/\sim}$ with $\bar{x} \leq \bar{y}$. Then,

$$I(1,x) \leq I(1,y).$$

Since I satisfies (OP), we have that

$$I(I(1,x), I(1,y)) = 1.$$

Then, we obtain that

$$\bar{I}(\bar{x}, \bar{y}) = \overline{I(I(1,x), I(1,y))} = \bar{1}.$$

Conversely, let $\bar{I}(\bar{x}, \bar{y}) = \bar{1}$, for $\bar{x}, \bar{y} \in L_{/\sim}$. Then,

$$\overline{I(I(1,x), I(1,y))} = \bar{1},$$

whence $I(1, I(I(1,x), I(1,y))) = I(1,1) = 1$. Since I satisfies (OP), we have that

$$1 \leq I(I(1,x), I(1,y)),$$

whence $I(I(1,x), I(1,y)) = 1$. Then, $I(1,x) \leq I(1,y)$. Thus, $\bar{x} \leq \bar{y}$.

(ii) Let $\bar{x} \in L_{/\sim}$. since I satisfies (IP), we have that

$$\bar{I}(\bar{x}, \bar{x}) = \overline{I(I(1,x), I(1,x))} = \overline{1}.$$

(iii) Let $\bar{x}, \bar{y}, \bar{z} \in L_{/\sim}$. Since I satisfies (EP), we have that

$$\begin{aligned}
\bar{I}(\bar{x}, \bar{I}(\bar{y}, \bar{z})) &= \bar{I}(\bar{x}, \overline{I(I(1,y), I(1,z))}) \\
&= \overline{I(I(1,x), I(1, I(I(1,y), I(1,z))))} \\
&= \overline{I(1, I(I(1,x), I(I(1,y), I(1,z))))} \\
&= \overline{I(1, I(I(1,y), I(I(1,x), I(1,z))))} \\
&= \overline{I(I(1,y), I(1, I(I(1,x), I(1,z))))} \\
&= \bar{I}(\bar{y}, \overline{I(I(1,x), I(1,z))}) \\
&= \bar{I}(\bar{y}, \bar{I}(\bar{x}, \bar{z})).
\end{aligned}$$

(iv)

$$\begin{aligned}
N_{\bar{I}}(\bar{x}) = \bar{I}(\bar{x}, \overline{0}) &= \overline{I(I(1,x), I(1,0))} \\
&= \overline{I(I(1,x), 0)} = \overline{I(x,0)} = \overline{N_I(x)}.
\end{aligned}$$

The converse of the statements in Proposition 3.7 may not be true. Let us investigate the following example. □

Example 3.8. (i) Take the Reichenbach implication I_{RC} on $[0,1]$. It is clear that I_{RC} doesn't satisfy (OP) (see [1]). For any $x \in [0,1]$, since

$$\begin{aligned}
\bar{x} &= \{y \in [0,1] \mid I_{RC}(1,x) = I_{RC}(1,y)\} \\
&= \{y \in [0,1] \mid 1 - x + x = 1 - y + y\} \\
&= [0,1]
\end{aligned}$$

we have that $[0,1]_{/\sim} = \{[0,1]\}$. Thus, $\bar{I}_{RC}$ satisfies (OP).

(ii) Take again the Reichenbach implication I_{RC}. It is clear that I_{RC} doesn't satisfy (IP). Although, $\bar{I}_{RC}$ satisfies (IP), since $[0,1]_{/\sim} = \{[0,1]\}$.

(iii) Take the Rescher implication I_{RS}. It is clear that I_{RS} doesn't satisfy (NP). Also, for all $x < 1$, we have that

$$\begin{aligned}
\bar{x} &= \{y \in [0,1] \mid I_{RS}(1,x) = I_{RS}(1,y)\} \\
&= \{y \in [0,1] \mid 0 = I_{RS}(1,y)\} \\
&= \{y \in [0,1] \mid 1 > y\} = [0,1)
\end{aligned}$$

and

$$\begin{aligned}
\overline{1} &= \{y \in [0,1] \mid I_{RS}(1,1) = I_{RS}(1,y)\} \\
&= \{y \in [0,1] \mid 1 = I_{RS}(1,y)\} \\
&= \{y \in [0,1] \mid 1 \leq y\} = \{1\}
\end{aligned}$$

Thus, $[0,1]_{/\sim} = \{\{1\}, [0,1)\}$. Since $\bar{I}_{RS}$ is an implication on $[0,1]_{/\sim}$, $\bar{I}_{RS}(\overline{1}, \overline{1}) = \overline{1}$. For all $x < 1$,

$$\begin{aligned}
\bar{I}_{RS}(\overline{1}, \bar{x}) &= \overline{I_{RS}\left(I_{RS}(1,1), I_{RS}(1,x)\right)} \\
&= \overline{I_{RS}(1,0)} = \overline{0} = [0,1) = \bar{x}.
\end{aligned}$$

Then, $\bar{I}_{RS}$ satisfies (NP).

4. The set defined by the linear combination of implications

In this section, we introduce a set by the linear combination of implications on bounded lattices and investigate the properties of the set in the term of the lattice theoretical structure.

Proposition 4.1. Let L be a bounded lattice, I and J be two implications on L. Let $K^{I,J}_{a,T,S,N}$ be the linear combination of I and J. Define the set

$$\mathcal{I} = \Big\{x \in L \mid K^{I,J}_{a,T,S,N}(x,x) = 1\Big\}.$$

(i) Let $K^{I,J}_{a,T,S,N}$ be $\wedge$-distributive. Then, $\mathcal{I}$ be a meet sublattice iff $K^{I,J}_{a,T,S,N}(x,y) = K^{I,J}_{a,T,S,N}(y,x) = 1$ for all $x, y \in \mathcal{I}$.

(ii) If $K^{I,J}_{a,T,S,N}$ is V-distributive, then $\mathcal{I}$ is a join sublattice.

(iii) If L is a complete lattice and $K^{I,J}_{a,T,S,N}$ is infinitely $\vee$ -distributive, then $\mathcal{I}$ is a complete lattice.

Proof. (i) Let $\mathcal{I}$ be a meet sublattice. Then, for every $x, y \in \mathcal{I}, x \wedge y \in \mathcal{I}$. Thus,

$$K^{I,J}_{a,T,S,N}(x \wedge y, x \wedge y) = 1.$$

Since

$$\begin{aligned} 1 = K^{I,J}_{a,T,S,N}(x \wedge y, x \wedge y) &= K^{I,J}_{a,T,S,N}(x,x) \wedge K^{I,J}_{a,T,S,N}(x,y) \wedge K^{I,J}_{a,T,S,N}(y,x) \\ &\wedge K^{I,J}_{a,T,S,N}(y,y) = K^{I,J}_{a,T,S,N}(x,y) \wedge K^{I,J}_{a,T,S,N}(y,x), \end{aligned}$$

we have that $K^{I,J}_{a,T,S,N}(x,y) = K^{I,J}_{a,T,S,N}(y,x) = 1$.

The converse of the claim is clear.

(ii) Let $K^{I,J}_{a,T,S,N}$ be $\vee$ -distributive and $x, y \in \mathcal{I}$. Since

$$\begin{aligned} K^{I,J}_{a,T,S,N}(x \vee y, x \vee y) &= K^{I,J}_{a,T,S,N}(x,x) \vee K^{I,J}_{a,T,S,N}(x,y) \\ &\vee K^{I,J}_{a,T,S,N}(y,x) \vee K^{I,J}_{a,T,S,N}(y,y) \\ &= 1 \vee K^{I,J}_{a,T,S,N}(x,y) \vee K^{I,J}_{a,T,S,N}(y,x) \vee 1 = 1, \end{aligned}$$

we obtain that $x \vee y \in \mathcal{I}$. Thus, $\mathcal{I}$ is a join sublattice.

(iii) Let L be a complete lattice and $K^{I,J}_{a,T,S,N}$ be infinitely $\vee$ -distributive. since $K^{I,J}_{a,T,S,N}(0,0) = 1, 0 \in \mathcal{I}$ Also, since

$$K^{I,J}_{a,T,S,N}\left(\vee_i x_i, \vee_i x_i\right) = \vee_i \vee_j K^{I,J}_{a,T,S,N}\left(x_i, x_j\right) = 1$$

we see that $\mathrm{V}_i x_i \in \mathcal{I}$. Thus, $\mathcal{I}$ is a complete lattice. □

Remark 4.2. In Proposition 4.1 (i), if $K^{I,J}_{a,T,S,N}$ is the g-convex combination of I and J, then $\mathcal{I}$ can not be an $\wedge$ -semi lattice. Suppose that $\mathcal{I}$ is an $\wedge$ -semi lattice. Since $K^{I,J}_{a,T,S,N}(0,0) = K^{I,J}_{a,T,S,N}(1,1) = 1$, it is clear that $0, 1 \in \mathcal{I}$. By Proposition 4.1 (i), it must be $K^{I,J}_{a,T,S,N}(x,y) = K^{I,J}_{a,T,S,N}(y,x) = 1$ for all $x, y \in \mathcal{I}$. That is, $K^{I,J}_{a,T,S,N}(1,0) = K^{I,J}_{a,T,S,N}(0,1) = 1$, which is a contradiction.

Proposition 4.3. Let L be a bounded lattice, I and J two implications on L. Let $K^{I,J}_{a,T,S,N}$ be a $\vee$-distributive linear combination of I and J. If $x \in \mathcal{I}$ and $y \in L$ with $x \leq y$, then $y \in \mathcal{I}$.

Proof. Let $x \in \mathcal{I}$ and $y \in L$ with $x \leq y$. Since $K^{I,J}_{a,T,S,N}$ is $\vee$ -distributive and $K^{I,J}_{a,T,S,N}(x,x) = 1$, we have that

$$\begin{aligned} K^{I,J}_{a,T,S,N}(y,y) &= K^{I,J}_{a,T,S,N}(x \vee y, x \vee y) \\ &= K^{I,J}_{a,T,S,N}(x,x) \vee K^{I,J}_{a,T,S,N}(x,y) \vee K^{I,J}_{a,T,S,N}(y,x) \vee K^{I,J}_{a,T,S,N}(y,y) \\ &= 1, \end{aligned}$$

whence $y \in \mathcal{I}$. □

Corollary 4.4. Let L be a bounded lattice, I and J two implications on L. Let $K^{I,J}_{a,T,S,N}$ be a $\vee$-distributive g -convex combination of I and J. Then, $\mathcal{I} = L$.

Proof. It is clear that $\mathcal{I} \subseteq L$. since $K^{I,J}_{a,T,S,N}$ is the g-convex combination of I and J, $K^{I,J}_{a,T,S,N}(0,0) = 1$, whence $0 \in \mathcal{I}$. By Proposition 4.3, for any element $x \in L$ with $0 \leq x$, it must be $x \in \mathcal{I}$. Then, $L \subseteq \mathcal{I}$. Thus, $\mathcal{I} = L$.

If $K^{I,J}_{a,T,S,N}$ is a linear combination of two implications on a bounded lattice, the set $\mathcal{I}$ given in Proposition 4.1 may be an empty set. The following proposition verifies this fact. □

Proposition 4.5. Let L be a chain, I and J be two implications on a bounded lattice L. If $S = \vee, T = \wedge$, N is a strong negation on L and $a \in L \backslash \{0,1\}$, then the set $\mathcal{I}$ is empty.

Proof. Suppose that there exists an element x of $\mathcal{I}$. Then,

$$1 = K^{I,J}_{a,\wedge,\vee,N}(x,x) = (a \wedge I(x,x)) \vee (N(a) \wedge J(x,x)).$$

Since L is a chain, it is clear that $a \wedge I(x,x) = 1$ or $N(a) \wedge J(x,x) = 1$. If $a \wedge I(x,x) = 1$, we would have $a = 1$, contradiction. If $N(a \wedge J(x,x)) = 1$, we would obtain that $N(a) = 1$, which is a contradiction again. Thus, it must be $\mathcal{I} = \emptyset$. □

Proposition 4.6. Let L be a lattice of finite length, I and J be two implications on L. Let $K^{I,J}_{a,T,S,N}$ be the $\vee$-distributive linear combination of I and J. If $\mathcal{I} \neq \emptyset$, then $\mathcal{I}$ is in shape of an interval. Moreover, there exists an element $x_0 \in \mathcal{I}$ such that $\mathcal{I} = [x_0, 1]$.

Proof. Let $x, y \in \mathcal{I}$ and z be any arbitrary element with $x \leq z \leq y$. By Proposition 4.3, since $x \in \mathcal{I}$, we have that $z \in \mathcal{I}$. Thus, $\mathcal{I}$ is in shape of an interval. Since $\mathcal{I} \subseteq L$ and L is a lattice of finite length, $\mathcal{I}$ has a finite length. Thus, there must exist the least element of this interval, that is, there exists $x_0 \in \mathcal{I}$ such that $x_0 \leq y$ for all element $y \in \mathcal{I}$. Since $x_0 \leq 1$ and $x_0 \in \mathcal{I}$, by Proposition 4.3, it must be $1 \in \mathcal{I}$. Thus, we have that $x_0 \leq y \leq 1$ for all element $y \in \mathcal{I}$, that is, $\mathcal{I} = [x_0, 1]$. □

Remark 4.7. Because $K^{I,J}_{a,T,S,N}$ coincides with I when $a = 1$, the obtained results in Proposition 4.1 and Proposition 4.3 are also true for implication on a bounded lattice. That is,

(i) If an implication I is $\vee$ -distributive, then the set $\mathcal{I}$ given in Proposition 4.1 is a join sublattice.

(ii) If I is infinitely $\vee$-distributive, then the set $\mathcal{I}$ given in Proposition 4.1 forms of a complete lattice.

(iii) If the set $\mathcal{I}$ given in Proposition 4.1 is non-empty, then it is in shape of the closed interval.

(iv) Also, if $K^{I,J}_{a,T,S,N}$ is a g-convex combination of two implications, then it is clear that the set $\mathcal{I}$ is not empty.

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Commutativity Associated with Confluent Hypergeometric Differential Equation

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Abstract

In this study, commutativity conditions of general Confluent hypergeometric differential equation are considered. In the sense of theoretical results for the commutativity of second-order analogue time-varying linear systems, it is proved that the system described by a general Confluent hypergeometric differential equation has commutative pairs which depend on the parameters of the equation.

1. Introduction

The commutativity of linear time-varying analog systems has gained attraction in the last few decades. This concept has been studied by Marshal [1] for the first time for first-order systems. In addition to the exhaustive study [2] which sets the basic theory of commutativity, there is a large cycle of works in the literature on higher-order systems [3], studying commutativity of most of the second-order linear time-varying systems [4, 5].

Although most of the famous second-order linear time-varying differential systems were subjected from the commutativity point of view in [4, 5], general Confluent hypergeometric differential equation [6] is not among them. So, the purpose of this study is to study the commutativity conditions of the following general Confluent hypergeometric differential equation

$$y''(t)+\left(\frac{2A}{t}+2f'+\frac{bh'}{h}-h'+\frac{h''}{h'}\right)y'(t)$$
$$+\left[\left(\frac{bh'}{h}-h'+\frac{h''}{h'}\right)\left(\frac{A}{t}+f'\right)+\frac{A(A-1)}{t^2}+\frac{2Af'}{t}+f''+f'^2-\frac{ah'^2}{h}\right]y(t)=0. \tag{1}$$

Here, A and b are constants, f and h depend on time variable..

2. Commutativity Conditions of Second-order Systems

We assume that A be linear time-varying second-order differential system described by the following second-order differential equations:

$$a_2(t)\ddot{y}_A(t)+a_1(t)\dot{y}_A(t)+a_0(t)y_A(t)=x_A(t);\, t\geq 0. \tag{2}$$

Here, $x_A(t)$ is the independent excitation and $y_A(t)$ is the resulting response. For the unique solution of Eq. (2) for $t\geq t_0$, it is sufficient that the excitation and the time-varying coefficients $a_2(t),\ a_1(t),\ a_0(t)$ be piece-wise continuous functions of time with $a_2(t)\neq 0$.

The system modeled by Eq. (2) has a commutative pair if and only if the coefficients of the system satisfy the following equation [7]:

$$-a_2^{0.5}\frac{d}{dt}\left[a_0-\frac{1}{16a_2}(4a_1^2+3\dot{a}_2^2-8a_1\dot{a}_2+8\dot{a}_1a_2-4a_2\ddot{a}_2)\right]k=0, \tag{3}$$

where k is a constant.

3. Commutativity of General Confluent Hypergeometric Differential Equation

For Eq. (1), $a_0=\left(\frac{bh'}{h}-h'+\frac{h''}{h'}\right)\left(\frac{A}{t}+f'\right)+\frac{A(A-1)}{t^2}+\frac{2Af'}{t}+f''+f'^2-\frac{ah'^2}{h}$ and $a_1=\left(\frac{2A}{t}+2f'+\frac{bh'}{h}-h'+\frac{h''}{h'}\right)$. Because of the fact that $a_2=1,\dot{a}_2=\ddot{a}_2=0$. So, Eq. (3) becomes

$$-a_2^{0.5}\frac{d}{dt}\left[a_0-\frac{a_1^2+2\dot{a}_1}{4}\right]k=0. \tag{4}$$

Excluding the special case $k=0$, the term in the square bracket of Eq. (4) must be equal to a constant. Thus, using the values of $a_0,a_1,\dot{a}_1=\frac{-2A}{t^2}$, we obtain following expression:

$$\frac{A(A-1)-A^2+A}{t^2}+\frac{A\left(\frac{bh'}{h}-h'+\frac{h''}{h'}+2f'-2f'-\frac{bh'}{h}+h'-\frac{h''}{h'}\right)}{t}$$
$$+\left(\frac{bh'}{h}-h'+\frac{h''}{h'}\right)f'+f''+f'^2-\frac{ah'^2}{h}-\frac{1}{4}\left(2f'+\frac{bh'}{h}-h'+\frac{h''}{h'}\right)^2=constant.$$

The expressions in the numerator of rational expressions with t^2 and t in the denominator are already equal to zero. The remaining part can be written in the following form:

$$(h')^2\left(\frac{b-2a}{h}-\frac{b^2}{4h^2}-\frac{1}{4}\right)+h''\left[-\frac{2b}{4h}-\frac{h''}{4(h')^2}+\frac{2}{4}\right]+f''=constant.$$

For the special case, the above expression can be satisfied for $a=b=0,h=e^x$ and $f=\frac{1}{16}(e^{2x}-8e^x+2x^2)$. For these values, the constant is equal to zero and the main Eq. (1) can be written in the following form:

$$y''+\left(\frac{e^{2x}}{4}-2e^x+\frac{x}{2}+1+\frac{2A}{x}\right)y'+\left[(1-e^x)\left(\frac{e^{2x}}{8}-\frac{e^x}{2}+\frac{x}{4}+\frac{A}{x}\right)+\right.$$
$$\left.+\frac{A(A-1)}{x^2}+\frac{e^{2x}}{4}-\frac{e^x}{2}+\frac{1}{4}+\left(\frac{e^{2x}}{8}-\frac{e^x}{2}+\frac{x}{4}\right)\left(\frac{e^{2x}}{8}-\frac{e^x}{2}+\frac{x}{4}+\frac{2A}{x}\right)\right]y=0.$$

Acknowledgments: This study was supported by the Scientific and Technological Research Council of Turkey (TUBITAK) under the project no. 115E952.

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Rate of Convergence of Some Positive Linear Operators

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Abstract

In this work, we combine some recent studies about rates of convergence and some other approximation properties of certain recent defined linear positive operators for functions of one and two variables. We extensively give a related survey.

1. Introduction

Acu et al. [8] introduced the following integral modification of Bernstein operators:

$$K_{n,\lambda}(h;z) = (n+1)\sum_{i=0}^{n} \tilde{b}_{n,i}(\lambda;z) \int_{\frac{i}{n+1}}^{\frac{i+1}{n+1}} h(t)dt. \tag{1}$$

They generalized both Bernstein and λ-Bernstein operators and obtain some asymptotic type results. Also Cai proposed λ-Kantorovich operators (1) and the following Bézier variant of Kantorovich type λ-Bernstein operators [6]:

$$L_{n,\lambda,\beta}(h;z) = (n+1)\sum_{i=0}^{n} Q_{n,i}^{(\beta)}(\lambda;z) \int_{\frac{i}{n+1}}^{\frac{i+1}{n+1}} h(t)dt,$$

where

$$Q_{n,i}^{(\beta)}(\lambda;z) = \left[J_{n,i}^{(\beta)}(\lambda;z)\right]^{\beta} - \left[J_{n,i+1}^{(\beta)}(\lambda;z)\right]^{\beta} \text{ and} J_{n,i}^{(\beta)}(\lambda;z) = \sum_{j=i}^{n} \tilde{b}_{n,j}(\lambda;z).$$

Approximation properties of λ Kantorovich operators with shifted knots were introduced by Rahman et al. [7] as

$$K_{n,\lambda}^{(\alpha,\beta)}(h;z) = \left(\frac{n+\beta}{n}\right)^{n} (n+\beta+1)\sum_{i=0}^{n} \hat{b}_{n,i}^{(\alpha,\beta)}(\lambda;z) \int_{\frac{i+\alpha}{n+\beta+1}}^{\frac{i+\alpha+1}{n+\beta+1}} h(t)dt.$$

A family of GBS operators of bivariate tensor product of λ-Kantorovich type was constructed in [18]. Many researchers established some Kantorovich type operators by modifying Bernstein type operators to have better error estimation [7, 9, 14].

Acu et al. extented λ-Bernstein operators to introduce a new type genuine Bernstein-Durrmeyer operators in [2]:

$$U_{n,\lambda}^{\rho}(h;z) = \sum_{i=1}^{n-1} \int_0^1 \left[\frac{t^{i\rho-1}(1-t)^{(n-i)\rho-1}}{B(i\rho,(n-i)\rho)} h(t)dt\right] \tilde{b}_{n,i}(\lambda;z)$$

$$+h(0)\tilde{b}_{n,0}(\lambda;z) + h(1)\tilde{b}_{n,n}(\lambda;z).$$

In [3], Qi et al. constructed λ-Szász-Mirakian operators

$$M_{n,\lambda}(h;z) = \sum_{i=0}^{n} \widetilde{m}_{n,i}(\lambda;z)\, h\left(\frac{i}{n}\right),\ z \in [0,\infty),$$

The Schurer polynomials $s_{n,i}(z)$ were introduced by Frans Schurer in [4] as

$$s_{n,i}(z) = \binom{n+\alpha}{i} z^i\, (1-z)^{n+\alpha-i} \qquad (i = 0,1,\dots,n+\alpha),$$

where α is a non-negative integer. The operators generated by these polynomials, which are called Schurer operators, were introduced to extend the domain of function from $C[0,1]$ to $C[0,1+\alpha]$. Some relevant work about Schurer polynomials and Schurer operators may be found in [11, 4]. λ-bases modified by adding parameter α to introduce the following new bases in [12]:

$$\tilde{s}_{n,0}(\lambda;z) = s_{n,0}(z) - \frac{\lambda}{n+\alpha+1} s_{n+1,1}(z),$$

$$\tilde{s}_{n,i}(\lambda;z) = s_{n,i}(z) + \frac{\lambda}{(n+\alpha)^2-1}\big[(n+\alpha-2i+1)s_{n+1,i}(z)$$

$$-(n+\alpha-2i-1)s_{n+1,i+1}(z)\big]\ (i = 1,2\dots,n+\alpha-1),$$

$$\tilde{s}_{n,n+\alpha}(\lambda;z) = s_{n,n+\alpha}(z) - \frac{\lambda}{n+\alpha+1} s_{n+1,n+\alpha}(z),$$

where shape parameter $\lambda \in [-1,1]$. In the same work, the λ-Schurer operators were introduced and some approximation and statistical approximation properties were studied.

2. Main Approximation Results

The results that are fiven given in this part are from the papers [12,16]. We remember the following Lemma [12].

$$S_{n,\alpha}^{\lambda}(1;z) = 1;$$

$$S_{n,\alpha}^{\lambda}(t;z) = \frac{n+\alpha}{n}z + \frac{1-2z+z^{n+\alpha+1}-(1-z)^{n+\alpha+1}}{n(n+\alpha-1)}\lambda;$$

$$S_{n,\alpha}^{\lambda}(t^2;z) = \frac{(n+\alpha)^2}{n^2}z^2 + \frac{n+\alpha}{n^2}(z-z^2)$$

$$+\frac{2(n+\alpha)z-1-4(n+\alpha)z^2+(2(n+\alpha)+1)z^{n+\alpha+1}+(1-z)^{n+\alpha+1}}{n^2(n+\alpha-1)}\lambda.$$

Let $z \in [0,1]$, α be a non-negative integer, $\lambda \in [-1,1]$ and $\psi_z = t - z$, then we have the following central moments:

$$K_{n,\alpha}^{\lambda}(\psi_z;z) = \frac{1-2z}{2(n+\alpha+1)} + \frac{1-2z+z^{n+\alpha+1}-(1-z)^{n+\alpha+1}}{(n+\alpha)^2-1}\lambda;$$

$$K_{n,\alpha}^{\lambda}(\psi_z^2;z) = \frac{z-z^2}{n+\alpha+1} + \frac{1+6z\lambda}{3(n+\alpha+1)^2}$$

$$-\frac{z(1-2z)+z^{n+\alpha+1}(z-1)-z(1-z)^{n+\alpha+1}}{(n+\alpha)^2-1}2\lambda - \frac{4(n+\alpha)z^2}{(n+\alpha+1)((n+\alpha)^2-1)}\lambda.$$

Let $\mathscr{J} = [0,1] \times [0,1]$ and $(x,y) \in \mathscr{J}$, then we consider bivariate λ-Kantorovich operators

$$\overline{K}_{n,m}^{\lambda_1,\lambda_2}(f;x,y) =$$
$$(n+1)(m+1)\sum_{k_1=0}^{n}\sum_{k_2=0}^{m}\tilde{b}_{n,k_1}(\lambda_1;x)\tilde{b}_{m,k_2}(\lambda_2;y)\int_{\frac{k_1}{n+1}}^{\frac{k_1+1}{n+1}}\int_{\frac{k_2}{m+1}}^{\frac{k_2+1}{m+1}} f(u,v)dudv$$

for $f(x,y) \in C(\mathscr{J})$, where Bézier bases $\tilde{b}_{n,k_1}(\lambda_1;x)$ and $\tilde{b}_{m,k_2}(\lambda_2;x)$ $(\lambda_1,\lambda_2 \in [-1,-1])$.

Theorem 1. Let $f(x,y) \in C(I)$, then the following inequality holds

$$\left|\overline{K}_{n,m}^{\lambda_1,\lambda_2}(f;x,y) - f(x,y)\right| \le 2\left[\omega_1\left(f;C^{1/2}(n,\lambda_1)\right) + \omega_2\left(f;C^{1/2}(m,\lambda_2)\right)\right],$$

where $C(n,\lambda_1)$ and $C(m,\lambda_2)$.

Let $\mathbb{K}_{n,n}^{\lambda_1,\lambda_2}(f;x,y)$ be a sequence of positive linear operators acting from $C_B(\mathscr{J})$ into $C(\mathscr{J})$ defined by $\mathbb{K}_{n,n}^{\lambda_1,\lambda_2}(f;x,y) = (1+x_n)\overline{K}_{n,n}^{\lambda_1,\lambda_2}(f;x,y)$.

Theorem 2. Let $f \in Lip_M(\hat{\beta}_1,\hat{\beta}_2)$. Then, for all $(x,y) \in \mathscr{J}_{ab}$, we have

$$|\overline{K}_{n,m}^{\lambda_1,\lambda_2}(f;x,y) - f(x,y)| \le MC^{\hat{\beta}_1/2}(n,\lambda_1)C^{\hat{\beta}_2/2}(m,\lambda_2),$$

where $C(n,\lambda_1)$ and $C(m,\lambda_2)$ are defined in [12].

Let $f(x,y) \in C^1(\mathcal{J}_{ab})$, then we have

$$\left|\bar{K}_{n,m}^{\lambda_1,\lambda_2}(f;x,y) - f(x,y)\right| \leq C^{\frac{1}{2}}(n,\lambda_1) \parallel f_x(x,y) \parallel_{C(\mathcal{J}_{ab})} + C^{\frac{1}{2}}(m,\lambda_2) \parallel f_y(x,y) \parallel_{C(\mathcal{J}_{ab})}.$$

Theorem 3. Let $A = (a_{nk})$ be a weighted non-negative regular summability matrix. Assume that following condition yields:

$$w(f,\varphi_n) = [stat_A, q_n] - o\,(u_n) \text{ on } [0,1], \text{ where } \varphi_n = \sqrt{\parallel K_{n,\lambda}((s-x)^2;x) \parallel_{C[0,1]}}.$$

Then for every bounded $f \in C[0,1]$ we have $\parallel K_{n,\lambda}(f;x) - f(x) \parallel_{C[0,1]} = [stat_A, q_n] - o\,(u_n)$.

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Some λ Bernstein type Operators

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Abstract

Parametric representations of surfaces and curves have extensively been used in computer aided geometric design and computer graphics for modeling surfaces. It is important which basis functions are used if we want to preserve the shape of the curve or surface when we demonstrate a parametric surface or curve. This is why Bezier curve and surface representation have an important role in computer graphics. Bernstein basis functions are used to construct classical Bezier curves since they have a simple structure to use. A new type Bezier bases with λ shape parameters have recently been defined [4, Cai et al., 2018]. In this work we summarize some recent studies about λ Bernstein type operators.

1. Introduction

For any continuous function $f(x)$ defined on $C[0,1]$, Bernstein polynomials of order n are given by

$$B_n(f;x) = \sum_{i=0}^{n} f\left(\frac{i}{n}\right) b_{n,i}(x) \qquad (x \in [0,1]), \tag{1}$$

where the Bernstein basis functions $b_{n,i}(x)$ are defined by

$$b_{n,i}(x) = \binom{n}{i} x^i (1-x)^{n-i} \qquad (i = 0, \ldots, n) \ [6].$$

Many generalizations of these operators have been introduced in literature. Stancu [2] constructed a generalization of Bernstein polynomials with the help of two parameters α and β such that $0 \leq \alpha \leq \beta$, as follows:

$$S_{n,\alpha,\beta}(f;x) = \sum_{i=0}^{n} f\left(\frac{i+\alpha}{n+\beta}\right) \binom{n}{i} x^i (1-x)^{n-i} \qquad (x \in [0,1]). \tag{2}$$

When $\alpha = \beta = 0$, then we get the classical Bernstein polynomials. The operators defined in (2) are called Bernstein–Stancu operators. For some recent work, we refer to [1-7].

In the recent past, Cai et al. [2] presented a new construction of Bernstein operators with the help of Bézier bases with shape parameter λ and called it λ-Bernstein operators, which are defined by

$$B_n^{\lambda}(f;x) = \sum_{i=0}^{n} f\left(\frac{i}{n}\right) \tilde{b}_{n,i}(\lambda;x) \qquad (n \in \mathbb{N}) \tag{3}$$

where $\tilde{b}_{n,i}(\lambda;x)$ are Bézier bases with shape parameter λ (see [3]), defined by

$$\tilde{b}_{n,0}(\lambda;x) = b_{n,0}(x) - \frac{\lambda}{n+1} b_{n+1,1}(x),$$

$$\tilde{b}_{n,i}(\lambda;x) = b_{n,i}(x) + \frac{n-2i+1}{n^2-1} \lambda b_{n+1,i}(x) - \frac{n-2i-1}{n^2-1} \lambda b_{n+1,i+1}(x), \qquad i = 1,2 \dots, n-1,$$

$$\tilde{b}_{n,n}(\lambda;x) = b_{n,n}(x) - \frac{\lambda}{n+1} b_{n+1,n}(x), \tag{4}$$

in this case $\lambda \in [-1,1]$ and $b_{n,i}(x)$ are the Bernstein basis functions. By taking the above operators into account, they established various approximation results, namely, Korovkin- and Voronovskaja-type Theorems, rate of convergence via Lipschitz continuous functions, local approximation and other related results. In the same year, Cai [4] generalized λ-Bernstein operators by constructing the Kantorovich-type λ-Bernstein operators, as well as its Bézier variant, and studied several approximation results. Later, various approximation properties and asymptotic type results of the Kantorovich-type λ-Bernstein operators have been studied by Acu et al. [5]. Very recently, Özger [14] obtained statistical approximation for λ-Bernstein Kantorovich operators including a Voronovskaja-type Theorem in statistical sense.

The Bernstein operators are some of the most studied positive linear operators which were modified by many authors, and we are mentioning some of them [10-17].

Suppose that α and β are two non-negative parameters such that $0 \leq \alpha \leq \beta$. Then, the Stancu-type modification of λ-Bernstein operators $B_{n,\alpha,\beta}^{\lambda}(f;x): C[0,1] \longrightarrow C[0,1]$ is defined by

$$B_{n,\alpha,\beta}^{\lambda}(f;x) = \sum_{i=0}^{n} f\left(\frac{i+\alpha}{n+\beta}\right) \tilde{b}_{n,i}(\lambda;x) \tag{5}$$

for any $n \in \mathbb{N}$ and we call it Stancu-type λ-Bernstein operators or λ-Bernstein–Stancu operators.

We calculate the moments, prove global approximation formula for some λ-Bernstein type operators in [1,10]. The local direct estimate of the rate of convergence by Lipschitz-type function involving two parameters for λ-Bernstein–Stancu operators is investigated. We establish quantitative Voronovskaja-type Theorem for our operators.

Kantorovich type operators have been widely studied by many researchers. λ-Bernstein operators were also modified to define λ-Kantorovich operators by Acu et al. [5] as

$$K_{n,\lambda}(f;x) = (n+1)\sum_{i=0}^{n} \tilde{b}_{n,i}(\lambda;x)\int_{i/(n+1)}^{(i+1)/(n+1)} f(t)dt \tag{6}$$

2. Main Results

The following results were obtain in [1,4,5,14].

We have following equalities for λ-Kantorovich operators:

$$K_{n,\lambda}(1;x) = 1;$$

$$K_{n,\lambda}(t;x) = x + \frac{1-2x}{2(n+1)} + \frac{1-2x+x^{n+1}-(1-x)^{n+1}}{n^2-1}\lambda;$$

$$K_{n,\lambda}(t^2;x) = x^2 + \frac{3nx(2-3x)-3x^2+1}{3(n+1)^2} + \frac{x^{n+1}-x+n(x^{n+1}+x-2x^2)}{(n-1)(n+1)^2}2\lambda.$$

Consider λ-Kantorovich operators in (6), λ-Bernstein operators defined in [4]. Consider λ-Bernstein, λ-Kantorovich and λ-Durrmeyer operators as

$$B_{n,\lambda}(f;x) = \sum_{i=0}^{n} \tilde{b}_{n,i}(\lambda;x)B_{i,n}(f); \quad B_{i,n}(f) = f\left(\frac{i}{n}\right);$$

$$K_{n,\lambda}(f;x) = \sum_{i=0}^{n} \tilde{b}_{n,i}(\lambda;x)K_{i,n}(f); \quad K_{i,n}(f) = (n+1)\int_{i/(n+1)}^{(i+1)/(n+1)} f(t)dt;$$

$$D_{n,\lambda}(f;x) = \sum_{i=0}^{n} \tilde{b}_{n,i}(\lambda;x)D_{i,n}(f); \quad D_{i,n}(f) = (n+1)\int_0^1 b_{n,i}(t)f(t)dt.$$

Let $F: C(I) \to \mathbb{R}$ be a positive linear functional such that $F(1) = 1$. If we denote $b^F = F(x)$ and

$$\mu_i^F = \frac{1}{i}F(e_1 - b^F e_0)^i,$$

then we have $\mu_0^F = 1$, $\mu_1^F = 0$ and $\mu_2^F = \frac{1}{2}[F(e_2) - (b^F)^2]$.

Let $f, f'' \in C[0,1]$. We have the following estimates for difference of λ-Bernstein and λ-Kantorovich operators:

- $\left|B_{n,\lambda}(f;x) - K_{n,\lambda}(f;x)\right| \le \|f''\|\alpha_n(x,\lambda) + \omega_1(f, 1/(2n+2))$;
- $\left|B_{n,\lambda}(f;x) - K_{n,\lambda}(f;x)\right| \le 3\omega_2\big(f, 1/(2\sqrt{6}n + 2\sqrt{6})\big) + 5\sqrt{6}\omega_1\big(f, 1/(2\sqrt{6}n + 2\sqrt{6})\big)$.

Let $f, f'' \in C[0,1]$. We have the following estimates for difference of λ-Bernstein and λ-Durrmeyer operators:

- $\left|B_{n,\lambda}(f;x) - D_{n,\lambda}(f;x)\right| \leq \|f''\| \beta_n(x,\lambda) + \omega_1(f, 1/(n+2))$;
- $\left|B_{n,\lambda}(f;x) - D_{n,\lambda}(f;x)\right| \leq 3\omega_2\left(f, \beta_n^{1/2}(x,\lambda)\right) + 5(n+2)\beta_n^{-1/2}(x,\lambda)\omega_1\left(f, \beta_n^{1/2}(x,\lambda)\right)$.

For $x \in [0,1]$, the moments of Stancu type λ-Bernstein operators are given as:

$$B_{n,\alpha,\beta}^{\lambda}(1;x) = 1;$$

$$B_{n,\alpha,\beta}^{\lambda}(t;x) = \frac{\alpha+nx}{n+\beta} + \lambda\left[\frac{1-2x+x^{n+1}+(\alpha-1)(1-x)^{n+1}}{(n+\beta)(n-1)} + \frac{\alpha x(1-x)^n}{n+\beta}\right];$$

$$B_{n,\alpha,\beta}^{\lambda}(t^2;x) = \frac{1}{(n+\beta)^2}\{n(n-1)x^2 + (1+2\alpha)nx + \alpha^2\}$$

Theorem. Let $C[0,1]$ denote the space of all real-valued continuous functions on $[0,1]$ endowed with the supremum norm. Then

$$\lim_{n\to\infty} B_{n,\alpha,\beta}^{\lambda}(f;x) = f(x) \qquad (f \in C[0,1])$$

$$\lim_{n\to\infty} K_{n,\lambda}(f;x) = f(x) \qquad (f \in C[0,1])$$

uniformly in [0,1].

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Fractional-Order PID speed controller based on the conformable derivative for the DC motor

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Abstract

In this study, fractional-order PID speed controller based on the conformable derivative for the DC motor is proposed and studied in this paper. A fractional-order PID is a PID whose derivative order is fractional number rather than integer. The simulation comparisons presented in this study is shown that the fractional order PID controller has proven to perform better than a traditional integer order PID controller if properly designed.

Keywords: DC motor, fractional-order PID (FOPID), PID, controller, conformable derivative

1. Introduction

DC motors have various applications such as printers, fuel pumps, and electric vehicles in our daily life and solar airplane, electric aircraft, and propulsion systems in spacecraft applications [1]. Also, DC motors are commonly used in the industry. Therefore, position and speed control of these systems are required to high accuracy [2].

In the literature, speed control of DC Motor has been commonly applied in a various applications. To control the speed of a DC motor, there are many methods that are available. These controllers can be: proportional integral (PI), proportional integral derivative (PID) Fuzzy Logic Controller (FLC) or sliding mode controller (SMC) [4]. Through current control method, the proportional – integral – derivative (PID) controller is one of the most preferred methods because of easy handling and simple dynamics. PID controller has taken a significant part in solution of control theory problems.

Fractional calculus is a subject of calculus that involves noninteger order differential and integral operators. Many mathematicians contributed to the development of fractional calculus, therefore many definitions for the fractional derivative are available. The most popular definitions are Riemann-Liouville and Caputo definition of fractional derivatives.

In this study, we use a new definition of fractional derivative called the conformable fractional derivative given by paper in [11].

2. Mathematical Modeling of DC Motor:

The electrical and mechanical equations of the DC motor that is used in this study can be given below:

$$T_m = K_T i, \tag{1.1}$$

$$v_{emf} = K_B \frac{d\theta}{dt}, \tag{1.2}$$

$$u - v_{emf} = L\frac{di}{dt} + Ri, \tag{1.3}$$

$$J\frac{d^2\theta}{dt^2} = \sum T_{m_i} = -B\frac{d\theta}{dt} + K_T i. \tag{1.4}$$

If we substitute (1.2) in (1.3), we get

$$u = L\frac{di}{dt} + Ri + K_B \frac{d\theta}{dt} \tag{1.5}$$

Equations 1.4 and 1.5 generate the two fundamental differential equations. State space form is

$$\begin{cases} \dfrac{d^2\theta}{dt^2} = -\dfrac{B}{J}\dfrac{d\theta}{dt} + \dfrac{K_T}{J}i \\ \dfrac{di}{dt} = -\dfrac{R}{L}i - \dfrac{K_B}{L}\dfrac{d\theta}{dt} + \dfrac{1}{L}u \end{cases} \tag{1.6}$$

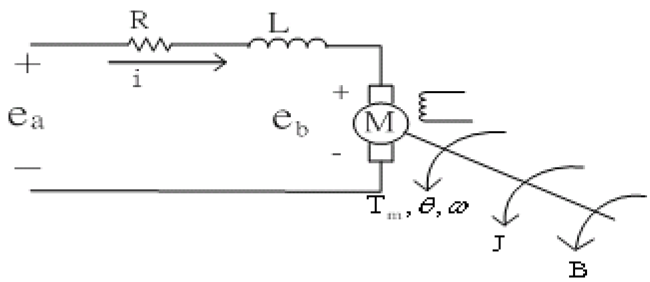

Figure 1 The control Equivalent Circuit of DC Motor

where i is motor current T_m is the torque, K_T is the torque constant, v_{emf} is the induced electromotive force, $d\theta\ /\ dt$ is the angular shaft velocity, K_B is a motor property dependent constant related to physical properties of the motor, B is the viscous friction, u is the control signal, R is the resistance, L is the inductance, and J is the moment of inertia. [3]

Table 1. System Parameters

Parameter	Value
J	5.0872e-07$kg.m^2$
K_T	0.00535Nm/A
K_B	0.00535 VS/rad
B	$1.4782e-06Nms/rad$
R	2.5760 Ω
L	0.014499 mH

2. Conformable Derivative

Definition 1. Let $a \in (n, n+1]$ and f be an $n-$ differentiable function at t, where $t > 0$ Then the conformable fractional derivative of f of order α is defined as

$$T_a(f)(t) = \lim_{\varepsilon \to 0} \frac{f^{([a]-1)}\left(t+\varepsilon t^{([a]-a)}\right) - f^{([a]-1)}(t)}{\varepsilon}$$

where $[\alpha]$ is the smallest integer greater than or equal to α.

Remark 1. As a consequence of Definition1, one can easily show that

$$T_a(f)(t) = t^{([a]-a)} f^{[a]}(t)$$

where $a \in (n, n+1]$ and f is $(n+1)$ differentiable at $t > 0$.

Theorem 2. Let $a \in (n, n+1]$ and $f; g$ be $\alpha -$ differentiable at a $t > 0$. Then

1. $T_a(af + bg) = aT_a(f) + bT_a$(g), for all $a, b \in R$.
2. $T_a(t^p) = pt^{p-a}$, for all $p \in R$.
3. $T_a(\lambda) = 0$, for all constant functions $f(t) = \lambda$.
4. $T_a(fg) = fT_a(g) + gT_a(f)$.
5. $T_a\left(\frac{f}{g}\right) = \frac{gT_a(f)+fT_a(g)}{g^2}$.

3. Fractional-Order PID Controller Design

The common form of a fractional order PID controller is the PID^{μ} controller. Involving a differentiator of order μ where μ can be any real number. The transfer function of such a controller has the form

$$G(s) = K_p + {}^{K_i}/_{s} + K_t s^{\mu} \qquad \mu > 0$$

The control input can be formulated in the following,

$$u(t) = K_p e(t) + K_i \int_0^t e(t) + K_t D^{\mu} e(t)$$

where $e(t)$ is tracking error and;

$$e(t) = y_r(t) - y(t)$$

where $e(t)$ is tracking error, $y_r(t)$ is command signal and $y(t)$ is output signal.

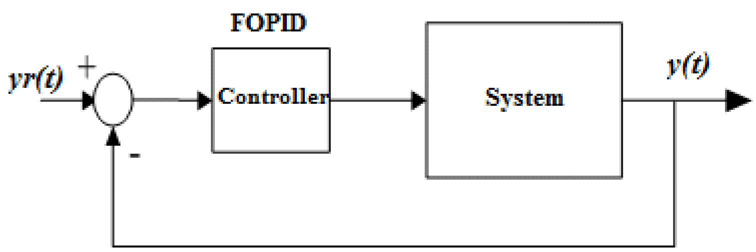

Fig. 2. FOPID DC motor speed control system block diagram

4. Results and Discussion

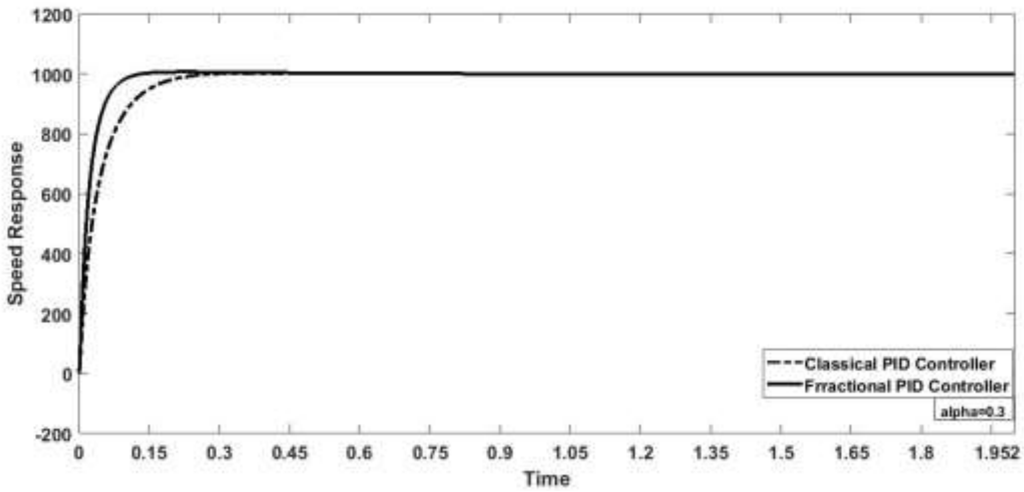

Figure 3. Step response for classical and fractional PID controller and set point 1000rpm, alpha=0,3

In Figure (3), the response of the speed response is shown for the case where the alpha is 0.3. The classical and fractional PID controller have reached the desired reference value as seen in the figure. Although, the fractional PID controller reached the desired reference value at the time of t = 0, 13, the classical PID controller reached the desired reference value at the time of t = 0:35. These results show that the designed fractional PID controller performs better and faster than the classical PID controller.

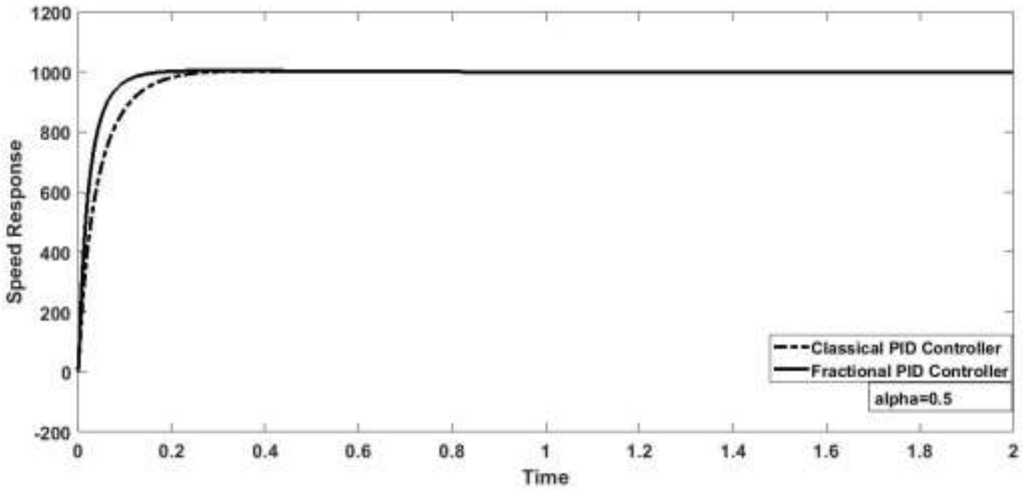

Figure 4. Step response for classical and fractional PID controller and set point 1000rpm, alpha=0,5

In Figure (4), the response of the speed response is shown for the case where the alpha is 0.5. The classical and fractional PID controller, as in the case where the load value is 0.3, have reached the desired reference value. As shown in the same way, performance of second order ADFTSMC is better than first order ADFTSMC.

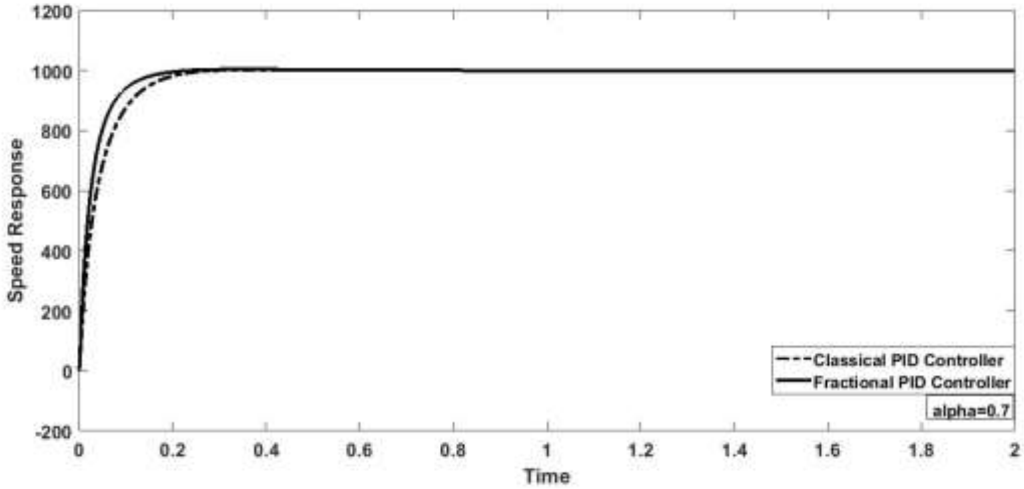

Figure 5. Step response for classical and fractional PID controller and set point 1000rpm, alpha=0,7

In Figure (5), the response of the speed response is shown for the case where the alpha is 0.7. The classical and fractional PID controller, designed as in the case where the alpha is 0,3 and 0,5, have reached the desired reference value. As shown in the same way, performance of second order ADFTSMC is better than first order ADFTSMC.

6. Conclusion

In recent years, DC motors are commonly used in industry. Therefore, position and speed control of DC motors are required to high accuracy. For controlling a DC motor speed, FOPID controller is designed. The experimental results confirm that the designed controller provides suitable tracking performance.

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A Study On The Trigonometric Approximation in Morrey Spaces Using Matrix Methods

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Abstract

In this study, we investigate the error of trigonometric approximation using different two matrix methods and prove some trigonometric approximation theorems with degree of $n^{-\beta}$ in Morrey spaces. Furthermore, we present an examples explaining the error of approximation in these spaces.

Key Words: Trigonometric approximation, matrix method, error of approximation, Morrey space.

1. Introduction

Morrey space, introduced by C. B. Morrey in [1], is one of the important generalizations of the well-known Lebesgue space. In Lebesgue spaces, several researchers obtained results related to approximation of trigonometric polynomials using matrix methods in [2]–[5]. In Morrey spaces, some results about approximation theory were obtained using different methods in [6]–[10]. The direct and inverse theorems and their improved versions were obtained by Israfilov and Tozman in [7], [8]. In [9], the problem of the deviation of a function f from de la Vallée-Poussin sums of its Fourier series were investigated in Morrey spaces. In [10], some results on the structure, regularity, and density properties of the generalized Morrey space were given. Morrey norm given in [10] have an important role in approximation theory. This approximation was sketched in [10].

In these studies, error of approximation to functions in Morrey spaces using Matrix methods were not examined.

In the present study, we focus on some approximation problems by means of special summability methods and obtain the trigonometric approximation results using different two matrix methods in Morrey spaces. Also, we discuss the error of approximation for given methods with an application.

We will give some basic definitions and notations before moving on to the main results.

Let $\mathrm{T} = [0,2\pi], 0 \leq \alpha \leq 2$ and $1 < p < \infty$. Then the Morrey space $L^{p,\alpha}(\mathrm{T})$ is defined as the set of functions $f \in L^p_{loc}(\mathrm{T})$ such that $\|f\|_{L^{p,\alpha}(\mathrm{T})} < \infty$, where

$$\|f\|_{L^{p,\alpha}(\mathrm{T})} := \left\{ \sup_I \left(\frac{1}{|I|^{1-\frac{\alpha}{2}}} \right) \int_I |f(t)|^p |dt| \right\}^{\frac{1}{p}}.$$

Here the supremum is taken with respect to all the intervals $I \subset \mathrm{T}$ and $|I|$ denotes the length of I.

This space turns into Lebesgue space $L^p(\mathrm{T})$ if we choice $\alpha = 2$. Detailed information about the Morrey spaces, basic properties of these spaces and including some generalizations can be found in [11]–[16].

Let $f \in L^{p,\alpha}(\mathrm{T}), 0 \leq \alpha \leq 2$ and $1 < p < \infty$. Then the modulus of continuity $\omega_{p,\alpha}(f,\delta): [0,\infty) \to [0,\infty)$ of the function $f \in L^{p,\alpha}(\mathrm{T})$ is defined as

$$\omega_{p,\alpha}(f,\delta) = sup\|\Delta_h f\|_{L^{p,\alpha}(\mathrm{T})}, \qquad \delta \geq 0,$$

where

$$\Delta_h(f;x) = \frac{1}{h}\int_0^h |f(x+t) - f(x)|dt.$$

The Hardy Littlewood maximal function of f is defined as

$$M(f;x) := \sup_I \frac{1}{I}\int_I |f(t)|dt, \qquad x \in \mathrm{T}$$

where $I \subset \mathrm{T}$ is any open interval.

Let $0 < \beta \leq 1, 0 \leq \alpha \leq 2$ and $1 < p < \infty$. Then, in Morrey spaces, Lipschitz class $Lip_{\alpha,p}(\beta)$ is defined as

$$Lip_{\alpha,p}(\beta) := \{f \in Lip_{\alpha,p}(\mathrm{T}): \omega_{p,\alpha}(f,\delta) = O(\delta^\beta), \delta > 0\}.$$

Let $0 \leq \alpha \leq 2, 1 < p < \infty$ and $f \in L^{p,\alpha}(\mathrm{T})$.

The Fourier series and the conjugate Fourier series of $f \in L^{p,\alpha}(\mathrm{T})$ can be given as

$$f(x) \sim \left(\frac{a_0(f)}{2}\right) + \sum_{k=1}^{\infty}(a_k(f)coskx + b_k(f)sinkx) \tag{1}$$

and

$$\tilde{f}(x) \sim \left(\frac{a_0(f)}{2}\right) + \sum_{k=1}^{\infty}(a_k(f)coskx - b_k(f)sinkx)$$

respectively. Here, $\tilde{f}$ is defined as

$$\tilde{f}(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{f(t)}{2tan\left(\frac{t-x}{2}\right)}dt, \tag{2}$$

where $a_0(f), a_k(f), b_k(f), k = 1,2,\ldots,$ are Fourier coefficients of f. The nth partial sum of the series (1) will be denoted by $S_n(f;x), (n = 0,1,2,\ldots)$ at the point x, that is,

$$S_n(f;x) = \sum_{k=0}^{n} U_k(f;x), \; n = 0,1,2, \ldots,$$

where

$$U_0(f;x) := \frac{a_0}{2} \quad \text{and} \quad U_k(f;x) := a_k(f)coskx + b_k(f)sinkx, \quad k = 1,2, \ldots.$$

We denote by $A \equiv (a_{n,k})$ a lower triangular regular matrix with nonnegative entries and let $S_n^{(A)}$ $(n = 0,1,\ldots)$ be the row sums of this matrix, that is

$$S_n^{(A)} = \sum_{k=0}^{n} a_{n,k}.$$

The matrix methods τ_n and T_n used in this study, are defined

$$\tau_n(f;x) := \sum_{k=0}^{n} a_{n,k}\, S_k(f;x), n = 0,1,2,\ldots \quad (3)$$

and

$$T_n(f;x) = \sum_{k=0}^{n} a_{n,n-k}\, S_k(f;x), n = 0,1,2,\ldots \quad (4)$$

respectively.

A nonnegative sequence $u := (u_n)$ is called almost monotone decreasing (increasing), if there exists a constant $K := K(u)$, depending on the sequence u only, such that $u_n \leq K u_m$ $(K u_n \geq u_m)$ for all $n \geq m$. Such sequences will be denoted by $u \in AMDS$ $(u \in AMIS)$.

Let

$$A_{n,k} := \frac{1}{k+1} \sum_{i=0}^{k} a_{n,i}.$$

If $\{A_{n,k}\} \in$ AMDS $(\{A_{n,k}\} \in AMIS)$, then we will say that $\{a_{n,k}\}$ is an almost monotone decreasing (increasing) mean sequence with respect to $k = 1, 2, \ldots, n$ for all n. Briefly, we will write $\{a_{n,k}\} \in$ AMDMS $(\{a_{n,k}\} \in AMIMS)$.

Let

$$A^{*}_{n,k} := \frac{1}{k+1} \sum_{i=n-k}^{n} a_{n,i}.$$

If $\{A_{n,k}\} \in$ AMDS $(\{A_{n,k}\} \in AMIS)$, then we will say that $\{a_{n,k}\}$ is an almost monotone decreasing (increasing) upper mean sequence with respect to $k = 1, 2, \ldots, n$ for all n. Briefly, we will write $\{a_{n,k}\} \in$ AMDUMS $(\{a_{n,k}\} \in AMIUMS)$.

The operator Δ_k is defined by $\Delta_k a_{n,k} = a_{n,k} - a_{n,k+1}$. The relation $\preccurlyeq$ is defined as "$A \preccurlyeq B$ $\Leftrightarrow$there exists a positive constant C, independent of essential parameters, such that $A \leq CB$". The notation O $(big\ O)$ is defined as "$f = O(g) \Leftrightarrow f \leq cg$, here c is a positive constant".

2. Auxiliary Results

Lemma 1. Let $0 \leq \alpha \leq 2, 1 < p < \infty$ and $f \in L^{p,\alpha}(\mathrm{T})$. Then the estimate

$$\|\tilde{f}\|_{L^{p,\alpha}(\mathrm{T})} \preccurlyeq \|f\|_{L^{p,\alpha}(\mathrm{T})}$$

is valid.

Lemma 2. Let $0 \leq \alpha \leq 2$ and $1 < p < \infty$. If $f \in Lip_{\alpha,p}(1)$ then f is absolutely continuous and $f' \in L^{p,\alpha}(\mathrm{T})$.

Lemma 3. [7] Let $0 \leq \alpha \leq 2, 1 < p < \infty$ and $f \in L^{p,\alpha}(\mathrm{T})$. Then the estimate

$$\|S_n(f)\|_{L^{p,\alpha}} \preccurlyeq \|f\|_{L^{p,\alpha}}$$

is valid.

Lemma 4. Let $0 \leq \alpha \leq 2$ *and* $1 < p < \infty$ *and* $f \in Lip_{\alpha,p}(1)$. Then the estimate
$$\|S_n(f) - \sigma_n(f)\|_{L^{p,\alpha}} = O(n^{-1})$$
holds for $n = 1,2,\ldots$.

Lemma 5. Let $0 < \beta \leq 1$ *and* $1 < p < \infty$ *and* $f \in Lip_{\alpha,p}(\beta)$. Then, the estimate
$$\|f - S_n(f)\|_{L^{p,\alpha}} = O(n^{-\beta})$$
is valid $n = 1,2,\ldots$.

Lemma 6. [5] Let $0 < \beta < 1$ *and* $A = (a_{n,k})$ be a lower triangular matrix with $\left|S_n^{(A)} - 1\right| = O(n^{-\beta})$. $(a_{n,k}) \in AMIMS$ or $(a_{n,k}) \in AMDMS$ and $(n+1)a_{n,0} = O(1)$, then
$$\sum_{k=0}^{n}(k+1)^{-\beta}a_{n,k} = O\left((n+1)^{-\beta}\right).$$

Lemma 7. [3] Let $0 < \beta < 1$ *and* $A = (a_{n,k})$ be a lower triangular matrix with $\left|S_n^{(A)} - 1\right| = O(n^{-\beta})$. $(a_{n,k}) \in AMDUMS$ or $(a_{n,k}) \in AMIUMS$ and $(n+1)a_{n,n} = O(1)$, then
$$\sum_{k=0}^{n}(k+1)^{-\beta}a_{n,n-k} = O\left((n+1)^{-\beta}\right).$$

3. Main Results

Theorem 8. Let $f \in Lip_{\alpha,p}(\beta), 0 < \beta < 1$ *and* $A = (a_{n,k})$ be a lower triangular matrix with $|S_n^{(A)} - 1| = O(n^{-\beta})$. If one of the following conditions

(i) $(a_{n,k}) \in AMDUMS,$

(ii) $(a_{n,k}) \in AMIUMS$ *and* $(n+1)(a_{n,n}) = O(1),$

holds, then
$$\|f - T_n(f)\|_{L^{p,\alpha}(\mathrm{T})} = O\left(n^{-\beta}\right).$$

Remark. When $a_{n,n-k} = \frac{p_{n,n-k}}{P_k}$, the method T_n turns to Nörlund method given as
$$N_n(f;x) := \frac{1}{P_n}\sum_{k=0}^{n} p_{n,n-k}S_k(f;x).$$
Here
$$P_n = p_0 + p_1 + p_2 + \ldots + p_n \neq 0 \quad (n \geq 0) \ \ and\ p_{-1} = P_{-1} = 0.$$

Corollary 9. Let $f \in Lip_{\alpha,p}(\beta), 0 < \beta < 1$ *and* (p_n) be a positive sequence. If one of the following conditions

$(i)(p_n) \in AMDUMS,$

(i) $(p_n) \in AMIUMS$ and $(n+1)p_n = O(P_n)$,

holds, then

$$\|f - N_n(f)\|_{L^{p,\alpha}(\mathrm{T})} = O\left(n^{-\beta}\right).$$

Theorem 10. Let $f \in Lip_{\alpha,p}(1)$ and $A = (a_{n,k})$ be a lower triangular matrix with $|S_n^{(A)} - 1| = O(n^{-\beta})$. If following condition

$$\sum_{k=0}^{n-2} \left|\Delta_k A_{n,k}\right| = O(n^{-1})$$

holds, then

$$\|f - T_n(f)\|_{L^{p,\alpha}(\mathrm{T})} = O(n^{-1}).$$

Corollary 11. Let $f \in Lip_{\alpha,p}(1)$ and (p_n) be a positive sequence. If following condition

$$\sum_{k=0}^{n-2} \left|\Delta_k P_{n,k}\right| = O\left(\frac{P_n}{n}\right)$$

holds, then

$$\|f - N_n(f)\|_{L^{p,\alpha}(\mathrm{T})} = O(n^{-1}).$$

Theorem 12. Let $f \in Lip_{\alpha,p}(\beta), 0 < \beta < 1$ and $A = \left(a_{n,k}\right)$ be a lower triangular matrix with nonnegative entries and row sums 1. If one of the conditions

(i) $\left(a_{n,k}\right) \in AMIMS$,

(ii) $\left(a_{n,k}\right) \in AMDMS$ and $(n+1)a_{n,0} = O(1)$,

is valid, then

$$\|f - \tau_n(f)\|_{L^{p,\alpha}(\mathrm{T})} = O\left(n^{-\beta}\right).$$

Remark. When $a_{n,k} = \frac{p_{n,k}}{P_k}$, the method τ_n turns to Riesz method given as

$$R_n(f;x) := \frac{1}{P_n} \sum_{k=0}^{n} p_{n,k} S_k(f;x).$$

Here

$$P_n = p_0 + p_1 + p_2 + \ldots + p_n \neq 0 \quad (n \geq 0) \;\; and \; p_{-1} = P_{-1} = 0.$$

Corollary 13. Let $f \in Lip_{\alpha,p}(\beta), 0 < \beta < 1$ and (p_n) be a positive sequence. If one of the conditions

$(i)(p_n) \in AMIMS$,

(ii) $(p_n) \in AMDMS$ and $(n+1)p_n = O(P_n)$,

is valid, then

$$\|f - R_n(f)\|_{L^{p,\alpha}(\mathrm{T})} = O\left(n^{-\beta}\right).$$

Theorem 14. Let $f \in Lip_{\alpha,p}(1)$ and $A = (a_{n,k})$ be a lower triangular matrix with nonnegative entries and row sums 1. If following condition

$$\sum_{k=0}^{n-1} |\Delta_k A_{n,k}| = O(n^{-1})$$

holds, then

$$\|f - \tau_n(f)\|_{L^{p,\alpha}(\mathrm{T})} = O(n^{-1}).$$

Corollary 15. Let $f \in Lip_{\alpha,p}(1)$ and (p_n) be a positive sequence. If following condition

$$\sum_{k=0}^{n-1} \left|\Delta_k \frac{P_k}{k+1}\right| = O\left(\frac{P_n}{n}\right)$$

holds, then

$$\|f - R_n(f)\|_{L^{p,\alpha}(\mathrm{T})} = O(n^{-1}).$$

4. Applications

In this section, we will give an application about the error of approximation between a function and some summability methods obtained through its Fourier series. Also, we sketch the figures of these methods under assumptions of given norm.

If a periodic function f has a jump discontinuity, then this function is represented using a Fourier series. In (5), function f is a square wave function. It is a simple example that has been used in books of applied mathematics to illustrate Gibbs phenomenon analytically [17], [18].

Example. The function f defined by

$$f(t) = \begin{cases} 1, & -\pi \le t \le 0, \\ -1, & 0 < t < -\pi. \end{cases} \tag{5}$$

For all real values of t, we have $f(t + 2\pi) = f(t)$. Since it is an odd function, its Fourier series is given as

$$f(t) = \sum_{n=1}^{\infty} b_n sin\, nt,$$

where

$$b_n = \left(\frac{2}{\pi}\right) \int_0^{\pi} f(t) sin\, ntdt = \left(\frac{2}{\pi}\right)\left(\frac{-1 + (-1)^{\mathrm{n}}}{n}\right).$$

Therefore, the Fourier series of $f(t)$ is given by

$$f(t) = \left(\frac{2}{\pi}\right) \sum_{n=1}^{\infty} \left(\frac{-1 + (-1)^{\mathrm{n}}}{n}\right) sin\, nt. \tag{6}$$

The partial sum $S_n(t)$ of the series (6) is given as

$$S_n(t) = \frac{4}{\pi}\left(\sin t + \left(\frac{1}{3}\right)\sin 3t + \ldots + \left(\frac{1}{n}\right)\sin(nt)\right).$$

Cesàro mean for (6) is

$$C_n(f) = \sum_{k=1}^{n}\left(\frac{n-k}{n}\right)\left(\frac{-2}{\pi}\right)\left(\frac{-1+(-1)^k}{k}\right)\sin(kt).$$

In (3), taking $(a_{n,k})$ as below, we will calculate the matrix mean for the series (6).

$$(a_{n,k}) = \begin{bmatrix} 1 & & & & & & & & \\ 1/2 & 1/2 & & & & & & & \\ 1/3 & 1/3 & 1/3 & & & & 0 & & \\ 1/4 & 1/4 & 1/4 & 1/4 & & & & & \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 & & & & \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & \\ \frac{1}{n+1} & \frac{1}{n+1} & \frac{1}{n+1} & \frac{1}{n+1} & \cdots & \frac{1}{n+1} & \frac{1}{n+1} & \cdots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \ddots \end{bmatrix}_{nxk}$$

In $(a_{n,k})$, $\left(\frac{1}{n+1}\right)$ repeats $(n+1)$ times in the n^{th} row. For (6) and lower triangular matrix given above, the matrix mean is

$$\tau_n(f;t) := \sum_{k=0}^{n}\left(\frac{1}{n+1}\right)S_k(f;t), n = 0,1,2,\ldots.$$

For $n = 10,20$ and 40, using the definitions of $f(t), S_n(t), C_n(f;t)$ and $\tau_n(f;t)$ we plot the following figures.

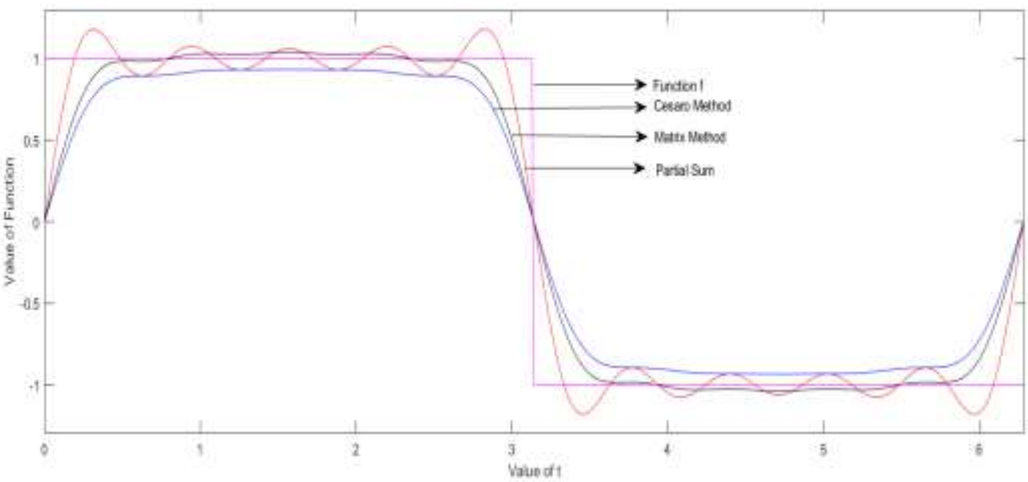

Figure 1: The figure above shows the oscillation and overshoots for n=10.

If we take $n = 20, 40$ instead of $n = 10$, then we have the following figures: note that the errors of approximation decrease.

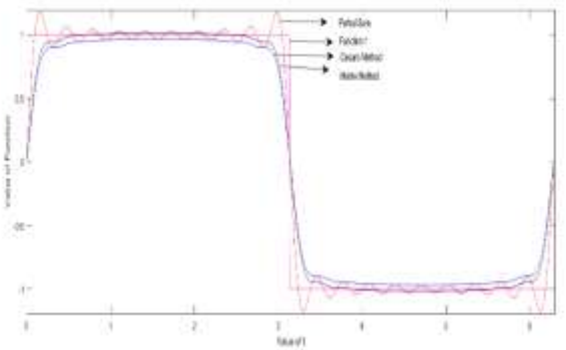

Figure 2: The figure above shows the oscillation and overshoots for n=20.

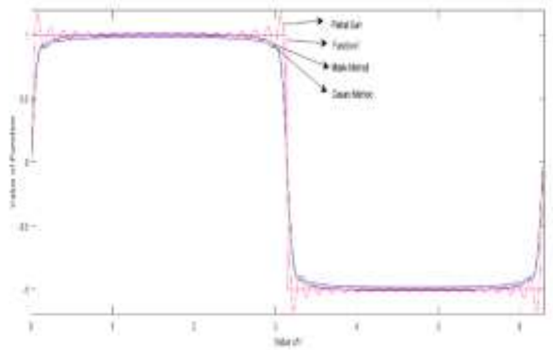

Figure 3: The figure above shows the oscillation and overshoots for n=40.

These figures inform to us about oscillation and overshoots occurred when we approximate a periodic function having a jump discontinuity by a finite sum of Fourier series.

Note that, according to Gibbs phenomenon, $S_n(f;t)$ have peaks in the neighborhood of discontinuity and its oscillation moves towards discontinuity points $(-\pi, 0, -\pi)$.

The Cesàro mean $C_n(f;t)$ and matrix mean $\tau_n(f;t)$ have a straightening effect on the oscillation of partial sum.

The oscillation amplitude on graphics tends to decrease when n is increases.

When using the Cesàro mean $C_n(f;t)$ and matrix mean $\tau_n(f;t)$ instead of partial sum or increasing the value n, we will have better results in the sense of straightening effect. So, the peaks become flatter.

Example. Then, we write that

$$v = \|f - \tau_n(f,\cdot)\|_{L^{p,\alpha}(\mathrm{T})} := \left\{ sup \left(\frac{1}{|I|^{\left\{1-\left(\frac{\alpha}{2}\right)\right\}}} \right) \int |f - \tau_n(f,\cdot)|^p |dt| \right\}^{\frac{1}{p}} < \infty.$$

Therefore, we get that

	$0 \le \alpha \le 2, 1 < p < \infty$, n=10	$0 \le \alpha \le 2, 1 < p < \infty$ n=20	$0 \le \alpha \le 2, 1 < p < \infty$ n=40
$\|f - \tau_n(f,\cdot)\|_{L^{p,\alpha}(\mathrm{T})}$	$v = 0.12075$	$v = 0.07433$	$v = 0.04460$
$\|f - C_n(f,\cdot)\|_{L^{p,\alpha}(\mathrm{T})}$	$v = 0.18818$	$v = 0.10890$	$v = 0.06217$
$\|f - s_n(f,\cdot)\|_{L^{p,\alpha}(\mathrm{T})}$	$v = 0.01136$	$v = 0.06655$	$v = 0.03846$

Table 1: Note that: the deviations tends to decrease to zero when n is increases. Deviation can be considered as error of approximation.

We say that the deviation of $\|f - \tau_n(f,\cdot)\|_{L^{p,\alpha}(\mathrm{T})}$ is better than the deviation of $\|f - C_n(f,\cdot)\|_{L^{p,\alpha}(\mathrm{T})}$ in the sense of approximation error in norm of Morrey spaces.

6. Acknowledgement

This study was supported by the grant of Science Research Project of Balikesir University, Project No.:2019-113.

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A Study On The Trigonometric Approximation In Weighted Orlicz Spaces Using Sub-Matrix Methods

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Abstract

In this study, the error of approximation of trigonometric polynomials by means of some special summability methods based on Cesàro sub-method are investigated in Orlicz spaces with weights satisfying Muckenhoupt condition. Also, the error of approximation is investigated for derivatives of function f in these spaces with degree of ${\lambda_n}^{-\alpha}$. In addition to this, we present the examples explaining the error of approximation taking into account the norm of weighted Orlicz spaces.

Key Words: Trigonometric approximation, matrix method, error of approximation, weighted Orlicz Spaces.

1. Introduction

The theory of trigonometric approximation has been largely studied in the past 100 years. Several researchers investigated the problems about approximation by means of different trigonometric polynomials in Banach function spaces. Some trigonometric approximation results in Lebesgue spaces and Orlicz spaces in the case of weighted or non-weighted case were obtained in [1]–[8].

Note that, approximation results were not investigated using sub-matrix methods in weighted Orlicz spaces with Muckenhoupt weights.

In present paper, we proved the approximation theorems in weighted Orlicz spaces using different two sub-matrix methods based on sub-Nörlund and sub-Riesz methods.

To formulate the main results obtained in this work we need some definitions and notations.

If $\Phi: [0,\infty) \to [0,\infty)$ be a convex and continuous function for which $\Phi(0) = 0, \Phi(u) > 0$ for $u > 0$. If $\Phi(u)$ is an even function and it satisfies the conditions $\lim_{u\to 0} \frac{\Phi(u)}{u} = 0$ and $\lim_{u\to\infty} \frac{\Phi(u)}{u} = \infty$, then $\Phi(u)$ is called a Young function [11, p.9].

Let Φ be a Young function. The function Ψ defined as

$$\Psi(v) := \max\{uv - \Phi(u): u \geq 0\}$$

is called the complementary Young function for $v \geq 0$ [11, p.13].

Let $\mathbb{T} = [0,2\pi]$. The class of measurable function $f: \mathbb{T} \to \mathbb{R}$ satisfying the condition

$$\int_0^{2\pi} \Phi(|f(x)|)dx < \infty$$

will be denoted by $\tilde{L}_\Phi(\mathbb{T})$.

The linear span of $\tilde{L}_{\Phi}(\mathbb{T})$ equipped with the norm

$$\|f\|_{\Phi} := sup\left\{\int_{0}^{2\pi} |f(x)g(x)|dx : g \in \tilde{L}_{\Psi}(\mathbb{T}), \int_{0}^{2\pi} \Psi(|f(x)|)dx \leq 1\right\},$$

or with the Luxemburg norm

$$\|f\|_{\Phi}^{*} := inf\left\{k > 0: \int_{0}^{2\pi} \Phi\left(\frac{|f(x)|}{k}\right)dx \leq 1\right\}$$

becomes a Banach space. This Banach space is called the Orlicz space generated by Φ [9, pp. 60–69][9, pp. 60–69].

Note that, the Orlicz spaces $L_{\Phi}(\mathbb{T})$ is known as one of the generalizations of the Lebesgue space $L^{p}(\mathbb{T})$, $1 < p < \infty$. If we take $\Phi_{p}(u) := \frac{u^{p}}{u}, 1 < p < \infty$, then the space $L_{\Phi}(\mathbb{T})$ turns into the Lebesgue space $L^{p}(\mathbb{T})$.

The Orlicz and Luxemburg norms satisfy the inequalities

$$\|f\|_{\Phi}^{*} \leq \|f\|_{\Phi} \leq 2\|f\|_{\Phi}^{*}, \qquad f \in L_{\Phi}(\mathbb{T}),$$

and hence they are equivalent. Furthermore, the Orlicz norm can be determined by means of the Luxemburg norm [9, p. 79-80] as

$$\|f\|_{\Phi} := \sup\left\{\int_{0}^{2\pi} |f(x)g(x)|dx : \|g\|_{\Psi}^{*} \leq 1\right\}.$$

Let $\Phi^{-1}: [0, \infty) \to [0, \infty)$ be the inverse of the Young function Φ and

$$h(t) := \limsup_{x\to\infty} \frac{\Phi^{-1}(x)}{\Phi^{-1}(tx)}, \; t > 0.$$

The numbers α_{Φ} and β_{Φ} defined by

$$\alpha_{\Phi} := \lim_{t\to\infty}\left(\frac{-log\ h(t)}{log\ t}\right) \text{ and } \beta_{\Phi} := \lim_{t\to\infty}\left(\frac{-log\ h(t)}{log\ t}\right)$$

are called the lower and upper Boyd indices [10]. It is known that these indices satisfy the conditions

$$0 \leq \alpha_{\Phi} \leq \beta_{\Phi} \leq 1$$

and

$$\alpha_{\Phi} + \beta_{\Psi} = 1 \text{ and } \alpha_{\Psi} + \beta_{\Phi} = 1.$$

These indices were firstly considered by Matuszewska and Orlicz [11].

A measurable 2π-periodic function $\omega: \mathbb{T} \to [0, \infty]$ is called a weight function if the set $\omega^{-1}(\{0, \infty\})$ has the Lebesgue measure zero.

The class of measurable functions f defined on $\mathbb{T}$ and satisfying the condition $\omega f \in L_{\Phi}(\mathbb{T})$ is called weighted Orlicz space $L_{\Phi}(\mathbb{T}, \omega)$ with the norm $\|f\|_{\Phi,\omega} := \|\omega f\|_{\Phi}$.

Let $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. A weight functions ω used in the paper belong to the Muckenhoupt class $A_{p}(\mathbb{T})$ which is defined by

$$\left(\frac{1}{|I|}\int_{I} \omega^{p}(x)dx\right)^{\frac{1}{p}} \left(\frac{1}{|I|}\int_{I} \omega^{-q}(x)dx\right)^{\frac{1}{q}} \leq C$$

with a finite constant C independent of I, where I is any subinterval of $\mathbb{T}$ and $|I|$ denotes the length of I.

Let $L_{\Phi}(\mathbb{T},\omega)$ be a weighted Orlicz space with Boyd indices $0<\alpha_{\Phi}\leq\beta_{\Phi}<1$ and $\omega\in A_{\frac{1}{\alpha_{\Phi}}}(\mathbb{T})\cap A_{\frac{1}{\beta_{\Phi}}}(\mathbb{T})$. The k-modulus of smoothness of the function $L_{\Phi}(\mathbb{T},\omega)$ is defined as

$$\Omega^{k}_{\Phi,\omega}(f;\delta)=\sup_{\substack{|h_i|\leq\delta\\ 1\leq i\leq k}}\left\|\prod_{i=1}^{k}\left(I-\sigma_{h_i}\right)f\right\|_{\Phi,\omega},\quad \delta>0,$$

where I is the identity operator and

$$(\sigma_h)(f;x):=\frac{1}{2h}\int_{-h}^{h}f(x+t)dt\,,0<h<\pi,x\in\mathbb{T}.$$

If $k=0$ we write $\Omega^{k}_{\Phi,\omega}(f;\delta)=\|f\|_{\Phi,\omega}$ and if k=1 we write $\Omega_{\Phi,\omega}(f;\delta)=\Omega^{1}_{\Phi,\omega}(f;\delta)$. The modulus of smoothness $\Omega^{k}_{\Phi,\omega}(f;\cdot)$ is a nondecreasing, nonnegative, continuous function and

$$\Omega^{k}_{\Phi,\omega}(f+g,\delta)\leq\Omega^{k}_{\Phi,\omega}(f;\delta)+\Omega^{k}_{\Phi,\omega}(g;\delta)$$

for $f,g\in L_{\Phi}(\mathbb{T},\omega)$.

For $0<\alpha$, let $k=\left[\frac{\alpha}{2}\right]+1$. The generalized Lipschitz class $Lip\big(\alpha,L_{\Phi}(\mathbb{T},\omega)\big)$ is defined as

$$Lip\big(\alpha,L_{\Phi}(\mathbb{T},\omega)\big)=\left\{f\in L_{\Phi}(\mathbb{T},\omega):\Omega^{k}_{\Phi,\omega}(f;\delta)=O(\delta^{\alpha}),\delta>0\right\}.$$

The Fourier series of $f\in L_{\Phi}(\mathbb{T},\omega)$ can be given as

$$f(x)\backsim\frac{a_0(f)}{2}+\sum_{k=1}^{\infty}(a_k(f)coskx+b_k(f)sinkx)\tag{1}$$

and the conjugate Fourier series

$$\tilde{f}(x)\backsim\frac{a_0(f)}{2}+\sum_{k=1}^{\infty}(a_k(f)sinkx-b_k(f)coskx).$$

Here $a_0(f),a_k(f),b_k(f),k=1,\ldots$, are Fourier coefficients of f. Let $S_n(f;x),(n=0,1,2,\ldots)$ be the nth partial sum of the series (1) at the point x, that is,

$$S_n(f;x):=\sum_{k=0}^{n}U_k(f;x),$$

where

$$U_0(f;x)=\left(\frac{a_0}{2}\right),\quad U_k(f;x)=a_k(f)coskx+b_k(f)sinkx,\quad k=1,2,\ldots.$$

1.1. Cesàro Submethod

Let $(\lambda_n)_{n=1}^{\infty}$ be a strictly increasing sequence of positive integers. For a sequence (x_k) of the real or complex numbers, the sub-Cesàro method (trigonometric polynomial) C_λ is defined by

$$(C_\lambda x)_n:=\frac{1}{\lambda_n}\sum_{k=1}^{\lambda_n}x_k.$$

Sub-Nörlund method $N_n^{\lambda}(f;x)$ is defined as

$$N_n^{\lambda}(f;x):=\frac{1}{P_{\lambda_n}}\sum_{m=0}^{\lambda_n} p_{\lambda_n-m}S_m(f;x)$$

and sub-Riesz method $R_n^{\lambda}(f;x)$ is defined as

$$R_n^{\lambda}(f;x):=\frac{1}{P_{\lambda_n}}\sum_{m=0}^{\lambda_n} p_m S_m(f;x)$$

where

$$P_{\lambda_n}=p_0+p_1+p_2+\ldots+p_{\lambda_n}\neq 0 \quad (n\geq 0)$$

and by convention, $p_{-1}=P_{-1}=0$.

We denote by $A\equiv(a_{n,k})$ a lower triangular regular matrix with nonnegative entries and let $S_n^{(A)}$ $(n=0,1,\ldots)$ be the row sums of this matrix, that is $S_n^{(A)}=\sum_{k=0}^{n} a_{n,k}$.

The sub-matrix methods τ_n^{λ} and T_n^{λ} used in this study, are defined

$$\tau_n^{\lambda}(f;x):=\sum_{k=0}^{\lambda_n} a_{\lambda_n,k}\,S_k(f;x),\ \lambda_n=0,1,2,\ldots \tag{2}$$

and

$$T_n^{\lambda}(f;x)=\sum_{k=0}^{\lambda_n} a_{\lambda_n,\lambda_n-k}\,S_k(f;x),\lambda_n=0,1,2,\ldots \tag{3}$$

respectively.

A nonnegative sequence $u:=(u_n)$ is called almost monotone decreasing (increasing), if there exists a constant $K:=K(u)$, depending on the sequence u only, such that $u_n\leq Ku_m$ $(Ku_n\geq u_m)$ for all $n\geq m$. Such sequences will be denoted by $u\in AMDS$ $(u\in AMIS)$.

Let

$$A_{n,k}:=\frac{1}{k+1}\sum_{i=0}^{k} a_{n,i}.$$

If $\{A_{n,k}\}\in$ AMDS $(\{A_{n,k}\}\in AMIS)$, then we will say that $\{a_{n,k}\}$ is an almost monotone decreasing (increasing) mean sequence with respect to $k=1,2,\ldots,n$ for all n. Briefly, we will write $\{a_{n,k}\}\in$ AMDMS $(\{a_{n,k}\}\in AMIMS)$.

Let

$$A_{n,k}^{*}:=\frac{1}{k+1}\sum_{i=n-k}^{n} a_{n,i}.$$

If $\{A_{n,k}\}\in$ AMDS $(\{A_{n,k}\}\in AMIS)$, then we will say that $\{a_{n,k}\}$ is an almost monotone decreasing (increasing) upper mean sequence with respect to $k=1,2,\ldots,n$ for all n. Briefly, we will write $\{a_{n,k}\}\in$ AMDUMS $(\{a_{n,k}\}\in AMIUMS)$.

The following inclusions are valid $AMDS\subset AMIUMS$ and $AMIS\subset AMDUMS$.

We will use sums up to λ_n in S_n and σ_n write these sums as S_n^λ and σ_n^λ, respectively

2. Auxiliary Results

Lemma 1. [8] Let $0 \leq \alpha_\Phi \leq \beta_\Phi \leq 1$, $\omega \in A_{\frac{1}{\alpha_\Phi}}(\mathbb{T}) \cap A_{\frac{1}{\beta_\Phi}}(\mathbb{T})$ and $f \in Lip\big(1, L_\Phi(\mathbb{T},\omega)\big)$. Then the estimate

$$\|S_n(f) - \sigma_n(f)\|_{\Phi,\omega} = O(n^{-1})$$

holds for $n = 1,2,\ldots$.

Lemma 2. [8] Let $0 < \alpha \leq 1, 0 \leq \alpha_\Phi \leq \beta_\Phi \leq 1$, $\omega \in A_{\frac{1}{\alpha_\Phi}}(\mathbb{T}) \cap A_{\frac{1}{\beta_\Phi}}(\mathbb{T})$ and $f \in Lip\big(\alpha, L_\Phi(\mathbb{T},\omega)\big)$. Then the estimate

$$\|f - S_n(f)\|_{\Phi,\omega} = O(n^{-\alpha})$$

is valid $n = 1,2,\ldots$.

Lemma 3. [12] Let $0 < \alpha < 1$ *and* $A = \big(a_{n,k}\big)$ be a lower triangular matrix with $\left|S_n^{(A)} - 1\right| = O(n^{-\alpha})$. $\{a_{n,k}\} \in AMIMS$ or $\{a_{n,k}\} \in AMDMS$ and $(\lambda_n + 1)a_{\lambda_n,0} = O(1)$, then

$$\sum_{k=0}^{\lambda_n}(k+1)^{-\alpha}a_{\lambda_n,k} = O({\lambda_n}^{-\alpha}).$$

Lemma 4. [7] Let $0 < \alpha < 1$ *and* $A = \big(a_{\lambda_n,k}\big)$ be a lower triangular matrix with $\left|S_n^{(A)} - 1\right| = O({\lambda_n}^{-\alpha})$. $\{a_{\lambda_n,k}\} \in AMDUMS$ or $\{a_{\lambda_n,k}\} \in AMIUMS$ and $(\lambda_n + 1)a_{\lambda_n,\lambda_n} = O(1)$, then

$$\sum_{k=0}^{\lambda_n}(k+1)^{-\alpha}a_{\lambda_n,\lambda_n-k} = O({\lambda_n}^{-\alpha}).$$

3. Main Results

Theorem 5. Let $f \in Lip\big(\alpha, L_\Phi(\mathbb{T},\omega)\big), 0 < \alpha < 1, \omega \in A_{\frac{1}{\alpha_\Phi}}(\mathbb{T}) \cap A_{\frac{1}{\beta_\Phi}}(\mathbb{T})$ and $A = (a_{\lambda_n,k})$ be a infinite lower triangular matrix with nonnegative entries and row sums 1. If one of the following conditions

(i) $\{a_{\lambda_n,k}\} \in AMDUMS,$

(ii) $\{a_{\lambda_n,k}\} \in AMIUMS$ *and* $(\lambda_n + 1)(a_{\lambda_n,\lambda_n}) = O(1),$

holds, then

$$\left\|f - T_n^\lambda(f)\right\|_{\Phi,\omega} = O({\lambda_n}^{-\alpha}).$$

Remark. When $a_{\lambda_n,\lambda_n-k} = \frac{p_{\lambda_n,\lambda_n-k}}{P_k}$, the method T_n^λ turns to sub-Nörlund method given as

$$N_n^\lambda(f;x) := \frac{1}{P_{\lambda_n}} \sum_{k=0}^{\lambda_n} p_{\lambda_n,\lambda_n-k} S_k(f;x).$$

Here

$$P_{\lambda_n} = p_0 + p_1 + p_2 + \ldots + p_{\lambda_n} \neq 0 \quad (\lambda_n \geq 0) \ and \ p_{-1} = P_{-1} = 0. \tag{4}$$

Corollary 6. $f \in Lip(\alpha, L_\Phi(\mathbb{T},\omega)), 0 < \alpha < 1, \omega \in A_{\frac{1}{\alpha_\Phi}}(\mathbb{T}) \cap A_{\frac{1}{\beta_\Phi}}(\mathbb{T})$ and (p_{λ_n}) be a positive sequence. If one of the following conditions

$(i) (p_{\lambda_n}) \in AMDUMS,$

$(ii)\ (p_{\lambda_n}) \in AMIUMS \ and \ (\lambda_n + 1) p_{\lambda_n} = O(P_{\lambda_n}),$

holds, then

$$\left\| f - N_n^\lambda(f) \right\|_{\Phi,\omega} = O({\lambda_n}^{-\alpha}).$$

Theorem 7. Let $f \in Lip(1, L_\Phi(\mathbb{T},\omega))$, $\omega \in A_{\frac{1}{\alpha_\Phi}}(\mathbb{T}) \cap A_{\frac{1}{\beta_\Phi}}(\mathbb{T})$ and $A = (a_{\lambda_n,k})$be a lower triangular matrix with $|S_n^{(A)} - 1| = O(n^{-1})$. If following condition

$$\sum_{k=0}^{\lambda_n - 2} \left| \Delta_k A^*_{\lambda_n,k} \right| = O\left({\lambda_n}^{-1}\right)$$

holds, then

$$\left\| f - T_n^\lambda(f) \right\|_{\Phi,\omega} = O\left({\lambda_n}^{-1}\right).$$

Corollary 8. Let $f \in Lip(1, L_\Phi(\mathbb{T},\omega))$, $\omega \in A_{\frac{1}{\alpha_\Phi}}(\mathbb{T}) \cap A_{\frac{1}{\beta_\Phi}}(\mathbb{T})$ and (p_{λ_n}) be a positive sequence. If following condition

$$\sum_{k=0}^{\lambda_n - 2} \left| \Delta_k P_{\lambda_n,k} \right| = O\left(\frac{P_{\lambda_n}}{\lambda_n}\right),$$

holds, then

$$\left\| f - N_n^\lambda(f) \right\|_{\Phi,\omega} = O\left({\lambda_n}^{-1}\right).$$

Theorem 9. Let $f \in Lip(\alpha, L_\Phi(\mathbb{T},\omega)), 0 < \alpha < 1, \omega \in A_{\frac{1}{\alpha_\Phi}}(\mathbb{T}) \cap A_{\frac{1}{\beta_\Phi}}(\mathbb{T})$ and $A = (a_{\lambda_n,k})$ be a lower triangular matrix with nonnegative entries and row sums 1. If one of the conditions

$(i) \left\{a_{\lambda_n,k}\right\} \in AMIMS,$

$(ii) \left\{a_{\lambda_n,k}\right\} \in AMDMS \ and \ (\lambda_n + 1) a_{\lambda_n,0} = O(1),$

is valid, then

$$\left\| f - \tau_n^\lambda(f) \right\|_{\Phi,\omega} = O({\lambda_n}^{-\alpha}). \tag{5}$$

Remark. When $a_{\lambda_n,k} = \frac{p_{\lambda_n,k}}{P_k}$, the method τ_n^λ turns to Riesz method given as

$$R_n^\lambda(f;x) := \frac{1}{P_{\lambda_n}} \sum_{k=0}^{\lambda_n} p_{\lambda_n,k} S_k(f;x).$$

Here

$$P_{\lambda_n} = p_0 + p_1 + p_2 + \ldots + p_{\lambda_n} \neq 0 \quad (\lambda_n \geq 0) \ and \ p_{-1} = P_{-1} = 0.$$

Remark. Also, in the case $p_n = 1, n \geq 0, \lambda_n = n$, both of $N_n^{\lambda}(f;x)$ and $R_n^{\lambda}(f;x)$ are equal to the classical Cesàro method

$$\sigma_n(f;x) = \frac{1}{n+1} \sum_{m=0}^{n} S_m(f;x).$$

The relation $\preccurlyeq$ is defined as "A$\preccurlyeq$B $\Leftrightarrow$ there exists a positive constant C, independent of essential parameters, such that A≤CB". The operator Δ_k is defined by $\Delta_k \varsigma_{n,k} = \varsigma_{n,k} - \varsigma_{n,k+1}$.

Corollary 10. Let $f \in Lip\big(\alpha, L_{\Phi}(\mathbb{T}, \omega)\big), 0 < \alpha < 1, \omega \in A_{\frac{1}{\alpha_{\Phi}}}(\mathbb{T}) \cap A_{\frac{1}{\beta_{\Phi}}}(\mathbb{T})$ and (p_{λ_n}) be a positive sequence. If one of the conditions

$(i)\big(p_{\lambda_n}\big) \in AMIMS,$

$(ii)\big(p_{\lambda_n}\big) \in AMDMS \ and \ (\lambda_n + 1)p_{\lambda_n} = O(P_{\lambda_n}),$

is valid, then

$$\big\|f - R_n^{\lambda}(f)\big\|_{\Phi,\omega} = O({\lambda_n}^{-\alpha}).$$

Theorem 11. Let $f \in Lip\big(1, L_{\Phi}(\mathbb{T}, \omega)\big), \omega \in A_{\frac{1}{\alpha_{\Phi}}}(\mathbb{T}) \cap A_{\frac{1}{\beta_{\Phi}}}(\mathbb{T})$ and $A = (a_{\lambda_n,k})$ be a lower triangular matrix with nonnegative entries and row sums 1. If following condition

$$\sum_{k=0}^{\lambda_n - 1} \big|\Delta_k A_{\lambda_n,k}\big| = O\big({\lambda_n}^{-1}\big)$$

holds, then

$$\big\|f - \tau_n^{\lambda}(f)\big\|_{\Phi,\omega} = O\big({\lambda_n}^{-1}\big).$$

Corollary 12. Let $f \in Lip\big(1, L_{\Phi}(\mathbb{T}, \omega)\big), \omega \in A_{\frac{1}{\alpha_{\Phi}}}(\mathbb{T}) \cap A_{\frac{1}{\beta_{\Phi}}}(\mathbb{T})$ and and (p_{λ_n}) be a positive sequence. If following condition

$$\sum_{k=0}^{\lambda_n - 1} \left|\Delta_k \frac{P_k}{k+1}\right| = O\left(\frac{P_{\lambda_n}}{\lambda_n}\right)$$

holds, then

$$\big\|f - R_n^{\lambda}(f)\big\|_{\Phi,\omega} = O\big({\lambda_n}^{-1}\big).$$

4. Applications

In this section, we will give an application about the error of approximation among a function and some summability methods obtained through its Fourier series. Also, we sketch the figures of these methods under assumptions of given norm.

If a periodic function f has a jump discontinuity, then this function is represented using a Fourier series. In (6), function f is a square wave function. We will use this function to obtain the error of approximation.

The function f defined by

$$f(t) = \begin{cases} 1, & -\pi \le t \le 0, \\ -1, & 0 < t < -\pi. \end{cases} \tag{6}$$

For all real values of t, we have $f(t + 2\pi) = f(t)$.

Since it is an odd function, its Fourier series is given as

$$f(t) = \sum_{n=1}^{\infty} b_n sin\, nt,$$

where

$$b_n = \left(\frac{2}{\pi}\right)\int_0^{\pi} f(t) sin\, ntdt = \left(\frac{2}{\pi}\right)\left(\frac{-1 + (-1)^n}{n}\right).$$

Therefore, the Fourier series of $f(t)$ is given by

$$f(t) = \left(\frac{2}{\pi}\right)\sum_{n=1}^{\infty}\left(\frac{-1 + (-1)^n}{n}\right) sin\, nt. \tag{7}$$

The partial sum $S_{\lambda_n}(t)$ of the series (7) is given as

$$S_{\lambda_n}(t) = \frac{4}{\pi}\left(sin\, t + \left(\frac{1}{3}\right) sin3t + \ldots + \left(\frac{1}{n}\right) sin(\lambda_n t)\right).$$

The sub-Cesàro mean for (7) is

$$C_{\lambda_n}(f) = \sum_{k=1}^{\lambda_n}\left(\frac{\lambda_n - k}{\lambda_n}\right)\left(\frac{-2}{\pi}\right)\left(\frac{-1 + (-1)^k}{k}\right) sin(kt).$$

In (2), taking $(a_{n,k})$ as below, we will calculate the matrix mean for the series (7).

$$(a_{n,k}) = \begin{bmatrix} 1 & & & & & & & & \\ 1/2 & 1/2 & & & & & & & \\ 1/3 & 1/3 & 1/3 & & & & 0 & & \\ 1/4 & 1/4 & 1/4 & 1/4 & & & & & \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 & & & & \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & \\ \frac{1}{n+1} & \frac{1}{n+1} & \frac{1}{n+1} & \frac{1}{n+1} & \cdots & \frac{1}{n+1} & \frac{1}{n+1} & \cdots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \ddots \end{bmatrix}_{n \times k}$$

In $(a_{n,k})$, $\left(\frac{1}{n+1}\right)$ repeats $(n + 1)$ times in the n^{th} row. For lower triangular matrix given above, we can write the sub-matrix mean as

$$\tau_n^{\lambda}(f;t) := \sum_{k=0}^{\lambda_n} \left(\frac{1}{\lambda_n + 1}\right) S_k(f;t)\,, n = 0,1,2,\ldots.$$

For $\lambda_n = 10,20$ and 40, using the definitions of $f(t), S_{\lambda_n}(t), C_{\lambda_n}(f;t)$ and $\tau_n^{\lambda}(f;t)$ we plot the following figures.

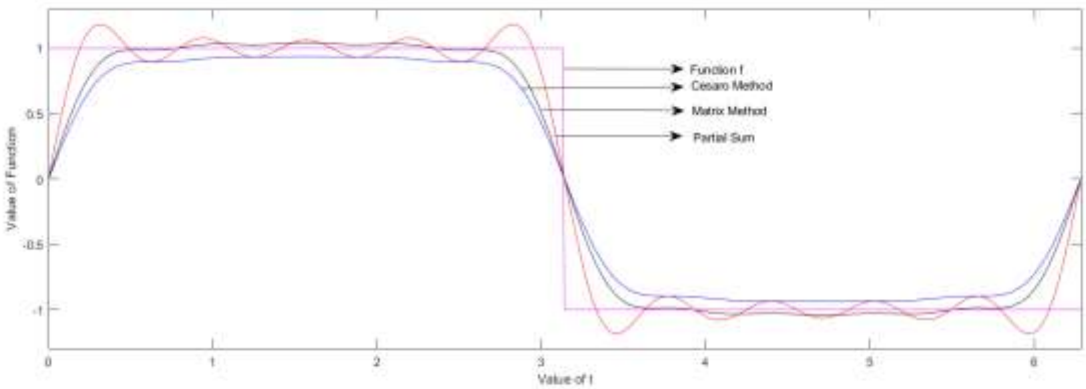

Figure 1: The figure above shows the oscillation and overshoots for λ_n=10.

If we take $\lambda_n = 20, 40$ instead of $\lambda_n = 10$, then we have the following figures: note that the errors of approximation decrease.

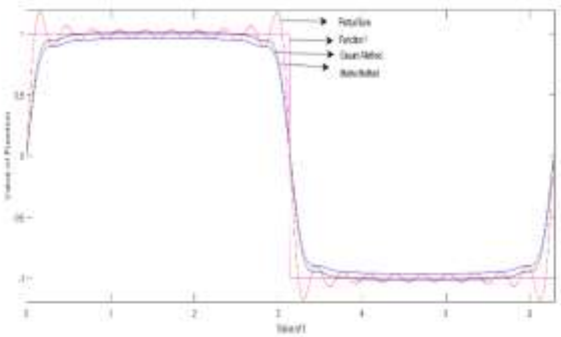

Figure 2: The figure above shows the oscillation and overshoots for λ_n=20.

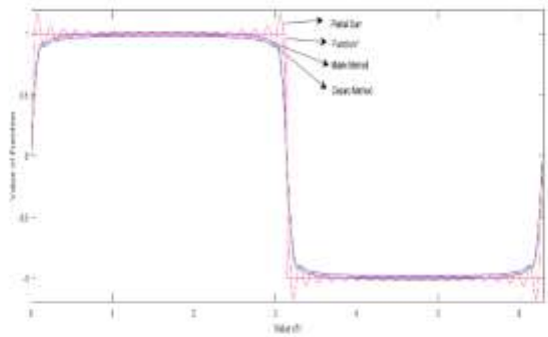

Figure 3: The figure above shows the oscillation and overshoots for λ_n=40.

These figures inform to us about oscillation and overshoots occurred when we approximate a periodic function having a jump discontinuity by a finite sum of Fourier series. Note that $S_{\lambda_n}(f;t)$ have peaks in the neighborhood of discontinuity and its oscillation moves towards discontinuity points $(-\pi, 0, -\pi)$. The oscillation amplitude on graphics tends to decrease when n is increases.

We summarized a conclusion related to the approximation errors by following example. Note that, we calculated the errors of approximation in Orlicz norm.

Example. We have that

$$\|f\|_{\Phi}^{*} := inf\left\{k > 0\colon \int_0^{2\pi} \Phi\left(\frac{|f(x)|}{k}\right) dx \le 1\right\}.$$

If we take $\Phi(x) = e^x - x - 1$ as young function, then we get that

$$\left\|f-\tau_n^{\lambda}(f)\right\|_{\Phi}^{*} := inf\left\{k>0: \int_0^{2\pi}\left(e^{\left(\frac{\left|f-\tau_n^{\lambda}(f)\right|}{k}\right)}-\left(\frac{\left|f-\tau_n^{\lambda}(f)\right|}{k}\right)-1\right)dx \le 1\right\}.$$

The following table gives the norm values of Orlicz spaces for different summability methods.

	λ_n=10	λ_n=20	λ_n=40
$\left\|f-S_{\lambda_n}(f,\cdot)\right\|_{\Phi}^{*}$	k =0.0183	k =0.0308	k =0.0610
$\left\|f-C_{\lambda_n}(f,\cdot)\right\|_{\Phi}^{*}$	k =0.0360	k =0.0515	k =0.0764
$\left\|f-\tau_n^{\lambda}(f,\cdot)\right\|_{\Phi}^{*}$	$k = 0.0624$	$k = 0.1424$	$k = 0.3039$

Table 1: The norm values of Orlicz spaces for λ_n=10,20,40.

5. Acknowledgement

This study was supported by the grant of Science Research Project of Balikesir University, Project No.:2019-113.

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A Numerical Approach to Solution of Nonlinear Riccati Differential Equation

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Abstract

In this study, a numerical approach for the solution of Riccati differential equation is investigated. Nonlinear Riccati differential equations have been used in many fields in science, engineering and especially in applied mathematics. A numerical solutions are obtained with regard to a matrix method and compared with other techniques in literature. Besides, error analysis is given in order to obtain more efficient results for its approximation.

Keywords: Riccati differential equation, matrix method, orthogonal polynomials.

1. Introduction

Riccati differential equations are of great importance in many areas in control theory, supply-demand relationship, social practise theory, biology, free vibration theory, forecasting and some other applications in science and engineering fields [1]-[3]. Both analytical and numerical solutions of different types of Riccati differential equation have been investigated by many techniques. These applications are important due to its support to other research areas. There has been many well-known techniques such as Padé approximation method [4], operation matrix method [5], Taylor matrix method [6], decomposition method [7], Bernstein polynomial approach [8], [9], Fourier polynomial approximation [10], classical fourth order Runge Kutta method [11], variational iteration algorithm [12] and so on.

In this study, the following type Riccati differential equation is defined in the form:

$$\begin{cases} u'(t) = A(t) + B(t)u(t) + C(t)u^2(t), \quad t_0 \le t \le T, \\ u(t_0) = c, \end{cases} \tag{1}$$

where c is an arbitrary constant, $A(t), B(t)$ and $C(t)$ are continuous given functions.

The organization of the paper is the following. Initially, preliminaries about polynomials are given. Then the numerical method for finding the approximate solution of the problem is proposed. Afterward, an error analysis is introduced. Accordingly, numerical results of the problem are given by tables and figures. The paper finalizes with the concluding remarks and brief discussion of results.

2. Preliminaries

Properties of Jacobi Polynomials

Jacobi polynomials are commonly known as hypergeometric polynomials since they play an important role in rotation groups and classification of molecular rotors in quantum mechanics. Jacobi polynomials are defined in the form:

$$P_n^{(\alpha,\beta)}(t)=\frac{(-2)^n}{n!}(1-t)^{-\alpha}(1+t)^{-\beta}\frac{d^n}{dt^n}\left[(1+t)^{n+\beta}(1-t)^{n-\alpha}\right], \tag{2}$$

where α and β are parameters with the relation $\alpha,\beta>-1$ [13]-[20]. Then the first few Jacobi polynomials are given from Eq. (2) as

$$\begin{aligned}
&P_0^{(\alpha,\beta)}(t)=1,\\
&P_1^{(\alpha,\beta)}(t)=\frac{1}{2}\left[2(\alpha+1)+(\alpha+\beta+2)(t-1)\right],\\
&P_2^{(\alpha,\beta)}(t)=\frac{1}{8}\left[4(\alpha+1)(\alpha+2)+4(\alpha+\beta+3)(\alpha+2)(t-1)+(\alpha+\beta+3)(\alpha+\beta+4)(t-1)^2\right],\\
&\quad\vdots
\end{aligned} \tag{3}$$

Definition: Jacobi polynomials are have the cases within the special concept of α and β parameters as

Case 1. For $\alpha=\beta=0$, we have the relation

$$P_n^{(0,0)}(t)=\frac{1}{2^n n!}\frac{d^n}{dt^n}\left(t^2-1\right)^n, \tag{4}$$

which is called as the *Legendre polynomials* (Fig. 1).

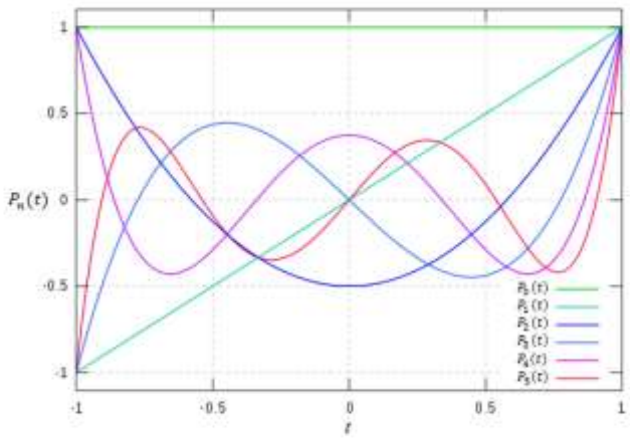

Figure 1. Legendre polynomials for $n = 0,1,2,3,4$ and 5 [21].

Case 2. For $\alpha = \beta = -\frac{1}{2}$, we have the relation

$$T_n(t) = 2^{2n} \binom{2n}{n}^{-1} P_n^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(t), \tag{5}$$

which is called as the *Chebyshev polynomials of the first kind* (Fig. 2).

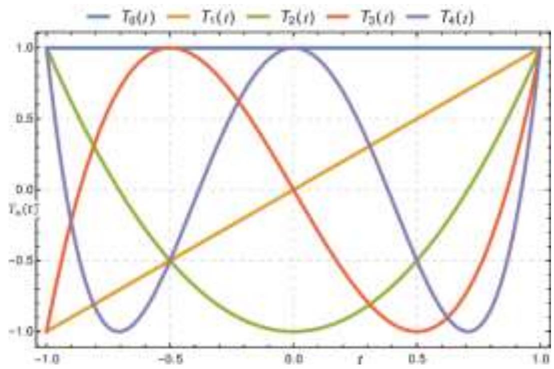

Figure 2. Chebyshev polynomials of the first kind for $n = 0,1,2,3$ and 4 [22].

Case 3. For $\alpha = \beta$, we have the relation

$$C_n^{(\lambda)}(t) = \binom{2\alpha}{\alpha}^{-1} \binom{n+2\alpha}{\alpha} C_n^{(\alpha,\alpha)}(t), \quad \alpha = \lambda - \frac{1}{2} \neq \frac{1}{2}, \tag{6}$$

which is called as the *Gegenbauer polynomials* (Fig. 3).

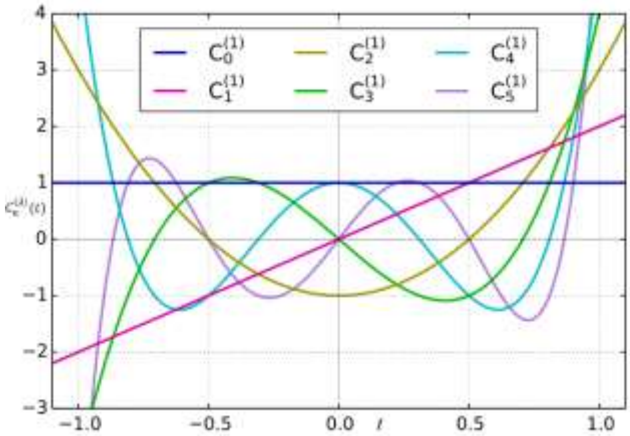

Figure 3. Gegenbauer polynomials for $\lambda = 1$ and $n = 0,1,2,3,4$ and 5 [23], [24].

3. Numerical Technique

In this section, an approximate solution in terms of orthogonal Jacobi polynomials is presented in the form:

$$u(t) \cong u_N^{(\alpha,\beta)}(t) = \sum_{n=0}^{N} a_n P_n^{(\alpha,\beta)}(t), \tag{7}$$

where a_n are unknown coefficients, $u_N^{(\alpha,\beta)}(t)$ is approximate solution for any integer N and $P_n^{(\alpha,\beta)}(t)$ are the orthogonal Jacobi polynomials defined in Eq.(2). For this purpose, Eq. (1) is considered in the form

$$\left[u_N^{(\alpha,\beta)}(t)\right] = \mathbf{P}_n^{(\alpha,\beta)}(t)\mathbf{A}, \tag{8}$$

where

$$\mathbf{P}_n^{(\alpha,\beta)}(t) = \left[\mathbf{P}_0^{(\alpha,\beta)}(t) \quad \mathbf{P}_1^{(\alpha,\beta)}(t) \quad \dots \quad \mathbf{P}_N^{(\alpha,\beta)}(t)\right], \ \mathbf{A} = \left[a_0 \quad a_1 \quad \dots \quad a_N\right]^T.$$

Then the orthogonal Jacobi polynomials are transformed into the matrix form as

$$\mathbf{P}_n^{(\alpha,\beta)}(t) = \mathbf{X}(t)\mathbf{J}^{(\alpha,\beta)}, \tag{9}$$

where

$$\mathbf{X}(t)=\begin{bmatrix}1 & (t-1) & \dots & (t-1)^N\end{bmatrix},\ \mathbf{J}^{(\alpha,\beta)}=\left[J_{i,k}^{(\alpha,\beta)}\right],\ 1\le i,\quad k\le N+1, \tag{10}$$

and

$$\left[J_{i,k}^{(\alpha,\beta)}\right]=\begin{cases}2^{1-i}\binom{\alpha+\beta+i-2+k}{i-1}\binom{\alpha+k-1}{k-i}, & i\le k,\\ 0 & , i>k.\end{cases} \tag{11}$$

By using the relations from (8) and (9),

$$\left[u_N^{(\alpha,\beta)}(t)\right]=\mathbf{X}(t)\mathbf{J}^{(\alpha,\beta)}\mathbf{A}, \tag{12}$$

Now, the relation between $\mathbf{X}(t)$ and its first derivative $\mathbf{X}'(t)$ is shown as

$$\mathbf{X}'(t)=\mathbf{X}(t)\mathbf{B}, \tag{13}$$

where

$$\mathbf{B}=\begin{bmatrix}0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 2 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & N\\ 0 & 0 & 0 & \cdots & 0\end{bmatrix}.$$

Then the derivative part of the problem is defined in the matrix form as

$$\left[\left(u_N^{(\alpha,\beta)}\right)'(t)\right]=\mathbf{X}'(t)\mathbf{J}^{(\alpha,\beta)}\mathbf{A}=\mathbf{X}(t)\mathbf{B}\mathbf{J}^{(\alpha,\beta)}\mathbf{A}, \tag{13}$$

Besides, the nonlinear term of the Riccati equation has a matrix form as

$$\left[\left(u_N^{(\alpha,\beta)}\right)^2(t)\right]=\mathbf{X}(t)\mathbf{J}^{(\alpha,\beta)}\overline{\mathbf{X}}(t)\overline{\mathbf{J}}^{(\alpha,\beta)}\overline{\mathbf{A}}, \tag{14}$$

where

$$\overline{\mathbf{X}}(t) = diag\left[\mathbf{X}(t) \quad \mathbf{X}(t) \quad \cdots \quad \mathbf{X}(t)\right],$$
$$\overline{\mathbf{J}}^{(\alpha,\beta)} = diag\left[\mathbf{J}^{(\alpha,\beta)} \quad \mathbf{J}^{(\alpha,\beta)} \quad \cdots \quad \mathbf{J}^{(\alpha,\beta)}\right],$$
$$\overline{\mathbf{A}} = diag\left[a_0\mathbf{A} \quad a_1\mathbf{A} \quad \cdots \quad a_N\mathbf{A}\right]^T.$$

The coefficients $A(t), B(t)$ and $C(t)$ of Eq. (1) are also defined in matrix form as

$$\mathbf{A}(t) = diag\left[A(t) \quad A(t) \quad \cdots \quad A(t)\right],$$
$$\mathbf{B}(t) = diag\left[B(t) \quad B(t) \quad \cdots \quad B(t)\right],$$
$$\mathbf{C}(t) = diag\left[C(t) \quad C(t) \quad \cdots \quad C(t)\right].$$

So that the matrix system of the nonlinear Riccati differential equation in Eq.(1) is described as

$$\begin{aligned} &\mathbf{X}(t)\mathbf{B}\mathbf{J}^{(\alpha,\beta)}\mathbf{A} - \mathbf{B}(t)\mathbf{X}(t)\mathbf{J}^{(\alpha,\beta)}\mathbf{A} - \mathbf{C}(t)\mathbf{X}(t)\mathbf{J}^{(\alpha,\beta)}\overline{\mathbf{X}}(t)\overline{\mathbf{J}}^{(\alpha,\beta)}\overline{\mathbf{A}} = \mathbf{A}(t), \\ &\left(\mathbf{X}(t)\mathbf{B}\mathbf{J}^{(\alpha,\beta)} - \mathbf{B}(t)\mathbf{X}(t)\mathbf{J}^{(\alpha,\beta)}\right)\mathbf{A} - \mathbf{C}(t)\mathbf{X}(t)\mathbf{J}^{(\alpha,\beta)}\overline{\mathbf{X}}(t)\overline{\mathbf{J}}^{(\alpha,\beta)}\overline{\mathbf{A}} = \mathbf{A}(t). \end{aligned} \tag{15}$$

Definition: The collocation method is a numerical method which is applicable pointwise in order to get the numerical solution of the problem i.e. a finite-dimensional space of candidate solutions. This is accepted that "selected points" or collocation principle over the set of points are called as *collocation points* in order to implement the collocation based methods [25]-[27]. Here, the collocations points are defined as

$$t_s = t_0 + \frac{T - t_0}{N}s, \quad s = 0,1,2,...,N. \tag{16}$$

The approximation of the problem is found pointwise and we use the collocation points which is defined in Eq.(16). Substituting the collocation points into Eq.(15),

$$\left(\mathbf{X}(t_s)\mathbf{B}\mathbf{J}^{(\alpha,\beta)} - \mathbf{B}(t_s)\mathbf{X}(t_s)\mathbf{J}^{(\alpha,\beta)}\right)\mathbf{A} - \mathbf{C}(t_s)\mathbf{X}(t_s)\mathbf{J}^{(\alpha,\beta)}\overline{\mathbf{X}}(t_s)\overline{\mathbf{J}}^{(\alpha,\beta)}\overline{\mathbf{A}} = \mathbf{A}(t_s). \tag{17}$$

where

$$\mathbf{A}(t_s) = diag\left[A(t_0) \quad A(t_1) \quad \cdots \quad A(t_N)\right],$$
$$\mathbf{B}(t_s) = diag\left[B(t_0) \quad B(t_1) \quad \cdots \quad B(t_N)\right],$$
$$\mathbf{C}(t_s) = diag\left[C(t_0) \quad C(t_1) \quad \cdots \quad C(t_N)\right].$$

Eq.(17) is obtained as the fundamental matrix relation which can be also written as

$$\left(\mathbf{XBJ}^{(\alpha,\beta)}-\mathbf{B}^{*}\mathbf{XJ}^{(\alpha,\beta)}\right)\mathbf{A}-\mathbf{C}^{*}\mathbf{XJ}^{(\alpha,\beta)}\overline{\mathbf{X}}\overline{\mathbf{J}}^{(\alpha,\beta)}\overline{\mathbf{A}}=\mathbf{A}^{*}, \tag{18}$$

where " * " is the sign to separate the coefficient matrices. Let us consider Eq.(18) in a brief form

$$\mathbf{W}=\mathbf{XBJ}^{(\alpha,\beta)}-\mathbf{B}^{*}\mathbf{XJ}^{(\alpha,\beta)} \text{ and } \mathbf{V}=-\mathbf{C}^{*}\mathbf{XJ}^{(\alpha,\beta)}\overline{\mathbf{X}}\overline{\mathbf{J}}^{(\alpha,\beta)}. \tag{19}$$

and then

$$\mathbf{WA}+\mathbf{V}\overline{\mathbf{A}}=\mathbf{A}^{*} \quad \text{or} \left[\mathbf{W};\mathbf{V}:\mathbf{A}^{*}\right]. \tag{20}$$

Besides, matrix representation of the initial condition is obtained by using the similar procedure. Then it is found as

$$\mathbf{UA}=\mathbf{C}\Rightarrow\left[\mathbf{U};\boldsymbol{\lambda}\right]. \tag{21}$$

By replacing the matrices (21) into the system (20) then new augumented matrix system is obtained in the form $\left[\tilde{\mathbf{W}};\tilde{\mathbf{V}}:\tilde{\mathbf{A}}^{*}\right]$. Then with the solution of the system by using Gauss Elimination procedure, the unknown coefficients are found and replaced in Eq. (7). Consequently, numerical solution is obtained [28]-[31].

4. Accuracy of Solution

In this section, we check the accuracy of the present method. The approximate solutions $u_N(t)$ of Eq. (1), and their first derivatives are considered and substituted into Eq. (1). Then we obtain approximate results for $t=t_r\in\left[0,R\right], r=0,1,....$

$$E_N(t_r)=\left|u(t_r)-u_N'(t_r)+A(t_r)+B(t_r)u_N(t_r)+C(t_r)u_N{}^2(t_r)\right|\cong 0, \tag{22}$$

where $E_N(t_r)\le 10^{-k_{\alpha,\beta}}=10^{-k}$ (k is a positive integer) is prescribed, then the truncation limit N is increased until difference $E_N(t_r)$ becomes smaller than the prescribed 10^{-k} at each points [32].

5. Numerical Experiments

In this section, to show the accuracy and efficiency of the presented method, for the problem given at (1), is solved with it. Numerical calculations and plottings were performed using Maple and Matlab softwares, respectively.

Example 1. Let us first consider the following Riccati differential equation [33]

$$\begin{aligned} &u'(t)=16t^2-5+8tu(t)+u^2(t),\\ &u(0)=1, \quad t\in[0,1]. \end{aligned} \tag{23}$$

The numerical technique which is introduced at Section 3 applied on the problem (23) for $N=3$. Then the collocation points are shown as

$$t_s=\frac{1}{3}s, \quad s=0,1,2,3; \quad t_0=0, t_s=\frac{1}{3}, t_s=\frac{2}{3}, t_s=1,\ 0\le t\le 1. \tag{24}$$

So that we have the following matrices

$$\mathbf{X}(t_s)=\begin{bmatrix} 1 & 1 & 1 & 1\\ 1 & 2/3 & 4/9 & 16/81\\ 1 & 1/3 & 1/9 & 1/27\\ 1 & 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{B}=\begin{bmatrix} 0 & 1 & 0 & 0\\ 0 & 0 & 2 & 0\\ 0 & 0 & 0 & 3\\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{J}^{(\alpha,\beta)}=\begin{bmatrix} \binom{\alpha+\beta}{0}\binom{\alpha}{0} & \binom{\alpha+\beta+1}{0}\binom{\alpha+1}{1} & \binom{\alpha+\beta+2}{0}\binom{\alpha+2}{2} & \binom{\alpha+\beta+3}{0}\binom{\alpha+3}{3}\\ 0 & 2^{-1}\binom{\alpha+\beta+2}{1}\binom{\alpha+1}{0} & 2^{-1}\binom{\alpha+\beta+3}{1}\binom{\alpha+2}{1} & 2^{-1}\binom{\alpha+\beta+4}{1}\binom{\alpha+3}{2}\\ 0 & 0 & 2^{-2}\binom{\alpha+\beta+4}{2}\binom{\alpha+2}{0} & 2^{-2}\binom{\alpha+\beta+5}{2}\binom{\alpha+3}{1}\\ 0 & 0 & 0 & 2^{-3}\binom{\alpha+\beta+6}{3}\binom{\alpha+3}{0} \end{bmatrix},$$

$$\mathbf{A}(t_s)=\begin{bmatrix}-5 & 0 & 0 & 0\\ 0 & -29/9 & 0 & 0\\ 0 & 0 & 19/9 & 0\\ 0 & 0 & 0 & 11\end{bmatrix},\ \mathbf{B}(t_s)=\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 8/3 & 0 & 0\\ 0 & 0 & 16/3 & 0\\ 0 & 0 & 0 & 8\end{bmatrix},\ \mathbf{C}(t_s)=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix},$$

and the fundamental matrix relation as

$$\left(\mathbf{X}(t_s)\mathbf{B}\mathbf{J}^{(\alpha,\beta)}-\mathbf{B}(t_s)\mathbf{X}(t_s)\mathbf{J}^{(\alpha,\beta)}\right)\mathbf{A}-\mathbf{C}(t_s)\mathbf{X}(t_s)\mathbf{J}^{(\alpha,\beta)}\overline{\mathbf{X}}(t_s)\overline{\mathbf{J}}^{(\alpha,\beta)}\overline{\mathbf{A}}=\mathbf{A}(t_s),\ \text{i.e.}$$

$$\mathbf{W}\mathbf{A}+\mathbf{V}\overline{\mathbf{A}}=\mathbf{A}^*$$

together with the matrix relation of the condition, $\mathbf{U}=\begin{bmatrix}0 & 0 & 0 & 0\end{bmatrix}$ we have the new augumented matrix form as $\left[\tilde{\mathbf{W}};\tilde{\mathbf{V}}:\tilde{\mathbf{A}}^*\right]$. Finally, we obtain the approximate solution in the form:

$$u(t)\cong u_N^{(\alpha,\beta)}(t)=\sum_{n=0}^{3}a_nP_n^{(\alpha,\beta)}(t)=a_0P_0^{(\alpha,\beta)}(t)+a_1P_1^{(\alpha,\beta)}(t)+a_2P_2^{(\alpha,\beta)}(t)+a_3P_3^{(\alpha,\beta)}(t).$$

Then we investigate the approximate solutions for different polynomials with regard to the pair $(\alpha,\beta)=(0.5,-0.5)$ Jacobi polynomials and $(\alpha,\beta)=(0.5,0.5)$ Gegenbauer polynomials. Finally, the exact solution $u(t)=1-4t$ is obtained for $t\geq 0$.

Example 2. Here, Riccati differential equation is considered [33], [34]

$$\begin{aligned}&u'(t)=e^t-e^{3t}+2e^{2t}u(t)-e^tu^2(t),\\ &u(0)=1,\quad t\in[0,1].\end{aligned}\tag{24}$$

The exact solution is $u(t)=e^t$, for $t\geq 0$.

By using the procedure, we obtain the approximate solutions. The approximate solutions for different N values can be seen in Table 1 and Fig. 1. The comparison of approximate solutions with different techniques and the exact solution is shown by Fig. 2.

Table 1. Absolute errors for N=3, 4 and 8 of Example 2.

t	E_3	E_4	E_8
0.1	0.1510E-03	0.9475E-04	0.1200E-05
0.2	0.1144E-04	0.2319E-05	0.9025E-06
0.5	0.1491E-04	0.2467E-05	0.9866E-06
0.6	0.1789E-04	0.2546E-05	0.9157E-06
1	0.1431E-04	0.2422E-05	0.9692E-06

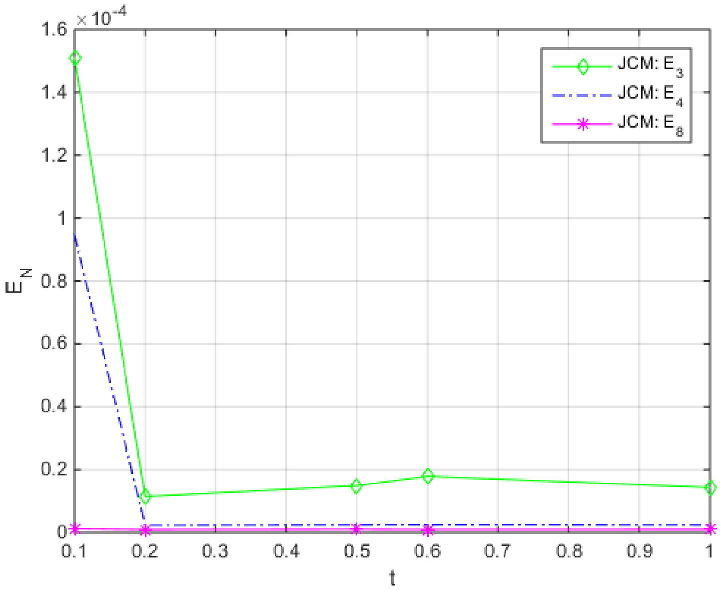

Figure 1. Comparison of absolute errors for N= 3,4 and 8 of Jacobi polynomial solution.

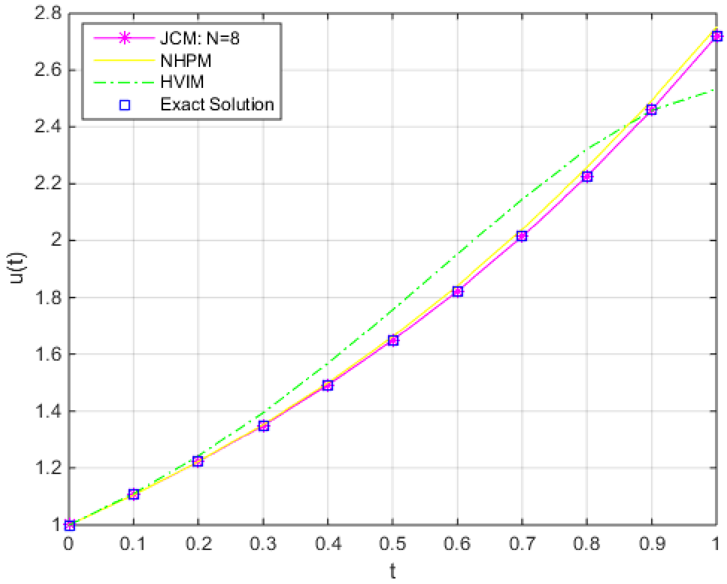

Figure 2. Comparison of approximate solutions between Jacobi Collocation Method (JCM) for N=8, He's variational iteration method (HVIM) and new homotopy perturbation method (NHPM).

6. Conclusion

In this study, we introduce a matrix method depending on Jacobi polynomials in order to solve nonlinear Riccati type differential equation with initial condition numerically. Furthermore, the accuracy of the solution is obtained by an error analysis. The present method and its error analysis are applied on the example which have been shown by figures and table. The method has significant advantages such as: straightforward computation procedure together with some computer programme algorithm in Maple. We obtain satisfactory results whenever we have N is chosen large enough. The method is applicable for further problems and their applications at interdiciplinary area [35]-[37].

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A Numerical Investigation on a Neural Field Model

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Abstract

In this study, a dynamical system based on neural field model is studied. Numerical investigation is reached and stability of the system including approximated results of the solution are introduced. Some numerical results are given.

Keywords: Neural field model, numerical method, stability analysis.

1. Introduction

Dynamical systems are of importance in many fields such as engineering, chemistry, biology, medicine and so on. In applied mathematics, analysis of such systems are investigated with different perspectives from numerical methods to stability. On the other hand, including time delay to dynamical systems gives us a chance to show a real-world approximation and more appropriate results in their applications. Besides, neural network models are one of the real-world applications of such systems. Recently, such models have been reached by many scientists [1]-[4]. A well-known neural network model is presented as

$$C_i u_i'(t) = -\frac{1}{R_i} u_i(t) + \sum_{j=1}^{n} T_{ij} f_j(u_j(t-\tau_j)),\ i = 1, 2, ..., n. \tag{1}$$

where $u_i(t)$ shows a voltage of unit i, C_i is the input capacitance, T_{ij} is the connection matrix and f is a continuous, differentiable, odd and strictly increasing function. This network model is called as "*Hopfield's circuit equation*" which is including time delay but not external input.

In this study, a neural network model is considered in the form:

$$u_i'(t) = -u_i(t) + \sum_{j=1}^{2} p_{ij} f_j(u_j(t-\tau_j)), \quad i = 1, 2,\ 0 \le t \le 1, \tag{2}$$

with the initial condition

$$u_i(t_0) = \lambda_i, \quad i = 1, 2, \tag{3}$$

where t_0 is the initial value, λ_i are suitable numbers, the units are identical $\tau_1 = \tau_2$ and the initial conditions are considered as continuous functions on the interval $[-\max(\tau_1, \tau_2), 0]$. Moreover, f_j are continuous functions and p_{ij} is the symmetric connection matrix with the property $p_{12} = p_{21}$. On the other hand, existence and uniqueness of the problem has been investigated by many authors [5]-[7]. Besides, Fig. 1 shows Hopfield's network together with three components: neuron drive circuitry, connection matrix circuitry and comparator circuitry [8].

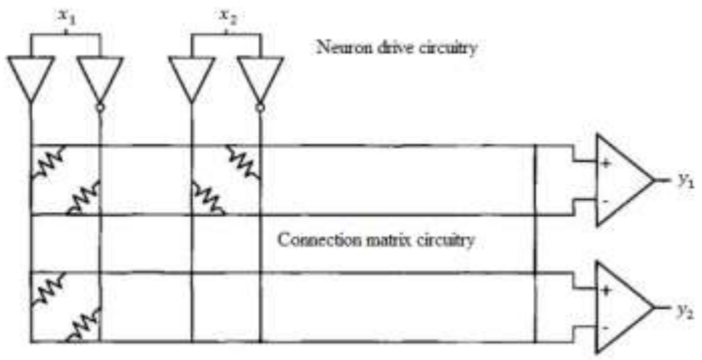

Fig. 1. A simple circuit diagram of Hopfield's network

2. Numerical Technique

In this section, a numerical solution of the problem (2)-(3) is obtained by using a collocation technique. The numerical technique is based on matrix relations of the approximate solution which is defined in the form:

$$u_i(t) \cong u_{i,N}(t) = \sum_{n=0}^{N} a_{i,n} L_n(t), \quad i = 1,2, \qquad 0 \le t \le 1 \tag{4}$$

where $a_{i,n}$ are unknown coefficients of the system approximation and $L_n(t)$ are the Laguerre polynomials in order to solve the system with regard to a series which is defined in the form:

$$L_n(t) = \sum_{r=0}^{n} \frac{(-1)^r}{r!} \binom{n}{r} t^r. \tag{5}$$

where n and r are the suitable numbers in order to represent Laguerre series in Eq. (5) [9]. Now, we are ready to apply the technique. Firstly, the matrix form of the approximation is defined in the form:

$$\left[u_i(t)\right] = \mathbf{L}(t)\mathbf{A}_i = \mathbf{X}(t)\mathbf{H}\mathbf{A}_i, \quad i = 1,2, \tag{6}$$

where

$\mathbf{L}(t) = \left[L_0(t) \quad L_1(t) \quad L_2(t) \quad \ldots \quad L_N(t)\right]$, $\mathbf{X}(t) = [1 \; t \ldots t^N]$, $\mathbf{A}_i = [a_{i,0} \; a_{i,1} \ldots a_{i,N}]^T$, $i = 1,2$,

$$\mathbf{H}=\begin{bmatrix} \frac{(-1)^0}{0!}\binom{0}{0} & \frac{(-1)^0}{0!}\binom{1}{0} & \frac{(-1)^0}{0!}\binom{2}{0} & \cdots & \frac{(-1)^0}{0!}\binom{N}{0} \\ 0 & \frac{(-1)^1}{1!}\binom{1}{1} & \frac{(-1)^1}{1!}\binom{2}{1} & \cdots & \frac{(-1)^1}{1!}\binom{N}{1} \\ 0 & 0 & \frac{(-1)^2}{2!}\binom{2}{2} & \cdots & \frac{(-1)^2}{2!}\binom{N}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{(-1)^N}{N!}\binom{N}{N} \end{bmatrix}.$$

So that we have

$$\left[u_i^{(1)}(t)\right]=\mathbf{L}^{(1)}(t)\mathbf{A}_i=\mathbf{X}(t)\mathbf{BHA}_i, \qquad (7)$$

where

$$\mathbf{B}=\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & N & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

In order to obtain the matrices for the delay term, $t \to (t-\tau_j)$, $j=1,2$ is given [10]. By using (6) and (7) then

$$\left[u_i(t-\tau_j)\right]=\mathbf{L}(t-\tau_j)\mathbf{A}_i=\mathbf{X}(t-\tau_j)\mathbf{HA}_i, \quad j=1,2. \qquad (8)$$

where $\mathbf{X}(t-\tau_j)=\mathbf{X}(t)\mathbf{B}_{(-\tau_j)}$ for $j=1,2$ and also

$$\mathbf{B}_{(-\tau_j)}=\begin{bmatrix} \binom{0}{0}(-\tau_j)^0 & \binom{1}{0}(-\tau_j)^1 & \binom{2}{0}(-\tau_j)^2 & \cdots & \binom{N}{0}(-\tau_{js})^N \\ 0 & \binom{1}{1}(-\tau_j)^0 & \binom{2}{1}(-\tau_j)^1 & \cdots & \binom{N}{1}(-\tau_j)^{N-1} \\ 0 & 0 & \binom{2}{2}(-\tau_j)^0 & \cdots & \binom{N}{2}(-\tau_j)^{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{N}{N}(-\tau_j)^0 \end{bmatrix}.$$

Then Eq. (8) is written as

$$\left[u_i(t-\tau_j)\right] = \mathbf{X}(t)\mathbf{B}_{(-\tau_j)}\mathbf{H}\mathbf{A}_i, \quad j = 1,2. \tag{9}$$

Secondly, from (6), (7) and (9) the system matrices are presented as

$$\mathbf{u}(t) = \bar{\mathbf{X}}(t)\bar{\mathbf{H}}\mathbf{A}, \tag{10}$$

$$\mathbf{u}^{(1)}(t) = \bar{\mathbf{X}}(t)\bar{\mathbf{B}}\bar{\mathbf{H}}\mathbf{A}, \tag{11}$$

$$\mathbf{u}(t-\tau_j) = \bar{\mathbf{X}}(t)\bar{\mathbf{B}}_{(-\tau_j)}\bar{\mathbf{H}}\mathbf{A}, \quad j = 1,2, \tag{12}$$

where

$$\mathbf{u}(t) = \begin{bmatrix} \mathbf{u}_1(t) \\ \mathbf{u}_2(t) \\ \vdots \\ \mathbf{u}_N(t) \end{bmatrix}, \mathbf{u}^{(1)}(t) = \begin{bmatrix} \mathbf{u}_1^{(1)}(t) \\ \mathbf{u}_2^{(1)}(t) \\ \vdots \\ \mathbf{u}_N^{(1)}(t) \end{bmatrix}, \mathbf{u}(t-\tau_j) = \begin{bmatrix} \mathbf{u}_1(t-\tau_j) \\ \mathbf{u}_2(t-\tau_j) \\ \vdots \\ \mathbf{u}_N(t-\tau_j) \end{bmatrix}, \bar{\mathbf{X}}(t) = \begin{bmatrix} \mathbf{X}(t) & 0 & \dots & 0 \\ 0 & \mathbf{X}(t) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{X}(t) \end{bmatrix},$$

$$\bar{\mathbf{B}}_{(-\tau_j)} = \begin{bmatrix} \mathbf{B}_{(-\tau_j)} & 0 & \dots & 0 \\ 0 & \mathbf{B}_{(-\tau_j)} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{B}_{(-\tau_j)} \end{bmatrix}, \bar{\mathbf{B}} = \begin{bmatrix} \mathbf{B} & 0 & \dots & 0 \\ 0 & \mathbf{B} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{B} \end{bmatrix}, \bar{\mathbf{H}} = \begin{bmatrix} \mathbf{H} & 0 & \dots & 0 \\ 0 & \mathbf{H} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{H} \end{bmatrix},$$

and $\mathbf{A} = [\mathbf{A}_1 \ \mathbf{A}_2 \ \dots \ \mathbf{A}_N]^T$.

Thirdly, the matrix representation of the system (2) is shown by using (10), (11) and (12) as

$$\bar{\mathbf{X}}(t)\bar{\mathbf{B}}\bar{\mathbf{H}}\mathbf{A} + \bar{\mathbf{X}}(t)\bar{\mathbf{H}}\mathbf{A} = \sum_{j=1}^{2} \mathbf{p}_{ij}\mathbf{f}_j(\mathbf{u}(t-\tau_j)), \ i = 1,2, \tag{13}$$

where we describe $\mathbf{f}_j$ according to the matrices including the delay term in (12). Here

$$\mathbf{p}_{ij} = \begin{bmatrix} [\mathbf{p}_{11} \quad \mathbf{p}_{12}] & \mathbf{0} \\ \mathbf{0} & [\mathbf{p}_{21} \quad \mathbf{p}_{22}] \end{bmatrix}, \ \mathbf{f}_j(\mathbf{u}(t-\tau_j)) = \begin{bmatrix} [\mathbf{f}_1(\mathbf{u}(t-\tau_j))] \\ [\mathbf{f}_2(\mathbf{u}(t-\tau_j))] \end{bmatrix}.$$

Then the collocation points are defined as

$$t_l = \frac{1}{N} l, \quad l = 0,1,...,N. \tag{14}$$

where N is the number for the truncation of the series and $t \in [0,1]$. So that the collocation points are replaced into the Eq.(13)

$$\bar{\mathbf{X}}(t_l)\bar{\mathbf{B}}\bar{\mathbf{H}}\mathbf{A} + \bar{\mathbf{X}}(t_l)\bar{\mathbf{H}}\mathbf{A} = \sum_{j=1}^{2} \mathbf{p}_{ij}\mathbf{f}_j(\mathbf{u}(t_l - \tau_j)), \; i = 1,2, \tag{15}$$

then the fundamental matrix system is obtained as

$$\bar{\mathbf{X}}\bar{\mathbf{B}}\bar{\mathbf{H}}\mathbf{A} + \bar{\mathbf{X}}\bar{\mathbf{H}}\mathbf{A} = \sum_{j=1}^{2} \mathbf{p}_{ij}\mathbf{f}_j, \; i = 1,2, \tag{16}$$

Briefly, it is shown in the augumented matrix form as

$$\mathbf{WA} = \mathbf{F} \Rightarrow [\mathbf{W};\mathbf{F}] \tag{17}$$

At the fourth step of the technique, similarly, the matrix representations of the initial conditions are given as

$$\mathbf{u}_i(t_0) = \bar{\mathbf{X}}(t_0)\bar{\mathbf{H}}\mathbf{A} \text{ and } \boldsymbol{\lambda} = \begin{bmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_N \end{bmatrix}^T, \tag{18}$$

With the help of (10) $\mathbf{UA} = \boldsymbol{\lambda} \Rightarrow [\mathbf{U};\boldsymbol{\lambda}]$ is obtained. Finally, by replacing the rows in (17) we obtain

$$\tilde{\mathbf{W}}\tilde{\mathbf{A}} = \tilde{\mathbf{F}} \Rightarrow [\tilde{\mathbf{W}};\tilde{\mathbf{F}}]. \tag{19}$$

As a last step, we solve the new augumented system by using the Gauss Elimination method and we obtain the approximate solution in the series form which is defined in (4) [9]-[15].

3. Stability Analysis

In this section, stability analysis of the model is considered. The stability analysis of the system shows the properties of the dynamics [16].

Theorem: Let us consider the system (2) and the coordinates x and y. Then the stationary solutions of the system exist which is $(u_1, u_2) = (0,0)$. Moreover, we have the steady states (x, y) of the system as following

$$u_1 = p_{11}\tanh(u_1) + p_{12}\tanh(u_2) \text{ and } u_2 = p_{21}\tanh(u_1) + p_{22}\tanh(u_2). \tag{20}$$

Proof: First we consider the system (2) in explicit form, then

$$\begin{aligned} u_1'(t) &= -u_1(t) + p_{11} f_1(u_1(t-\tau_1)) + p_{12} f_2(u_2(t-\tau_2)), \\ u_2'(t) &= -u_2(t) + p_{21} f_1(u_1(t-\tau_1)) + p_{22} f_2(u_2(t-\tau_2)), \end{aligned} \tag{21}$$

where we consider $f(u) = \tanh(u)$. Then we can have it easily as

$$u_1(t) = p_{11}\tanh(u_1) + p_{12}\tanh(u_2),$$

$$u_2(t) = p_{21}\tanh(u_1) + p_{22}\tanh(u_2),$$

[16]-[18]. Then the characteristic equation is found as

$$(\lambda+1)^2 - (\lambda+1)\left(\alpha_{11}e^{-\lambda\tau_1} + \alpha_{22}e^{-\lambda\tau_2}\right) + \left(\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}\right)e^{-\lambda(\tau_1+\tau_2)} = 0. \tag{22}$$

4. Illustrative Example

Let us consider

$$\mathbf{u}^{(1)}(t) = \begin{bmatrix} 0 & 1 \\ -4 & -2 \end{bmatrix}\mathbf{u}(t) + \begin{bmatrix} 0 & 0 \\ -0.1 & 0 \end{bmatrix}\mathbf{u}(t-1) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\mathbf{f}(t), \quad t \in [0,1], \tag{23}$$

with the initial conditions $u_1(0)=1$ and $u_2(0)=1$, the delay terms $\tau_1 = \tau_2 = 1$ and $\mathbf{f}(t) = \exp(-t)$. Then we apply the technique for $N=8$ and $N=10$. The numerical technique is applied on the problem and it is obtained. Laguerre polynomial solution of the problem for $u_1(t)$, $N=8$ and $N=10$ is given in Table 1. Similarly, the numerical solution of the problem with the same technique for $u_2(t)$, $N=8$ and $N=10$ is given in Table 2. Then absolute error comparisons are shown in Table 3 and Table 4 for $u_1(t)$ and $u_2(t)$ $N=8$ and $N=10$, respectively.

Table 1. Numerical solution of the problem for $u_1(t)$, $N=8$ and $N=10$.

t	$N=8$	$N=10$
0.0	1.00000	1.00000
0.1	1.05315	1.05315
0.2	1.08801	1.08801
0.3	1.10620	1.10620
0.4	1.10890	1.10890
1.0	0.00921	0.00089

Table 2. Numerical solution of the problem for $u_2(t)$, $N=8$ and $N=10$.

t	$N=8$	$N=10$
0.0	1.00000	1.00000
0.1	0.06980	0.06980
0.2	0.41751	0.41751
0.3	0.16068	0.16068
0.4	-0.07165	-0.07165
1.0	-0.00112	-0.00112

Table 3. Absolute error comparison of $u_1(t)$, $N=8$ and $N=10$.

t	$E_N=E_8$	$E_N=E_{10}$
0.0	0.00000	0.00000
0.1	0.1695E-7	0.2103E-8
0.2	0.1594E-7	0.2218E-8
0.3	0.1382E-7	0.2195E-8
0.4	0.1819E-7	0.1820E-8
1.0	0.3702E-6	0.3605E-7

Table 4. Absolute error comparison of $u_2(t)$, $N=8$ and $N=10$.

t	$E_N=E_8$	$E_N=E_{10}$
0.0	0.00000	0.00000
0.1	0.1969E-7	0.2562E-8
0.2	0.1978E-7	0.2579E-8
0.3	0.1688E-7	0.1991E-8
0.4	0.1675E-7	0.1923E-8
1.0	0.1992E-6	0.1300E-7

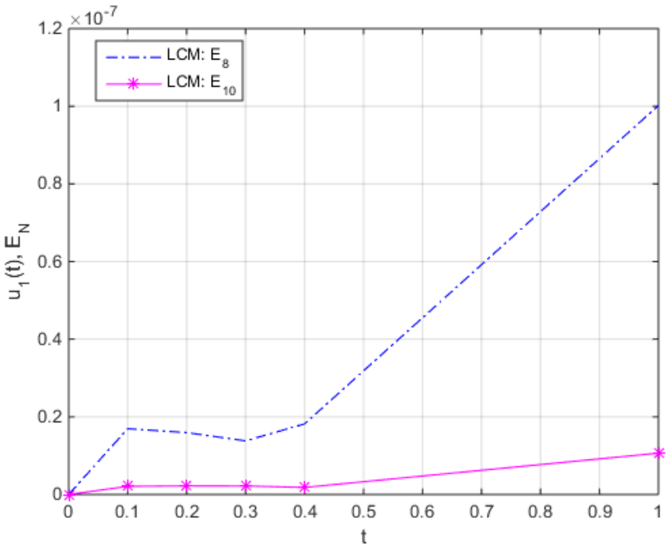

Figure 2. Comparison of the absolute errors for $u_1(t)$, $N=8$ and $N=10$.

In Fig. 2, absolute errors for the function $u_1(t)$ are given for the truncations $N=8$ and $N=10$. Similarly, the comparison for the function $u_2(t)$ with the same truncations is shown in Fig. 3.

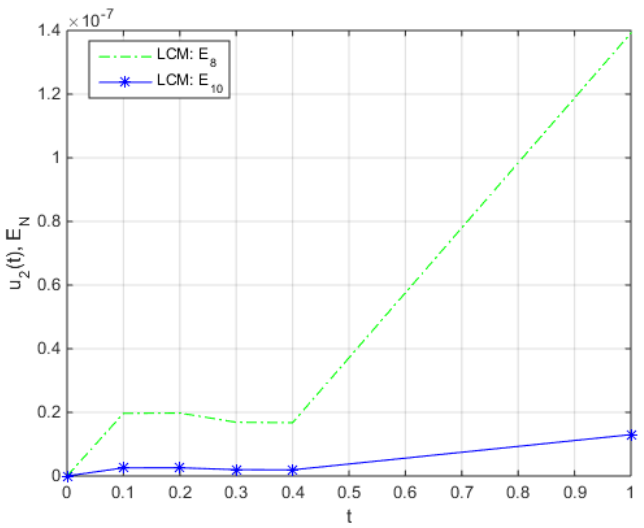

Figure 3. Comparison of the absolute errors for $u_2(t)$, $N=8$ and $N=10$.

The comparison of the Laguerre polynomial solution of the problem for $u_1(t)$, $N=8$ with Haar wavelet and fourth order Runge Kutta method solutions is shown in Fig. 4.

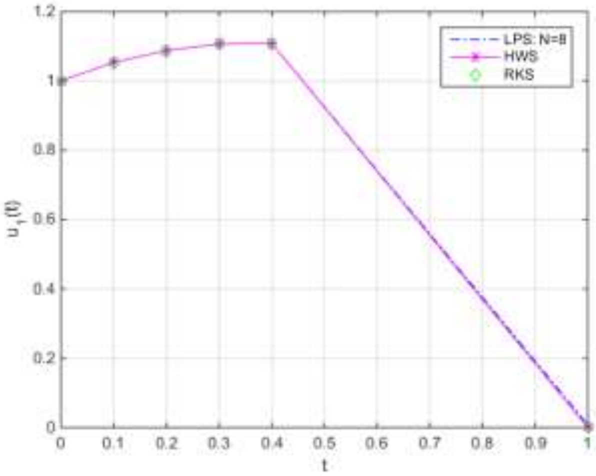

Figure 4. Comparison of the approximate solutions as Laguerre polynomial solution (LPS) for $N=8$, Haar wavelet solution (HWS) and Runge Kutta solution (RKS) of $u_1(t)$.

5. Conclusion

In this study, a numerical investigation on neural network model is considered. A numerical technique is presented and the stability analysis of the model is also given. Accuracy of the solution is reached and the results are given by tables and figures.

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