
# 3. International Conference on Mathematics:"An Istanbul Meeting for World Mathematicians"

**ICOM 2019 Conference Proceedings Book**

**ISBN: 978-605-67964-3-2**

**Editor**
**Kenan Yıldırım**

**Special Thanks to Our Supporters:**

## Chairman's Welcome Speech

Dear Colleagues ans Dear Guests,

On behalf of the organizing committee, welcome to 3. International Conference on Mathematics: *An Istanbul Meeting for World Mathematicians*, 3-5 July 2019, Istanbul, Turkey. First of all, we present our deepest thanks to Fatih Sultan Mehmet Vakif University Management due to their great hospitality and understanding. Also, we present our deepest thanks to our supporters; Muş Alparslan University, Çobanpınar Drinking Water, Zeytinburnu Municipality(in Istanbul) and Istanbul Metropolitan Municipality, Turkish Airlines and others.

The conference aims to bring together leading academic scientists, researchers and research scholars to exchange and share their experiences and research results about mathematical sciences.

Besides these academic aims, we also have some social programs for introducing our culture and Istanbul to you. We hope that you will have nice memories in Istanbul for conference days.

We wish to all participants efficient conference and nice memories in Istanbul.

Thank you very much for your interest in International Conference on Mathematics: *An Istanbul Meeting for World Mathematicians*.

Kenan YILDIRIM, Ph. D.

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Kenan Yıldırım

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**INTERNATIONAL CONFERENCE ON MATHEMATICS**
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# Parsing and Checking Mathematics Written in a Morse Language

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**Abstract**

When mathematics is written formally it becomes possible to machine check the validity of its syntax and logic. ProofCheck is a Python package, available since 2009 at www.proofcheck.org, which parses mathematical texts and checks mathematical proofs and inference rule derivations contained in a plain TeX or a LaTeX source file. Its mathematical syntax requirements are based on rules developed by A. P. Morse, which allow great flexibility in the mode of mathematical expression. Checkable proofs must also conform to rules of proof syntax developed by the authors. A summary of texts parsed and proofs and derivations checked is given.

**Keywords:** Proof checking, Commutative-Associative Unification, Derived Rules of Inference, Morse language, Python, TeX.

## 1. Introduction

One of the first forays into machine checking of mathematics was Bledosoe's work, [3], in checking proofs of the theorems in A.P. Morse's *A Theory of Sets* [9]. It remains relatively unique in that it was applied to pre-existing theorems, if not proofs, written without machine checking in mind. Proofs today are often checked by systems such as Mizar [13], HOL [7], and Isabelle [12]. These are big systems, with current download sizes of 35 Meg, 12.5 Meg, and 196 Meg respectively. By contrast ProofCheck, [10], which in many ways continues Bledsoe's early work, has a download size of 200K. We show here that in spite of its small size it can and has done useful work.

## 2. Preliminaries

Presented with a mathematical text in the form of a TeX file, ProofCheck constructs a grammar for the given text based on definitions encountered in the text, which must be of the form:

$$(x = y)$$

where 'x' is the defined term or:

$$(p \leftrightarrow q)$$

where 'p' is the defined formula.

Theorems whose proofs are to be checked or referred to in proofs of other theorems, must be numbered in the form **num.num**. A proof is a series of **notes**, numbered in the form **.num** directly following the numbered theorem,each of which is supplied with a justification which is an expression referring

to previous notes in the same proof or to other numbered theorems. This justification is combined with the note being checked to form an expression which in order to check successfully must "match" a stored rule of inference. An example of a rule of inference is the modus ponens rule of inference which is simply the following expression:

$$(p \rightarrow q) ; p \mid\!- q$$

Rules of Inference are stored simply as lines of a TeX file. The checking of a note consists simply of a sequential search through this file. A justification for a note 'q' matching this rule might be '1.2;.3' referring to a theorem 1.2 of the form $(p \rightarrow q)$ and to a note 3 of the form p. A proof is checked when all its notes check. This proof syntax can be used to check either proofs of theorems, or derivations of derived rules of inference. This is valuable especially in the case of a non-standard logic whose theorems and rules of inference can be jointly derived in a single **logic development file**.

A note with its justification matches a rule of inference if there is a **unification** of the two. A unification of two expressions A and B is a substitution s such that s(A) = s(B). In ordinary first order unification the substitution, s, simply replaces variables by terms. If the variables occurring in the note being checked are treated as constants, ineligible for replacement, then a successful unification shows that the note can be obtained by substitutions from the other formulas referred to in the justification.

ProofCheck's unification algorithm is a second order algorithm. As such in addition to replacing ordinary first order variables it also replaces second order expressions such as those we write here as f(x) and f(y) occurring in

$$(f(x) < f(y))$$

to forrm instances such as

$$(x + 1 < y + 1)$$

The algorithm also unifies bound variables so that terms such as {x:p(x)} and {y:p(y)} are not distinguished. An important feature is that operators which are known to be both commutative and associative can be treated as such. The '+' operator for example is known to be commutative and associative and if so marked it allows (x + y + z) to unify immediately with (z + x + y), for example. Without this feature checkable proofs would quickly become prohibitively lengthy. Commutative associative unification is a special case of Gallier's E-unification, [6].

To flesh out the very sketchy description just given, the reader may consult the documentation located on the website: http://www.proofcheck.org or download the package itself.

## 3. Main Results

A compendium of A.P. Morse's works comprising the bulk of his entire mathematical output including all of [9] and currently exceeding 400 pages in length, is being prepared for publication by the second author. This work in progress has been completely checked for the correctness of its mathematical syntax. Numerous typos were eliminated and a mathematical error was discovered in one of Morse's published papers, (Theorem 6.24 on p270 of [5]).

A book on Constructive Morse Set Theory which is currently in preparation by Douglas Bridges, well-known among constructivists for [4], and R.A. Alps, had its logic chapter completely checked. This check included verification of the derivations of 95 rules of inference as well as of the proofs of its 150 theorems of logic. With no details omitted, the chapter still amounted to less than 40 pages. Constructive logic is, of course, non-standard, eschewing the law of the excluded middle.

A proof by R.A. Alps of the relative consistency of Morse's set theory with Kelley-Morse set theory, (see the Appendix of [8]), was checked.

Over 500 proofs in a mathematical manuscript consisting of an exploratory development of combinatorial topology and all of the 71 proofs in a submitted paper dealing with induction were checked. The proofs were based on the predicate logic, [2], and a file of over 1500 rules of inference.

A file consisting of a development of relevant logic by Viraga Perera was checked. Relevant logic is a non-standard sentential logic which disallows inferences such as (Not $p \rightarrow (p \rightarrow q)$) and demands that hypotheses of an inference be relevant to the conclusion.

The proof checker has also used to teach proof construction in a discrete mathematics class.

The just mentioned developments of Bridges' logic and relevant logic used only rules of inference, except for a small number of primitive rules, whose derivations were checked. The other mathematical proofs were checked using a large file of over 1500 rules of inference, which we claim are derivable but are yet without formal derivations.

## 4. Conclusion

The ProofCheck system represents a different approach from other computer proof assistants such as Mizar, HOL, and Isabelle in that it is less aimed at adapting mathematics to the computer and more towards adapting the computer to the mathematics. The fact that its source files are TeX files testifies to this. So too does the fact that its syntactical basis, [11], derived from that of a practicing mathematician, is open to the syntax and logic desired by the user, at least to the extent that we have been able to make it so.

The fact that the mathematical proofs that we have so far checked tend to be from 3 to 10 times longer than informal proofs shows that there is still plenty of room for improvement. The fact that this ratio is much less in the logic development files so far checked, suggests that allowing derived rules of inference to contain mathematical, not just logical content, and allowing them to be interspersed with theorems, suggests itself as a possible way forward.

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# Controlled Hodgkin-Huxley Network Model for Suppressing Epileptic Seizures Detected from the Intracranial EEG Signals

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**Abstract**

Hodgkin-Huxley (HH) mathematical neuron model describes the formation and propagation of action potentials along the axons due to the change of intra- and extracellular concentration of ions passing through the membrane lipid bilayer. HH cells demonstrate the variety of nonlinear dynamical regimes including single spikes, spiking trains and bursting.

The individual neuron-based control elements in small HH clusters developed by Borisenok, Çatmabacak, Ünal (2018) are capable to detect a hypersynchronized epileptiform behavior of the cluster and efficiently suppress it by sending feedback signals to other network elements. Here we extend our approach with the more realistic case of network links where the neurons form a memory trace with the effect of synaptic consolidation, covering the contribution from the neurotransmitter release.

We discuss the perspectives of the intracranical electroencephalogram (EEG) modeling. EEG is recorded from the electrodes implanted into the epileptic focus and represent the total activity for a large number of neurons rather than the impulse activity of stand-alone ones. The EEG data we use include both pre-ictal and ictal phases. The goal of modeling is to identify the pre-ictal phase (or, at least, its beginging) in the observed EEG and design the electrode control stimulus acting on the tissue to suppress the seizure.

Thus, the controlled Hodgkin-Huxley network proposed here can be implemented for modeling the epileptic seizures suppressing by electrical stimulating through the electrode from which the intracranial EEG signal is observed.

**Keywords: Keywords: Epileptiform Behavior, Hodgkin-Huxley Neurons, Control in Small Networks, Intracranical EEG**

## 1. Introduction

Modern neuroscience demonstrates a great progress in study of the epileptic dynamics in neurons. Experimental methods allow now to make an efficient control of the neural population even at the single cell level. Nevertheless, the mathematical modeling of an epileptiform behavior is mostly focused on the upper scales of the brain network. The applied control theory still needs a sufficient improvement of algorithms for detection and suppression of the epileptiform behavior in the populations of mathematical neurons modeling a real ictal phase of epilepsy.

We discuss here the perspectives of the intracranical electroencephalogram (EEG) modeling. EEG is recorded from the electrodes implanted into the epileptic focus and represents the total activity of a large number of neurons rather than the impulse activity of stand-alone ones. The EEG data we use include both pre-ictal and ictal phases. The goal of modeling is to identify the ictal phase (or, at least, its beginning) in the observed EEG and design the electrode control stimulus acting on the tissue to suppress the seizure.

## 2. Mathematical Network Model for the Epileptiform Behavior

Hodgkin-Huxley (HH) mathematical neuron model describes the formation and propagation of action potentials along the axons due to the change of intra- and extracellular concentration of ions passing through the membrane lipid bilayer. The microscopic detailed mathematical modeling of real neural cells can be represented with the phenomenological nonlinear systems of ordinary differential equations. Each HH element contains four independent variables, one for the spiking action potential and three for the probabilities of the membrane ion gates to be open or closed [1]:

$$\begin{aligned} C_M \cdot \frac{dv}{dt} &= -g_{Na} m^3 h \cdot (v - E_{Na}) - g_K n^4 \cdot (v - E_K) - g_{Cl} \cdot (v - E_{Cl}) + I(t); \\ \frac{dm}{dt} &= \alpha_m \cdot (1-m) - \beta_m \cdot m ; \\ \frac{dn}{dt} &= \alpha_n \cdot (1-n) - \beta_n \cdot n ; \\ \frac{dh}{dt} &= \alpha_h \cdot (1-h) - \beta_h \cdot h . \end{aligned} \tag{1}$$

Here $v(t)$ stands for the membrane potential, $m(t)$, $n(t)$, $h(t)$ are the membrane gate variables, and the control signal $I(t)$ is the sum of currents stimulating the cell. $\alpha_{m,n,h}$, $\beta_{m,n,h}$ are experimentally found functions related to the gate probabilities and given by [1]:

$$\begin{aligned}
&\alpha_m(v) = \frac{0.1\cdot(25-v)}{\exp\left\{\frac{25-v}{10}\right\}-1}\ ;\ \beta_m(v) = 4\cdot\exp\left\{-\frac{v}{18}\right\};\\
&\alpha_n(v) = \frac{0.01\cdot(10-v)}{\exp\left\{\frac{10-v}{10}\right\}-1}\ ;\ \beta_n(v) = 0.125\cdot\exp\left\{-\frac{v}{80}\right\};\\
&\alpha_h(v) = 0.07\cdot\exp\left\{-\frac{v}{20}\right\};\ \beta_h(v) = \frac{1}{\exp\left\{\frac{30-v}{10}\right\}+1}.
\end{aligned}\tag{2}$$

The set of constants in (1) includes the potentials $E_{Na}$ (the equilibrium potential at which the net flow of $Na$ ions is zero), $E_K$ (the equilibrium potential at which the net flow of $K$ ions is zero), $E_{Cl}$ (the equilibrium potential at which the leakage is zero) in mV, the membrane capacitance $C_M$ and the conductivities $g_{Na}$ (the sodium channel conductivity), $g_K$ (potassium channel conductivity), $g_{Cl}$ (leakage channel conductivity) in mS/cm$^2$:

$$\begin{aligned}
&g_{Na} = 120;\ E_{Na} = 115;\\
&g_K = 36;\quad E_K = -12;\\
&g_{Cl} = 0.3;\ E_{Cl} = 10.36.
\end{aligned}\tag{3}$$

HH cells demonstrate the variety of nonlinear dynamical regimes including single spikes, spiking trains and bursting.

Different models could be applied for the synaptic links in the HH network, like the model od synaptic plasticity and synaptic consolidation [2]. It may cover the time delay in the process of transmission of signals in synapses and integration and accumulation of the large number of stimulating signals entering the control neuron.

The individual neuron-based control elements in small HH clusters are capable to detect and suppress a hypersynchronized epileptiform behavior of the cluster and efficiently suppress it by sending feedback signals to other network elements to force them to switch their bursting behavior to resting via the speed-gradient based control algorithm [3]. Such control element can work autonomously at the each phase of pre-ictal and ictal epileptiform dynamics. The control algorithm is numerically simple and robust, i.e. it is stable under the external noisy perturbation and does not depend on the initial states of the differential equations representing HH neurons.

### 3. Modeling of Ictal Phase Recorded by Intracranical Electroencephalogram

Intracranical electroencephalogram records the signals made from electrodes implanted into the epileptic focus. It integrates the total activity of a large neuron population rather than impulse activity of individual neurons. Each patient has an EEG record in the seizure and between the seizures.

The ictal phase detection could be performed by implementing the Neural Clouds (NC), a data encapsulation method, which provides a confidence measure regarding classification of the complex system conditions. It has been already successfully applied for surface EEG analysis [4]. The NC method may be considered as a detector of the critical changes in the dynamics detected by intracranical EEG records.

The mathematical model developing in our approach covers few basic features of the control over the ictal phase in the recorded intracranical EEG:

• The transition from neuronal impulse activity to EEG signals recorded from the implanted electrodes and back.

• Modeling of seizures suppressing via electrical stimulating trough the electrode from which the intracranial EEG signal is observed.

Thus, the control electrode plays a double rope in our approach, like in the mathematical model [3] for small neural clusters: it detects the ictal phase and generates the signal suppressing the focus of seizure via the feedback links. For the practical clinic application, it is much easier to learn how to link impulse activity from the implanted electrode rather than from the surface EEG stimulation.

### 4. Conclusions

The open question of our approach for the present moment is to check whether there are any differences in the models obtained for different groups of patients. In this case, we will single out the groups according to results of other clinical observations.

Thus, the controlled Hodgkin-Huxley network proposed here can be implemented for modeling the epileptic seizures suppressing by electrical stimulating through the electrode from which the intracranial EEG signal is observed. The extended version of our model approach will lead us to the development of a simulator for epileptiform dynamics to minimize risks of experiments over real epileptic patients.

## 5. Acknowledgement

Study was supported by the Russian Foundation for Basic Research (RFBR), grant # 18-013-01086.

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# Solution of the Rational Difference Equation $x_{n+1} = \dfrac{x_{n-11}}{1 + x_{n-3}x_{n-7}}$

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**Abstract**

Recently, a high attention to studying the periodic nature of nonlinear difference equations has been attracted. Difference equations are used in a variety of contexts, such as in economics to model the evolution through time of variables such as gross domestic product, the inflation rate, the exchange rate, etc. They are used in modeling such time series because values of these variables are only measured at discrete intervals. In econometric applications, linear difference equations are modeled with stochastic terms in the form of autoregressive (AR) models and in models such as vector autoregression (VAR) and autoregressive moving average (ARMA) models that combine AR with other features.

The behaviour of the solutions of the following system of difference equations is examined,

$$x_{n+1} = \frac{x_{n-11}}{1 + x_{n-3}x_{n-7}}$$

where the initial conditions are positive real numbers. The initial conditions of the equation are arbitrary positive real numbers.

**Keywords:** Difference equation, rational difference equation, period 4 solutions.

## 1. Introduction

Difference equations appear naturally as discrete analogs and as numerical solutions of differential and delay differential equations, having applications in biology, ecology, physics.

Recently, a high attention to studying the periodic nature of nonlinear difference equations has been attracted. For some recent results concerning the periodic nature of scalar nonlinear difference equations, among other problems, see, for example, [1-33].

Agarwal et al. [1], investigated the global stability, periodicity character and gave the solution of some special cases of the difference equation

$$x_{n+1} = a + \frac{dx_{n-l}x_{n-k}}{b - cx_{n-s}}.$$

Elsayed [15], studied the global result, boundedness, and periodicity of solutions of the difference equation

$$x_{n+1} = a + \frac{bx_{n-l} + cx_{n-k}}{dx_{n-l} + ex_{n-k}},$$

where the parameters a; b; c; d and e are positive real numbers and the initial conditions are positive real numbers where $t = \max\{l, k\}, l \neq k$.

Simsek et. al. [21, 22, 23, 26], studied the following problems with positive initial values

$$x_{n+1} = \frac{x_{n-3}}{1 + x_{n-1}},$$

$$x_{n+1} = \frac{x_{n-5}}{1 + x_{n-2}},$$

$$x_{n+1} = \frac{x_{n-5}}{1 + x_{n-1}x_{n-3}},$$

respectively.

In this work the following nonlinear difference equation was studied,

$$x_{n+1} = \frac{x_{n-11}}{1 + x_{n-3}x_{n-7}}, \tag{1.1}$$

where the initial conditions are positive real numbers.

## 2. Main Results

Let $\overline{x}$ be the unique positive equilibrium of the equation (1.1), then clearly,

$$\overline{x} = \frac{\overline{x}}{1 + \overline{x}\overline{x}} \Rightarrow \overline{x} + \overline{x}^3 = \overline{x} \Rightarrow \overline{x}^3 = 0 \Rightarrow \overline{x} = 0,$$

so, $\overline{x} = 0$ can be obtained.

**Theorem 1:** Consider the difference equation (1.1). Then the following statements are true.

a) The sequences,

$$(x_{12n-11}),(x_{12n-10}),\ \ldots,\ (x_{12n-1}),\ (x_{12n}),$$

are being decreasing and,

$$a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9,a_{10},a_{11},a_{12}\geq 0,$$

are existed and such that,

$$\lim_{n\to\infty} x_{12n-11+k}=a_{1+k}\ \text{ for }\ k=\overline{0,11}.$$

b) $(a_1,a_2,a_3,a_4,a_5,\mathrm{a}_6,a_7,a_8,a_9,a_{10},a_{11},a_{12},...)$ is a solution of equation (1.1) having period twelve.

c) $\prod_{k=0}^{2}\lim_{n\to\infty} x_{12n-11-j+4k}=0,\ j=\overline{0,3}$ or $\prod_{k=0}^{2} a_{4k+i}=0,\ i=\overline{1,4}$.

d) If there exist $n_0\in\mathbb{N}$ such that $x_{n-7}\geq x_{n+1}$ for all $n\geq n_0$ , then,

$$\lim_{n\to\infty} x_n=0.$$

e) The following formulas below are hold:

$$x_{12n+k+1}=x_{-11+k}\left(1-\frac{x_{-3+k}x_{-7+k}}{1+x_{-3+k}x_{-7+k}}\sum_{j=0}^{n}\prod_{i=1}^{3j}\frac{1}{1+x_{4i-7+k}x_{4i-3+k}}\right),$$

$$x_{12n+k+5}=x_{-7+k}\left(1-\frac{x_{-3+k}x_{-11+k}}{1+x_{-3+k}x_{-7+k}}\sum_{j=0}^{n}\prod_{i=1}^{3j+1}\frac{1}{1+x_{4i-7+k}x_{4i-3+k}}\right),$$

$$x_{12n+k+9}=x_{-3+k}\left(1-\frac{x_{-7+k}x_{-11+k}}{1+x_{-3+k}x_{-7+k}}\sum_{j=0}^{n}\prod_{i=1}^{3j+2}\frac{1}{1+x_{4i-7+k}x_{4i-3+k}}\right),$$

$k=\overline{0,3}$ holds.

f) If $x_{12n+1+k}\to a_{1+k}\neq 0$, $x_{12n+5+k}\to a_{5+k}\neq 0$ then $x_{12n+9+k}\to a_{9+k}=0$ as $n\to\infty$, $k=\overline{0,3}$.

**Proof:**

a) Firstly, from the equation (1.1)

$$x_{n+1}\left(1+x_{n-3}x_{n-7}\right)=x_{n-11},$$

is obtained. If $x_{n-3}x_{n-7}\in\left(0,\infty\right)$ then $1+x_{n-3}x_{n-7}\in\left(1,\infty\right)$. Since,

$$x_{n+1}<x_{n-11},$$

$n\in\mathbb{N}$,

$$\lim_{n\to\infty}x_{12n-11+k}=a_{1+k}\ \text{ for }\ k=\overline{0,11},$$

existed formulas are obtained.

b) $\left(a_1,a_2,a_3,a_4,a_5,\mathrm{a}_6,a_7,a_8,a_9,a_{10},a_{11},a_{12},\ ...\right)$ is a solution of equation (1.1) having period twelve.

c) In wiew the equation (1.1),

$$n=12n\Rightarrow x_{12n+1}=\frac{x_{12n-11}}{1+x_{12n-3}x_{12n-7}}$$

is obtained. Similarly for $n=12n+1$,

$$n=12n+1\Rightarrow x_{12n+2}=\frac{x_{12n-10}}{1+x_{12n-2}x_{12n-6}},$$

is obtained. Similarly for $n=12n+2$,

$$n=12n+2\Rightarrow x_{12n+3}=\frac{x_{12n-9}}{1+x_{12n-1}x_{12n-5}},$$

is obtained. Similarly for $n=12n+3$,

$$n=12n+3\Rightarrow x_{12n+4}=\frac{x_{12n-8}}{1+x_{12n}x_{12n-4}},$$

is obtained. If the limits are put on both sides of the above equality,

$$\prod_{k=0}^{2}\lim_{n\to\infty}x_{12n-11-j+4k}=0,\ j=\overline{0,3}\ \text{ or }\ \prod_{k=0}^{2}a_{4k+i}=0,\ i=\overline{1,4}.$$

d) If there exist $n_0\in\mathbb{N}$ such that $x_{n-7}\ge x_{n+1}$ for all $n\ge n_0$, then,

$a_1\le a_5\le a_9\le a_1$, $a_2\le a_6\le a_{10}\le a_2$, $a_3\le a_7\le a_{11}\le a_3$, $a_4\le a_8\le a_{12}\le a_4$. Using (c), we get

$$\prod_{k=0}^{2}a_{4k+i}=0,\ i=\overline{1,4}.$$

Then we see that

$$\lim_{n\to\infty}x_n=0.$$

Hence the proof of (d) completed.

e) Subracting $x_{n-11}$ from the left and right-hand sides in equation (1.1),

$$x_{n+1}-x_{n-11}=\frac{1}{1+x_{n-3}x_{n-7}}\left(x_{n-3}-x_{n-15}\right),$$

and the following formula: for $n\ge 4$

$$\left.\begin{aligned}
x_{4n-15}-x_{4n-27}&=(x_1-x_{-11})\prod_{i=1}^{n-4}\frac{1}{1+x_{4i-3}x_{4i-7}}\\
x_{4n-14}-x_{4n-26}&=(x_2-x_{-10})\prod_{i=1}^{n-4}\frac{1}{1+x_{4i-2}x_{4i-6}}\\
x_{4n-13}-x_{4n-25}&=(x_3-x_{-9})\prod_{i=1}^{n-4}\frac{1}{1+x_{4i-1}x_{4i-5}}\\
x_{4n-12}-x_{4n-24}&=(x_4-x_{-8})\prod_{i=1}^{n-4}\frac{1}{1+x_{4i}x_{4i-4}}
\end{aligned}\right\}\tag{1.2}$$

hold. Replacing n by $3j$ in (1.2) and summing from $j=0$ to $j=n$, we obtain:

$$x_{12n+1+k}-x_{-11+k}=\left(x_{1+k}-x_{-11+k}\right)\sum_{j=0}^{n}\prod_{i=1}^{3j}\frac{1}{1+x_{4i-3+k}x_{4i-7+k}},\ k=\overline{0,3}.$$

Also, $3j+1$ inserted in (1.2) by replacing $n$, $j=0$ to $j=n$ is obtained by summing,

$$x_{12n+5+k} - x_{-7+k} = \left(x_{5+k} - x_{-7+k}\right)\sum_{j=0}^{n}\prod_{i=1}^{3j+1}\frac{1}{1+x_{4i-3+k}x_{4i-7+k}},\ k=\overline{0,3}.$$

Also, $3j+2$ inserted in (1.2) by replacing $n$, $j=0$ to $j=n$ is obtained by summing,

$$x_{12n+9+k} - x_{-3+k} = \left(x_{9+k} - x_{-3+k}\right)\sum_{j=0}^{n}\prod_{i=1}^{3j+2}\frac{1}{1+x_{4i-3+k}x_{4i-7+k}},\ k=\overline{0,3}.$$

Now we obtained of the above formulas:

$$x_{12n+k+1} = x_{-11+k}\left(1-\frac{x_{-3+k}x_{-7+k}}{1+x_{-3+k}x_{-7+k}}\sum_{j=0}^{n}\prod_{i=1}^{3j}\frac{1}{1+x_{4i-7+k}x_{4i-3+k}}\right),$$

$$x_{12n+k+5} = x_{-7+k}\left(1-\frac{x_{-3+k}x_{-11+k}}{1+x_{-3+k}x_{-7+k}}\sum_{j=0}^{n}\prod_{i=1}^{3j+1}\frac{1}{1+x_{4i-7+k}x_{4i-3+k}}\right),$$

$$x_{12n+k+9} = x_{-3+k}\left(1-\frac{x_{-7+k}x_{-11+k}}{1+x_{-3+k}x_{-7+k}}\sum_{j=0}^{n}\prod_{i=1}^{3j+2}\frac{1}{1+x_{4i-7+k}x_{4i-3+k}}\right),$$

$k=\overline{0,3}$ holds.

f) Suppose that $a_{1+k}=a_{5+k}=a_{9+k}=0$. By e) we have, for $k=\overline{0,3}$

$$\lim_{n\to\infty} x_{12n+1+k} = \lim_{n\to\infty} x_{-11+k}\left(1-\frac{x_{-3+k}x_{-7+k}}{1+x_{-3+k}x_{-7+k}}\sum_{j=0}^{n}\prod_{i=1}^{3j}\frac{1}{1+x_{4i-3+k}x_{4i-7+k}}\right),$$

$$a_{1+k} = x_{-11+k}\left(1-\frac{x_{-3+k}x_{-7+k}}{1+x_{-3+k}x_{-7+k}}\sum_{j=0}^{\infty}\prod_{i=1}^{3j}\frac{1}{1+x_{4i-3+k}x_{4i-7+k}}\right),$$

$$a_{1+k}=0 \Rightarrow \frac{1+x_{-3+k}x_{-7+k}}{x_{-3+k}x_{-7+k}} = \sum_{j=0}^{\infty}\prod_{i=1}^{3j}\frac{1}{1+x_{4i-3+k}x_{4i-7+k}}. \tag{1.3}$$

Similarly,

$$a_{5+k}=0 \Rightarrow \frac{1+x_{-3+k}x_{-7+k}}{x_{-3+k}x_{-11+k}} = \sum_{j=0}^{\infty}\prod_{i=1}^{3j+1}\frac{1}{1+x_{4i-3+k}x_{4i-7+k}}. \tag{1.4}$$

Similarly,

$$a_{9+k} = 0 \Rightarrow \frac{1+x_{-3+k}x_{-7+k}}{x_{-7+k}x_{-11+k}} = \sum_{j=0}^{\infty}\prod_{i=1}^{3j+2}\frac{1}{1+x_{4i-3+k}x_{4i-7+k}}. \tag{1.5}$$

From the (1.3) and (1.4),

$$\frac{1+x_{-3+k}x_{-7+k}}{x_{-3+k}x_{-7+k}} = \sum_{j=0}^{\infty}\prod_{i=1}^{3j}\frac{1}{1+x_{4i-3+k}x_{4i-7+k}} > \frac{1+x_{-3+k}x_{-7+k}}{x_{-3+k}x_{-11+k}} = \sum_{j=0}^{\infty}\prod_{i=1}^{3j+1}\frac{1}{1+x_{4i-3+k}x_{4i-7+k}},$$

thus, $x_{-11+k} > x_{-7+k}$. From the (1.4) and (1.5),

$$\frac{1+x_{-3+k}x_{-7+k}}{x_{-3+k}x_{-11+k}} = \sum_{j=0}^{\infty}\prod_{i=1}^{3j+1}\frac{1}{1+x_{4i-3+k}x_{4i-7+k}} > \frac{1+x_{-3+k}x_{-7+k}}{x_{-7+k}x_{-11+k}} = \sum_{j=0}^{\infty}\prod_{i=1}^{3j+2}\frac{1}{1+x_{4i-3+k}x_{4i-7+k}},$$

thus, $x_{-7+k} > x_{-3+k}$.

From here we obtain $x_{-11+k} > x_{-7+k} > x_{-3+k}$ for $k = \overline{0,3}$. We arrive at a contradiction which completes the proof of theorem.

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**INTERNATIONAL CONFERENCE ON MATHEMATICS**
**"An Istanbul Meeting for World Mathematicians"**
**3-5 July 2019, Istanbul, Turkey**

# Solution of the Maximum of Difference Equation

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**Abstract**

There has been a huge interest in max type of, nonlinear difference equations or systems in recent years. Difference equations are used in a variety of contexts, such as in economics to model the evolution through time of variables such as gross domestic product, the inflation rate, the exchange rate, etc. They are used in modeling such time series because values of these variables are only measured at discrete intervals. In econometric applications, linear difference equations are modeled with stochastic terms in the form of autoregressive (AR) models and in models such as vector autoregression (VAR) and autoregressive moving average (ARMA) models that combine AR with other features.

We study the behaviour of the solutions of the following system of difference equation with the max operator:

$$x_{n+1} = \max\left\{\frac{A}{x_{n-1}}, \frac{y_n}{x_n}\right\}; y_{n+1} = \max\left\{\frac{A}{y_{n-1}}, \frac{x_n}{y_n}\right\} \tag{1.1}$$

where the parameter $A$ and initial conditions are positive real numbers.

**Keywords:** Difference equations, Maximum Operators, Semi-cycle.

## 1. Introduction

Recently, there has been a great concern in studying nonlinear difference equations since many models describing real life situations in population biology, economics, probability theory, genetics, psychology, sociology etc. are represented by these equations. See for example [1-28].

## 2. Preliminaries

**Definition 1:** Let $I$ be an interval of real numbers and let $f : I^{s+1} \to I$ be a continuously differentiable function where s is a non-negative integer. Consider the difference equation

$$x_{n+1} = f\left(x_n, x_{n-1}, ..., x_{n-s}\right) \text{ for } n = 0,1,..., \tag{1.2}$$

with the initial values $x_{-s}, ..., x_0 \in I$. A point $\overline{x}$ called an equilibrium point of equation.(2) if $\overline{x} = f\left(\overline{x}, ..., \overline{x}\right)$.

**Definition 2:** A positive semi-cycle of a solutions $\{x_n\}_n^{\infty} = -s$ of equation (1.2) consist of a string of terms $\{x_l, x_{l+1}, ..., x_m\}$ all greater than or equal to equilibrium $\bar{x}$ with $l \geq -s$ and $m \leq \infty$ such that either $l = -s$ or $l > -s$ and $x_{l-1} < \bar{x}$ and either $m = \infty$ or $m < \infty$ and $x_{m+1} < \bar{x}$.

**Definition 3:** A negative semicycle of a solutions $\{x_n\}_n^{\infty} = -s$ of equation (1.2) consist of a string of terms $\{x_l, x_{l+1}, ..., x_m\}$ all less than or equal to equilibrium $\bar{x}$ with $l \geq -s$ and $m \leq \infty$ such that either $l = -s$ or $l > -s$ and $x_{l-1} \geq \bar{x}$ and either $m = \infty$ or $m \leq \infty$ and $x_{m+1} \geq \bar{x}$.

## 3. Main Results

Let $\bar{x}$ and $\bar{y}$ be the unique positive equilibrium of equation.(1), then clearly,

$$\bar{x} = \max\left\{\frac{A}{\bar{x}}, \frac{\bar{y}}{\bar{x}}\right\}; \bar{y} = \max\left\{\frac{A}{\bar{y}}, \frac{\bar{x}}{\bar{y}}\right\},$$

Since parameter $A$ is the greatest value in all initial conditions, so

$$\bar{x} = \frac{A}{\bar{x}} \Rightarrow \bar{x}^2 = A \Rightarrow \bar{x} = \pm\sqrt{A},$$

$$\bar{y} = \frac{A}{\bar{y}} \Rightarrow \bar{y}^2 = A \Rightarrow \bar{y} = \pm\sqrt{A},$$

we interested positive results so, we can obtain $\bar{x} = \sqrt{A}$ and $\bar{y} = \sqrt{A}$.

**Lemma1:** Assume that**,** $A \in (1, \infty)$ $x_0, x_{-1}, y_0, y_{-1} \in (1, \infty)$ integer sequence for equation (1) $A > x_{-1} > y_0 > y_{-1} > x_0$, $A > x_0 > y_{-1} > y_0 > x_0$, $A > y_0 > x_{-1} > x_0 > y_{-1}$, $A > y_{-1} > x_{-1} > y_0 > x_0$,

Then the following statements are true:

$n \geq 0$ for $x_n$ and $y_n$,

a) Every positive semi-cycle consist two term.

b) Every negative semi-cycle consist two term.

c) Every positive semi-cycle of length two is followed by a negative semi-cycle of length two.

d) Every negative semi-cycle of length two is followed by a positive semi-cycle of length two.

**Proof:**

$A > x_{-1} > y_0 > y_{-1} > x_0$, $A > x_0 > y_{-1} > y_0 > x_0$, $A > y_0 > x_{-1} > x_0 > y_{-1}$, $A > y_{-1} > x_{-1} > y_0 > x_0$,

The solution $x_n$ and $y_n$ can be obtained as follows:

$$x_1 = \max\left\{\frac{A}{x_{-1}}, \frac{y_0}{x_0}\right\} = \frac{y_0}{x_0} < \bar{x},$$

$$y_1 = \max\left\{\frac{A}{y_{-1}}, \frac{x_0}{y_0}\right\} = \frac{A}{y_{-1}} < \bar{y},$$

$$x_2 = \max\left\{\frac{A}{x_0}, \frac{y_1}{x_1}\right\} = \max\left\{\frac{A}{x_0}, \frac{Ax_0}{y_0 y_{-1}}\right\} = \frac{A}{x_0} < \bar{x},$$

$$y_2 = \max\left\{\frac{A}{y_0}, \frac{x_1}{y_1}\right\} = \max\left\{\frac{A}{y_0}, \frac{y_0 y_{-1}}{Ax_0}\right\} = \frac{A}{y_0} < \bar{y},$$

$$x_3 = \max\left\{\frac{A}{x_1}, \frac{y_2}{x_2}\right\} = \max\left\{\frac{Ax_0}{y_0}, \frac{x_0}{y_0}\right\} = \frac{Ax_0}{y_0} > \bar{x},$$

$$y_3 = \max\left\{\frac{A}{y_1}, \frac{x_2}{y_2}\right\} = \max\left\{y_{-1}, \frac{y_0}{x_0}\right\} = y_{-1} > \bar{y},$$

$$x_4 = \max\left\{\frac{A}{x_2}, \frac{y_3}{x_3}\right\} = \max\left\{x_0, \frac{y_0 y_{-1}}{Ax_0}\right\} = x_0 > \bar{x},$$

$$y_4 = \max\left\{\frac{A}{y_2}, \frac{x_3}{y_3}\right\} = \max\left\{y_0, \frac{Ax_0}{y_0 y_{-1}}\right\} = y_0 > \bar{y},$$

$$x_5 = \max\left\{\frac{A}{x_3}, \frac{y_4}{x_4}\right\} = \max\left\{\frac{y_0}{x_0}, \frac{y_0}{x_0}\right\} = \frac{y_0}{x_0} < \bar{x},$$

$$y_5 = \max\left\{\frac{A}{y_3}, \frac{x_4}{y_4}\right\} = \max\left\{\frac{A}{y_{-1}}, \frac{x_0}{y_0}\right\} = \frac{A}{y_{-1}} < \bar{y},$$

$$x_6 = \max\left\{\frac{A}{x_4}, \frac{y_5}{x_5}\right\} = \max\left\{\frac{A}{x_0}, \frac{Ax_0}{y_0 y_{-1}}\right\} = \frac{A}{x_0} < \bar{x},$$

$$y_6 = \max\left\{\frac{A}{y_4}, \frac{x_5}{y_5}\right\} = \max\left\{\frac{A}{y_0}, \frac{y_0 y_{-1}}{Ax_0}\right\} = \frac{A}{y_0} < \bar{y},$$

$$x_7 = \max\left\{\frac{A}{x_5}, \frac{y_6}{x_6}\right\} = \max\left\{\frac{Ax_0}{y_0}, \frac{x_0}{y_0}\right\} = \frac{Ax_0}{y_0} > \bar{x},$$

$$y_7 = \max\left\{\frac{A}{y_5}, \frac{x_6}{y_6}\right\} = \max\left\{y_{-1}, \frac{y_0}{x_0}\right\} = y_{-1} > \bar{y},$$

$$x_8 = \max\left\{\frac{A}{x_6}, \frac{y_7}{x_7}\right\} = \max\left\{x_0, \frac{y_0 y_{-1}}{Ax_0}\right\} = \frac{y_0 y_{-1}}{x_0} > \bar{x},$$

$$y_8 = \max\left\{\frac{A}{y_6}, \frac{x_7}{y_7}\right\} = \max\left\{y_0, \frac{Ax_0}{y_0 y_{-1}}\right\} = y_0 > \bar{y},$$

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Hence we obtained.

$x_1 < \bar{x}$, $x_2 < \bar{x}$, $x_3 > \bar{x}$, $x_4 > \bar{x}$, $x_5 < \bar{x}$, $x_6 < \bar{x}$, $x_7 > \bar{x}$, $x_8 > \bar{x}$, $\ldots$

$y_1 < \bar{y}$, $y_2 < \bar{y}$, $y_3 > \bar{y}$, $y_4 > \bar{y}$, $y_5 < \bar{y}$, $y_6 < \bar{y}$, $y_7 > \bar{y}$, $y_8 > \bar{y}$, $\ldots$

Hence, the solution $n \geq 0$ for $x_n$ and $y_n$, every positive semi-cycle consists of two terms, every negative semi-cycle consists of two terms.

**Lemma2:** Assume that, $A \in (1, \infty)$ $x_0, x_{-1}, y_0, y_{-1} \in (1, \infty)$ integer sequence for equation (1.1) $x_{-1} > A > y_{-1} > y_0 > x_0$, $x_{-1} > y_0 > A > x_0 > y_{-1}$,

Then the following statements are true:

$n \geq 0$ for $x_n$ and $y_n$,

a) Every positive semi-cycle consist two term.

b) Every negative semi-cycle consist two term.

c) Every positive semi-cycle of length two is followed by a negative semi-cycle of length two.

d) Every negative semi-cycle of length two is followed by a positive semi-cycle of length two.

**Proof:** Lemma 2 proof’s can be obtained similarly Lemma 1.

**Lemma3:** Assume that, $A \in (1, \infty)$ $x_0, x_{-1}, y_0, y_{-1} \in (1, \infty)$ integer sequence for equation (1) $x_{-1} > A > y_0 > x_0 > y_{-1}$,

Then the following statements are true:

$n \geq 0$ for $x_n$ and $y_n$,

a) Every positive semi-cycle consist two term.

b) Every negative semi-cycle consist two term.

c) Every positive semi-cycle of length two is followed by a negative semi-cycle of length two.

d) Every negative semi-cycle of length two is followed by a positive semi-cycle of length two.

**Proof:**

Lemma 3 proof’s can be obtained similarly Lemma 1.

**Theorem 1:** Let $(x_n, y_n)$ be a solution of equation (1) for $A > x_{-1} > y_0 > y_{-1} > x_0$, $A > x_0 > y_{-1} > y_0 > x_0$, $A > y_0 > x_{-1} > x_0 > y_{-1}$, $A > y_{-1} > x_{-1} > y_0 > x_0$,

Then for $n = 0,1,...$ we have,

$$x(n) = \left\{ \frac{y_0}{x_0}; \frac{A}{x_0}; \frac{Ax_0}{y_0}; x_0;\ ... \right\},$$

$$y(n) = \left\{ \frac{A}{y_{-1}}; \frac{A}{y_0}; y_{-1}; y_0;\ ... \right\}.$$

**Proof:** We obtain

$$x_1 = \max\left\{ \frac{A}{x_{-1}}, \frac{y_0}{x_0} \right\} = \frac{y_0}{x_0},$$

$$y_1 = \max\left\{ \frac{A}{y_{-1}}, \frac{x_0}{y_0} \right\} = \frac{A}{y_{-1}},$$

$$x_2 = \max\left\{ \frac{A}{x_0}, \frac{y_1}{x_1} \right\} = \max\left\{ \frac{A}{x_0}, \frac{Ax_0}{y_0 y_{-1}} \right\} = \frac{A}{x_0},$$

$$y_2 = \max\left\{ \frac{A}{y_0}, \frac{x_1}{y_1} \right\} = \max\left\{ \frac{A}{y_0}, \frac{y_0 y_{-1}}{Ax_0} \right\} = \frac{A}{y_0},$$

$$x_3 = \max\left\{ \frac{A}{x_1}, \frac{y_2}{x_2} \right\} = \max\left\{ \frac{Ax_0}{y_0}, \frac{x_0}{y_0} \right\} = \frac{Ax_0}{y_0},$$

$$y_3 = \max\left\{ \frac{A}{y_1}, \frac{x_2}{y_2} \right\} = \max\left\{ y_{-1}, \frac{y_0}{x_0} \right\} = y_{-1},$$

$$x_4 = \max\left\{ \frac{A}{x_2}, \frac{y_3}{x_3} \right\} = \max\left\{ x_0, \frac{y_0 y_{-1}}{Ax_0} \right\} = x_0,$$

$$y_4 = \max\left\{\frac{A}{y_2}, \frac{x_3}{y_3}\right\} = \max\left\{y_0, \frac{Ax_0}{y_0 y_{-1}}\right\} = y_0,$$

$$x_5 = \max\left\{\frac{A}{x_3}, \frac{y_4}{x_4}\right\} = \max\left\{\frac{y_0}{x_0}, \frac{y_0}{x_0}\right\} = \frac{y_0}{x_0},$$

$$y_5 = \max\left\{\frac{A}{y_3}, \frac{x_4}{y_4}\right\} = \max\left\{\frac{A}{y_{-1}}, \frac{x_0}{y_0}\right\} = \frac{A}{y_{-1}},$$

$$x_6 = \max\left\{\frac{A}{x_4}, \frac{y_5}{x_5}\right\} = \max\left\{\frac{A}{x_0}, \frac{Ax_0}{y_0 y_{-1}}\right\} = \frac{A}{x_0},$$

$$y_6 = \max\left\{\frac{A}{y_4}, \frac{x_5}{y_5}\right\} = \max\left\{\frac{A}{y_0}, \frac{y_0 y_{-1}}{Ax_0}\right\} = \frac{A}{y_0},$$

$$x_7 = \max\left\{\frac{A}{x_5}, \frac{y_6}{x_6}\right\} = \max\left\{\frac{Ax_0}{y_0}, \frac{x_0}{y_0}\right\} = \frac{Ax_0}{y_0},$$

$$y_7 = \max\left\{\frac{A}{y_5}, \frac{x_6}{y_6}\right\} = \max\left\{y_{-1}, \frac{y_0}{x_0}\right\} = y_{-1},$$

$$x_8 = \max\left\{\frac{A}{x_6}, \frac{y_7}{x_7}\right\} = \max\left\{x_0, \frac{y_0 y_{-1}}{Ax_0}\right\} = \frac{y_0 y_{-1}}{x_0},$$

$$y_8 = \max\left\{\frac{A}{y_6}, \frac{x_7}{y_7}\right\} = \max\left\{y_0, \frac{Ax_0}{y_0 y_{-1}}\right\} = y_0,$$

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Thus,

$$x(n)=\left\{\frac{y_0}{x_0};\frac{A}{x_0};\frac{Ax_0}{y_0};x_0;\ \ldots\right\},$$

$$y(n)=\left\{\frac{A}{y_{-1}};\frac{A}{y_0};y_{-1};y_0;\ \ldots\right\}.$$

the solutions are shown to be 4-period.

**Theorem 2:** Let $(x_n, y_n)$ be a solution of equation (1) for $x_{-1} > A > y_{-1} > y_0 > x_0$, $x_{-1} > y_0 > A > x_0 > y_{-1}$,

Then for $n = 0,1,\ldots$ we have,

$$x(n)=\left\{\frac{y_0}{x_0};\frac{A}{x_0};\frac{Ax_0}{y_0};x_0;\ \ldots\right\},$$

$$y(n)=\left\{\frac{A}{y_{-1}};\frac{A}{y_0};y_{-1};y_0;\ \ldots\right\}.$$

**Proof:** Proof of the Theorem 2 can be obtain similar way to the Theorem 1.

**Theorem 3:** Let $(x_n, y_n)$ be a solution of equation (1) for $A > x_{-1} > y_{-1} > x_0 > y_0$, $x_{-1} > A > y_0 > x_0 > y_{-1}$,

Then for $n = 0,1,\ldots$ we have,

$$x(n)=\left\{\frac{x_0}{y_{-1}};\frac{Ax_0}{y_0y_{-1}};\frac{Ay_{-1}}{x_0};\frac{y_0y_{-1}}{x_0};\ \ldots\right\},$$

$$y(n)=\left\{\frac{A}{y_{-1}};\frac{A}{y_0};y_{-1};y_0;\ \ldots\right\}.$$

**Proof:**

Proof of the Theorem 3 can be obtain similar way to the Theorem 1.

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INTERNATIONAL CONFERENCE ON MATHEMATICS
"An Istanbul Meeting for World Mathematicians"
3-5 July 2019, Istanbul, Turkey

# Super decisions for the influenza activity viruses using AHP and ANP methods

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**Abstract**

Nowadays we are very interested about the infections caused by different viruses, to know the most activity spread during years and to make predictions for the future. According to the world health organization WHO online data updated every week, we can evaluate the seasonal influence activity of viruses A, B and their subtypes. We will propose a decision-making model based on two methods AHP and ANP. According to the decision-maker Goal, we can choose the most spread virus by his activity. There are 5 types of activity according to one year of study: no activity, sporadic, local outbreak, widespread outbreak, regional outbreak. The software used is "Super Decision" version 2.10. In fact we can`t agree that one method is better than another because it depends on the purpose of the problem. We will see results in both methods and we will make their comparisons in each case. Generally in the last 10 years, the two methods show that the priority activity in general in each season is no activity, and the more spread virus is $AH_1N_1$.

**Keywords: Super decisions, AHP, ANP, influenza virus activity, pairwise comparisons.**

## 1. Introduction

In 1952 the World Health Organization (WHO) Executive Board decided to have a system for the influenza surveillance in order to collect some data regarding occurrence, epidemiology, viruses etc. The laboratory is called "The Global Influenza Surveillance and Response Systems" (GISRS). It includes 143 institutions and 113 member states, as a network built on voluntary collaboration and real time reporting. In 11 March 2019 GISRS launched the strategy for 2019-2030 in order to protect people from the threat of influenza. The goal of the strategy is to prevent seasonal influenza, in order to prevent the next influenza from animals to humans. Regarding the situation about the predictive modeling, we have used the multi criteria decision making (MCDM) models as: analytic hierarchic process (AHP) and analytic network process (ANP) for the data collected from WHO European Region, United Kingdom of Great Britain and North Ireland from 2010 till nowadays. The report is updated every week, and data are at real time collected. The aim of this study is to compare the two models. T. L. Saaty developed the AHP in 1971- 1975 (University of Pennsylvania, Philadelphia). AHP is used to determine relative priorities on absolute scales from both discrete and continuous cases of the paired comparisons in hierarchic structures (Saaty and Vargas, 1996). The importance measurement has been developed by Saaty (1980, 1996) to represent the relative importance of the criteria, known as Saaty Scale. Pairwise comparisons matrices of these factors provide the means for calculation of importance (Sharma et al., 2008). AHP is a hierarchic decision model with a Goal (Main scope), then next level is Criteria as a cluster of nodes that are being pairwise compared for their importance to the goal, next level of criteria are alternatives also evaluated for the preference to each criterion. In other hand ANP does not have a Goal, it has only Criteria and Alternatives. The ANP method is a mathematical theory for evaluating a network and all kinds of dependence and feedback on it, by priorities as ratio scales of criterion and alternatives. The connection between nodes of each cluster is anyway for the inner and outer dependence. The AHP model is an hierarchic structure that rank the alternatives according to the Goal, while ANP compares the dependence between the nodes of criteria cluster and nodes of alternatives cluster called outer dependence, and the inner dependence between nodes to a cluster [6].

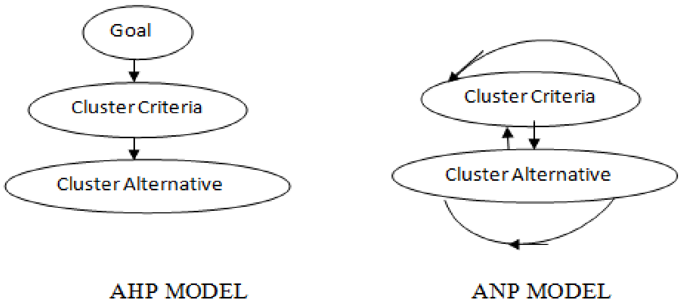



Fig. 1 Structure models

## 2. Materials and Methods

The data used in this paper are from World Health Organization (WHO) from GISRS, Flu-Net functions online data for United Kingdom of Great Britain and North Ireland, from 01.01.2010 to 29.04.2019, week by week all these years. They are organized with type A viruses including subtypes AH1, AH1N1, AH3, A and B viruses that are B Yamangata Lineage, B Victoria Lineage, B Lineage. For every week, we have a column named "ILI activity" for each virus with types: no activity, sporadic, local outbreak, widespread outbreak, regional outbreak. We have formulated a Goal Cluster named "the most spread virus" for the decision-maker. The Goal: Which is the most spread virus over these years for these "ILI activities"? According to the data we will built a hierarchy with AHP method by taking as a first level a cluster that will be called Criteria, and the next level a cluster that will be Alternative. The Criteria cluster will have nodes of five activities, and Alternative cluster will have seven nodes of types of viruses. While the ANP process will be the same hierarchy without the goal, and the clusters will be the same with their nodes as AHP Hierarchy.

### AHP Method

In the literature AHP, has been widely used in solving many decision making problems, in many areas and applications. Kangas et al., 2001, Kajanusa et al., 2004; Arslan and Turan, 2009; Kandakoğlu et al., 2009; Dinçer and Görener, 2011; Lee and Walsh, 2011; Saaty and Vargas, L.G. (1982, 1991, 2000, 2006); Dinçer and Görener, 2011; Lee and Walsh, 2011; Amir Azizi 2014; Naila Jan 2018; Luis G Vargas, H. J. Zoffer 2019. Clusters are connected by a line, we say nodes in them are connected and it means that the criteria must be pairwise compared for their importance with respect to the Goal, similarly Alternatives must be pairwise compared for their importance to Criteria. Clusters are connected by a line, we say nodes in them are connected and it means that the criteria must be pairwise compared for their importance with respect to the Goal, similarly Alternatives must be pairwise compared to Criteria for their importance. In order to determine the relative importance we have used Saaty's scale. Many questionnaire have been formulated to answer by experts in health based on Saaty scale, evaluations are made from mathematicians [5],[11].

**Table 1**

| Relative importance value | Importance | Explanation |
|---|---|---|
| 1 | Equal | Two nodes have equal importance. |
| 3 | Moderate | Experience moderately favors one node over the other. |
| 5 | Strong | Experience strongly favors one node over the other. |
| 7 | Demonstrated | A node is strongly favored and has a demonstrated dominance. |
| 9 | Extreme strong | A node is on the highest possible order domination. |
| 2,4,6,8 | Intermediate values | A node with compromise intermediate value. |

The relative weights were measured using the Super Decision Software. The instructions on how to use the Super Decisions software were prepared by Rozann W. Saaty,wife of Thomas L. Saaty of the Creative Decisions Foundation. The software that implements the Analytic Network Process, Super Decisions, was developed by William J. Adams of Embry Riddle Aeronautical University, Daytona Beach, Florida, working with Rozann W. Saaty. The dictionary of ANP applications, the Encyclicon, included here as an appendix, was compiled from materials by Thomas L. Saaty and his students, Luis Vargas etc [3].

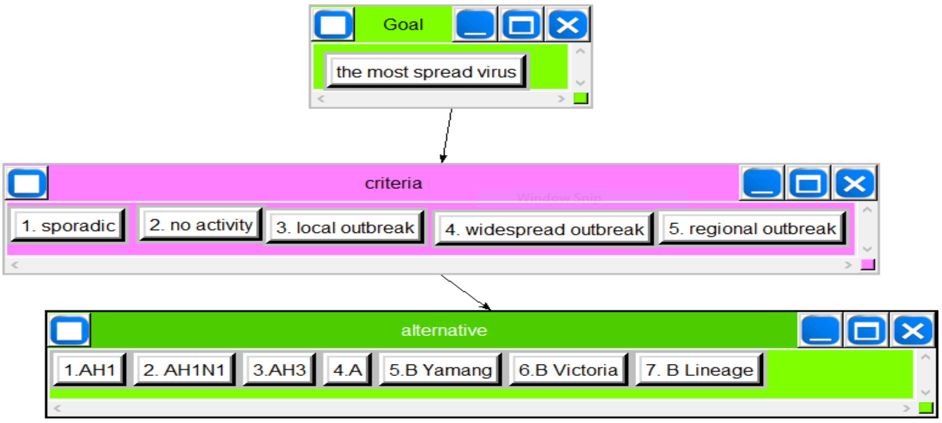



Fig.2 AHP model with Super Decision

The matrix of pairwise comparisons of Criteria cluster is a matrix with elements 1-9 according to the data obtained for the ILI Activity (Figure 2) of the viruses as:

$$A = (a_{ij})_{n \times n} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \text{ (1).}$$

The relative weights are given by the right eigenvector $\omega$, corresponding to the largest eigenvalue $\lambda_{max}$ where $A \cdot \omega = \lambda_{max} \cdot \omega$ [4]. Decision makers can weight the elements at each level using Saaty's scale from 1 to 9 and then calculate the global weights at the bottom level using pairwise comparisons (2).

$$\omega_i = \frac{1}{n}\sum_{j=1}^{n}\frac{a_{ij}}{\sum_{j=1}^{n} a_{ij}} \ , \quad \sum_{i=1}^{n} w_i = 1 \quad (2)$$

The inconsistency index is associated with matrix of the weights $CI = \frac{\lambda_{max}-n}{n-1}$. The consistence ratio is $CR = CI/RI$, where RI is the average of the eigenvalues as shown in the table nr 2 below. In order to improve the consistency of the pairwise comparisons CR, we need to adjust CI, but not larger as the judgment is, and thus the overall inconsistency should be less than 10%. [11].

**Table nr 2**

| Order | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| R.I | 0 | 0 | 0.52 | 0.89 | 1.11 | 1.25 | 1.35 | 1.40 | 1.45 | 1.49 |

**ANP Method**

Everything we decide to do, and the decisions we make, in essence we are all decision-makers. To improve our understanding and judgments is not useful all the information. In some papers authors say that too much information is as bad as little information. The information that we have to use for the judgments is to help us understand occurrences. There are many uses of the ANP model. Dağdeviren and Yüksel (2007) developed an ANP-based personel selection system and weighted personel selection factors.Yang et al. (2009) developed a manufacturing evaluation system model with ANP approach for wafer fabricating industry. Valmohammadi (2010) used the ANP to identify specific resources and capabilities of an Iranian dairy products firm and to develop an evaluation framework of business strategy, Amir Azizi (2014) proposed a paper in comperative study of AHP and ANP on multi automotive suppliers with Multi Criteria, Feibert (2016) used the ANP to assess the distribution of pharmaceuticals in hospitals, Sajad Zare (2018) used the ANP method for prioritizing and weighting shift work disorders among the personnel of hospitals of Kerman University of medical Science. The ANP model does not have the top-bottom form of the AHP hierarchy. ANP structure seems like a network with cycles connecting the criteria level itself and with the alternative level, and vice versa. ANP consist of four steps (Satty, 1996), [8],[9].

**Step 1**. The problem have to be construct like a network with connections and loops. We have criteria cluster with five nodes as subcriteria, and alternative cluster with seven nodes connected respectively to each other and with the loops itself.
**Step 2**. Perform pairwise comparisons on the clusters connected to each others, evaluating their importance respectively to criteria and alternative.
**Step 3**. Compute the limit supermatrix. Synthesize to obtan the limit priority and ideal alternative.
**Step 4**. Create a ratings model and conduct a sensitivity analysis for the final outcame.

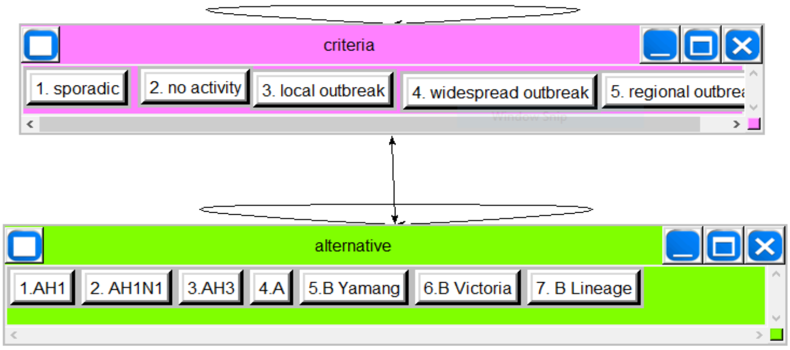



Fig.3 ANP model with Super Decision

The ANP provides a way to judge and measure ratio scales priorities for the distribution of influenza data. In fact the AHP theory is a special case of ANP theory. It's not true that an ANP model always outputs better results than the AHP. ANP is a theory that extends the AHP to a structure of dependence and feedback and generalizes on the supermatrices approach introduced in Thomas Saaty's 1980 book. It allows interactions and feedback to all nodes of the cluster as inner dependence and between clusters outer dependence. Similarly as the AHP method the pairs of comparisons for each cluster are being compared respectively to their importance within the nodes of the cluster and between the clusters [7]. A questionnaire was made to the decision makers to respond for the Saaty scale of two comparisons.

## 3. Main Results

**AHP Method.** [9],[10] Firstly we have to construct the A matrix of comparisons to criteria cluster by Saaty scale. Using the super decision software we have these values for our data base [2]:

| | Sporadic | No activity | Local | Widespread | Regional |
|---|---|---|---|---|---|
| Sporadic | 1 | 0.2 | 0.33 | 3.03 | 1.85 |
| No activity | 5 | 1 | 1.11 | 2 | 3 |
| Local | 3 | 0.9 | 1 | 4 | 2 |
| Widespread | 0.33 | 0.5 | 0.25 | 1 | 0.5 |
| Regional | 0.54 | 0.33 | 0.5 | 2 | 1 |

The weights of the $\omega$ vector of pairwise comparisons are:

$\omega = (sporadic, no\ activity,\ local, widespread, regional)$

$\omega_{criteria} = (0.14464; 0.35739; 0.29971; 0.08189; 0.11636)$ with $CI = 0.08714 < 0.1 = 10\%$,

If $CI$ is larger than 10%, the input data have to be reconsider by Saaty scale to explain better the problem decision making. The next step is the pairwise comparison between each node of criteria cluster to all nodes of alternative cluster. The consistence ratio for cluster of 5 nodes is

$$CR = {CI}/{RI} = {0.087}/{1.11} = 0.078$$

The values of a higher CR also depend on the specific decision making problem, the out coming priorities and the required accuracy. The perfectly priorities are being selected well if the number of the criteria is 5-9 nodes. This is because the human limits on our capacity for generating information, published by George A. Miller in 1956, and taken-up Saaty and Ozdemir in 2003. For our data we have to do the pairwise comparisons for each node of Criteria to the alternative cluster.

The $\omega$ normalized vector for ***Sporadic*** node, by computing the comparisons with alternative nodes is:

$\omega_{sporadic} = (AH_1,\ AH_1N_1,\ AH_3, A\ , B\ Yamagata, B\ Victoria, B\ lineage) =$

$(0.092; 0.322; 0.077; 0.207; 0.122; 0.086; 0.091)$ with $CI = 0.085 < 0.1$.

$$CR = {CI}/{RI} = {0.085}/{1.11} = 0.076$$

The $\omega$ normalized vector for ***No Activity*** node, by computing the pairwise comparisons with alternative nodes is: $\omega_{no\ activity} = (AH_1,\ AH_1N_1,\ AH_3, A\ , B\ Yamagata, B\ Victoria, B\ lineage)$

$\omega_{no\ activity} = (0.159; 0.165; 0.067; 0.238; 0.179; 0.08; 0.108)$ with $CI = 0.0852 < 0.1$

$$CR = {CI}/{RI} = {0.0852}/{1.11} = 0.077$$

The $\omega$ normalized vector for ***Local outbreak*** node, by computing the pairwise comparisons with alternative nodes is: $\omega_{local\ outbreak} = (AH_1,\ AH_1N_1,\ AH_3, A\ , B\ Yamagata, B\ Victoria, B\ lineage)$

$\omega_{local\ outbreak} = (0.176; 0.343; 0.062; 0.087; 0.086; 0.082; 0.162)$ with $CI = 0.089 < 0.1$

$$CR = {CI}/{RI} = {0.089}/{1.11} = 0.08$$

The $\omega$ normalized vector for ***Widespread outbreak*** node, by computing the comparisons with alternative nodes is: $\omega_{widespread\ outbreak} = (AH_1,\ AH_1N_1,\ AH_3, A\ , B\ Yamagata, B\ Victoria, B\ lineage)$

$\omega_{widespread\ outbreak} = (0.130; 0.271; 0.112; 0.083; 0.163; 0.111; 0.127)$ with $CI = 0.0992 < 0.1$

$$CR = {CI}/{RI} = {0.0992}/{1.11} = 0.089$$

The $\omega$ normalized vector for ***Regional outbreak*** node, by computing the pairwise comparisons with alternative nodes is:

$\omega_{regional\ outbreak} = (AH_1,\ AH_1N_1,\ AH_3, A\ , B\ Yamagata, B\ Victoria, B\ lineage) =$

$\omega_{regional\ outbreak} = (0.128; 0.267; 0.118; 0.148; 0.121; 0.115; 0.098)$ with $CI = 0.093 < 0.1$

$$CR = {CI}/{RI} = {0.093}/{1.11} = 0.083$$

The matrix of the $\omega$ weights normalized for the alternatives have to be multiplicative with the $\omega$ global weights of the criteria cluster.

$$\begin{pmatrix} 0.092 & 0.159 & 0.176 & 0.130 & 0.128 \\ 0.322 & 0.165 & 0.343 & 0.271 & 0.267 \\ 0.077 & 0.067 & 0.062 & 0.112 & 0.118 \\ 0.207 & 0.238 & 0.087 & 0.083 & 0.148 \\ 0.122 & 0.179 & 0.086 & 0.163 & 0.121 \\ 0.086 & 0.08 & 0.082 & 0.111 & 0.115 \\ 0.091 & 0.108 & 0.162 & 0.127 & 0.098 \end{pmatrix} \cdot \begin{pmatrix} 0.14464 \\ 0.35739 \\ 0.29971 \\ 0.08189 \\ 0.11636 \end{pmatrix} = \begin{pmatrix} 0.145 \\ 0.257 \\ 0.104 \\ 0.163 \\ 0.132 \\ 0.0754 \\ 0.12 \end{pmatrix} \quad \text{Rank} \begin{pmatrix} 3 = AH_1 \\ 1 = AH_1N_1 \\ 6 = AH_3 \\ 2 = A \\ 4 = Byam \\ 7 = Bvict \\ 5 = Bline \end{pmatrix}$$

Ranking the most spread virus we find out that the most spread is $AH_1N_1$ about 25.7%, then the second virus is A with its subtypes about 16.3%, then virus $AH_1$ about 14.5%, virus B Yamagata about 13.2%, virus B lineage 12%, virus $AH_3$ 10.4%, the last virus B Victoria 7.54%.

## ANP Method

ANP method is composed as a network, in which we have to compare the dependences in the same level and between levels [9]. So the calculations have to be double compared to AHP method. Since there are many calculations for ANP, we better share the nodes in the cluster for having the efficient results. There are three supermatrices with the network [1]: The Unweighted Supermatrix contains the priorities from the pairwise comparisons, the Weighted supermatrix obtains the multiplications of all the elements in a component of the unweighted supermatrix by the corresponding cluster weight, and Limit Supermatrix is obtained by raising the weighted supermatrix to powers until the column of numbers is the same for every column, in alphabetical order for their nodes inner dependence is for the same nodes of the have results for Criteria and Alternatives

| | |
|---|---|
| Local outspread | 0.29971 |
| No activity | 0.35739 |
| Regional | 0.11636 |
| Sporadic | 0.14464 |
| Widespread | 0.08189 |

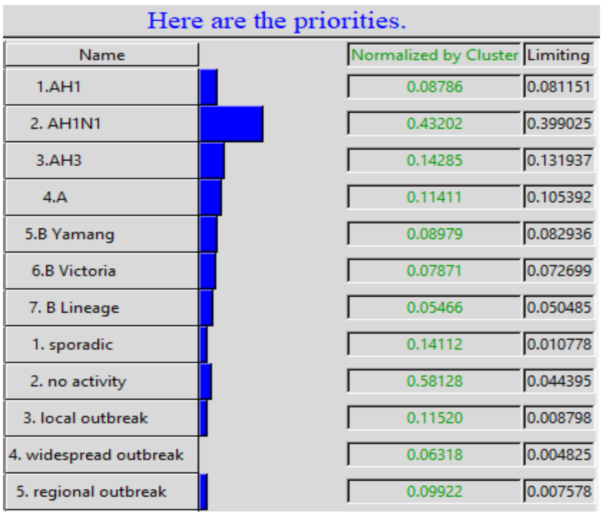
| | |
|---|---|
| $AH_1$ | 0.09246 |
| $AH_1N_1$ | 0.32266 |
| $AH_3$ | 0.07708 |
| A | 0.20768 |
| B Yamagata | 0.12258 |
| B Victoria | 0.08649 |
| B Lineage | 0.09194 |

of comparisons [3]. The clusters [10]. So we nodes:

IC=0.08714

IC=0.08533

After comparing the outer dependence and constructing the supermatrices, we have the priorities: [3]


Here are the priorities.

| Name | Normalized by Cluster | Limiting |
|---|---|---|
| 1.AH1 | 0.08786 | 0.081151 |
| 2. AH1N1 | 0.43202 | 0.399025 |
| 3.AH3 | 0.14285 | 0.131937 |
| 4.A | 0.11411 | 0.105392 |
| 5.B Yamang | 0.08979 | 0.082936 |
| 6.B Victoria | 0.07871 | 0.072699 |
| 7. B Lineage | 0.05466 | 0.050485 |
| 1. sporadic | 0.14112 | 0.010778 |
| 2. no activity | 0.58128 | 0.044395 |
| 3. local outbreak | 0.11520 | 0.008798 |
| 4. widespread outbreak | 0.06318 | 0.004825 |
| 5. regional outbreak | 0.09922 | 0.007578 |

Fig. 4 Priorities ANP

For the whole network the most spread virus is $AH_1N_1$ with 0.43=43% priority value for the alternatives, and the best activity node NO Activity with 0.58=58% priority value for criteria cluster. The importance is ranked as follows: $AH_3$=14.28%, A=11.4%, B Yamagata=8.9%, $AH_1$=8.7%, B Victoria=7.8%, B lineage=5.4%. Comparing to AHP we have:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| **Rank AHP** | 1.$AH_1N_1$ | 2.A | 3.$AH_1$ | 4.B Yamagata | 5.B Lineage | 6.$AH_3$ | 7.B Victoria |
| **Rank ANP** | 1.$AH_1N_1$ | 2.$AH_3$ | 3.A | 4. B Yamagata | 5. $AH_1$ | 6.B Victoria | 7. B Lineage |

## 4. Conclusion

The final ranking for the most spread virus during the application of the AHP and ANP methods are significantly the same for the best alternative node $AH_1N_1$, but different for the other nodes. The reason is that AHP is a hierarchy model with a main goal, but ANP a network with inner and outer dependence. Is better using AHP method instead of ANP wherever possible, trying to keep the nodes in a cluster between 5-9 for both methods. Always use AHP as a method to get consolidated results in ranking alternatives and use ANP as a tool to gain deeper inside into a decision problem, evaluated its ranking by decision makers main scopus.

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## On The Stability of a Nonlinear Difference Equation

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**Abstract**

In this paper, our aim is to study the stability of following difference equation $x_{n+1} = \frac{\gamma x_{n-2}}{Bx_n + Dx_{n-2}}$. Moreover we investigate the boundedness and convergence of solutions of related difference equation.

**Keywords: Difference equations, stability, boundedness**

## 1. Introduction

The dynamic analysis of difference equations (or recursive sequences) has an important place in applied sciences. Because applied sciences need many mathematical models for real life situations. Many real life problems consist of discrete variables like numbers of bacteria. The solutions of mathematical models created by difference equations are used for this problems. From this reason, difference equations have been huge attention by a lot of researchers for the last years. Although this attention have many reasons, the most important of this reasons implementation to different fields of science such as biology, economy and genetic. In literature, there are many papers and books related to difference equations (see [1]-[9]).

Moreover, in [1], Camouzis and Ladas investigated the global behaviors of solutions of the following difference equations

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{A + Bx_n + Cx_{n-1} + Dx_{n-2}}, n = 0,1,\cdots. \quad (1)$$

with nonnegative parameters $\alpha, \beta, \gamma, \delta, A, B, C, D$ and with arbitrary nonnegative initial conditions $x_{-2}, x_{-1}, x_0$ such that the denominator is always positive.

In this study we investigate the stability of following rational difference equations:

$$x_{n+1} = \frac{\gamma x_{n-2}}{Bx_n + Dx_{n-2}}, n = 0,1, \ldots \qquad (2)$$

where the initial conditions are nonnegative numbers.

## 2. Preliminaries

**Definition 1.** Let $I$ be some interval of real numbers and let $f: I^{k+1} \to I$ be a continuously differentiable function. A difference equation of order ($k$+1) is an equation of the form

$$x_{n+1} = f(x_n, x_{n-1}, \cdots, x_{n-k}), \qquad n = 0,1,\cdots. \qquad (3)$$

A solution of Eq.(3) is a sequence $\{x_n\}_{n=-k}^{\infty}$ that satisfies Eq.(3) for all $n \geq -k$.

As a special case of Eq.(3), for every set of initial conditions $x_{-2}, x_{-1}, x_0 \in I$, the third order difference equation

$$x_{n+1} = f(x_n, x_{n-1}, x_{n-2}), \qquad n = 0,1,\cdots. \qquad (4)$$

has a unique solution $\{x_n\}_{n=-2}^{\infty}$.

**Definition 2.** A solution of Eq.(3) that is constant for all $n \geq -k$i s called an equilibrium solution of Eq.(3). If

$$x_n = \bar{x}, for\ all\ n \geq -k$$

is an equilibrium solution of Eq.(3), then $\bar{x}$ is called an equilibrium point, or simply an equilibrium of Eq.(3).

So a point $\bar{x} \in I$ is called an equilibrium point of Eq.(3) if

$$\bar{x} = f(\bar{x}, \bar{x}, \cdots, \bar{x}),$$

that is,

$$x_n = \bar{x}, for\ n \geq -k$$

is a solution of Eq.(3).

**Definition 3.** Suppose that the function f is continuously differentiable in some open neighborhood of an equilibrium point $\bar{x}$. Let

$$q_i = \frac{\partial f}{\partial u_i}(\bar{x}, \bar{x}, \cdots, \bar{x}), for\ i = 0,1,2,\cdots,k$$

denote the partial derivative of $f(u_0, u_1, \cdots, u_k)$ with respect to $u_i$ evaluated at the equilibrium point $\bar{x}$ of Eq.(3).

The equation

$$z_{n+1} = q_0 z_n + q_1 z_{n-1} + \cdots, q_k z_{n-k}, k = 0,1,\cdots. \qquad (5)$$

is called the linearized equation of Eq.(3) about the equilibrium point $\bar{x}$.

**Definition 4.** The equation

$$\lambda^{k+1} - q_0 \lambda^k + q_1 \lambda^{k-1} + \cdots, q_k = 0 \qquad (6)$$

is called the characteristic equation of Eq.(5) about $\bar{x}$.

**Definition 5.** Let $\bar{x}$ an equilibrium point of Eq.(3).

**(a)** An equilibrium point $\bar{x}$ of Eq.(3) is called locally stable if, for every $\varepsilon > 0$; there exists $\delta > 0$ such that if $\{x_n\}_{n=-k}^{\infty}$ is a solution of Eq.(3) with

$$|x_{-k} - \bar{x}| + |x_{1-k} - \bar{x}| + \cdots + |x_0 - \bar{x}| < \delta$$

then

$$|x_n - \bar{x}| < \varepsilon, for\ all\ n \geq -k.$$

**(b)** An equilibrium point $\bar{x}$ of Eq.(3) is called locally asymptotically stable if, it is locally stable, and if in addition there exists $\gamma > 0$ such that if $\{x_n\}_{n=-k}^{\infty}$ is a solution of Eq.(3) with

$$|x_{-k} - \bar{x}| + |x_{1-k} - \bar{x}| + \cdots + |x_0 - \bar{x}| < \delta$$

then we have

$$\lim_{n\to\infty} x_n = \bar{x}.$$

**(c)** An equilibrium point $\bar{x}$ of Eq.(3) is called a global attractor if, for every solution $\{x_n\}_{n=-k}^{\infty}$ of Eq.( 3), we have

$$\lim_{n\to\infty} x_n = \bar{x}.$$

**(d)** An equilibrium point $\bar{x}$ of Eq.(3) is called globally asymptotically stable if it is locally stable, and a global attractor.

**(e)** An equilibrium point $\bar{x}$ of Eq.(3) is called unstable if it is not locally stable.

**Theorem 1 (The Linearized Stability Theorem).** Assume that the function $F$ is a continuously differentiable function defined on some open neighborhood of an equilibrium point $\bar{x}$. Then the following statements are true:

**(a)** When all the roots of Eq.(6) have absolute value less than one, then the equilibrium point $\bar{x}$ of Eq.(3) is locally asymptotically stable. Moreover, in this here the equilibrium point $\bar{x}$ of Eq.(3) is called sink.

**(b)** If at least one root of Eq.(6) has absolute value greater than one, then the equilibrium point $\bar{x}$ of Eq.(3) is unstable.

**(i)** The equilibrium point $\bar{x}$ of Eq.(3) is called hyperbolic if no root of Eq.(6) has absolute value equal to one.

**(ii)** If there exists a root of Eq.(6) with absolute value equal to one, then the equilibrium $\bar{x}$ is called nonhyperbolic.

**(iii)** An equilibrium point $\bar{x}$ of Eq.(3) is called a saddle point if it is hyperbolic and if there exists a root of Eq.(6) with absolute value less than one and another root of Eq.(6) with absolute value greater than one.

**(iv)** An equilibrium point $\bar{x}$ of Eq.(3) is called a repeller if all roots of Eq.(6) have absolute value greater than one.

**Theorem 2 ([3], p.207).** Suppose that $f: [a,b]^3 \to [a,b]$ is a continuous function and let $[a,b]$ be an interval of real numbers.

**(i)** $f(x,y,z)$ is non-increasing in $x$ and $y$ for each $z \in [a,b]$ and is non-decreasing in $z$ for each $x$ and $y \in [a,b]$ of its arguments;

**(ii)** If $(m,M) = [a,b] \times [a,b]$ is a solution of the system

$$M = f(m,m,M) \ and \ m = f(M,M,m)$$

then $m = M$.

If (i) and (ii) hold then Eq.(4) has a unique equilibrium $\bar{x} \in [a, b]$ and every solution of Eq.(4) converges to $\bar{x}$.

## 3. Main Results

Firstly we present some results given by Camouzis and Ladas in [1], p184.

The normalized form of Eq.(2) is

$$x_{n+1} = \frac{x_{n-2}}{Bx_n + x_{n-2}}, n = 0,1, \dots \qquad (7)$$

with positive parameter B and arbitrary nonnegative initial conditions. Moreover the equilibrium point of Eq.(7) is $\bar{x} = \frac{1}{B+1}$ and if $B < 1 + \sqrt{2}$ holds then the equilibrium point $\bar{x} = \frac{1}{B+1}$ of Eq.(7) is locally asymptotically stable.

**Theorem 3.** Let the initial conditions $x_{-2}, x_{-1}, x_0 \geq 0$ and $B \in (0,1 + \sqrt{2})$. Hence Eq.(7) is bounded from below and above with $0 \leq x_n \leq 1$ for all $n \geq 1$.

**Proof.** Let $x_{-2}, x_{-1}, x_0 \geq 0$ and $0 < B < 1 + \sqrt{2}$. We have from Eq.(7),

$$x_1 = \frac{x_{-2}}{Bx_0 + x_{-2}} \geq 0,$$

$$x_2 = \frac{x_{-1}}{Bx_1 + x_{-1}} = \frac{x_{-1}}{B\left(\frac{x_{-2}}{Bx_0 + x_{-2}}\right) + x_{-1}} \geq 0,$$

and by induction, we get

$$x_n = \frac{x_{n-3}}{Bx_{n-1} + x_{n-3}} \geq 0. \qquad (8)$$

From Eq.(7) and $B \in (0,1 + \sqrt{2})$, we obtain

$$x_{n+1} = \frac{x_{n-2}}{Bx_n + x_{n-2}} \leq \frac{x_{n-2}}{x_{n-2}} = 1,$$

for all $n \geq 1$. So, $x_n \in [0,1]$.

**Theorem 4 (Conjecture 5.39.1 of [1]).** Assume that $B < 1 + \sqrt{2}$ holds. Then every positive solutions of Eq.(7) converges to positive equilibrium, $\bar{x} = \frac{1}{B+1}$.

**Proof.** Now we apply to Theorem 2 for Eq.(7). Let $a$ and $b$ are real numbers and assume that

$$f: [a, b]^3 \to [a, b]$$

a function defined by

$$f(x, y, z) = \frac{z}{Bx + z}. \qquad (8)$$

**(i)** According to Theorem 2, we calculate the partial derivative of (8):

$$\frac{\partial f(x, y, z)}{\partial x} = -\frac{Bz}{(Bx + z)^2} < 0,$$

$$\frac{\partial f(x, y, z)}{\partial y} = 0,$$

$$\frac{\partial f(x, y, z)}{\partial z} = \frac{Bz}{(Bx + z)^2} > 0.$$

So, $f(x, y, z)$ is non-decreasing in $z$ for each $x$ and $y \in [a, b]$ and is non-increasing in $x$ and $y$ for each $z \in [a, b]$.

**(ii)** Assume that $(m, M) = [a, b] \times [a, b]$ is a solution of the system

$$M = f(m, m, M) \; and \; m = f(M, M, m)$$

then from Eq.(7), we have following equalities:

$$M = \frac{M}{Bm + M}, \quad m = \frac{m}{BM + m}.$$

From this we obtain $m = M$. According to Theorem 2, every solutions of Eq.(7) converges to $\bar{x}$ and the proof completed.

**Example 1.** Consider the Eq.(7) with initial conditions $x_{-2} = 1, x_{-1} = 2, x_0 = 4$ and $B = 1.5$. Thus every positive solutions of Eq.(7) converges to $\bar{x} = \frac{1}{B+1} = 0.4$.The following figure verifies our results.

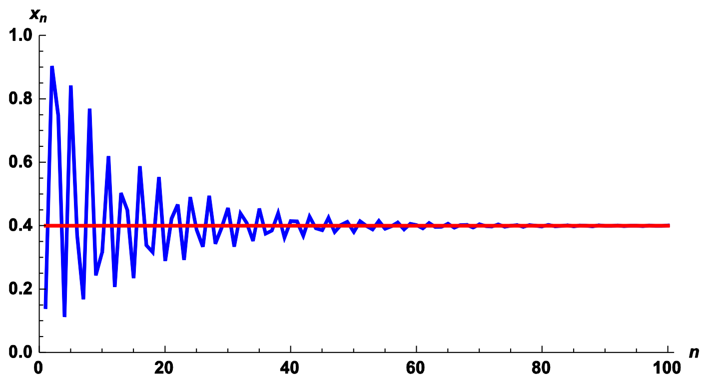


Figure 1: Plot of Eq.(7) for $B = 1.5$.

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**INTERNATIONAL CONFERENCE ON MATHEMATICS**
**“An Istanbul Meeting for World Mathematicians”**
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# On The Stability Analysis of a System of Difference Equations

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**Abstract**

During this study we investigate the stability analysis of following system of difference equations $x_{n+1} = x_{n-1}y_n - 1, y_{n+1} = y_{n-1}x_n - 1$. Furthermore we research the boundedness of solutions of this system.

**Keywords: Difference equations, stability, boundedness, dynamical systems**

## 1. Introduction

Discrete dynamic systems attract great attention among researchers. The reasons for this attention are the applications of these dynamic systems to different fields of science such genetics, economy, biology. Over the last years there are many paper to difference equations and dynamical systems for examples:

In [1] Kurbanlı et al studied behaviour of positive solutions of system of difference equations

$$x_{n+1} = \frac{x_{n-1}}{y_n x_{n-1} + 1}, y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1} + 1}.$$

In [2] Kent et al studied studied long-term behaviour of solutions of difference equation

$$x_{n+1} = x_n x_{n-1} - 1.$$

Moreover, in [3] Wang et al and in [4] Liu et al obtained some important results about related difference equation.

In this study we investigate the stability of following system of difference equations:

$$x_{n+1} = x_{n-1}y_n - 1, y_{n+1} = y_{n-1}x_n - 1, n = 0,1, \dots \tag{1}$$

where all initial values are real numbers.

## 2. Preliminaries

Let us introduce a four-dimensional discrete dynamical system of the form

$$x_{n+1} = f(x_n, x_{n-1}, y_n, y_{n-1}), y_{n+1} = g(x_n, x_{n-1}, y_n, y_{n-1}), \tag{2}$$

$n = 0,1,2, \ldots$, where $f: I^4 \times J^4 \to I$ and $g: I^4 \times J^4 \to J$ are continuously differentiable functions and $I, J$ are some intervals of real numbers. Moreover, a solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$ of system (2) is uniquely determined by initial values $(x_i, y_i) \in I \times J$ for $i \in \{-1,0\}$.

**Definition 1.** Along with the system (2), we consider the corresponding vector map $F = \{f, x_n, x_{n-1}, g, y_n, y_{n-1}\}$. A point $(\bar{x}, \bar{y})$ is called an equilibrium point of the system (2) if

$$\bar{x} = f(\bar{x}, \bar{x}, \bar{y}, \bar{y}), \qquad \bar{y} = g(\bar{x}, \bar{x}, \bar{y}, \bar{y}).$$

The point $(\bar{x}, \bar{y})$ is also called a fixed point of the vector map $F$.

**Definition 2.** Let $(\bar{x}, \bar{y})$ be an equilibrium point of system (2).

**i.** An equilibrium point $(\bar{x}, \bar{y})$ of system (2) is called stable if, for every ε>0, there exists δ>0 such that, for every initial value $(x_i, y_i) \in I \times J$, with

$$\sum_{i=-1}^{0} |x_i - \bar{x}| < \delta, \qquad \sum_{i=-1}^{0} |y_i - \bar{y}| < \delta,$$

implying $|x_n - \bar{x}| < \varepsilon$ and $|y_n - \bar{y}| < \varepsilon$ for $n \in \mathbb{N}$.

**ii.** An equilibrium point $(\bar{x}, \bar{y})$ of system (2) is called unstable, if it is not stable.

**iii.** An equilibrium point $(\bar{x}, \bar{y})$ of system (2) is called locally asymptotically stable if it is stable and if, in addition, there exists γ>0 such that

$$\sum_{i=-1}^{0} |x_i - \bar{x}| < \gamma, \qquad \sum_{i=-1}^{0} |y_i - \bar{y}| < \gamma,$$

and $(x_n, y_n) \to (\bar{x}, \bar{y})$ as $n \to \infty$.

**iv.** An equilibrium point $(\bar{x}, \bar{y})$ of system (2) is called a global attractor if $(x_n, y_n) \to (\bar{x}, \bar{y})$ as $n \to \infty$.

**v.** An equilibrium point $(\bar{x}, \bar{y})$ of system (2) is called globally asymptotically stable if it is stable and a global attractor.

**Definition 3.** Let $(\bar{x}, \bar{y})$ be an equilibrium point of the map $F$ where $f$ and $g$ are continuously differentiable functions at $(\bar{x}, \bar{y})$. The linearized system of system (2) about the equilibrium point $(\bar{x}, \bar{y})$ is

$$X_{n+1} = F(X_n) = BX_n,$$

where

$$X_n = \begin{pmatrix} x_n \\ x_{n-1} \\ y_n \\ y_{n-1} \end{pmatrix}$$

and $B$ is a Jacobian matrix of system (2) about the equilibrium point $(\bar{x}, \bar{y})$.

**Linearized Stability Theorem (see [5], p 11).** Assume that

$$X_{n+1} = F(X_n), n = 0,1, \ldots,$$

is a system of difference equations such that $\bar{X}$ is a fixed point of $F$.

**i.** If all eigenvalues of the Jacobian matrix $B$ about $\bar{X}$ lie inside the open unit disk $|\lambda|<1$, that is, if all of them have absolute value less than one, then $\bar{X}$ is locally asymptotically stable.

**ii.** If at least one of them has a modulus greater than one, then $\bar{X}$ is unstable.

**3. Main Results**

Firstly we study the equilibrium points of system (1).

**Theorem 1.** System (1) has two equilibrium points which are

$$(\bar{x}_1, \bar{y}_1) = \left(\frac{1-\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right),$$

$$(\bar{x}_2, \bar{y}_2) = \left(\frac{1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right).$$

Due to $\frac{1+\sqrt{5}}{2} \approx 1.618$, the elements of second equilibrium point equal to golden ratio.

**Proof.** We can easily seen for the equilibrium points of system (1):

$$\bar{x} = \bar{x}\bar{y} - 1,$$

$$\bar{y} = \bar{y}\bar{x} - 1.$$

From this system we obtain the followings

$$\bar{x} = \bar{x}\bar{y} - 1 = \bar{y},$$

$$\bar{x} = \bar{x}\bar{x} - 1.$$

Then we have $\bar{x} = \bar{y} = \frac{1 \pm \sqrt{5}}{2}$.

Now, we investigate the stability of first equilibrium point $(\bar{x}_1, \bar{y}_1)$ of system (1).

**Theorem 2.** The first equilibrium point $(\bar{x}_1, \bar{y}_1)$ of system (1) is locally asymptotically stable.

**Proof.** System (1) is equivalent to following system of difference equations:

$$t_{n+1} = 1 - t_{n-1}w_n,\ w_{n+1} = 1 - w_{n-1}t_n, n = 0,1, \dots \qquad (3)$$

with change to variables $x_n = -t_n$ and $y_n = -w_n$. From this, equilibrium point $(\bar{x}_1, \bar{y}_1)$ of system turn to positive equilibrium point $(\bar{t}, \bar{w})$ of system (3). We can clearly seen that

$$(\bar{t}, \bar{w}) = \left(\frac{\sqrt{5} - 1}{2}, \frac{\sqrt{5} - 1}{2}\right).$$

For this, we consider the transformation:

$$(t_n, t_{n-1}, w_n, w_{n-1}) \to (h, h_1, k, k_1),$$

where

$$h = 1 - t_{n-1}w_n,$$

$$h_1 = t_n,$$

$$k = 1 - w_{n-1}t_n,$$

$$k_1 = w_n.$$

Therefore we have the Jacobian matrix about equilibrium point $(\bar{t}, \bar{w})$:

$$B(\bar{t}, \bar{w}) = \begin{pmatrix} 0 & -\bar{w} & -\bar{t} & 0 \\ 1 & 0 & 0 & 0 \\ -\bar{w} & 0 & 0 & -\bar{t} \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Thus, the linearized system about the equilibrium point $(\bar{t},\bar{w}) = \left(\frac{\sqrt{5}-1}{2},\frac{\sqrt{5}-1}{2}\right)$ is $X_{N+1} = B(\bar{t},\bar{w})X_N$, where $X_N = \left((t_n,t_{n-1},w_n,w_{n-1})\right)^T$ and

$$B(\bar{t},\bar{w}) = \begin{pmatrix} 0 & \frac{1-\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} & 0 \\ 1 & 0 & 0 & 0 \\ \frac{1-\sqrt{5}}{2} & 0 & 0 & \frac{1-\sqrt{5}}{2} \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

So, the characteristic equation of $B(\bar{t},\bar{w})$ is

$$\lambda^4 - \left(\frac{5-3\sqrt{5}}{2}\right)\lambda^2 + \frac{3-\sqrt{5}}{2} = 0. \qquad (4)$$

Hence, we have four roots of Eq.(4):

$$|\lambda_1| = |\lambda_2| = |\lambda_3| = |\lambda_4| = 0.78615 < 1.$$

From linearized stability theorem, all roots of the characteristic equation lie inside the unit disk. So, the positive equilibrium of system (3) is locally asymptotically stable.

**Theorem 3.** The equilibrium point $(\bar{x}_2,\bar{y}_2)$ of system (1) is locally unstable.

**Proof.** Firstly we study linearized form of system (1). For this, we consider the transformation:

$$(x_n,x_{n-1},y_n,y_{n-1}) \to (f,f_1,g,g_1),$$

where $f = x_{n-1}y_n - 1$, $f_1 = x_n$, $g = y_{n-1}x_n - 1$ and $g_1 = y_n$. Therefore we have the Jacobian matrix about equilibrium point $(\bar{x},\bar{y})$:

$$B(\bar{x},\bar{y}) = \begin{pmatrix} 0 & \bar{y} & \bar{x} & 0 \\ 1 & 0 & 0 & 0 \\ \bar{y} & 0 & 0 & \bar{x} \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Thus, the linearized system about the equilibrium point $(\bar{x},\bar{y}) = \left(\frac{1+\sqrt{5}}{2},\frac{1+\sqrt{5}}{2}\right)$ is $X_{N+1} = B(\bar{x},\bar{y})X_N$, where $X_N = \left((x_n,x_{n-1},y_n,y_{n-1})\right)^T$ and

$$B(\bar{x},\bar{y}) = \begin{pmatrix} 0 & \frac{1+\sqrt{5}}{2} & \frac{1+\sqrt{5}}{2} & 0 \\ 1 & 0 & 0 & 0 \\ \frac{1+\sqrt{5}}{2} & 0 & 0 & \frac{1+\sqrt{5}}{2} \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

So, the characteristic equation of $B(\bar{x},\bar{y})$ is

$$\lambda^4 - \left(\frac{5+3\sqrt{5}}{2}\right)\lambda^2 + \frac{3+\sqrt{5}}{2} = 0. \qquad (5)$$

Hence, we have four roots of Eq.(5) such that $|\lambda_{1,2}| < 1 < |\lambda_{3,4}|$. From linearized stability theorem, two roots of the characteristic equation lie inside the unit disk but the other roots lie outside the unit disk. So, the positive equilibrium of system (1) is locally unstable.

## 4. Conclusion

Throughout this paper, we investigate the stability of system (1). Further, we find out negative equilibrium point of system (1) is locally asymptotically stable. But we discover the positive equilibrium point of system (1) is unstable.

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# The use of partial least squares structural equation modeling approach for analysis of the dimensions of poverty, a case study of Albania.

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**Abstract:** Partial Least Squares Structural Equation Modeling (PLS-SEM) is a multivariate analysis technique for modeling the relations in several fields of knowledge including dimensions of poverty. The purpose of this article is to operationalize living conditions, social inclusion, education, expenditures as poverty dimensions with a view to understanding the links between them. The data is derived from Living Standards Measurement Survey (LSMS) 2012. The results show that education has a positive impact on expenditures and social inclusion. Relationships in structural models between latent dimensions are significant. Measurement models indicate allowed values for internal consistency, reliability, validity. Our findings support as instruction for PLS-SEM implementation in multidimensional poverty analysis.
**Keywords: partial least squares structural equation modeling, LSMS, latent dimension, measurement models.**

## 1. Introduction

Nowadays it is necessary to study the phenomenon of poverty, well-being and the factors that cause them as the most important goal of development policies. Researchers studying the causes of poverty are mindful of the fact that the concept of poverty above all is a complex and multidimensional concept, has different meanings, multiple causes that cannot easily be distinguished [1], [2] . Poverty is a complex concept that we need to understand the ties between its dimensions. Partial least squares structural equation modeling (PLS-SEM) is a widely used method to analyze interaction between dimensions or constructs. In his study of Nepal, Wagle 2008 [3] explores the relationship between latent poverty dimensions which in this case are considered well-being, capability, social inclusion, and set of observed indicators for each dimension. By PLS-SEM it is shown how these dimensions are interconnected with each other so that the results obtained can be used by policy makers.

## 2. Materials and Methods

Structural equation modeling (SEM) is a multivariate analysis technique for operationalizing latent variables, and describes the relationship between latent variables (dimensions) and its indicators. SEM includes two models: the measurement model or external model and structural model or inner model. Among the different approaches to estimating the model parameters in SEM are the covariance-based model and the variance-based model or partial least square (PLS) path model for which there has been a growing interest in recent decades [4]. The PLS-SEM ability is that it does not make assumptions about the distribution of data, it is used when distributions are highly skewed used for metric data, nominal, ordinal data [5], is used in small samples and finally builds more complex models with many latent variables, indicators.

In this study, data is derived from Living Standards Measurement Survey 2012 (LSMS 2012) which includes 2000 households. Partial Least Squares Structural Equation Modeling has been used through the SmartPLS3 program. So, in our study we have used the multidimensional approach of poverty it is necessary to quickly determine the dimensions to be taken into account and their corresponding variables. For the dimensions we have taken in the study we are based on available data, expert knowledge and, on the review of the literature on multidimensional poverty, where the latter includes Multidimensional Poverty Index [6].

### 2.1 The variables selected in the study are:

**Educational Level, ED**

The father's educational level is ordinal variable, the values it receives are from 1 to 5 (four-year-old school, four-year high school, high school, some high school, university), $\mathbf{Ed_1}$

The mother's educational level is ordinal variable, the values it takes are from 1 to 5 (four-year-old school, primary school, high school, some high school, university), $\mathbf{Ed_2}$

**Expenditures Household, EX**

Family expenses are taken into account.

**Social Inclusion, SI**

Cinema is the ordinal variable, the values it receives are from 1 to 5 (never, 1 to 6 times, more than 6 times, every month, every week), $\mathbf{SI_1}$

Live is the ordinal variable, the values it receives are from 1 to 5 (never, 1 to 6 times, more than 6 times, every month, every week), $\mathbf{SI_2}$

Cultural Sites is the ordinal variable, the values it receives are from 1 to 5 (never, 1 to 6 times, more than 6 times, every month, every week), $\mathbf{SI_3}$.

**Living Condition, LC**

The condition of dwelling type is variable ordinal, the values it receives are from 1 to 3 (inappropriate for living, suitable for living, very good condition), $\mathbf{LC_1}$

Outside apparence of building is the ordinal variable, the values that are taken are from 1 to 3 (plastered, partially plastered, not plastered), $\mathbf{LC_2}$

### 2.2 Conceptual Model

The proposed model for our work includes four latent dimensions which include: **ED, EX, SI, LC**. Below are represented casual relationships between dimensions.

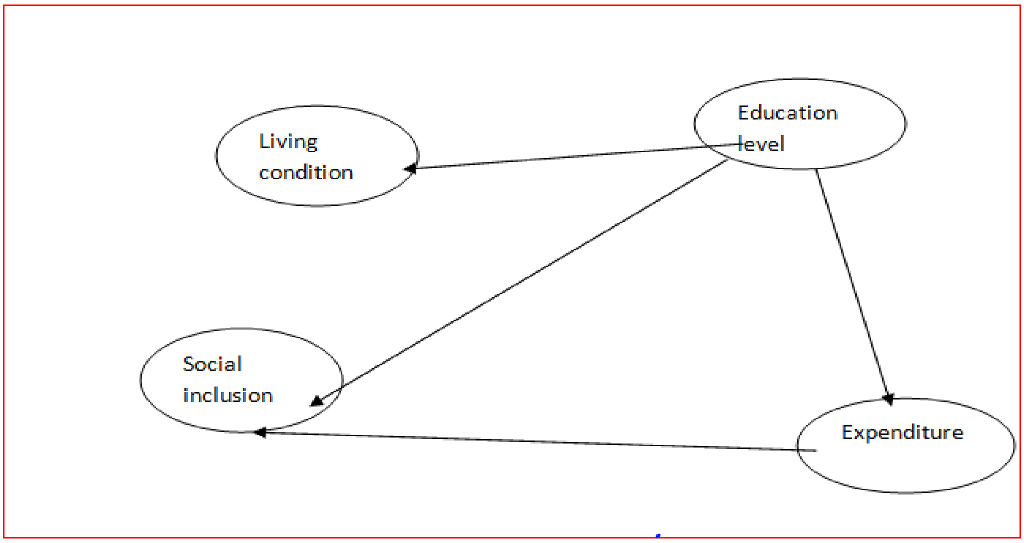


**Figure 1. Conceptual Model**

## 3. Main Results

### 3.1 Assessment of Measurement Model

Our reflective dimension is used in our model. Reflective measurement models are evaluated based on the internal consistency reliability that includes the composite reliability statistic. The composite reliability values (**CR**) should be between 0.7 and 0.9 because values above this limit are problematic due to excessive indicators [4], [7]. Validity, that includes the convergent validity indicator, and, discriminant validity [5]. To estimate convergent validity, we should consider the indicator load and the average variance extracted (AVE), each having at least the value of 0.7 and, 0.5 respectively [5]. To study discriminant validity, consider the Fornell and Larcker's criterion [8].

**Table 1. Reflective measurement model**

| **Dimensions and indicators** | **Loadings** | **CR** | **AVE** |
|---|---|---|---|
| **Educational level** | | 0.868 | 0.767 |
| Father educational level | 0.893 | | |
| Mother educational level | 0.857 | | |
| **Expenditures** | 1 | 1 | 1 |
| **Social Inclusion** | | 0.881 | 0.713 |
| Cinema | 0.875 | | |
| Live | 0.886 | | |
| Cultural sites | 0.776 | | |
| **Living condition** | | 0.918 | 0.849 |
| Condition of dwelling type | 0.907 | | |
| Outside apparence of building | 0.935 | | |

By the results reflected in Table 1, it is shown that all dimensions of this study are within the permissible parameters for Loading, CR, AVE. To evaluate discriminant validity we have used the Forner and Larcker’s criterion that requires the condition to be met:
which requires all the square root of AVE to be higher than their inter-correlations dimensions. The values that are placed in the diagonal of table 2 indicate the AVE square root, and the other values inter-correlations between dimensions.

**Table 2. Forner and Larcker criterion**

| **Dimensions** | **Expenditure** | **Living Condition** | **Educational level** | **Social Inclusion** |
|---|---|---|---|---|
| **Expenditure** | **1** | | | |
| **Living Condition** | **0.098** | **0.921** | | |
| **Educational level** | **0.156** | **0.578** | **0.876** | |
| **Social Inclusion** | **0.185** | **0.180** | **0.238** | **0.844** |

It is noted that all values outside the diagonal are smaller than those in the main diagonal, therefore Forner and Larcker criteria are met.

### 3.2 Assessment of structural model

Assessment of structural model includes the significance of the structural relations, the coefficient of determination $R^2$. Table 3 shows the path coefficients, p-value, t-statistics, significance level for all paths. The analysis shows that the educational level has a significantly positively correlated impact on expenditures, also has a significant positive impact on social inclusion. Household expenditures have a positive impact on social inclusion. Educational level has a positive impact on living condition. Ultimately, all path coefficients are significant.

**Table 3.** Path Coefficients of the Structural Model

| **Path** | **Path Coefficient** | **t-statistics** | **p-value** |
|---|---|---|---|
| **ED→EX** | 0.156 | 12.378 | 0.000 |
| **ED→ SI** | 0.214 | 10.859 | 0.000 |
| **EX→ SI** | 0.152 | 12.734 | 0.000 |
| **ED→ LC** | 0.578 | 32.187 | 0.000 |

For $R^2$ values, it is difficult to set a lower limit of its values because it depends on the complexity of the model and field of study [5]. Based on the study of Falk and Miller (1992) [9] it is considered as a criterion that the value of $R^2$ should not be less than 0.1.

**Table 4. Coefficient of determination for dimension, $R^2$**

| Dimension | $R^2$ |
|---|---|
| Expenditure | 0.03 |
| Living Condition | 0.33 |
| Social Inclusion | 0.09 |

Another size that is used for structural model estimation is effect size $f^2$. The latter shows the effect of removing a dimension in the value of $R^2$. As a rule, the values of $f^2$ 0.02, 0.14, 0.35 are respectively considered small, medium and large [10]. Table 5 shows the effect size values for each dimensional connection. The level of education has a substantial effect on living condition and a small size effect on expenditure and social inclusion.

**Table 5. Effect size**

| Dimension | Expenditure | Living Condition | Level Education | Social Inclusion |
|---|---|---|---|---|
| Expenditure | | | | 0.024 |
| Living Condition | | | | |
| Level Education | 0.025 | 0.501 | | 0.049 |
| Social Inclusion | | | | |

## 4. Conclusions

Using PLS-SEM helps in analyzing the dimensions of poverty by understanding how these dimensions are related to one another. From the results of the model, we draw conclusions about the impact that have dimensions with each other that serve to improve social policies. Specifically, our study confirmed the positive impact of education on social inclusion, expenditures and living conditions. In further studies it is thought that the model will expand and with other latent dimensions.

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## On Some Characterizations of Prime N-groups

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**Abstract**

Prime near-rings and prime near-ring modules ( $N-$groups) have been studied by various authors [1-7]. The various notions of primeness that were defined in a near-ring $N$ are generalized to the $N-$group $\Gamma$ [3,6-7]. In [7] the authors gave some characterizations of completely prime (completely semiprime) and 3-prime (3-semiprime) $N-$groups.

In this study, we give another new characterizations of completely prime $N-$group and 3-prime $N-$group. Then we prove that they are equivalent.

**Keywords:** near-ring, completely prime N-group, 3-prime N-group

## 1. Introduction

Near-rings, different from rings as they do not need to be abelian for first operation and they satisfy only one-side distribution law, are generalized rings. So, many concepts which are known in ring-theory are different for near-rings due to these two properties. Especially, the concept of primeness in ring theory appears as different types in near-ring theory. Holcombe [2] defined the notions of 0-prime (prime), 1-prime and 2-prime. In [5] the authors gave the definitions of 3-prime and completely prime(c-prime) ideals.

In [3] the authors generalized the various notions of primeness that were defined in a near-ring to the near-ring module. In [7] the authors gave some characterizations of completely prime (completely semiprime) and 3-prime (3-semiprime) $N-$groups.

In this study we give some new characterizations of completely prime and 3-prime $N-$groups.

## 2. Preliminaries

**Definition 1** An ideal $I$ of a near-ring $N$ is called a 3-prime (3-semiprime) ideal if for all $x, y \in N$, $xNy \subseteq I$ ( $xNx \subseteq I$ ) implies $x \in I$ or $y \in I$ ( $x \in I$ ).

**Definition 2** If for all $x, y \in N$, $xy \in I$ ($x^2 \in I$) implies $x \in I$ or $y \in I$ ($x \in I$), then $I \lhd N$ is called a completely prime (completely semiprime) ideal.

$N$ is called a completely prime (3-prime) near-ring if $0_N$ is completely prime (3-prime).

**Definition 3** If $\Gamma$ is an additive group, then it is called an $N-$group (near-ring module) if for all $x, y \in N$ and $\gamma \in \Gamma$,
a) $x\gamma \in \Gamma$,
b) $(x+y)\gamma = x\gamma + y\gamma$,
c) $(xy)\gamma = x(y\gamma)$.

**Definition 4** A normal subgroup $P$ of $\Gamma$ is called an $N-$ideal of $\Gamma$ ($P \lhd_N \Gamma$) if $\forall \gamma \in \Gamma$, $\forall p \in P$, $\forall n \in N$: $n(\gamma + p) - n\gamma \in P$.

The Noetherian quotient $(A:B)_N$ is defined as the set $\{n \in N : nB \subseteq A\}$ where $A$, $B$ be subsets of $\Gamma$.

**Definition 5** [3] Let $P \lhd_N \Gamma$ be such that $N\Gamma \not\subseteq P$ and $n \in N$ and $\gamma \in \Gamma$.
a) If $nN\gamma \subseteq P$ implies that $n\Gamma \subseteq P$ or $\gamma \in P$, then $P$ is called 3-prime.
b) If $n\gamma \in P$ implies that $n\Gamma \subseteq P$ or $\gamma \in P$, then $P$ is called completely prime (c-prime).

If $N\Gamma \neq 0$ and $0_\Gamma$ is a 3-prime (completely prime) $N-$ideal of $\Gamma$, then $\Gamma$ is called a 3-prime (completely prime) $N-$group.

## 3. Main Results

**Theorem 1** Let $N \neq P \lhd N$. Then the followings are equivalent:
$i$) $P$ is a completely prime ideal of $N$.
$ii$) There is a completely prime $N-$group $\Gamma$ with $P = (0_\Gamma : \Gamma)_N$.
$iii$) There is an $N-$group $\Gamma$ with $P = (0_\Gamma : \Gamma)_N$ where $xy\gamma = 0_\Gamma$ implies $x\gamma = 0_\Gamma$ or $y\gamma = 0_\Gamma$ for all $x, y \in N$ and $\gamma \in \Gamma$.

**Proof**

$(i) \Rightarrow (ii)$ Let $P$ be a completely prime ideal of $N$ and $\Gamma = N/P$. Then $\Gamma$ is an $N-$group with the natural operations. Let $p \in P$ and $x \in N$. Then $p(x+P) = px + P = P$, since $px \in P$. Hence, $P \subseteq (0_\Gamma : \Gamma)_N$. Now assume that $x \in (0_\Gamma : \Gamma)_N$. Then $x(n+P) = xn + P = P$ for all $n \in N$, whence

$xN \subseteq P$. Since $P$ is also a completely semiprime ideal of $N$, $x \in P$. Hence, $P = (0_\Gamma : \Gamma)_N$. We need to show that $\Gamma$ is a completely prime $N-$group. Let $x(m+P) = P$ for $x \in N$ and $m+P \in N/P$. We must show that $xN/P = P$ or $m+P = P$. If $m \in P$, then the proof complete. So suppose $m \notin P$. Since $x(m+P) = P$ whence $xm \in P$ and since $P$ is a completely prime ideal of $N$, $x \in P$. Then $xn \in P$ for all $n \in N$, since $P$ is an ideal of $N$. Hence $x(n+P) = P$ for all $n \in N$, i.e. $xN/P = P$.

$(ii) \Rightarrow (i)$ Suppose $\Gamma$ is a completely prime $N-$group and $P = (0_\Gamma : \Gamma)_N$. Let $x, y \in N$ such that $xy \in P$. Then $xy\gamma = 0_\Gamma$ for all $\gamma \in \Gamma$. Since $\Gamma$ is completely prime and $y\gamma \in \Gamma$, $xy\gamma = 0_\Gamma$ implies $x\Gamma = 0_\Gamma$ or $y\gamma = 0_\Gamma$. If $x\Gamma = 0_\Gamma$, then $x \in (0_\Gamma : \Gamma)_N = P$. If $y\gamma = 0_\Gamma$ for all $\gamma \in \Gamma$, then $y\Gamma = 0_\Gamma$ i.e. $y \in (0_\Gamma : \Gamma)_N = P$. Therefore $P$ is completely prime.

$(i) \Rightarrow (iii)$ Assume $P$ is a completely prime ideal of $N$ and $\Gamma = N/P$. Then $\Gamma$ is an $N-$group and $P = (0_\Gamma : \Gamma)_N$ as in the proof of $(i) \Rightarrow (ii)$. Now let $x, y \in N$ and $n+P \in N/P = \Gamma$ such that $xy(n+P) = P$. Then $xyn \in P$. Since $P$ is completely prime, it follows that $x \in P$ or $yn \in P$. If $x \in P$, then we get $xn \in P$ i.e. $x(n+P) = P$. If $yn \in P$, then $y(n+P) = P$. So we are done.

$(iii) \Rightarrow (i)$ Let $x, y \in N$ such that $xy \in P = (0_\Gamma : \Gamma)_N$. Then $xy\gamma = 0_\Gamma$ for all $\gamma \in \Gamma$. Hence $x\gamma = 0_\Gamma$ or $y\gamma = 0_\Gamma$ for all $\gamma \in \Gamma$ by assumption; whence $x \in (0_\Gamma : \Gamma)_N = P$ or $y \in (0_\Gamma : \Gamma)_N = P$. Thus, $P$ is a completely prime ideal of $N$.

**Corollary** Let $\Gamma$ be an $N-$group. Then $\Gamma$ is completely prime if and only if $xy\gamma = 0_\Gamma$ implies that $x\gamma = 0_\Gamma$ or $y\gamma = 0_\Gamma$ for all $x, y \in N$ and $\gamma \in \Gamma$.

The following proposition shows that similar charaterization exists for 3-prime $N-$groups.

**Theorem 2** Let $N \neq P \triangleleft N$. Then the followings are equivalent:
$i$) $P$ is a 3-prime ideal of $N$.
$ii$) There is a 3-prime $N-$group $\Gamma$ with $P = (0_\Gamma : \Gamma)_N$.
$iii$) There is an $N-$group $\Gamma$ with $P = (0_\Gamma : \Gamma)_N$ where $xNy\gamma = 0_\Gamma$ implies that $x\gamma = 0_\Gamma$ or $y\gamma = 0_\Gamma$ for all $x, y \in N$ and $\gamma \in \Gamma$.

**Proof.** $(i) \Leftrightarrow (ii)$ ([3], Proposition 1.25)

$(i) \Rightarrow (iii)$ Assume $P$ is a 3-prime ideal of $N$ and $\Gamma = N/P$. Then $\Gamma$ is an $N-$group and $P = (0_\Gamma : \Gamma)_N$ as in the proof of Teorem 1 $(i) \Rightarrow (ii)$. Also, let $xNy\left(a+P\right) = P$ for $x, y \in N$ and $a+P \in N/P$. Then $xnya \in P$ for all $n \in N$. Since $P$ is 3-prime, $x \in P$ or $ya \in P$. If $x \in P$, then $xa \in P$ since $P \lhd N$. Hence $x\left(a+P\right) = P$. If $ya \in P$, then $y\left(a+P\right) = ya + P = P$.

$(iii) \Rightarrow (i)$ Let $x, y \in N$ such that $xNy \subseteq P = (0_\Gamma : \Gamma)_N$. Then $xNy\gamma = 0_\Gamma$ for all $\gamma \in \Gamma$. Furthermore, by assumption, $x\gamma = 0_\Gamma$ or $y\gamma = 0_\Gamma$ for all $\gamma \in \Gamma$. It follows that $x \in (0_\Gamma : \Gamma)_N = P$ or $y \in (0_\Gamma : \Gamma)_N = P$. Hence, $P$ is a 3-prime ideal of $N$.

**Corollary** Let $\Gamma$ be an $N-$group. Then $\Gamma$ is 3-prime if and only if $xNy\gamma = 0_\Gamma$ implies that $x\gamma = 0_\Gamma$ or $y\gamma = 0_\Gamma$ for all $x, y \in N$ and $\gamma \in \Gamma$.

## 4. Conclusion

We obtain two new characterizations of prime $N-$groups. Thanks to these characterizations, one can study with elements instead of studying with sets.

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# Bell polynomial solutions of high order linear Volterra integro differential equtions with functional delays

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**Abstract**

In this study, a combining matrix-collocation method based on Bell series and collocation points is presented to find the solution of Volterra integro-differential equations with variable coefficients involving functional delays under the mixed conditions. This method reduces the mentioned problem to a matrix equation corresponding to the system of linear algebraic equations with unknown Bell coefficients. Thereby, the solutions of the problem are obtained in terms of Bell polynomials.Some illustrative examples, which arise in physics, biology, chemistry, mechanics and so on, are included to indicate the reliability and applicability of the method. Also, an error analysis technique based on residual functions is performed to check the accuracy of the solutions.

**Keywords:** Volterra integro differential equaton, Bell polynomials and series, matrix - collocation method, functional delays, residual error.

## 1. Introduction

Volterra delay-integro-differential equations (VDIDEs) are combination of delay differential equation and Volterra integral equation.This class of equations plays an important role scientific fiels and engineering such as physics, biology, chemistry, mechanics and so on[1,2]. Since the mentioned equations are usually difficult to solve analytically; therefore, numerical methods are needed. In recent years, for solving these equations, numerical methods have been developed. For example, numerical time-integration techniques of one-step collocation and Runge– Kutta methods [3-6], numerical treatments [7], Bernoulli collocation method [8], Taylor collocation method [9], Legendre spectral-collocation method [10] and etc.

In this study, we consider the approximate solutions of high order linear Volterra integro-differential equtions with variable coefficients involving functional delays in the form

$$\sum_{k=0}^{m}\sum_{j=0}^{J}P_{kj}(x)y^{(k)}\left(x+\tau_{kj}(x)\right)=g(x)+\int_{a}^{x}\sum_{l=0}^{m_1}K_l\,(x,t)y^{(l)}(t)dt, a\le x,t\le b; \mathrm{m}_1\le m \qquad (1)$$

with the mixed conditions

$$\sum_{\mathrm{k}=0}^{\mathrm{m}-1}(a_{jk}y^{(k)}(a)+b_{jk}y^{(k)}(b))=\lambda_{\mathrm{j}}, \qquad \mathrm{j}=0,1,\dots,\mathrm{m}-1 \qquad (2)$$

where $P_{kj}(x), \tau_{kj}(x), g(x)$ and the kernel function $K_l(x,t)$ are known functions defined on the interval $a \le x, t \le b; a_{jk}, b_{jk}$ and $\lambda_j$ are suitable constants.
Our aim is to obtain an approximate solution of (1) in the following truncated Bell series form

$$y(x) \cong y_N(x) = \sum_{n=0}^{N} a_n B_n(x) \tag{3}$$

here $a_n, n = 0,1, \cdots N$ are unknown Bell coefficients and $B_n(x), n = 0,1, \cdots, N$ ,are Bell polynomial defined by

$$B_n(x) = \sum_{k=0}^{n} S(n,k)\, x^k \tag{4}$$

where

$$S(n,k) = \sum_{j=0}^{k} \frac{(-1)^{k-j}}{k!} \binom{k}{j} . j^n$$

is Stirling numbers of the second kind [9 -12].

## 2. Fundamental Matrix Relations

In this part we construct the fundamental matrix relations. Let us consider Eq.(1) and find the matrix forms. Firstly, we can convert the Bell polynomials (4) to the matrix form

$$\mathbf{B}(\mathrm{x}) = \mathbf{X}(\mathrm{x})\mathbf{S} \tag{5}$$

where

$$\mathbf{B}(\mathrm{x}) = [B_0(x)\ \ B_1(x) \ldots\ B_N(x)] \quad , \mathbf{X}(\mathrm{x}) = [1\ \ x\ \ x^2 \ldots\ x^N]$$

and

$$\mathbf{S} = \begin{bmatrix} \mathrm{S}(0,0) & \mathrm{S}(1,0) & \mathrm{S}(2,0) & \cdots & \mathrm{S}(\mathrm{N},0) \\ 0 & \mathrm{S}(1,1) & \mathrm{S}(2,1) & \cdots & \mathrm{S}(\mathrm{N},1) \\ 0 & 0 & \mathrm{S}(2,2) & \cdots & \mathrm{S}(\mathrm{N},2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mathrm{S}(\mathrm{N},\mathrm{N}) \end{bmatrix}$$

Let us show Eq. (1) in the form

$$D(x) = g(x) + V(x) \tag{6}$$

where the functional differential part is

$$D(x) = \sum_{k=0}^{m} \sum_{j=0}^{J} P_{kj}(x) y^{(k)}\left(x + \tau_{kj}(x)\right)$$

and Volterra integral part is

$$V(x)=\int_a^x\sum_{l=0}^{m_1}K_l\,(x,t)y^{(l)}(t)dt.$$

Now we transform the parts $D(x)$,$V(x)$ and the conditions (2) to matrix forms.

**2.1Matrix relation for the differential part D(x)**

Let us consider the solution $y(x)$ of (1) can be written in the matrix form

$$y(x)=\mathbf{B}(x)\mathbf{A}\,;\mathbf{A}=[a_0\quad a_1\quad\cdots\quad a_N]^{\mathbf{T}} \tag{7}$$

By substituting (5) into (7) we obtain that

$$y(x)=\mathbf{X}(x)\mathbf{SA}. \tag{8}$$

In addition to this, it is cleary seen [13] that the relation between the matrix $\mathbf{X}$(x) and its $k$th derivative $\mathbf{X}^{(k)}$(x) is

$$\mathbf{X}^{(k)}(x)=\mathbf{X}(x)\mathbf{M}^k \tag{9}$$

where

$$\mathbf{M}=\begin{bmatrix}0&1&0&\cdots&0\\0&0&2&\cdots&0\\\vdots&\vdots&\vdots&\cdots&\vdots\\0&0&0&\cdots&N\\0&0&0&\cdots&0\end{bmatrix},\mathbf{M}^0=\begin{bmatrix}1&0&0&\cdots&0\\0&1&0&\cdots&0\\0&0&1&\cdots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\cdots&1\end{bmatrix}.$$

Therefore by using relations (8) and (9), we can write the following matrix form

$$y^{(k)}(x)=\mathbf{B}^{(\mathrm{k})}(x)\mathbf{A}=\mathbf{X}^{(\mathrm{k})}(x)\mathbf{SA}=\mathbf{X}(x)\mathbf{M}^{\mathrm{k}}\mathbf{SA}\,. \tag{10}$$

Similarly, if we put $x\to x+\tau_{kj}(x)$ into (9), [14,15] we obtain the matrix relation

$$y^{(k)}\big(x+\tau_{kj}(x)\big)=\mathbf{X}^{(\mathrm{k})}\big(x+\tau_{kj}(x)\big)\mathbf{SA}=\overline{\mathbf{X}}(x)\mathrm{M}\big(\tau_{kj}(x)\big)\mathbf{M}^{\mathrm{k}}\mathbf{SA} \tag{11}$$

where

$$\mathrm{M}\Big(\tau_{kj}(x)\Big)=\begin{bmatrix}\binom{0}{0}(\tau_{kj}(x))^0&\binom{1}{0}(\tau_{kj}(x))^1&\binom{2}{0}(\tau_{kj}(x))^2&\cdots&\binom{N}{0}(\tau_{kj}(x))^N\\0&\binom{1}{1}(\tau_{kj}(x))^0&\binom{2}{1}(\tau_{kj}(x))^1&\cdots&\binom{N}{1}\tau_{kj}(x))^{N-1}\\0&0&\binom{2}{2}(\tau_{kj}(x))^0&\cdots&\binom{N}{2}(\tau_{kj}(x))^{N-2}\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\cdots&\binom{N}{N}(\tau_{kj}(x))^0\end{bmatrix}.$$

By substituting the expression (11) into Eq. (6), we get the matrix relation

$$D(x) = \sum_{k=0}^{m} \sum_{j=0}^{J} P_{kj}(x) y^{(k)} \left(x + \tau_{kj}(x)\right)$$

### 2.2 Matrix Representation for Volterra Integral Part

Let us find the matrix form for the Volterra integral part $V(x)$.The kernel function can be $K_l(x,t)$ approximated by the truncated Maclaurin series [16]

$$\mathrm{K}_l(x,t) = \sum_{p=0}^{N} \sum_{q=0}^{N} \mathrm{k}_{pq} \mathrm{x}^p \mathrm{t}^q \qquad (12)$$

where

$$\mathrm{k}_{pq} = \frac{1}{p!q!} \frac{\partial^{p+q} \mathrm{K}(0,0)}{\partial \mathrm{x}^p \partial \mathrm{t}^q}\ ,\ p,q = 0,1,\dots,N$$

The expression (12) convert to the matrix form

$$\mathrm{K}_l(x,t) = \mathbf{X}(x)\mathbf{K}_l\mathbf{X}^{\mathrm{T}}(\mathrm{t})\ ;\ \mathbf{K}_l = \left[\mathrm{k}_{pq}\right]. \qquad (13)$$

Substituting the relations (10) and (13) in the Volterra integral part, we obtain

$$V(x) = \int_a^x \sum_{l=0}^{m_1} \mathbf{X}(\mathrm{x})\mathbf{K}_l\mathbf{X}^{\mathrm{T}}(\mathrm{t})\mathbf{X}(\mathrm{t})\mathbf{M}^l\mathbf{SA}\ \ dt \qquad (14)$$

$$= \sum_{l=0}^{m_1} \overline{\mathbf{X}}(x)\mathbf{K}_l \Theta(x)\mathbf{M}^l\mathbf{SA}$$

where

$$\Theta(x) = \left[\varphi_{ij}(x)\right] = \int_{\mathrm{a}}^{\mathrm{x}} \mathbf{X}^{\mathrm{T}}(\mathrm{t})\mathbf{X}(t)\mathrm{dt}\ \ and\ \ \varphi_{ij}(x) = \frac{\mathrm{x}^{\mathrm{i+j+1}} - \mathrm{a}^{\mathrm{i+j+1}}}{\mathrm{i+j+1}}\ \ where\ i,j = 0,1,2,\dots,N.$$

### 2.3 Matrix relation for the conditions

We can write the corresponding matrix forms for the conditions (2), using the relations (7) and (10), as

$$\left\{ \sum_{k=0}^{m-1} (a_{jk}\mathbf{X}(\mathrm{a})\mathbf{M}^{\mathrm{k}}\mathbf{S} + b_{jk}\mathbf{X}(b)\ \mathbf{M}^k\ \mathbf{S} \right\} \mathbf{A} = \lambda_j\ , j = 0,1\ ,\dots\ ,m-1 \qquad (15)$$

**3. Method of Solution**

By substituting the matrix relations (11) and (14) into (1) we construct the matrix equation

$$\sum_{k=0}^{m}\sum_{j=0}^{J} P_{kj}(x)y^{(k)}\,\overline{\mathbf{X}}(x)\text{Ɱ}(x)\mathbf{M}^{\mathrm{k}}\mathbf{SA}-\sum_{l=0}^{m_1}\overline{\mathbf{X}}(x)\mathbf{K}_l\Theta(x)\mathbf{M}^l\mathbf{SA}=g(x) \tag{16}$$

The collocation points $x_i$ are defined by

$$x_i=a+\frac{b-a}{N}i\ ,\ i=0,1,\ldots,N. \tag{17}$$

and by using the points (17), it is obtained the system of the matrix equations

$$\sum_{k=0}^{m}\sum_{j=0}^{J} P_{kj}(x_{\mathrm{i}})\,\overline{\mathbf{X}}(x_{\mathrm{i}})\text{Ɱ}(x_{\mathrm{i}})\mathbf{M}^{\mathrm{k}}\mathbf{SA}-\sum_{l=0}^{m_1}\overline{\mathbf{X}}(x_{\mathrm{i}})\mathbf{K}_l\Theta(x_{\mathrm{i}})\mathbf{M}^l\mathbf{SA}=g(x_{\mathrm{i}}) \tag{18}$$

or shortly the fundamental matrix equation

$$\left\{\sum_{k=0}^{m}\sum_{j=0}^{J}\mathbf{P}_{\mathrm{kj}}\,\overline{\mathbf{X}}\overline{\text{Ɱ}}\mathbf{M}^{\mathrm{k}}\mathbf{S}-\sum_{l=0}^{m_1}\overline{\mathbf{X}}\overline{\mathbf{K}}_l\overline{\Theta}\mathbf{M}^l\mathbf{S}\right\}\mathbf{A}=\mathbf{G}$$

where

$$\mathbf{P}_{kj}=\begin{bmatrix}P_{kj}(x_0) & 0 & \cdots & 0\\ 0 & P_{kj}(x_1) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & P_{kj}(x_N)\end{bmatrix}_{(N+1)\times(N+1)}\qquad \overline{\mathbf{K}}_l=\begin{bmatrix}\mathrm{K} & 0 & \cdots & 0\\ 0 & \mathrm{K} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \mathrm{K}\end{bmatrix}_{(N+1)^2\times(N+1)^2}$$

$$\mathbf{X}=\begin{bmatrix}1 & x_0 & \cdots & {x_0}^N\\ 1 & x_1 & \cdots & {x_1}^N\\ \vdots & \vdots & \ddots & \vdots\\ 1 & x_N & \cdots & {x_N}^N\end{bmatrix}_{(N+1)\times(N+1)}\ ,\overline{\mathbf{X}}=\begin{bmatrix}\mathrm{X}(x_0) & 0 & \cdots & 0\\ 0 & \mathrm{X}(x_1) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \mathrm{X}(x_N)\end{bmatrix}_{(N+1)\times(N+1)^2}$$

$$\overline{\text{Ɱ}}=\begin{bmatrix}\text{Ɱ}(\tau_{kj}(x_0))\\ \text{Ɱ}(\tau_{kj}(x_1))\\ \vdots\\ \text{Ɱ}(\tau_{kj}(x_N))\end{bmatrix}_{(N+1)^2\times(N+1)}\ ,\overline{\mathbf{\Theta}}=\begin{bmatrix}\Theta(x_0)\\ \Theta(x_1)\\ \vdots\\ \Theta(x_N)\end{bmatrix}_{(N+1)^2\times(N+1)}\ ,\mathbf{G}=\begin{bmatrix}g(x_0)\\ g(x_1)\\ \vdots\\ g(x_N)\end{bmatrix}_{(N+1)\times1}$$

The fundamental matrix Eq. (18) for (1) corresponds to a system of $(N+1)$ algebraic equation for the $(N+1)$ unknown coefficients $a_0,a_1,a_2,\ldots,a_N$ . Concisely we can write as

$$\mathbf{WA}=\mathbf{G}\quad or\ \ [\mathbf{W};\mathbf{G}] \tag{19}$$

where

$$\mathbf{W}=\sum_{k=0}^{m}\sum_{j=0}^{J}\mathbf{P}_{\mathrm{kj}}\,\overline{\mathbf{X}}\overline{\mathbf{M}}_{\mathbf{J}}\mathbf{M}^{\mathrm{k}}\mathbf{S}-\sum_{l=0}^{m_1}\overline{\mathbf{X}}\overline{\mathbf{K}}_{l}\overline{\Theta}\mathbf{M}^{l}\mathbf{S}$$

On the other hand, the matrix form (15) for the conditions can be written as

$$\mathbf{U}_{\mathrm{j}}\mathbf{A}=\lambda_j \text{ ya da } [\mathbf{U};\lambda_j], \qquad j=0,1,\dots,m-1 \tag{20}$$

where

$$\mathbf{U}_j=[u_{j0} \quad u_{j1} \quad \cdots \quad u_{jN}]=\sum_{k=0}^{m-1}a_{jk}\mathbf{X}(\mathrm{a})\mathbf{M}^{\mathrm{k}}\mathbf{S}+b_{jk}\mathbf{X}(b)\mathbf{M}^{\mathrm{k}}\mathbf{S}\ , j=0,1,\dots,m-1$$

To obtain the solution of (1) under conditions (2), by replacing the m rows in matrix equation (20) into the matrix equation (19), we have the required new augmented matrix system

$$\widetilde{\mathbf{W}}\mathbf{A}=\widetilde{\mathbf{G}} \quad or \quad [\widetilde{\mathbf{W}};\widetilde{\mathbf{G}}]$$

If $rank(\widetilde{\mathbf{W}})=rank[\widetilde{\mathbf{W}}\,;\,\widetilde{\mathbf{G}}]=N+1$ ,then we can write

$$\mathbf{A}=(\widetilde{\mathbf{W}})^{-1}\widetilde{\mathbf{G}}.$$

Thus the matrix **A** (there by the coefficients $a_0,a_1,a_2,\dots,a_N$ ) is uniquely determined and the Eq. (1) under the coefficient equation (2) has unique solution. This solution is given by truncated Bell series

$$y(x)\cong y_N(x)=\sum_{n=0}^{N}a_nB_n(x).$$

### 3.2 Residual Correction and Error Estimation

We can easily check the accuracy of the obtained solutions as follows. Since the truncated Bell series (3) is the approximate solution of (1), when the function $y_N(x)$ and its derivatives are substituted in (1), the resulting equation must be satisfied approximately; that is, for $x=x_q\in[a,b]\,,q=0,1,\dots$

$$R_N(x_q)=\sum_{k=0}^{m}\sum_{j=0}^{J}P_{kj}(x_q){y_N}^{(k)}\left(x_q+\tau_{kj}(x_q)\right)-\int_a^{x_q}\sum_{l=0}^{m_1}K_l\left(x_q,t\right){y_N}^{(l)}(t)dt\ -g(x_q)\cong 0$$

or

$$R_N(x_q)\leq 10^{-k_q},\left(k_q \text{ is any positive integer}\right).$$

İf $max\ 10^{-k_q}=10^{-k}$( $k$ is a positive integer) is prescribed, then the truncation limit N is increased until the difference $R_N(x_q)$ at each of the points becomes smaller than the prescribed $10^{-k}$ . Therefore, if $R_N(x_q)\to 0$ when N is sufficiently large enough, then the error decreases.

On the other hand, by means of the residual function defined by $R_N(x)$ and the mean value of the function $|R_N(x)|$ on the interval [a, b], the accuracy of the solution can be controlled and the error can be estimated [17]. Thus, we can estimate the upper bound of the mean error $\overline{R_N}$ as follows:

$$\left|\int_a^b R_N(x)dx\right| \leq \int_a^b |R_N(x)|dx$$

and

$$\int_a^b |R_N(x)|dx = (b-a)|R_N(c)|\,, a \leq c \leq b$$

$$\Rightarrow \left|\int_a^b R_N(x)dx\right| = (b-a)|R_N(c)| \Rightarrow (b-a)|R_N(c)| \leq \int_a^b |R_N(x)|dx$$
$$\Downarrow$$
$$|R_N(c)| \leq \frac{\int_a^b |R_N(x)|dx}{b-a} = \overline{R_N}$$

## 3.2 Numerical Examples

Using exact solution $y(x)$ and the approximate solution $y_N(x)$, the error function $e_N$ is calculated by the following form.

$$e_N = y(x) - y_N(x)$$

**Example 1**. Let us first consider the first order linear Volterra type integro-differential equation with functional delay given by

$$y'(x) - y(x-x^2) = \frac{x^2}{2} - 2x + \int_0^x y(x) \quad 0 \leq x, t \leq 1$$

with initial conditions $y(0) = 1$ and the approximate solution $y(x)$ by the truncated Bell series

$$y(x) \cong y_N(x) = \sum_{n=0}^{N} a_n B_n(x).$$

Here $P_{00}(x) = -1, P_{10}(x) = 1, \tau_{10}(x) = -x^2,\ ,K_0(x,t) = 1$ and $g(x) = x^2/2 - 2x$.Then for $N = 2$,the collotion points are

$$\{x_0 = 0, x_1 = 1/2\,, x_2 = 1\}$$

and from Eq. (18), the fundamental matrix equation of the problem is

$$\{\mathrm{P}_{10}\mathbf{XMS} + \mathrm{P}_{00}\mathbf{X}\overline{\mathrm{M}}\mathbf{M}^0\mathbf{S} - \overline{\mathbf{X}}\overline{\mathbf{K}}_{00}\overline{\Theta}\mathbf{M}^0\mathbf{S}\}\mathbf{A} = \mathbf{G}$$

$$\mathbf{P}_{00} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \mathbf{P}_{10} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \mathbf{S} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \mathbf{M} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{X} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1/2 & 1/4 \\ 1 & 1 & 1 \end{bmatrix}, \overline{\mathbf{X}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}, \mathbf{G} = \begin{bmatrix} 0 \\ -9/8 \\ -5/2 \end{bmatrix}$$

$$\overline{\Theta} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1/2 & 1/8 & 1/24 \\ 1/8 & 1/24 & 1/64 \\ 1/24 & 1/64 & 1/160 \\ 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}, \overline{\mathrm{M}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1/4 & 1/16 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}, \overline{\mathbf{K}}_{00} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The augmented matrix for this fundamental matrix equation is calculated as

$$[\mathbf{W}; \mathbf{G}] = \begin{bmatrix} -1 & -1 & 1 & ; & 0 \\ -3/2 & 5/8 & 73/8 & ; & -9/8 \\ -2 & 1/2 & 13/6 & ; & -5/2 \end{bmatrix}$$

From Eq. (20), the matrix forms for the initial condition is

$$[\mathbf{U}_0; \lambda_0] = [1 \quad 0 \quad 0 \quad ; \quad 1]$$

From system (21), the new augmented matrix based on conditions can be obtained as follows:

$$[\widetilde{\mathbf{W}}\, ;\, \widetilde{\mathbf{G}}] = \begin{bmatrix} -1 & -1 & 1 & ; & 0 \\ 1 & 0 & 0 & ; & 1 \\ -2 & 1/2 & 13/6 & ; & -5/2 \end{bmatrix}$$

Solving this system, the unknown Bell coefficient matrix is obtained as

$$\mathbf{A} = [1 \quad 1 \quad 0]^{\mathrm{T}}$$

By substituting the above Bell coefficient matrix into equation (7), we obtain the approximate solution $y(x) = \mathrm{x} + 1$ which is the exact solution.

**Example 2.** Let us consider the first order linear Volterra type integro-differential equation with functional delay given by

$$y'(x) - y(x - x^2) = e^{(x-x^2)} + x + 1 + \int_0^x (x - t)y'(t)dt \quad 0 \leq x, t \leq 1$$

with initial condition $y(0) = 1$. The exact solution of problem is $y(x) = e^x$ . The fundamental matrix equation of the problem is

$$\{\mathrm{P}_{10}\mathbf{XMS} + \mathrm{P}_{00}\mathbf{X}\overline{\mathrm{M}}\mathbf{M}^0\mathbf{S} - \overline{\mathbf{X}}\overline{\mathbf{K}}_{00}\overline{\Theta}\mathbf{M}^0\mathbf{S}\}\mathbf{A} = \mathbf{G}$$

When necessary operations are taken, approximate solutions are calculated for

$$N = 4, N = 5 \; and \; N = 6$$

$y_4(x) = 1 + x + 0.5026x^2 + 0.1542x^3 + 0.0610x^4$
$y_5(x) = 1 + x + 0.49946x^2 + 0.1702x^3 + 0.0345x^4 + 0.0141x^5$
$y_6(x) = 1 + x + 0.5001x^2 + 0.1661x^3 + 0.0431x^4 + 0.0066x^5 + 0.0024x^{6.}$

The exact solution and approximate solutions of the equation are shown in Figure 1.1 and their absolute errors are shown in Table 1. Furthermore, graphs of the residual error functions of the calculated numerical results are shown in Figure 1.2.

Table 1. Comparison of the absolute errors of Example 2 for N= 4, 5,6.

| $x_i$ | $y(x) = e^{x_i}$ | $\lvert e_4(x_i)\rvert$ | $\lvert e_5(x_i)\rvert$ | $\lvert e_6(x_i)\rvert$ |
|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 |
| 0.2 | 1.2214 | 3.2442e-05 | 5.4462e-06 | 1.2674e-06 |
| 0.4 | 1.4918 | 2.1702e-05 | 3.1364e-07 | 2.4768e-06 |
| 0.6 | 1.8221 | 3.0000e-05 | 3.9844e-06 | 5.7500e-06 |
| 0.8 | 2.2255 | 5.9072e-05 | 3.1040e-05 | 1.1865e-05 |
| 1 | 2.7183 | 4.8183e-04 | 8.1828e-05 | 1.8172e-05 |

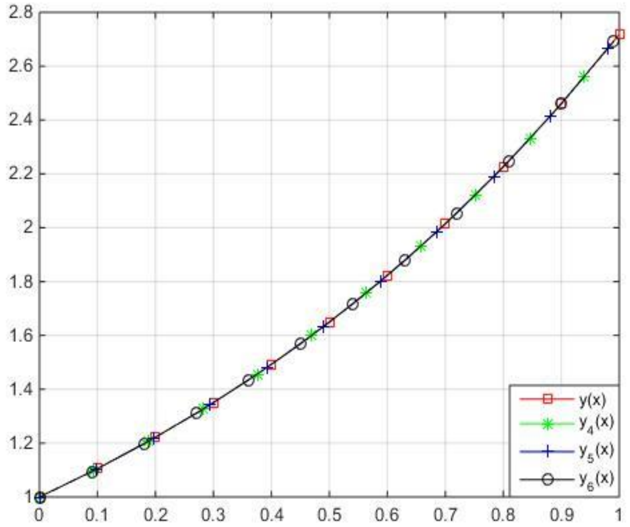


**Figure 1. 1** Numerical and Exact Solutions of Example 2 for N = 4,5,6.

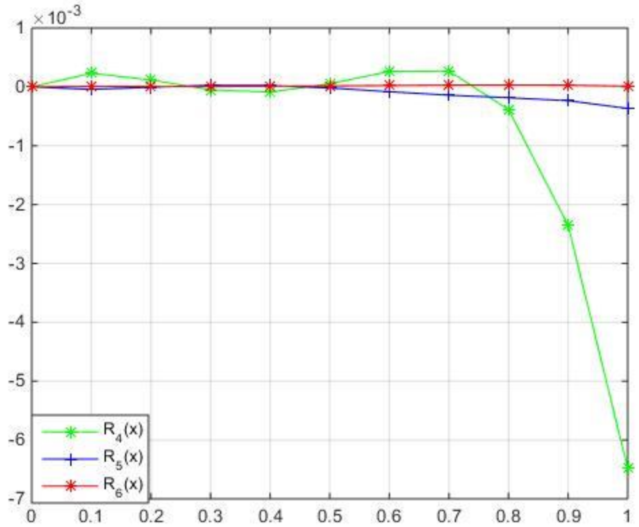


**Figure 1. 2.** Residual Error Functions of Example 2 for N =4,5,6.

## 4.Conclusion

In this study, a new method was developed by using Bell polynomials for the solution of high order linear Volterra integro-differential equations with functional delay. To illustrate the validity and applicability of this method, explanatory examples were solved, and an error analysis based on the residual function was performed to show the accuracy of the results. These comparisons and error estimates show that the proposed method is highly effective. We have calculated the solutions with the help of MATLAB.

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**INTERNATIONAL CONFERENCE ON MATHEMATICS**
**"An Istanbul Meeting for World Mathematicians"**
**3-5 July 2019, Istanbul, Turkey**

## A New Class of Soft Topological Space via (1,2)*- Soft b-open Set

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**Abstract**

In this study, firstly, the properties of soft sets, (1,2)*- soft b-open sets and (1,2)*- soft b-closed sets, which are needed for the definition of (1,2)*- soft b-extremally disconnected space, are studied for soft bitopological spaces and some important results are given. Secondly, the characterization of (1,2)*-soft b-extremally disconnected bitopological spaces is presented. In particular, the relation between (1,2)*-soft dense and (1,2)*- soft b-open set is obtained. In the last section of this text, the properties of (1,2)*- soft b-submaximal space are studied using by (1,2)*- soft dense sets.

**Keywords:** (1,2)*- soft b-open sets, (1,2)*- soft dense sets, (1,2)*- soft b-submaximal space, (1,2)*- soft b-extremally disconnected space.

## 1. Introduction

One of the very natural trends of mathematical research is to define the framework of the unknown theorems and results. For instance, Molodtsov [5] introduced soft set theory in the setting of not clearly defined objects. Immediate extension of this theorem was given by Molodtsov who observed the application of soft set theory in the context of concepts like game theory, operations research, theory of probability, Riemann-integration, Perron-integration, smoothness of functions and so on. After Molodtsov, research on the soft set theory has been accelerated rapidly and several analogs of the soft set principle have been reported that can be in the references. Among them I can underline some of the interesting structures such as topological structures of set theories dealing with uncertainities were first defined by Chang [9] and soft topological spaces introduced by Shabir and Naz [3]. In recent years, the topological space theory has been embedding in the soft set theory to obtain some interesting applications that is coused to introduce the theory of soft bitopological spaces [2]. In the view of the definition of bitopological spaces [4] in general topology, the application area of bitopological spaces by soft sets is expanded in a short time. Nowadays, new concepts of soft sets are introduced in soft bitopological spaces such as (1,2)*- soft b- open sets, (1,2)*- soft b –closed sets, (1,2)*- soft regular - open, (1,2)*- soft preopen, (1,2)*-soft semi open, (1,2)*- soft $\alpha$ - open and (1,2)*- soft $\beta$ - open sets [1]. In this text, these soft sets are examined in detail and moreover, (1,2)*- soft b-extremally disconnected space is studied and the relation between (1,2)*-soft dense and (1,2)*- soft b-open set is obtained. In this paper, motivated and inspired by the above literature, I try to develop a point of view to the soft sets that are defined in soft bitopological spaces.

## 2. Preliminaries

I now provide some basic concepts, definitions and lemmas which will be used in the sequel which may be found in [1, 2, 3, 4, 5] for further details.

Throughout this work, U refers to an initial universe, E is a set of parameters and P(U) is the power set of U.

**Definition 2.1 : [6]** (Soft Set) A soft set $F_A$ on the universe X is defined by the set of ordered pairs

$$F_A = \{( x, f_A(x)) : x \in E\}$$

where $f_A : E \to P(X)$ such that $f_A(x) = \phi$ if $x \notin A$. Here $f_A$ is called approximate function of the soft set $F_A$ . The value of $f_A(x)$ may arbitrary, some of them maybe empty, some may have non empty intersection. The set of all soft sets over X will be denoted by S(X).

From now on, we will use the definitions and operations of soft sets are written with the form of ([6]).

**Definition 2.2: [6]** Let $F_A \tilde{\in}$ S(X). If $f_A(x) = \phi$ for all $x \in E$, then $F_A$ is called an empty set, denoted by $F_\phi$. $f_A(x) = \phi$ means that there is no element in X related to the parameter $x \in E$. Therefore, we do not display such elements in the soft sets, as it is meaningless to consider such parameters.

**Definition 2.3: [6]** Let $F_A \tilde{\in}$ S(X). If $f_A(x) = X$ for all $x \in A$, then $F_A$ is called an A-universal soft set, denoted by $F_{\tilde{A}}$. If A = E, then the A- universal soft set is called a universal soft set, denoted by $\tilde{X}$ .

**Definition 2.4: [6]** Let $F_A$ , $F_B \tilde{\in}$ S(X). Then $F_A$ is a soft subset of $F_B$, denoted by $F_A \tilde{\subseteq} F_B$ if $f_A(x) \subseteq f_B(x)$ for all $x \in E$. Let $F_A$ and $F_B$ are soft equal denoted by $F_A = F_B$ if $f_A(x) = f_B(x)$ for all $x \in E$.

**Definition 2.5:** **[7]** Let $F_A \tilde{\in}$ S(X). A soft topology on $F_A$, denoted by $\tilde{\tau}$ , is a collection of soft subsets of $F_A$ having following properties: collection of soft sets over $\tilde{X}$ ,

(i) $F_\phi$ , $F_A$ belong to $\tilde{\tau}$

(ii) Union of any number of soft sets in $\tilde{\tau}$ belongs to $\tilde{\tau}$

(iii) Intersection of two soft sets in $\tilde{\tau}$ belongs to $\tilde{\tau}$

The pair ( $F_A$ , $\tilde{\tau}$) is called a soft topological space.

**Definition 2.6:** **[4]** Let $X \neq \emptyset$, $\tau_1$ and $\tau_2$ are two different topologies on X. Then (X, $\tau_1$, $\tau_2$)is called a bitopological space. Throughout this paper (X, $\tau_1$, $\tau_2$) [or simply X] denote bitopological space on which no seperation axioms are assumed unless explicitly stated.

**Definition 2.7:** **[4]** A subset S of X is called $\tau_1\tau_2$-open if S = H $\cup$ K such that H $\in\tau_1$ and K $\in\tau_2$ and the complement of $\tau_1\tau_2$-open is $\tau_1\tau_2$-closed.

**Definition 2.8:** **[2]** Let $F_A$ be a nonempty soft set on the universe U, $\tilde{\tau}_1$ and $\tilde{\tau}_2$ be two different soft topologies on $F_A$. Then, ($F_A$, $\tilde{\tau}_1$ , $\tilde{\tau}_2$, E) is called a soft bitopological space.

## 3. Main Results

The main aim of this paper is to give an improvement of the recent result on the soft bitopological spaces. The results given here are obtained using different soft sets such as (1,2)*- soft b-open sets and (1,2)*- soft b-closed sets and spaces such as (1,2)*- soft b-extremally disconnected space and (1,2)*-soft b-extremally disconnected bitopological spaces. Apart from definitions and theorems are numbered, known concepts are mentioned in the text along with the reference [1].

**Definition 3.1: [2]**

i. The $\tilde{\tau}_{1,2}$ - closure of $F_E$, denoted by $\tilde{\tau}_{1,2}$ - $\mathrm{cl}(F_E)$, is defined by

$$\tilde{\tau}_{1,2}\text{ - cl}(F_E) = \cap\{H_E : H_E \tilde{\supseteq} F_E; H_E \text{ is a } \tilde{\tau}_{1,2}\text{- open set}\}$$

ii. The $\tilde{\tau}_{1,2}$ - interior of $F_E$, denoted by $\tilde{\tau}_{1,2}$ - $\mathrm{int}(F_E)$, is defined by

$$\tilde{\tau}_{1,2}\text{ - int}(F_E) = \tilde{\cup}\{G_E : G_E \tilde{\subseteq} F_E : G_E \text{ is a } \tilde{\tau}_{1,2}\text{- soft open set}\}$$

**Definition 3.2 [1]:** Let $(\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2, E)$ be a soft bitopological space and $F_A \tilde{\subseteq} \tilde{X}$.

Then $F_A$ is called $(1,2)*$ - soft b-open set (briefly $(1,2)*$-sb-open) if
$F_A \tilde{\subseteq} \tilde{\tau}_{1,2}\text{- int}(\tilde{\tau}_{1,2}\text{- cl}(F_A)) \tilde{\cup} \tilde{\tau}_{1,2}\text{- cl}(\tilde{\tau}_{1,2}\text{- int}(F_A))$.

The set of all $(1,2)*$ - soft b-open sets are denoted by $(1,2)*$-SbO$(\tilde{X})$.

**Definition 3.3 [1]:** Let $(\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2, E)$ be a soft bitopological space and $F_A$ be a soft set over $\tilde{X}$

i. $(1,2)*$ - soft b-closure (briefly $(1,2)*$ - sbcl$(F_A)$) of a set $F_A$ in $\tilde{X}$ defined as the soft intersection of all $(1,2)*$ - soft b-closed supersets of $F_A$.
ii. $(1,2)*$ - soft b-interior (briefly $(1,2)*$ - sbint$(F_A)$) of a set $F_A$ in $\tilde{X}$ defined as the soft union of all $(1,2)*$ - soft b-open supersets of $F_A$.

$F_A \tilde{\subseteq} \tilde{X}$. Then $F_A$ is called $(1,2)*$ - soft b-open set (briefly $(1,2)*$-sb-open) if
$F_A \tilde{\subseteq} \tilde{\tau}_{1,2}\text{- int}(\tilde{\tau}_{1,2}\text{- cl}(F_A)) \tilde{\cup} \tilde{\tau}_{1,2}\text{- cl}(\tilde{\tau}_{1,2}\text{- int}(F_A))$.

**Definition 3.4 [1]:** A soft bitopological space $(\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2, E)$ is said to be $(1,2)*$ - soft extremally disconnected space if $\tilde{\tau}_{1,2}$ - closure of every $\tilde{\tau}_{1,2}$ - open set of $\tilde{X}$ is $\tilde{\tau}_{1,2}$ - open set in $\tilde{X}$.

**Definition 3.5 [1]:** A soft bitopological space $(\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2, E)$ is said to be $(1,2)*$ - soft b-extremally disconnected space if $(1,2)*$ - b-closure of every $\tilde{\tau}_{1,2}$ -b-open set of $\tilde{X}$ is $(1,2)*$ - b-open set in $\tilde{X}$.

**Example 3.6:** Let $X=\{x,y,z,a\}$, $E=\{e_1\}$. $\tilde{X}=\{(e_1,\{x,y,z,a\})\}$.

$G_{E1}=\tilde{X}$, $G_{E2}=\phi$,

$G_{E3}=\{(e_1,\{x\})\}$, $G_{E4}=\{(e_1,\{y\})\}$, $G_{E5}=\{(e_1,\{z\})\}$, $G_{E6}=\{(e_1,\{a\})\}$,
$G_{E7}=\{(e_1,\{x,y\})\}$, $G_{E8}=\{(e_1,\{x,z\})\}$, $G_{E9}=\{(e_1,\{x,a\})\}$, $G_{E10}=\{(e_1,\{y,z\})\}$,

$G_{E11}=\{(e_1,\{y,a\})\}$, $G_{E12}=\{(e_1,\{z,a\})\}$, $G_{E13}=\{(e_1,\{x,y,z\})\}$, $G_{E14}=\{(e_1,\{x,y,a\})\}$,
$G_{E15}=\{(e_1,\{y,z,a\})\}$, $G_{E16}=\{(e_1,\{x,a,z\})\}$.

Consider the soft bitopological space $(\tilde{X},\tilde{\tau}_1,\tilde{\tau}_2,E)$, where
$\tilde{\tau}_1=\{\tilde{X},\varphi,G_{E6},G_{E7},G_{E14}\}$, $\tilde{\tau}_2=\{\tilde{X},\varphi,G_{E3}\}$. Then $\tilde{\tau}_{1,2}$ - open set are
$\{\tilde{X},\varphi,G_{E3},G_{E6},G_{E7},G_{E14},G_{E9}\}$, $(1,2)^*$ - soft b-open sets are
$\{\tilde{X},\varphi,G_{E3},G_{E6},G_{E7},G_{E8},G_{E9},G_{E12},G_{E3},G_{E14},G_{E16},G_{E4}\}$. Then $(1,2)^*$ -soft b-closure of every $(1,2)^*$ - soft b-open set of $\tilde{X}$ is $(1,2)^*$ - soft b-open set in $\tilde{X}$. Hence $(\tilde{X},\tilde{\tau}_1,\tilde{\tau}_2,E)$ is $(1,2)^*$ - soft b-extremally disconnected space.

**Remark 3.7 [1]:** Every $(1,2)^*$ - soft extremally disconnected space is $(1,2)^*$ - soft b-extremally disconnected space but not conversely as shown in the following example:

**Example 3.8:** Consider the soft bitopological space $(\tilde{X},\tilde{\tau}_1,\tilde{\tau}_2,E)$ given in Example 3.6, where
$\tilde{\tau}_1=\{\tilde{X},\varphi,G_{E6},G_{E7},G_{E14}\}$, $\tilde{\tau}_2=\{\tilde{X},\varphi,G_{E3}\}$. Then $\tilde{\tau}_{1,2}$ - open set are
$\{\tilde{X},\varphi,G_{E3},G_{E6},G_{E7},G_{E14},G_{E9}\}$, $\tilde{\tau}_{1,2}$ - open set are $\{\tilde{X},\varphi,G_{E13},G_{E15},G_{E5},G_{E12},G_{E10}\}$.

Here $\tilde{\tau}_{1,2}$ - cl$(G_{E6})=\tilde{\tau}_{1,2}$ - cl$(\{(e_1,\{a\})\})=G_{E12}=\{(e_1,\{z,a\})\}$, which is not a $\tilde{\tau}_{1,2}$ - open set.

Therefore, $(\tilde{X},\tilde{\tau}_1,\tilde{\tau}_2,E)$ is a $(1,2)^*$ - soft b-extremally disconnected space but not $(1,2)^*$ - soft extremally disconnected space.

**Definition 3.9 [1]:** A soft subset $F_E$ of a soft bitopological space $(\tilde{X},\tilde{\tau}_1,\tilde{\tau}_2,E)$ is called (1,2)*-soft dense if $\tilde{\tau}_{1,2}$-cl$(F_E)=\tilde{X}$.

**Definition 3.10 [1]:** A soft subset $F_E$ of a soft bitopological space $(\tilde{X},\tilde{\tau}_1,\tilde{\tau}_2,E)$ is called (1,2)*-soft b-dense if (1,2)*-sbcl$(F_E)=\tilde{X}$.

**Proposition 3.11 [1]:** Every (1,2)*- soft b-dense set is (1,2)*-soft dense.

Proof. Let $F_E$ be (1,2)*- soft b-dense set. Then (1,2)*-sbcl$\left(F_E\right)=\tilde{X}$. Since (1,2)*-sbcl$\left(F_E\right)\tilde{\subseteq}\tilde{\tau}_{1,2}$-cl$\left(F_E\right)$, we have $\tilde{\tau}_{1,2}$-cl$\left(F_E\right)=\tilde{X}$ and so $F_E$ is (1,2)*-soft dense.

The converse of the Proposition 4.3 need not be true as can be seen from the following example:

**Example 3.12 [1]:** Consider the soft bitopological space $\left(\tilde{X},\tilde{\tau}_1,\tilde{\tau}_2,E\right)$, where $\tilde{\tau}_1=\left\{\tilde{X},\varphi,F_{E_4},F_{E_{10}}\right\}$, $\tilde{\tau}_2=\left\{\tilde{X},\varphi,F_{E_1},F_{E_7},F_{E_{13}}\right\}$ and the soft subsets are as in the above Example. Then $\tilde{\tau}_{1,2}$-open set $\left\{\tilde{X},\varphi,F_{E_1},F_{E_4},F_{E_7},F_{E_{10}},F_{E_{13}}\right\}$ and $\tilde{\tau}_{1,2}$-closed set are $\left\{\tilde{X},\varphi,F_{\phi},F_{E_{12}},F_{E_{14}},F_{E_{11}},F_{E_8},F_{E_5}\right\}$. (1,2)*-soft b-open sets are $\left\{\tilde{X},\varphi,F_{E_1},F_{E_4},F_{E_7},F_{E_6},F_{E_8},F_{E_9},F_{E_{12}},F_{E_{10}},F_{E_{13}}\right\}$ and (1,2)*-soft b-closed sets are $\left\{\tilde{X},\varphi,F_{E_1},F_{E_2},F_{E_7},F_{E_3},F_{E_8},F_{E_5},F_{E_{12}},F_{E_{10}},F_{E_{14}}\right\}$. Take the soft subset $F_{E_7}=\left\{\left(e_1,\{x_1\}\right),\left(e_2,\{x_1\}\right)\right\}$ and $\tilde{\tau}_{1,2}$-cl$\left(F_{E_7}\right)=\left\{\left(e_1,\{x_1,x_2\}\right)\right\}=F_{E_6}\neq\tilde{X}$.

Thus $F_{E_7}$ is (1,2)*-soft dense but not (1,2)*-soft b-dense set.

**Definition 3.13 [1]:** A soft bitopological space $\left(\tilde{X},\tilde{\tau}_1,\tilde{\tau}_2,E\right)$ is called (1,2)*-soft submaximal if every (1,2)*-soft dense subset is $\tilde{\tau}_{1,2}$- open set in $\tilde{X}$ .

**Definition 3.14 [1]:** A soft bitopological space $\left(\tilde{X},\tilde{\tau}_1,\tilde{\tau}_2,E\right)$ is called (1,2)*-soft b-submaximal if every (1,2)*-soft dense subset is (1,2)*-soft b-open set in $\tilde{X}$ .

**Example 3.15 [1]:** Let us consider the soft bitopological space $\left(\tilde{X},\tilde{\tau}_1,\tilde{\tau}_2,E\right)$.

Define $\tilde{\tau}_1=\left\{\tilde{X},\varphi,F_{E_1},F_{E_7}\right\}$, $\tilde{\tau}_2=\left\{\tilde{X},\varphi,F_{E_3}\right\}$ $\tilde{\tau}_{1,2}$-open set are $\left\{\tilde{X},\varphi,F_{E_1},F_{E_7},F_{E_8},F_{E_{13}}\right\}$ and $\tilde{\tau}_{1,2}$-closed set are $\left\{\tilde{X},\varphi,F_{E_{11}},F_{E_6},F_{E_5},F_{E_{12}}\right\}$.

Then the collection of (1,2)*-soft open sets $\left\{\tilde{X},\varphi,F_{E_1},F_{E_3},F_{E_7},F_{E_8},F_{E_9},F_{E_{14}},F_{E_{13}}\right\}$.

The collection of (1,2)*-soft dense sets of $\tilde{X}$ are $\left\{\tilde{X},\varphi,F_{E_1},F_{E_3},F_{E_7},F_{E_8},F_{E_9},F_{E_{14}},F_{E_{13}}\right\}$.

Here all (1,2)*-soft dense sets are (1,2)*-soft b-open set and so $\left(\tilde{X},\tilde{\tau}_1,\tilde{\tau}_2,E\right)$ is a (1,2)*-soft b-submaximal space.

## 4. Conclusion

I would like to note that this paper convinces us to consider the future research directions, for example, to consider the more general cases of soft sets in multi topological structures; one may see [1, 2, 4] for more inspiration.

**Acknowledgement**: Known concepts are mentioned in the text along with appropriate references.

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## A Study On Some New Soft Sets In Soft Bitopological Spaces

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**Abstract**

This disquisition, which consists of interrelating sections is devoted to the definition and study of the concepts of (1,2)*- soft b- open sets, (1,2)*- soft b –closed sets, (1,2)*- soft regular - open, (1,2)*- soft preopen, (1,2)*-soft semi open, (1,2)*- soft $\alpha$ - open, (1,2)*- soft $\beta$ - open in soft bitopological spaces and exhibit the properties of them. Following these definitions of soft set properties are made and the relations between these soft sets are presented by theorems and remarks.

**Keywords:** (1,2)*- softb- open set, (1,2)*- soft b –closed set, (1,2)*-soft regular- open, (1,2)*- soft pre open, (1,2)*-soft semi open, (1,2)*- soft $\alpha$ - open, (1,2)*-soft $\beta$ -open, (1,2)*- soft b-interior, (1,2)*- soft b- closure.

**Mathematics Subject Classification:** 54E55, 54C08, 54C19.

## 1. Introduction

Soft set theory [1] was firstly introduced by Molodtsov in 1999 named as a general mathematical tool for dealing with uncertainty. In the near future, on set theories dealing with uncertainities, the topological structures has been studied increasingly: fuzzy topology [14], topological spaces by generalizing rough set theory [15] and soft topology [16], respectively. Soft topology, the main theme of this text, is introduced over an initial universe with a fixed set of parameters by Shabir and Naz in 2011. Cağman et al. [6] introduced a topology on a soft set called "soft topology" and presented the foundations of the theory of soft topological spaces. Moreover, many authors studied soft topology and its applications that are given in the references.

In 1963, Kelly [7] was defined bitopological space as an original and fundamental work by using two different topologies on a set. The notion of bitopological space (X, $\delta_1$ , $\delta_2$) which is a nonempty set X endowed with two topologies $\delta_1$ and $\delta_2$ is introduced in this Pioneer work. Also in [7], some basic results of separation axioms in topological spaces are extended to bitopological spaces. The notion of semi-open sets in bitopological spaces was initated by Ravi and Thivagar [17] in 2004. The set theory on bitopological spaces was studied by many authors: The concept of $\alpha$-closed sets, semi-closed sets, g-closed sets and sg-closed sets were some of them. Based on Çağman et al.[6]'s soft topology, Şenel and Çağman [18] define a bitopology on a soft set, called "soft bitopology". In this work, its related properties are proved and some relations between soft topology and soft bitopology are obtained. In this disquisition, which consists of interrelating sections is devoted to the definition and study of the concepts of (1,2)*-soft b- open sets, (1,2)*- soft b –closed sets, (1,2)*- soft regular - open, (1,2)*- soft preopen, (1,2)*-soft semi open, (1,2)*- soft $\alpha$- open, (1,2)*- soft $\beta$- open in soft bitopological spaces and exhibit the properties of them. Following these definitions of soft set properties are made and the relations between these soft sets are presented by theorems and remarks.

## 2. Preliminaries

In this section, we have presented the basic definitions and results of soft set theory, soft topology, bitopological space and soft bitopological space to use in the sequel. Throughout this paper, U is an initial universe, E is a set of parameters, P(U) is the power set of U, and $A \subseteq E$.

**Definition 2.1 : [6]** (Soft Set) A soft set $F_A$ on the universe X is defined by the set of ordered pairs

$$F_A = \{( x, f_A(x)) : x \in E\}$$

where $f_A : E \to P(X)$ such that $f_A(x) = \phi$ if $x \notin A$. Here $f_A$ is called approximate function of the soft set $F_A$ . The value of $f_A(x)$ may arbitrary, some of them maybe empty, some may have non empty intersection. The set of all soft sets over X will be denoted by S(X).

**Definition 2.2: [6]** Let $F_A \tilde{\in} S(X)$. If $f_A(x) = \phi$ for all $x \in E$, then $F_A$ is called an empty set, denoted by $F_\phi$. $f_A(x) = \phi$ means that there is no element in X related to the parameter $x \in E$. Therefore, we do not display such elements in the soft sets, as it is meaningless to consider such parameters.

**Definition 2.3: [6]** Let $F_A \tilde{\in} S(X)$. If $f_A(x) = X$ for all $x \in A$, then $F_A$ is called an A-universal soft set, denoted by $F_{\tilde{A}}$. If A = E, then the A- universal soft set is called a universal soft set, denoted by $\tilde{X}$ .

**Definition 2.4: [6]** Let $F_A$ , $F_B$ $\tilde{\subseteq}$ S(X). Then $F_A$ is a soft subset of $F_B$, denoted by $F_A \tilde{\subseteq} F_B$ if $f_A(x) \subseteq f_B(x)$ for all $x \in E$. Let $F_A$ and $F_B$ are soft equal denoted by $F_A = F_B$ if $f_A(x) = f_B(x)$ for all $x \in E$.

**Definition 2.5: [6]** Let $F_A$ $\tilde{\in}$ S(X). A soft topology on $F_A$, denoted by $\tilde{\tau}$ , is a collection of soft subsets of $F_A$ having following properties: collection of soft sets over $\tilde{X}$ ,

(i) $F_\phi$ , $F_A$ belong to $\tilde{\tau}$

(ii) Union of any number of soft sets in $\tilde{\tau}$ belongs to $\tilde{\tau}$

(iii) Intersection of two soft sets in $\tilde{\tau}$ belongs to $\tilde{\tau}$

The pair ( $F_A$ , $\tilde{\tau}$) is called a soft topological space.

**Definition 2.6: [7]** Let $X \neq \emptyset$, $\tau_1$ and $\tau_2$ are two different topologies on X. Then $(X, \tau_1, \tau_2)$is called a bitopological space. Throughout this paper $(X, \tau_1, \tau_2)$ [or simply X] denote bitopological space on which no seperation axioms are assumed unless explicitly stated.

**Definition 2.7: [7]** A subset S of X is called $\tau_1\tau_2$-open if $S = H \cup K$ such that $H \in \tau_1$ and $K \in \tau_2$ and the complement of $\tau_1\tau_2$-open is $\tau_1\tau_2$-closed.

**Definition 2.8: [18]** Let $F_A$ be a nonempty soft set on the universe U, $\tilde{\tau}_1$ and $\tilde{\tau}_2$ be two different soft topologies on $F_A$. Then, $(F_A, \tilde{\tau}_1 , \tilde{\tau}_2)$ is called a soft bitopological space.

## 3. Some New Soft Sets In Soft Bitopological Spaces

In this section, the concepts of (1,2)*- soft b- open sets, (1,2)*- soft b –closed sets, (1,2)*- soft regular - open, (1,2)*- soft preopen, (1,2)*-soft semi open, (1,2)*- soft $\alpha$ - open, (1,2)*- soft $\beta$ - open in soft bitopological spaces are introduced and the properties of them are exhibited. Apart from definitions and theorems are numbered, known concepts are mentioned in the text along with the reference [19].

**Lemma 3.1: [19]** Let $\left(\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2\right)$ be a soft bitopological space and $F_A \tilde{\in} \tilde{X}$. Then,
(i) Every (1,2)*-soft preopen set is (1,2)*-soft $\beta$ -open.

(ii) Every (1,2)*-soft semi open set is (1,2)*-soft $\beta$ -open.

(iii) Every (1,2)*-soft $\alpha$ open set is (1,2)*-soft preopen.

**Proof.** (i) Let $\left(\tilde{X},\tilde{\tau}_1,\tilde{\tau}_2\right)$ be a soft bitopological space and $F_A \tilde{\in} \tilde{X}$. Suppose $F_A$ be a (1,2)*-soft preopen set. This implies $F_A \tilde{\subseteq} \tilde{\tau}_{1,2}\text{-int}\left(\tilde{\tau}_{1,2}\text{-cl}\left(F_A\right)\right) \tilde{\subseteq} \tilde{\tau}_{1,2}\text{-cl}\left(\tilde{\tau}_{1,2}\text{-int}\left(\tilde{\tau}_{1,2}\text{-cl}\left(F_A\right)\right)\right)$. Thus, (i) is proved.

(ii) Let $F_A$ be a (1,2)*-soft semi open set. This implies
$F_A \tilde{\subseteq} \tilde{\tau}_{1,2}\text{-cl}\left(\tilde{\tau}_{1,2}\text{-int}\left(F_A\right)\right) \tilde{\subseteq} \tilde{\tau}_{1,2}\text{-cl}\left(\tilde{\tau}_{1,2}\text{-int}\left(\tilde{\tau}_{1,2}\text{-cl}\left(F_A\right)\right)\right)$. Thus, (ii) proved.

(iii) Let $F_A$ be a (1,2)*-soft $\alpha$-open set. This implies
$F_A \tilde{\subseteq} \tilde{\tau}_{1,2}\text{-int}\left(\tilde{\tau}_{1,2}\text{-cl}\left(\tilde{\tau}_{1,2}\text{-int}\left(F_A\right)\right)\right) \tilde{\subseteq} \tilde{\tau}_{1,2}\text{-int}\left(\tilde{\tau}_{1,2}\text{-cl}\left(F_A\right)\right)$. Thus, (iii) proved.

**Remark 3.2:** The converse of the above lemma is need not be true as seen in the following example:

**Example 3.3: [19]** Let $\left(\tilde{X},\tilde{\tau}_1,\tilde{\tau}_2\right)$ be a soft bitopological space, where
$\tilde{\tau}_1=\left\{\tilde{X},F_\phi,F_{E_4},F_{E_{10}}\right\}$, $\tilde{\tau}_2=\left\{\tilde{X},F_\phi,F_{E_1},F_{E_7},F_{E_{13}}\right\}$. Then $\tilde{\tau}_{1,2}$ - soft open set are
$\left\{\tilde{X},F_\phi,F_{E_1},F_{E_4},F_{E_7},F_{E_{10}},F_{E_{13}}\right\}$ and $\tilde{\tau}_{1,2}$ - soft closed set are $\left\{\tilde{X},F_\phi,F_{E_{12}},F_{E_{14}},F_{E_{11}},F_{E_8},F_{E_5}\right\}$

$F_{E_9}$ is a (1,2)*-soft $\alpha$ open but not (1,2)*-soft open. $F_{E_6}$ is a (1,2)*-soft semi open but not (1,2)*-soft $\alpha$ open and $F_{E_6}$ is a (1,2)*-soft $\beta$ open but not (1,2)*-soft preopen.

The proof of (iii) is interminable. Fort his proof firstly we get a soft bitopological space:

Let $X=\{x_1,x_2,x_3\}, E=\{e_1,e_2,e_3\}$ and
$\tilde{X}=\left\{\left(e_1,\{x_1,x_2,x_3,x_4\}\right),\left(e_2,\{x_1,x_2,x_3,x_4\}\right),\left(e_3,\{x_1,x_2,x_3,x_4\}\right)\right\}$.
Then $\tilde{\tau}_1=\left\{\tilde{X},F_\phi,F_{E_1},F_{E_2},F_{E_3},F_{E_4},F_{E_5},F_{E_6},F_{E_7},F_{E_8},F_{E_9},F_{E_{10}},F_{E_{11}},F_{E_{12}},F_{E_{13}},F_{E_{14}},F_{E_{15}}\right\}$
$\tilde{\tau}_2=\left\{\tilde{X},F_\phi\right\}$. Where $F_{E_1}=\left\{\left(e_1,\{x_1\}\right),\left(e_2,\{x_2,x_3\}\right),\left(e_3,\{x_1,x_4\}\right)\right\}$.
$F_{E_2}=\left\{\left(e_1,\{x_2,x_4\}\right),\left(e_2,\{x_1,x_3,x_4\}\right),\left(e_3,\{x_1,x_2,x_4\}\right)\right\}$.
$F_{E_2}=\left\{\left(e_1,\{x_2,x_4\}\right),\left(e_2,\{x_1,x_3,x_4\}\right),\left(e_3,\{x_1,x_2,x_4\}\right)\right\}$. $F_{E_3}=\left\{\left(e_2,\{x_3\}\right),\left(e_3,\{x_1\}\right)\right\}$.
$F_{E_4}=\left\{\left(e_1,\{x_1,x_2,x_4\}\right),\left(e_2,X\right),\left(e_3,X\right)\right\}$. $F_{E_5}=\left\{\left(e_1,\{x_1,x_3\}\right),\left(e_2,\{x_2,x_4\}\right),\left(e_3,X\right)\right\}$.
$F_{E_6}=\left\{\left(e_1,\{x_1\}\right),\left(e_2,\{x_2\}\right)\right\}$. $F_{E_7}=\left\{\left(e_1,\{x_1,x_3\}\right),\left(e_2,\{x_2,x_3,x_4\}\right),\left(e_3,\{x_1,x_2,x_4\}\right)\right\}$.

$F_{E_8}=\{(e_2,\{x_4\}),(e_3,\{x_2\})\}$. $F_{E_9}=\{(e_1,X),(e_2,X),(e_3,\{x_1,x_2,x_3\})\}$.
$F_{E_{10}}=\{(e_1,\{x_1,x_3\}),(e_2,\{x_2,x_3,x_4\}),(e_3,\{x_1,x_2\})\}$. $F_{E_{11}}=\{(e_1,\{x_2,x_3,x_4\}),(e_2,X),(e_3,\{x_1,x_2,x_3\})\}$.
$F_{E_{12}}=\{(e_1,\{x_1\}),(e_2,\{x_2,x_3,x_4\}),(e_3,\{x_1,x_2,x_4\})\}$. $F_{E_{13}}=\{(e_1,\{x_1\}),(e_2,\{x_2,x_4\}),(e_3,\{x_2\})\}$.
$F_{E_{14}}=\{(e_1,\{x_3,x_4\}),(e_2,\{x_1,x_2\})\}$. $F_{E_{15}}=\{(e_1,\{x_1\}),(e_2,\{x_2,x_3\}),(e_3,\{x_1\})\}$. Then $(\tilde{X},\tilde{\tau}_1,\tilde{\tau}_2)$ is a soft bitopological space.

(iii) Consider $F_E$, the soft subset of $\tilde{X}$. Where
$F_E=\{(e_1,\{x_4\}),(e_2,\{x_1,x_2,x_3\}),(e_3,\{x_2,x_4\})\}$. $\tilde{\tau}_{1,2}\text{-int}(\tilde{\tau}_{1,2}\text{-cl}(F_E))=\tilde{X}$ and $F_E\tilde{\subseteq}\tilde{X}$. But
$\tilde{\tau}_{1,2}\text{-int}(\tilde{\tau}_{1,2}\text{-cl}(\tilde{\tau}_{1,2}\text{-int}(F_E)))=F_\phi$. Hence $F_E$ is (1,2)*-soft preopen set but not (1,2)*-soft $\alpha$ open.

**Definition 3.4: [19]** Let $(\tilde{X},\tilde{\tau}_1,\tilde{\tau}_2)$ be a soft bitopological space and $F_A\tilde{\subseteq}\tilde{X}$. Then, $F_A$ is called **(1,2)*-soft b-open set** (briefly (1,2)*-sb-open) if $F_A\tilde{\subseteq}\tilde{\tau}_{1,2}\text{-int}(\tilde{\tau}_{1,2}\text{-cl}(F_A))\tilde{\cup}\tilde{\tau}_{1,2}\text{-cl}(\tilde{\tau}_{1,2}\text{-int}(F_A))$.

**Theorem 3.5: [19]** Let $(\tilde{X},\tilde{\tau}_1,\tilde{\tau}_2)$ be a soft bitopological space. Then

(i) Every (1,2)*-soft preopen set is (1,2)*-soft b-open set.

(ii) Every (1,2)*-soft b-open set is (1,2)*-soft $\beta$-open set.

(iii) Every (1,2)*-soft semi open set is (1,2)*-soft b-open set.

**Proof.** Let $(\tilde{X},\tilde{\tau}_1,\tilde{\tau}_2)$ be a soft bitopological space and $F_A\in\tilde{X}$. Let $F_A$ is a (1,2)*-soft preopen set. Then $F_A\tilde{\subseteq}\tilde{\tau}_{1,2}\text{-int}(\tilde{\tau}_{1,2}\text{-cl}(F_A))\tilde{\subseteq}\tilde{\tau}_{1,2}\text{-int}(\tilde{\tau}_{1,2}\text{-cl}(F_A))\tilde{\cup}\tilde{\tau}_{1,2}\text{-int}(F_A)$
$\tilde{\subseteq}\tilde{\tau}_{1,2}\text{-int}(\tilde{\tau}_{1,2}\text{-cl}(F_A))\tilde{\cup}\tilde{\tau}_{1,2}\text{-cl}(\tilde{\tau}_{1,2}\text{-int}(F_A))$. Thus (i) proved.

Let $F_A$ be a (1,2)*-soft b-open set. Then $F_A\tilde{\subseteq}\tilde{\tau}_{1,2}\text{-cl}(\tilde{\tau}_{1,2}\text{-int}(F_A))\tilde{\cup}\tilde{\tau}_{1,2}\text{-int}(\tilde{\tau}_{1,2}\text{-cl}(F_A))$
$\tilde{\subseteq}\tilde{\tau}_{1,2}\text{-cl}(\tilde{\tau}_{1,2}\text{-int}(\tilde{\tau}_{1,2}\text{-cl}(F_A)))\tilde{\cup}\tilde{\tau}_{1,2}\text{-int}(\tilde{\tau}_{1,2}\text{-cl}(F_A))\tilde{\subseteq}\tilde{\tau}_{1,2}\text{-cl}(\tilde{\tau}_{1,2}\text{-int}(\tilde{\tau}_{1,2}\text{-cl}(F_A)))$. Thus (ii) proved.

Let $F_A$ is a (1,2)*-soft semi open set. This implies
$F_A\tilde{\subseteq}\tilde{\tau}_{1,2}\text{-cl}(\tilde{\tau}_{1,2}\text{-int}(F_A))$
$\tilde{\subseteq}\tilde{\tau}_{1,2}\text{-cl}(\tilde{\tau}_{1,2}\text{-int}(F_A))\tilde{\cup}\tilde{\tau}_{1,2}\text{-int}(F_A)\tilde{\subseteq}\tilde{\tau}_{1,2}\text{-cl}(\tilde{\tau}_{1,2}\text{-int}(F_A))\tilde{\cup}\tilde{\tau}_{1,2}\text{-int}(\tilde{\tau}_{1,2}\text{-cl}(F_A))$. Thus (iii) proved.

**Remark 3.6: [19]** The converse of the above lemma is need not be true as seen in the following example.

**Example 3.7: [19]** Let us consider the soft subsets of $\tilde{X}$ that are given in Example 3.3. Let $\left(\tilde{X}, \tilde{\tau}_1, \tilde{\tau}_2\right)$ be a soft bitopological space, where $\tilde{\tau}_1 = \left\{\tilde{X}, F_\phi, F_{E_1}\right\}$, $\tilde{\tau}_2 = \left\{\tilde{X}, F_\phi, F_{E_2}\right\}$. Then $\tilde{\tau}_{1,2}$ - soft open set are $\left\{\tilde{X}, F_\phi, F_{E_1}, F_{E_2}, F_{E_3}\right\}$, $\tilde{\tau}_{1,2}$ - soft closed set are $\left\{\tilde{X}, F_\phi, F_{E_{12}}, F_{E_9}, F_{E_6}\right\}$.

(i) The soft set $F_{E_7}$ in $\tilde{X}$ is (1,2)*-soft b-open set but not (1,2)*-soft preopen set.

(ii) The soft set $F_{E_5}$ in $\tilde{X}$ is (1,2)*-soft $\beta$ -open set but not (1,2)*-soft b-open set.

(iii) The soft set $F_{E_4}$ in $\tilde{X}$ is (1,2)*-soft b-open set but not (1,2)*-soft semi open set.

**Remark 3.8: [19]** The above discussions are summarized in the following diagrams:

$$\begin{tikzcd}
\text{(1,2)*-soft regular open} \arrow[r] & \text{(1,2)*-soft open} \arrow[r] & \text{(1,2)*-soft } \alpha\text{-open} \arrow[dl] \arrow[dr] & \\
& \text{(1,2)*-soft semi open} \arrow[dr] & & \text{(1,2)*-soft pre open} \arrow[dl] \\
& & \text{(1,2)*-soft b-open} \arrow[d] & \\
& & \text{(1,2)*-soft } \beta\text{-open} &
\end{tikzcd}$$

**Acknowledgements:** The author gratefully acknowledge the comments of the chairman which led to the improvement of this paper.

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## Boole Polynomail Solutions of Linear Volterra Integro-Differential Equations

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**Abstract**

This study aims to develop a numerical method which is used for the approximate solution of the Volterra integro-differantial equation under mixed conditions. For this purpose, the problem is reduced from the linear algebraic equation system to the matrix equation. This system is solved by using Boole polynomials, their derivatives and collocation points. From this solution, the Boole coefficients are obtained for the approximate solution. Numerical examples are given for to demonstrate the validity and applicability of the technique. Also, the results are compared with the graphs and tables.

**Keywords: Boole polynomials, collocation points, approximate solutions, numerical, integro-differential equation.**

## 1. Introduction

Some problems encountered in real life are issues of science. These issues are frequently encountered in applied areas such as physics, biology, chemistry, mathematics, engineering, electrostatic. Integro-differential equations have a large role in these applied areas. Therefore, numerical methods such as the Bernoulli matrix-collocation method [1], the meshless method [2], Bessel matrix method [3], operational Tau method [4], Legendre collocation method [5], homotopy perturbation method [6], Spectral collocation method [7], the variational Adomian decomposition method [8], modified Taylor expansion method [9], Tau method [10], Taylor matrix method [11] and improved Legendre method [12] have been used for the solution of integro-differential equations [14]-[18].

The aim of this study is to develop a method using Boole polynomials, derivatives and collocation points to find approximate solution of linear Volterra integro-differential equation

$$\sum_{k=0}^{m} P_k(x)y^{(k)}(x) = g(x) + \lambda \int_a^x K(x,t)y(t)dt, \quad \mathrm{a} \le \mathrm{x,t} \le \mathrm{b} \tag{1}$$

under mixed conditions

$$\sum_{k=0}^{m-1} \left(a_{jk}y^k(a) + b_{jk}y^k(b)\right) = \lambda_j; \quad j = 0,1,2,3,\dots m-1 \tag{2}$$

where $y^{(k)}(x)$ is an unknown function, the known function $P_k(x), \mathrm{g}(\mathrm{x})$ and the kernel function $K(x,t)$ are defined in the interval $\mathrm{a} \leq \mathrm{x,t} \leq \mathrm{b}$. $a_{jk}$, $b_{jk}$ and $\lambda_j$ fixed numbers. The approximate solution of the Eq. (1) is obtained in the truncated Boole series form

$$y(x) \cong y_N(x) = \sum_{n=0}^{N} a_n R_n(x) \tag{3}$$

where $R_n(x)$, $n = 0,1,2,\dots,N$ is the Boole polynomial and $a_n$, $n = 0,1,2,\dots,N$ is the unknown coefficients of the Boole polynomials. Charles Jordan has defined the general equation of Boole polynomials as follows [13].

$$R_n(x) = \sum_{m=0}^{n+1} \frac{(-1)^m}{2^m} \binom{x}{n-m} \tag{4}$$

The defined form of the Boole polynomial is

$$\sum_{n=0}^{\infty} \frac{R_n(x)}{n!} t^n = \frac{2(1+t)^x}{2+t}. \tag{5}$$

**2. Fundamental Matrix Relation**

Primarily, the matrix relation for Boole polynomial is as follows

$$\mathbf{R}(\mathrm{x}) = \mathbf{X}(x)\mathbf{H}^{\mathbf{T}} \tag{6}$$

where $\mathbf{H}^{\mathbf{T}}$ is coefficient matrix and also

$$\mathbf{R}(x) = \begin{bmatrix} 1 & x - \frac{1}{2} & x^2 - 2x + \frac{1}{2} & x^3 - \frac{9}{2}x^2 + 5x - \frac{3}{4} & \dots & R_N(x) \end{bmatrix},$$

$$\mathbf{X}(\mathrm{x}) = [1 \quad x \quad x^2 \quad x^3 \quad \dots \quad x^N],$$

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & \dots \\ -\frac{1}{2} & 1 & 0 & \dots \\ \frac{1}{2} & -2 & 1 & \dots \\ \vdots & \vdots & \vdots & \dots \\ \vdots & \vdots & \vdots & \dots \end{bmatrix} \text{ and } \mathbf{H}^{\mathbf{T}} = \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} & \dots \\ 0 & 1 & -2 & \dots \\ 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \dots \\ \vdots & \vdots & \vdots & \dots \end{bmatrix}.$$

The Eq. (1) is shown as follows

$$L(x) = g(x) + \lambda I(x) \tag{7}$$

where

$$L(x) = \sum_{k=0}^{m} P_k(x)\, y^{(k)}(x) \text{ and } I(x) = \int_a^x K(x,t)y(t)dt.$$

### 2.1. Matrix relation for the differential part L(x)

The approximate solution of the Eq. (1) in truncated Boole series (3) is written in matrix form

$$y(x) = \mathbf{R}(x)\mathbf{A} \tag{8}$$

and $k$th derivative of the solution $y(x)$ is

$$y^{(k)}(x) = \mathbf{R}^{(k)}(x)\mathbf{A}. \tag{9}$$

According to the matrix relation (6) this matrix form is written as

$$y^{(k)}(x) = \mathbf{X}^{(k)}(x)\mathbf{H}^{\mathbf{T}}\mathbf{A} \tag{10}$$

where $\mathbf{X}^{(k)}(x)$ is defined as

$$\mathbf{X}^{(k)}(x) = \mathbf{X}(x)\mathbf{E}^{k}.$$

From this form the following matrix relation is obtained.

$$y^{(k)}(x) = \mathbf{X}(x)\mathbf{E}^{k}\mathbf{H}^{\mathbf{T}}\mathbf{A} \tag{11}$$

where matrix $\mathbf{E}$ is derivatives transition matrix of Taylor polynomial,

$$\mathbf{E} = \begin{bmatrix} 0 & 1 & 0 & 0 & ... & 0 \\ 0 & 0 & 2 & 0 & ... & 0 \\ . & . & . & . & ... & . \\ . & . & . & . & ... & . \\ . & . & . & . & ... & . \\ 0 & 0 & 0 & 0 & ... & k \\ 0 & 0 & 0 & 0 & ... & 0 \end{bmatrix} \text{ and } \mathbf{A} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ . \\ . \\ . \\ a_N \end{bmatrix}.$$

According to the matrix form (6) the matrix relation is written as

$$y^{(k)}(x) = \mathbf{R}(x)(\mathbf{H}^{\mathbf{T}})^{-1}\mathbf{E}^{k}\mathbf{H}^{\mathbf{T}}\mathbf{A} \tag{12}$$

or

$$y^{(k)}(x) = \mathbf{R}(x)\mathbf{D}^{k}\mathbf{A} \tag{13}$$

where

$$\mathbf{D}^k = (\mathbf{H}^\mathbf{T})^{-1}\mathbf{E}^k\mathbf{H}^\mathbf{T}.$$

The matrix $\mathbf{D}$ is derivatives transition matrix of Boole polynomial. That time, the general matrix form of the differential part is written the following.

$$L(x) = \sum_{k=0}^{m} P_k(x)\mathbf{R}(x)\mathbf{D}^k\mathbf{A} \tag{14}$$

### 2.2. Matrix relation for the integral part I(x)

According to the Eq. (8) the integral part of Eq. (1) is written as

$$I(x) = \int_a^x K(x,t)\mathbf{R}(t)\mathbf{A}dt. \tag{15}$$

The matrix form of the kernel function is defined as follows for the Taylor polynomial and Boole polynomial, respectively

$$K(x,t) = \mathbf{X}(x)^T\mathbf{K}_{rs}\mathbf{X}^T(t) \text{ and } K(x,t) = \mathbf{R}(x)^R\mathbf{K}_{rs}\mathbf{R}^T(t). \tag{16}$$

From the matrix form, the following matrix relation is obtained.

$${}^R\mathbf{K}_{rs} = \ (\mathbf{H}^\mathbf{T})^{-1}\,{}^T\mathbf{K}_{rs}\mathbf{H}^{-1} \tag{17}$$

where

$${}^TK_{rs}(x,t) = \sum_{r=0}^{N}\sum_{s=0}^{N} {}^Tk_{mn}^{rs}\, x^m t^n\,, {}^RK_{rs}(x,t) = \sum_{r=0}^{N}\sum_{s=0}^{N} {}^Rk_{mn}^{rs}\,\mathrm{R_m(x)R_n(t)},$$

$${}^Tk_{mn}^{rs} = \frac{1}{m!\,n!}\frac{\partial^{m+n}k_{rs}(0,0)}{\partial x^m \partial t^n} \quad m,n = 0,1,2,\dots,N$$

and

$${}^T\mathbf{K}_{rs} = \begin{bmatrix} k_{00} & k_{01} & k_{02} & \cdots & k_{0n} \\ k_{10} & k_{11} & k_{12} & \cdots & k_{1n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k_{m0} & k_{m1} & k_{m2} & \cdots & k_{mn} \end{bmatrix}.$$

By substituting the kernel function for Boole polynomial into Eq. (15), the matrix relation is obtained as

$$\mathrm{I}(x) = \int_a^x \mathbf{R}(x)^R\mathbf{K}_{rs}\mathbf{R}^T(t)\mathbf{R}(t)\mathbf{A}dt = \mathbf{R}(x)^R\mathbf{K}_{rs}\mathbf{Q}\mathbf{A} \tag{18}$$

where

$$\mathbf{Q}(\mathrm{x}) = \int_a^x \mathbf{R}^T(t)\mathbf{R}(t)dt.$$

If the matrix relation (6) is written in the this equation, the following equation is obtained.

$$\mathbf{Q}(\mathrm{x}) = \int_a^x \mathbf{H}\mathbf{X}^T(t)\mathbf{X}(t)\mathbf{H}^T dt = \mathbf{H}\mathbf{C}(x)\mathbf{H}^T \tag{19}$$

where

$$\mathbf{C}(\mathrm{x}) = \int_a^x \mathbf{X}^T(t)\mathbf{X}(t)dt = [c_{i,j}(x)], \quad c_{i,j}(x) = \frac{x^{i+j+1} - a^{i+j+1}}{i+j+1}, i,j = 0,1,2,\dots,N.$$

If the matrix form (6), matrix relation (17) and Eq. (19) are placed in the expression (18), the following matrix relation is obtained.

$$\mathbf{I}(x) = \mathbf{X}(x)^T\mathbf{K}_{rs}\boldsymbol{C}(x)\mathbf{H}^T\boldsymbol{A} \tag{20}$$

### 2.3. Matrix relation for the conditions

The corresponding matrix form of conditions (2) is obtained by Eq. (13) as

$$\sum_{k=0}^{m-1}(a_{jk}\mathbf{R}(a) + b_{jk}\,\mathbf{R}(b))\mathbf{D}^k\mathbf{A} = \lambda_k; \;\; j = 0,1,2,\dots,m-1. \tag{21}$$

### 2.4. Collocation method

In the previous sections, the matrix relation (14) and (20) obtained for the approximate solution of the Eq. (1) are placed in the Eq. (7) and obtained the matrix equation

$$\sum_{k=0}^{m} P_k(x)\,\mathbf{R}(x)\mathbf{D}^k\mathbf{A} = g(x) + \lambda\,\mathbf{X}(x)^T\mathbf{K}_{rs}\boldsymbol{C}(x)\mathbf{H}^T\boldsymbol{A}. \tag{22}$$

The collocation points $x_i$ is defined by

$$x_i = a + \frac{b-a}{N}i, \qquad i = 0,1,\dots,N. \tag{23}$$

This collocation points are applied in the matrix relation (22) and obtained the matrix relation

$$\sum_{k=0}^{m} P_k(x_i)\mathbf{R}(x_i)\mathbf{D}^k\mathbf{A} = g(x_i) + \lambda\,\mathbf{X}(x_i)^T\mathbf{K}_{rs}\boldsymbol{C}(x_i)\mathbf{H}^T\boldsymbol{A} \tag{24}$$

or briefly shown as

$$\left\{\sum_{k=0}^{m} \mathbf{P}_k \mathbf{R} D^k - \lambda\, \overline{\mathbf{X}}\overline{\mathbf{K}}\overline{\mathbf{C}}\overline{\mathbf{H}^T}\right\}\mathbf{A} = \mathbf{G} \tag{25}$$

where

$$\mathbf{P}_k(x_i) = \begin{bmatrix} \mathbf{P}_k(x_0) & 0 & \cdots & 0 \\ 0 & \mathbf{P}_k(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{P}_k(x_N) \end{bmatrix}_{(N+1)x(N+1)}, \mathbf{G} = \begin{bmatrix} g(x_0) \\ g(x_1) \\ \vdots \\ g(x_N) \end{bmatrix}_{(N+1)x1}, \mathbf{R} = \begin{bmatrix} \mathbf{R}(x_0) \\ \mathbf{R}(x_1) \\ \vdots \\ \mathbf{R}(x_N) \end{bmatrix}_{(N+1)x(N+1)},$$

$$\overline{\mathbf{X}} = \begin{bmatrix} \mathbf{X}(x_0) & 0 & \cdots & 0 \\ 0 & \mathbf{X}(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{X}(x_N) \end{bmatrix}_{(N+1)x(N+1)^2}, \overline{\mathbf{K}} = \begin{bmatrix} \mathbf{K} & 0 & \cdots & 0 \\ 0 & \mathbf{K} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{K} \end{bmatrix}_{(N+1)^2x(N+1)^2},$$

$$\overline{\mathbf{C}} = \begin{bmatrix} \mathbf{C}(x_0) & 0 & \cdots & 0 \\ 0 & \mathbf{C}(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{C}(x_N) \end{bmatrix}_{(N+1)^2x(N+1)^2} \text{ and } \overline{\mathbf{H}^T} = \begin{bmatrix} \mathbf{H}^T \\ \mathbf{H}^T \\ \vdots \\ \mathbf{H}^T \end{bmatrix}_{(N+1)^2x(N+1)}.$$

Eq. (25) is written as

$$\mathbf{WA} = \mathbf{G} \text{ or } [\mathbf{W};\mathbf{G}] \tag{26}$$

where

$$\mathbf{W} = \sum_{k=0}^{m} \mathbf{P}_k \mathbf{R} D^k - \lambda\, \overline{\mathbf{X}}\overline{\mathbf{K}}\overline{\mathbf{C}}\overline{\mathbf{H}^T}.$$

Of all these operations, a system of $(N+1)$ linear algebraic equation is obtained with unknown Boole coefficients $a_0, a_1, \dots, a_N$ as follows

$$[\mathbf{W};\mathbf{G}] = \begin{bmatrix} w_{00} & w_{01} & w_{02} & \cdots & w_{0N} & ; & g(x_0) \\ w_{10} & w_{11} & w_{12} & \cdots & w_{1N} & ; & g(x_1) \\ \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \vdots \\ w_{N0} & w_{N1} & w_{N2} & \cdots & w_{NN} & ; & g(x_N) \end{bmatrix}.$$

The matrix relation of conditions (21) are written as

$$\mathbf{U}_j\mathbf{A} = [\lambda_i] \text{ or } [U_i;\, \lambda_i]; \quad \mathrm{j} = 0,1,2,\dots,\mathrm{m}-1 \tag{27}$$

where

$$\mathbf{U}_j = \sum_{k=0}^{m-1} (a_{jk}\mathbf{R}(a) + b_{jk}\,\mathbf{R}(b))\mathbf{D}^k = [u_{j0} \quad u_{j1} \quad u_{j2} \quad \cdots \quad u_{jN}], \quad j = 0,1,2,\dots,m-1.$$

Finally, to reach the approximate solution of the Eq. (1) under mixed conditions, the $m$ row matrices are deleted and the row matrices (27) are added. Thus the augmented matrix system is gained as

$$[\widetilde{\mathbf{W}};\widetilde{\mathbf{G}}] = \begin{bmatrix} w_{00} & w_{01} & w_{02} & \cdots & w_{0N} & ; & g(x_0) \\ w_{10} & w_{11} & w_{12} & \cdots & w_{1N} & ; & g(x_1) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ w_{(N-m)0} & w_{(N-m)1} & w_{(N-m)2} & \cdots & w_{(N-m)N} & ; & g(x_{N-m}) \\ u_{00} & u_{01} & u_{02} & \cdots & u_{0N} & ; & \lambda_0 \\ u_{10} & u_{11} & u_{12} & \cdots & u_{1N} & ; & \lambda_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ u_{(m-1)0} & u_{(m-1)1} & u_{(m-1)2} & \cdots & u_{(m-1)N} & ; & \lambda_{m-1} \end{bmatrix}. \tag{28}$$

If $rank\widetilde{\mathbf{W}} = rank[\widetilde{\mathbf{W}};\widetilde{\mathbf{G}}] = N+1$ is, the augmented matrix is obtained as

$$\mathbf{A} = (\widetilde{\mathbf{W}})^{-1}\widetilde{\mathbf{G}}. \tag{29}$$

As a result, unknown Boole coefficients $a_0, a_1, \dots, a_N$ are found and placed in the Eq. (3), $y_N(x)$ is obtained. Since $y_N(x)$ is a solution of the Eq. (1), $y_N(x)$ and its derivatives are provide the Eq. (1) approximately. So the absolute error function can be obtained as follows

$$e(x_q) = |y_N(x_q) - y(x_q)| \cong 0 \tag{30}$$

for $\forall x = x_q \in [a,b]$.

## 3. Numerical examples

***Example 1.*** Firstly, linear Volterra integro-differential equation given as

$$y''(x) - y(x) = \frac{x^5}{4} + \frac{x^3}{2} - x^2 + 1 - \int_0^x xty(t)dt, \quad 0 \le x,t \le 2 \tag{31}$$

with the boundary conditions $y(0) = 1, y'(0) = 0$ and the approximate solution by the truncated Boole series

$$y_4(x) = \sum_{n=0}^{4} a_n R_n(x)$$

where $m = 2, N = 4, P_0(x) = 1, P_1(x) = -1, g(x) = \frac{x^5}{4} + \frac{x^3}{2} - x^2 + 1, \lambda = -1$ and $K(x,t) = xt$. The collocation points (23) for $N = 4$ are calculated as

$$\left\{x_0 = 0, x_1 = \frac{1}{2}, x_2 = 1, x_3 = \frac{3}{2}, x_4 = 2\right\}$$

and the fundamental matrix equation of the problem is

$$\left[\mathbf{P}_0\mathbf{R}\mathbf{D}^2 + \mathbf{P}_1\mathbf{R} - \lambda\overline{\mathbf{X}}\,\overline{\mathbf{K}}\,\overline{\mathbf{C}}\,\overline{\mathbf{H}^{\mathbf{T}}}\right]\mathbf{A} = \mathbf{G}$$

where

$$\mathbf{R} = \begin{bmatrix} \mathbf{R}(0) \\ \mathbf{R}\left(\frac{1}{2}\right) \\ \mathbf{R}(1) \\ \mathbf{R}\left(\frac{3}{2}\right) \\ \mathbf{R}(2) \end{bmatrix}_{5x5} = \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} & -\frac{3}{4} & \frac{3}{2} \\ 1 & 0 & -\frac{1}{4} & \frac{3}{4} & -\frac{39}{16} \\ 1 & \frac{1}{2} & -\frac{1}{2} & \frac{3}{4} & -\frac{3}{2} \\ 1 & 1 & -\frac{1}{4} & 0 & \frac{9}{16} \\ 1 & \frac{3}{2} & \frac{1}{2} & -\frac{3}{4} & \frac{3}{2} \end{bmatrix}_{5x5}, \mathbf{G} = \begin{bmatrix} g(0) \\ g\left(\frac{1}{2}\right) \\ g(1) \\ g\left(\frac{3}{2}\right) \\ g(2) \end{bmatrix}_{5x1} = \begin{bmatrix} 1 \\ \frac{105}{128} \\ \frac{3}{4} \\ \frac{299}{128} \\ 9 \end{bmatrix}_{5x1},$$

$$\mathbf{P}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{5x5}, \mathbf{P}_1 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}_{5x5}, \mathbf{D} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{5x5},$$

$$\overline{\mathbf{X}} = \begin{bmatrix} \mathbf{X}(0) & 0 & 0 & 0 & 0 \\ 0 & \mathbf{X}\left(\frac{1}{2}\right) & 0 & 0 & 0 \\ 0 & 0 & \mathbf{X}(1) & 0 & 0 \\ 0 & 0 & 0 & \mathbf{X}\left(\frac{3}{2}\right) & 0 \\ 0 & 0 & 0 & 0 & \mathbf{X}(2) \end{bmatrix}_{5x25}, \mathbf{K} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{5x5},$$

$$\overline{\mathbf{K}} = \begin{bmatrix} \mathbf{K} & 0 & 0 & 0 & 0 \\ 0 & \mathbf{K} & 0 & 0 & 0 \\ 0 & 0 & \mathbf{K} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{K} & 0 \\ 0 & 0 & 0 & 0 & \mathbf{K} \end{bmatrix}_{25x25}, \overline{\mathbf{C}} = \begin{bmatrix} \mathbf{C}(0) & 0 & 0 & 0 & 0 \\ 0 & \mathbf{C}\left(\frac{1}{2}\right) & 0 & 0 & 0 \\ 0 & 0 & \mathbf{C}(1) & 0 & 0 \\ 0 & 0 & 0 & \mathbf{C}\left(\frac{3}{2}\right) & 0 \\ 0 & 0 & 0 & 0 & \mathbf{C}(2) \end{bmatrix}_{25x25},$$

$$\overline{\mathbf{H}^T} = \begin{bmatrix} \mathbf{H}^T \\ \mathbf{H}^T \\ \mathbf{H}^T \\ \mathbf{H}^T \\ \mathbf{H}^T \end{bmatrix}_{25x5} \quad \text{and } \mathbf{H}^T = \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} & -\frac{3}{4} & \frac{3}{2} \\ 0 & 1 & -2 & 5 & -16 \\ 0 & 0 & 1 & -\frac{9}{2} & 20 \\ 0 & 0 & 0 & 1 & -8 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

After the fundamental matrix equation is calculated, the augmented matrix is found as

$$[\mathbf{W}; \mathbf{G}] = \begin{bmatrix} -1 & \frac{1}{2} & \frac{3}{2} & -\frac{9}{4} & \frac{9}{2} & ; & 1 \\ -\frac{15}{16} & -\frac{1}{96} & \frac{863}{384} & -\frac{2783}{3840} & -\frac{857}{1280} & ; & \frac{105}{128} \\ -\frac{1}{2} & -\frac{5}{12} & \frac{7}{3} & \frac{157}{60} & -\frac{331}{60} & ; & \frac{3}{4} \\ \frac{11}{16} & -\frac{5}{32} & \frac{207}{128} & \frac{8841}{1280} & -\frac{6963}{1280} & ; & \frac{299}{128} \\ 3 & \frac{11}{6} & \frac{5}{6} & \frac{613}{60} & \frac{41}{10} & ; & 9 \end{bmatrix}.$$

From Eq. (27), the matrix form for the boundary conditions becomes

$$[\mathbf{U_0}; \lambda_0] = \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} & -\frac{3}{4} & \frac{3}{2} & ; & 1 \end{bmatrix} \text{ and } [\mathbf{U_1}; \lambda_1] = [0 \quad 1 \quad -2 \quad 5 \quad -16 \quad ; \quad 0].$$

From Eq. (28), the new augmented matrix based on the conditions is calculated as

$$[\widetilde{\mathbf{W}}; \widetilde{\mathbf{G}}] = \begin{bmatrix} -1 & \frac{1}{2} & \frac{3}{2} & -\frac{9}{4} & \frac{9}{2} & ; & 1 \\ -\frac{15}{16} & -\frac{1}{96} & \frac{863}{384} & -\frac{2783}{3840} & -\frac{857}{1280} & ; & \frac{105}{128} \\ -\frac{1}{2} & -\frac{5}{12} & \frac{7}{3} & \frac{157}{60} & -\frac{331}{60} & ; & \frac{3}{4} \\ 1 & -\frac{1}{2} & \frac{1}{2} & -\frac{3}{4} & \frac{3}{2} & ; & 1 \\ 0 & 1 & -2 & 5 & -16 & ; & 0 \end{bmatrix}.$$

Finally, this system is solved and the unknown Boole coefficients are obtained becomes

$$\boldsymbol{A} = \begin{bmatrix} \frac{3}{2} & 2 & 1 & 0 & 0 \end{bmatrix}.$$

As a result, if the Boole coefficients found are placed in the Eq. (3), the approximate solution is obtained as $y(x) = x^2 + 1$ and this is the exact solution.

***Example 2.*** [20] In this example, Volterra integro-differential equation is given as

$$y'(x) = 1 - \int_0^x y(t)dt, \quad 0 \leq x,t \leq 1 \tag{32}$$

with the boundary conditions $y(0) = 0$. The exact solution of this problem is $y(x) = sinx$. The approximate solutions of this equation for $N = 5, N = 10$ and $N = 13$ are as follows, respectively.

$$y_5(x) = 0.00762454157655x^5 + 0.000815190187764x^4 - 0.167011007897x^3 + 0.0000519273964109x^2 + x$$

$$y_{10}(x) = -0.00000011940869022x^{10} + 0.00000300738456215x^9 - 0.000000304038190892x^8 - 0.000198181763416x^7 - 0.00000114100245679x^6 + 0.00833336999246x^5 - 0.000000000740691670152x^4 - 0.166666665804x^3 - 0.000000000000456965423116x^2 + x$$

$$y_{13}(x) = -0.0000000000222162790114x^{13} + 0.000000000299675955759x^{12} - 0.00000000348549017797x^{11} + 0.000000000179590151595x^{10} + 0.00000273505863464x^9 + 0.00000000157953384959x^8 - 0.000198420906373x^7 + 0.0000000000291129107609x^6 + 0.00833333263991x^5 + 0.0000000000106199114123x^4 - 0.166666666676x^3 + 3.83771490063e - 13x^2 + x$$

The error functions of the numerical results are given in Table 1 and its graphic is shown in Fig. 1. Also in Fig. 2, the exact solution and Boole solutions are shown.

**Tablo 1.** The Comparison of Error Function $N = 5, N = 10$ and $N = 13$

| | Absolute Error Function | | |
|---|---|---|---|
| $x_i$ | $\lvert e_5(x_i)\rvert$ | $\lvert e_{10}(x_i)\rvert$ | $\lvert e_{13}(x_i)\rvert$ |
| 0 | 0.0 | 0.0 | 0.0 |
| 0.2 | 4.0240e-07 | 5.1098e-14 | 1.3878e-16 |
| 0.4 | 2.0574e-07 | 4.9738e-14 | 5.5511e-17 |
| 0.6 | 3.7576e-07 | 4.4520e-14 | 2.2204e-16 |
| 0.8 | 1.8172e-07 | 3.1197e-14 | 0 |
| 1.0 | 9.6665e-06 | 2.3728e-12 | 2.7423e-14 |

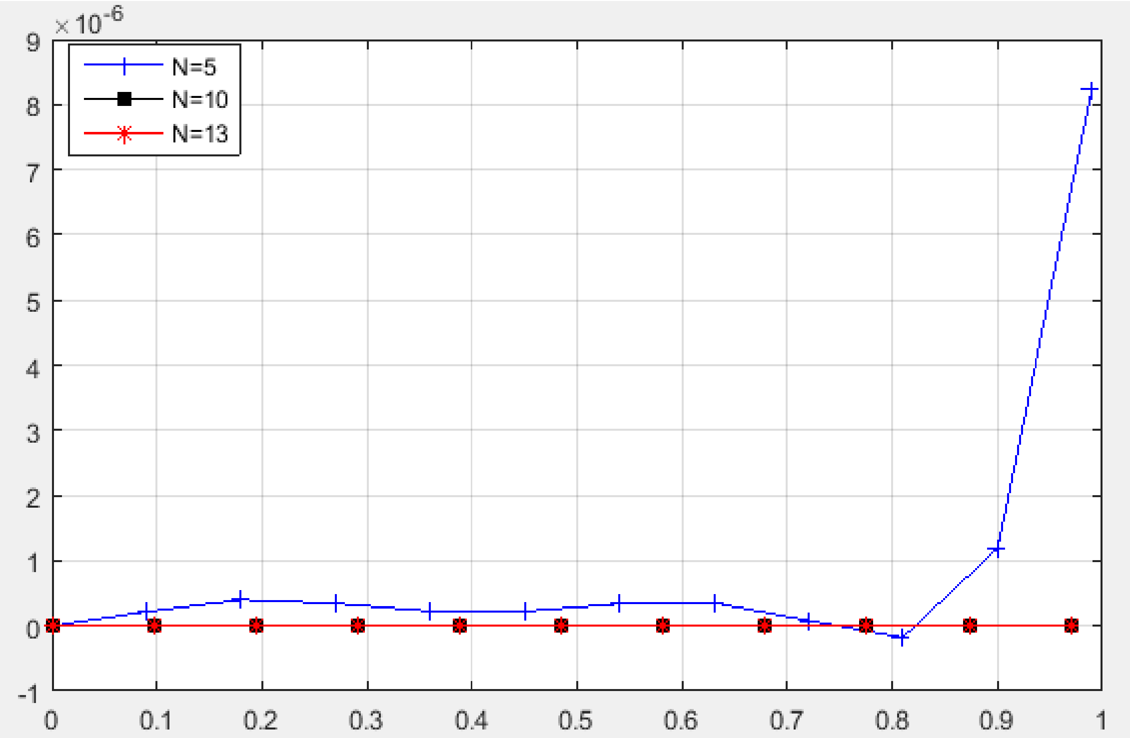


**Figure 1**. The Comparison of Error Function $N = 5, N = 10$ and $N = 13$

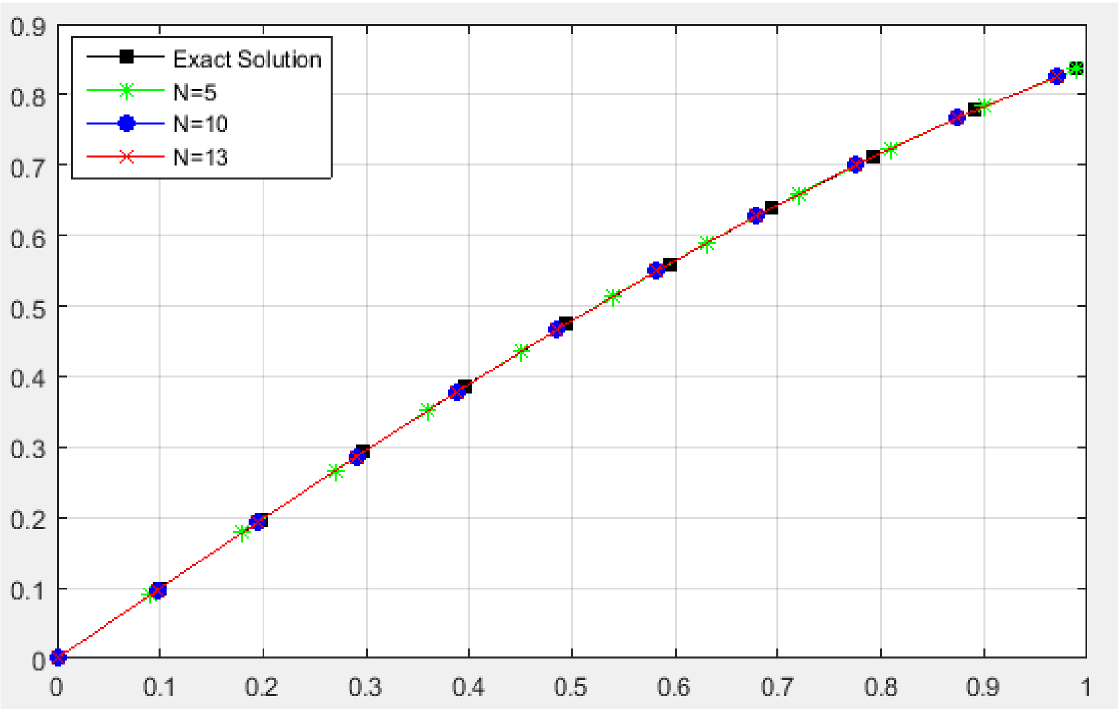


**Figure 2.** The Comparison of Exact Solution and Boole Solutions

***Example 3.*** [19] Finally, the following order linear Volterra integro-differential equation

$$y''(x) + xy'(x) - xy(x) = e^x + \frac{1}{2}xcosx - \frac{1}{2}\int_0^x cosx\, e^{-t}y(t)dt, \quad 0 \leq x, t \leq 1 \tag{33}$$

with the boundary conditions $y(0) = 1, y'(0) = 1$ and the exact solution is $y(x) = e^x$. For $N = 4, N = 7, N = 8$ and $N = 10$, the graphic of the exact solution and Boole solutions is shown in Fig. 3. In graphic, the error functions of numerical results are shown in Fig. 4.

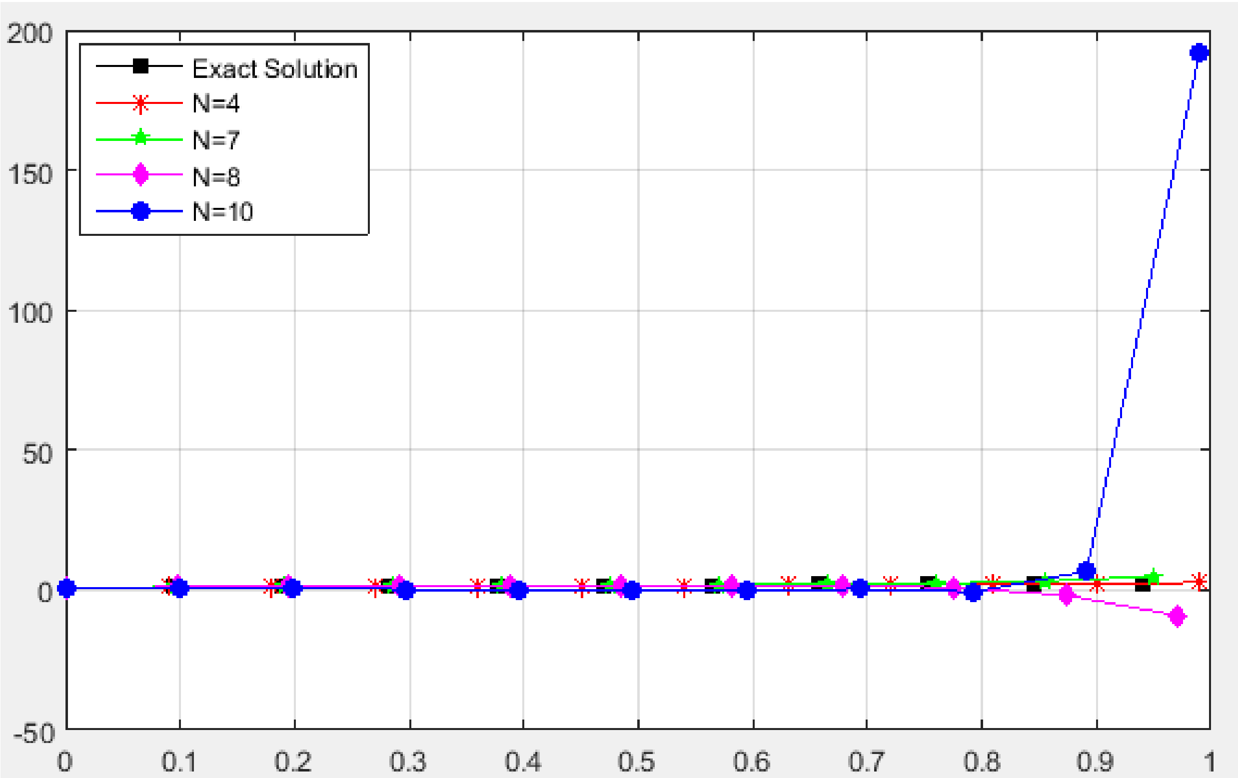


**Figure 3.** The Comparison Exact Solution and Boole Solutions

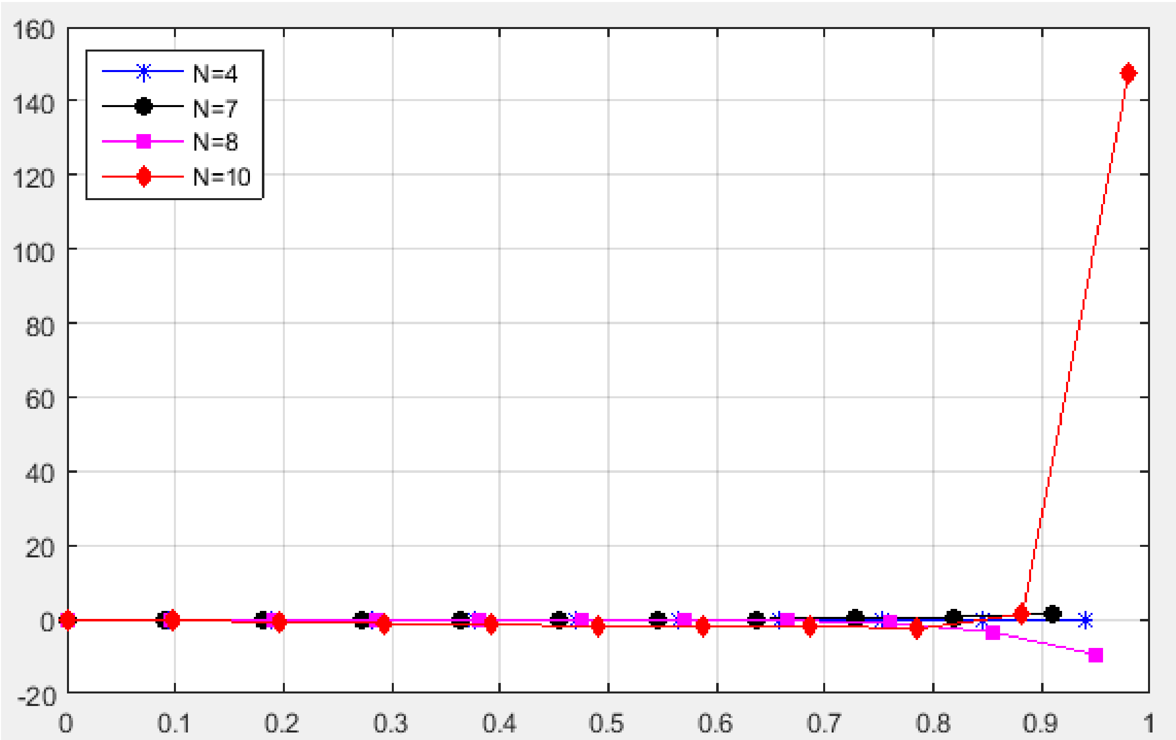


**Figure 4.** The Comparison of Error Function $N = 4, N = 7, N = 8$ and $N = 10$

## 4. Conclusion

In order to demonstrate the validity and applicability of this technique which is developed for numerical solution of Volterrra integro-differential equations, the example of the exact and the approximate solutions are given. The computer code written in MATLAB 2015 was used to calculate the exact solution, the approximate solutions and error function. The result of the calculations show this new method developed solve the problem. This method can be developed for other equation systems.

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## Estimation of conditional hazard function in the single functional index model under random censorship

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This work presents a nonparametric estimate of the conditional hazard function, when the covariate is functional and when the sample is considered as an -mixing sequence. We prove consistency properties (with rates) in various situations, including censored and uncensored variables. The pointwise almost complete convergence and the uniform almost complete convergence (with rate) of the kernel estimate of this model are established.

**Keyword(s):** Censored data, conditional hazard function, functional variable, nonparametric estimation, single functional index process, small ball probability, strong mixing processes..

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# Domain decomposition method for an image processing problem by the non linear partial differential equation

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**Abstract**

*We presented our contribution resumed in a domain decomposition method for an image processing problem by the non linear partial differential equation. As a first step, we described in details, the implementation of the domain decomposition and coarse mesh correction techniques. Then by several numerical simulations we gave useful guidelines for the choice of parameters through such quantitative studies, and demonstrate the efficiency of the implemented methods in CPU time and memory saving. Show the advantages for our non linear partial differential equation and the subdomain technique used.*

**Keywords:** domain decomposition method, image processing

**The minimization problem:** Assume $u$ is a piecewise constant function as given in [4]. The multiphases piecewise constant Tixotrop model [5] is to solve the following minimization problem:

$$\min_u F(u) = \frac{\varepsilon}{p}\int_\Omega |\nabla u|^p \, dx + g\int_\Omega |\nabla (u-u_0)| \, dx + \frac{\lambda}{2}\int_\Omega |u-u_0|^2 \, dx \quad (1)$$

Where $p$ can be adaptively selected based on the local gradient image features that is, away from edges, $p$ tend to 2 to preserve edges. Therefore this new model where $p = 2$ can effectively reduce the staircase effect in TV model whereas it can still retain the sharp edges [1].

**Domain decomposition based subspace correction method:**

We put the method in a more general setting and start with the description of the subspace correction algorithm of [4]. Given a reflexive Banach space $V$ and a convex, Gateaux differentiable functional $F : V \to \mathbb{R}$; we consider the minimization problem:

$$\min_{u\in V} F(u) \quad (2)$$

Under the notion of space correction, we first decompose the space $V$ into a sum of smaller subspaces:

$$V = V_1 + V_2 + \cdots + V_m, \quad (3)$$

which means that for any $v \in V$ , there exists $v_j \in V_j$ such that $v = \sum_{i=1}^{m} vj$:

Following the framework of [2,6] for linear problems, we solve a finite sequence of sub-minimization problems over the subspaces:

$$\min_{e \in V_j} F(u^n + e), \tag{4}$$

where $u^n$ denotes a previous approximation, to resolve (3). Two types of subspace correction methods based on (4)-(5), known as the parallel subspace correction (PSC) and successive subspace correction (SSC) method, were proposed in [7, 3]. Here, we adopt the latter, which can be described as follows:

Algorithm **SSC**. Choose an initial value $u_0 \in V$ .
For $n = 0$,
while $j = 1, \ldots, m$ do

Find $e_j^n \in V_j$ such that

$$F\left(u^{n+(j-1)/m} + e_j^n\right) \le F\left(u^{n+(j-1)/m} + v_j\right), \quad \forall v_j \in V_j \tag{5}$$

set

$$u^{n+j/m} = u^{n+(j-1)/m} + e_j^n$$

end
Go to next iteration for $n$.

As an illustrative example, we apply the algorithm to the (regularized) Tixotrop denoising model with the cost functional:

$$F(u) = \frac{1}{2}\int_{\Omega} |\nabla u|^2 \, dxdy + \alpha \int_{\Omega} \sqrt{\beta + |\nabla (u - u_0)|^2} dxdy \tag{6}$$
$$+ \frac{1}{2}\int_{\Omega} |u - u_0|^2 \, dxdy, \quad \alpha, \beta > 0$$

where $u_0$ is a given noisy image defined on $\Omega = (0,1) * (0,1)$. Here, $F$ is differentiable and it also avoids the division by zero in the corresponding Euler-Lagrange equation:

$$\Delta u - \alpha div\left(\frac{\nabla (u - u_0)}{\sqrt{\beta + |\nabla (u - u_0)|^2}}\right) + u = u_0 \tag{7}$$

with a homogenous Neumann boundary condition $\partial u / \partial n = 0$ along the boundary. Recall that the lagged diffusivity fixed-point iteration for (8) is to solve the linearized equation:

$$\Delta u^{k+1} - \alpha div\left(\frac{\nabla (u^{k+1} - u_0)}{\sqrt{\beta + |\nabla (u^k - u_0)|^2}}\right) + u^{k+1} = u_0, \quad k = 0, 1, \ldots., \tag{8}$$

with the initial value $u_0$. We see that each iteration involves all the pixel values in the image domain, so it will be costly and usually the system is not in good conditioning when the size of images is large. The domain decomposition based SSC algorithm will overcome the difficulties by breaking down the whole problem into sub-problems of much smaller size. In the first place, we use an overlapping domain decomposition to decompose the solution space $V = H^1(\Omega)$. More precisely, we partition $\Omega$ into $m$ overlapping subdomains

$$\Omega = \bigcup_{j=1}^{m} \Omega_j, \quad \Omega_j \cap \Omega_k \neq \emptyset, \quad k \neq j, \tag{9}$$

For clarity, the subdomain $\Omega_j$ is colored with a color $j$, and $\Omega_j$ consists of $m_j$ subdomains (assumed to be "blocks" for simplicity), which are not intersected. Hence, the total number of blocks that cover $\Omega$ is :

$$M = \sum_{j=1}^{m} m_j \tag{10}$$

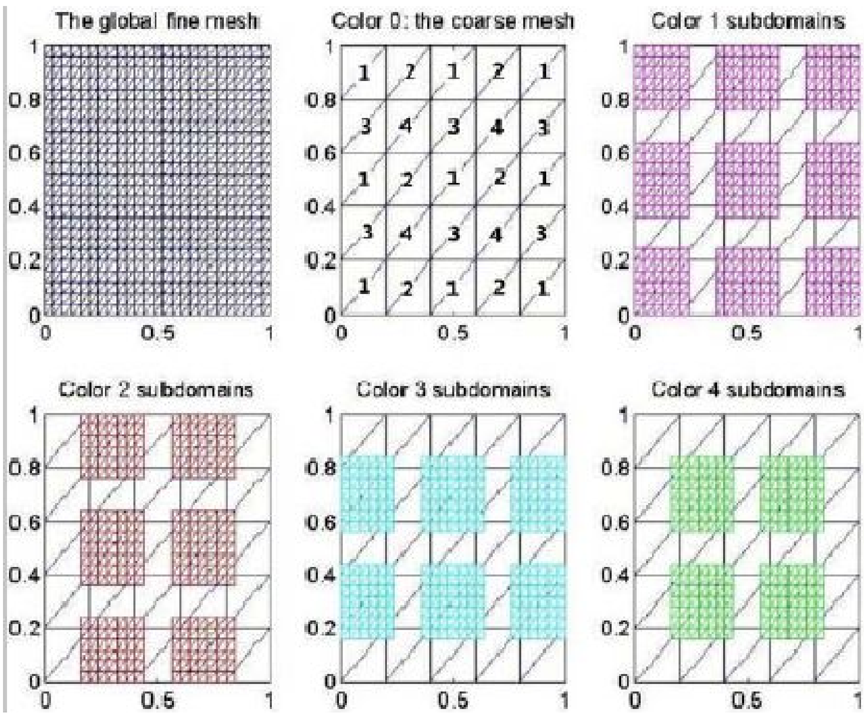


Fig.1 Illustrates schematically the decomposition of $\Omega$ into four colored subdomains with 25 blocks.

Based on the above domain decomposition, we decompose the space $V = H^1(\Omega)$ as

$$V = \sum_{j=1}^{m} V_j, \quad V_j = H_0^1(\Omega_j), \tag{11}$$

where $H_0^1(\Omega_j)$ denotes the subspace of $H^1(\Omega_j)$ with zero traces on the "interior" boundaries $\partial\Omega_j / \partial\Omega$ .

Applying the SSC algorithm to the Tixotrop-denoising model leads to an iterative method.
Given an initial value $u_0 \in V$ , Algorithm SSC needs us to solve $u^n$ from

$$\begin{cases} F\left(u^{n+\frac{j-1}{m}}+e_j^n\right) \le F\left(u^{n+\frac{j-1}{m}}+v_j\right), \forall v_j \in V_j = H_0^1(\Omega_j) \\ u^{n+\frac{j}{m}} = u^{n+\frac{j-1}{m}} + e_j^n, \qquad 1 \le j \le m. \end{cases} \tag{12}$$

Here, we notice that $e_j^n$ is the solution of the subproblem over $\Omega_j$ . It is also easy to see that $u^{n+\frac{j}{m}}$ satisfies the associated Euler-Lagrange equations for $1 \le j \le m$;

$$\begin{cases} \Delta u^{n+\frac{j}{m}} - \alpha div\left(\frac{\nabla\left(u^{n+\frac{j}{m}}-u_0\right)}{\sqrt{\beta+\left|\nabla\left(u^{n+\frac{j-1}{m}}-u_0\right)\right|^2}}\right) + u^{n+\frac{j}{m}} = u_0, \text{ in } \Omega_j \\ \frac{\partial u^{n+\frac{j}{m}}}{\partial n} = 0, \qquad \text{on } \partial\Omega_j \cap \partial\Omega, \\ u^{n+\frac{j}{m}} = u^{n+\frac{j-1}{m}}, \qquad \text{on } \partial\Omega_j \backslash \partial\Omega. \end{cases} \tag{13}$$

Outside $\Omega_j$, we have $u^{n+\frac{j}{m}} = u^{n+\frac{j-1}{m}}$ . Thus, there is no need to solve $u^{n+\frac{j-1}{m}}$ outside $\Omega_j$. As the subdomain $\Omega_j$ may contain many disjoint "block", the values of $u^{n+\frac{j-1}{m}}$ can be obtained in parallel in these "blocks" by solving (13).

**Numerical discrete algorithm for Tixotrop denoising**

We next present the full two-level algorithm formulated in the previous section for the Tixotrop denoising model. We partition the image domain $\Omega = (0,1) * (0,1)$ into $N \times N$ uniform cells with mesh size $h = {}^1/_N$. The cell centers are

$$(x_i, y_j) = ((i-1)h, (j-1)h), \qquad 1 \le i, j \le N+1. \tag{14}$$

Hereafter, let $u_{i,j}^0$ be the pixel value of the original image $u^0$ at $(x_i;\ y_j)$, and let $u_{i;j}$ be the finite difference solution at $(x_i;\ y_j)$. Denote

$$\delta_x^{\pm} u_{i,j} = \pm\left(u_{i\pm1,j} - u_{i,j}\right), \qquad \delta_y^{\pm} u_{i,j} = \pm\left(u_{i,j\pm1} - u_{i,j}\right), \tag{15}$$

$$\delta_x^{c} u_{i,j} = \left(u_{i+1,j} - u_{i-1,j}\right), \quad \delta_y^{c} u_{i,j} = \left(u_{i,j+1} - u_{i,j-1}\right). \tag{16}$$

The finite difference approximation of (7) is:

$$u_{i,j} - \delta_x^+\left(\delta_x^- u_{i,j}\right) - \delta_y^+\left(\delta_y^- u_{i,j}\right) - \alpha_h \left\{ \begin{array}{c} \delta_x^- \left[\frac{\delta_x^+ u_{i,j}}{\sqrt{\left(\delta_x^+ u_{i,j}\right)^2 + \left(\delta_y^c u_{i,j}\right)^2 + \beta_h}}\right] + \\ \delta_y^- \left[\frac{\delta_y^+ u_{i,j}}{\sqrt{\left(\delta_x^c u_{i,j}\right)^2 + \left(\delta_y^+ u_{i,j}\right)^2 + \beta_h}}\right] \end{array} \right\} = u_{i,j}^0 \tag{17}$$

where $\alpha_h = \frac{\alpha}{h}$ and $\beta_h = 4h\beta\_$.

## 4. Conclusion

We use in this paper fast algorithms for nonlinear minimization problems with particular applications to TV-image denoising. We described the very detailed implementation of the domain decomposition and coarse mesh correction techniques.

## 5. Simulation and result analysis

**Simulation and result analysis**

*Fig2. Simulation and result analysis*

**1**. Original image Lenna, **2**. noise image with 15% noise multiplicative, **3**. Restored image obtained by DD with subdomain size $d = 32$, overlapping size $\delta = 4$,and $PSNR = 25{:}9388$., **4.** Original image color, **5**. noise image color with $15\ {}^{0}/_{0}$ noise multiplicative, **6**. Restored image color obtained by DD with subdomain size $d = 64$, overlapping size $\delta = 4$,and $PSNR = 27{:}303$.

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# Prediction Model of TENSILE PROPERTIES of Burnished S355JR Steel

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**Abstract**

Various statistical and mathematical model including regression, Taguchi method, response surface methodology (RSM), and the analysis of variance (ANOVA) have been used to predict and optimize the output burnishing response.

This work focuses on the application of burnishing S355JR steel by diamond-tip active part. The considered process parameters are burnishing force (P) and number of tool passes (i). The experiment was performed using full factorial methodology to develop a mathematical model and optimize the parameters for the tensile properties such as yield strength (Re), tensile strength (Rm) and ultimate elongation (A%).

A combination of the two parameters was released according to the full factorial methodology with a complete $2^2$ type design. A linear model for predicting output responses was also established.

Linear regression model was used to predict the output responses.

The effect of each of both input factors (P) and (i) as well as their interactions were investigated and analyzed. Results show that burnishing has a beneficial effect on the physical state of the material given the increase in the tensile strength of the material despite a modest reduction of the yield strength up to 11.87% and a reduction from the ultimate elongation of 12.3% to the worst case.

An optimal solution combining burnishing force P = 10 kgf with a number of tool passes i = 3 resulted in an increase in the tensile strength (Rm) of 4.22% without greatly affecting the ductility of the material

**Keywords: Steel S355 JR, burnishing, tensile properties, full factorial methodology**

## 1. Introduction

To ensure a good surface condition some manufacturing processes use mechanical surface treatments (MST) as cold working processes in which surface plastic deformation (SPD) generate a uniform and work-hardened surface. These treatments appreciably modify the basic properties of material. [1-2]. An example of the SPD process is ball burnishing process [3,4] which does not involve material removal, but improves the surface properties by deforming the surface plastically. It strengthens the metal surface through the application of pressure through a hard diamond ball and it is often performed on high-strength steel and alloys with a surface hardness up to 60 HRC [5]. Burnishing is a low-cost surface treatment and

an environment friendly green process (where skilled operators are not required). This process can be effectively used in many fields such as aerospace industries, automobiles manufacturing sector etc.
Microscopically, this Mechanical surface treatment (MST) induces high dislocation densities (cold work) in near-surface regions. The consolidation of material is the result of a movement and proliferation of dislocations sweeping plan slip and intersects them by increasing density [6,7]. Due to the local plastic deformation, the surface topography is changed [8] and the superficial layers are work-hardened [9]. It results an improvement in surface roughness [10], an increase in hardness [11] and a development of macroscopic compressive residual stresses [12], which leads to improve, among others, tensile properties [13,14,15].
From the literature review, it can be concluded that the most studies conducted on the surface layer ball-burnishing process have been focused on the most important classical factors like burnishing speed, burnishing feed, the burnishing load and the number of tool passes on the treated surfaces quality enhancement.
Some researchers have developed and manufactured different tools [16,17] to produce good quality of burnished surfaces. Even if the use of deep hydrostatic ball-burnishing tool [1], an hybrid tool that combines both function namely milling and burnishing [18] or roller burnishing assisted with ultrasonic vibration is more effective[19].
Various statistical and mathematical model including regression, Taguchi method, response surface methodology (RSM), and the analysis of variance (ANOVA) have been used to predict and optimize the output burnishing response [2,14,20].
This work focuses on the application of burnishing S355JR steel by diamond-tip active part. The input parameters are burnishing force (P) and number of tool passes (i). The experiment was performed using full factorial methodology to develop a mathematical model and optimize the parameters for the tensile properties (such as yield strength (Re), tensile strength (Rm) and ultimate elongation (A%).

## 2. Procedure and equipment

### 2.1. Material

In this study, Steel S355JR with chemical composition according the standard ISO 1424 and given in Table1 was used as workpiece material. Steel S355JR was selected because of its machinability and its range of applications in the industry. Workpieces were received from initially turned into cylindrical rod of 12 mm diameter and 130 mm length.

**Table 1**. Chemical composition of steel S355 JR

| C | S | Al | Si | P | V | Cr | Mn | Ni | Cu | Mo |
|---|---|---|---|---|---|---|---|---|---|---|
| 0,188 | 0,003 | 0,0273 | 0,2314 | 0,0051 | 0 ,00327 | 0,1571 | 1,053 | 0,0548 | 0,0575 | 0,0297 |

The mechanical characteristics of the material were evaluated using a tensile test performed on cylindrical specimens taken from the above-mentioned metal. The specimens were machined from cylindrical rods according to the standard ISO 6892-1. An ALMO model lathe and M20 carbide cutting tool were used under condition of cutting oil lubrication. Dimensions of rod flat are shown in Fig. 1. All specimens were machined from the same batch to minimize possible deviation in experimental data. The gauge of the specimens was polished with fine sandpaper up to grade 400.

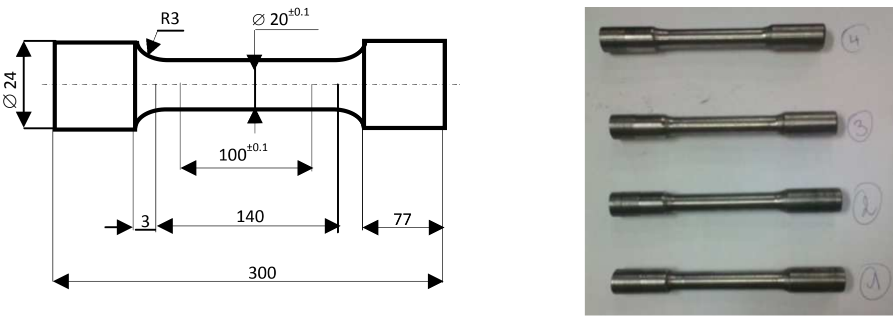



**Figure 1.** Dimensions (mm) of ISO 6892-1tensile specimen

Before the tensile test, the specimens have according to the case been undergone an appropriate burnishing treatment or remaining with machining state. After machining and burnishing of the specimens no further treatment was performed.

### 2.3. Burnishing tool

A burnishing tool with interchangeable adapter for roller, ball or diamond tip burnishing (DTB) were designed and manufactured [11] for the purpose of the experimental tests. Fig. 2 shows a schematic representation of the burnishing tool in which the active part (here DTB) fixed to a shank which is to be firmly clamped on the lathe machine.

**Figure 2.** Burnishing tool

Burnishing speed (V) is obtained by the rotation of the spindle whereas the burnishing feed (f) is communicated to the tool by means of the longitudinal carriage. SPD results in the penetration of the diamond tip which acts with burnishing force (P) exerted with a helical compression spring during burnishing operations. A calibration process was conducted using the actual burnishing operation setting to obtain a relationship between the burnishing force and the corresponding axial displacement.
Two (2) burnishing parameters (P) and (i), have significant effect on the tensile properties. The burnishing parameters considered are based on previous work conducted by the first author [9] and given in Table 2.

**Table 2**. Summary of burnishing parameters

| Burnishing parameters | Values |
|---|---|
| Burnishing speed (rpm) | 560 |
| Burnishing force (kgf) | 10, 20 |
| Number of tool passes | 3, 5 |
| Diamond tip radius (mm) | 3 |
| Burnishing feed rate (mm/rev) | 0.054 |
| Lubricant | Oil SAE20 |

The surface properties of the pre-machined and ball burnished specimens were determined by roughness and hardness measurements using respectively a digital display profilometer-profilograph instrument brand: Roughpocket and universal hardness tester LEICA Wetzlar Vickers indenter with indentation load of 200 gr. At the turned state, roughness and hardness average values are Ra = 5.5µm and Hv=237.3 respectively.

### 2.3. Tensile tests

To assess strength of material, monotonic tension test according the standard ISO 6892-1 were performed with 0.005 mm/sec displacement rate at room temperature on steel S355 JR specimens. The experiment were carried out with a universal testing machine hydraulically MFL type ZWICK 1476 equipped with hydraulic wedge grips and 10kN force transducer. Data acquisition was carried out using Zwick software Test-Expert V5. Mechanical properties for the machined and burnished material were given in Table 3.

### 2.4. Experimental Procedure

Fig. 3 **represents** the parameters of controlled burnishing. Burnishing force (P), and the number of tool passes (i) were accepted as the variable (input) factors of the process. Surface microhardness (Hv) and surface roughness (Ra) was as a surface layer response to an input action on which tensile properties depend. Burnishing speed (N) and Burnishing feed rate (f) within the frames of this experiment were considered as constant magnitudes.

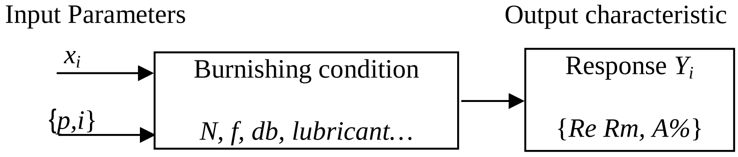



**Figure 3.** Process diagram and controlling parameters of burnishing

The full factorial experimental design for the factors (P and i) which varies at two levels each of which can assume two discrete values (conditionally let us denote them $X_1$, $X_2$). The upper limit of the parameters was coded as +1 and the lower limit was coded as -1. A standard matrix $2^2$ is used, which structure in simple terms is given in Table 3. Consequently, according to the full factorial experimental method, it is enough to make only four experiments to establish the degree of impact of each of two variable parameters on the output characteristic of the material. The other coded values were calculated using Eq. (1).

$$X_i = \frac{2[2X-(X_{max}-X_{min})]}{X_{max}-X_{min}} \quad (1)$$

Where $X_i$ in the above equation is the resulting coded value of a variable X, X is any value of the variable from $X_{min}$ to $X_{max}$, and $X_{min}$ is the lower and $X_{max}$ is the upper limit of the variable. A mathematical model was developed to predict mechanical properties including yielding stress, ultimate tensile strength and elongation of steel S355 JR at different burnishing conditions. The linear regression equation to represent the predicting output responses is given by Eq. (2):

$$Y = a_0 + a_1X_1 + a_2X_2 + a_{12}X_1X_2 \quad (2)$$

Where Y is the response; the term $a_0$ is the mean of responses; and the terms $a_1$, $a_2$, and $a_{12}$ are the coefficients of responses and it depends on the respective main and interaction effects of the parameters. $X_1$ and $X_2$ are the coded independent variables. The calculations were carried out by means of the software for processing statistics, Microsoft Excel. The results are given in Table 3.

**Table 3**. Experimental design matrix

| N° | | input factors | | | | Output responses | | |
|---|---|---|---|---|---|---|---|---|
| | | Coded | | Natural | | $Y_1$ | $Y_2$ | $Y_3$ |
| | | P $X_1$ | i $X_2$ | P (Kgf) | (i) | Re (MPa) | Rm (MPa) | A% |
| Turned | | - | - | - | - | 365,6 | 523,6 | 30.0 |
| DTB1 | Burnished | +1 | -1 | 20 | 3 | 334,2 | 544,2 | 27.5 |
| DTB 2 | Burnished | +1 | +1 | 20 | 5 | 322,2 | 531,6 | 26.6 |
| DTB 3 | Burnished | -1 | -1 | 10 | 3 | 339,6 | 545,7 | 26.3 |
| DTB 4 | Burnished | -1 | +1 | 10 | 5 | 329,4 | 535,8 | 25.0 |

The final regression models for tensile properties (Re, Rm and A%) are given in Eqs. (3), (4), and (5) respectively.

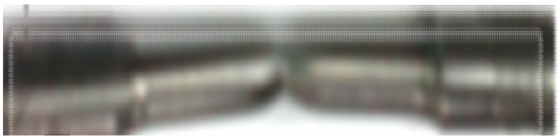
$$Re = 331.35 - 3.15\,P + 5.6\,i - 0.450\,P\,i \tag{3}$$

$$Rm = 539.325 - 1.425\,P - 5.625\,i - 0.675\,P\,i \tag{4}$$

$$A\% = 26.350 + 0.700\,P - 0.550\,i - 0.100\,P\,i \tag{5}$$

## 3. Results and Discussions

During the tensile test, the specimen develops an important necking phenomenon (Fig. 4a) whereas the fracture surface often tilted or mixed exhibits a rupture at nerve (Fig. 4b) whose fracture surface testifies the ductile character of tested material.



a)

b)

**Figure 4.** a)Necking and b)Fracture surface of steel S355JR

Tensile strength is improved in the sequential turning and burnishing process because of plastic deformation induced by the diamond tip. In the other hand yield strength (Re) and ultimate elongation (A%) are reduced due to the generation of compressive residual stress. The tensile tests on the four burnished work-pieces revealed a reduction of Re (Fig. 5) which can fall by 11.87 % when performing burnishing with a force of 20 kgf in 5 passes.

This can be explained by the fact that a large load (P = 20Kgf) associated with a high number of passes (i = 5) accentuates the hardening of the surface layers and consequently reduces the ductility and the yield strength of the material. The beneficial effect of the surface treatment has been noted on the tensile strength (Rm) which is for all burnishing conditions, several times higher than that of removal-chip process. The most favorable working regime characterized by (P = 10Kgf and i = 3) favors a Rm = 545.7 MPa. Such a combination of the two parameters seems to ensure the best hardening and consolidation of superficial layers, hence the increase in Rm. However, the elongation (A%) experiencing a decrease in particular for burnishing with a large number of passes (i = 5), where the phenomenon of strain hardening is feared.

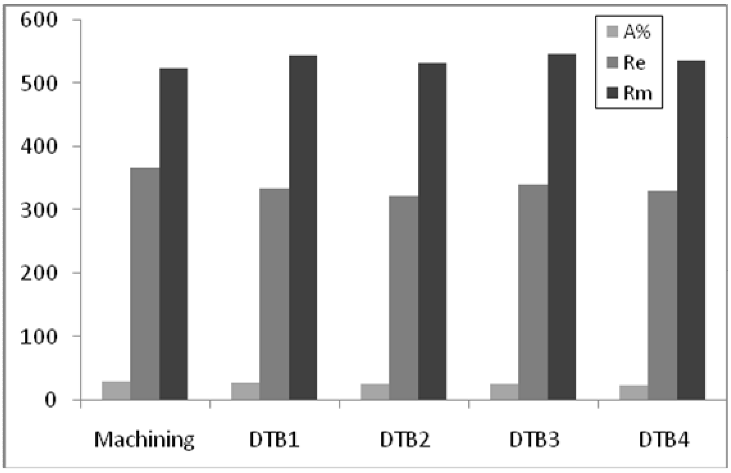



**Figure 5.** Effect of burnishing parameters on tensile properties

Fig. 6 shows the influence of the burnishing parameters on the yield strength (Re). Even if a reduction of this property is observed, whatever the combination of the two parameters envisaged, the increase of the burnishing force from 10 to 20 kgf results in a reduction of 0.16% on Re when number of tools passes (i) is 3. When increasing this number (i) to 5, the fall of Re is more increased since the difference increases to 2.18%. The interaction of number of tools passes is therefore stronger when the latter is taken to its high level. When both factors (P and i) are at their highest levels, the surface layers are very affected by hardening which increases their hardness and consequently reduces Re to its lowest estimated rate at 11.87% compared to the machining state.

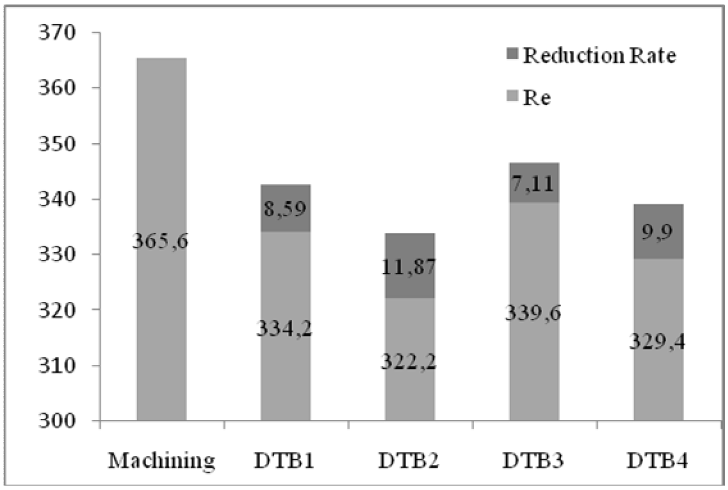



**Figure 6.** Reduction rate of yield strength under DTB effect

Overall, the hardening induced by burnishing is positively reflected in the tensile strength (Rm) which gains up to 4.22% compared to the turning state for the low levels of each of the two factors (Fig. 7).

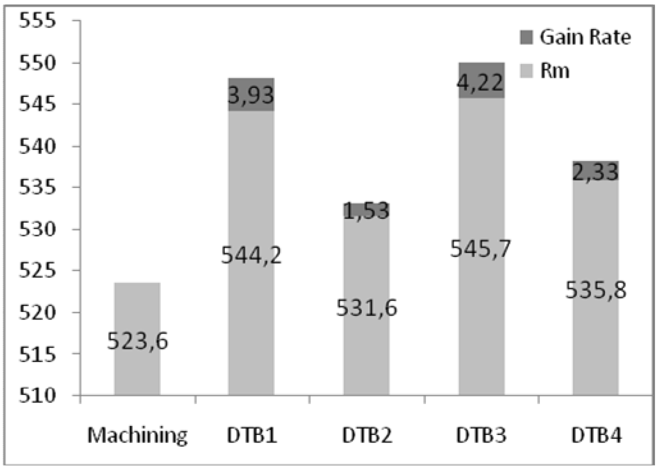



**Figure 7.** Gain rate of tensile strength under DTB effect

The interaction of these two factors is not negligible in this case; by fixing the burnishing force at its low level (10kgf), the increase in the factor (i) reduces the burnishing efficiency with respect to Rm which drops by 1.81%. It is the same when the treatment is done at the high level of factor (P) i.e 20kgf since Rm falls by 2.3% when factor (i) goes from 3 to 5.
Burnishing seems generate compressive residual stress layer. This effect results in a ductility drop of material that loses up to 16.6% of its ultimate elongation compared to the machining state (Fig. 8).

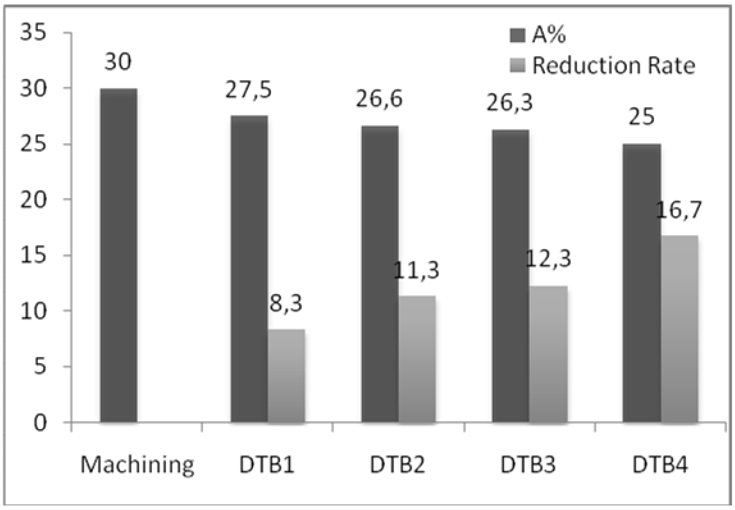



**Figure 8.** Reduction rate of ultimate elongation under DTB effect
The influence of the burnishing force (P) is more striking when the number of passes (i) is used at its high level since by increasing the force (P) from 10 to 20 kgf, the ductility increases by 6.4% for (i) = 5 then it

increases by 4.5% when it is 3. On the other hand, the interaction of the number of passes is stronger when the burnishing force is taken at a low level; for P = 20kgf, the ultimate elongation (A%) is reduced by 3.2% when (i) goes from 3 to 5 whereas for P = 10kgf, the elongation is reduced by 4.94% for the same variation of ( i). Combination of high burnishing forces (P) with large numbers of passes (i) is to be avoided since it provides the least desirable output responses.

## 4. Conclusion

Ball-burnishing process has been commonly used to improve the quality of finished surfaces. The effects of diamond tip burnishing process on yield strength, tensile strength and ultimate elongation of the S355 JR Steel have been investigated in this research.
The following conclusions can be drawn.
-Burnishing process shows a dependency on the parameters (P) and (i) whose the high level would weaken the burnishing effect induced by the diamond tip.
-Using high force and number of passes increase the cold work dramatically and may consequently decrease tensile properties. On the other hand, low burnishing forces agree well with low numbers of passes to provide the best yield strength (Re), and tensile strength (Rm) even if the ultimate elongation is more decreased. Thus for optimal Rm we recommend P = 10kgf and i = 3.
-The results showed that after burnishing, tensile strength was higher than 545.7 MPa and the yield strength fall down at 339.6 MPa while using the best burnishing condition.
-The best combination condition of process parameters was obtained after a full factorial design analysis. The quality reproducibility was verified to be excellent through confirmation experiments. Thus, using the proposed procedure, the optimal diamond ball burnishing conditions should be obtained to control the surface responses of other materials.

**Acknowledgements**
A part of this work was supported by the IndustrialTechnologies Research Center URASM-(CRTI) Annaba. All the members are gratefully acknowledged.

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# A Goodness-of-fit tests based on the empirical distribution function for the AFT-Bertholon model

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**Abstract**

In this work, we consider some nonparametric goodness-of-fit tests for AFT-Bertholon distribution. Kolmogorov-Smirnov, Anderson-Darling are proposed.

We use the Monte Carlo simulation method for calculate the critical values for each test for several sample sizes and significance levels. The power of the proposed tests can be specified for different sample sizes and considering diverse alternatives.

**Keyword(s):** Accelerated failure time model, Bertholon model, Kolmogorov-Smirnov statistic, Anderson-Darling statistic.

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## A Modified Chi-squared Goodness-of-fit test for the AFT-Bertholon and its Applications

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**Abstract**

In this work, we propose firstly the construction of a new model called AFT-Bertholon. This new model combines the accelerated failure time model with a competing risks model proposed by Bertholon et al. (2004).

Next, we suggest the construction of a modified chi-squared goodness-of-fit test for AFT-Bertholon model. We use the NRR (Nikulin-Rao-Robson) statistic based on maximum likelihood estimation for ungrouped data.

We applied this new model and the corresponding statistic test to numerical examples from simulated samples and real data.

**Keyword(s):** Accelerated failure time model, Bertholon model, Competing risks model, Chi-squared goodness-of-fit test.

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## Prediction Model of TENSILE PROPERTIES of Burnished S355JR Steel

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**Abstract**

Various statistical and mathematical model including regression, Taguchi method, response surface methodology (RSM), and the analysis of variance (ANOVA) have been used to predict and optimize the output burnishing response.

This work focuses on the application of burnishing S355JR steel by diamond-tip active part. The considered process parameters are burnishing force (P) and number of tool passes (i). The experiment was performed using full factorial methodology to develop a mathematical model and optimize the parameters for the tensile properties such as yield strength (Re), tensile strength (Rm) and ultimate elongation (A%).

A combination of the two parameters was released according to the full factorial methodology with a complete $2^2$ type design. A linear model for predicting output responses was also established.

Linear regression model was used to predict the output responses.

The effect of each of both input factors (P) and (i) as well as their interactions were investigated and analyzed. Results show that burnishing has a beneficial effect on the physical state of the material given the increase in the tensile strength of the material despite a modest reduction of the yield strength up to 11.87% and a reduction from the ultimate elongation of 12.3% to the worst case.

An optimal solution combining burnishing force P = 10 kgf with a number of tool passes i = 3 resulted in an increase in the tensile strength (Rm) of 4.22% without greatly affecting the ductility of the material

**Keywords: Steel S355 JR, burnishing, tensile properties, full factorial methodology**

## 1. Introduction

To ensure a good surface condition some manufacturing processes use mechanical surface treatments (MST) as cold working processes in which surface plastic deformation (SPD) generate a uniform and work-hardened surface. These treatments appreciably modify the basic properties of material. [1-2]. An example of the SPD process is ball burnishing process [3,4] which does not involve material removal, but improves the surface properties by deforming the surface plastically. It strengthens the metal surface through the application of pressure through a hard diamond ball and it is often performed on high-strength steel and alloys with a surface hardness up to 60 HRC [5]. Burnishing is a low-cost surface treatment and an environment friendly green process (where skilled operators are not required). This process can be effectively used in many fields such as aerospace industries, automobiles manufacturing sector etc.

Microscopically, this Mechanical surface treatment (MST) induces high dislocation densities (cold work) in near-surface regions. The consolidation of material is the result of a movement and proliferation of dislocations sweeping plan slip and intersects them by increasing density [6,7]. Due to the local plastic deformation, the surface topography is changed [8] and the superficial layers are work-hardened [9]. It results an improvement in surface roughness [10], an increase in hardness [11] and a development of macroscopic compressive residual stresses [12], which leads to improve, among others, tensile properties [13,14,15].

From the literature review, it can be concluded that the most studies conducted on the surface layer ball-burnishing process have been focused on the most important classical factors like burnishing speed, burnishing feed, the burnishing load and the number of tool passes on the treated surfaces quality enhancement.

Some researchers have developed and manufactured different tools [16,17] to produce good quality of burnished surfaces. Even if the use of deep hydrostatic ball-burnishing tool [1], an hybrid tool that combines both function namely milling and burnishing [18] or roller burnishing assisted with ultrasonic vibration is more effective[19].

Various statistical and mathematical model including regression, Taguchi method, response surface methodology (RSM), and the analysis of variance (ANOVA) have been used to predict and optimize the output burnishing response [2,14,20].

This work focuses on the application of burnishing S355JR steel by diamond-tip active part. The input parameters are burnishing force (P) and number of tool passes (i). The experiment was performed using full factorial methodology to develop a mathematical model and optimize the parameters for the tensile properties (such as yield strength (Re), tensile strength (Rm) and ultimate elongation (A%).

## 2. Procedure and equipment

### 2.1. Material

In this study, Steel S355JR with chemical composition according the standard ISO 1424 and given in Table1 was used as workpiece material. Steel S355JR was selected because of its machinability and its range of applications in the industry. Workpieces were received from initially turned into cylindrical rod of 12 mm diameter and 130 mm length.

**Table 1**. Chemical composition of steel S355 JR

| C | S | Al | Si | P | V | Cr | Mn | Ni | Cu | Mo |
|---|---|---|---|---|---|---|---|---|---|---|
| 0,188 | 0,003 | 0,0273 | 0,2314 | 0,0051 | 0 ,00327 | 0,1571 | 1,053 | 0,0548 | 0,0575 | 0,0297 |

The mechanical characteristics of the material were evaluated using a tensile test performed on cylindrical specimens taken from the above-mentioned metal. The specimens were machined from cylindrical rods according to the standard ISO 6892-1. An ALMO model lathe and M20 carbide cutting tool were used under condition of cutting oil lubrication. Dimensions of rod flat are shown in Fig. 1. All specimens were machined from the same batch to minimize possible deviation in experimental data. The gauge of the specimens was polished with fine sandpaper up to grade 400.

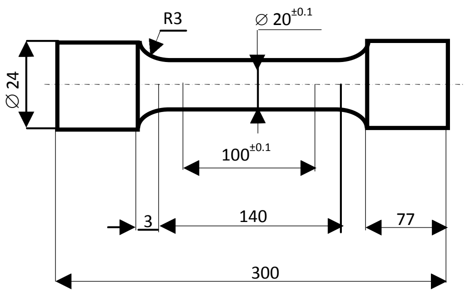


**Figure 1.** Dimensions (mm) of ISO 6892-1tensile specimen

Before the tensile test, the specimens have according to the case been undergone an appropriate burnishing treatment or remaining with machining state. After machining and burnishing of the specimens no further treatment was performed.

### 2.3. Burnishing tool

A burnishing tool with interchangeable adapter for roller, ball or diamond tip burnishing (DTB) were designed and manufactured [11] for the purpose of the experimental tests. Fig. 2 shows a schematic representation of the burnishing tool in which the active part (here DTB) fixed to a shank which is to be firmly clamped on the lathe machine.

**Figure 2.** Burnishing tool

Burnishing speed (V) is obtained by the rotation of the spindle whereas the burnishing feed (f) is communicated to the tool by means of the longitudinal carriage. SPD results in the penetration of the diamond tip which acts with burnishing force (P) exerted with a helical compression spring during burnishing operations. A calibration process was conducted using the actual burnishing operation setting to obtain a relationship between the burnishing force and the corresponding axial displacement.
Two (2) burnishing parameters (P) and (i), have significant effect on the tensile properties. The burnishing parameters considered are based on previous work conducted by the first author [9] and given in Table 2.

**Table 2**. Summary of burnishing parameters

| Burnishing parameters | Values |
|---|---|
| Burnishing speed (rpm) | 560 |
| Burnishing force (kgf) | 10, 20 |
| Number of tool passes | 3, 5 |
| Diamond tip radius (mm) | 3 |
| Burnishing feed rate (mm/rev) | 0.054 |
| Lubricant | Oil SAE20 |

The surface properties of the pre-machined and ball burnished specimens were determined by roughness and hardness measurements using respectively a digital display profilometer-profilograph instrument brand: Roughpocket and universal hardness tester LEICA Wetzlar Vickers indenter with indentation load of 200 gr. At the turned state, roughness and hardness average values are Ra = 5.5µm and Hv=237.3 respectively.

## 2.3. Tensile tests

To assess strength of material, monotonic tension test according the standard ISO 6892-1 were performed with 0.005 mm/sec displacement rate at room temperature on steel S355 JR specimens. The experiment were carried out with a universal testing machine hydraulically MFL type ZWICK 1476 equipped with hydraulic wedge grips and 10kN force transducer. Data acquisition was carried out using Zwick software Test-Expert V5. Mechanical properties for the machined and burnished material were given in Table 3.

## 2.4. Experimental Procedure

Fig. 3 **represents** the parameters of controlled burnishing. Burnishing force (P), and the number of tool passes (i) were accepted as the variable (input) factors of the process. Surface microhardness (Hv) and surface roughness (Ra) was as a surface layer response to an input action on which tensile properties depend. Burnishing speed (N) and Burnishing feed rate (f) within the frames of this experiment were considered as constant magnitudes.

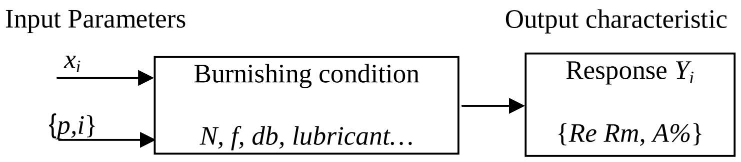


**Figure 3.** Process diagram and controlling parameters of burnishing

The full factorial experimental design for the factors (P and i) which varies at two levels each of which can assume two discrete values (conditionally let us denote them $X_1$, $X_2$). The upper limit of the parameters was coded as +1 and the lower limit was coded as -1. A standard matrix $2^2$ is used, which structure in simple terms is given in Table 3. Consequently, according to the full factorial experimental method, it is enough to make only four experiments to establish the degree of impact of each of two variable parameters on the output characteristic of the material. The other coded values were calculated using Eq. (1).

$$X_i = \frac{2\,[2X-(X_{max}-X_{min})]}{X_{max}-X_{min}} \tag{1}$$

Where $X_i$ in the above equation is the resulting coded value of a variable X, X is any value of the variable from $X_{min}$ to $X_{max}$, and $X_{min}$ is the lower and $X_{max}$ is the upper limit of the variable. A mathematical model was developed to predict mechanical properties including yielding stress, ultimate tensile strength and elongation of steel S355 JR at different burnishing conditions. The linear regression equation to represent the predicting output responses is given by Eq. (2):

$$Y = a_0 + a_1X_1 + a_2X_2 + a_{12}X_1X_2 \tag{2}$$

Where Y is the response; the term $a_0$ is the mean of responses; and the terms $a_1$, $a_2$, and $a_{12}$ are the coefficients of responses and it depends on the respective main and interaction effects of the parameters. $X_1$ and $X_2$ are the coded independent variables. The calculations were carried out by means of the software for processing statistics, Microsoft Excel. The results are given in Table 3.

**Table 3**. Experimental design matrix

| **N°** | | **input factors** | | | | **Output responses** | | |
|---|---|---|---|---|---|---|---|---|
| | | **Coded** | | **Natural** | | ***Y₁*** | ***Y₂*** | ***Y₃*** |
| | | P $X_1$ | i $X_2$ | P (Kgf) | (i) | Re (MPa) | Rm (MPa) | **A%** |
| Turned | | - | - | - | - | 365,6 | 523,6 | 30.0 |
| DTB1 | Burnished | +1 | -1 | 20 | 3 | 334,2 | 544,2 | 27.5 |
| DTB 2 | Burnished | +1 | +1 | 20 | 5 | 322,2 | 531,6 | 26.6 |
| DTB 3 | Burnished | -1 | -1 | 10 | 3 | 339,6 | 545,7 | 26.3 |
| DTB 4 | Burnished | -1 | +1 | 10 | 5 | 329,4 | 535,8 | 25.0 |

The final regression models for tensile properties (Re, Rm and A%) are given in Eqs. (3), (4), and (5) respectively.

$$Re = 331.35 - 3.15\,P + 5.6\,i - 0.450\,P\,i \tag{3}$$

$$Rm = 539.325 - 1.425\,P - 5.625\,i - 0.675\,P\,i \tag{4}$$

$$A\% = 26.350 + 0.700\,P - 0.550\,i - 0.100\,P\,i \tag{5}$$

## 3. Results and Discussions

During the tensile test, the specimen develops an important necking phenomenon (Fig. 4a) whereas the fracture surface often tilted or mixed exhibits a rupture at nerve (Fig. 4b) whose fracture surface testifies the ductile character of tested material.

a) b)

**Figure 4.** a)Necking and b)Fracture surface of steel S355JR

Tensile strength is improved in the sequential turning and burnishing process because of plastic deformation induced by the diamond tip. In the other hand yield strength (Re) and ultimate elongation (A%) are reduced due to the generation of compressive residual stress. The tensile tests on the four burnished work-pieces revealed a reduction of Re (Fig. 5) which can fall by 11.87 % when performing burnishing with a force of 20 kgf in 5 passes.

This can be explained by the fact that a large load (P = 20Kgf) associated with a high number of passes (i = 5) accentuates the hardening of the surface layers and consequently reduces the ductility and the yield strength of the material. The beneficial effect of the surface treatment has been noted on the tensile strength (Rm) which is for all burnishing conditions, several times higher than that of removal-chip process. The most favorable working regime characterized by (P = 10Kgf and i = 3) favors a Rm = 545.7 MPa. Such a combination of the two parameters seems to ensure the best hardening and consolidation of superficial layers, hence the increase in Rm. However, the elongation (A%) experiencing a decrease in particular for burnishing with a large number of passes (i = 5), where the phenomenon of strain hardening is feared.

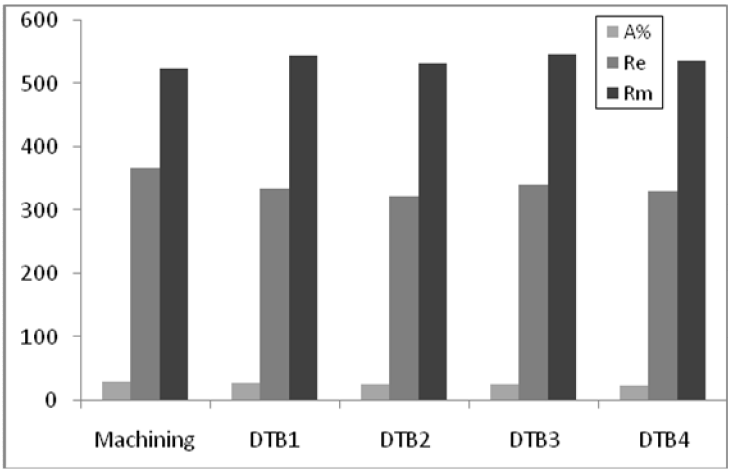


**Figure 5.** Effect of burnishing parameters on tensile properties

Fig. 6 shows the influence of the burnishing parameters on the yield strength (Re). Even if a reduction of this property is observed, whatever the combination of the two parameters envisaged, the increase of the burnishing force from 10 to 20 kgf results in a reduction of 0.16% on Re when number of tools passes (i) is 3. When increasing this number (i) to 5, the fall of Re is more increased since the difference increases to 2.18%. The interaction of number of tools passes is therefore stronger when the latter is taken to its high level. When both factors (P and i) are at their highest levels, the surface layers are very affected by hardening which increases their hardness and consequently reduces Re to its lowest estimated rate at 11.87% compared to the machining state.

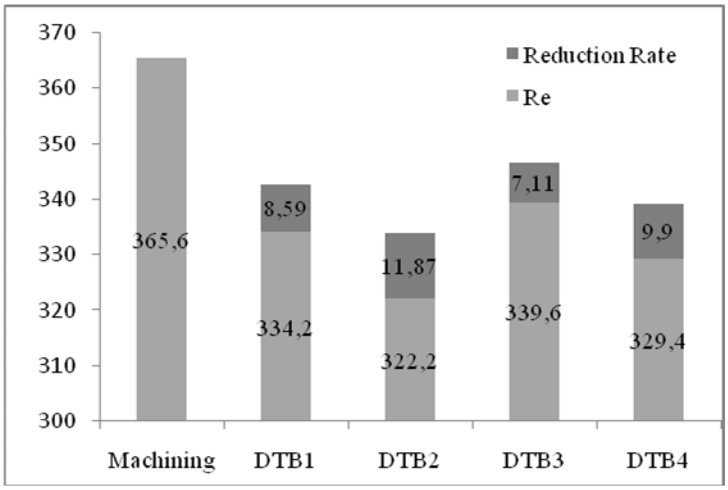


**Figure 6.** Reduction rate of yield strength under DTB effect

Overall, the hardening induced by burnishing is positively reflected in the tensile strength (Rm) which gains up to 4.22% compared to the turning state for the low levels of each of the two factors (Fig. 7).

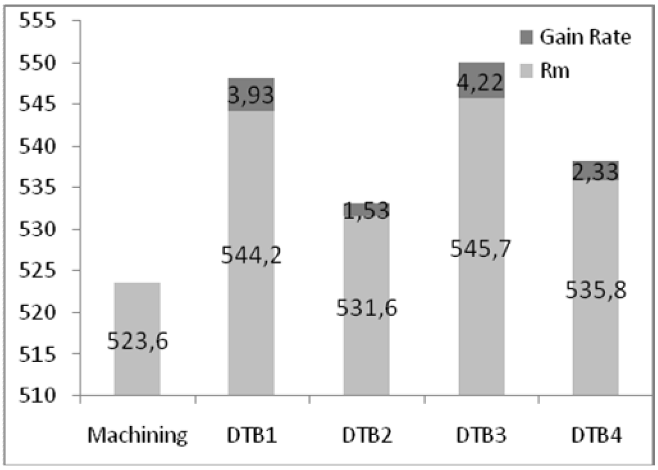


**Figure 7.** Gain rate of tensile strength under DTB effect

The interaction of these two factors is not negligible in this case; by fixing the burnishing force at its low level (10kgf), the increase in the factor (i) reduces the burnishing efficiency with respect to Rm which drops by 1.81%. It is the same when the treatment is done at the high level of factor (P) i.e 20kgf since Rm falls by 2.3% when factor (i) goes from 3 to 5.
Burnishing seems generate compressive residual stress layer. This effect results in a ductility drop of material that loses up to 16.6% of its ultimate elongation compared to the machining state (Fig. 8).

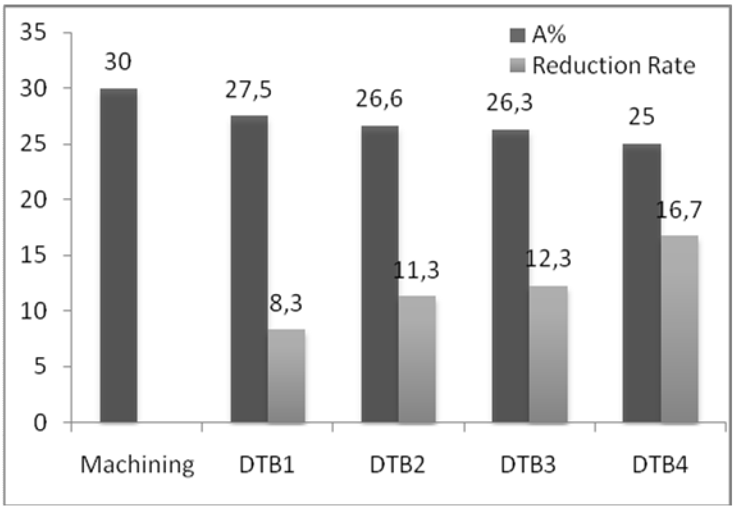


**Figure 8.** Reduction rate of ultimate elongation under DTB effect

The influence of the burnishing force (P) is more striking when the number of passes (i) is used at its high level since by increasing the force (P) from 10 to 20 kgf, the ductility increases by 6.4% for (i) = 5 then it increases by 4.5% when it is 3. On the other hand, the interaction of the number of passes is stronger when the burnishing force is taken at a low level; for P = 20kgf, the ultimate elongation (A%) is reduced by 3.2% when (i) goes from 3 to 5 whereas for P = 10kgf, the elongation is reduced by 4.94% for the same variation of ( i). Combination of high burnishing forces (P) with large numbers of passes (i) is to be avoided since it provides the least desirable output responses.

## 4. Conclusion

Ball-burnishing process has been commonly used to improve the quality of finished surfaces. The effects of diamond tip burnishing process on yield strength, tensile strength and ultimate elongation of the S355 JR Steel have been investigated in this research.
The following conclusions can be drawn.
-Burnishing process shows a dependency on the parameters (P) and (i) whose the high level would weaken the burnishing effect induced by the diamond tip.
-Using high force and number of passes increase the cold work dramatically and may consequently decrease tensile properties. On the other hand, low burnishing forces agree well with low numbers of passes to provide the best yield strength (Re), and tensile strength (Rm) even if the ultimate elongation is more decreased. Thus for optimal Rm we recommend P = 10kgf and i = 3.
-The results showed that after burnishing, tensile strength was higher than 545.7 MPa and the yield strength fall down at 339.6 MPa while using the best burnishing condition.
-The best combination condition of process parameters was obtained after a full factorial design analysis. The quality reproducibility was verified to be excellent through confirmation experiments. Thus, using the proposed procedure, the optimal diamond ball burnishing conditions should be obtained to control the surface responses of other materials.

**Acknowledgements**
A part of this work was supported by the IndustrialTechnologies Research Center URASM-(CRTI) Annaba. All the members are gratefully acknowledged.

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# Two-Parameter Xgamma Distribution: Different Methods of Estimation

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**Abstract**

Two-parameter Xgamma distribution is relatively a new probability distribution in the vast literature of distribution theory for modeling the positive-valued and skewed data. The aim of the current study is to estimate parameters of the two-parameter Xgamma distribution employing different estimation methods such as maximum likelihood, moments, L-moments, least-squares, and maximum spacing. We compare the estimation performances of these estimation methods by comprehensive Monte-Carlo simulation studies performed on the different sample of sizes small, moderate and large.

**Keywords:** Lifetime distributions, Statistical inference, L-moments estimation, Maximum likelihood estimation, Least-square estimation.

## 1. Introduction

The Gamma and Exponential are two popular probability distribution models used to statistically modeling of lifetime data. Recently, Sen et al. [1] introduced a two parameters lifetime distribution called two-parameter Xgamma distribution as a special mixture of the Exponential and Gamma distributions with mixing ratios $\frac{\theta}{\alpha+\theta}$ and $\frac{\alpha}{\alpha+\theta}$, respectively. The probability density function (pdf) of the Xgamma distribution is

$$f(x,\alpha,\theta)=\frac{\theta^2}{\alpha+\theta}\left(1+\frac{1}{2}\alpha\theta x^2\right)e^{-\theta x},\ \ x>0 \tag{1}$$

and the corresponding cumulative distribution function (cdf) is

$$F(x,\alpha,\theta)=1-\frac{\left(\alpha+\theta+\alpha\theta x+\frac{1}{2}\alpha\theta^2x^2\right)e^{-\theta x}}{\alpha+\theta},\ \ x>0\,, \tag{2}$$

where $\alpha$ and $\theta$ are the positive valued parameters of the distribution. When the parameter $\alpha=1$ then the distribution reduces to the Xgamma distribution with parameter $\theta$. From now on, we will use the $TPXG(\alpha,\theta)$ notation to indicate the two-parameter Xgamma distribution with parameters $\alpha$ and $\theta$.

The survival function of the two-parameter Xgamma distribution is

$$S(x,\alpha,\theta)=\frac{\left(\alpha+\theta+\alpha\theta x+\frac{1}{2}\alpha\theta^2x^2\right)e^{-\theta x}}{\alpha+\theta} \tag{3}$$

and its hazard rate function is

$$h(x,\alpha,\theta)=\frac{\theta^2\left(1+\frac{\alpha\theta}{2}x^2\right)}{\left(\alpha+\theta+\alpha\theta x+\frac{1}{2}\alpha\theta^2x^2\right)}. \tag{4}$$

Sen et al. [1] have showed the hazard rate function of the distribution is increasing for $x>\sqrt{\frac{2}{\alpha\theta}}$, decreasing otherwise. Therefore, we can conclude the distribution is an alternative probability distribution for modeling the data with a decreasing-increasing hazard rate. Some other basic features of the two-parameter Xgamma distribution studied by Sen et at. [1] are tabulated by the following table.

Table 1: Some basic features of the two-parameter Xgamma distribution

| | |
|---|---|
| Characteristic Function | $\frac{\theta^2}{(\alpha+\theta)}\left[(\theta-it)^{-1}+\alpha\theta(\theta-it)^{-3}\right]$ |
| Moment Generating Function | $\frac{\theta^2}{(\alpha+\theta)}\left[(\theta-t)^{-1}+\alpha\theta(\theta-t)^{-3}\right]$ |
| *r*-th Moment | $\frac{r!}{2\theta^r(\alpha+\theta)}\left[2\theta+\alpha(1+r)(2+r)\right]$ |
| Variance | $\frac{2\left(\theta^2+8\alpha\theta+3\alpha^2\right)}{\theta^2(\alpha+\theta)}$ |
| Mode | $\frac{1+\sqrt{1-\frac{2\theta}{\alpha}}}{\theta},\theta<\frac{\alpha}{2};\quad 0,\text{otherwise}$ |
| Mean Residual Life Function | $\frac{1}{\theta}+\frac{\alpha(2+\theta x)}{\theta\left(\alpha+\theta+\alpha\theta x+\frac{1}{2}\alpha\theta^2x^2\right)}$ |
| Shannon entropy | $\left(\frac{3\alpha+\theta}{\alpha+\theta}\right)-\ln\frac{\theta^2}{\alpha+\theta}-\frac{\theta^2}{\alpha+\theta}\sum_{j=1}^{\infty}(-1)^{j+1}\frac{(\alpha/2)^j}{j\theta^{j+1}}\left[\Gamma(2j+1)+\frac{\alpha}{2\theta}\Gamma(2j+3)\right]$ |

The main purpose of this study is to obtain the different estimators of the two-parameter Xgamma distribution parameters and to show how they behave at small, moderate and large sample sizes and different parameter values. In the recent, Sen et al. [1] studied the maximum likelihood (ML) estimators and the method of moments (MOM) estimators of two-parameter Xgamma distribution parameters. By this study, we discuss the ML and MOM estimators of the two-parameter Xgamma distribution, and also studied the least-squares (LS), the L-moments (L-MOM) and the maximum spacing (MSP) estimators of the unknown parameters of the two-parameter Xgamma distribution as the different estimators, which have not been studied yet.

The remaining sections of this study is organized as follows: in section 2, we study the different estimators for the unknown parameters of the two-parameter Xgamma distribution by employing the different methodologies such as the ML, the MOM, the LS, the L-MOM, and the MSP. Some numerical study results are provided in section 3 for comparing the estimation efficiencies of the estimators obtained in section 2. Finally, section 4 concludes the study.

## 2. Statistical Inference for Xgamma Parameters

In this section, we investigate the solution of the estimation problem for the Xgamma parameters $\alpha$ and $\theta$. In order to estimate the parameters $\alpha$ and $\theta$, we obtain several prominent estimators by using the different estimation methodologies such as the ML, MOM, LS, L-MOM, and MSP.

### 2.1. ML Estimation

Let $X_1, X_2, ..., X_n$ be a random sample taken from two-parameter Xgamma distribution with parameters $\alpha$ and $\theta$. By considering the pdf (1), the log-likelihood function for the sample $X_1, X_2, ..., X_n$ is easily written as

$$L(\alpha,\theta) = 2n\ln(\theta) - n\ln(\alpha+\theta) - \theta\sum_{j=1}^{n} x_j + \sum_{j=1}^{n} \ln\left(1 + \frac{1}{2}\alpha\theta x_j^2\right). \tag{5}$$

If we derive the log-likelihood function given by equation (5) with respect to parameters $\alpha$ and $\theta$, the two likelihood equations become

$$\frac{\partial L}{\partial \alpha}=-\frac{n}{\alpha+\theta}+\sum_{j=1}^{n}\frac{\theta x_j^2}{2\left(\frac{1}{2}\alpha\theta x_j^2+1\right)}=0, \tag{6}$$

and

$$\frac{\partial L}{\partial \theta}=\frac{2n}{\theta}-\frac{n}{\alpha+\theta}-\sum_{j=1}^{n}x_j+\sum_{j=1}^{n}\frac{\alpha x_j^2}{2\left(\frac{1}{2}\alpha\theta x_j^2+1\right)}=0. \tag{7}$$

Thus, the ML estimators of the parameters can be obtained from the solution of the nonlinear system given by equations (6) and (7). Unfortunately, explicit form of the ML estimators cannot be obtained from this system. However, nonlinear system given by equations (6) and (7) can be solved numerically by employing a numerical method such as Newton's method and we can obtain the ML estimates of the parameters. Now, let us investigate the ML estimates of the parameters $\alpha$ and $\theta$ employing the Newton's method.

The Newton's iterative formula is given by

$$\hat{\Lambda}_{j+1}=\hat{\Lambda}_j-H^{-1}\left(\hat{\Lambda}_j\right)\nabla\left(\hat{\Lambda}_j\right), \tag{8}$$

where $j$ is the iteration number, $\hat{\Lambda}_j$ is the estimation of parameter vector in the $j$th iteration, $\nabla\left(\hat{\Lambda}_j\right)$ is the corresponding gradient, $H\left(\hat{\Lambda}_j\right)$ is the corresponding Hessian matrix. Here, to obtain the numerical solutions of the likelihood equations given by equations (6) and (7), $\hat{\Lambda}_j$, $\nabla\left(\hat{\Lambda}_j\right)$ and $H\left(\hat{\Lambda}_j\right)$ are defined as

$$\hat{\Lambda}_j=\begin{bmatrix}\hat{\alpha}_j\\ \hat{\theta}_j\end{bmatrix}, \tag{9}$$

$$\nabla\left(\hat{\Lambda}_j\right)=\begin{bmatrix}\dfrac{\partial \ln L(\alpha,\theta)}{\partial \alpha}\\ \dfrac{\partial \ln L(\alpha,\theta)}{\partial \theta}\end{bmatrix}_{\alpha=\hat{\alpha}_j,\theta=\hat{\theta}_j} \tag{10}$$

and

$$H\left(\hat{\Lambda}_j\right)=\begin{bmatrix}\dfrac{\partial^2 \ln L(\alpha,\theta)}{\partial\alpha^2} & \dfrac{\partial^2 \ln L(\alpha,\theta)}{\partial\alpha\partial\theta}\\ \dfrac{\partial^2 \ln L(\alpha,\theta)}{\partial\alpha\partial\theta} & \dfrac{\partial^2 \ln L(\alpha,\theta)}{\partial\theta^2}\end{bmatrix}_{\alpha=\hat{\alpha}_j,\theta=\hat{\theta}_j}, \tag{11}$$

here, the elements of $H\left(\Lambda\right)$, say $h_{ij}$ $\left(i, j=1,2\right)$, can be easily obtained from the log-likelihood equation given by equation (5) as

$$h_{11}=\frac{\partial^2 \ln L}{\partial\alpha^2}=-\sum_{j=1}^{n}\left(\frac{\theta^2}{\left(\theta\alpha x_j^2+2\right)^2}x_j^4\right), \tag{12}$$

$$h_{12}=h_{21}=\frac{\partial^2 L}{\partial\alpha\partial\theta}=\frac{n}{\left(\theta+\alpha\right)^2}+\sum_{j=1}^{n}\frac{2}{\left(\theta\alpha x_j^2+2\right)^2}x_j^2 \tag{13}$$

and

$$h_{22}=\frac{\partial^2 \ln L}{\partial\theta^2}=-\frac{2n}{\theta^2}+\frac{n}{\left(\alpha+\theta\right)^2}-\sum_{j=1}^{n}\left(\frac{\alpha^2 x_j^4}{\left(\theta\alpha x_j^2+2\right)^2}\right). \tag{14}$$

Thus, by employing the iterative rule given by equation (8) with an initial estimation of the $\Lambda$ like $\hat{\Lambda}_0$, we can easily obtain the ML estimates of the parameters.

## 2.2. MOM Estimation

In this subsection, we discuss method of moments estimators of two-parameter Xgamma distribution. Let us assume that $X_1, X_2,\ldots, X_n$ be a random sample drawn from $TPXG(\alpha,\theta)$ distribution. If $X_i$, $\left(i=1,2,\ldots,n\right)$ random variables follow $TPXG(\alpha,\theta)$ distribution, by using the expression for the *r*-th moment of the two-parameter Xgamma distribution given in Table 1, the first and second population moments can easily be written as

$$\mu_1 = E(X) = \frac{\theta + 3\alpha}{\theta(\alpha + \theta)} \tag{15}$$

and

$$\mu_2 = E\left(X^2\right) = \frac{2(\theta + 6\alpha)}{\theta^2(\alpha + \theta)}, \tag{16}$$

respectively [1]. On the other hands, the first and second sample moments are described as

$$m_1 = \frac{1}{n}\sum_{i=1}^{n} x_i \tag{17}$$

and

$$m_2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 . \tag{18}$$

In a general point of view, the moment estimators are obtained by equating the first and second population moments to the corresponding sample moments for a family of distribution with two unknown parameters. Following the general definition of the method of moments estimator, we can write the following system.

$$\frac{\theta + 3\alpha}{\theta(\alpha + \theta)} - m_1 = 0 \tag{19}$$

$$\frac{2(\theta + 6\alpha)}{\theta^2(\alpha + \theta)} - m_2 = 0 \tag{20}$$

Hence, the MOM estimators of the parameter $\alpha$ and $\theta$, say $\hat{\alpha}_{MOM}$ and $\hat{\theta}_{MOM}$, respectively, are obtained as

$$\hat{\alpha}_{MOM} = \frac{\dfrac{2m_1^2\left(\sqrt{25m_1^2 - 12m_2} + 5m_1\right)}{m_2} - \dfrac{1}{2}\sqrt{25m_1^2 - 12m_2} - \dfrac{9m_1}{2}}{3m_2 - 4m_1^2} \tag{21}$$

and

$$\hat{\theta}_{MOM} = \frac{\sqrt{25m_1^2 - 12m_2} + 5m_1}{2m_2}. \tag{22}$$

### 2.3. LS estimation

The least squares estimators of the paramerers of $TPXG(\alpha,\theta)$ distribution are obtained in this subsection. Suppose $X_1, X_2, \ldots, X_n$ be a random sample from the $TPXG(\alpha,\theta)$ distribution and $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$ shows ordered measurements. By following these assumptions and the notations given by Swain et al. [2], the LS estimators of the parameters $\alpha$ and $\theta$, say $\hat{\alpha}_{LS}$ and $\hat{\theta}_{LS}$, respectively, can be easily obtained by minimizing

$$Q(\alpha,\beta,\theta,\lambda) = \sum_{i=1}^{n} (F(X_{(i)}) - \hat{F}(x_{(i)}))^2 \tag{23}$$

with respect to $\alpha$ and $\theta$.

By considering $F(x) = 1 - S(x)$, the quadratic form $Q(\alpha,\theta)$ can also be written as

$$Q(\alpha,\theta) = \sum_{i=1}^{n} \left( S(X_{(i)}) - \hat{S}(x_{(i)}) \right)^2. \tag{24}$$

where $\hat{S}(x_{(i)})$ can be estimated by

$$\hat{S}(x_{(i)}) = 1 - \frac{i}{n+1} = u_{(i)}, i = 1, 2, \ldots, n.. \tag{25}$$

By deriving the quadratic form $Q(\alpha,\theta)$ with respect to parameters and equated them to zero, we achieve the following nonlinear equations:

$$\frac{\partial Q(\alpha,\theta)}{\partial \alpha} = 2\sum_{i=1}^{n} \left[ S(x_{(i)}) - u_{(i)} \right] \frac{\partial S(x_{(i)})}{\partial \alpha} = 0 \tag{26}$$

$$\frac{\partial Q(\theta,\lambda)}{\partial \theta} = 2\sum_{i=1}^{n} \left[ S(x_{(i)}) - u_{(i)} \right] \frac{\partial S(x_{(i)})}{\partial \theta} = 0, \tag{27}$$

where

$$\frac{\partial S\left(x_{(i)}\right)}{\partial \alpha} = \frac{\theta^2 x e^{-\theta x}(\theta x + 2)}{2(\alpha+\theta)^2}, \tag{28}$$

$$\frac{\partial S\left(x_{(i)}\right)}{\partial \theta} = -\frac{\theta x e^{-\theta x}(2\theta + \alpha(\theta x(x(\alpha+\theta)+1)+4))}{2(\alpha+\theta)^2}. \tag{29}$$

Therefore, the least-square estimates $\hat{\alpha}_{LS}$ and $\hat{\theta}_{LS}$ are obtained from numerical solution of the equations (26) and (27).

### 2.4. L-MOM estimation

The L-moments estimation method is a more robust estimation technique than the method of moments. It was introduced using the linear combinations of the order statistics by Hosking [3]. The L-moment estimators are obtained by using the same main idea of the ordinary moments estimation methods, namely, by equating the sample L-moments with the population L-moments. In order to estimate the unknown parameters of the two-parameter Xgamma distribution according to the L-moments method, we need to the first two sample and population L-moments.

The first and the second sample L-moments are

$$l_1 = \frac{1}{n}\sum_{i=1}^{n} x_{(i)} \tag{30}$$

and

$$l_2 = \frac{2}{n(n-1)}\sum_{i=1}^{n}(i-1)\,x_i^2 - l_1, \tag{31}$$

respectively, see [3]. On the other hands, follows the definition of population L-moments given in [3], first two population L-moments, say $L_1$ and $L_2$, of the two-parameter Xgamma distribution are obtaine as

$$L_1 = E\left(X_{1:1}\right) = E\left(X\right) = \frac{3\alpha+\theta}{\alpha\theta+\theta^2} \tag{32}$$

and

$$L_2 = E\left(X_{2:2}\right) - E\left(X_{1:2}\right) = \frac{15\alpha^2 + 36\alpha\theta + 8\theta^2}{16\theta(\alpha+\theta)^2},. \tag{33}$$

respectively, here $E\left(X_{2:2}\right)$ denotes the *m*-th order statistic of a sample of size *n*. Therefore, the L-MOM estimators of the parameters $\alpha$ and $\theta$, say $\hat{\alpha}_{L-MOM}$ and $\hat{\theta}_{L-MOM}$, respectively, are obtained from solution of the equation system

$$L_1 - l_1 = 0. \tag{34}$$

$$L_2 - l_2 = 0 \tag{35}$$

as

$$\hat{\theta}_{L-MOM} = \frac{4\sqrt{51l_1^2 - 132l_1l_2 + 64l_2^2} + 33l_1 - 32l_2}{13l_1^2}. \tag{36}$$

and

$$\hat{\alpha}_{L-MOM} = \frac{l_1\left(896l_2 - 46\sqrt{51l_1^2 - 132l_1l_2 + 64l_2^2}\right) + 64l_2\left(\sqrt{51l_1^2 - 132l_1l_2 + 64l_2^2} - 8l_2\right) - 334l_1^2}{13l_1^2(5l_1 - 16l_2)}. \tag{37}$$

### 2.5. MSP estimation

In this subsection we study the maximum spacing estimator of the unkonown parameters $\alpha$ and $\theta$. Maximum spacing estimation method was introduced by Ranneby [4]. The method is also known as maximum product space estimation, see [5]. The MSP estimators have nice properties such as consistency and asymptotically unbiasedness.

Suppose that *X* is a random variable with the $TPXG\left(\alpha,\theta\right)$ distribution and $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$ shows ordered observations. By these assumptions, the MSP estimators of parameters $\alpha$ and $\theta$ are obtained maximizing

$$\sum_{j=1}^{n+1} \ln\left[F\left(x_{(j)},\alpha,\theta\right)-F\left(x_{(j-1)},\alpha,\theta\right)\right]$$
$$=\sum_{j=1}^{n+1} \ln\left[\frac{\left(\alpha+\theta+\alpha\theta x_{(j-1)}+\frac{1}{2}\alpha\theta^2 x_{(j-1)}^2\right)e^{-\theta x_{(j-1)}}}{\alpha+\theta}-\frac{\left(\alpha+\theta+\alpha\theta x_{(j)}+\frac{1}{2}\alpha\theta^2 x_{(j)}^2\right)e^{-\theta x_{(j)}}}{\alpha+\theta}\right]. \tag{38}$$

with respect to $\alpha$ and $\theta$, where $F(.,\alpha,\theta)$ is cdf of the two-parameter Xgamma distribution given by equation (2) and $F\left(x_{(0)},\alpha,\theta\right)=0$, and $F\left(x_{(n+1)},\alpha,\theta\right)=1$.

### 3. Simulation Study

Comparing the performances of the estimators studied in the previous section is theoretically a difficult task. Therefore, to compare the performance of these estimators, some Monte-Carlo simulation studies are carried out in this section. In the simulation studies, we consider the four cases of the parameter values:

- $\alpha=0.5$ and $\theta=0.5$
- $\alpha=0.5$ and $\theta=1.5$
- $\alpha=1.5$ and $\theta=0.5$
- $\alpha=1.5$ and $\theta=1.5$

Performances of the estimators are compared by using the bias and mean squared errors (MSE) criteria. For the different sample of sizes *n*=30,50,100 and 200, the simulated results by 1000 replicated simulations are given by Tables 2-5.

When the Monte-Carlo simulation study results given by Tables 2-5 are examined, it is seen that the estimation performance of all estimators is quite satisfactory. As the sample size *n* increases, both bias and MSE values of all estimators decrease. Therefore, it can be said that these estimators are asymptotically unbiased and consistent. In addition, we can conclude that MSP estimators have outperformed the other estimators with smaller bias and MSE values in estimating the parameter $\alpha$, and also ML estimators have outperformed the other estimators with smaller bias and MSE values in estimating the parameter $\theta$.

Table 1: Simulated results for $\alpha = 0.5$ and $\theta = 0.5$

| $n$ | Method | $\hat{\alpha}$ | Bias($\hat{\alpha}$) | MSE($\hat{\alpha}$) | $\hat{\theta}$ | Bias($\hat{\theta}$) | MSE($\hat{\theta}$) |
|---|---|---|---|---|---|---|---|
| 30 | ML | 0.6247 | 0.1247 | 0.1622 | 0.5059 | 0.0059 | 0.0061 |
| | LS | 0.5866 | 0.0866 | 0.1680 | 0.4945 | -0.0055 | 0.0066 |
| | MOM | 0.6668 | 0.1668 | 0.2194 | 0.5112 | 0.0112 | 0.0061 |
| | L-MOM | 0.5835 | 0.0835 | 0.1429 | 0.4993 | -0.0007 | 0.0056 |
| | MSP | 0.5113 | 0.0113 | 0.1210 | 0.4713 | -0.0287 | 0.0104 |
| 50 | ML | 0.5462 | 0.0462 | 0.0841 | 0.4945 | -0.0055 | 0.0050 |
| | LS | 0.5242 | 0.0242 | 0.1056 | 0.4873 | -0.0127 | 0.0049 |
| | MOM | 0.5840 | 0.0840 | 0.1185 | 0.5018 | 0.0018 | 0.0044 |
| | L-MOM | 0.5163 | 0.0163 | 0.0787 | 0.4907 | -0.0093 | 0.0040 |
| | MSP | 0.4502 | -0.0498 | 0.0751 | 0.4625 | -0.0375 | 0.0096 |
| 100 | ML | 0.5473 | 0.0473 | 0.0530 | 0.5027 | 0.0027 | 0.0029 |
| | LS | 0.5209 | 0.0209 | 0.0582 | 0.4940 | -0.0060 | 0.0038 |
| | MOM | 0.5868 | 0.0868 | 0.0856 | 0.5079 | 0.0079 | 0.0030 |
| | L-MOM | 0.5288 | 0.0288 | 0.0517 | 0.4995 | -0.0005 | 0.0027 |
| | MSP | 0.4786 | -0.0214 | 0.0499 | 0.4809 | -0.0191 | 0.0050 |
| 200 | ML | 0.5328 | 0.0328 | 0.0274 | 0.5013 | 0.0013 | 0.0013 |
| | LS | 0.5227 | 0.0227 | 0.0374 | 0.4961 | -0.0039 | 0.0021 |
| | MOM | 0.5440 | 0.0440 | 0.0347 | 0.5028 | 0.0028 | 0.0015 |
| | L-MOM | 0.5196 | 0.0196 | 0.0263 | 0.4987 | -0.0013 | 0.0013 |
| | MSP | 0.4906 | -0.0094 | 0.0247 | 0.4896 | -0.0104 | 0.0017 |

Table 2: Simulated results for $\alpha = 0.5$ and $\theta = 1.5$

| $n$ | Method | $\hat{\alpha}$ | Bias($\hat{\alpha}$) | MSE($\hat{\alpha}$) | $\hat{\theta}$ | Bias($\hat{\theta}$) | MSE($\hat{\theta}$) |
|---|---|---|---|---|---|---|---|
| 30 | ML | 1.0609 | 0.5609 | 1.9773 | 1.6051 | 0.1051 | 0.1972 |
| | LS | 1.0578 | 0.5578 | 2.1876 | 1.6290 | 0.1290 | 0.1798 |
| | MOM | 1.1094 | 0.6094 | 2.5756 | 1.6047 | 0.1047 | 0.2271 |
| | L-MOM | 0.8216 | 0.3216 | 1.5337 | 1.5435 | 0.0435 | 0.1795 |
| | MSP | 0.6343 | 0.1343 | 0.9498 | 1.3872 | -0.1128 | 0.1989 |
| 50 | ML | 0.7139 | 0.2139 | 0.5091 | 1.5280 | 0.0280 | 0.0958 |
| | LS | 0.7630 | 0.2630 | 0.7315 | 1.5527 | 0.0527 | 0.1225 |
| | MOM | 0.7338 | 0.2338 | 0.5581 | 1.5411 | 0.0411 | 0.1083 |
| | L-MOM | 0.6281 | 0.1281 | 0.5022 | 1.5146 | 0.0146 | 0.0989 |
| | MSP | 0.4861 | -0.0139 | 0.3546 | 1.3757 | -0.1243 | 0.1278 |
| 100 | ML | 0.7019 | 0.2019 | 0.3416 | 1.5355 | 0.0355 | 0.0789 |
| | LS | 0.7627 | 0.2627 | 0.5305 | 1.5651 | 0.0651 | 0.1023 |
| | MOM | 0.7138 | 0.2138 | 0.3628 | 1.5468 | 0.0468 | 0.0794 |
| | L-MOM | 0.6436 | 0.1436 | 0.3549 | 1.5248 | 0.0248 | 0.0754 |
| | MSP | 0.5197 | 0.0197 | 0.2559 | 1.4098 | -0.0902 | 0.0993 |
| 200 | ML | 0.5886 | 0.0886 | 0.1706 | 1.5005 | 0.0005 | 0.0571 |
| | LS | 0.6798 | 0.1798 | 0.3060 | 1.5547 | 0.0547 | 0.0715 |
| | MOM | 0.5844 | 0.0844 | 0.1889 | 1.5026 | 0.0026 | 0.0548 |
| | L-MOM | 0.5441 | 0.0441 | 0.2005 | 1.4924 | -0.0076 | 0.0520 |
| | MSP | 0.4566 | -0.0434 | 0.1581 | 1.4001 | -0.0999 | 0.0856 |

Table 3: Simulated results for $\alpha = 1.5$ and $\theta = 0.5$

| $n$ | Method | $\hat{\alpha}$ | Bias($\hat{\alpha}$) | MSE($\hat{\alpha}$) | $\hat{\theta}$ | Bias($\hat{\theta}$) | MSE($\hat{\theta}$) |
|---|---|---|---|---|---|---|---|
| 30 | ML | 1.9189 | 0.4189 | 1.6386 | 0.5042 | 0.0042 | 0.0020 |
| | LS | 1.7951 | 0.2951 | 1.7592 | 0.4934 | -0.0066 | 0.0023 |
| | MOM | 1.9673 | 0.4673 | 2.8649 | 0.5115 | 0.0115 | 0.0028 |
| | L-MOM | 1.9405 | 0.4405 | 2.7474 | 0.5034 | 0.0034 | 0.0022 |
| | MSP | 1.6035 | 0.1035 | 0.8165 | 0.4890 | -0.0110 | 0.0021 |
| 50 | ML | 1.7965 | 0.2965 | 0.9965 | 0.5026 | 0.0026 | 0.0018 |
| | LS | 1.8369 | 0.3369 | 2.5179 | 0.4968 | -0.0032 | 0.0021 |
| | MOM | 1.9468 | 0.4468 | 2.8683 | 0.5065 | 0.0065 | 0.0024 |
| | L-MOM | 1.9283 | 0.4283 | 2.7319 | 0.5007 | 0.0007 | 0.0021 |
| | MSP | 1.5421 | 0.0421 | 0.6190 | 0.4884 | -0.0116 | 0.0019 |
| 100 | ML | 1.6201 | 0.1201 | 0.4227 | 0.4996 | -0.0004 | 0.0011 |
| | LS | 1.5743 | 0.0743 | 0.4824 | 0.4950 | -0.0050 | 0.0014 |
| | MOM | 1.7389 | 0.2389 | 1.7457 | 0.5021 | 0.0021 | 0.0014 |
| | L-MOM | 1.6555 | 0.1555 | 0.8300 | 0.4981 | -0.0019 | 0.0012 |
| | MSP | 1.4624 | -0.0376 | 0.3213 | 0.4895 | -0.0105 | 0.0013 |
| 200 | ML | 1.6193 | 0.1193 | 0.2932 | 0.5015 | 0.0015 | 0.0007 |
| | LS | 1.5831 | 0.0831 | 0.2996 | 0.4989 | -0.0011 | 0.0008 |
| | MOM | 1.7663 | 0.2663 | 0.9965 | 0.5024 | 0.0024 | 0.0009 |
| | L-MOM | 1.6313 | 0.1313 | 0.4485 | 0.5005 | 0.0005 | 0.0008 |
| | MSP | 1.5206 | 0.0206 | 0.2388 | 0.4951 | -0.0049 | 0.0007 |

Table 4: Simulated results for $\alpha = 1.5$ and $\theta = 1.5$

| $n$ | Method | $\hat{\alpha}$ | Bias($\hat{\alpha}$) | MSE($\hat{\alpha}$) | $\hat{\theta}$ | Bias($\hat{\theta}$) | MSE($\hat{\theta}$) |
|---|---|---|---|---|---|---|---|
| 30 | ML | 1.9054 | 0.4054 | 2.6984 | 1.4949 | -0.0051 | 0.0791 |
| | LS | 1.6128 | 0.1128 | 2.0198 | 1.4225 | -0.0775 | 0.0915 |
| | MOM | 2.2990 | 0.7990 | 4.6030 | 1.5404 | 0.0404 | 0.0824 |
| | L-MOM | 1.7124 | 0.2124 | 2.0507 | 1.4588 | -0.0412 | 0.0832 |
| | MSP | 1.3820 | -0.1180 | 1.6929 | 1.3395 | -0.1605 | 0.1421 |
| 50 | ML | 1.8169 | 0.3169 | 1.2603 | 1.5132 | 0.0132 | 0.0416 |
| | LS | 1.6869 | 0.1869 | 1.1905 | 1.4768 | -0.0232 | 0.0551 |
| | MOM | 1.9599 | 0.4599 | 1.9271 | 1.5311 | 0.0311 | 0.0406 |
| | L-MOM | 1.7024 | 0.2024 | 1.1074 | 1.4930 | -0.0070 | 0.0400 |
| | MSP | 1.4843 | -0.0157 | 1.0452 | 1.4074 | -0.0926 | 0.0867 |
| 100 | ML | 1.6651 | 0.1651 | 0.6046 | 1.4973 | -0.0027 | 0.0317 |
| | LS | 1.6055 | 0.1055 | 0.6415 | 1.4779 | -0.0221 | 0.0367 |
| | MOM | 1.7493 | 0.2493 | 0.7669 | 1.5132 | 0.0132 | 0.0282 |
| | L-MOM | 1.5964 | 0.0964 | 0.5600 | 1.4863 | -0.0137 | 0.0277 |
| | MSP | 1.4621 | -0.0379 | 0.5220 | 1.4368 | -0.0632 | 0.0453 |
| 200 | ML | 1.6682 | 0.1682 | 0.3592 | 1.5124 | 0.0124 | 0.0144 |
| | LS | 1.5996 | 0.0996 | 0.3727 | 1.4896 | -0.0104 | 0.0200 |
| | MOM | 1.7539 | 0.2539 | 0.5085 | 1.5245 | 0.0245 | 0.0163 |
| | L-MOM | 1.6478 | 0.1478 | 0.3700 | 1.5076 | 0.0076 | 0.0149 |
| | MSP | 1.5410 | 0.0410 | 0.3137 | 1.4775 | -0.0225 | 0.0173 |

## 4. Conclusion

In this study, we have considered the two-parameter Xgamma distribution and estimation problem of its parameters. The Xgamma distribution introduced by Sen et al.[A] is a relatively new distribution in the vast literature of lifetime distributions, which have nice features. To estimate the unknown parameters of the two-parameter Xgamma distribution, we have obtained several estimators such as the ML, MOM, LS, L-MOM, and MSP. By through the different sample of sizes small, moderate, and large, the estimation performances of these estimators are demonstrated by a series of Monte-Carlo simulation studies, where estimators have been compared regarding biases and MSE. Performed Monte-Carlo simulations have shown that all estimators provide satisfactory estimation performances in all sample sizes. In addition, ML and MSP estimators have provided the best performances in estimating the parameters $\theta$ and $\alpha$, respectively. Therefore, it can be recommended that use of the MSP or ML estimators in practical applications.

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# CONCAVE OPERATORS VIA FRACTIONAL DIFFERENTIAL EQUATIONS

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**Abstract**
It is well known that a concave operator is important for studying positive solutions of nonlinear differential and integral equations. In this paper, we study a class of mixed monotone operators with convexity, also the properties of monotone iterative technique in ordered Banca spaces, some new existence and uniqueness theorems by fixed points of operators are investigated. Finally, as applications, we apply the results obtained in this paper to study the existence and uniqueness of positive solutions for nonlinear fractional differential equation boundary value problems.

**Keywords**: Fixed point; normal cone; positive solution; fractional differential equation.

## 1. Introduction

Suppose $(E, \|.\|)$ is a Banach space which is partially ordered by a cone $P \subseteq E$, that is, $x \leq y$ if and only if $y - x \in P$. We denote the zero element of $E$ by $\theta$. Recall that a non-empty closed convex set $P \subset E$ is a cone if it satisfies$(i)$ $x \in P, \lambda \geq 0 \Rightarrow \lambda x \in P$; $(ii)$ $x \in P, -x \in P \Rightarrow x = \theta$. $A$ cone $P$ is called normal if there exists a constant $N > 0$ such that $\theta \leq x \leq y$ implies $\| x \| \leq N \| y \|$. For all $x, y \in E$, the notation $x \sim y$ means that there exist $\lambda > 0$ and $\mu > 0$, such that $\lambda x \leq y \leq \mu x$. Clearly, $\sim$ is an equivalent relation. Given $e > \theta$, we denote by $Pe$ the set $P_e = \{x \in E | x \sim e\}$. It is easy to see that $P_e \subset P$ is convex and $\lambda P_e = P_e$ for all $\lambda > 0$. If $P \neq \phi$ and $e \in P$, it is clear that $P_e = P$.
Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on $X$ such that $(X, d)$ is a complete metric space.

**Definition 1.1.** [2] Let $(X, \leq)$ be a partially ordered set and $F : X \times X \times X \to X$. We say $F$ has the mixed monotone property if for any $x, y, z \in X$,

$$x_1, x_2 \in X, x_1 \leq x_2 \; implies \; F(x_1, y, z) \leq F(x_2, y, z),$$

$$y_1, y_2 \in X, y_1 \leq y_2 \; implies \; F(x, y_1, z) \geq F(x, y_2, z),$$

$$z_1, z_2 \in X, z_1 \leq z_2 \; implies \; F(x, y, z_1) \leq F(x, y, z_2).$$

**Definition 1.2.** [2] An element $(x, y, z) \in X \times X \times X$ is called a tripled fixed point of a mapping $F : X \times X \times X \to X$ if $F(x, y, z) = x, F(y, x, y) = y$ and $F(z, y, x) = z$.

## 2. Main Results

**Definition 2.1.** We say an operator $A : P_e \times P_e \times P_e \to P_e$ is an e-concave-convex operator if there exists one positive function $\eta(t, u, v, w)$ such that

(1) $$A\left(tu, \frac{1}{t}v, tw\right) \geq t\big(1 + \eta(t,u,v,w)\big)A(u,v,w), \forall\, u,v,w \in P_e, 0 < t < 1.$$

**Theorem 2.2.** Let $P$ be a normal cone of $E$, and let $A : P_e \times P_e \times P_e \to P_e$ be a mixed monotone and e-concave-convex operator. In addition, suppose that there exist $u_0, v_0, \xi_0 \in P_e, u_0 \leq v_0, \xi_0 \leq v_0$ (since $u_0, v_0, \xi_0 \in P_e$, we can choose a sufficiently small $\epsilon \in (0,1)$ such that $u_0 \geq \epsilon v_0, \xi_0 \geq \epsilon v_0$) such that

$$(H_1)\ u_0 \leq A(u_0, v_0, u_0), A(v_0, u_0, v_0) \leq v_0;$$

$$(H_2)\ \forall t \in (0,1), \overline{\lim_{n\to+\infty}}\, n\eta(t, v_n, u_n, v_n) \geq \frac{1}{\epsilon} - 1,$$

hold, where

$$u_n = A(u_{n-1}, v_{n-1}, \xi_{n-1}), v_n = A(v_{n-1}, u_{n-1}, v_{n-1}), \xi_n = A(\xi_{n-1}, v_{n-1}, u_{n-1}), n = 1, 2, \ldots.$$

Then $A$ has exactly one fixed point $x_n$ in $P_e$. Moreover, constructing successively the sequences

$$x_n = A(x_{n-1}, y_{n-1}, z_{n-1}),\ y_n = A(y_{n-1}, x_{n-1}, y_{n-1}), z_n = A(z_{n-1}, y_{n-1}, x_{n-1}), n = 1, 2, \ldots,$$

for any initial values $x_0, y_0, z_0 \in P_e$, we have,

(2) $$\| x_n - u^* \| \to 0, \| y_n - u^* \| \to 0, \| z_n - u^* \| \to 0 \ as\ n \to \infty.$$

## 3. Application

We study the existence and uniqueness of a solution for the following fractional differential equation

$$\frac{D_v}{Dt}u(r,s,t) + f(r,s,t,u(r,s,t)) = 0,$$

(3) $$(0 < \epsilon < T, T \geq 1, t \in [\epsilon, T], 0 < v < 1, s \in [a,b], t \in [c,d])$$

subject to condition

(4) $$u(r,s,\zeta) = u(r,s,T), (r,s,\zeta) \in [a,b] \times [c,d] \times (\epsilon, t),$$

where $D^v$ is the Riemann-Liouville fractional derivative of order $v$ and $a, b, c, d \in (0, \infty)$ with $a < b$ and $c < d$. Let

$$E = C([a,b] \times [c,d] \times [\epsilon, T]).$$

Consider the Banach space of continuous functions on $[a,b] \times [c,d] \times [\epsilon, T]$ with sup norm and set

$$P = \{y \in C([a,b] \times [c,d] \times [\epsilon,T]) : \min_{(r,s,t)\in[a,b]\times[c,d]\times[\epsilon,T]} y(r,s,t) \geq 0\}.$$

Then $P$ is a normal cone.

**Lemma 3.1.** [1] Let $(r,s,t) \in [a,b] \times [c,d] \times [\epsilon,T], (r,s,\zeta) \in [a,b] \times [c,d] \times (\epsilon,t)$ and $0 < \alpha < 1$, then the problem

$$\frac{D^{\nu}}{Dt} u(r,s,t) + f(r,s,t,u(r,s,t)) = 0$$

with the boundary value condition $u(r,s,\zeta) = u(r,s,T)$ has a solution $u_0$ if and only if $u_0$ is a solution of the fractional integral equation

$$u(r,s,t) = \int_{\epsilon}^{T} G(t,\xi) f(r,s,\xi,u(r,s,\xi)) d\xi,$$

Where

$$G(t,\xi) = \begin{cases} \dfrac{t^{\upsilon-1}(\zeta - \xi)^{\upsilon-1} - t^{\upsilon-1}(T - \xi)^{\upsilon-1}}{(\zeta^{\upsilon-1} - T^{\upsilon-1})\Gamma(\nu)} - \dfrac{(t - \xi)^{\upsilon-1}}{\Gamma(\nu)}, & \epsilon \leq \xi \leq \zeta \leq t \leq T, \\ \dfrac{-t^{\upsilon-1}(T - \xi)^{\upsilon-1}}{(\zeta^{\upsilon-1} - T^{\upsilon-1})\Gamma(\nu)} - \dfrac{(t - \xi)^{\upsilon-1}}{\Gamma(\nu)}, & \epsilon \leq \zeta \leq \xi \leq t \leq T, \\ \dfrac{-t^{\upsilon-1}(T - \xi)^{\upsilon-1}}{(\zeta^{\upsilon-1} - T^{\upsilon-1})\Gamma(\nu)}, & \epsilon \leq \zeta \leq t \leq \xi \leq T. \end{cases}$$

**Theorem 3.2.** Let $0 < \epsilon < T$ be given and

$f(r,s,t,u,v,\eta) \in C([a,b],[c,d],[\epsilon,T],[0,\infty],[0,\infty],[0,\infty])$ is increasing in $u, w$

and decreasing in v and for $l \in (0,1), (r,s,t) \in [a,b] \times [c,d] \times [\epsilon,T]$, there exists $\varphi(l,s,r,t) \in (1,\infty)$ such that

$$f(r,s,t,lu(r,s,t),\frac{1}{l}v(r,s,t),l\eta(r,s,t))) \geq l(1+\varphi(l,r,s,t))f(r,s,t,u(r,s,t),v(r,s,t),\eta(r,s,t))$$

and $f(r,s,t,u(r,s,t),v(r,s,t),\eta(r,s,t)) = 0$ whenever $G(s,t) < 0$. Suppose that there exist $u_0,\ v_0,\eta_0 \in P, r \in (0,1)$ such that

$$u_0(r,s,t) \leq v_0(r,s,t)$$

$$\int_{\epsilon}^{T} G(t,\xi) f(r,s,\xi,u_{\mathrm{n}}(r,s,\xi),\ v_0(r,s,\xi),u_0(r,s,\xi)) d\xi \geq u_0(r,s,t),$$

$$\int_{\epsilon}^{T} G(t,\xi)f(r,s,\xi,\ v_0(r,s,\xi),u_0(r,s,\xi),\ v_0(r,s,\xi))d\xi\ \geq\ v_0(r,s,t),$$

for $(r,s,t)\ \in\ ([a,b]\ \times\ [c,d]\ \times\ [\epsilon,T])$, where $u_0,\eta_0$ are comparable. Also suppose

$$\forall t\ \in\ (0,1),\qquad \overline{\lim_{n\to+\infty}} n\varphi(t,v_n,u_n,v_n) \geq \frac{1}{\epsilon} - 1$$

Then the problem (3) with the boundary condition (4) has a unique solution in $P$. Moreover, for the sequences

$$u_{n+1}\ =\ \int_{\epsilon}^{T} G(t,\xi)f(r,s,\xi,u_n(r,s,\xi),v_n(r,s,\xi),\eta_n(r,s,\xi))d\xi$$

$$v_{n+1}\ =\ \int_{\epsilon}^{T} G(t,\xi)f(r,s,\xi,v_n(r,s,\xi),u_n(r,s,\xi),v_n(r,s,\xi))d\xi$$

$$\eta_{n+1}\ =\ \int_{\epsilon}^{T} G(t,\xi)f(r,s,\xi,\eta_n(r,s,\xi),v_n(r,s,\xi),u_n(r,s,\xi))d\xi$$

we have $\| u_n - u^* \| \to 0, \| v_n - u^* \| \to 0, \| \eta_n - u^* \| \to 0.$

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# A TOPOLOGICAL PROPERTY OF THE COMMON FIXED POINTS SET OF MULTI-FUNCTIONS IN B-METRIC SPACES

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**Abstract**
In 2008, Sintamarian has proved some results on absolute retractivity of the common fixed points set of two multi-functions. On the other hand, Suzuki generalized the notion of contractive mappings in 2008. Since then, there has been a lot of activity and a number of results have appeared on Suzuki's method for mappings and multi-functions. Recently Mirmostafaee has established set-valued version of Suzuki's fixed point theorem when the underling space is a complete b-metric space. In this paper, by using the upper methods, we study some new consequences on absolute retractivity of the common fixed points set of multi-valued Suzuki type contractions.

**Keywords**: Absolute retract, Fixed point, Suzuki contractive multifunction. 2010 Mathematics Subject Classification: 74H10, 54H25.

## 1. Introduction

In 2008, Sintamarian proved some results on absolute retractivity of the common fixed points set of two multi-valued operators ([3]). On the other hand, Suzuki generalized the notion of contractive mappings in [4]. In 2011, Aleomraninejad et al.developed a new method to prove Suzukis fixed point theorem for set-valued mapping([1]). The method was extended by Yingtaweesittikul for set-valued functions in general b-metric spaces ([5]). In this paper, we investigate some new conclusions on absolute retractivity of the fixed points set of Suzuki type contractive multifunctions.
Let $X$ be a nonempty set. We denote by $P(X)$ the set of all nonempty subsets of $X$, i.e. $\{P(X) = Y \mid \emptyset \neq Y \subseteq X\}$. Let $F : X \to P(X)$ be a multi-valued operator.We denote by $\mathcal{F}_F$ the fixed points set of $F$, i.e. $\mathcal{F}_F = \{x \in X | x \in F(x)\}$.
Let $F_1, F_2 \colon X \to P(X)$ be two multi-valued operators. We denote by $(\mathcal{CF})_{F_1,F_2}$ the common fixed points set of $F_1$and $F_2$, i.e. $(\mathcal{CF})_{F_1,F_2} = \{x \in X | x \in F_1(x) \cap F_2(x)\}$. Let $X$ and $Y$ be two nonempty sets, $P(Y)$ the set of all nonempty subsets of $Y$ and $F : X \to P(Y)$ a multifunction. $A$ mapping $\varphi : X \to Y$ is called a selection of $F$ whenever $\varphi(x) \in Fx$ for all $x \in X$. Throughout the paper, we denote the set of all nonempty closed and bounded subsets of $X$ by $P_{b,cl}(X)$ when $X$ is a metric space. $B(x_0, r) = \{x \in X : d(x_0, x) < r\}$. For $x \in X$ and $A, B \subseteq X$, set $d(x, A) = inf_{y\in A}\, d(x, y)$ and

$$H(A,B) = max\{\sup_{x\in A} d(x,B)\ , \sup_{y\in B} d(y,A)\}.$$

It is known that $H$ is a metric on closed bounded subsets of $X$ which is called the Hausdorff metric (for more details see [3]). We say that a topological space $X$ is an absolute retract for metric spaces whenever for each metric space $Y$ , $A \in P_{cl}(Y)$ and continuous function $\psi : A \to X$, there exists a continuous function $\varphi : Y \to X$ such that $\varphi|_A = \psi$.
Let $\mathcal{M}$ be the set of all metric spaces, $X \in \mathcal{M}$, $\mathcal{D} \in P(\mathcal{M})$ and $F : X \to P_{b,cl}(X)$ a lower semi-continuous multifunction. We say that $F$ has the selection property with respect to $\mathcal{D}$ if for each $Y \in \mathcal{D}$, continuous function f : Y → X and continuous functional $g : Y \to (0, \infty)$ such that $G(y) := \overline{F(f(y)) \cap B(f(y), g(y))} \neq \emptyset$ for all $y \in Y$ , $A \in P_{cl}(Y)$, every continuous selection $\psi : A \to X$ of $G|_A$ admits a continuous extension $\varphi : Y \to X$, which is a selection of $G$. If $\mathcal{D} = \mathcal{M}$, then we say that $F$ has the selection property and we denote this by $F \in S_p(X)$. Define $\theta : [0,1) \to (\frac{1}{2}, 1]$ as

$$\theta(r) = \begin{cases} 1 & if \ \ 0 \le r \le \dfrac{\sqrt{5}-1}{2}, \\ (1-r)r^{-2} & if \ \ \dfrac{\sqrt{5}-1}{2} \le r \le \dfrac{1}{\sqrt{2}}, \\ (1+r)^{-1} & if \ \ \dfrac{1}{\sqrt{2}} \le r \le 1. \end{cases}$$

The function $\theta$ is non-increasing.
We consider a continuous function $e : [0, \infty)^5 \to [0, \infty)$ satisfying the following conditions:

a) $e(1,1,1,2,0) = e(1,1,1,0,2) = h \in (0,1)$,

b) $e$ is sub-homogeneous, that is, $e(\gamma x_1, \gamma x_2, \gamma x_3, \gamma x_4, \gamma x_5) \le \gamma e(x_1, x_2, x_3, x_4, x_5)$ for all $\gamma \ge 0$ and all $(x_1, x_2, x_3, x_4, x_5) \in [0, \infty)^5$,

c) If $x_i$, $y_i \in [0, \infty)$ and $xi < yi$ for $i = 1, \dots, 4$, then

$e(x_1, x_2, x_3, x_4, 0) < e(y_1, y_2, y_3, y_4, 0)$ and $e(x1, x2, x3, 0, x4) < e(y_1, y_2, y_3, 0, y_4)$ (see [1]).

In this case, we write $e \in \mathcal{R}$. We appeal the following results in the sequel.

**Proposition 1.1.** [1] If $e \in R$ and $u, v \in [0, \infty)$ are such that

$$u \le max\{e(v, v, u, v + u, 0), e(v, u, v, v + u, 0), e(v, u, v, 0, v + u)\},$$

then $u \le hv$.

**Theorem 1.2.** [1] Let $(X, d)$ be a complete metric space and, $F_1, F_2 : X \to P_{b,cl}(X)$ a multi-functions. Suppose that there exist $\gamma \in (0, \infty)$ and $e \in \mathcal{R}$ such that $\gamma(h + 1) \le 1$ and $\gamma d(x, F_1 x) \le d(x, y)$ or $\gamma d(x, F_2 x) \le d(x, y)$ implies

$$H(F_1 x, F_2 y) \le e(d(x, y), d(x, F_1 x), d(y, F_2 y), d(x, F_2 y), d(y, F_1 x))$$

for all $x, y \in X$. Then $\mathcal{F}(F_1) = \mathcal{F}(F_2)$ and $(\mathcal{CF})_{F_1,F_2}$ is non-empty.

**Theorem 1.3.** [1] Let (X, d) be a complete metric space, $F : X \to P_{b,cl}(X)$ a multifunction. Assume that there exists $r \in [2^{\frac{-1}{2}}, 1)$ such that $\theta(r)d(x, Fx) \leq d(x, y)$ implies $H(Fx, Fy) \leq r\, max\{d(x, y), d(x, Fx), d(y, Fy)\}$ for all $x, y \in X$. Then $\mathcal{F}_F$ is non-empty.

**Theorem 1.4.** [1] Let $(X, d)$ be a complete metric space, $F : X \to P_{b,cl}(X)$ a multifunction. Assume that there exists $\beta, \gamma \in [0,1)$ such that $\frac{1}{2\beta+\gamma+1}d(x, Fx) \leq d(x, y)$ implies $H(Fx, Fy) \leq \gamma d(x, y) + \beta d(x, Fx) + \beta d(y, Fy)$ for all $x, y \in X$. Then $\mathcal{F}_F$ is non-empty.

In the all over this paper let $\Psi$ be set of all increasing and continuous functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following property: $\psi(ct) \leq c\psi(t)$ for all $c > 1$ and $\psi(0) = 0$. We denote by $\Theta$ the family of all increasing functions $\beta : [0, \infty) \to (0,1)$.

**Definition 1.5.** Let $F : X \to P_{b,cl}(X)$ be a multivalued mapping and $\alpha : X \times X \to [0, \infty)$ be a given function. Then $F$ is said to be $\alpha$-admissible if

$(T3)$ $\alpha(x, y) \geq 1$ for all $y \in Fx, \Rightarrow \alpha(y, z) \geq 1$, for all $z \in Fy$.

**Definition 1.6.** Let $(X, d)$ be a b-metric space and $F : X \to P_{b,cl}(X)$ be a multi-valued mapping. We say that F is an $\alpha - \psi$-Suzuki-Geraghty multi-valued type contraction whenever there exist $\alpha : X \times X \to [0, \infty), a \in [0,1)$ such that

$$\eta(a)D(x, F(x)) \leq d(x, y) \Rightarrow \psi(s^3 d(Tx, Ty)) \leq \beta(\psi(d(x, y)))\psi(d(x, y)),$$

for all $x, y \in X$, where $\eta(a) = \frac{1}{1+a}$ $, \beta \in \Theta\ and\ \psi \in \Psi$.

## 2. Main Results

**Theorem 2.1.** Let $(X, d)$ be a complete b-metric space and absolute retract for b-metric spaces, $F : X \to P_{b,cl}(X)$ be an $\alpha - \psi$-Suzuki-Geraghty multi-valued type contraction, $F$ is continuous and $F \in SP(X)$. If $\alpha(x, y) \geq 1$ for all $x \in X$ and $y \in F(x)$, then $\mathcal{F}_F$ is an absolute retract for b-metric spaces.

**Corollary 2.2.** Let $(X, d)$ be a complete b-metric space and absolute retract for b-metric spaces, $f : X \to X$ an $\alpha - \psi$-Suzuki-Geraghty type contraction, f is continuous and $f \in SP(X)$. If $\alpha(x, y) \geq 1$ for all $x \in X$ and $y = f(x)$, then $\mathcal{F}_f$ is an absolute retract for b-metric spaces.

**Theorem 2.3.** Let $(X, d)$ be a complete metric space and absolute retract for metric spaces, $F_1, F_2 : X \to P_{b,cl}(X)$ are multi-functions and $F_1, F_2 \in Sp(X)$. Suppose that there exist $\gamma \in (0, \infty)$ and $e \in \mathcal{R}$ such that $\gamma(h + 1) \leq 1$ and the conditions $\gamma d(x, F1x) \leq d(x, y)$ and $\gamma d(x, F_2 x) \leq d(x, y)$ implies

$$H(F_1(x), F_2(y)) \leq e(d(x, y), d(x, F_1(x)), d(y, F_2(y)), d(x, F_2(y)), d(y, F_1(x))),$$

for all $x, y \in X$. Then$(\mathcal{CF})_{F_1,F_2}$ is an absolute retract for metric spaces.

**Theorem 2.4.** Let $(X, d)$ be a complete metric space and absolute retract for metric spaces, $F : X \to P_{b,cl}(X)$ a multifunction and $F \in Sp(X)$. Suppose that there exist $\gamma \in (0, \infty)$ and $e \in \mathcal{R}$ such that $\gamma(h + 1) \leq 1$ and $\gamma d(x, Fx) \leq d(x, y)$ implies

$$H(Fx, Fy) \leq e(d(x, y), d(x, Fx), d(y, Fy), d(x, Fy), d(y, Fx))$$

for all $x, y \in X$. Then $\mathcal{F}_F$ is an absolute retract for metric spaces.

**Corollary 2.5.** Let $(X, d)$ be a complete metric space and absolute retract for metric spaces, $F : X \to P_{b,cl}(X)$ a multifunction and $F \in Sp(X)$. Assume that there exists $r \in [2^{\frac{-1}{2}}, 1)$ such that $\theta(r)d(x, Fx) \leq d(x, y)$ implies $H(Fx, Fy) \leq r\, max\{d(x, y), d(x, Fx), d(y, Fy)\}$ for all $x, y \in X$. Then $\mathcal{F}_F$ is an absolute retract for metric spaces.

**Theorem 2.6.** [2] Let $X$ be a complete b-metric space and let $T, S : X \to P_{b,cl}(X)$ be two set-valued functions and there exist some $\lambda \in (0, 1)$ and $e \in \mathcal{R}$ such that $2\lambda(1 + h) < 1$ and $\lambda d(x, Tx) \leq \lambda d(x, y)$ or $\lambda d(x, Sx) \leq d(x, y)$ implies that

$$H(Tx, Sy) \leq e(d(x, y), d(x, Tx), d(y, Sy), d(x, Sy), d(y, Tx), \qquad (x, y \in X).$$

Then $F(T) = F(S)$ and $F(T)$ is non-empty.

**Theorem 2.7.** Let X be a complete b-metric space and absolute retract for metric spaces, $F : X \to P_{b,cl}(X)$ a multifunction and $F \in Sp(X)$. let $T, S : X \to P_{b,cl}(X)$ be two set-valued functions and there exist some $\lambda \in (0, 1)$ and $e \in \mathcal{R}$ such that $2\lambda(1 + h) < 1$ and $\lambda d(x, Tx) \leq \lambda d(x, y)$ or $\lambda d(x, Sx) \leq d(x, y)$ implies that

$$H(Tx, Sy) \leq e(d(x, y), d(x, Tx), d(y, Sy), d(x, Sy), d(y, Tx), \qquad (x, y \in X).$$

Then $(\mathcal{CF})_{F_1,F_2}$ is an absolute retract for metric spaces.

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**"An Istanbul Meeting for World Mathematicians"**
**3-5 July 2019, Istanbul, Turkey**

# Improving Election Algorithm for numerical function optimization

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In this research, an improved version of election algorithm is presented. Election algorithm is a swarm intelligence and population-based approach, which inspired from the presidential election process. This algorithm starts its optimization process with a population of candidate solutions, which every solution is called a person. The individuals in the population are divided into voters and candidates, which form some electoral parties in the solution space. Then, using three operators of positive advertisement, negative advertisement and coalition, the electoral parties explore the solution to reach a global optimum point. The election algorithm suffers from an important problem: no convergence to the global optimal point due to inappropriate exploration of the search space in the positive advertisement stage. In this research, to alleviate the problem of low speed, a new version of positive advertisement is introduced. In addition, a new migration operator is proposed to empower the algorithm to escape from local optima. The proposed algorithm is evaluated on ten test functions and is compared with other algorithms. The experimental results show that the proposed algorithm has better performance in comparison to the counterpart algorithms.

**Keyword(s): Optimization, election algorithm, improved election algorithm, positive advertisement**

## 1. Introduction

In this paper, we introduced Improved Election Algorithm (IEA), which is an improved version of the traditional Election Algorithm (EA) [1]. The EA is a novel swarm intelligent algorithm inspired by the behaviors of candidates and voters in presidential election process. The advertising campaign forms the basis of the EA and consists of three main phases: *positive advertisement*, *negative advertisement* and *coalition*. These operators causes the individuals converge to the optimal solution in the solution space. The EA is easy to implement, simple in the concept, has fewer control parameters and shows superior performance in solving numerical optimization problems. However, the EA algorithm deals with an important challenge: no guaranty to converge to the global optimum point due to inappropriate exploration of the search space in the positive advertisement stage. To alleviate this issue, we proposed IEA, which improves the traditional EA algorithm in two folds: (*i*) introducing new positive advertisement operator to searching efficiently the entire solution space, and (*ii*) introducing migration operator to enhance the diversity of population and preventing early convergence of the algorithm. The new version of positive advertisement is introduced to enhance the social information exchange between voters and candidates within electoral parties. This enables the individuals to efficiently explore the entire solution space and converge to the global optimum. The migration operator is introduced to increases the diversity in the population by generating new individuals on different areas of solution space and keeps away the IEA to get stuck in local optima.

Ten benchmark functions are employed to test the performance of IEA algorithm compared with several other state-of-the-art algorithms. The results show that the IEA algorithm often outperforms the other algorithms in terms of solution quality and convergence speed.

After this short introduction, the rest of this paper is organized as follows. Section 2 presents the traditional EA algorithm. Section 3 presents the detail of IEA algorithm. In Section 4, the IEA algorithm is tested on ten benchmark functions, and is compared with several state-of-the-art algorithms. Finally, in Section 5, conclusions are drawn and some possible future works are suggested.

## 2. Election Algorithm (EA)

EA is a novel stochastic optimization algorithm that uses presidential election as a source of inspiration [1]. Figure 1 shows the pseudo code of the algorithm. EA starts with an initial population. Each individual in the population is called a person. For a problem with $x_1, x_2, \ldots, x_{N_{\text{var}}}$ variables, the initial population consists of $N_{pop}$ persons. Each person $P_i$ is an $1 \times N_{\text{var}}$ array of variables values and defined as follows:

$$P_i = \left[x_1, x_2, \ldots, x_{N_{\text{var}}}\right] \tag{1}$$

The eligibility of a person $P_i$ is found by evaluation of the eligibility function $E$ at the variables $x_1, x_2, \ldots, x_{N_{\text{var}}}$ considering objective function of the problem. The eligibility function is defined as follows:

$$E(P_i) = E(x_1, x_2, \ldots, x_{N_{\text{var}}}) \tag{2}$$

Some of the best persons in the population are selected to be the candidates and the remaining persons form the voters. All the voters of initial population are divided among the mentioned candidates based on their similarity in opinions. To do this, voter $v_k$ is considered as a supporter of candidate $c_i$, if the following predicate holds

$$P_i = \left\{ v_k : \left\| E_{v_k} - E_{c_i} \right\| < \left\| E_{v_k} - E_{c_j} \right\| \ \forall\ 1 \le j \le N_c \right\} \tag{3}$$

$P_i$ is the $i$th party, and $N_c$ is the number of initial candidates. $E_{c_i}$ and $E_{v_k}$ present the eligibility of candidate $c_i$ and voter $v_k$, respectively. In the party formation process, each voter is assigned to exactly one party.

After dividing the voters among candidates and forming the initial parties, the candidates start advertising campaign. The advertising campaign consists of three main phases: *positive advertisement*, *negative advertisement* and *coalition*.

The positive advertisement is modeled by conveying some variables of the candidate to its voters inside a party. To do this task, in each party, $N_s$ variables of the target candidate are randomly selected and replaced with the selected variables of the voters. $N_s$ is computed as follows:

$$N_s = \lceil X_s \times S_c \rceil \tag{4}$$

$S_c$ is the number of candidate's variables and $X_s$ is the selection rate. The selected variables $N_s$ are weighted with a coefficient $\omega$ and then embedded in voters. The new value for the $i$th variable of a voter after positive advertisement is given by:

$$x_{i_{new}} = \omega . x_{i_{old}} \tag{5}$$

$\omega$ is defined as follows:

$$\omega = \frac{1}{\left| E_{c_i} - E_{v_k} \right| + 1} \tag{6}$$

In negative advertisement, powerful candidates try to attract voters of weak candidates toward themselves. A party is weak, if its candidate to be the weakest compared to other parties' candidates. To model the negative advertisement, first, a number of voters from the weakest party are selected. Then, a race is taken place among powerful parties to possess these voters. To select the weakest voters form the weakest party, the eligibility distance between the voters and the weakest candidate is computed, and then 5% of the farthest voters are selected. The distance between selected voters and the powerful candidates are computed, and the voters are assigned to the closest candidates.

In coalition phase, several candidates join together and form a new party. Among the candidates that wish to collate, a candidate is picked at random to be the leader candidate and the remaining is considered as the followers. In coalition, all of the follower candidates and their voters become the voters of the leader one.

Until termination conditions are not satisfied, the advertising campaign operators are iteratively applied to update the population. Finally, the update process stops and the candidate with the majority of votes announces as the winner. The winner is equal to the best solution found for the optimization problem.

```
Population ← Create random population (Problem);
Costs ← Evaluate population (Population);
Parties ← Create initial parties (Population, Costs);
Repeat
        For number of candidates do
                Parties ← Positive advertisement (Parties);
                Parties ← Negative advertisement (Parties, Costs);
                Parties ← Coalition (Parties, Costs);
                Costs ← Evaluation (Parties);
        End For
Until termination conditions are satisfied
```

Figure 1. Pseudo code of the EA algorithm

## 3. Improved Election Algorithm (IEA)

Advertising campaign of the EA suffers from two problems: getting stuck at local optima and inefficiency of positive advertisement phase. In advertising campaign, after several iterations, diversity in the population may decrease. As a result, the candidates and their voters cannot explore the entire solution space and get stuck at local optima. To alleviate these issues, we proposed an improved EA, denoted as IEA. Figure 2 shows the pseudo code of the IEA. The IEA enhances the canonical EA algorithm in two folds: introducing migration operator, and introducing a new positive advertisement operator.

```
Population ← Create random population (Problem);
Costs ← Evaluate population (Population);
Parties ← Create initial parties (Population, Costs);
Repeat
        For number of candidates do
                Parties ← Positive advertisement (Parties);
                Parties ← Negative advertisement (Parties, Costs);
                Parties ← Coalition (Parties, Costs);
                Parties ← Migration (Parties, Costs);
                Costs ← Evaluation (Parties);
        End For
Until termination conditions are satisfied
```

Figure 2. Pseudo code of the IEA algorithm

### 3.1. Migration

Migration keeps the EA away from converging too fast before exploring the entire solution space. İn real-world elections, some individuals can travel from other countries to the target country and vote to their favorite candidate. The *travelers* are referred as *migrants*, which can increase the popularity of some candidates. To model migration, first the number of migrants in every generation of the algorithm is calculated as follows:

$$M = \lceil \mu \times N_{pop} \rceil \tag{7}$$

*M* is the number of new migrants, $\mu$ is the migration coefficient and $N_{pop}$ is the population size.

Then, *M* voters are randomly generated on different areas of the solution space. Here, the new generated voters referred as migrants. The migration in every generation of the algorithm adds *M* new individuals to the population.

### 3.2. Positive advertisement

Let $v_i(t)$ denote the position of voter *i* in the search space at time step *t*, $l_i$ denote the personal best position in the search space, which it has achieved by voter $v_i$ since the first generation, and $g_i$ denote the social best position achieved so far by candidate $c_i$. In positive advertisement, the position of voter $v_i$ at iteration *t*+1 is computed as

$$v_i(t+1) = v_i(t) + L_1 + L_2, \quad \text{where } v_i(0) \in U(lb, ub) \tag{8}$$

*t* indicates the current iteration, $L_1$ resembles the movement of voter $v_i$ toward his personal best experience obtained so far, and $L_2$ resembles the movement of voter $v_i$ toward the social best knowledge. $U(lb, ub)$ generates random number with uniform distribution in the range $[lb, ub]$. $L_1$ is defined as

$$L_1 = \alpha \times r_1 \times \left(l_i(t) - v_i(t)\right) \times \{V_1\} \tag{9}$$

$\alpha$ is a constant value indicating the weight of personal experience, $r_1 \in [0, 1]$ is a random value regenerated every iteration, and $V_1$ is a vector which its start point is the current position of the voter $v_i$ and its direction is toward the personal best position, $l_i$. $V_1$ is considered to prevent the voter from drastically changing its direction and alignment it to its current direction. The length of this vector is set to unity. The effect of term $L_1$ is to linearly attract the voter toward its own best experience. $L_2$ is defined as

$$L_2 = \beta \times r_2 \times \left(g_i(t) - v_i(t)\right) \times \{V_2\} \tag{10}$$

$\beta$ is a constant value indicating the effect of social experience on the voter $v_i$ during advertisement, and $r_2 \in \mathrm{U}(-\lambda,+\lambda)$ is a random number with uniform distribution regenerated every iteration. Parameter $r_2$ adds a random amount of deviation to locating the final position of the voter $v_i$ in its movement toward $g_i$. By this way, different points around the candidate $g_i$, are explored. $\lambda$ is a parameter that adjusts the deviation of voter $v_i$ from its original direction. In our implementation, $\lambda = \pi/4$ is used that resulted in good convergence of individuals to the global optimum. $V_2$ is a unit vector which its start point is the current position of the voter $v_i$ and its direction is toward the social best position $g_i$. The effect of $L_2$ is to attract the voter $v_i$ toward the social knowledge which is the best position found by the candidate $c_i$ in the party $P_i$. Considering minimization problems, the personal best position at the iteration $t$+1, is updated as

$$l_i(t+1) = \begin{cases} l_i(t) & if \;\; \mathrm{e}_{\mathrm{v}_i}(t+1) \geq e_{l_i(t)} \\ v_i(t+1) & if \;\; \mathrm{e}_{\mathrm{v}_i}(t+1) < e_{l_i(t)} \end{cases} \tag{11}$$

The global best position discovered by any of the individuals so far in the party $P_i$, $g_i(t+1)$ is updated as

$$g_i(t+1) = \min\{l_0(t),\, l_1(t),\, ...,\, l_{n_m}(t)\} \tag{12}$$

$n_m$ is the total number of voters in the party $P_i$.

## 4. Experiments

The proposed IEA algorithm is tested on ten benchmark functions. Four algorithms are used as a comparison. The algoithms include Particle Swarm Optimization (PSO) [2], Artificial Bee Colony (ABC) [3], Grey Wolf Optimizer (GWO) [4] and canonical EA [1]. Tables 1 lists the characteristic of benchmark functions used in the test. A detailed description of the test functions are available in [5] and [6].

In experiments, the algorithms ran for 30 times for all test functions, each time using a different initial population. The statistical results are reported in Table 2. In these tables, *min* and *mean* are respectively the minimum and the mean function values obtained by the algorithms over 30 simulation runs. *Std* indicates the standard deviation of the results. In Table 2, in order to make comparison clear, the values below $10^{-16}$ are assumed to be 0. From numerical simulations, it is obvious that all algorithms have almost consistent behavior on all benchmark functions. It can be seen that the solution quality and convergence accuracy obtained on most test functions using the IEA in 30 independent simulation runs are almost exceeding or matching the best performance obtained by other algorithms. This testifies that the IEA has better stability behavior on most test functions rather than other algorithms. IEA has the ability of getting out of local optima in the search space and finding the global optima.

Table 1. Test functions used in the test

| No. | Function | Formulation | Dimension | Range | Global minimum |
|---|---|---|---|---|---|
| $f_1$ | Beale | $f_4(x)=(1.5-x_1+x_1x_2)^2+(2.25-x_1+x_1x_2^2)^2+(2.625-x_1+x_1x_2^3)^2$ | 5 | [-4.5, 4.5] | 0 |
| $f_2$ | Stepint | $f_5(x)=25+\sum_{i=1}^{5}\lfloor x_i \rfloor$ | 5 | [-5.12, 5.12] | 0 |
| $f_3$ | Sphere | $f_8(x)=\sum_{i=1}^{n} x_i^2$ | 30 | [-100, 100] | 0 |
| $f_4$ | Quartic | $f_{10}(x)=\sum_{i=1}^{n} ix_i^2+random[0,1)$ | 30 | [-1.28, 1.28] | 0 |
| $f_5$ | Zakharov | $f_{11}(x)=\sum_{i=1}^{n} x_i^2+\left(\sum_{i=1}^{n} 0.5ix_i^{2}\right)+\left(\sum_{i=1}^{n} 0.5ix_i\right)^4$ | 10 | [-5, 10] | 0 |
| $f_6$ | Powell | $f_{12}(x)=\sum_{i=1}^{n} x_i^2+\left(\sum_{i=1}^{n} 0.5ix_i^{2}\right)+\left(\sum_{i=1}^{n} 0.5ix_i\right)^4$ | 24 | [-4, 5] | 0 |
| $f_7$ | Rosenbrock | $F_{15}(x)=\sum_{i=1}^{n-1}\left[100(x_{i+1}-x_i^2)^2+(x_i-1)^2\right]$ | 30 | [-30, 30] | 0 |
| $f_8$ | Rastrigin | $f_{27}(x)=10n+\sum_{i=1}^{n}\left(x_i^2-10cos(2\pi x_i)\right)$ | 30 | [-5.12, 5.12] | 0 |
| $f_9$ | Griewank | $F_{28}(x)=\frac{1}{4000}\left(\sum_{i=1}^{n} x_i^2\right)-\left(\prod_{i=1}^{n}\cos\left(\frac{x_i}{\sqrt{i}}\right)+1\right)$ | 30 | [-600, 600] | 0 |
| $f_{10}$ | Ackley | $f_{29}(x)=-20\exp\left(-0.02\sqrt{\frac{1}{n}\sum_{i=1}^{n} x_i^2}\right)-\exp\left(\frac{1}{n}\sum_{i=1}^{n}\cos(2\pi x_i)\right)+20+e$ | 30 | [-32, 32] | 0 |

Table 2. Results of test functions

| Function | Statistics | PSO | ABC | GWO | EA | IEA |
|---|---|---|---|---|---|---|
| $f_1$ | Min | 0 | 1.675e-11 | 3.8675e-07 | 1.113e-14 | 9.5644e-15 |
| | Mean | 4.1234e-8 | 1.9362e-09 | 5.79e-06 | 8.35e-11 | 2.1224e-12 |
| | Std | 2.5107e-8 | 4.2205e-09 | 5.546e-06 | 1.1286e-11 | 1.9246e-12 |
| $f_2$ | Min | 0 | 0 | 0 | 0 | 0 |
| | Mean | 0.3 | 0 | 0 | 0 | 0 |
| | Std | 0.45826 | 0 | 0 | 0 | 0 |
| $f_3$ | Min | 0 | 0 | 0 | 0 | 0 |
| | Mean | 0 | 0 | 0 | 0 | 0 |
| | Std | 0 | 0 | 0 | 0 | 0 |
| $f_4$ | Min | 6.8999e-07 | 0.020563 | 2.4034e-05 | 0.00036265 | 4.7863e-07 |
| | Mean | 0.000419 | 0.29495 | 0.00014029 | 0.018397 | 0.00016881 |
| | Std | 6.1983e-05 | 0.10335 | 9.1414e-05 | 43.0045 | 8.9064e-05 |
| $f_5$ | Min | 0 | 0.03363 | 0 | 0 | 0 |
| | Mean | 0 | 0.088016 | 0 | 0 | 0 |
| | Std | 0 | 0.065937 | 0 | 0 | 0 |
| $f_6$ | Min | 0 | 0.019457 | 1.5268e-07 | 1.2095e-9 | 8.2758e-10 |
| | Mean | 1.2409e-06 | 0.037274 | 6.6783e-06 | 0.00027755 | 1.4444e-07 |
| | Std | 5.3059e-07 | 0.012143 | 7.9129e-06 | 0.00056147 | 2.9883e-07 |
| $f_7$ | Min | 0.00010204 | 0.079482 | 25.9125 | 0.0037742 | 0 |
| | Mean | 1.966733 | 0.43175 | 26.4394 | 22.4021 | 18.4507 |
| | Std | 2.1342 | 0.21306 | 0.47338 | 7.4159 | 13.0478 |
| $f_8$ | Min | 6.2881 | 4.2165e-09 | 0 | 0 | 0 |
| | Mean | 11.4352 | 1.6453e-06 | 0 | 9.4739e-15 | 0 |
| | Std | 2.2662 | 4.0684e-06 | 0 | 2.1184e-14 | 0 |
| $f_9$ | Min | 0 | 0 | 0 | 0 | 0 |
| | Mean | 0.0005169481 | 0 | 0 | 0 | 0 |
| | Std | 0.00000042 | 0 | 0 | 0 | 0 |
| $f_{10}$ | Min | 0 | 1.7507e-06 | 2.6645e-15 | 6.9427e-15 | 6.7098e-15 |
| | Mean | 1.0481e-14 | 4.5265e-06 | 5.5067e-15 | 9.3969e-14 | 4.4631e-14 |
| | Std | 3.4809e-15 | 2.4668e-06 | 1.4211e-15 | 7.7099e-14 | 7.1143e-15 |

## 5. Conclusion

This paper presents Improved Election Algorithm (IEA) to improve the canonical Election Algorithm (EA). The IEA improves the EA in two folds: introducing migration operator, and improving positive advertisement phase. With the improve positive advertisement, the information is exchange between candidates and voters efficiently, and improves the search ability. With the migration operator, diversity in the population is maintained, which keeps the IEA away from converging too fast before exploring the entire solution space. IEA algorithm is evaluated on ten benchmarks, and is compared with PSO, ABC, GWO and EA. The results show that the proposed IEA algorithm outperforms the canonical EA and other compared counterparts in terms of solution quality and convergence speed. There remain several points to improve our research. First, the IEA will trap in local optimums on few functions, which can be seen from the benchmark functions. We can combine the IEA with some local search strategies or other meta-heuristics to enhance its optimization ability. Second, we can apply the proposed IEA algorithm to solve more practical optimization problems to accurately identify its weaknesses and merits.

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## A new bio-inspired optimization algorithm for data clustering

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In this paper, we present Seasons Algorithm (SA), a new stochastic globally search and optimization strategy. The SA is inspired by the natural growth cycle of trees in the different seasons of a year. It is an iterative population-based algorithm working on a set of solutions known as population. In the SA terminology, each candidate solution in the population is referred to as a tree and the population is called forest. Until the termination condition is satisfied, the population of solutions is updated to a new generation by applying four main operators similar to the trees life cycles in the seasons of a year including: (*i*) renew in spring, (*ii*) growth- competition in summer, (*iii*) reinforcement in autumn, and (*iv*) resistance in winter. These operators hopefully cause the trees to converge to a state of solution space that is the global optimum. The SA algorithm is employed for data clustering task. Experiments on real benchmarks for data clustering indicate that the SA algorithm is encouraging and outperformed several other comparison algorithms.

**Keyword(s): Bio-inspired optimization, evolutionary algorithms, meta-heuristics, global optimization**

### 1. Introduction

Clustering is the process of partitioning a set of unlabeled data objects into clusters such that data objects within a cluster shares a high degree of similarity while being dissimilar from data objects of other clusters [1]. Clustering needed in a large variety of applications including data mining [2], information retrieval [3], [4], information extraction [5], pattern recognition [6], machine learning [4], bioinformatics [7] and other fields of science and engineering.

Clustering algorithms can roughly be classified into two main classes [1], [8], [9], [10]: hierarchical and partitional. The goal of hierarchical clustering is dividing the data objects into a hierarchy of nested clusters, while partitional clustering is partitioning the data objects into a set of disjoint clusters based on a pre-defined objective function [10]. Partitional algorithms initially obtain a set of disjoint clusters and iteratively refine them to minimize the objective function. The objective is to minimize the inter-cluster connectivity and maximize the intra-cluster compactness [8].

The majority of researchers often formulated the partitional clustering problem as an optimization problem. From an optimization viewpoint, clustering can be considered as a particular kind of NP-hard problem [11]. Researchers proposed different evolutionary, and swarm intelligence meta-heuristics for solving the clustering problem; for survey papers see [3], [11], [12], [13]. Evolutionary algorithms mimic the concepts of evolution in nature [14]. The first effort to develop an evolutionary based clustering algorithm using genetic algorithm (GA) was proposed by Bezdek et al. [15] in 1994. Then Sarkar et al. [16] proposed a clustering method based on evolutionary programming.

Swarm intelligence algorithms models the intelligent behavior of social insects such as ants, birds, and bees or flocks of creatures in nature for the purpose of optimization and search. A particle swarm optimization (PSO) algorithm, which models the intelligent behavior of birds was used for clustering by Omran et al. [17]. The ant colony optimization (ACO) which is inspired by foraging behavior of real ants was applied for clustering by Shelokar et al. [18]. ACO models the way real ants find a shortest path from the nest to food source. Karaboga and Ozturk [9] proposed an artificial bee colony (ABC) algorithm for clustering.

In order to improve the quality of solutions and the performance of meta-heuristic based clustering methods, some work has been directed towards hybridizing different meta-heuristics. Yan et al. [10] proposed HABC algorithm to improve the optimization ability of canonical ABC. In HABC, the crossover operator of GA is introduced to enhance the social learning between bees. Yanga et al. [19] proposed PSOKHM algorithm based on PSO and K-Harmonic Means (KHM), which makes full use of the merits of both algorithms. Emami and Derakhshan [20] proposed two hybrid clustering algorithms namely ICAFKM and PSOFKM. ICAFKM combined the advantages aspects of ICA and Fuzzy K-Means (FKM) algorithm. Similarly, PSOFKM makes full use of the merits of both PSO and FKM.

According to NFL principle [21], there is no meta-heuristic best suited for solving all optimization problems. This is true for clustering; a particular meta-heuristic may show very promising performance on a set of datasets, but the same meta-heuristic may show poor results on a different set of datasets. However, the performance far from ideal state and there is a room for more improvement. To alleviate this issue, researchers improve the current approaches or introduce new meta-heuristics. This also motivates our attempts to introduce a new meta-heuristic with inspiration from trees' growth cycle in a forest.

The rest of this paper is organized as follows. Section 2 presents a brief review of clustering problem. In Section 3, we introduce the proposed SA algorithm. In Section 4, the proposed methods are evaluated through benchmark problems, and the experimental results are compared to counterpart optimization algorithms. Finally, in Section 5, conclusions are drawn and some possible future works are suggested.

## 2. Data clustering

Let $D=\{O_1, O_2, ..., O_N\} \subset R^P$ be a collection of $N$ data objects to be clustered, where $P$ is the number of features in the dataset $D$. Each data object $O_i \in D$ is an $1\times P$ array of real value features. The goal of partitional clustering is to find a partition set $C=\{C_1,\ C_2,\ ...,\ C_K\}$ such that

$$\begin{array}{l} C_k \neq \varnothing \quad \forall\, k=1,\ 2,\ ...,\ K; \quad \bigcup_{k=1}^{K} C_k = D \\ C_p \cap C_q = \varnothing \quad \forall\, p \neq q, \text{ and } p,\ q=1,\ 2,\ ...,\ K \end{array} \tag{1}$$

Objects in every cluster $C_k \in C$ are more similar to each other than different clusters based on some prescribed criteria. In order to measure the quality of clustering, an adequacy criterion must be defined. The simplest and most widely used criterion to measure the quality of a clustering algorithm is the sum of mean-squared error or the total within-cluster variance, which is defined as follows:

$$J=\sum_{i=1}^{N} \min\left\{D\left(O_i,\ C_j\right),\ j=1,\ 2,\ ...,\ K\right\} \tag{2}$$

where $D(O_i,\ C_j)$ denotes the similarity between the object $O_i$ and the center of cluster $C_j$. The most popular used similarity metric in clustering is Euclidean distance [10], as Eq. (3).

$$D\left(O_i,\ C_j\right)=\sqrt{\sum_{f=1}^{P}\left(O_{if}-C_{jf}\right)^2} \tag{3}$$

where $O_i$ and $C_j$ are the vectors with P dimension, $O_{if}$ and $C_{jf}$ are the *f*th feature of $O_i$ and $C_j$, respectively. The optimal partitioning is defined as the solution that minimizes the cost function **J**.

## 3. Seasons optimization algorithm

Figure 1 shows the flowchart of the SA algorithm. In the following, the components of the algorithm are described.

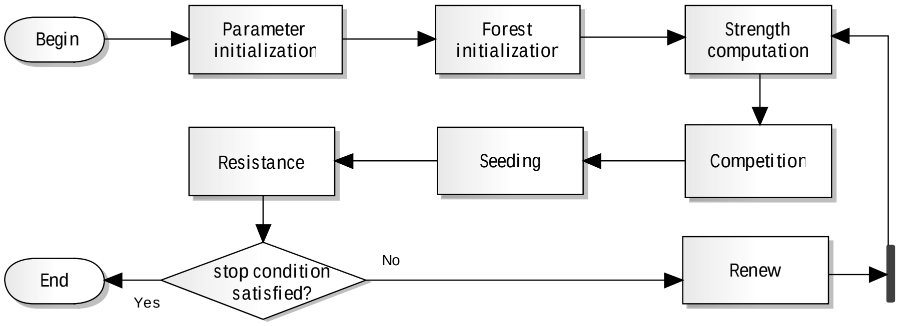


Figure 1. Flowchart of the SA algorithm

### 3.1 Create initial forest

The Seasons algorithm starts its search process with a set of candidate solutions collectively known as *forest*. Each individual in the forest is named a tree. For an D-dimensional optimization problem with $x_1, x_2, ..., x_D$, the forest is a $T_N \times D$ matrix, which consists of $T_N$ trees of dimension *D*. Every tree $T_i$ in the forest is an array of variable values, which formalized as

$$T_i = [t_1,\ t_2,\ ...,\ t_D] \tag{4}$$

$t_i$ denotes a candidate value assigned to variable $x_i$. Each $t_i$ can be interpreted as a natural characteristic of tree $T_i$ such as height, structural strength, roots and the diameter of the trunk.

### 3.2 Renew

Let $F^t$ denote the forest at generation *t* and *r* denote the renew rate. In order to mathematically model the behavior of trees in the spring, we defined a simple formula as

$$F^{t+1} = F^t + R(\lceil r \times S^t \rceil) \tag{5}$$

$S^t$ denotes the number of seeds that fall on the land in the last autumn, and $T_N^{t+1}$ indicates the forest at generation *t*+1. The function *R*(.) randomly generates $\lceil r \times S_N^t \rceil$ new trees in different areas of the solution space. The function R(.) simulates the natural sprout of several seeds that fall on the land in the past autumn and turning into young trees. It is obvious that only few seeds have the chance to grow up and become a tree in the forest.

### 3.3 Competition

In order to mathematically model the competition process, trees are randomly divided into two groups: the trees that are in non-dense areas and the trees in dense areas. The number of trees that are in dense areas is computed by

$$N_1 = round\left(c \times N\right) \tag{6}$$

$N_1$ indicates the number of tress in dense area, $c$ is the competition that generates randomly every iteration, and $N$ is the number of all trees in the forest. The remaining trees are in non-dense areas

$$N_2 = N - N_1 \tag{7}$$

In order to mathematically model the growth of the trees in non-dense areas, first $N_2$ trees are randomly selected from the forest. Then, from each selected tree, some variable are randomly selected and their values are added with small random values. The value of each selected variable $t_j \in T_i$ is computed by

$$t_j^{t+1} = t_j^t + dx; \quad dx \in U\left(-\Delta x,\ \Delta x\right) \tag{8}$$

$\Delta x$ is a random number with uniform distribution regenerated every iteration. The value of $\Delta x$ is arbitrary, in our implementation, a value of $\pi/4$ is used that resulted in good convergence of trees to the global optimum.

In order to mathematically model the competition of trees in dense area, first, the trees in dense area are sorted in terms of strength in ascending order and then $N_c$ of these trees are selected to be cored trees. The remaining Ng of the trees will be the neighbors each of which belongs to a cored tree. Having the cored trees, its neighbors are identified and its neighborhood zone is formed. The number of neighbors for cored tree $t_f$ is determined by multiplying its normalized strength by the number of all trees in dense areas:

$$Ni_f = \left|NZ_f\right| = round\left(ns_{t_f} \times Ng\right) \tag{9}$$

$Ni_f$ is the number of neighbors around cored tree $t_f$, $NZ_f$ denotes neighbors of *f*-th cored tree such that $t_f \notin NZ_f$, Ng is the number of all trees in dense area, and $ns_{t_f}$ is the normalized strength of cored tree. $ns_{t_f}$ is defined by

$$ns_{t_f} = \frac{s_{t_f} - \min(I)}{\sum_{k=1}^{N_2} s_{t_k}}, \qquad \text{where } I = \left\{s_{t_k} \mid k = 1,\ 2,\ \ldots,\ N_2\right\} \tag{10}$$

Next, *Ni* trees are selected randomly from the neighbor trees to form the neighborhood zone of cored tree $t_f$. Once the neighbors are identified, the competition begins among the cored tree $t_f$ and its neighbors. In order to model the influence of neighbors and cored tree on each other, a simple measure of crowding or competition index (W) was calculated. W is defined as a function of the number, distance, and size of neighbors:

$$W = \sum_{i=1}^{Ni_f} k \times S_i \times d_i^{-1} \tag{11}$$

$Ni_f$ is the number of neighbors within the neighborhood zone of cored tree $t_f$ ; $S_i$ is the strength of the ith neighbor; $d_i$ is the Euclidean distance between the ith neighbor and cored tree $t_f$ , and k is the effect of neighbor $t_i$, which is computed as

$$k = \begin{cases} 1 & if \left(S_f \leq S_i\right) \\ 1-\lambda & f\left(S_f > S_i\right) \end{cases} \tag{12}$$

$\lambda \in rand\left(0,1\right)$ is an asymmetry coefficient which varies from 0 (completely symmetric competition) to 1 (completely asymmetric competition), and represents the degree to which the effects of relatively smaller neighbors are discounted. The smaller trees that have less strength, will have less impact on the cored tree, but the stronger trees will weaken the cored tree. The new position of neighbor tree $t_i$ after competition is as follows:

$$t_i^{t+1} = t_i^t + W \times \{rand\} \tag{13}$$

The *rand* function generates random numbers in range [0, 1]. Rand is added to search different points around the neighbor trees. At final, among the neighbor trees, a tree with the highest strength is selected and replaced with the cored tree. Then algorithm will continue by the cored tree in a new position.

### 3.4. Seeding

In the fall, the seeds may fall near the trees or distributed in wide areas of landscape. In order to mathematically model the seeding process, at first a number of S best trees are randomly selected to form seeding list $Sl = \{T_1,\ T_2,\ ...,\ T_S\}$. These trees would have chance to disperse their seeds in the forest. S is computed as

$$S = round\left(s \times N\right) \tag{14}$$

*s* is seeding rate and *N* is the number of all trees in the forest. In order to prevent an ever-increasing population size, it is assumed that each tree generates only one seed. Thus, the number of seeds at every generation of the algorithm is equal to *S*. Then, from each selected tree, some variables are randomly selected and their values are exchanged with new randomly generated values in the related variable's range. Let $\{t_1,\ t_2,\ ...,\ t_D\}$ are the variables selected from tree $T_i \in Sl$. The value of each selected variable $t_j \in T_i$ is computed by

$$t_j^{t+1} = r_2 + U\left(-1,\ 1\right) \tag{15}$$

$r_2$ is a random number with uniform distribution regenerated every iteration in range [l, u]. $U\left(-1,\ 1\right)$ is a uniformly distributed random number in range [-1,1], which is considered to change the direction of seed dispersal by using the negative values.

### 3.5. Resistance

To mathematically model the impact of winter on trees, we introduced resistance operator. Resistance operator removes the weakest trees from the forest. The weakest trees are those that cannot stand against harsh cold weather and deem to collapse. In order to determine the weakest trees that will remove from the forest, the following equations are proposed

$$W = \lceil \rho \times T_N \rceil \tag{16}$$

where $\rho$ is the critical temperature at which the trees may injure and collapse, and $W$ denotes the number of weakest trees that will remove from the forest. The algorithm sorts the trees according to their strength value in ascending order, and removes $W$ weakest trees from the forest.

## 4. Applying the SA Algorithm to Clustering

SA algorithm is a capable algorithm and can be successfully applied to practical and engineering applications. In this section, SA is applied for clustering. Similar to the other evolutionary algorithms applied on clustering, it is easy to apply the SA algorithm for clustering. For this purpose, just two changes need to be done in SA algorithm, which include: (*i*) solution representation and (*ii*) strength calculation.

### 4.1 Solution Representation

In SA for data clustering, each individual presents a candidate solution for the clustering problem. Each individual $P_i$ in initial population is encoded as a set of cluster centers:

$$P_i = \{C_1,\ C_2,\ ...,\ \mathrm{C}_K\} \tag{17}$$

$K$ presents the number of clusters and $C_i$ presents center of cluster $i$, which is defined as follows:

$$C_i = \{m_{i,\,1},\ m_{i,\,2},\ ...,\ m_{i,\,p}\} \tag{18}$$

$p$ presents the number of features of the dataset to be clustered and $m_{i,\,j}$ presents the data sample $i$ in cluster $j$.

### 4.2 Strength Calculation

When solving data clustering problem, the quality of each individual is measured using the strength function, which is defined as follows.

$$E_i = \frac{1}{1 + J(O,\ P_i)} \tag{19}$$

where $O$ presents the input dataset to be clustered, $P_i$ presents a candidate solution produced by clustering algorithm, $E_i$ is the strength value of the solution which is inversely proportional to the $J(O,\ P_i)$ as given in Eq. (2). $J(O,\ P_i)$ is the objective function of the clustering problem. It is obvious that the smaller the objective function value is, the higher the strength value is.

## 5. Experiments and results

To evaluate the performance of SA algorithm for data clustering, we compared it with K-Means (KM) [19], canonical GA [22], canonical PSO [19], canonical ABC [9], canonical EA [23], and ICAFKM [20] on five real datasets [24]. These datasets include: *Iris*, *Ionosphere*, *CMC*, *Balance-Scale* and *Glass*. Detailed information about these datasets is provided in [24]. These datasets are varying in characteristics and come from different real-world problems. *Iris* is from the domains of pattern recognition, *Ionosphere* from image recognition domain, *CMC* from medical diagnosis, *Balance-Scale* and *Glass* are from various control applications. Table 1 summarizes the characteristics of these datasets in alphabetic order.

Table 1. Characteristics of datasets

| **Dataset** | **# class** | **# features** | **#Instances** |
|---|---|---|---|
| Ionosphere | 2 | 34 | 351 |
| Iris | 3 | 4 | 150 |
| Glass | 6 | 10 | 214 |
| CMC | 3 | 9 | 1473 |
| Balance-Scale | 3 | 4 | 625 |

In the experiments, the evaluation is conducted using s*uccess rate* measure. We defined success rate as follows:

$$s_r = \frac{C}{T} \times 100\% \qquad (20)$$

*C* is the total number of correctly classified data objects, and *T* is the number of all objects in dataset. The quality of the results is evaluated by comparing the objects classified by the algorithm and those objects in the ground truth annotated by annotators. A data object is regarded as a correctly classified example, if after applying clustering algorithm the assigned label for that object is exactly same as the desired class label in ground truth. In Eq. (20), $s_r$ indicates the percentage of correctly classified examples of entire dataset. It is clear that, the bigger $s_r$ is, the higher the quality of clustering algorithm is.

Since the performance of stochastic algorithms is largely dependent on the generation of initial population so for every dataset, algorithms are executed 10 times, each time with randomly generated initial population.

In the experiments, for each benchmark, the algorithms were run for 10 times and clustering results are given in Table 2. $\mu$ presents the average total success rate for 10 independent runs and the $\delta$ indicates the standard deviation. In Table 2, the best algorithm in terms of success rate is determined by putting * mark beside it.

As shown in Table 2, SA algorithm relatively obtained the best mean and standard deviation of success rate on most datasets.

Table 2. Average and variance of the success rate of KM, GA, PSO, ABC, EA, ICAFKM and SA algorithms on five datasets.

| | | **KM** | **GA** | **PSO** | **ABC** | **EA** | **ICAFKM** | **SA** |
|---|---|---|---|---|---|---|---|---|
| Balance-Scale | $\mu$ | 19.04 | **20.60*** | 20.44 | 20.54 | 20.38 | 19.74 | 19.32 |
| | $\delta$ | 5.10 | 0.89 | **0.80** | 1.43 | 1.07 | 2.32 | 1.71 |
| CMC | $\mu$ | 33.57 | 36.27 | 37.93 | 35.37 | 37.94 | 35.87 | **40.11*** |
| | $\delta$ | 7.70 | 7.66 | **2.01** | 3.36 | 3.47 | 7.44 | 4.03 |
| Glass | $\mu$ | 32.24 | 13.31 | 41.49 | 14.76 | 41.57 | 41.71 | **43.05*** |
| | $\delta$ | 12.73 | 7.95 | 8.23 | 13.47 | **8.13** | 9.18 | 7.16 |
| Ionosphere | $\mu$ | 59.48 | 47.46 | 63.38 | 48.83 | 63.15 | 60.71 | **63.44*** |
| | $\delta$ | 14.37 | 14.18 | **3.81** | 15.10 | 9.79 | 10.43 | 11.26 |
| Iris | $\mu$ | 68.86 | 34.6 | 88.93 | 31.60 | 90.46 | 91.05 | **91.32*** |
| | $\delta$ | 12.74 | 23.11 | 6.34 | 25.34 | 6.17 | 9.65 | 7.81 |

## 6. Conclusion

In this paper, a new optimization algorithm is presented, which is called Seasons Algorithm (SA). The SA algorithm is inspired by the growth cycle of the trees in different seasons of a year. The SA algorithm includes four operators including renew, competition, seeding and resistance. SA algorithm is tested on five well-known real datasets selected from the UCI machine-learning repository and compared with baseline and counterpart algorithms. The results show that SA algorithm achieved encouraging results. This indicates the SA algorithm is a competitive approach for data clustering. There remain several points to improve our research. First, we can combine the SA algorithm with some local search strategies or other meta-heuristic algorithms. Furthermore, we can apply the proposed SA algorithm to solve some of more practical optimization problems. Also in some specific applications, some components of the algorithm can be modified in order to improve the algorithm processing time.

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**INTERNATIONAL CONFERENCE ON MATHEMATICS**
**"An Istanbul Meeting for World Mathematicians"**
**3-5 July 2019, Istanbul, Turkey**

# On the bounds for the spectral norms of geometric and r-circulant matrices with Biperiodic Jacobsthal numbers

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**Abstract**

The study is about the bounds of the spectral norms of r-circulant and geometric circulant matrices with the sequences called biperiodic Jacobsthal numbers. Then we give bounds for the spectraal norms of Kronecker and Hadamard products of these r-circulant matrices and geometric circulant matrices**.**

**Keywords:** Biperiodic Jacobsthal numbers; geometric circulant matrix; Norms

## 1 Introduction

Special integer sequences are encountered in many areas such as architucture, nature, in human body, computer programmming. There have been several studies on properties of different special integer sequences. For example in [1] the author investigated the Fibonacci and Lucas sequences in detail. From these sequences, Jacobsthal and Jacobsthal Lucas numbers are given by the recurrence relations $\mathrm{j_n = j_{n-1} + 2j_{n-2}}$, with the initial values of $\mathrm{j_0 = 0}$ , $\mathrm{j_1 = 1}$ and $\mathrm{c_n = c_{n-1} + 2c_{n-2}}$ ,with the initial values of $\mathrm{c_0 = 2}$ , $\mathrm{c_1 = 1}$ , $n \geq 2$ respectively in [2]. The reason of the importance of integer sequences, there have been many generalizations of these sequences. For example Edson, Yayenie defined a new generalization of Fibonacci sequences called biperiodic Fibonacci sequences in [3]. Then Bilgici gave the properties of biperiodic Lucas sequence in [4]. The authors studied biperiodic Jacobsthal sequences in [5].

Circulant matrices are important for various reasons since they are widely used in probability, coding theory and numerical analysis. There are many articles in the literature that study on the norms of circulant, r-circulant matrices with different sequences and their generalized sequences. For example, in [10], Solak studied the spectral norms of circulant matrices with Fibonacci and Lucas numbers. In [11], Kocer et al. obtained the norms of circulant and semicurculant matrices whose entries are Horadam numbers. In [12], Shen and Cen found bounds for the spectral norms of *r*-circulant matrices with Fibonacci and Lucas numbers. Uslu and Uygun in [13] have given the relation among k-Fibonacci, k-Lucas and generalized *k*-Fibonacci numbers and the spectral norms of the matrices involving these numbers. In [14], Yazlık and Taskara calculated upper and lower bounds for r-circulant matrices with generalized k-Horadam numbers. In [15], Bahsi and Solak computed the norms of *r*-circulant matrices with the hyper-Fibonacci and Lucas numbers. In [16], the authors established the upper bound estimation on the spectral norm of r-circulant matrices with the Fibonacci and Lucas numbers. In [17], Tuğlu and Kızılateş studied the spectral norms of circulant and *r*-circulant matrices with the hyperharmonic Fibonacci numbers. In [18,19], Bahsi gave the spectral norms of circulant and r-circulant matrices with the hyperharmonic numbers and generalized Fibonacci numbers. In [20], Uygun computed some bounds for the norms of circulant matrices with the k-Jacobsthal and *k*-Jacobsthal Lucas numbers. In [21], Kızılateş, Tuğlu calculated bounds for the spectral norms of geometric circulant matrices with generalized Fibonacci, Lucas and hyperharmonic Fibonacci numbers. The bounds for the spectral norms of geometric circulant matrices with the generalized k-Horadam numbers are studied in [22]. The authors studied the norms of some special matrices with generalized Fibonacci sequence. The authors gave.the norms of some special matrices with generalized Fibonacci sequence in [23]. In [24]. Kome and Yazlık gave bounds spectral norms of r-circulant matrices with the biperodic Fibonacci and Lucas numbers.

Inspired by these studies, in this study we give upper and lower bounds for the spectral norms of *r*-circulant and geometric circulant matrices whose entries are the biperiodic Jacobsthal numbers. Then we give bounds for the spectral norms of Kronecker and Hadamard products of these *r*-circulant matrices and geometric circulant matrices.

## 2. Preliminaries

**Definition 1** For any two non-zero real numbers *a* and *b*, the bi-periodic Jacobsthal sequence is defined as

$$j_0 = 0\,, j_1 = 1, \quad j_n = \begin{cases} a j_{n-1} + 2 j_{n-2} & \text{if n is even} \\ b j_{n-1} + 2 j_{n-2} & \text{if n is odd} \end{cases} \qquad n \geq 2$$

The Binet formula for the bi-periodic Jacobsthal sequence is

$$j_m = \left( \frac{a^{1-\varepsilon(m)}}{(ab)^{\left\lfloor \frac{m}{2} \right\rfloor}} \right) \frac{\alpha^m - \beta^m}{\alpha - \beta} \tag{1}$$

where $\alpha$ and $\beta$ are the roots of the nonlinear quadratic equation for the biperiodic Jacobsthal sequence, which is given as $x^2 - abx - 2ab = 0$, and $\lfloor a \rfloor$ is the floor function of *a* and

$\varepsilon(n) = n - 2\left\lfloor \frac{n}{2} \right\rfloor$ is the parity function [5].

**Definition 2** For any two non-zero real numbers *a* and *b*, the bi-periodic Jacobsthal Lucas sequence is defined as [26]

$$c_0 = 2\,, c_1 = a, \quad c_n = \begin{cases} b c_{n-1} + 2 c_{n-2} & \text{if n is even} \\ a c_{n-1} + 2 c_{n-2} & \text{if n is odd} \end{cases} \qquad n \geq 2 \tag{2}$$

The Binet formula for the bi-periodic Jacobsthal Lucas sequence is in the following [5].

$$c_m = \left( \frac{a^{\varepsilon(m)}}{(ab)^{\left\lfloor \frac{m+1}{2} \right\rfloor}} \right) (\alpha^m + \beta^m)$$



**Definition 3** *r*-circulant matrix $C_r$ is defined as the following

$$C_r = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ rc_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\ rc_{n-2} & rc_{n-1} & c_0 & \cdots & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r\,c_1 & r\,c_2 & r\,c_3 & \cdots & c_0 \end{bmatrix} \tag{3}$$

For brevity, we denote the *r*-circulant matrix with $C_r = \text{circ}(c_0, c_1, \ldots, c_{n-1})$ If we choose *r* = 1, we get the circulant matrix. the matrix was first defined by Davis in [25]. Then the researchers found different properties of this matrix. Then it became a popular study in applied and pure mathematics. You can encounter different papers about it in references.

**Definition 4** *nxn* geometric circulant matrix $C_{r^*}$ is defined as the following

$$C_{r^*} = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ rc_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\ r^2c_{n-2} & rc_{n-1} & c_0 & \cdots & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r^{n-1}c_1 & r^{n-2}c_2 & r^{n-3}c_3 & \cdots & c_0 \end{bmatrix} \tag{4}$$

by Kızılates, and Tuğlu in [21]. For brevity, we denote the geometric circulant matrix with $C_{r^*} = \text{circ}(c_0, c_1, \ldots, c_{n-1})$. If we choose *r = 1*, we get the circulant matrix.

**Lemma 5** The summation of the squares of the first *n* terms of bi-periodic Jacobsthal sequences is given as the following:

$$\sum_{i=1}^{n} \left(\frac{2b}{a}\right)^{\varepsilon(i+1)} \left(\frac{j_i}{2^{\lfloor \frac{i}{2} \rfloor}}\right)^2 = \frac{1}{a} \frac{j_m j_{m+1}}{2^{m-1}} \tag{5}$$

**Proof.** By using Binet forms of bi-periodic Jacobsthal sequences we have

$$\left(\frac{2b}{a}\right)^{\varepsilon(i+1)} \left(\frac{j_i}{2^{\lfloor \frac{i}{2} \rfloor}}\right)^2 = \frac{2ab}{(\alpha-\beta)^2}\left[\left(\frac{\alpha^2}{2ab}\right)^k - \left(\frac{\beta^2}{2ab}\right)^k - 2(-1)^k\right]$$

Using the properties $ab(\alpha+2) = \alpha^2$ and $ab(\beta+2) = \beta^2$

$$\sum_{i=1}^{n} \left(\frac{2b}{a}\right)^{\varepsilon(i+1)} \left(\frac{j_i}{2^{\lfloor \frac{i}{2} \rfloor}}\right)^2 = \frac{2ab}{(\alpha-\beta)^2}\left[\sum_{i=1}^{n}\left(\frac{\alpha^2}{2ab}\right)^k - \sum_{i=1}^{n}\left(\frac{\beta^2}{2ab}\right)^k - 2\sum_{i=1}^{n} (-1)^k\right]$$

$$=\frac{\left(\frac{\alpha^2}{2ab}\right)^{m+1}-\left(\frac{\alpha^2}{2ab}\right)}{\left(\frac{\alpha^2}{2ab}\right)-1}-\frac{\left(\frac{\beta^2}{2ab}\right)^{m+1}-\left(\frac{\beta^2}{2ab}\right)}{\left(\frac{\beta^2}{2ab}\right)-1}+(-1)^m-1$$

$$=\frac{j_n j_{n-1}}{a\,2^{n-2}}$$

■

**Lemma 6** The following property is hold for the bi-periodic Jacobsthal sequences

$$\sum_{i=1}^{m}\left(\frac{2b}{a}\right)^{\varepsilon(i+1)}\left(\frac{j_i}{|r|^i\,2^{\left\lfloor\frac{i}{2}\right\rfloor}}\right)^2$$
$$=\frac{n\,|r|^{2n}}{a^2b^2+8ab}\left[\frac{ab(2c_n-|r|c_{2n+2})+2ab(|r|(a^2b^2+8ab-1))}{(1+|r|^2-\frac{|r|}{2}(ab+4))(2|r|)^n}\right.$$
$$\left.+\,2ab[1-(-1)^n]\right] \tag{6}$$

**Proof**. The proof is made by using similar procedure with the proof of the previous Lemma.

■

For any $A=[a_{ij}]\epsilon M_{m,n}(C)$, the Frobenious (or Euclidean) norm of matrix A is displayed by the following equality

$$\|A\|_F=\left(\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2\right)^{\frac{1}{2}}, \tag{7}$$

and the spectral norm of matrix A is also shown as

$$\|A\|_2=\sqrt{\boxed{\frac{\max}{1\le i\le n}}\lambda_i(A^HA)}, \tag{8}$$

where $A^H$ is the conjugate transpose of matrix A and $\lambda_i(A^HA)$ is an eigenvalue of $A^HA$

**Lemma 7** Suppose that $A\,\epsilon M_{m,n}(C)$, then the following inequalities are held [7]

$$\frac{1}{\sqrt{n}}\|A\|_F\le\|A\|_2\le\|A\|_F, \tag{9}$$

$$\|A\|_2 \leq \|A\|_F \leq \sqrt{n}\|A\|_2$$

**Lemma 8** Suppose that $A, B \epsilon M_{m,n}(C)$, and the Hadamard product of A,B is entrywise product and defined by [6,7,8]

$$AoB = (a_{ij}b_{ij})$$

that has the following properties

$$\|AoB\|_2 \leq \|A\|_2 \ \|B\|_2, \tag{10}$$

and $r_1(A)$, the maximum row length norm, $c_1(B)$ , the maximum column length norm are given as $r_1(A) = \max_{1\leq i\leq n} \sqrt{\sum_{j=1}^{n} |b_{ij}|^2}$ and $c_1(B) = \max_{1\leq j\leq n} \sqrt{\sum_{i=1}^{n} |c_{ij}|^2}$ with the following property

$$\|AoB\|_2 \leq r_1(A)c_1(B), \tag{11}$$

**Lemma 9** Let $A \in M_{m,n}(C)$, and $B \in M_{p,q}(C)$, be given, then the Kronecker product of A ,B is defined by

$$\|A \otimes B\| = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}$$

and has the following property [7]

$$\|A \otimes B\|_2 = \|A\|_2 \ \|B\|_2 \tag{12}$$

## 3. Main Results

### Lower and Upper Bounds of r- Circulant Matrices Involving bi-periodic Jacobsthal Numbers

**Theorem 10** Let $r \in C$ and $J_r = circ_r\left(\left(\frac{2b}{a}\right)^{\frac{\varepsilon(1)}{2}} j_0, \left(\frac{2b}{a}\right)^{\frac{\varepsilon(2)}{2}} j_1, \ldots, \left(\frac{2b}{a}\right)^{\frac{\varepsilon(n)}{2}} \frac{j_{n-1}}{2^{\lfloor\frac{n-1}{2}\rfloor}}\right)$ be a *r*-circulant matrix with bi-periodic Jacobsthal numbers, then the upper and lower bounds for the spectral norm of *Jr* are obtained as:

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*(i)* If $|r| \geq 1$, then

$$\sqrt{\frac{j_n j_{n-1}}{a\,2^{n-2}}} \le \|J_r\|_2 \le \sqrt{(n-1) r \frac{j_n j_{n-1}}{a\,2^{n-2}}}$$

*(ii)* If $|r| < 1$, then

$$|r|\sqrt{\frac{j_n j_{n-1}}{a\,2^{n-2}}} \le \|J_r\|_2 \le \sqrt{(n-1)\frac{j_n j_{n-1}}{a\,2^{n-2}}}$$

**Proof.** The *r*- circulant matrix $J_r$ is of the form

$$J_r = \begin{bmatrix} \left(\frac{2b}{a}\right)^{\frac{\varepsilon(1)}{2}} j_0 & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(2)}{2}} j_1 & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(3)}{2}} \frac{j_2}{2} & \cdots & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(n)}{2}} \frac{j_{n-1}}{2^{\left\lfloor\frac{n-1}{2}\right\rfloor}} \\ r\left(\frac{2b}{a}\right)^{\frac{\varepsilon(n)}{2}} \frac{j_{n-1}}{2^{\left\lfloor\frac{n-1}{2}\right\rfloor}} & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(1)}{2}} j_0 & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(2)}{2}} j_1 & \cdots & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(n-1)}{2}} \frac{j_{n-2}}{2^{\left\lfloor\frac{n-2}{2}\right\rfloor}} \\ r\left(\frac{2b}{a}\right)^{\frac{\varepsilon(n-1)}{2}} \frac{j_{n-2}}{2^{\left\lfloor\frac{n-2}{2}\right\rfloor}} & r\left(\frac{2b}{a}\right)^{\frac{\varepsilon(n-2)}{2}} \frac{j_{n-1}}{2^{\left\lfloor\frac{n-1}{2}\right\rfloor}} & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(1)}{2}} j_0 & \cdots & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(n-2)}{2}} \frac{j_{n-3}}{2^{\left\lfloor\frac{n-3}{2}\right\rfloor}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r\left(\frac{2b}{a}\right)^{\frac{\varepsilon(2)}{2}} j_1 & r\left(\frac{2b}{a}\right)^{\frac{\varepsilon(3)}{2}} \frac{j_2}{2} & r\left(\frac{2b}{a}\right)^{\frac{\varepsilon(4)}{2}} \frac{j_3}{2} & \cdots & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(1)}{2}} j_0 \end{bmatrix}$$

For $|r| \ge 1$ by using (5), (7) we have

$$\|J_r J_{r^*}\|_F^2 = \sum_{k=0}^{n-1} (n-k)\left(\frac{2b}{a}\right)^{\varepsilon(k+1)} \left(\frac{j_k}{2^{\left\lfloor\frac{k}{2}\right\rfloor}}\right)^2 + \sum_{k=1}^{n-1} k\,|r|^2 \left(\frac{2b}{a}\right)^{\varepsilon(k+1)} \left(\frac{j_k}{2^{\left\lfloor\frac{k}{2}\right\rfloor}}\right)^2$$

$$\ge \sum_{k=0}^{n-1} (n-k)\left(\frac{2b}{a}\right)^{\varepsilon(k+1)} \left(\frac{j_k}{2^{\left\lfloor\frac{k}{2}\right\rfloor}}\right)^2 + \sum_{k=1}^{n-1} k \left(\frac{2b}{a}\right)^{\varepsilon(k+1)} \left(\frac{j_k}{2^{\left\lfloor\frac{k}{2}\right\rfloor}}\right)^2$$

$$= n\sum_{k=0}^{n-1} \left(\frac{2b}{a}\right)^{\varepsilon(k+1)} \left(\frac{j_k}{2^{\left\lfloor\frac{k}{2}\right\rfloor}}\right)^2$$

$$= n\left(\frac{j_n j_{n-1}}{a\,2^{n-2}}\right)$$

From the equality (9),

$$\|J_r\|_2 \;\geq\; \frac{\|J_r\|_F}{\sqrt{n}} \;\geq\; \sqrt{\frac{j_n j_{n-1}}{a\,2^{n-2}}}$$

On the other hand, let $J_r = BoC$ where B = $[b_{ij}]$ and C = $[c_{ij}]$ are defined as

$$B = \begin{bmatrix} j_0 & 1 & 1 & \cdots & 1 \\ r & j_0 & 1 & \cdots & 1 \\ r & r & j_0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r & r & r & \cdots & j_0 \end{bmatrix}$$

and

$$C = \begin{bmatrix} \left(\frac{2b}{a}\right)^{\frac{\varepsilon(1)}{2}} j_0 & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(2)}{2}} j_1 & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(3)}{2}} \frac{j_2}{2} & \cdots & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(n)}{2}} \frac{j_{n-1}}{2^{\lfloor\frac{n-1}{2}\rfloor}} \\ \left(\frac{2b}{a}\right)^{\frac{\varepsilon(n)}{2}} \frac{j_{n-1}}{2^{\lfloor\frac{n-1}{2}\rfloor}} & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(1)}{2}} j_0 & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(2)}{2}} j_1 & \cdots & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(n-1)}{2}} \frac{j_{n-2}}{2^{\lfloor\frac{n-2}{2}\rfloor}} \\ \left(\frac{2b}{a}\right)^{\frac{\varepsilon(n-1)}{2}} \frac{j_{n-2}}{2^{\lfloor\frac{n-2}{2}\rfloor}} & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(n)}{2}} \frac{j_{n-1}}{2^{\lfloor\frac{n-1}{2}\rfloor}} & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(1)}{2}} j_0 & \cdots & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(n-2)}{2}} \frac{j_{n-3}}{2^{\lfloor\frac{n-3}{2}\rfloor}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \left(\frac{2b}{a}\right)^{\frac{\varepsilon(2)}{2}} j_1 & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(3)}{2}} \frac{j_2}{2} & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(4)}{2}} \frac{j_3}{2} & \cdots & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(1)}{2}} j_0 \end{bmatrix}$$

By the maximum row and column length norm of these matrices,

$$r_1(B) = \max_{1\leq i\leq n} \sqrt{\sum_{j=1}^{n} |b_{ij}|^2} = \sqrt{\sum_{j=1}^{n} |b_{nj}|^2} = \sqrt{j_0^2 + (n-1)r} = \sqrt{(n-1)r}$$

$$c_1(C) = \max_{1\leq j\leq n} \sqrt{\sum_{i=1}^{n} |c_{ij}|^2} = \sqrt{\frac{j_n j_{n-1}}{a\,2^{n-2}}}$$

By using (11) we obtain

$$\|J_r\|_2 \leq\; r_1(B)c_1(C) = \sqrt{(n-1)r\,\frac{j_n j_{n-1}}{a\,2^{n-2}}}$$

The proof is completed fort he first past.

(ii) For $|r| \leq 1$, by using (5), (7) we have ,

$$\| J_{r^*}\|_F^2 = \sum_{k=0}^{n-1}(n-k)\left(\frac{2b}{a}\right)^{\varepsilon(k+1)}\left(\frac{j_k}{2^{\left\lfloor\frac{k}{2}\right\rfloor}}\right)^2 + \sum_{k=1}^{n-1} k\,|r|^2\left(\frac{2b}{a}\right)^{\varepsilon(k+1)}\left(\frac{j_k}{2^{\left\lfloor\frac{k}{2}\right\rfloor}}\right)^2$$

$$\geq \sum_{k=0}^{n-1}(n-k+k)\,|r|^2\left(\frac{2b}{a}\right)^{\varepsilon(k+1)}\left(\frac{j_{k+1}}{2^{\left\lfloor\frac{k+1}{2}\right\rfloor}}\right)^2 = n|r|^2\left(\frac{j_n j_{n-1}}{a\,2^{n-2}}\right)$$

From (9), we get

$$\| J_{r^*}\|_2 \geq \frac{\| J_{r^*}\|_F}{\sqrt{n}} \geq |r|\sqrt{\frac{j_n j_{n-1}}{a\,2^{n-2}}}$$

On the other hand, let $J_r = B \circ C$ where B ,C are given in the following forms

$$B = \begin{bmatrix} j_0 & 1 & 1 & \cdots & 1 \\ r & j_0 & 1 & \cdots & 1 \\ r & r & j_0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r & r & r & \cdots & j_0 \end{bmatrix}$$

and

$$C = \begin{bmatrix} \left(\frac{2b}{a}\right)^{\frac{\varepsilon(1)}{2}} j_0 & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(2)}{2}} j_1 & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(3)}{2}} \frac{j_2}{2} & \cdots & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(n)}{2}} \frac{j_{n-1}}{2^{\left\lfloor\frac{n-1}{2}\right\rfloor}} \\ \left(\frac{2b}{a}\right)^{\frac{\varepsilon(n)}{2}} \frac{j_{n-1}}{2^{\left\lfloor\frac{n-1}{2}\right\rfloor}} & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(1)}{2}} j_0 & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(2)}{2}} j_1 & \cdots & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(n-1)}{2}} \frac{j_{n-2}}{2^{\left\lfloor\frac{n-2}{2}\right\rfloor}} \\ \left(\frac{2b}{a}\right)^{\frac{\varepsilon(n-1)}{2}} \frac{j_{n-2}}{2^{\left\lfloor\frac{n-2}{2}\right\rfloor}} & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(n)}{2}} \frac{j_{n-1}}{2^{\left\lfloor\frac{n-1}{2}\right\rfloor}} & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(1)}{2}} j_0 & \cdots & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(n-2)}{2}} \frac{j_{n-3}}{2^{\left\lfloor\frac{n-3}{2}\right\rfloor}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \left(\frac{2b}{a}\right)^{\frac{\varepsilon(2)}{2}} j_1 & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(3)}{2}} \frac{j_2}{2} & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(4)}{2}} \frac{j_3}{2} & \cdots & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(1)}{2}} j_0 \end{bmatrix}$$

By the maximum row and column length norm of these matrices,

$$r_1(B) = \max_{1\leq i\leq n}\sqrt{\sum_{j=1}^{n}\left|b_{ij}\right|^2} = \sqrt{\left(\frac{2b}{a}\right)^{\frac{\varepsilon(1)}{2}} j_0^2 + (n-1)}$$

$$c_1(C) = \max_{1\leq j\leq n}\sqrt{\sum_{i=1}^{n}\left|c_{ij}\right|^2} = \sqrt{\sum_{k=0}^{n-1}\left[\left(\frac{2b}{a}\right)^{\frac{\varepsilon(n)}{2}} \frac{j_i}{2^{\left\lfloor\frac{i}{2}\right\rfloor}}\right]^2} = \sqrt{\frac{j_n j_{n-1}}{a\,2^{n-2}}}$$

By using (11) we obtain the second part of the proof

$$\|J_r\|_2 \leq r_1(B)c_1(C) = \sqrt{\frac{(n-1)\, j_n j_{n-1}}{a\, 2^{n-2}}}$$

Therefore the proof is completed ■

**Corollary 11** Let $A = B = J_r = \mathrm{circ}_r\left(\left(\frac{2b}{a}\right)^{\frac{\varepsilon(1)}{2}} j_0\ , \left(\frac{2b}{a}\right)^{\frac{\varepsilon(2)}{2}} j_1\,, \ldots, \left(\frac{2b}{a}\right)^{\frac{\varepsilon(n)}{2}}\ \frac{j_{n-1}}{2^{\left\lfloor\frac{n-1}{2}\right\rfloor}}\right)$ be an r- circulant matrix with biperiyodic Jacobsthal numbers, then the upper bounds for spectral norm of Kronecker product of A and B are demonstrated by

*(i)* If $|r| \geq 1$, then

$$\|A \otimes B\|_2 \leq (n-1)r\, \frac{j_n j_{n-1}}{a\, 2^{n-2}}$$

*(ii)* If $|r| \leq 1$, then

$$\|A \otimes B\|_2 \leq (n-1)\, \frac{j_n j_{n-1}}{a\, 2^{n-2}}$$

**Proof**. The proof is easily seen by

$$\|A \otimes B\|_2 = \|A\|_2\ \|B\|_2$$

■

**Corollary 12** Let $A = B = J_r = \mathrm{circ}_r\left(\left(\frac{2b}{a}\right)^{\frac{\varepsilon(1)}{2}} j_0\ , \left(\frac{2b}{a}\right)^{\frac{\varepsilon(2)}{2}} j_1\,, \ldots, \left(\frac{2b}{a}\right)^{\frac{\varepsilon(n)}{2}}\ \frac{j_{n-1}}{2^{\left\lfloor\frac{n-1}{2}\right\rfloor}}\right)$ be an r-circulant matrix with bi-periodic Jacobsthal numbers, then the upper bounds for spectral norm of Hadamard product of A and B are demonstrated by

*(i)* If $|r| \geq 1$, then

$$\|A \circ B\|_2 \leq (n-1)r\, \frac{j_n j_{n-1}}{a\, 2^{n-2}}$$

*(ii)* If $|r| \leq 1$, then

$$\|A \circ B\|_2 \leq (n-1)\, \frac{j_n j_{n-1}}{a\, 2^{n-2}}$$

**Proof.** The proof is easily seen by

$$\|A \circ B\|_2 \leq \|A\|_2\ \|B\|_2$$

■

**Lower and Upper Bounds of Geometric Circulant Matrices Involving bi-periodic Jacobsthal Numbers**

**Theorem 13** Let $r\epsilon C$ and $J_{r^*} = circ_r\left(\left(\frac{2b}{a}\right)^{\frac{\varepsilon(1)}{2}} j_0 \ , \left(\frac{2b}{a}\right)^{\frac{\varepsilon(2)}{2}} j_1 \ , \dots , \left(\frac{2b}{a}\right)^{\frac{\varepsilon(n)}{2}} \frac{j_{n-1}}{2^{\lfloor\frac{n-1}{2}\rfloor}}\right)$ be a geometric circulant matrix with bi-periodic Jacobsthal numbers, then the upper and lower bounds for the spectral norm of $J_r$ are obtained as :

*(i)* If $|r| \geq 1$, then

$$\sqrt{\frac{j_n j_{n-1}}{a\,2^{n-2}}} \leq \| J_{r^*} \|_2 \leq \sqrt{\frac{j_n j_{n-1}}{a\,2^{n-2}} \frac{1-|r|^{2n}}{1-|r|^2}}$$

*(ii)* If $|r| < 1$, then

$$\frac{n|r|^{2n}}{a^2b^2+8ab}\left[\frac{ab(2c_{2n}-|r|c_{2n+2})+2ab(|r|(a^2b^2+8ab-1))}{1+|r|^2-\frac{|r|}{2}(ab+4)(2|r|)^n}+2ab[1-(-1)^m]\right] \leq \| J_{r^*} \|_2$$

$$\| J_{r^*} \|_2 = \sqrt{(n-1)\frac{j_n j_{n-1}}{a\,2^{n-2}}}$$

**Proof.** The geometric circulant matrix $J_{r^*}$ is of the form

$$J_{r^*} = \begin{bmatrix} \left(\frac{2b}{a}\right)^{\frac{\varepsilon(1)}{2}} j_0 & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(2)}{2}} j_1 & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(3)}{2}} \frac{j_2}{2} & \cdots & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(n)}{2}} \frac{j_{n-1}}{2^{\lfloor\frac{n-1}{2}\rfloor}} \\ r\left(\frac{2b}{a}\right)^{\frac{\varepsilon(n)}{2}} \frac{j_{n-1}}{2^{\lfloor\frac{n-1}{2}\rfloor}} & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(1)}{2}} j_0 & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(2)}{2}} j_1 & \cdots & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(n-1)}{2}} \frac{j_{n-2}}{2^{\lfloor\frac{n-2}{2}\rfloor}} \\ r^2\left(\frac{2b}{a}\right)^{\frac{\varepsilon(n-1)}{2}} \frac{j_{n-2}}{2^{\lfloor\frac{n-2}{2}\rfloor}} & r\left(\frac{2b}{a}\right)^{\frac{\varepsilon(n)}{2}} \frac{j_{n-1}}{2^{\lfloor\frac{n-1}{2}\rfloor}} & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(1)}{2}} j_0 & \cdots & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(n-2)}{2}} \frac{j_{n-3}}{2^{\lfloor\frac{n-3}{2}\rfloor}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r^{n-1}\left(\frac{2b}{a}\right)^{\frac{\varepsilon(2)}{2}} j_1 & r^{n-2}\left(\frac{2b}{a}\right)^{\frac{\varepsilon(3)}{2}} \frac{j_2}{2} & r^{n-3}\left(\frac{2b}{a}\right)^{\frac{\varepsilon(4)}{2}} \frac{j_3}{2} & \cdots & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(1)}{2}} j_0 \end{bmatrix}$$

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For $|r| \geq 1$, by using the definiton of Frebinous norm, we have

$$\| J_{r^*} \|_F^2 = \sum_{k=0}^{n-1} (n-k) \left(\frac{2b}{a}\right)^{\varepsilon(k+1)} \left(\frac{j_k}{2^{\left\lfloor \frac{k}{2} \right\rfloor}}\right)^2 + \sum_{k=1}^{n-1} k \left|r^{n-k}\right|^2 \left(\frac{2b}{a}\right)^{\varepsilon(k+1)} \left(\frac{j_k}{2^{\left\lfloor \frac{k}{2} \right\rfloor}}\right)^2$$

$$\geq \sum_{k=0}^{n-1} (n-k) \left(\frac{2b}{a}\right)^{\varepsilon(k+1)} \left(\frac{j_k}{2^{\left\lfloor \frac{k}{2} \right\rfloor}}\right)^2 + \sum_{k=1}^{n-1} k \left(\frac{2b}{a}\right)^{\varepsilon(k+1)} \left(\frac{j_k}{2^{\left\lfloor \frac{k}{2} \right\rfloor}}\right)^2$$

$$= n \sum_{k=0}^{n-1} \left(\frac{2b}{a}\right)^{\varepsilon(k+1)} \left(\frac{j_k}{2^{\left\lfloor \frac{k}{2} \right\rfloor}}\right)^2$$

$$= n \left(\frac{j_n j_{n-1}}{a\, 2^{n-2}}\right)$$

From the equality (9),

$$\sqrt{\left(\frac{j_n j_{n-1}}{a\, 2^{n-2}}\right)} \leq \| J_{r^*} \|_2$$

On the other hand, let the matrices B and C be presented by

$$B = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ r & 1 & 1 & \cdots & 1 \\ r^2 & r & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r^{n-1} & r^{n-2} & r^{n-3} & \cdots & 1 \end{bmatrix}$$

and

$$C = \begin{bmatrix} \left(\frac{2b}{a}\right)^{\frac{\varepsilon(1)}{2}} j_0 & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(2)}{2}} j_1 & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(3)}{2}} \frac{j_2}{2} & \cdots & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(n)}{2}} \frac{j_{n-1}}{2^{\left\lfloor \frac{n-1}{2} \right\rfloor}} \\ \left(\frac{2b}{a}\right)^{\frac{\varepsilon(n)}{2}} \frac{j_{n-1}}{2^{\left\lfloor \frac{n-1}{2} \right\rfloor}} & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(1)}{2}} j_0 & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(2)}{2}} j_1 & \cdots & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(n-1)}{2}} \frac{j_{n-2}}{2^{\left\lfloor \frac{n-2}{2} \right\rfloor}} \\ \left(\frac{2b}{a}\right)^{\frac{\varepsilon(n-1)}{2}} \frac{j_{n-2}}{2^{\left\lfloor \frac{n-2}{2} \right\rfloor}} & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(n)}{2}} \frac{j_{n-1}}{2^{\left\lfloor \frac{n-1}{2} \right\rfloor}} & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(1)}{2}} j_0 & \cdots & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(n-2)}{2}} \frac{j_{n-3}}{2^{\left\lfloor \frac{n-3}{2} \right\rfloor}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \left(\frac{2b}{a}\right)^{\frac{\varepsilon(2)}{2}} j_1 & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(3)}{2}} \frac{j_2}{2} & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(4)}{2}} \frac{j_3}{2} & \cdots & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(1)}{2}} j_0 \end{bmatrix}$$

In this case $J_{r^*} = B \circ C$. By the maximum row and column length norm of these matrices,

$$r_1(B) = \max_{1\le i\le n} \sqrt{\sum_{j=1}^{n} |b_{ij}|^2} = \sqrt{\sum_{j=1}^{n} |b_{nj}|^2} = \sqrt{\frac{1-|r|^{2n}}{1-|r|^2}}$$

$$c_1(C) = \max_{1\le j\le n} \sqrt{\sum_{i=1}^{n} |c_{ij}|^2} = \sqrt{\frac{j_n j_{n-1}}{a\,2^{n-2}}}$$

By using (11) we obtain

$$\| J_{r^*}\|_2 \le r_1(B)c_1(C) = \sqrt{\frac{j_n j_{n-1}}{a\,2^{n-2}}\,\frac{1-|r|^{2n}}{1-|r|^2}}$$

From $|r| < 1$,

$$\| J_{r^*}\|_F^2 = \sum_{k=0}^{n-1} (n-k) \left(\frac{2b}{a}\right)^{\varepsilon(k+1)} \left(\frac{j_k}{2^{\lfloor\frac{k}{2}\rfloor}}\right)^2 + \sum_{k=1}^{n-1} k\,|r^{n-k}|^2 \left(\frac{2b}{a}\right)^{\varepsilon(k+1)} \left(\frac{j_k}{2^{\lfloor\frac{k}{2}\rfloor}}\right)^2$$

$$\ge n \sum_{k=0}^{n-1} |r^{n-k}|^2 \left(\frac{2b}{a}\right)^{\varepsilon(k+1)} \left(\frac{j_k}{2^{\lfloor\frac{k}{2}\rfloor}}\right)^2$$

$$= n|r|^{2n} \sum_{k=0}^{n-1} \left(\frac{2b}{a}\right)^{\varepsilon(k+1)} \left(\frac{j_k}{2^{\lfloor\frac{k}{2}\rfloor}|r|^k}\right)^2$$

$$= \frac{n|r|^{2n}}{a^2b^2 + 8ab}\left[\frac{ab(2c_{2n} - |r|c_{2n+2}) + 2ab(|r|(a^2b^2 + 8ab - 1))}{1 + |r|^2 - \frac{|r|}{2}(ab+4)(2|r|)^n} + 2ab[1-(-1)^m]\right]$$

For the matrices B and C as mentioned above we have

$$B = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ r & 1 & 1 & \cdots & 1 \\ r^2 & r & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r^{n-1} & r^{n-2} & r^{n-3} & \cdots & 1 \end{bmatrix}$$

and

$$C = \begin{bmatrix} \left(\frac{2b}{a}\right)^{\frac{\varepsilon(1)}{2}} j_0 & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(2)}{2}} j_1 & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(3)}{2}} \frac{j_2}{2} & \cdots & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(n)}{2}} \frac{j_{n-1}}{2^{\left\lfloor\frac{n-1}{2}\right\rfloor}} \\ \left(\frac{2b}{a}\right)^{\frac{\varepsilon(n)}{2}} \frac{j_{n-1}}{2^{\left\lfloor\frac{n-1}{2}\right\rfloor}} & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(1)}{2}} j_0 & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(2)}{2}} j_1 & \cdots & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(n-1)}{2}} \frac{j_{n-2}}{2^{\left\lfloor\frac{n-2}{2}\right\rfloor}} \\ \left(\frac{2b}{a}\right)^{\frac{\varepsilon(n-1)}{2}} \frac{j_{n-2}}{2^{\left\lfloor\frac{n-2}{2}\right\rfloor}} & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(n)}{2}} \frac{j_{n-1}}{2^{\left\lfloor\frac{n-1}{2}\right\rfloor}} & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(1)}{2}} j_0 & \cdots & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(n-2)}{2}} \frac{j_{n-3}}{2^{\left\lfloor\frac{n-3}{2}\right\rfloor}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \left(\frac{2b}{a}\right)^{\frac{\varepsilon(2)}{2}} j_1 & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(3)}{2}} \frac{j_2}{2} & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(4)}{2}} \frac{j_3}{2} & \cdots & \left(\frac{2b}{a}\right)^{\frac{\varepsilon(1)}{2}} j_0 \end{bmatrix}$$

In this case $J_{r^*} = B \circ C$

$$r_1(B) = \max_{1\le i\le n} \sqrt{\sum_{j=1}^{n} \left|b_{ij}\right|^2} = \sqrt{\sum_{j=1}^{n} \left|b_{1j}\right|^2} = \sqrt{n}$$

$$c_1(C) = \max_{1\le j\le n} \sqrt{\sum_{i=1}^{n} \left|c_{ij}\right|^2} = \sqrt{\frac{j_n j_{n-1}}{a\, 2^{n-2}}}$$

By using (11) we obtain

$$\| J_{r^*} \|_2 \le r_1(B) c_1(C) = \sqrt{n \; \frac{j_n j_{n-1}}{a\, 2^{n-2}}}$$

■

**Corollary 12** Let $A = B = J_r = circ_r\left(\left(\frac{2b}{a}\right)^{\frac{\varepsilon(1)}{2}} j_0\ , \left(\frac{2b}{a}\right)^{\frac{\varepsilon(2)}{2}} j_1\ , \ldots , \left(\frac{2b}{a}\right)^{\frac{\varepsilon(n)}{2}} \frac{j_{n-1}}{2^{\left\lfloor\frac{n-1}{2}\right\rfloor}}\right)$ be a geometric circulant matrix with bi-periodic Jacobsthal numbers, then the upper bounds for spectral norm of Kronecker product of A and B are demonstrated by

*(i)* If $|r| \ge 1$, then

$$\|A \otimes B\|_2 \le \frac{j_n j_{n-1}}{a\, 2^{n-2}} \frac{1 - |r|^{2n}}{1 - |r|^2}$$

*(ii)* If $|r| \le 1$, then

$$\|A \otimes B\|_2 \le n \frac{j_n j_{n-1}}{a\, 2^{n-2}}$$

**Corollary 13** Let $A = B = J_r = circ_r\left(\left(\frac{2b}{a}\right)^{\frac{\varepsilon(1)}{2}} j_0 \;, \left(\frac{2b}{a}\right)^{\frac{\varepsilon(2)}{2}} j_1 \,, \ldots , \left(\frac{2b}{a}\right)^{\frac{\varepsilon(n)}{2}} \frac{j_{n-1}}{2^{\left\lfloor\frac{n-1}{2}\right\rfloor}}\right)$ be a geometric circulant matrix with bi-periodic Jacobsthal numbers, then the upper bounds for spectral norm of Hadamard product of A and B are demonstrated by

*(i)* If $|r| \geq 1$, then

$$\|A \text{ o } B\|_2 \leq \frac{j_n j_{n-1}}{a\, 2^{n-2}} \frac{1 - |r|^{2n}}{1 - |r|^2}$$

*(ii)* If $|r| \leq 1$, then

$$\|A \text{ o } B\|_2 \leq n \frac{j_n j_{n-1}}{a\, 2^{n-2}}$$

**Proof.** The proof is easily seen by

$$\|A \text{ o } B\|_2 \leq \|A\|_2 \; \|B\|_2$$

■

## 4. Conclusion

In this study we compute upper and lower bounds of spectral norms of *r*-circulant and geometric circulant matrices with the bi-periodic Jacobsthal numbers. If we take a = b = 1, we get upper and lower bounds of spectral norms of *r*-circulant and geometric circulant matrices with the Jacobsthal numbers. And also ıf we take *r = 1* in geometric and r-circulant matrices we get upper and lower bounds of spectral norms of circulant matrices with the Jacobsthal numbers

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**INTERNATIONAL CONFERENCE ON MATHEMATICS**
**"An Istanbul Meeting for World Mathematicians"**
**3-5 July 2019, Istanbul, Turkey**

# Bazı Ultra Normlu Uzaylar

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**Özet**

Bu çalışmada metrik uzayları, ultra metrik uzayları, normlu uzayları ve ultra normlu uzayları tanıtıp aralarındaki ilişkileri ve örneklerini gösterdik. Ultra normlu uzaylarda izometri ve ultra izometri tanımlarını yaptık. Sonra bazı ultra normlu uzaylara örnek verip ultra Banach uzay olduğunu gösterdik.

**Anahtar kelimeler: Ultra Metric, Ultra Norm, İzometri, Krull Sharpening, Ultra izometri, Ultra Banach Uzayı.**

## 1. Giriş

Bu çalışmada metrik uzay aksiyomlarından $d(x,y) \leq d(x,z) + d(z,y)$ $(\forall x, y, z \in X)$ (Üçgen eşitsizliği aksiyomu) yerine $d(x,y) \leq \text{maks}\{ d(x,z) , d(z,y)\}$ (Güçlü üçgen eşitsizliği aksiyomu) alınarak elde edilen metriğe ultra metrik denmiş ve bu sayede pek çok şaşırtıcı, doğal olmayan sonuçlar ortaya çıkmıştır. Ultra metrik ile metrik uzayları, ultra normlu ile normlu uzayları ve ilişkilerini gösterdik. Biz burada güçlü üçgen eşitsizliğini kullanarak ultra metrik uzay örneklerini, ultra normu, ultra metrik Banach uzaylarını bu uzayların çeşitli özelliklerini inceledik. Dolayısıyla karşımıza farklı uzaylar ve yapılar ortaya çıktı. Ayrıca Diagana, Krull, Ludkovsky ve Şanlıbaba [1,2,3,4] ultrametrik uzayların yapılarını, bu uzaylara örnekleri ve kapsama durumlarını göstermişlerdir.

## 2. Temel Tanım ve Teoremler

**Tanım 2.1**. Bir $X$ boş olmayan bir küme ve $X$ üzerinde $d: X \, x \, X \to \mathbb{R}$ şeklinde tanımlı fonksiyonu $d$ verilmiş olsun. $\forall x, y, z \in X$ için aşağıdaki aksiyomları sağlayan $d$ fonksiyonuna metrik, $(X, d)$ ikilisine metrik uzay denir.

(m1) $d(x,y) = 0 \Leftrightarrow x = y$,

(m2) $d(x,y) = d(y,x)$,

(m3) $d(x,y) \leq d(x,z) + d(z,y)$ .

Negatif olmayan $d(x,y)$ reel sayısına $x$ ile $y$ elemanları arasındaki uzaklık denir [5].

Metrik uzay tanımının özel hali olan ultra metrik uzay tanımı aşağıda verilmiştir:

**Tanım 2.2.** $X$ bir küme ve $d_u: X \text{ x } X \rightarrow \mathbb{R}$ fonksiyonu verilsin. $\forall x, y, z \in X$ için $d_u$ fonksiyonu

(um1) $d_u(x,y) = 0 \;\Leftrightarrow\; x = y$

(um2) $d_u(x,y) = d_u(y,x)$

(um3) $d_u(x,y) \leq \text{maks}\{ d_u(x,z), d_u(z,y)\}$

şartlarını sağlıyorsa $d_u{}'$ ya ultrametrik, $(X, d_u)$ ya da ultra metrik uzay, (um1), (um2) ve (um3) ye de ultra metrik uzay aksiyomları denir.

Dikkat edilirse, (um1) şartı (m1) şartı ile, (um2) şartı da (m2) şartı ile aynıdır. $d(x,y) \leq \text{maks}\{d(x,z), d(z,y)\}$ eşitsizliğinde eğer $\text{maksimum}$ değer $d(x,z)$ ise açık olarak $d(x,y) \leq d(x,z) + d(z,y)$ eşitsizliği sağlanır. Eğer $\text{maksimum}$ değer $d(z,y)$ ise yine $d(x,y) \leq d(x,z) + d(z,y)$ eşitsizliği sağlanır. Şu halde (um3) den (m3) eşitliği elde edilir. Dolayısıyla her ultra metrik uzay aynı zamanda klâsik anlamda metrik uzay olur.

Şimdi $d: \mathbb{R} \text{ x } \mathbb{R} \rightarrow \mathbb{R}$, $d(x,y) = |x - y|$ ile tanımlı $\mathbb{R}$ üzerindeki doğal metriği göz önüne alalım. $d$'nin m1-m3 şartlarını sağladığı açıktır. Fakat $d(x,z) + d(z,y) = d(x,y) \leq \text{maks}\{ d(x,z), d(z,y)\}$ yazılamayacağından $d$, $\mathbb{R}$ de bir metrik olmasına rağmen (um3) şartını sağlamadığından ultra metrik değildir. Dolayısıyla $(\mathbb{R}, d)$ ikilisi ultra metrik uzay olmaz.

Şu halde aşağıdaki teoremi verebiliriz:

**Teorem 2.1.** Her ultra metrik uzay bir metrik uzaydır ama her metrik uzay ultra metrik uzay olmak zorunda değildir.

**Örnek 2.1.** Boş olmayan her hangi bir $X$ kümesi verilsin. $X \text{ x } X$ den $\mathbb{R}$ ye tanımlanan

$$d_u(x,y) = \begin{cases} 0, x = y \\ 1, x \neq y \end{cases} \tag{2.1}$$

dönüşümü $X$ için bir ultra metriktir [6], [7].

**Tanım 2.3.** $X$ bir küme ve $X$ üzerinde toplama ve skalarla çarpma işlemleri tanımlanmış olsun. $\forall x \in X$ için $0 + x = 0$ olacak şekilde $X$ in birimi $\theta$ ile gösterelim.

Eğer $g: X \to \mathbb{R}$ fonksiyonu $\forall x, y \in X$ ve $\forall \alpha \in \mathbb{C}$ için

(un1) $g(x) = 0 \Leftrightarrow x = \theta$

(un2) $g(\alpha x) = |\alpha|\, g(x)$

(un3) $g(x + y) \leq \text{maks}\{g(x), g(y)\}$

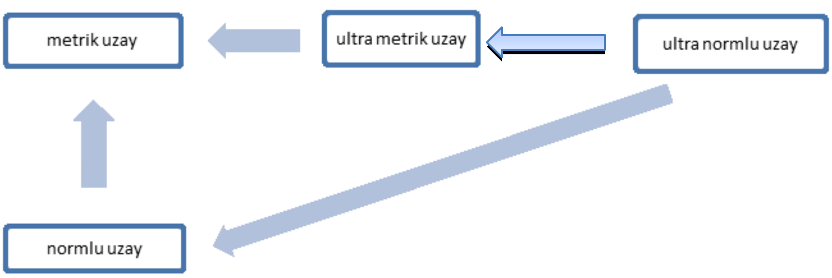
şartlarını sağlıyorsa $g$ ye $X$ üzerinde ultra norm denir. n1-n3 ile un1- un3 karşılaştırıldığında (un3) ile (n3) ün farklı olduğu görülür.

$g(x + y) \leq g(x) + g(y)$ eşitsizliği $g(x + y) \leq \text{maks}\{g(x), g(y)\}$ eşitsizliği mevcut iken mevcut fakat $g(x + y) \leq g(x) + g(y)$ olmasının $g(x + y) \leq \text{maks}\{g(x), g(y)\}$ olmasını gerektirmediğinden her ultra norm bir normdur, tersi doğru değildir.

Biliyoruz ki her norm bir metriktir. Bu önerme ultra metrik ve ultra norm bakımından da geçerlidir [6].

Açık olarak bir $X$ normlu uzayının ilk iki şartı ultra normlu uzayların ilk iki şartı ile aynıdır. Yine normlu uzaylardaki üçgen eşitsizliği şartının $\|x + y\| \leq \text{maks}\{\|x\|, \|y\|\}$ mevcut iken mevcut ama $\|x + y\| \leq \|x\| + \|y\|$ olmasının $\|x + y\| \leq \text{maks}\{\|x\|, \|y\|\}$ olmasını gerektirmeyeceği görülebilir. Demek ki her ultra normlu uzay normlu uzay, tersine her normlu uzay ultra normlu uzay olmak zorunda değildir, önermesini verebiliriz.

Şimdiye kadar söylenenleri aşağıdaki diyagramla özetleyebiliriz:



Şekil 1. Metrik Uzay, Ultra Metrik Uzay, Normlu Uzay, Ultra Normlu Uzay İlişkilerini Açıklayan Diyagram

**Tanım 2.4.** Bir Arşimedyan olmayan yani ultra normlu $X$ lineer uzayı tam ise bu uzaya Arşimedyan olmayan Banach uzayı veya ultra normlu Banach uzayı denir.

**Tanım 2.5.** $X$ bir normlu uzay olsun. $\forall x, y \in X$ ve $\|x\| \neq \|y\|$ olmak üzere $\|x+y\| \leq \text{maks}(\|x\|, \|y\|)$ eşitsizliği sağlanıyorsa $\|.\|$ ye *Krull* özelliğine sahiptir denir [2].

## 3. Bazı Ultra Normlu Uzaylar ve İzomorfiklikleri

**Tanım 3.1.** (Ultra izometri) $X$ ve $Y$, $K$ cismi üzerinde ultra normlu uzaylar olsun ve $T: X \to Y$ dönüşümü verilsin. Eğer $\forall x \in X$ için

$$\|T(x)\|_Y = \|x\|_X$$

ise yani ultra normu koruyorsa $T$ ye $X$ den $Y$ ye bir ultra izometri denir.

**Tanım 3.2.** $X$ ve $Y$, aynı $K$ cismi üzerinde ultra normlu uzaylar olsun. Eğer $T: X \to Y$ lineer, 1-1, örten izometrisi varsa $X$ ve $Y$ uzayları ultra izometrik olarak ultra izomorfiktir denir, bu husus $X_u \cong Y_u$ ile gösterilir.

$K$ bir cisim olsun. $|.|: K \to \mathbb{R}$ fonksiyonu $\forall x, y \in K$ için $(d3)'$ yani $|x+y| \leq \text{maks}\{|x|, |y|\}$ eşitsizliğini sağlamak üzere her $k \in \mathbb{N}$ için $(\rho_k)$ dizisi $(0 \neq \rho_k)$ ve sınırlı olacak şekilde bir dizi, $I$ da bir indis kümesi olmak üzere, $K$ üzerinde

$$l_\infty(I, K, \rho) = \{x = (x_k) \in K : \sup_{k \in \mathbb{N}} |x_k| \rho_k < \infty\}$$

cümlesini tanımlayalım.

$$\|.\|: l_\infty(I, K, \rho) \to \mathbb{R}$$

$$x \to \|x\| = \sup_k |x_k| \rho_k$$

ile tanmlı $\|.\|$ fonksiyonu verilsin. $(l_\infty(I, K, \rho), \|.\|)$ ikilisi Arşimedyan olmayan Banach uzayıdır.

Benzer olarak,

$c(I,K,\rho) = \{x = (x_k) \in K: \lim_k |x_k|\rho_k \text{ mevcut}\}$ ve

$c_0(I,K,\rho) = \{x = (x_k) \in K: \lim_k |x_k|\rho_k = 0\}$

kümeleri de $\|x\| = \sup_k |x_k|\rho_k$ ile beraber Arşimedyan olmayan Banach uzaylarıdır [1,8,9,10].

$l_\infty(I,K,\rho)$, $c(I,K,\rho)$ ve $c_0(I,K,\rho)$ kümelerine sırası ile ultra sınırlı (veya Arşimedyan olmayan sınırlı), ultra yakınsak (veya Arşimedyan olmayan yakınsak) ve ultra sıfıra yakınsak (veya Arşimedyan olmayan sıfıra yakınsak ) dizilerin uzayları adı verilir.

Biz bunlardan sadece $l_\infty(I,K,\rho)$ nin ultra normlu Banach uzayı olduğunu göstereceğiz, diğerleri de benzer olarak gösterilebilir.

(un1) $\|x\| = \sup_k |x_k|\rho_k = 0 \Leftrightarrow |x_k|\rho_k = 0 \Leftrightarrow |x_k| = 0 \Leftrightarrow x_k = 0 \Leftrightarrow x = 0,$

(un2) $\|\alpha x\| = \sup_k |\alpha x_k|\rho_k = |\alpha| \sup_k |x_k|\rho_k = |\alpha|\|x\|$ dir.

(un3)

$$\begin{aligned}\|x+y\| = \sup_k |x_k + y_k| \rho_k &\leq \sup_k \{\text{maks}\{|x_k|\rho_k, |y_k| \rho_k\} \\ &= \sup_k \{\text{maks}\{|x_k + \theta|\rho_k, |y_k + \theta| \rho_k\} \\ &\leq \sup_k \{\text{maks}\{\text{maks}\{|x_k| \rho_k, \theta\}, \text{maks}\{|y_k| \rho_k, \theta\}\}\} \\ &= \text{maks}\{\sup_k \{\text{maks}\{|x_k| \rho_k, \theta\}, \sup_k \{\text{maks}\{|y_k| \rho_k, \theta\}\}\} \\ &= \text{maks}\{\sup_k d(x_k, \theta), \sup_k d(y_k, \theta)\} = \text{maks}\{\|x\|, \|y\|\}.\end{aligned}$$

Demek ki $l_\infty(I,K,\rho)$ ultra normlu uzaydır.

$ii)$ Şimdi tamlığı ispatlayalım.

Kabul edelim ki $(x^n)$, $l_\infty(I,K,\rho)$ de bir Cauchy dizisi olsun. Eğer $(x^n)$ sabit bir dizi ise durum açıktır.

Eğer $(x^n)$ sabit bir dizi değilse bu takdirde $m,n \geq n_0$ olacak şekilde

$$\|x^m - x^n\| = sup_{k\in I} |x_k{}^m - x_k{}^n|\rho_k < \varepsilon \qquad (*)$$

olacak şekilde bir $n_0$ tam sayısı vardır. Buradan $\varepsilon > 0$ ve $k = 1,2, \dots$ olmak üzere keyfi fakat sabit her $k$ için $m, n \geq n_0$ olduğunda

$|x_k{}^m - x_k{}^n| < \varepsilon$ elde edilir. O halde her sabit $k$ için $(x_k^1, x_k^2, x_k^3, \dots)$ $K$ ' de bir Cauchy dizisidir. $K$ tam olduğundan $x_k^m \to x_k \in K$. Her $k$ doğal sayısı için elde edeceğimiz bu limitler yardımıyla $K$ ' de $x = (x_1, x_2, \dots)$ dizisini teşkil edelim.

$$x_1 = (x_1^1, x_2^1, \dots, x_n^1, \dots)$$

$$x_2 = (x_1^2, x_2^2, \dots, x_n^2, \dots)$$

$$x_3 = (x_1^3, x_2^3, \dots, x_n^3, \dots)$$

$$\vdots$$

$$x_k = (x_1^k, x_2^k, \dots, x_n^k, \dots)$$

$$\vdots$$

$$x_m = (x_1^m, x_2^m, \dots, x_n^m, \dots)$$

$$\downarrow \quad \downarrow \quad \downarrow \cdots \quad \downarrow \cdots$$

$$x = (x_1, x_2, \ \dots, x_n, \dots)$$

Şimdi $x \in l_\infty(I, K, \rho)$ ve $x_n \to x, n \to \infty$ olduğunu göstereceğiz. (*) ifadesinde $n \to \infty$ yapılırsa $\|x_k{}^n - x_k\|_{l_\infty(I,K,\rho)} < \varepsilon$ elde edilir.

$x_n = (x_k^n) \in l_\infty(I, K, \rho)$ olduğundan $k = 1,2, \dots$ için $\|x_k{}^n\| \leq t_n$ olacak şekilde bir $t_n$ reel dizisi vardır. Kuvvetli üçgen eşitsizliğinden

$$\|x_k\|\rho_k = \|x_k - x_k{}^n + x_k{}^n\|\rho_k \leq maks\{\|x_k - x_k{}^n\|, \|x_k{}^n\|\}\rho_k \leq maks\{\varepsilon, t_n\}M < \infty.$$

Bu eşitsizlik her $k$ için geçerli ve sağ tarafta $k$' yi ihtiva etmediğinden $(x_k) \in l_\infty(I, K, \rho)$ dir.□

Her ultra normlu uzay, normlu uzaydır (Şekil 1’e bakınız). Yukarıdaki teoremde eğer $\rho_k = 1$ alınırsa $l_\infty(I,K,\rho)$ bildiğimiz anlamda $l_\infty$ uzayına, $c(I,K,\rho)$ uzayı *c* yakınsak dizilerin uzayına $c_0(I,K,\rho)$ de $c_0$ uzayına dönüşür.

**4. Sonuç**

Bu çalışmada ultra metrik, ultra norm, ultra metrik Banach uzayları incelenmiş, teoremler ve örnekler verilmiştir. Sonuç olarak ultra metrik uzaylar metrik uzayların kapsadığı daha özel yapılar olduğu görülmüştür ve metrik uzaylarda geçerli olan tüm özelliklerin ultra metrik uzayda olmadığı fark edilmiştir. Ultra metrikler daha spesifik oldukları için bazen şaşırtıcı sonuçlar elde edilmiştir. Ayrıca Ultra normlu Banach uzayına örnekler verilmiş, ispatlanmış ve ultra izomorfiklik tanımı yapılmıştır.

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INTERNATIONAL CONFERENCE ON MATHEMATICS
"An Istanbul Meeting for World Mathematicians"
3-5 July 2019, Istanbul, Turkey

**The Spectrum of New Type Boudary Value Problems**

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**Abstract**

Sturm-Liouville type boundary value problems arise throughout in physics, engineering, electromagnetics, elasticity and other branches of natural sciences. For example, they describe the energy eigenfunctions of a quantum mechanical oscillator, the vibrational modes of a various systems, the heat and mass transfer problems and etc. Usually, the eigenvalue parameter appears linearly only in the differential equation of the classic Sturm-Liouville problems. However, in solving of many significant physics problems the eigenvalue parameter appear also in the boundary conditions. There is quite substantial literature on such type problems. Moreover, Sturm-Liouville problems with the eigenvalue parameter in boundary conditions together with additional transmission conditions at some interior points of interaction arise in various problems of applied mathematics. Note that boundary-value-transmission problems arise in diffraction problems, in vibrating string problems, when the string loaded additionally with point masses and etc. This study deal with a new type boundary-value-transmission problems for many-interval Sturm-Liouville equations. For self-adjoint realization of the considered problem we define alternative inner product in the Lebesgue space of square-integrable functions. We shall establish some properties of the spectrum.

**Keywords :** Boundary-value problem, eigenvalue, eigenfunction, Hilbert space.

# 1 Introduction

Many subjects in physics and engineering are developing depending on the studies in partial differential equations. It is not possible to list all these. For example, there are many topics, such as acoustics, aerodynamics, elastic, electrodynamics, fluid dynamics, geophysics (seismic wave propagation), heat transfer, meteorology, ocean science, optics, petroleum engineering, plasma physics (liquid and gas ionization), quantum mechanics which are largely dependent on the studies in partial differential equations (see, for example, [10, 11, 15, 18]).

In order to apply the Fourier method (i.e. the method of separation of variables) in the solution of many problems of mathematics physics, it is necessary to express the given function as series expansion in terms of eigenfunctions of boundary-value problem for ordinary differential equations. In addition, for justification of this method it is needed investigation also the eigenvalues of the considered boundary value problem.

This study is concerned with the theory of boundary value problems with supplementary transmission conditions, the spectral theory of linear differential operators and connections between the two fields. In this paper we consider a Sturm-Liouville equation together with eigenparameter dependent boundary conditions and transmission conditions. By using a new approach we obtain that the eigenvalues of the pure differential part of the considered problem are real and the corresponding eigenfunctions are orthogonal with respect to modified inner-product. By modifying some classic methods we obtain asymptotic formulas for eigenvalues and corresponding eigenfunctions.

In the present work we consider a new class of boundary value problems which consists of a Sturm-Liouville equation involving an abstract linear operator $\mathcal{B}$, namely the equation

$$\ell u \ := \ -u''(x) \ + \ q(x)u(x) + (\mathcal{B}u)(x) \ = \ \lambda u(x) \tag{1.1}$$

on $(-1,0)\cup(0,1)$, together with eigen-dependent boundary conditions;

$$\ell_1(\lambda)u \ := \ \alpha_1 u'(-1) \ - \ \alpha_2 u(-1) \ + \ \lambda\alpha_3 u(-1) \ = \ 0 \tag{1.2}$$

$$\ell_2(\lambda)u \ := \ \beta_1 u'(1) \ - \ \beta_2 u(1) \ + \ \lambda\beta_3 u(1) \ = \ 0, \tag{1.3}$$

and with eigen-dependent transmission conditions at point of discontinuity $x=0$;

$$\ell_3 u \ := \ u(0^+) \ - \ u(0^-) \ = \ 0, \tag{1.4}$$

$$\ell_4(\lambda)u \ := \ u'(0^+) \ - \ u'(0^-) \ + \ \lambda\delta_1 u(0) \ = \ 0 \tag{1.5}$$

where $\alpha_i, \beta_i, \delta_1,\ (i = 1,2,3)$ are real numbers, the real-valued function $q(x)$ continuous in each of $[-1,0)$ and $(0,1]$ and has a finite limits $q(\mp 0) = \lim_{x\to\mp 0} q(x)$ , $\lambda$ is a complex spectral parameter, and $\mathcal{B}$ is an abstract linear operator (unbounded and non-self-adjoint in general) in the Hilbert space $L_2(-1,0)\oplus L_2(0,1)$. Naturally, everywhere we will assume that $\delta_1 > 0$, $D(\mathcal{B}) \subset W_2^2(-1,0)\oplus W_2^2(0,1)$, $\alpha_1\alpha_3 > 0$ and $\beta_1\beta_3 > 0$.

Since the values of the solutions and their derivatives at the interior point $x = 0$ is not defined, an important question is how to introduce a new Hilbert space such a way that the considered problem can be interpreted as self-adjoint problem in this space. Note that boundary value problems together with supplementary transmission conditions appear frequently in various fields of physics and technics. For example, in electrostatics and magnetostatics the model problem which describes the heat transfer through an infinitely conductive layer is a transmission problem (see, [8] and the references listed therein). Another completely different field is that of hydraulic fracturing (see, [6]) used in order to increase the flow of oil from a reservoir into a producing oil well. Boundary value problems with transmission conditions (but without abstract operator $\mathcal{B}$ in the equation) were investigate extensively in the resent years (see, for example [1, 2, 3, 4, 5, 7, 9, 13, 14, 16, 17]).

# 2 Some preliminary results in according Hilbert space

For spectral analysis of the Sturm-Liouville problem (1.1)-(1.5) we construct an adequate Hilbert space and define a symmetric linear operator in such a way that the considered problem can be interpreted as the eigenvalue problem of this operator. Namely, in the Hilbert space $(L_2(-1,0) \oplus L_2(0,1)) \oplus \mathbb{C}^3$ introduce a new inner product by

$$\begin{aligned}\langle U,V\rangle_{H_0} &= \int_{-1}^{0} u(x)\overline{v(x)}dx + \int_{0}^{1} u(x)\overline{v(x)}dx \\ &+ \frac{u_1\overline{v_1}}{\alpha_1\alpha_3} + \frac{u_2\overline{v_2}}{\beta_1\beta_3} + \frac{u_3\overline{v_3}}{\delta_1}\end{aligned}$$

for $F = \big( u(x), u_1, u_2, u_3 \big),\quad V = \big( v(x), v_1, v_2, v_3 \big) \in (L_2(-1,0) \oplus L_2(0,1)) \oplus \mathbb{C}^3$.

**Theorem 2.1.** $H_0 := \Big(\big(L_2(-1,0) \oplus L_2(0,1)\big) \oplus \mathbb{C}^3 \,,\, \langle\ ,\ \rangle_{H_0}\Big)$ *is a Hilbert space.*

Let us introduce an auxiliary linear operator $\mathcal{L} : H_0 \to H_0$ with action low

$$\mathcal{L}F = (-f'' + q(x)f, f'(1))$$

and domain $D(\mathcal{L})$ consisting of all $U = \big( u(x), u_1, u_2, u_3 \big) \in H_0$ which satisfy the following conditions:

i. $u, u'$ are absolutely continuous functions on $[-1,-\gamma]$ and $[\gamma, 1]$ for arbitrary $\gamma > 0$,

ii. There are finite one-hand limits $u(0^{\pm})$ and $u'(0^{\pm})$,

iii. $\ell u \in L_2(-1,0) \oplus L_2(0,1)$,

iv. $u_1 = \alpha_3 u(-1),\ u_2 = \beta_3 u(1),\ u_3 = \delta_1 u(0)$.

# 3 Main Results

**Theorem 3.1.** *The differential operator $\mathcal{L}$ is densely defined in the Hilbert space $H_0$, i.e., $D(\mathcal{L}) = H_0$.*

**Theorem 3.2.** *The domain of definition of the adjoint operator $\mathcal{L}^*$ coincide with the domain of definition of $\mathcal{L}$, i.e.,*

$$D(\mathcal{L}^*) = D(\mathcal{L}).$$

**Theorem 3.3.** *The operator $\mathcal{L}$ is self-adjoint in the Hilbert space $H_0$.*

*Proof.* Let $U, V \in D(\mathcal{L})$. By two partial integration we obtain

$$\begin{aligned}\langle \mathcal{L}U, V\rangle &= \langle U, \mathcal{L}V\rangle + W(u,\overline{v}\,;\,0^-) - W(u,\overline{v}\,;\,-1) + W(u,\overline{v}\,;\,1) - W(u,\overline{v}\,;\,0) \\ &+ \frac{1}{\alpha_1\alpha_3}\left\{\Big(\alpha_2 u(-1) - \alpha_1 u'(-1)\Big)\alpha_3\overline{v}(-1) - \alpha_3 u(-1)\Big(\alpha_2\overline{v}(-1) - \alpha_1\overline{v'}(-1)\Big)\right\} \\ &+ \frac{1}{\beta_1\beta_3}\left\{\beta_3 u(1)\Big(\beta_2\overline{v}(1) - \beta_1\overline{v'}(1)\Big) - \Big(\beta_2 u(1) - \beta_1 u'(1)\Big)\beta_3\overline{v}(1)\right\} \\ &+ \frac{1}{\delta_1}\left\{\Big(u'(0^+) - u'(0^-)\Big)(-\delta_1\overline{v}(0)) + \delta_1 u(0)\Big(\overline{v'}(0^+) - \overline{v'}(0^-)\Big)\right\}\end{aligned}$$

where, as usual, by $W(u, v\ ;\ x)$ we denote the Wronskian of the functions $u$ and $v$ :

$$W(u, v\ ;\ x) \ = \ u(x)v'(x) \ - \ u'(x)v(x).$$

Since $u$ and $\overline{v}$ are satisfied the boundary conditions (1.2)-(1.3) and transmission conditions (1.4)-(1.5) we get

$$\langle \mathcal{L}U, V\rangle \ = \ \langle U, \mathcal{L}V\rangle \quad , \quad (U, V \in D(\mathcal{L}))$$

so $\mathcal{L}$ is self-adjoint. □

**Corollary 3.4.** *The operator $\mathcal{L}$ has only real eigenvalues.*

**Corollary 3.5.** *Let $\lambda$ and $\mu$ be two eigenvalues of $\mathcal{L}$ with the corresponding eigenelements $u(x, \lambda)$ and $u(x, \mu)$ respectively. If $\lambda \neq \mu$ then*

$$\langle u(.\ ,\ \lambda)\ ,\ u(.\ ,\ \mu)\rangle_{H_0} \ = \ 0.$$

**Theorem 3.6.** *The operator $\mathcal{L}$ has an precisely denumerable many eigenvalues $\lambda_1 \ < \ \lambda_2 \ < \ ...$ with $\lambda_n \ \to \ \infty$ as $n \ \to \ \infty$.*

Let us introduce to the consideration the following operators in the Hilbert space $H_0$, as

$$\widetilde{\mathcal{B}}U \ := \ (\mathcal{B}u, 0, 0, 0) \quad \text{and} \quad \widetilde{\mathcal{L}}U \ := \ \mathcal{L}U \ + \ \widetilde{\mathcal{B}}U$$

for $U \ = \ (u(x), u_1, u_2, u_3) \in D(\mathcal{L})$.

**Definition 3.7.** *[12] Let $T$ be any closed linear operator which has at least one regular value $\mu$, and let $S$ is linear operator. If the operator $S(T \ - \ \mu I)^{-1}$ is compact, then the operator $S$ is said to be $S-$ compact.*

**Theorem 3.8.** *Let the operator $\widetilde{\mathcal{B}}$ be $\mathcal{L}-$ compact in the Hilbert space $H_0$. Then*

*i. The BVTP (1.1)-(1.5) has precisely denumerable many eigenvalues $\widetilde{\lambda}_n$,*

*ii. For any $\varepsilon \ > \ 0$ there is $n_0 = n_0(\varepsilon)$ such that $|arg\widetilde{\lambda}_n| \ < \ \varepsilon$ for all $n \ > \ n_0(\varepsilon)$.*

*iii. $|\widetilde{\lambda}_n \ - \ \lambda_n| \ = \ o(n^2)$.*

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INTERNATIONAL CONFERENCE ON MATHEMATICS
"An Istanbul Meeting for World Mathematicians"
3-5 July 2019, Istanbul, Turkey

**The Rayleigh Principle for Transmission Problems**

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**Abstract**

Transmission problems for the Sturm-Liouville equations with discontinuous coefficients arise in many problems of physics, such as in modeling toroidal vibrations of the earth, in vibrating of a loaded string, in diffraction problems and etc. It is important to find a complete set of eigenfunctions, or equivalently, to diagonalize the suitable differential operator in adequate infinite-dimensional Hilbert space. In the finite-dimensional case, the spectrum of a linear operator consists only of its eigenvalues. However, the linear operators on infinite-dimensional Hilbert spaces may have not only a point spectrum of eigenvalues, but also a continuous spectrum. For many applications in science and engineering it is required to determine the eigenvalues as well as the corresponding eigenfunctions. In fact, the general theory of eigenvalues and eigenfunctions is one of the deepest and richest parts of pure and applied mathematics, mathematical physics and engineering. This study devoted to the investigation of the Sturm-Liouville type boundary-value problems with supplementary transmission conditions. We derive some extremal properties of the eigenvalues and corresponding eigenfunctions of the considered boundary-value-transmission problems by using variational methods. Also, we shall modified the Rayleigh method for investigation some computational aspects of the eigenvalues.

**Keywords :** Boundary-value problem, Rayleigh quotient, eigenfunctions, eigenvalue.

# 1 Introduction

For many important applications in physics, engineering and other branches of natural science it is required to determine the eigenvalues as well as the corresponding eigenfunctions of corresponding boundary value problem for Sturm-Liouville equations. Therefore the Sturm-Liouville theory is one of the most actual and extensively developing fields in pure and applied mathematics. In fact, the general theory of eigenvalues and eigenfunctions is one of the deepest and richest parts of functional analysis, operator theory, spectral theory, mathematical physics and engineering. Several problems of physics and engineering are often stated as boundary value problems (BVP's for short). Among these BVP's, the Sturm-Liouville type BVTP's is a typical one, since Sturm-Liouville theory plays an important role in solving various problems in mathematical physics and engineering. The Sturm-Liouville problems that arise from diverse mechanical models and contain the eigenvalue parameter in the boundary conditions have been studied in various formulations by many authors (see, for instance, [5, 6, 14, 18, 19] and corresponding references cited therein). In different areas of applied mathematics and physics many problems arise in the form of boundary value problems involving not only eigen-parameter in the boundary conditions, but also supplementary transmission conditions at the interior singular points. This kind of problems are called boundary-value-transmission problems. For example, such type of problems arise in heat and mass transfer problems in diffraction problem, in vibrating string problems when the string loaded with additional point masses and etc. (see, for example, [11, 12, 17, 20]). In the recent years there is a growing interest in discontinuous Sturm-Liouville problems with the supplementary transmission conditions at the interior singular points (see, [1, 2, 3, 4, 9, 13, 15]). It is well-known that Sturm-Liouville problems with eigenparameter dependent boundary conditions can be interpreted as spectral problems for operator polynomials. The general spectral results about operator polynomials can be found in monographs by Gohberg and Krein [7], Ladyzhenskaia [10] and Rodman [16]. Completeness and eigenfunction expansions have been considered in [5, 8, 15].

In the present work we consider a Sturm-Liouville equation

$$\tau f \ := \ -f''(x) \ + \ q(x)f(x) \ = \ \lambda f(x) \tag{1.1}$$

on $[a, c_1 - 0) \cup (c_1 + 0, c_2 - 0) \cup (c_2 + 0, b]$, together with eigendependent boundary conditions at the end-points $x = a$ and $x = b$, given by

$$\tau_1(\lambda) f \ := \ \alpha_1 f(a) \ - \ \alpha_3 f'(a) \ - \ \lambda \alpha_2 f'(a) \ = \ 0 \tag{1.2}$$

$$\tau_2 f \ := \ f'(b) \ = \ 0 \tag{1.3}$$

and with transmission conditions at two interior points $x = c_1$ and $x = c_2$, given by

$$\tau_3 f \ := \ f(c_1 + 0) \ - \ \ell_1 f(c_1 - 0) \ = \ 0, \tag{1.4}$$

$$\tau_4 f \ := \ f'(c_1 + 0) \ - \ \frac{1}{\ell_1} f'(c_1 - 0) \ - \ f(c_1 - 0) \ = \ 0, \tag{1.5}$$

$$\tau_5 f \ := \ f(c_2 + 0) \ - \ \ell_2 f(c_2 - 0) \ = \ 0, \tag{1.6}$$

$$\tau_6 f \ := \ f'(c_2 + 0) \ - \ \frac{1}{\ell_2} f'(c_2 - 0) \ - \ f(c_2 - 0) \ = \ 0, \tag{1.7}$$

where $q(x)$ is a real valued function which is continuous in $J = J_1 \cup J_2 \cup J_3$ , $J_1 = (a, c_1)$ , $J_2 = (c_1, c_2)$ and $J_3 = (c_2, b)$ , and has finite limits $q(a+0), q(c_1 \pm 0), \ q(c_2 \pm 0), q(b-0), \ \lambda$ is a complex spectral parameter, $\alpha_i, \ell_j$ ($i = 1, 2, 3$ and $j = 1, 2$) are real numbers and $\rho = \alpha_1 \alpha_2 > 0$ and $\ell_j > 0$.

# 2 Definitions and Facts Related to Hilbert Spaces

Let $f = f(x)$ be any function defined on $J$, then by $f_i$ we shall denote the restriction of $f(x)$ on the interval $J_i$ ($i = 1, 2, 3$). Below, by $\mathbb{H}_0$ we denote the direct sum of the Hilbert spaces $L_2(J_1) \oplus L_2(J_2) \oplus L_2(J_3)$ with the inner product

$$\langle f, g \rangle_{\mathbb{H}_0} := \langle f_1, g_1 \rangle_{L_2(J_1)} + \langle f_2, g_2 \rangle_{L_2(J_2)} + \langle f_3, g_3 \rangle_{L_2(J_3)}.$$

We can prove that the linear subspace

$$\mathbb{H}_1 = \left\{ f(x) \in \mathbb{H}_0 \;\middle|\; f_i(x) \in W_2^1(J_i),\ \tau_3 f \ = \ 0\ ,\ \tau_5 f \ = \ 0 \right\} \tag{2.1}$$

equipped with the inner-product

$$\begin{aligned} \langle f, g \rangle_{\mathbb{H}_1} \ &:= \ \int_a^{c_1-0} \left( \frac{df_1}{dx}\frac{d\overline{g_1}}{dx} + f_1(x)\overline{g_1}(x) \right) dx + \int_{c_1+0}^{c_2-0} \left( \frac{df_2}{dx}\frac{d\overline{g_2}}{dx} + f_2(x)\overline{g_2}(x) \right) dx \\ &+ \ \int_{c_2+0}^{b} \left( \frac{df_3}{dx}\frac{d\overline{g_3}}{dx} + f_3(x)\overline{g_3}(x) \right) dx \end{aligned} \tag{2.2}$$

form a Hilbert space, where by $W_2^1(J_i)$ we denote the Hilbert space of all square-integrable complex valued functions $f$ having generalized derivatives $\frac{df}{dx} \in L_2(J_i)$ with the inner product $\langle f, g \rangle_{W_2^1(J_i)} = \int_{J_i} \left( f(x)\overline{g}(x) + \frac{df}{dx}\frac{d\overline{g}}{dx} \right) dx$.

By virtue of the embedding theorems for the Sobolev spaces ([7]) the functions in $\mathbb{H}_1$ are continuous on $J_1 \cup J_2 \cup J_3$, but their generalized derivatives can only be assumed to be elements of $\mathbb{H}_0$.

Everywhere in below we shall assume that $q(x)$ is bounded, positively defined and measurable function on $J \ = \ J_1 \cup J_2 \cup J_3$. Consequently, $q(x)$ is integrable on $J$.

Now, in the Hilbert space $\mathbb{H}_1$ we can introduce a new inner-product by

$$\langle f, g \rangle_{\mathbb{H}_{1,q}} := \langle f, qg \rangle_{\mathbb{H}_0} + \langle \frac{df}{dx}, \frac{dg}{dx} \rangle_{\mathbb{H}_0}. \tag{2.3}$$

Since $q(x)$ is bounded, positively defined and measurable function, there exist constants $m > 0$ and $M > 0$ such that

$$m \ \|f\|_{\mathbb{H}_1} < \|f\|_{\mathbb{H}_{1,q}} < M \ \|u\|_{\mathbb{H}_1}$$

for all $f \in \mathbb{H}_1$, so the inner-product $\langle .\ ,\ . \rangle_{\mathbb{H}_{1,q}}$ is equivalent to $\langle .\ ,\ . \rangle_{\mathbb{H}_1}$.

The concept of a weak solution is fundamental to this work. To define this concept let us introduce to the consideration the Hilbert space $\Xi$, consisting of all vector-functions $\left( \ \chi(x)\ ,\ \chi_1 \ \right) \in \mathbb{H}_1 \ \oplus \ \mathbb{C}$, equipped with the inner product

$$< \Gamma, \Psi >_{\Xi} \ := \ < \chi, \varphi >_{\mathbb{H}_1} + \chi_1 \ \overline{\varphi_1}, \tag{2.4}$$

where $\Gamma = \left( \ \chi\ ,\ \chi_1 \ \right)$ and $\Psi = \left( \ \varphi\ ,\ \varphi_1 \ \right) \in \Xi$.

By multiplying the differential equation (1.1) by the complex conjugate of an arbitrary function $\eta \in \mathbb{H}_1$ satisfying the transmission conditions $\tau_3 \eta \ = \ 0$ and $\tau_5 \eta \ = \ 0$ and integrate by parts over the intervals $J_i (i = 1, 2, 3)$, we get

$$\langle f\ ,\ \eta \rangle_{\mathbb{H}_1} \ + \ \ell_1 f(c_1 - 0)\overline{\eta}(c_1 - 0) \ + \ \ell_2 f(c_2 - 0)\overline{\eta}(c_2 - 0) \ + \ \frac{\omega \overline{\eta}(a)}{\alpha_2} \ = \ \lambda \langle f\ ,\ \eta \rangle_{\mathbb{H}_0} \tag{2.5}$$

$$-\frac{f(a)}{\alpha_2} \ - \ \frac{\alpha_3 \omega}{\alpha_2 \rho} \ = \ \lambda \frac{\omega}{\rho} \tag{2.6}$$

where $\omega \ := \ \alpha_2 f'(a)$.

Thus the BVTP's (1.1)-(1.7) is transformed into the system of equalities (2.5)-(2.6) all terms of which are defined for the $f, \eta \in \mathbb{H}_1$.

Now we are ready to define the concept of weak solution for the considered BVTP (1.1)-(1.7).

**Definition 2.1.** *The vector-function* $\Gamma = (f(x)\ ,\ \omega) \in \Xi$ *is said to be a weak solution of the BVTP (1.1)-(1.7) if the equations (2.5)-(2.6) are satisfied for any* $\eta \in \mathbb{H}_1$.

We can show that the concept of weak solution is an extension of a classical solution. The reduction of identities (2.5)-(2.6) to an operator equation is based on the following result.

**Lemma 2.2.** *The following representations hold*

$$\ell_1 f(c_1-0)\overline{\eta}(c_1-0) + \ell_2 f(c_2-0)\overline{\eta}(c_2-0) := \langle K_1 f\ ,\ \eta\rangle_{\mathbb{H}_{1,q}} \tag{2.7}$$

$$\langle f\ ,\ \eta\rangle_{\mathbb{H}_0} := \langle K_2 f\ ,\ \eta\rangle_{\mathbb{H}_{1,q}} \tag{2.8}$$

$$\frac{\omega\overline{\eta}(a)}{\alpha_2} := \langle K_3 \omega\ ,\ \eta\rangle_{\mathbb{H}_{1,q}} \tag{2.9}$$

*where the operators* $K_1 : \mathbb{H}_1 \to \mathbb{H}_1, K_2 : \mathbb{H}_1 \to \mathbb{H}_1, K_3 : \mathbb{C} \to \mathbb{H}_1, K_3^* : \mathbb{H}_1 \to \mathbb{C}$ *are compact, the operators* $K_1$ *and* $K_2$ *are self-adjoint and* $K_2$ *is positive and* $K_3^*$ *denotes the operator conjugated to* $K_3$.

Since $\eta \in \mathbb{H}_1$ is an arbitrary element, we have

$$f + K_1 f + K_3\omega = \lambda K_2 f. \tag{2.10}$$

**Lemma 2.3.** *Let*

$$\mathcal{L}_1 := \begin{pmatrix} I + K_1 & K_3 \\ K_3^* & -\frac{\alpha_3}{\alpha_2\ \rho} I \end{pmatrix}\ ,\ \mathcal{L}_2 := \begin{pmatrix} K_2 & 0 \\ 0 & \frac{1}{\rho} I \end{pmatrix} \text{ and } \mathcal{L}(\lambda) = \mathcal{L}_1 - \lambda\mathcal{L}_2.$$

*Then the weak eigenfunctions of the BVTP's (1.1)-(1.7) satisfy the equation*

$$\mathcal{L}(\lambda)\Phi = 0 \tag{2.11}$$

*Here* $I$ *is the identity operator and* $\Phi := \begin{pmatrix} f(x) \\ \omega \end{pmatrix} \in \Xi.$

**Theorem 2.4.** $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *are self-adjoint operators in the Hilbert space* $\Xi$.

**Theorem 2.5.** *The operator* $\mathcal{L}(-\lambda) = \mathcal{L}_1 + \lambda\mathcal{L}_2$ *is self-adjoint in the Hilbert space* $\Xi$ *for each real* $\lambda$.

**Theorem 2.6.** *There is* $c > 0$ *such that for all* $\lambda_0 > c$ *the operator polynomial* $\mathcal{L}(-\lambda_0)$ *is positive definite.*

Let us consider a new spectral parameter $\mu = \lambda_0 + \lambda$. Then the considered spectral problem can be written in the following form

$$\Big(\mathcal{L}(-\lambda_0) - \mu\mathcal{L}_2\Big)\Phi = 0. \tag{2.12}$$

# 3 The Rayleigh Principle associated with the BVTP (1.1)-(1.7)

By using the Rayleigh Principle we have the following variational problem; find a non-trivial element $\Phi \in \Xi$ such that the Rayleigh quotient

$$\mu \quad = \quad \frac{\langle \mathcal{L}_1\Phi \ , \ \Phi\rangle_\Xi \ + \ \lambda_0 \langle \mathcal{L}_2\Phi \ , \ \Phi\rangle_\Xi}{\langle \mathcal{L}_2\Phi \ , \ \Phi\rangle_\Xi} \tag{3.1}$$

yields the minimum value.

The Rayleigh quotient (3.1) also allows a lower bound estimation for eigenvalues to be found. Indeed, using equality (2.10) we have

$$\mu \quad \geq \quad \frac{C_1 \int_J |f'(x)|^2 dx + C_2(\lambda_0) \int_J q(x)|f(x)|^2 dx + \lambda_0 \int_J |f(x)|^2 dx + C_3(\lambda_0)|\omega|^2}{\int_J |f(x)|^2 dx + \frac{1}{\rho}|\omega|^2}. \tag{3.2}$$

Now by applying the inequality

$$C_4 \int_J |f(x)|^2 dx \leq \int_J q(x)|f(x)|^2 dx \ \leq \ C_5 \int_J |f(x)|^2 dx$$

we have

$$\mu \ \geq \ \min\left(C_2(\lambda_0)C_4 \ + \ \lambda_0 \ , \ C_3(\lambda_0)\rho\right).$$

Consequently the lower bound estimation for eigenvalues of the BVTP (1.1)-(1.7) has the form

$$\lambda_k \ \geq \ -\lambda_0 \ + \ \min\left(C_2(\lambda_0)C_4 \ + \ \lambda_0 \ , \ C_3(\lambda_0)\rho\right).$$

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INTERNATIONAL CONFERENCE ON MATHEMATICS
"An Istanbul Meeting for World Mathematicians"
3-5 July 2019, Istanbul, Turkey

**Parseval and Carleman equalities for Sturm-Liouville problems under jump conditions**

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**Abstract**

Sturm-Liouville problems are boundary-value problems that naturally arise when solving certain partial differential equation problems using the "separation of variables" method. The "Rayleigh quotient" is the basis of an important approximation method that is used in solid mechanics as well as in quantum mechanics. In the latter, it is used in the estimation of energy eigenvalues of nonsolvable quantum systems, e.g., many-electron atoms and molecules. The simplest applications lead to the various Fourier series, and less simple applications lead to generalizations of Fourier series. Often in physical problems, the sign of the eigenvalue is quite important. The Rayleigh quotient cannot be used to explicitly determine the eigenvalue since eigenfunction is unknown. However, interesting and significant results can be obtained from the Rayleigh quotient without solving the differential equation. Nonetheless, it can be quite useful in estimating the eigenvalue. For example, the equation $\frac{dh}{dt} + h = 0$ is certain heat flow problems. Here, positive corresponds to exponential decay in time $\frac{d^2h}{dt^2} + \lambda h = 0$, while negative corresponds to exponential growth. In the vibration problems only positive corresponds to "usually" expected oscillations. Namely, any eigenvalue can be related to its eigenfunction by the Rayleigh quotient. It is the purpose of this paper to extend and generalize such important spectral properties as Rayleigh quotient, eigenfunction expansion, Rayleigh-Ritz formula(minimization principle), Parseval's equality and Carleman equality for Sturm-Liouville problems with interior singularities.

**Keywords :** Sturm-Liouville Problems, Carleman Equalities, jump conditions.

# 1 Introduction

The solution of a partial differential equation which is appearing in many branches of natural science usually regarded as a finite expansion of eigenfunctions of a suitable Sturm-Liouville problem. The importance of Sturm-Liouville problems for spectral methods lies in this fact. The issue of expansion in generalized eigenfunctions is a classical one going back at least to Fourier. A relatively recent impact is due to the study of wave propagation in random media [4, 11], where eigenfunction expansions are an important input in the proof of localization. Eigenfunction expansions and the corresponding Parseval's equality problems for classical Sturm-Liouville problems have been investigated by many authors (see [5, 12, 13] and references cited therein). In this paper we shall investigate one non-classical eigenvalue problem which consists of a Sturm-Liouville equation,

$$-u'' + q(x)u = \lambda u, \qquad x \in [a, c) \cup (c, b] \tag{1.1}$$

together with transmission conditions at the interior point $x = c$

$$\delta_1 u(+c) + \delta_2 u(-c) = 0 \tag{1.2}$$

$$\gamma_1 u'(+c) + \gamma_2 u'(-c) = 0 \tag{1.3}$$

and the boundary conditions at the end points $x = a$ and $x = a$

$$\cos \alpha u(a) + \sin \alpha u'(a) = 0, \tag{1.4}$$

$$\cos \beta u(b) + \sin \beta u'(\beta) = 0 \tag{1.5}$$

where $a < c < b$ the potential $q(x)$ is real-valued, continuous in each of interval $[a, c)$ and $(c, b]$ and has finite limits $q(\mp c)$ ; $\delta_1, \delta_2, \gamma_1, \gamma_2$ are real numbers; $\alpha, \beta \in [0, \pi)$ $\delta_1^2 + \delta_2^2 \neq 0$, $\gamma_1^2 + \gamma_2^2 \neq 0$; $\lambda$ is a complex spectral parameter. Such problems often arise in varied assortment of transfer problems appearing in physics and engineering. Also, some problems with transmission conditions arise in thermal conduction problems for a thin laminated plate (i.e., a plate composed by materials with different characteristics piled in the thickness, see [6]). Some spectral properties of problems with transmission conditions are studied in [1, 2, 3, 7, 8, 9, 10].

In this paper by applying an our own approach we present a new expansion formula and modified Parseval and Carleman equalities for the problem (1.1)-(1.5).

**Assumption 1.1.** *Everywhere in below we will assume that* $\delta_1\delta_2 < 0$ *and* $\gamma_1\gamma_2 < 0$

# 2 The Green's Function

At first we shall define four one-sided solutions $\phi_1(x, \lambda)$, $\phi_2(x, \lambda)$, $\chi_1(x, \lambda)$ and $\chi_2(x, \lambda)$ of the equation

$$-u'' + q(x)u = \lambda u, \qquad x \in [a, c) \cup (c, b] \tag{2.1}$$

by our own procedure as follows. Let $\phi_1(x, \lambda)$ and $\chi_2(x, \lambda)$ are the solutions of this equation on the intervals $[a, c)$ and $(c, b]$, satisfying the conditions

$$u(-a) = \sin \alpha, \quad u'(a) = -\cos \alpha$$

and

$$u(b) = \sin \beta, \quad u'(b) = -\cos \beta$$

respectively. Now, we shall define the other solutions $\phi_2(x, \lambda)$ and $\chi_1(x, \lambda)$ of the equation (2.1) on the intervals $(c, b]$ and $[a, c)$ satisfying the initial conditions

$$\delta_1 \phi_2(c) + \delta_2 \phi_1(c) = 0, \tag{2.2}$$

and

$$\gamma_1\phi_2'(c)+\gamma_2'\phi_1(c)=0 \tag{2.3}$$

respectively. We can prove that each of these solutions is an entire function of the parameter $\lambda\in\mathbb{C}$ for each fixed $x$. and each of the Wronskians $\omega_i(\lambda):=W(\phi_i(x,\lambda),\chi_i(x,\lambda))$ is an entire function of the parameter $\lambda$. The eigenvalues of the problem (1.1)-(1.4) coincide with the zeros of $\omega_i(\lambda)$. Now let $\lambda\in\mathbb{C}$ be not an eigenvalue. Then the Green's function for our problem is the following function

$$G(x,\xi;\lambda)=\begin{cases}-\frac{\delta_1}{\delta_2}\frac{\phi_1(x,\lambda)\chi_1(\xi,\lambda)}{\omega_1(\lambda)}, a\le\xi\le x<c,\\ -\frac{\delta_1}{\delta_2}\frac{\phi_1(\xi,\lambda)\chi_1(x,\lambda)}{\omega_1(\lambda)}, a\le x\le\xi<c,\\ -\frac{\delta_1}{\delta_2}\frac{\phi_2(\xi,\lambda)\chi_1(x,\lambda)}{\omega_1(\lambda)}, a\le x<c,\quad c<\xi\le b,\\ -\frac{\gamma_2}{\gamma_1}\frac{\phi_2(x,\lambda)\chi_1(\xi,\lambda)}{\omega_2(\lambda)}, c<x\le b,\quad a\le\xi<c,\\ -\frac{\gamma_2}{\gamma_1}\frac{\phi_2(x,\lambda)\chi_2(\xi,\lambda)}{\omega_2(\lambda)}, c<\xi\le x\le b,\\ -\frac{\gamma_2}{\gamma_1}\frac{\phi_2(\xi,\lambda)\chi_2(x,\lambda)}{\omega_2(\lambda)}, c<x\le\xi\le b.\end{cases} \tag{2.4}$$

It is symmetric with respect to $x$ and $\xi$, and real-valued for real $\lambda$. We can show that the function

$$u(x,\lambda) \;=\; -\frac{\delta_2}{\delta_1}\int_a^{c-}G(x,\xi;\lambda)f(\xi)d\xi-\frac{\gamma_1}{\gamma_2}\int_{c+}^{b}G(x,\xi;\lambda)f(\xi)d\xi \tag{2.5}$$

satisfies the equation

$$u''+\{\lambda-q(x)\}u=f(x) \tag{2.6}$$

Below, without loss of generality we assume that $\lambda=0$ is not an eigenvalue. Let $\lambda_0,\lambda_1,\lambda_2,\cdots,\lambda_n,\cdots$ be the collection of all the eigenvalues of the problem (1.1)-(1.4), and let the corresponding eigenfunctions $\varphi_0,\varphi_1,\varphi_2,\cdots,\varphi_n,\cdots$ be normalized by

$$-\frac{\delta_2}{\delta_1}\int_a^{c-}\varphi_n(x)\varphi_m(x)dx+-\frac{\gamma_1}{\gamma_2}\int_{c+}^{b}\varphi_n(x)\varphi_m(x)=\delta_{nm} \tag{2.7}$$

where $\delta_{nm}=1$ for $n\neq m$ and $\delta_{nn}=0$ for all $n$. We can prove that the following expansion formula is hold for the Green's function

$$G(x,\xi)=-\sum_{n=0}^{\infty}(\frac{1}{\lambda_n}\varphi_n(x)\varphi_n(\xi)). \tag{2.8}$$

Let $f(x)$ be any function square-integrable in each of the intervals $[a,c)$ and $(c,b]$. From (2.8) it follows that

$$f(x)=-\sum_{n=0}^{\infty}\{\frac{\delta_2}{\delta_1}\int_a^{c-}f(x)\varphi_n(x)dx+\frac{\gamma_1}{\gamma_2}\int_{c+}^{b}f(x)\varphi_n(x)dx\}\varphi_n(x). \tag{2.9}$$

# 3 The modified Carleman equation for transmision problems

We can derive that the expansion of the resolvent is

$$y(x,\lambda) = -\frac{\delta_2}{\delta_1}\int_a^{c-} G(x,\xi;\lambda)f(\xi)d\xi - \frac{\gamma_1}{\gamma_2}\int_{c+}^{b} G(x,\xi;\lambda)f(\xi)d\xi = \sum_{n=0}^{\infty}\frac{c_n(f)\varphi_n(x)}{\lambda-\lambda_n}.$$

where $c_n(f)$ is the Fourier coefficients, given by

$$c_n(f) = -\frac{\delta_2}{\delta_1}\int_a^{c-} G(x,\xi;\lambda)f(\xi)d\xi - \frac{\gamma_1}{\gamma_2}\int_{c+}^{b} G(x,\xi;\lambda)f(\xi)d\xi$$

substituting into the right-hand side, we get

$$\begin{aligned}&\delta\int_{-1}^{0} G(x,\xi;\lambda)f(\xi)d\xi + \frac{1}{\gamma}\int_0^1 G(x,\xi;\lambda)f(\xi)d\xi \\ &= \sum_{n=0}^{\infty}\frac{\varphi_n(x)}{\lambda-\lambda_n}\{\delta\int_{-1}^{0} f(\xi)\varphi_n(\xi)d\xi + \frac{1}{\gamma}\int_0^1 f(\xi)\varphi_n(\xi)d\xi\}.\end{aligned}$$

Taking into account that this equality is satisfied for an arbitrary $f$ we find that

$$G(x,\xi;t) = \sum_{n=0}^{\infty}\frac{\varphi_n(x)\varphi_n(\xi)}{t-\lambda_n}.$$

Since the set of eigenfunctions $\varphi_n(x)$ is orthonormal, in the sense of (2.7), putting $\xi = x$ and integrating with respect to $x$ we obtain

$$\frac{\delta_2}{\delta_1}\int_a^{c-} G(x,x;t)dt + -\frac{\gamma_1}{\gamma_2}G(x,x;t)dx = \sum_{n=0}^{\infty}\frac{1}{t-\lambda_n} \tag{3.1}$$

Denote the number of eigenvalues $\lambda_n$ less than $\lambda$ by $N(\lambda)$ (i.e. $N(\lambda) = \sum\limits_{0\le\lambda_n\le\lambda} 1$) we get from (3.1) the modified Carleman equation

$$\delta\int_{-1}^{0} G(x,x;t)dx + \frac{1}{\gamma}\int_0^1 G(x,x;t)dx = \int\limits_0^{\infty}\frac{dN(\lambda)}{t-\lambda}.$$

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INTERNATIONAL CONFERENCE ON MATHEMATICS
"An Istanbul Meeting for World Mathematicians"
3-5 July 2019, Istanbul, Turkey

# The resolvent operator for two-interval Sturm-Liouville problems with eigenparameter depending transmission conditions

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**Abstract**

Sturm-Liouville type problems involving additional transmission conditions at some interior singular points has become an important area of research in recent years because of the needs of modern technology, engineering, physics and other branches of natural sciences. Many of the mathematical problems encountered in the study of transmission problem cannot be treated with the usual techniques within the standard framework of the classical theory of boundary value problems. In this study it is developed the operator-theoretical method to investigate a new type boundary value problems consisting of two-interval Sturm-Liouville problem together with additional transmission conditions at one interior point of interaction. Moreover, the eigenvalue parameter appear not only in the differential equation but also in the transmission conditions. By suggesting an our own approach we construct modified Hilbert spaces and a linear operator in it such a way that the considered problem can be interpreted as a spectral problem for this operator. Finally, we shall investigate some important properties of the resolvent operator.

**Keywords :** Sturm-Liouville problems, boundary and transmission conditions.

# 1 Introduction

The one-dimensional form of the advection-dispersion equation for a nonreactive dissolved solute in a saturated, homogeneous, isotropic porous medium under steady, uniform flow is

$$D\frac{\partial^2 w}{\partial x^2} = \frac{\partial w}{\partial t} + \nu\frac{\partial w}{\partial t}, \quad 0 < x < \ell, \ t > 0$$

where $w(x,t)$ is the concentration of the solute, $\nu$ is the average linear groundwater velocity, $D$ is the coefficient of hydrodynamic dispersion, and $\ell$ is the length of the aquifer. Using the Fourier's method of separation of variables the problem can be written in the classical Sturm-Liouville form

$$[p(x)u']' + \lambda r(x)u = 0, \ u(x) = 0, \quad u'(\ell) = 0, \quad 0 < x < \ell.$$

This example make it clear that the Sturm-Liouville problems is of broad interest. Note that many-important special equations which appear in physics such as airy equation, Bessel equation, wave equation, heat equation, Schrödinger equation, Heun equation, advection-dispersion equation, etc., are associated with Sturm-Liouville type operators.

There is a well-developed theory for classical Sturm-Liouville problems. Details of the derivation of the theory and of related background results can be found in the cited references. Although the subject of Sturm-Liouville problems is over 180 years old this theory is an intensely active field of research today. The main tool for solvability analysis of Sturm-Liouville problems is the concept of Green's function and corresponding resolvent-operator. It is well-known that the possibility of a transition from the problems in mathematical physics to integral equations is based on the fundamental concept of the Green's function and corresponding resolvent-operator. Therefore, Green's function method finds applications not only in standard physics but also at the forefront of current and, most likely, future developments(see [2]). In terms of resolvent function, the various type non-homogeneous initial and boundary-value problems with arbitrary data can be solved in a form that shows clearly the dependence of the solution on the data. Determination of resolvent-operator is also possible using Sturm-Liouville theory.

In this paper we shall consider a new type Sturm-Liouville problem consisting of two-interval Sturm-Liouville equation

$$\Xi(u) := -u''(x) + q(x)u(x) = \lambda u(x), \ x \in \Omega^- \cup \Omega^+ \tag{1.1}$$

together with eigenparameter-depending boundary conditions of the form

$$\mathfrak{B}_\alpha(u) := \cos\alpha u(a) + \sin\alpha u'(a) = 0, \tag{1.2}$$

$$\mathfrak{B}_\beta(u) := \cos\beta u(b) + \sin\beta u'(b) = 0, \tag{1.3}$$

and eigenparameter-depending transmission conditions at one interaction point $x = c$ of the form

$$t_1(u) := \cos\gamma u(c+) + \sin\gamma u(c-) = 0, \tag{1.4}$$

$$t_2(u) := \sum_{j=0}^{1}[u^{(j)}(c+0) - (a_j + \lambda b_j)u^{(j)}(c-0)] = 0, \tag{1.5}$$

where $\Omega^- = [a,c)$, $\Omega^+ = (c,b]$ the potential $q(x)$ is real-valued function which continuous in each of the intervals $\Omega^-$ and $\Omega^+$, and has a finite limits $q(c\pm) = \lim_{x\to c\pm 0} q(x)$, $\lambda$ is a complex spectral parameter, the coefficients $a_j, \ b_j, \ (j = 0,1)$ are real numbers. This Sturm-Liouville problem is a non-classical eigenvalue problem since the eigenvalue parameter $\lambda$ appears not only in the differential equation, but also in the transmission conditions. Boundary value problems with transmission conditions arise in heat and mass transfer problems [6], in vibrating string problems when the string loaded additionally with point masses [9], in quantum mechanics [4], in thermal conduction problems for a thin laminated plate [10] etc. For applications an boundary value transmission problem to different areas, we refer the reader to the well-known monographs and some recent contribution [1, 3, 5, 7, 8].

## 2 Operator treatment in modified Hilbert space

In classical point of view, our problem can not be characterized self-adjoint operator in the classical Hilbert spaces. For self-adjoint characterization of the considered problem (1.1) – (1.5) we shall define a modified Hilbert space as follows.

Through the paper we shall assume that the conditions

$$b_1 \tan\gamma + \theta > 0, \quad b_0 \cot\gamma < 0,$$

holds where $\theta := a_0 b_1 - a_1 b_0$. Define a new inner-product space $\mathcal{H}$ as direct sum space $(L_2(\Omega^-) \oplus L_2(\Omega^+)) \oplus \mathbb{C}$ which is identical with $L_2(a,b) \oplus \mathbb{C}$ equipped with the modified inner-product

$$\langle U, V\rangle_{\mathcal{H}} \;:=\; (b_1 \tan\gamma + \theta) \int_a^{c-} u(x)\overline{v(x)}dx + (-b_0 \cot\gamma) \int_{c+}^{b} u(x)\overline{v(x)}dx - u_1\overline{v_1} \tag{2.1}$$

for $U = \begin{pmatrix} u(x), u_1 \end{pmatrix}$, $V = \begin{pmatrix} v(x), v_1 \end{pmatrix} \in L_2(a,b) \oplus \mathbb{C}$. It is easy to see that the relation (2.1) really define a new inner product in the direct sum space $L_2(a,b) \oplus \mathbb{C}$.

**Lemma 2.1.** *$\mathcal{H}$ is an Hilbert space.*

*Proof.* Let $U_n = (u_n(x), u_{1n}), n = 1, 2, ...,$ be any Cauchy sequence in the inner-product space $\mathcal{H}$. Then by (2.1) the sequences $(u_n(x))$ and $(u_{1n})$ will be a Cauchy sequences in the Hilbert spaces $L_2(a,b)$ and $\mathbb{C}$ respectively and therefore they are convergent. Denote by $u_0(x)$ and $(u_{10})$ the limits of these sequences, respectively. Defining $U_0 = (u_0(x), u_{10})$ we have that $U_0 \in \mathcal{H}$ and $U_n \to U_0$ in $\mathcal{H}$, which completes the proof. $\square$

Let us define the linear operator $\mathfrak{S} : \mathcal{H} \to \mathcal{H}$ with the domain

$$\begin{aligned} dom(\mathfrak{S}) := \Big\{ & U = (u(x), u_1) : u(x), u'(x) \ \in AC_{loc}(\Omega^-) \oplus AC_{loc}(\Omega^+), \\ & \text{with a finite left and right hand limits } u(c \mp 0) \text{ and } u'(c \mp 0);\ \Xi(u) \in L_2[a,b]; \\ & \mathfrak{B}_\alpha(u) = 0, \mathfrak{B}_\beta(u) = 0, t_1(u) = 0, u_1 = b_0 u(c-0) + b_1 u'(c-0) \Big\} \end{aligned}$$

and action low

$$\mathfrak{S}\big(u(x), u_1\big) = \big(\Xi f, -(a_0 u(c-0) + u'(c+0)) - (a_1 u(c-0) + u'(c+0))\big).$$

Then the problem (1.1) – (1.5) can be written in the operator equation form as $\mathfrak{S}U = \lambda U, u \in\in dom(\mathfrak{S})$ in the Hilbert space $\mathcal{H}$.

**Theorem 2.2.** *The linear operator $\mathfrak{S}$ is symmetric in the Hilbert space $\mathcal{H}$.*

*Proof.* By applying the method of [5] it is not difficult to prove that the operator $\mathfrak{S}$ is densely defined in $\mathcal{H}$, i.e. $\overline{dom(\mathfrak{S})} = \mathcal{H}$. Further, taking in view the definition of $\mathfrak{S}$ we obtain that

$$\langle \mathfrak{S}U, V\rangle_{\mathcal{H}} = \langle U, \mathfrak{S}V\rangle_{\mathcal{H}} \text{ for all } U, V \in dom(\mathfrak{S}).$$

The proof is complete. $\square$

**Theorem 2.3.** *The linear operator $\mathfrak{S}$ is self-adjoint in $\mathcal{H}$.*

# 3 The Green's function and corresponding resolvent operator

Consider the equation

$$(\lambda I - \mathfrak{S})U = F \tag{3.1}$$

for arbitrary $F = (f(x), f_1) \in \mathcal{H}$. This operator- equation is equivalent to the following nonhomogeneous BVTP:

$$(\lambda - \mathfrak{S})u(x) = f(x), \quad x \in \Omega^- \cup \Omega^+ \tag{3.2}$$

$$\mathfrak{B}_\alpha(u) = 0, \quad \mathfrak{B}_\beta(u) = 0, t_1(u) = 0, \quad t_2(u) = f_1. \tag{3.3}$$

We shall search the resolvent function of this BVTP in the form

$$u(x,\mu) = \begin{cases} \hbar_{11}(x,\lambda)\upsilon^-(x,\lambda) + \hbar_{12}(x,\lambda)\omega^-(x,\lambda) \text{ for } x \in \Omega^- \\ \hbar_{21}(x,\lambda)\upsilon^+(x,\lambda) + \hbar_{22}(x,\lambda)\omega^+(x,\lambda) \text{ for } x \in \Omega^+ \end{cases} \tag{3.4}$$

where the functions $\hbar_{11}(x,\lambda)$ and $\hbar_{12}(x,\lambda)$ are the solutions of the system of equations

$$\begin{cases} \frac{\partial \hbar_{11}(x,\mu)}{\partial x}\upsilon^-(x,\lambda) + \frac{\partial \hbar_{12}(x,\lambda)}{\partial x}\omega^-(x,\lambda) = 0 \\ \frac{\partial \hbar_{11}(x,\lambda)}{\partial x}\frac{\partial \upsilon^-(x,\lambda)}{\partial x} + \frac{\partial \hbar_{12}(x,\lambda)}{\partial x}\frac{\partial \omega^-(x,\lambda)}{\partial x} = f(x) \end{cases} \tag{3.5}$$

and the functions $\hbar_{21}(x,\lambda)$, $\hbar_{22}(x,\lambda)$ are the solutions of the system of equations

$$\begin{cases} \frac{\partial \hbar_{21}(x,\lambda)}{\partial x}\upsilon^+(x,\lambda) + \frac{\partial d\hbar_{22}(x,\lambda)}{\partial x}\omega^+(x,\lambda) = 0 \\ \frac{\partial \hbar_{21}(x,\lambda)}{\partial x}\frac{\partial \upsilon^+(x,\lambda)}{\partial x} + \frac{\partial \hbar_{22}(x,\lambda)}{\partial x}\frac{\partial \omega^+(x,\lambda)}{\partial x} = f(x) \end{cases} \tag{3.6}$$

for $x \in \Omega^-$ and $x \in \Omega^+$ respectively. Here $(\upsilon^\pm, \omega^\pm)$ is the fundamental solution of the equation $(\lambda - t)u = 0$ on the interval $\Omega^\pm$. We can find that the Green's function has the form

$$G(x,s;\lambda) = \begin{cases} \frac{\upsilon^-(x,\lambda)\omega^-(s,\lambda)}{\Delta^-(\lambda)}, & if\ x,s \in \Omega^-, \quad s \geq x \\ \frac{\omega^-(x,\lambda)\upsilon^-(s,\lambda)}{\Delta^-(\lambda)}, & if\ x,s \in \Omega^-, \quad s < x \\ \frac{\upsilon^-(x,\lambda)\omega^+(s,\lambda)}{\Delta^-(\lambda)}, & if\ x \in \Omega^-, \quad s \in \Omega^+ \\ \frac{\omega^-(x,\lambda)\upsilon^+(s,\lambda)}{\Delta^+(\lambda)}, & if\ x \in \Omega^+, \quad s \in \Omega^- \\ \frac{\omega^+(x,\lambda)\upsilon^+(s,\lambda)}{\Delta^+(\lambda)}, & if\ x \in \Omega^+, \quad s \leq x \\ \frac{\upsilon^+(x,\lambda)\omega^+(s,\lambda)}{\Delta^+(\lambda)}, & if\ x \in \Omega^+, \quad s > x \end{cases} \tag{3.7}$$

where $\Omega^\pm = W(\upsilon^\pm, \omega^\pm)$ is the wronskian. We now shall define the Green's vector-function as

$$\widetilde{G}_{x,\lambda} := \left( \ G(x,.;\lambda), \widetilde{T}_c(G(x,.;\lambda)) \ \right) \tag{3.8}$$

where $\widetilde{T}_c := b_0 u(c-0) + b_1 u'(c-0)$ Consequently for the solution $U(F,\lambda)$ of the nonhomogeneous operator equation (3.2) we obtain the following formula

$$U(F,\lambda) = \big(\langle \widetilde{G}_{x,\lambda}, \overline{F}\rangle\big) \tag{3.9}$$

Using this, the resolvent function (3.4) can be written in the form

$$u(x,\lambda) = \langle \widetilde{G}_{x,\lambda}, \overline{F}\rangle_{\mathcal{H}} \tag{3.10}$$

where $\overline{F} = (\overline{f}(x), \overline{f}_1) \in \mathcal{H}$. Consequently we have the following

**Theorem 3.1.** *For the resolvent operator $R(\lambda, \mathfrak{S}) = (\lambda I - \mathfrak{S})^{-1}$ the formula*

$$R(\lambda,\mathfrak{S})F = \big(\langle \widetilde{G}_{x,\lambda}, \overline{F}\rangle, \widetilde{T}_c(\langle \widetilde{G}_{x,\lambda}, \overline{F}\rangle)\big) \tag{3.11}$$

*is hold, where $\overline{F} = (\overline{f}(x), \overline{f}_1) \in \mathcal{H}$.*

**Theorem 3.2.** *The estimation*

$$\|R(\lambda,\mathfrak{S})F\|_{\mathcal{H}} \leq |\ Im\lambda\ |^{-1}\ \|F\|_{\mathcal{H}}, \quad F \in \mathcal{H}$$

*holds for all regular value $\lambda$, such that $Im\lambda \neq 0$.*

*Proof.* Let $F = (f(x), f_1) \in \mathcal{H}$. Denote $U = R(\lambda,\mathfrak{S})F$. Since $\mathfrak{S}U = \lambda U - F$, taking into account that the operator $\mathfrak{S}$ is symmetric we have

$$\lambda\langle U,U\rangle_{\mathcal{H}} - \langle F,U\rangle_{\mathcal{H}} = \langle \mathfrak{S}U,U\rangle_{\mathcal{H}} = \langle U,\mathfrak{S}U\rangle_{\mathcal{H}} = \overline{\lambda\langle U,U\rangle_{\mathcal{H}} - \overline{\langle F,U\rangle}_{\mathcal{H}}}.$$

Using well-known Cauchy-Schwartz inequality we conclude that

$$|\ Im\lambda\ |\ \|U\|^2_{\mathcal{H}} = |Im\langle F,U\rangle|_{\mathcal{H}} \leq \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}.$$

Consequently,

$$\|R(\lambda,\mathfrak{S})F\|_{\mathcal{H}} \leq |\ Im\lambda\ |^{-1}\ \|F\|_{\mathcal{H}}.$$

The proof is complete. □

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**3-5 July 2019, Istanbul, Turkey**

# On Almost Prime and Almost Primary $R_\Gamma$-Submodules of $R_\Gamma$-Modules

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**Abstract**

Let $R$ be a commutative $\Gamma$-ring and $M$ an $R_\Gamma$-module which is unitary. In this paper, we aim to introduce the concepts of almost prime and almost primary $R_\Gamma$-submodules of $R_\Gamma$-module $M$. We research some basic properties of almost prime and almost primary $R_\Gamma$-submodules and obtain some characterizations of these concepts and the relations among these concepts.

**Keywords:** Almost prime $R_\Gamma$-submodules, almost primary $R_\Gamma$-submodules

## 1. Introduction

Let $R$ and $\Gamma$ be two additive abelian groups. We say that $R$ is a $\Gamma$-ring if there exists a mapping $\cdot : R \times \Gamma \times R \longrightarrow R$, written $(r, \gamma, s) \mapsto r\gamma s$ for each $r, s \in R, \gamma \in \Gamma$, the following hold for each $r, s, u \in R$; $\alpha, \gamma \in \Gamma$:

*i.* $r\gamma(s+u) = r\gamma s + r\gamma u$;

*ii.* $(r+s)\gamma u = r\gamma u + s\gamma u$;

*iii.* $r(\alpha+\gamma)s = r\alpha s + r\gamma s$;

*iv.* $(r\alpha s)\gamma u = r\alpha(s\gamma u)$.

Let $R$ be a $\Gamma$-ring. An additive abelian group $M$ is called left $R_\Gamma$-module (similarly one can defined right module) if with the map $\cdot : R \times \Gamma \times M \longrightarrow M$, written $(r, \gamma, m) \mapsto r\gamma m$ for each $r \in R, \gamma \in \Gamma, m \in M$, it hold the followings for every $r, r_1, r_2 \in R$; $\alpha, \gamma \in \Gamma$; $m, m_1, m_2 \in M$:

i. $r\gamma(m_1 + m_2) = r\gamma m_1 + r\gamma m_2$;

ii. $(r_1 + r_2)\gamma m = r_1\gamma m + r_2\gamma m$;

iii. $r(\alpha+\gamma)m = r\alpha m + r\gamma m$;

iv. $(r_1\alpha r_2)\gamma m = r_1\alpha(r_2\gamma m)$.

$M$ is called $R_\Gamma$-module if it is both left $R_\Gamma$-module and right $R_\Gamma$-module.

A left $R_\Gamma$-module is unitary if there exist elements, say $1_R$ in $R$ and $\gamma_0$ in $\Gamma$ such that $1_R\gamma_0 m = m$ for every $m \in M$ (for more information, see [1]). We give the following examples to understand these concepts. If $R$ is a $\Gamma$-ring, then every abelian group $M$ can be made into an $R_\Gamma$-module with trivial module structure by defining $r\gamma m = 0$ for each $r \in R$, $\gamma \in \Gamma, m \in M$. For other example: Every $\Gamma$-ring $R$ is an $R_\Gamma$-module with the mapping $\cdot: R \times \Gamma \times R \longrightarrow R$, written $(r, \gamma, s) \mapsto r\gamma s$ for each $r, s \in R$; $\gamma \in \Gamma$.

A non empty subset $N$ of $M$ is called a left (right) $R_\Gamma$-submodule if $N$ is a subgroup of $M$ and $R\Gamma N \subseteq N (N\Gamma R \subseteq N)$ where $\{r\gamma n : r \in R, \gamma \in \Gamma, n \in N\}(\{n\gamma r : r \in R, \gamma \in \Gamma, n \in N\})$. $N$ is both a right $R_\Gamma$-submodule and a left $R_\Gamma$-submodule, then we say that $N$ is an $R_\Gamma$-submodule of $M$. We have two statements with this definition: i). $n_1 - n_2 \in N$ for each $n_1, n_2 \in N$ and ii). $r\,\gamma n \in N$ for each $r \in R, \gamma \in \Gamma, n \in N$. It is defined $(N : M) = \{r \in R : r\gamma m\, for\ each\, \gamma \in \Gamma, m \in M\}$. An $R_\Gamma$-submodule is said to be a prime (primary) if $K\Gamma I \subseteq N$ for some $R_\Gamma$-submodule $K$ of $M$ and $\Gamma$-ideal $I$ of $R$ implies $K \subseteq N$ or $I \subseteq (N : M)$ ($K \subseteq N$ or $I \subseteq \sqrt{(N : M)}$ ). It can be seen that $\sqrt{(N : M)}$ is the intersection of the minimal primes of $(N: M)$ (See Theorem 7 in [6]). In this paper, we give basic results about prime and primary $R_\Gamma$-submodules. In Theorem 2, we have that a prime $R_\Gamma$-submodule is an almost prime $R_\Gamma$-submodule. Then in Example 1, we show that the reverse of Theorem 2 is not always true. In Theorem 3, we give that if an $R_\Gamma$-submodule N of M is an almost prime, then $K/N$ is an almost prime $R_\Gamma$-submodule of $M/N$ where $N \subseteq K$. Then from Proposition 1, we show that an almost prime $R_\Gamma$-submodule $N$ of $M$ is almost primary if and only if $(N : M) = \sqrt{(N : M)}$. Additionally, we obtain similar results to almost prime $R_\Gamma$-submodule for almost primary $R_\Gamma$-submodule of $M$.

## 3. Almost Prime $R_\Gamma$-submodules

Throughout this paper, we assume that $R$ is a commutative $R_\Gamma$-ring and $M$ is a unitary $R_\Gamma$-module.

**Definition 1.** Let $M$ be an $R_\Gamma$-module and $N$ an $R_\Gamma$-submodule of $M$. $N$ is called almost prime $R_\Gamma$-submodule if $K\Gamma I \subseteq N - N\Gamma(N: M)$ for some $R_\Gamma$-submodule $K$ of $M$ and $\Gamma$-ideal $I$ of $R$ implies $K \subseteq N$ or $I \subseteq (N : M)$.

**Theorem 1.** Let $M$ be an $R_\Gamma$-module and $N$ an $R_\Gamma$-submodule of $M$. $N$ is an almost prime $R_\Gamma$-submodule if and only if $(m)\Gamma(r) \subseteq N - N\Gamma(N: M)$ for some $r \in R, m \in M$ implies $m \in N$ or $r \in (N : M)$.

Proof. ($\Rightarrow$) : It is clear.

($\Leftarrow$) : Let $K\Gamma I \subseteq N - N\Gamma(N: M)$ and $K \nsubseteq N$ for some $R_\Gamma$-submodule $K$ of $M$. Then there exists $k \in K - N$ such that $(k)\Gamma I \subseteq N - N\Gamma(N: M)$. That is, $(k)\Gamma(r) \subseteq N - N\Gamma(N: M)$ for every $r \in R$. Then $r \in (N : M)$, that is, $I \subseteq (N : M)$ by our assumption.

**Theorem 2.** Every prime $R_\Gamma$-submodule of $M$ is an almost prime $R_\Gamma$-submodule.

Proof. It is obvious.

But the inverse of the claim in the previous theorem is not true. In the following, we give an example for this situation.

**Example 1.** Let $R = \mathbb{Z}$, $\Gamma = \mathbb{Z}_2$ and $\Gamma = \mathbb{Z}_{54}$. Note that $27\mathbb{Z} = (N : M)$. Then the $R_\Gamma$-submodule $N = (\overline{27})$ of $M$ is an almost prime submodule since $N\Gamma(N{:}M) = N$. But it is not a prime since $(\overline{3})\Gamma(9) \subseteq N$ but $\overline{3} \notin N$ and $9 \notin (N : M)$.

Let M be an $R_\Gamma$-module and $N$ an $R_\Gamma$-submodule of $M$. Note that $M/N$ is an $R_\Gamma$-module with the map$\cdot$ $: R \times \Gamma \times M/N \longrightarrow M/N$, is defined $(r, \gamma, m + N) \mapsto r\gamma m$ for each $r \in R, \gamma \in \Gamma, m + N \in M/N$. Recall that $K/N$ is an $R_\Gamma$-submodule of $M/N$ where K is an $R_\Gamma$-submodule of $M$, containing $N$ [2].

**Theorem 3.** Let $M$ be an $R_\Gamma$-module, $K$ and $N$ $R_\Gamma$-submodules of $M$ with $N \subseteq K$. If $N$ is an almost prime $R_\Gamma$-submodule of $M$, then $K/N$ is an almost prime $R_\Gamma$-submodule of $M/N$.

Proof. Let $L/N\Gamma\mathrm{I} \subseteq K/N - K/N\Gamma(K/N{:}\,\mathrm{M/N})$ and $L/N \nsubseteq K/N$ for some $R_\Gamma$-submodule $L$ of $M$ and $R_\Gamma$-ideal $I$ of $R$. Then we have $m + N \in L/N - K/N$, that is, $m \in L - N$. Note that $(m)\Gamma I \subseteq K - K\Gamma(K{:}M)$ and $(m) \nsubseteq K$. By assumption, we get $I \subseteq (K : M)$, and so $I \subseteq (K/N : M/N)$.

**Corollary 1.** If $N$ is an almost prime $R_\Gamma$-submodule of $M$, then $N/N\Gamma(N : M)$ is an almost prime $R_\Gamma$-submodule of $M/N\Gamma(N : M)$.

Let $N$, $N_1$ and $N_2$ be $R_\Gamma$-submodules of $M$. Note that $N \subseteq N_1 \cup N_2$ implies $N \subseteq N_1$ or $N \subseteq N_2$.

**Theorem 4.** Let $M$ be an $R_\Gamma$-module and $N$ an $R_\Gamma$-submodule of $M$. Then the followings are equivalent:

1) $N$ is an almost prime $R_\Gamma$-submodule.
2) $(N : I) = N \cup (N\Gamma(N : M) : I)$ for some $\Gamma$-ideals $I$ with $I \cap (N : M) = \emptyset$.
3) $(N : I) = N$ or $(N{:}I) = (N\Gamma(N : M) : I)$ for some $\Gamma$ -ideals $I$ with $I \cap (N : M) = \emptyset$.

Proof. (1 ⇒2): The claim $N \cup (N\Gamma(N : M) : I) \subseteq (N : I)$ always is satisfied. We show that the inverse inclusion is true. Let $m \in (N : I)$ for some $m \in M$. Then we have $(m)\Gamma I \subseteq N$. If $(m)\Gamma I \subseteq N\Gamma(N : M)$, then $m \in (N\Gamma(N : M): I)$. Let $(m)\Gamma I \nsubseteq N\Gamma(N : M)$. Thus we obtain $(m) \subseteq N$, that is, $m \in N$ by our assumptions.

(2 ⇒3): The inclusions $N \subseteq (N : I)$ and $(N\Gamma(N : M): I) \subseteq (N : I)$ always hold. The inverse inclusions are true from the above explanation.

(3 ⇒1): Let $K\Gamma I \subseteq N - N\Gamma(N: M)$ and $I \nsubseteq (N: M)$ for some $R_\Gamma$-submodule $K$ of $M$ and $\Gamma$-ideal $I$ of $R$. Then $K\Gamma I \subseteq N$ and $K\Gamma I \nsubseteq N\Gamma(N: M)$. Thus $(N : I) \neq (N\Gamma(N : M) : I)$ since $K\Gamma I \nsubseteq N\Gamma(N: M)$. Then it must be $(N : I) = N$ and so we obtain $K \subseteq (N : I) = N$.

**4. Almost Primary $R_\Gamma$-submodules**

**Definition 2.** Let $M$ be an $R_\Gamma$-module and $N$ an $R_\Gamma$-submodule of $M$. $N$ is called almost primary $R_\Gamma$-submodule if $K\Gamma I \subseteq N - N\Gamma(N: M)$ for some $R_\Gamma$-submodule $K$ of $M$ and $\Gamma$-ideal $I$ of $R$ implies $K \subseteq N$ or $I \subseteq \sqrt{(N : M)}$.

The following sequence is clear:

Prime $R_\Gamma$-submodule ⇒Almost Prime $R_\Gamma$-submodule ⇒Almost Primary $R_\Gamma$-submodule.

But the reverse sequence is always not true:

**Example 2.** Consider $R = \mathbb{Z}$, $\Gamma = \mathbb{Z}$ and $M = \mathbb{Z}$. Then note that $N = (8)$ is an almost primary $R_\Gamma$-submodule, however $(2)\Gamma(2) \subseteq N - N\Gamma(N: M)$, $(2) \nsubseteq N$ and $(2) \nsubseteq (N : M)$, that is, $N$ is not an

almost primary $R_\Gamma$-submodule.

We have the following proposition.

**Proposition 1.** Let $M$ be an $R_\Gamma$-module. If an $R_\Gamma$-submodule $N$ of $M$ is almost prime, then it is almost primary. But the converse is true if $(N : M) = \sqrt{(N: M)}$.

Proof. It is clear by the definitions of almost prime and almost primary $R_\Gamma$-submodule.

**Theorem 5.** Let $M$ be an $R_\Gamma$-module and $N$ an $R_\Gamma$-submodule of $M$. $N$ is an almost primary $R_\Gamma$-submodule if and only if $(m)\Gamma(r) \subseteq N - N\Gamma(N: M)$ for some $r \in R$, $m \in M$ implies $m \in N$ or $r \in \sqrt{(N : M)}$.

Proof. (⇒) : It is clear.

($\Leftarrow$) : Let $K\Gamma I \subseteq N - N\Gamma(N{:}M)$ and $K \nsubseteq N$ for some $R_\Gamma$-submodule $K$ of $M$. Then there exists $k \in K - N$ such that $(k)\Gamma I \subseteq N - N\Gamma(N{:}M)$. That is, $(k)\Gamma(r) \subseteq N - N\Gamma(N{:}M)$ for every $r \in R$. Then $r \in \sqrt{(N : M)}$, that is, $I \subseteq \sqrt{(N : M)}$ by our assumption.

**Theorem 6.** Let $M$ be an $R_\Gamma$-module, $K$ and $N$ $R_\Gamma$-submodules of $M$ with $N \subseteq K$. If $N$ is an almost primary $R_\Gamma$-submodule of $M$, then $K/N$ is an almost primary $R_\Gamma$-submodule of $M/N$.

Proof. Let $L/N\Gamma\mathrm{I} \subseteq K/N - K/N\Gamma(K/N{:}\mathrm{M/N})$ and $L/N \nsubseteq K/N$ for some $R_\Gamma$-submodule $L$ of $M$ and $R_\Gamma$-ideal $I$ of $R$. Then we have $m + N \in L/N - K/N$, that is, $m \in L - N$. Note that $(m)\Gamma I \subseteq K - K\Gamma(K{:}M)$ and $(m) \nsubseteq K$. By assumption, we get $I \subseteq \sqrt{(K : M)}$, and so $I \subseteq \sqrt{(K/N : M/N)}$.

**Corollary 2.** If $N$ is an almost primary $R_\Gamma$-submodule of $M$, then $N/N\Gamma(N : M)$ is an almost primary $R_\Gamma$-submodule of $M/N\Gamma(N : M)$.

**Theorem 7.** Let $M$ be an $R_\Gamma$-module and $N$ an $R_\Gamma$-submodule of $M$. Then the followings are equivalent:

1) $N$ is an almost prime $R_\Gamma$-submodule.
2) $(N : I) = N \cup (N\Gamma(N : M) : I)$ for some $\Gamma$-ideals $I$ with $I \cap \sqrt{(N : M)} = \emptyset$.
3) $(N : I) = N$ or $(N{:}I) = (N\Gamma(N : M) : I)$ for some $\Gamma$ -ideals $I$ with $I \cap \sqrt{(K : M)} = \emptyset$.

Proof. It is seen easily to be hold in a similar way to the proof of Theorem 4.

## 4. Conclusion

As the brief of the study, we introduce the concepts of almost prime $R_\Gamma$-submodules and almost primary $R_\Gamma$-submodules.

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# Solution of Third-order Boundary-Value-Transmission Problems by Differential Transform Method

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## Abstract

A great deal of interest has been focused on the applications of the differential transform method (DTM) to solve many problems appearing in physics and engineering. For example, DTM has been used to solve differential-difference equations, delay differential equations, differential algebraic equations, integro-differential systems and etc. A numerical method based on the differential transform method is introduced in this work for the approximate solution of one third-order boundary-value-transmission problem. Namely, we investigate the differential equation,

$$y'''(x) + y(x) = 0, \qquad x \in [0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$$

subject to boundary conditions at the end-points $x = 0, 1$ given by

$$y(0) = 0, \qquad y'(0) = 1, \qquad y(1) = 0$$

and additional transmission conditions at the interior singular point $x = \frac{1}{2}$, given by

$$y(\frac{1}{2} - 0) = y(\frac{1}{2} + 0), \quad y'(\frac{1}{2} - 0) = y'(\frac{1}{2} + 0), \quad y''(\frac{1}{2} - 0) = ky''(\frac{1}{2} + 0)$$

The main objective of this study is to present the usage of DTM to investigation of discontinuous problems involving an additional transmission conditions. First, we will find solution of the problem in the left interval $x \in (0, \frac{1}{2}]$ . Then, we will investigate solution of the problem in the right interval $(\frac{1}{2}, 1]$ . Finaly, we find the approximate solution of the main problem on whole interval.



**Keywords :** Differential transform method, transmission conditions, approximation solution.

# 1 Introduction

The method of the differential transform was first proposed by Zhou [1]. Zhou's aim was to solve initial value problems for some linear and non-linear differential equations appearing in electric circuit analysis.

Initial and boundary value problems for two order ordinary differential equations occur frequently in physics, engineering and different areas of natural science. For instance, such type of problems occur in quantum mechanics, fluid mechanics, optimal control, chemical reactor theory, aerodynamics, geophysics, reaction-diffusion process, and other related fields of applied sciences. Also, studies showed that higher order boundary- value problems arise in the areas of fluid dynamics, hydrodynamics and hydromagnetic stability and other applied sciences. Note that, fifth-order boundary-value problems arise in viscoelastic fluid [2], sixth-order boundary value problems occur in astrophysics [5], seventh-order boundary value problems arise in modeling induction motors [4] and eight-order boundary value problem occur in hydrodynamic and hydromagnetic stability [3]. Recently a great deal of interest has been focused on the applications of the Differential transform method (DTM) to solve various type of boundary-value problems which appears in physics and engineering. Moreover, DTM has been used to solve differential-difference equation, delay differential equations, integro-differential systems, differential algebraic equation.

A numerical method based on the DTM is introduced in this work for the approximate solution of the third-order boundary-value problems under additional transmission conditions at some interior singular points.

# 2 Analysis of Differential Transform Method

Recall that the differential transformation of the $k$ th derivative of function $y(x)$ is defined as $k$-th term of the Taylor's series given by

$$D(y,k) = \frac{1}{k!}[\frac{d^k y(x)}{dx^k}]_{x=x_0} \tag{2.1}$$

Obviously the corresponding inverse transformation of $D(Y,k)$ is defined by

$$y(x) = \sum_{k=0}^{\infty} D(y,k)(x-x_0)^k. \tag{2.2}$$

Naturally in real applications, the approximate value of the function $y(x)$ is expressed by a finite series given by

$$y(x) \approx \sum_{k=0}^{n} D(y,k)(x-x_0)^k. \tag{2.3}$$

It is ease to have the following properties of the DTM.

**Theorem 2.1.** $D(f \pm g, k) = D(f,k) \pm D(g,k)$ .

**Theorem 2.2.** $D(cf,k) = cD(f,k)$.



**Theorem 2.3.** $D(\frac{d^n g(x)}{dx^n}, k) = \frac{(k+n)!}{k!} D(f,k)$.

**Theorem 2.4.** $D(f(x)g(x),k) = \sum_{n=0}^{k} D(f,k)D(g,k-n)$.

**Theorem 2.5.** $D(x^n,k) = \delta(k-n)$ *where*

$$\delta(m) = \begin{cases} n, & for\ m = 0 \\ 0, & for\ m \neq 0. \end{cases}$$

# 3 Solution of the problem by using DTM

We shall consider the third-order differential equation,

$$y'''(x) + y(x) = 0, \qquad x \in [0, \frac{1}{2}) \cup (\frac{1}{2}, 1] \tag{3.1}$$

subject to boundary conditions,

$$y(0) = 0, \qquad y'(0) = 1, \qquad y(1) = 0 \tag{3.2}$$

and additionally transmission conditions at the interior singular point $x = \frac{1}{2}$, given by

$$y(\frac{1}{2}-0) = y(\frac{1}{2}+0), \quad y'(\frac{1}{2}-0) = y'(\frac{1}{2}+0), \quad y''(\frac{1}{2}-0) = ky''(\frac{1}{2}+0) \tag{3.3}$$

First, let's get the solution for the problem in the left interval $x \in [0, \frac{1}{2})$. If differential transform method is applied to the differential equation,

$$D^-(y,k+3) = \frac{-D^-(y,k)}{(k+3)(k+2)(k+1)} \tag{3.4}$$

is obtained. Then, using the representation $y^-(x) = \sum_{k=0}^{n}(x-x_0)^k D^-(y,k)|_{x=x_0}$, we get

$$\begin{aligned} y^-(x) &= \sum_{k=0}^{n} x^k D^-(y,k)|_{x=0} \\ &= D^-(y,0) + xD^-(y,1) + x^2 D^-(y,2) + ... + x^n D^-(y,n) \end{aligned} \tag{3.5}$$

From $y(1) = 0$, the following transformed initial condition at $x_0 = 0$ can be obtained as

$$D^-(y,0) = 0 \tag{3.6}$$

Taking in view

$$y'^-(x) = D^-(y,1) + 2xD^-(y,2) + ... + nx^{n-1}D^-(y,n) \tag{3.7}$$

and $y'(0) = 0$ we have

$$D^-(y,1) = 1. \tag{3.8}$$

Let us introduce to the consideration a parameter $c$ by



$$D^-(y,2) = c \tag{3.9}$$

Following the recursive procedure, we find that

$$\begin{aligned}
D^{-}(y,3) &= 0\\
D^{-}(y,4) &= \frac{-1}{4!}\\
D^{-}(y,5) &= \frac{-2c}{5!}\\
&\ \ .\\
&\ \ .\\
&\ \ .
\end{aligned}$$

Let's choose $n=5$. Then we get the following representations

$$y^{-}(x) = x + x^2c - \frac{x^4}{4!} - \frac{x^5 2c}{5!},\ y'^{-}(x) = 1 + 2xc - \frac{x^3}{3!} - \frac{x^4 2c}{4!},\ y''^{-}(x) = 2c - \frac{x^2}{2!} - \frac{x^3 2c}{3!}$$

Secondly, let's find the approximate solution of the considered problem in the right interval $(\frac{1}{2}, 1]$. If differential transform method is applied to the differential equation, we have

$$D^{+}(y,k+3) = \frac{-D^{+}(y,k)}{(k+3)(k+2)(k+1)} \tag{3.10}$$

Using

$$y^{+}(x) = D^{+}(y,0) + (x-1)D^{+}(y,1) + (x-1)^2 D^{+}(y,2) + ... + (x-1)^n D^{+}(y,n) \tag{3.11}$$

and following the similar recursive procedure, we find

$$y^{+}(x) = a(x-1) + b(x-1)^2 - \frac{a}{4!}(x-1)^3 - \frac{2b}{5!}(x-1)^5 \tag{3.12}$$

$$y'^{+}(x) = a + 2b(x-1) - \frac{3a}{4!}(x-1)^2 - \frac{2b}{4!}(x-1)^4 \tag{3.13}$$

$$y''^{+}(x) = 2b - \frac{6a}{4!}(x-1) - \frac{2b}{3!}(x-1)^3 \tag{3.14}$$

If $k=5$ are selected, then we find $a=-0.491316$, $b=-0.220106$, $c=-1.24034$. Finally, we can obtain the approximate solution which can be written as

$$y(x) = \begin{cases} x + (-1.24034)x^2 - \frac{1}{4!}x^4 + \frac{2.48068}{5!}x^5 + ...,\ x \in [0, \frac{1}{2}) \\ \\ (-0.491316)(x-1) - (0.220106)(x-1)^2 + \frac{0.491316}{4!}(x-1)^3 + ...,\quad x \in (\frac{1}{2}, 1] \end{cases}$$

# Conclusion

In this study, we found the approximate solution of one third-order boundary-value-transmission problem by using modified DTM.

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"An Istanbul Meeting for World Mathematicians"
3-5 July 2019, Istanbul, Turkey

# Solution of Initial Value Transmission Problem by the Adomian Decomposition Method

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**Abstract**

The Adomian decomposition method has been used widely in solving ordinary and partial differential equations, Sturm-Liouville problems, physical problems and stochastic problems. We will adapt the Adomian decomposition method to nonclassical boundary value problems the main feature of which is the nature of the equations and boundary conditions imposed. Namely, the boundary conditions contains not only end points of the considered interval, but also an interior point of singularity at which given additional so-called transmission conditions, so our problem is nonclassical once. Based on decomposition method and our approaches, a new analytical treatment is introduced for such type transmission problems. In this study, we examine the differential equation,

$$y''(t) - 2y'(t) + 5y(t) = 0, \qquad t \in [0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$$

together with initial conditions,

$$y(0) = -1, \quad y'(0) = 7$$

and additional transmission conditions at the point of singularity $t = \frac{1}{2}$, given by

$$y(\frac{1}{2} - 0) = k_1 y(\frac{1}{2} + 0), \quad y'(\frac{1}{2} - 0) = k_2 y'(\frac{1}{2} + 0)$$

Particularly, we shall find the Adomian polynomials for left-hand and right-hand solution of this problem. Afterwards, we will compare the approximate solution with the exact solution of the considered problem.



**Keywords :** Adomian Decomposition Method, transmission conditions, approximation solution.

# 1 Introduction

In the 1980t's, George Adomian (1923−1996) introduced a new method for solving linear or nonlinear differential and functional equations. His technique is known as the Adomian decomposition method (ADM), [1]- [3]. This method is based on the representation of a solution as series of functions, each term of which is obtained from a so-called Adomian polynomial generated by a power series expansion of an analytic function.

In this study, we consider a differential equation together with initial conditions and two supplementary transmission conditions at the point of discontinuity. We will adapt the Adomian decomposition method for solving initial value transmission problems. Comparison with the exact solutions and graphical illustration will also be presented.

# 2 Analysis of Adomian Decomposition Method

Let us describe the algoritm of ADM as it applies to the nonlinear differential equations of the form

$$My + Ny = g, \tag{2.1}$$

where the linear differential expression term is composed into $M = L + R$, where $L$ is the higher derivative differential operator which is easily invertible and the remainder of the linear operator of less order than $L$ is $R$, $Ny$ represents the nonlinear term and $g$ is the source term. Operating by the inverse operator $L^{-1}$ leads to

$$L^{-1}Ly = L^{-1}g - L^{-1}Ry - L^{-1}Ny. \tag{2.2}$$

If $L$ is a second-order operator $L = \frac{d^2}{dx^2}$, then $L^{-1}$ is a two-fold integration and therefore $L^{-1}Ly = y - y(0) - ty'(0)$. Using the given conditions we obtain that the apriori solution $y$ has the form

$$y = A + Bt + L^{-1}g - L^{-1}Ry - L^{-1}Ny. \tag{2.3}$$

Now, assume that the nonlinear term $Ny$ is an analytic function with respect to the variable $y$ and for a given $g$ there exists a unique solution $y$ of the equation (2.1). The Adomian decomposition method assumes that the solution $y$ can be decomposed into an infinite series, given by $y = y_0 + y_1 + y_2 + ...$ and the nonlinear analytic function can be decomposed into an infinite series.

$$Ny = \sum_{n=0}^{\infty} A_n(y_0, y_1, ..., y_n)$$

where the terms $A_n$ are specially generated so-called Adomian polinomials depending only on the variable $y_0, y_1, ..., y_n$, which are given by

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\right]_{\lambda=0}, \qquad n = 0, 1, ...$$

Putting these expression into (2.3) yields

$$y = \sum_{n=0}^{\infty} y_n = y_0 - L^{-1}\left(R\left(\sum_{n=0}^{\infty} y_n\right)\right) - L^{-1}\left(\sum_{n=0}^{\infty} A_n\right)$$

Thus we have the following iterative formulas

$$
\begin{aligned}
y_1 &= -L^{-1}(Ry_0) - L^{-1}(A_0(y_0)) \\
y_2 &= -L^{-1}(Ry_1) - L^{-1}(A_1(y_0, y_1)) \\
&. \\
&. \\
&. \\
y_{n+1} &= -L^{-1}(Ry_n) - L^{-1}(A_n(y_0, y_1, ..., y_n))
\end{aligned}
$$

for all $n \in N$. Consequently we can recursively determine all terms $y_0, y_1, y_2...$ of the series solution

$$y = \sum_{n=0}^{\infty} y_n.$$

# 3 Solution using the Adomian Decomposition Method

We shall consider the Sturm-Liouville equation,

$$y''(t) - 2y'(t) + 5y(t) = 0, \qquad t \in [0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$$

subject to initial conditions,

$$y(0) = -1$$

$$y'(0) = 7$$

and additional transmission conditions at the interior singular point $t = \frac{1}{2}$, given by

$$y(\frac{1}{2} - 0) = k_1 y(\frac{1}{2} + 0)$$

$$y'(\frac{1}{2} - 0) = k_2 y'(\frac{1}{2} + 0)$$

By applying an our own approach, at first we will consider some auxiliary initial-value problems on the left and right side of the considered interval:

Let us consider the auxiliary initial-value problem on the left interval $[0, \frac{1}{2})$, given by

$$y''(t) - 2y'(t) + 5y(t) = 0, \qquad t \in [0, \frac{1}{2}) \tag{3.1}$$

$$y(0) = -1, \qquad y'(0) = 7 \tag{3.2}$$

By virtue of existence and uniqueness theorem of differential equation theory, the problem (3.1)-(3.2) has a unique solution $\widetilde{y}(x)$ (see, for example [4]). By applying the decomposition method we have



$$\widetilde{y}_0(t) = -1 + 7t$$

$$
\begin{aligned}
\widetilde{y}_1(t) &= L^{-1}(2\widetilde{y'}_0(t) - 5\widetilde{y}_0(t)) \\
&= \int_0^t \int_0^t 14 - 5(-1+7t)dtdt \\
&= \frac{19t^2}{2!} - \frac{35t^3}{3!}
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{y}_2(t) &= L^{-1}(2\widetilde{y'}_1(t) - 5\widetilde{y}_1(t)) \\
&= \frac{19t^3}{3} - \frac{55t^4}{8} + \frac{35t^5}{24}
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{y}_3(t) &= L^{-1}(2\widetilde{y'}_2(t) - 5\widetilde{y}_2(t)) \\
&= \frac{19t^4}{6} - \frac{13t^5}{3} + \frac{235t^6}{144} - \frac{25t^7}{144}
\end{aligned}
$$

...................

Thus we get the fourth order approximation of the left solution as

$$
\widetilde{y}(t) = -1 + 7t + \frac{19t^2}{2!} - \frac{35t^3}{3!} + \frac{38t^3}{6} - \frac{165t^4}{24} + \frac{175t^5}{120} + \frac{19t^4}{6} - \frac{13t^5}{3} + \frac{235t^6}{144} - \frac{25t^7}{144} + ... \tag{3.3}
$$

Now we will consider the right-hand problem on the right interval $(\frac{1}{2}, 1]$ given by

$$
y''(t) - 2y'(t) + 5y(t) = 0, \qquad t \in (\frac{1}{2}, 1] \tag{3.4}
$$

$$
y(1) = a, \quad y'(1) = b \tag{3.5}
$$

We know that the problem (3.4)-(3.5) has a unique solution $\widetilde{\widetilde{y}}(t)$ (see [4] ). By using the same technique we can calculate the following terms of the series solution $\widetilde{\widetilde{y}}(t) = \sum_{n=0}^{\infty} \widetilde{\widetilde{y}}_n(t)$, as

$$
\widetilde{\widetilde{y}}_0(t) = a - b + bt
$$

$$
\widetilde{\widetilde{y}}_1(t) = -\frac{5a}{2} + \frac{31b}{6} + 5at - \frac{19bt}{2} - \frac{5at^2}{2} + \frac{7bt^2}{2} + \frac{5bt^3}{6}
$$

$$
\widetilde{\widetilde{y}}_2(t) = \frac{1}{24}(-1+t)^3(5a(3+5t) - b(31+60t+5t^2))
$$

..............................

Consequently, we have the following approximation of the right solution, as

$$
\begin{aligned}
\widetilde{\widetilde{y}}(t) &= a - b + bt - \frac{5a}{2} + \frac{31b}{6} + 5at - \frac{19bt}{2} - \frac{5at^2}{2} + \frac{7bt^2}{2} + \frac{5bt^3}{6} \\
&+ \frac{1}{24}(-1+t)^3(5a(3+5t) - b(31+60t+5t^2)) + ...
\end{aligned} \tag{3.6}
$$

Using the series solutions (3.3)-(3.6) and to satisfy the transmission conditions , we must solve the following system of equations:

$$\widetilde{y}(\frac{1}{2}) = k_1\widetilde{\widetilde{y}}(\frac{1}{2}) \tag{3.7}$$

$$\widetilde{y'}(\frac{1}{2}) = k_2\widetilde{\widetilde{y'}}(\frac{1}{2}) \tag{3.8}$$

If $k_1 = 1, \quad k_2 = 1$ are selected, then $a = 6.64943, \quad b = 3.8243$ are obtained as a result of the common solution of the equations (3.7)-(3.8).

Exact solution of the problem is $y(t) = -e^t \cos 2t + 4e^t \sin 2t$.

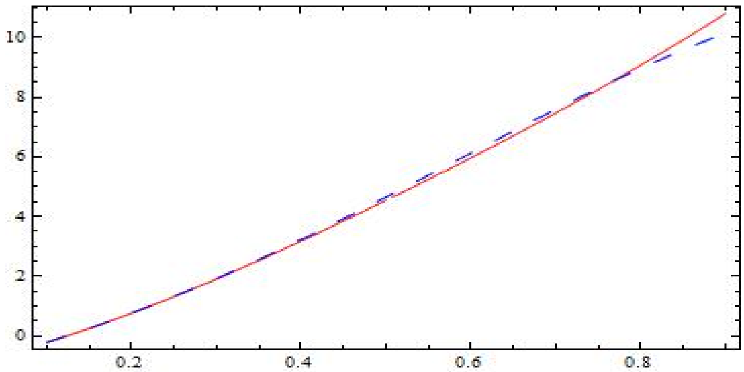


**Fig.1** Graph of the the approximate solution and exact solution.

# Conclusion

In this study, we found the approximate solution of one initial value transmission problem by using ADM. Also, we compared the approximate solution with exact solution. Consequently the Adomian decomposition method is effective and reliable for solving discontinuous boundary-value problems under additional transmission conditions at the point of discontinuity.

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An Istanbul Meeting for World Mathematicians
3-5 July 2019, Istanbul, Turkey

# Stabilization of the wave equation with Ventcel's boundary conditions

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**Abstract**

This work is concerned with stabilization of a wave equation stabilized by a boundary feedback. It was shown in [11] that this problem is not exponentially stable. The purpose of this work, is to show that the solution of this system decays polynomially by using a Fourier analysis combined with the multiplier techniques.

**Key words and phrases:** Stabilization, Ventcel's boundary conditions, Fourier analysis, The multiplier method, semigroups theory.

## 1 Introduction

In this paper, we are concerned by the stabilization of the wave equation posed in a disk $\Omega = \{x \in \mathcal{R}; 0 < |x| < 1\}$ subject to Ventcel boundary conditions on the boundary $\Gamma = \partial\Omega$. Ventcel boundary conditions arise naturally in many contexts. In the context of multidimentional diffusion processes, Ventcel boundary conditions were introduced in the pioneering work

of Ventcel [7, 14] (see also the work of Feller for one-dimensional processes [8]). It is known in [11] that the resolvent operator of such an equation must be unbounded on the imaginary axis. Hence we are interested in proving a weaker decay of the energy. More precisely we will furnish sufficient conditions and methods that guarantee a polynomial decay of the energy of our system.
The proof is based on a Fourier-Bessel analysis, spectral theory, the multiplier technique and a specific method of the obtained one-dimensional problem combining Ingham's inequality.

The paper is organized as follows: In Section 2 we present the problem and recall that it is well posed. Section 3 is devoted to the analysis of a one-dimensional wave dissipative equation depending on a parameter. These results allows to obtain in the last section, our polynomial stability of the Ventcel-problem using Fourier-Bessel analysis.

# 2 The problem

Let $\Omega = \{x \in \mathcal{R}^2; 0 < |x| < 1\} \subset \mathcal{R}^2$ be an open bounded domain with a smooth boundary $\Gamma$.

$$\Gamma = \{x \in \mathcal{R}^2 / |x| = 1\}$$

We consider the problem

$$\begin{cases} \psi_{tt} - \Delta\psi = 0 & in\ \Omega \times \mathcal{R}^+, \\ \partial_\nu \psi - \Delta_T \psi + \psi_t = 0 & on\ \Gamma \times \mathcal{R}^+, \\ \psi(.,0) = \psi_0,\ \psi_t(.,0) = \psi_1 & in\ \Omega, \end{cases} \tag{1}$$

$\triangle_T$ means the tangentiel Laplace operator on $\Gamma$ [14].
$\psi = \psi(x,y,t)$ where $(x,y) \in \Omega$ , $t \in \mathcal{R}^+$ , $\nu$ is the unit outward normal vector along the boundary.

The energy of the solution of (1) is defined by

$$E(t) = E(\psi, \psi_t)(t) = \frac{1}{2} \| (\psi, \psi_t)(t) \|_H^2 \tag{2}$$

Using some integration by parts, one can show that for all $(\psi_0, \psi_1) \in D(A)$ we have

$$\frac{d}{dt} E(t) = -\int_\Gamma |\psi_t|^2 \, d\Gamma \tag{3}$$

Hence the system is dissipative.

Problem (1) can be rewritten in the form

$$\begin{cases} U_t(t) = UY(t) \\ U(0) = U_0 \end{cases} \tag{4}$$

with $U = (\psi,\, \psi_t)$, $U_0 = (\psi_0,\, \psi_1)$ and $A$ was defined previously. Semi group theory allows the next existence and uniqueness results [6]

**Theorem 2.1.** *If* $(\psi_0, \psi_1) \in D\,(A)$*, then the problem (1) admits a unique solution*

$$(\psi, v) \in C([0,\infty[\,, D\,(A)) \cap C^1\,([0,\infty[\,, H)\,.$$

*Proof.* The proof is based on the use of Hille Yosida's theorem, see [11] □

# 3 The one-dimensional problem

We now consider the following expression of $\psi$

$$\psi = \sum_{\kappa\in\mathcal{Z}} \psi^{(2\kappa)}(r,t)e^{i2\kappa\theta}$$

This allows us to deduce the following $1-d$ model

$$\begin{cases} \psi_{tt}^{(k)} - \frac{1}{r^2}(r\frac{\partial}{\partial r})^2\psi^{(k)} + \frac{(k)^2}{r^2}\psi^{(k)} = 0 \qquad \text{in} \quad (0,1)\times\mathcal{R}^+ \\ \frac{\partial}{\partial r}\psi^{(k)}(1) + (k)^2\psi^{(k)}(1) + \psi_t^{(k)}(1) = 0 \quad \text{on} \quad \mathcal{R}^+ \\ \psi^{(k)}(0) = 0 \qquad\qquad\qquad\qquad \text{on} \quad \mathcal{R}^+ \\ \psi^{(k)}\,(0) = \psi_0^{(k)},\ \psi_t^{(k)}\,(0) = \psi_1^{(k)} \qquad\qquad \text{in} \quad (0,1) \end{cases} \tag{5}$$

## 3.1 Exponential Stabilization of a 1-d Model with a parameter.

The energy of the solution of the problem (5) is given by

$$E_k(t) = \frac{1}{2}\int_0^1 ((\frac{\partial\psi^{(k)}}{\partial r})^2 + \frac{k^2}{r^2}(\psi^{(k)})^2 + (\psi_t^{(k)})^2)rdr + \frac{k^2}{2}(\psi^{(k)}(1))^2. \tag{6}$$

**Lemma 3.1.** *For all regular solutions* $\psi^{(k)}$ *of (5) , it holds*

$$E_k^{'}(t) = -((\psi_t^{(k)}(1))^2 \tag{7}$$

*Proof.* We use the method of integration by parts and boundary conditions, we obtain

$$\begin{aligned} E_k^{'}(t) &= \int_0^1 \psi_t^{(k)}[\psi_{tt}^{(k)} - \frac{1}{r^2}(r\frac{\partial}{\partial r})^2\psi^{(k)} + \frac{(k)^2}{r^2}\psi^{(k)}]\, rdr + \psi_t^{(k)}(1)[-k^2\psi^{(k)}(1) - \psi_t^{(k)}(1) + k^2\psi^{(k)}(1)] \\ &= -((\psi_t^{(k)}(1))^2 \leq 0 \end{aligned}$$

□

In order to show the exponential stabilization of the problem (5), we write $\psi^{(k)}$ in the form

$$\psi^{(k)} = y + w$$

where $y$ is solution of the same problem as $\psi^{(k)}$ but without dissipation, $w$ is the remainder.

We introduce the operator $A_k$ defined on $L^2((0,1))$ by

$$A_k(\upsilon) = -\frac{1}{r^2}(r\frac{\partial}{\partial r})^2\upsilon + \frac{(k)^2}{r^2}\upsilon$$

and domain

$$D(A_k) = \{\upsilon \in H^2(0,1); \upsilon(0) = 0 \text{ and } \upsilon_r(1) = -k^2\upsilon(1)\}$$

**Lemma 3.2.** *The operator $A_k$ is positive, self-adjoint on $L^2((0,1))$, moreover its resolvent $R_\lambda(A_k)$ is compact.*

**Theorem 3.3.** *The roots of the equation*

$$\frac{J_k(\lambda)}{J_{k+1}(\lambda)} = \frac{1}{k^2+k}\lambda,\ k \in \mathcal{Z}^* \tag{8}$$

*are simple, strictly larger than $k^2$ and form the eigenvalues of $A_k$. $J_k$ is the Bessel functions of the first kind.*

*The associated normalized eigenvectors are given by*

$$\varphi_n(r) = AnJ_k(\lambda_n r),\ n \in IN \tag{9}$$

*where*

$$An \geq \frac{1}{\sqrt{2}} \tag{10}$$

*and*

$$\varphi_n(1) \geq \frac{C}{\sqrt{2}} \tag{11}$$

*$C$ is a positive constant*

We are now ready to bound the energy of $y$ with respect to an appropriate boundary term.

**Proposition 3.4.** *Let $E_y(t)$ be the energy of $y$ solution of $(p_y)$ defined by*

$$E_y(t) = \frac{1}{2}\{\int_0^1 ((\frac{\partial y}{\partial r})^2 + \frac{k^2}{r^2}(y)^2 + (y_t)^2)\, rdr + k^2(y(1))^2\}.$$

*Then there exist two positives constants $c_1$ and $c_2$ independent on $k$ such that for all $T > c_1$ one has*

$$E_y(0) \leq c_2 \int_0^T (y_t(1,t))^2\, dt \tag{12}$$

**Theorem 3.5.** *There exist two positive constants $M_1$, $M_2$ independent on $k$ such that*

$$E_k(t) \leq M_1 e^{-\frac{M_2}{k^3}t} E_k(0) \tag{13}$$

**Theorem 3.6.** *For all $m \in IN^\star$,there exists a positive constant $C_m > 0$ such that for the initial data $(\psi_0, \psi_1) \in H^{m+1}(\Omega) \times H^m(\Omega)$ the solution $\psi$ of (1) satisfies*

$$E(t) \leq \frac{C_m}{t^m} \sum_{\kappa=-\infty}^{\infty} k^{3m} E_k(0), \quad \forall\, t\, >\, 0, \tag{14}$$

*where*

$$\sum_{\kappa=-\infty}^{\infty} k^{3m} E_k(0) \leq \|\psi_0\|^2_{H^{m+1}(\Omega)} + \|\psi_{0|\Gamma_1}\|^2_{H^{m+1}(\Gamma_1)} + \|\psi_1\|^2_{H^m(\Omega)}.$$

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An Istanbul Meeting for World Mathematicians

3-5 July 2019, Istanbul, Turkey

# Fractional Optimal Control Problem for Differential System with mixed boundary conditions

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**Abstract**

In this paper, the fractional optimal control problem for differential system is considered.

**Key words and phrases:** Optimal control, Fractional Dirichlet problem, Fractional Neumann problem, Riemann- Liouville fractional derivative.

## 1 Introduction

In this paper, the fractional optimal control problem for differential system is considered. The fractional time derivative is considered in Riemann-Liouville sense. Necessary and sufficient optimality conditions for the fractional Dirichlet and Neumann problems with the quadratic performance functional are derived. Some examples are analyzed in details. We consider here a different type of equations, namely, fractional partial differential equations involving second order operators. The existence and uniqueness of solutions for such equations were proved. Fractional optimal control is characterized by the adjoint problem. By using this characterization, particular properties of fractional optimal control are proved.

## 2 Fractional Dirichlet Problem for Differential System

Let us consider the fractional partial differential equations:

$$D_+^\beta + \mathcal{A}y(t) = f(t) \tag{1}$$

$$I_+^{1-\beta}y(0^+) = y_0 \qquad ,x \in \Omega \tag{2}$$

$$y(x,t) = 0 \qquad ,x \in \Gamma \tag{3}$$

$$Ay = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial y}{\partial x_j}\right) + a_0(x)y, \tag{4}$$

$$a_0(x), a_{ij}(x) \in L^\infty(\Omega)$$

$$a_0(x) \geq \alpha > 0, \ \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \alpha(\xi_1^2 + ... + \xi_n^2), \quad \forall \xi \in \mathbb{R}^n,$$

$$\pi(t,y,\phi) = \int_\Omega \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial y}{\partial x_i}\frac{\partial \phi}{\partial x_j}dx \tag{5}$$

**Lemma 2.1**. The bilinear form $\pi(t,y,\phi)$ is coercive on $H_0^1(\Omega)$

$$\pi(t,y,y) \geq \lambda \|y\|_{H_0^1(\Omega)} \tag{6}$$

## 3 Result and discussions

**Lemma 2.2** If (2.6) and (2.7) hold, then the problem (2.1)-(2.3) admits a unique solution $y \in H_0^1(\Omega)$ Proof. See [Lions, 1971]. From the coerciveness condition (2.6) and using the Lax-Milgram lemma, there exists a unique element $y(t) \in H_0^1(\Omega)$ such that

$$\left(D_+^\beta y(t), \phi\right)_{L^2(Q)} + \pi(t,y,\phi) = L(\phi) \quad pour\ tout \ \ \phi \in H_0^1(\Omega) \tag{7}$$

which can be written as

$$\int_Q \left(D_+^\beta y(t) + \mathcal{A}y(t)\right)\phi(x)dxdt = L(\phi) \quad pour\ tout \ \ \phi \in H_0^1(\Omega). \tag{8}$$

This know as the variational fractional Dirichlet problem, where $L(\phi)$ is a continuous linear form on $H_0^1(\Omega)$ and takes the form

$$L(\phi) = \int_Q f\phi dxdt \ + \ \int_\Omega y_0\phi(x,0)dx, \quad f \in L^2(Q), y_0 \in L^2(\Omega) \tag{9}$$

Then equation (2.9) is equivalent to

$$D_+^\beta y(t) + \mathcal{A}y(t) = f, \tag{10}$$

# 4 Optimization Theorem and the Control Problem

For a control $u \in L^2(Q)$ the state $y(u)$ of the system is given by :

$$D_+^\beta y \ + \ \mathcal{A}y(u) \ = \ u, \quad (x,t) \in Q \tag{11}$$

$$y(u)|_\Sigma \ = \ 0 \tag{12}$$

$$I_+^{1-\beta} y(x,0,u) \ = \ y_0(x), \quad x \in \Omega \tag{13}$$

The observation equation is given by

$$z(u) = y(u), \tag{14}$$

The cost function is given by

$$J(v) = \int_Q (y(v) - z_d)^2 \, dxdt \ + \ (Nv, v)_{L^2(Q)} \tag{15}$$

$z_d \in L^2(\Sigma)$, $N$ is hermitian positive definite operator:

$$(Nu, u) \geq c||u||^2_{L^2(Q)}, \quad c > 0. \tag{16}$$

We define $U_{ad}$ (set of admissible controls) is closed, convex.

$$U_{ad} \subset U = L^2(Q)$$

# 5 Control Problem

**Definition 5.1.** *We want to minimize*

$$J(v) = \inf_{v \in U_{ad}} J(v) \tag{17}$$

Under the given considerations we have the following theorem:

**Theorem 5.2.** *The problem* (17) *admits a unique solution given by* (11) − (13)*, et on a :*

$$\int_Q (p(u) + Nu)\,(v - u)\,dxdt \geq 0, \tag{18}$$

where $p(u)$ is the adjoint state.

## 5.1 Neumann Problem

$$D_+^\beta y + \mathcal{A} = f, \qquad dans\ Q \tag{19}$$

$$\frac{\partial y}{\partial \nu_{\mathcal{A}}} = h, \qquad sur\ \Sigma \tag{20}$$

$$I_+^{1-\beta} y(0^+) = y_0(x), \qquad x \in \Omega \tag{21}$$

$$\pi(y, y) \ \geq \ c||y||^2_{H^1(\Omega)},\ pour\ tout\ y \in H^1(\Omega) \tag{22}$$

**Lemma 5.3.** *If* (22) *is satisfied then there exists a unique element* $y \in H^1(\Omega)$ *satisfying Neumann problem:* (19) − (21)

## 5.2 Fractional Neumann Problem for Differential System

We consider the space $U = L^2$,(the space of controls), for every control u $u \in U$ the state of the system $(y(u))$, $y(u) \in H^1(\Omega)$ is given by the solution of

$$D_+^\beta y(u) + \mathcal{A} y(u) = f \quad dans\ Q, \tag{23}$$

$$\frac{\partial y(u)}{\partial \nu_{\mathcal{A}}} = u \quad sur\ \Sigma, \tag{24}$$

### 5.3 Observation

For the observation, we consider the following two cases:

$$z(u) = y(u) \tag{25}$$

The cost function is given by:

$$J(v) = \int_Q (y(v) - z_d)^2 \, dxdt + (Nv, v)_{L^2(\Sigma)} \qquad z_d \in L^2(Q), \tag{26}$$

where $N$ is hermitian positive definite

$$(Nu, u)_{L^2(\Sigma)} \geq c||u||^2_{L^2(\Sigma)}, c > 0. \tag{27}$$

We define $U_{ad} \subset U = L^2(\Sigma)$ ( set of admissible controls) is closed, convex subset of $L^2(\Sigma)$

$$\inf_{v \in U_{ad}} J(v). \tag{28}$$

**Theorem 5.4.** *Assume that* (27) *holds and the cost function being given by* (26)*. The optimal control $u$ is characterized by* (23) − (24) *together with*

:

$$-D^\beta p(u) + \mathcal{A}^* p(u) \;=\; y(u) - z_d \quad \textit{dans } Q \tag{29}$$

$$\frac{\partial p(u)}{\partial \nu_{\mathcal{A}^*}} \;=\; 0 \tag{30}$$

$$p(x, T; u) \;=\; 0 \quad x \in \Omega. \tag{31}$$

and the optimality condition is

$$\int_\Sigma (p(u) + Nu)\,(v - u) d\Sigma \geq 0, \;\; \forall v \in U_{ad} \tag{32}$$

# 6 Summary and conclusions

An analytical scheme for fractional optimal control of differential systems is considered. The fractional derivatives was defined in the Riemann-Liouville sense. The analytical results were given in terms of Euler-Lagrange equations for the fractional optimal control problems. The formulation presented and the resulting equations are very similar to those for classical optimal control problems. The optimization problem presented in this paper constitutes a generalization of the optimal control problem of parabolic systems with Dirichlet and Neumann boundary conditions considered in (Lions, 1971) to fractional optimal control problem for second order systems.

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**3-5 July 2019, Istanbul, Turkey**

# Lie Symmetries for Variable Coefficient Compound KdV-Burgers Equations

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**Abstract**

We derive the Lie group classification for a class of KdV-Burgers equations with time dependent coefficients. The derived Lie symmetries are employed to transform boundary value problems with a partial differential equation to problems with corresponding ordinary differential equation. The list of Lie reductions is presented.

**Keywords:** KdV-Burgers equations, Lie group classification, similarity reductions, boundary value problems

## 1. Introduction

The compound KdV-Burgers equation

$$u_t + au_{xxx} + bu_{xx} + cu^2u_x + duu_x = 0, \qquad (1)$$

where $a, b, c$ and $d$ are real constants, has considerable interest in Mathematical Physics and in particular has significant applications in Fluid Mechanics, see for example in [3,23]. Explicit solutions for equation (1) can be found, for example, in [5,10,22].

In many cases the parameters in a differential equation can vary in time so the physical model is more accurate. Recently, the variable coefficient compound KdV-Burgers equation

$$u_t + h(t)u_{xxx} + g(t)u_{xx} + f(t)u^2u_x + k(t)uu_x = 0 \qquad (2)$$

was studied [1,7,24]. The functions $h(t), g(t), f(t)$ and $k(t)$ are smooth and we require that $hg(f^2 + k^2) \neq 0.$ The condition $f^2 + k^2 \neq 0$ is needed, so the class of equations (2) maintains a nonlinear form.

In the last years a large number of variable coefficient equations have appeared in the literature which are studied from the Lie symmetries point of view. The appearance of the coefficient functions which depend on spatial or/and time variables, make the classification problem a difficult task. It is for this reason that in a number of cases the group classifications that found in the literature are incomplete and sometimes the results are not completely correct. Finding Lie symmetries for a single (or a system) of partial differential equation with constant coefficients is a simple routine. The existence of algebraic packages, such as MAPLE, MATHEMATICA and REDUCE, make the search for Lie symmetries even easier and in fact in most cases provide the complete list of Lie symmetries. However for equations with variable coefficients, such algebraic packages are very helpful in the calculations, but they do not provide the complete classification. In order to overcome such difficult tasks, we use the equivalence transformations admitted by the class of equations under study. Such transformations enable us to reduce the number of arbitrary elements (coefficient functions) of the class of equations. This procedure was applied in a number of recent articles, see for example in [9,11-14,17-21], where exhaustive and complete group classifications were achieved.

Our main task is to present an enhanced group classification for the class (2). However, we can equivalently classify the Lie symmetries of the special case of the class (2)

$$u_t + u_{xxx} + g(t)u_{xx} + f(t)u^2u_x + k(t)uu_x = 0. \qquad (3)$$

In fact, the point transformation

$$\tilde{t} = \int h(t)\mathrm{d}t,\ \tilde{x} = x,\ \tilde{u} = u \qquad (4)$$

connects equation (2) and equation (3) (with the variables being tilded).

In the next section the Lie group classification is presented. In section 3, we demonstrate the application of Lie symmetries to boundary value problems. In section 4 we list the Lie mappings that reduce certain forms of the class (3) into an ordinary differential equation.

## 2. Lie Symmetries

We derive the group classification of the class (3) using the Lie classical method. We search for vector fields of the form

$$\Gamma = \tau(t,x,u)\partial_t + \xi(t,x,u)\partial_x + \eta(t,x,u)\partial_u$$

which generate one-parameter groups of point symmetry transformations of an equation from the class (3). Such vector fields satisfy the infinitesimal invariance criterion. Specifically, we require that the action of the third prolongation, $\Gamma^{(3)}$, of the vector field $\Gamma$ on equation (3) results in the condition that it is an identity for all solutions for this equation. Hence, the criterion takes the form

$$\Gamma^{(3)}\left[u_t + u_{xxx} + g(t)u_{xx} + f(t)u^2u_x + k(t)uu_x\right]\Big|_{u_{xxx}=-(u_t+g(t)u_{xx}+f(t)u^2u_x+k(t)uu_x)} = 0. \qquad (5)$$

Equation (5) is an identity in the variables $t, x, u$ and the derivatives of $u$ and more specifically, it is a multivariable polynomial in the variables $u_t, u_x, u_{tt}, u_{tx}, u_{xx}, u_{ttt}, u_{ttx}$ and $u_{txx}$. Coefficients of these variables are equal to zero, which provide the system of determining equations. Solution of these equations provide the forms of the coefficient functions $\tau, \xi, \eta$ and of the arbitrary elements $f, g, k$. Initially, we use certain results from references [6,14] to deduce that the coefficient functions have the simplified forms

$$\tau = \tau(t),\ \xi = \phi_1(t)x + \phi_2(t),\ \eta = \psi_1(t,x)u + \psi_2(t,x).$$

These forms simplify the identity (5). Coefficients of various derivatives of $u$ lead to the determining system which consists of the following equations:

$$\tau_t - 3\phi_1 = 0,$$
$$3\psi_{1x} + g\tau_t + \tau g_t - 2g\phi_1 = 0,$$
$$f\tau_t + \tau f_t + 2f\psi_1 - f\phi_1 = 0,$$
$$k\tau_t + \tau k_t + k\psi_1 + 2f\psi_2 - k\phi_1 = 0,$$
$$3\psi_{1xx} + 2g\psi_{1x} - \phi_{2t} - x\phi_{1t} + k\psi_2 = 0,$$
$$f\psi_{1x} = 0,$$
$$k\psi_{1x} + f\psi_{2x} = 0,$$
$$\psi_{1t} + \psi_{1xxx} + g\psi_{1xx} + k\psi_{2x} = 0,$$
$$\psi_{2t} + \psi_{2xxx} + g\psi_{2xx} = 0.$$

Solving the above system provide the forms of the functions $\tau(t), \phi_1(t), \phi_2(t), \psi_1(t,x)$ and $\psi_2(t,x)$ and also the forms of the arbitrary elements $g(t)$, $f(t)$ and $k(t)$. Consequently, the desired Lie symmetries are derived which are admitted by specific forms of the class of equations (3). Omitting the detailed analysis, we state that we find five cases. We tabulate the results in the table 1.

**Table 1.** The group classification of class (3).

| no. | $g(t)$ | $f(t)$ | $k(t)$ | Basis of $A^{\max}$ |
|---|---|---|---|---|
| 1 | $\forall$ | $\forall$ | $\forall$ | $\partial_x$ |
| 2 | $\epsilon_1$ | $e^{\lambda t}$ | $\delta_1 e^{\frac{\lambda}{2}t}$ | $\partial_x, \partial_t - \frac{\lambda}{2}u\partial_u$ |
| 3 | $\epsilon_1 t^{-\frac{1}{3}}$ | $t^n$ | $\delta_1 t^{\frac{3n-2}{6}}$ | $\partial_x, 6t\partial_t + 2x\partial_x - (3n+2)u\partial_u$ |
| 4 | $\epsilon_1 t^{-\frac{1}{3}}$ | $t^{-\frac{2}{3}}$ | $\epsilon_2 t^{-\frac{2}{3}} \ln\|t\|$ | $\partial_x, 6t\partial_t + [2x - 9\epsilon_2^2(\ln\|t\| - 3)t^{\frac{1}{3}}]\partial_x - 3\epsilon_2\partial_u$ |
| 5 | $G(t)$ | 0 | $\epsilon_1 G(t) e^{\delta_2 \int G^3 dt}$ | $\partial_x, (\int k(t)dt)\partial_x + \partial_u$, $[G(t)]^{-3}\partial_t - [G(t)]^{-4} G_t x\partial_x + [(\delta_1 \int k(t)dt - \delta_2)u - \delta_1 x]\partial_u$ |

Here $G(t)$ is a solution of the equation $G_t - \epsilon_1\delta_1 G^4 \int G e^{\delta_2 \int G^3 dt} dt = 0$ and $\epsilon_1 \neq 0, \epsilon_2 \neq 0, n, \lambda, \delta_1$ and $\delta_2$ are constants.

**Note 1.** Cases 1-3 in table 1 with $\delta_1 = 0$ give the results of the group classification for the modified KdV-Burgers with variable coefficients

$$u_t + u_{xxx} + g(t)u_{xx} + f(t)u^2 u_x = 0. \qquad (6)$$

The present group classification of the class (6) completes the results that appear recently in the literature [8].

We use the mapping (4) to obtain the group classification for the class (2). The desired results are tabulated in the table 2.

**Table 2.** The group classification of class (2).

| no. | $g(t)$ | $f(t)$ | $k(t)$ | Basis of $A^{\max}$ |
|---|---|---|---|---|
| 1 | $\forall$ | $\forall$ | $\forall$ | $\partial_x$ |
| 2 | $\epsilon_1 h(t)$ | $he^{\lambda\int h\mathrm{d}t}$ | $\delta_1 he^{\frac{\lambda}{2}\int h\mathrm{d}t}$ | $\partial_x, \frac{1}{h}\partial_t - \frac{\lambda}{2}u\partial_u$ |
| 3 | $\epsilon_1 h(\int h\mathrm{d}t)^{-\frac{1}{3}}$ | $h(\int h\mathrm{d}t)^n$ | $\delta_1 h(\int h\mathrm{d}t)^{\frac{3n-2}{6}}$ | $\partial_x, 6\frac{\int h\mathrm{d}t}{h}\partial_t + 2x\partial_x - (3n+2)u\partial_u$ |
| 4 | $\epsilon_1 h(\int h\mathrm{d}t)^{-\frac{1}{3}}$ | $h(\int h\mathrm{d}t)^{-\frac{2}{3}}$ | $\epsilon_2 h(\int h\mathrm{d}t)^{-\frac{2}{3}}\ln\lvert\int h\mathrm{d}t\rvert$ | $\partial_x, 6\frac{\int h\mathrm{d}t}{h}\partial_t + \left[2x - 9\epsilon_2^2(\ln\lvert\int h\mathrm{d}t\rvert - 3)(\int h\mathrm{d}t)^{\frac{1}{3}}\right]\partial_x - 3\epsilon_2\partial_u$ |
| 5 | $h(t)G(\xi)$, $\xi = \int h\mathrm{d}t$ | 0 | $\epsilon_1 h(t)G(\xi)e^{\delta_2\int G(\xi)^3\mathrm{d}\xi}$ | $\partial_x$, $(\int k(t)\mathrm{d}t)\partial_x + \partial_u$, $[h(t)G(\xi)]^{-3}\partial_t - [G(\xi)]^{-4}G_\xi x\partial_x + [(\delta_1\int k(t)\mathrm{d}t - \delta_2)u - \delta_1 x]\partial_u$ |

Here $G(\xi)$ is a solution of the equation $G_\xi - \epsilon_1\delta_1 G^4\int Ge^{\delta_2\int G^3\mathrm{d}\xi}\mathrm{d}\xi = 0$ and $\epsilon_1 \neq 0, \epsilon_2 \neq 0, n, \lambda, \delta_1$ and $\delta_2$ are constants, where $h(t)$ is every smooth function.

## 3. Boundary value problems

We apply the derived Lie symmetries to solve boundary value problems. The procedure is straightforward and it was applied recently in various problems [4,15,16]. We consider the problem of the form

$$u_t + f(t)u^2u_x + k(t)uu_x + g(t)u_{xx} + u_{xxx} = 0, \qquad t>0, \qquad x>0$$

$$\lim_{t\to\infty} u(t,x) = 0, \qquad x>0$$

$$u(t,0) = q_1(t), \qquad u_x(t,0) = q_2(t), \qquad u_{xx}(t,0) = q_3(t), \qquad t>0,$$

where $q_1(t), q_2(t)$ and $q_3(t)$ are smooth functions in their arguments. A Lie symmetry that leaves invariant the partial differential equation of the problem, must also leave invariant the given conditions. We use the Lie symmetries of the cases 2 and 3 in table 1 and apply a linear combination of them to initial and boundary conditions. This results to restriction of the forms $q_1(t), q_2(t)$ and $q_3(t)$. In fact, they are specific functions.

In the case where $g(t) = \epsilon_1,\ f(t) = \mathrm{e}^{\lambda t}$ and $k(t) = \delta_1 \mathrm{e}^{\frac{\lambda}{2}t}$, equation (3) admits the general Lie symmetry

$$\Gamma = \alpha_1 \partial_x + \alpha_2 \left( \partial_t - \frac{\lambda}{2} u \partial_u \right),$$

where $\alpha_1$ and $\alpha_2$ are arbitrary constants. Application of the symmetry $\Gamma$ to the first boundary condition $x=0$ and $u - q_1(t) = 0$ gives $\alpha_1 = 0$ and $2q_{1t} + \lambda q_1 = 0.$ Hence, $q_1(t) = \gamma_1 \mathrm{e}^{-\frac{\lambda}{2}t}$ and we take w.l.o.g. $\alpha_2 = 1.$ In order to examine invariance of the other two boundary conditions we require to have the extension of $\Gamma$ up to the second order,

$$\Gamma^{(1)} = \Gamma - \frac{\lambda}{2} u_t \partial_{u_t} - \frac{\lambda}{2} u_x \partial_{u_x},$$
$$\Gamma^{(2)} = \Gamma^{(1)} - \frac{\lambda}{2} u_{tt} \partial_{u_{tt}} - \frac{\lambda}{2} u_{tx} \partial_{u_{tx}} - \frac{\lambda}{2} u_{xx} \partial_{u_{xx}}.$$

For the second condition, we have

$$\Gamma^{(1)} \{u_x - q_2(t)\}\big|_{u_x = q_2(t)} = 0$$

which gives $q_2(t) = \gamma_2 e^{-\frac{\lambda}{2}t}$.

Similarly, the invariance condition

$$\Gamma^{(2)}\{u_{xx} - q_3(t)\}\big|_{u_{xx}=q_3(t)} = 0$$

gives $q_3(t) = \gamma_3 e^{-\frac{\lambda}{2}t}$. Now, solving the characteristic system that corresponds to the symmetry $\Gamma$, we find the similarity reduction

$$u(t,x) = e^{-\frac{\lambda}{2}t} F(x)$$

which reduces the above problem to

$$F_{xxx} + \epsilon_1 F_{xx} + (F + \delta_1) F F_x - \tfrac{1}{2}\lambda F = 0,\ F(0) = \gamma_1,\ F_x(0) = \gamma_2,\ F_{xx}(0) = \gamma_3.$$

Now, we repeat the same procedure for the case where $g(t) = \epsilon_1 t^{-\frac{1}{3}},\ f(t) = t^n, k(t) = \delta_1 t^{\frac{3n-2}{6}}$. We use the linear combination of the two Lie symmetries admitted by equation (3) to obtain the similarity mapping

$$u(t,x) = t^{-\frac{3n+2}{6}} F(\omega),\ \ \omega = x t^{-\frac{1}{3}}$$

that reduces the boundary value problem

$$u_t + t^n u^2 u_x + \delta_1 t^{\frac{3n-2}{6}} u u_x + \epsilon_1 t^{-\frac{1}{3}} u_{xx} + u_{xxx} = 0, \qquad t > 0, \qquad x > 0$$

$$\lim_{t \to 0} u(t,x) = 0, \qquad x > 0$$

$$u(t,0) = \gamma_1 t^{-\frac{3n+2}{6}}, \qquad u_x(t,0) = \gamma_2 t^{-\frac{3n+4}{6}}, \qquad u_{xx}(t,0) = \gamma_3 t^{-\frac{3n+6}{6}}, \qquad t > 0,$$

to the problem

$$F_{\omega\omega\omega} + \epsilon_1 F_{\omega\omega} + \tfrac{1}{3}(3F^2 + 3\delta_1 F - \omega) F_\omega - \tfrac{3n+2}{6} F = 0,\ F(0) = \gamma_1,\ F_\omega(0) = \gamma_2,\ F_{\omega\omega}(0) = \gamma_3,$$

where we assume that $n<-\frac{2}{3}$. In the case $n>-\frac{2}{3}$, we replace the initial condition with $\lim_{t\to\infty} u(t,x)=0.$

Although we have initial value problems with the differential equation being an ordinary, finding exact solutions is a very difficult task. In such cases, searching for numerical solutions is a possible direction. Such approach was used, for example, in references [2,15,16].

## 4. Lie reductions

The differential operator $\Gamma=\tau(t,x,u)\partial_t+\xi(t,x,u)\partial_x+\eta(t,x,u)\partial_u$, $(\tau,\xi)\neq(0,0)$, that corresponds to a Lie symmetry can be used to construct reduction mappings for the equation under consideration. They reduce the number of independent variables by one. For example, in the case of a partial differential equation in two independent variables, such mappings reduce it to an ordinary differential equation. In fact, construction of such mappings is one of the most important applications of Lie symmetries. Reduction operators are derived by solving the invariant surface condition $\tau u_t+\xi u_x-\eta=0.$

We present the non-trivial Lie reductions of the class (3). We refer to the results that presented in the table 1. In the case 2, we have two Lie reductions. The first one corresponds to the Lie symmetry $\partial_t-\frac{\lambda}{2}u\partial_u$ which was presented in the previous section and the second reduction corresponds to the linear combination of the two symmetries admitted by the equation, $\partial_t-\frac{\lambda}{2}u\partial_u+c\partial_x$. Solving the appropriate invariant surface condition, we find the mapping

$$u=e^{-\frac{\lambda}{2}t}F(\xi),\ \ \xi=x-ct$$

that reduces

$$u_t+u_{xxx}+\epsilon_1 u_{xx}+\mathrm{e}^{\lambda t}u^2u_x+\delta_1\mathrm{e}^{\frac{\lambda}{2}t}uu_x=0$$

to the ordinary differential equation

$$F_{\xi\xi\xi}+\epsilon_1 F_{\xi\xi}+(F^2+\delta_1 F-c)F_\xi-\frac{1}{2}\lambda F=0.$$

For the case 3, the only nontrivial reduction corresponds to the differential operator $6t\partial_t+2x\partial_x-(3n+2)u\partial_u$ and the corresponding result appears in the previous section. For the case 4, we find one reductions which corresponds to the differential operator $6t\partial_t+[2x-9\epsilon_2^2(\ln|t|-3)t^{\frac{1}{3}}]\partial_x-3\epsilon_2\partial_u$. We find the mapping

$$u=-\tfrac{1}{2}\epsilon_2\ln|t|+F(\xi),\ \ \xi=xt^{-\frac{1}{3}}+\tfrac{3}{4}\epsilon_2{}^2\left(\ln^2|t|-6\ln|t|\right)$$

that reduces

$$u_t + u_{xxx} + \epsilon_1 t^{-\frac{1}{3}} u_{xx} + t^{-\frac{2}{3}} u^2 u_x + \epsilon_2 t^{-\frac{2}{3}} \ln|t| u u_x = 0$$

to the ordinary differential equation

$$F_{\xi\xi\xi} + \epsilon_1 F_{\xi\xi} + (F^2 - \tfrac{1}{3}\xi - \tfrac{9}{2}\epsilon_2^2) F_\xi = \tfrac{1}{2}\epsilon_2.$$

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**3-5 July 2019, Istanbul, Turkey**

# Statistical Modeling of Contact Wear Polluted Elastohydrodynamics

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***Abstract:***

Progress in the mechanical industry increasingly requires the use of special parameters to improve the service life of the mechanisms. Among these operating conditions, solid pollution is one of the most difficult enemy.

Statistical modeling is a simplified way and mathematically formalized to approximate the reality, so it is a tool not to neglected to model the degradation of the lifetime of contact surfaces. Although these undesirable particles can lead to premature wear of the machine elements or even a total failure of the mechanisms.

The objective of this work is to statistically model the influence of lubricant contamination by solid pollution on an elastohydrodynamic contact. Based on TAGUCHI's experimental plans and analysis

**Keywords:** Contact, Lubrication, Solid pollution, Wear.

## 1. Introduction

Tribological studies and research contribute to the technological renovation of mechanical contacts. For that, this modeling study, will try to answer the thematic approach, and this by the use of a plan of experiences of Taguchi.

The main objective of this work is to establish a mathematical model that expresses the loss of dimension of an elastohydrodynamic contact under specific operating conditions. A series of experiments are carried out on a bench where two metal disks are in contact.

Lubricants in their use are already polluted even when new, they are more polluted during assembly, maintenance or operation. These undesirable particles can damage the surfaces [1] and lead to fatigue and wear of the machine elements, including several contributions in this field such as the work of [2, 3,4].

## 2. Preliminaries

The test device is taken in such a way as to respond to the smooth running of the experiments. Indeed, it is a device in which are mounted in contact a rotating steel disc and a cylindrical test piece.

### 2.1. Equipment used:

In order to carry out the experiments, we will use the following equipment

Triangological test bench

Cylindrical specimen (bronze).100Cr6 steel discs;

Oil ATF;

Comparator ;

Loads ;

Filtered sand at 63 microns;

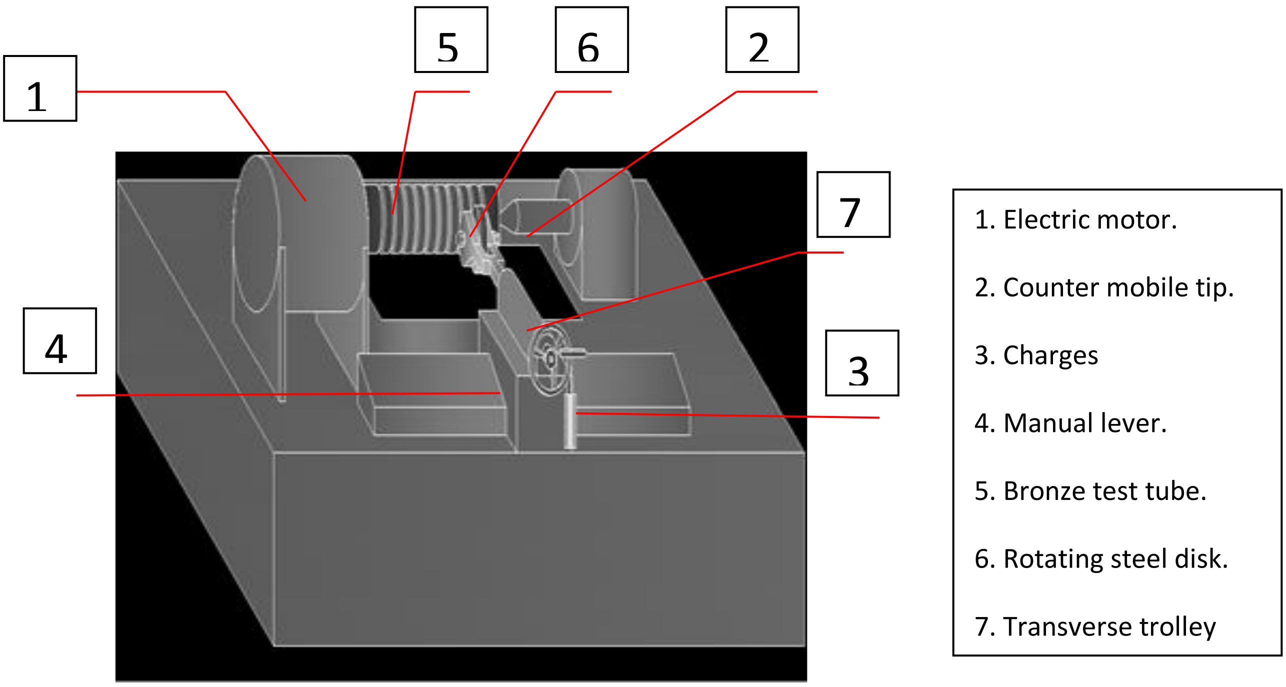


Fig. I.1 Triangological test bench

The experiments are planned by Taguchi L9 plan as shown in table 2.1

**Table.2.1** Taguchi L9 plan

| **Concentration (g/l)** | **Load (N)** | **Speed (tr/min°)** |
|---|---|---|
| 5,0 | 100 | 500 |
| 5,0 | 150 | 750 |
| 5,0 | 200 | 1000 |
| 7,5 | 100 | 750 |
| 7,5 | 150 | 1000 |
| 7,5 | 200 | 500 |
| 10,0 | 100 | 1000 |
| 10,0 | 150 | 500 |
| 10,0 | 200 | 750 |

## 3. Main Results

After carrying out the tests according to Taguchi orthogonal planes L9, wear (W) is measured. For each answer, five measurements were made and the average value was considered. Tables 3.1 show the measurement results for the experiment plan adopted.

**Table 3.1** Table of wear measurements

| **Concentration (g/l)** | **Load (N)** | **Speed (tr/min)** | **Wear (mm)** |
|---|---|---|---|
| 5,0 | 100 | 500 | 0,150 |
| 5,0 | 150 | 750 | 0,160 |
| 5,0 | 200 | 1000 | 0,190 |
| 7,5 | 100 | 750 | 0,180 |
| 7,5 | 150 | 1000 | 0,190 |
| 7,5 | 200 | 500 | 0,180 |
| 10,0 | 100 | 1000 | 0,222 |
| 10,0 | 150 | 500 | 0,190 |
| 10,0 | 200 | 750 | 0,200 |

### 3.1 Taguchi statistical analysis:

First, a Taguchi analysis of the wear for the experiment was made, whose objective is to better understand the impact of the operating parameters on the degradation of the contacts. EHD.

In a second step, mathematical models of regression, expressing the relation between the input parameters and the output parameters, are established:

$YP = f(x_1, x_2, x_3)$

$$YP = b_0 + \sum_{i=1}^{k} b_i x_i$$

For the experiment, the model sought is of linear type without interaction. It looks like this With:

$b_0$: free coefficient.

$b_1, b_2, \ldots b_k$: linear coefficients

$x_i$: represents the input parameters (factors).

In order to show that the predicted values of the responses studied approximate the experimental data in the best way, the coefficient of determination $R^2$ is calculated. This coefficient is a number that indicates how the observed data adjusts to the values predicted by the statistical model. Indeed, it is a parameter of good fit. We find that the value of $R^2$ is very high and tend towards 1 ($R^2 \approx 1$). This value gives an efficient prediction and therefore it can be assumed that the predicted and experimental values are in excellent agreement; Figure.I.2.

**Coefficient of determination $R^2$:**

**$R^2$ = 99,30%**

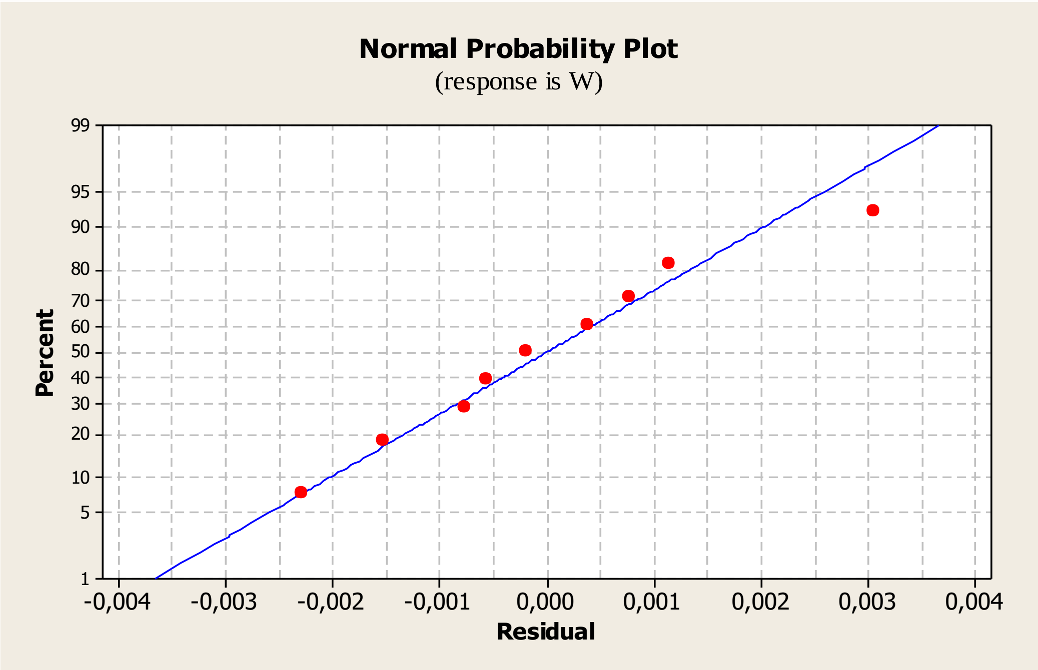


**Fig.I.2** Normal probability plot

### 3.2 Regression Equation :

The experimental regression equation is written as follows

**W** = 0,0733434 + 0,0221308 $\mathbf{X_1}$ + 0,00878677 $\mathbf{X_2}$ - 0,00024782 $\mathbf{X_3}$ - 0,00192503 $\mathbf{X_1 * X_3}$ + 1,65295e-005 $\mathbf{X_1 * X_2}$ + 1,14904e-005 $\mathbf{X_3 * X_2}$ + **e**

W : Wear

X1 : concentration

X2 : Speed

X3 : Load

e : error

Table.3.2 Response Table for Means

| Level | concentration | Load | speed |
|---|---|---|---|
| 1 | 0,1667 | 0,1840 | 0,1733 |
| 2 | 0,1833 | 0,1800 | 0,1800 |
| 3 | 0,2040 | 0,1900 | 0,2007 |
| Delta | 0,0373 | 0,0100 | 0,0273 |
| Rank | 1 | 3 | 2 |

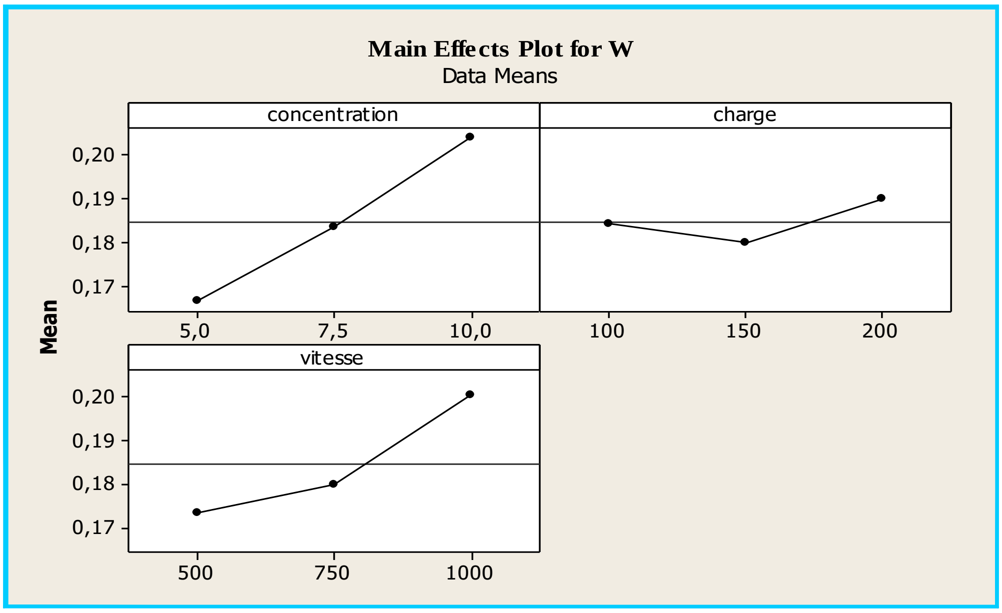


**Fig. 3.2** Main effect plot for W

According to Taguchi's analysis, the effect of the speed, load and concentration parameters is different for each answer, Tab.3.2.

From Fig.2 it is found that the first effect responsible for the degradation of the contact is the concentration of the pollutant in the lubricant.

Then, the speed and the load which causes the appearance of the wear expressed by the loss of dimension, which leads to the failure.

## 4. Conclusion

In this article it is evident that the effect of particle concentration of sand grains is dominant and followed by the other parameters of the study. It is concluded that the wear of the elestohydrodynamic contact studied is caused by abrasion by the pollutant levels in the lubricant, which normally has the protective role.

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# Analytical Solutions for Conformable Fractional Order (3+1)-Dimensional Telegraph Differential Equation

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**Abstract**

In this study an analytical solution procedure for the solution of conformable fractional order (3+1)-dimensional Telegraph differential equation is presented. For this aim the computer software called Mathematica is employed. Solution procedure is summarized with an example.

**Keywords:** Conformable fractional derivative, analytical solutions, fractional partial differential equations, Telegraph equation, quadruple Laplace transform.

## 1. Introduction

Fractional calculus, which is concluded as arbitrary order differentiation and integration, is as old as traditional derivation and integration. However, the value of the fractional derivative was better understood after it was discovered that it had better described events arising in nature. After then it attracted many researchers' attention who study on different natural sciences especially physics, engineering, chemistry, medical sciences social sciences etc. [1-4]. The main purpose of fractional analysis is to express all aspects of engineering and natural phenomena that the traditional derivative fails to explain. For this purpose, different types of derivative and integral definitions are stated. Nevertheless, much of them involve integral forms in their definitions. Because of the integral formed definition, the calculations became hard and complicated. However, scientists determined some deficiencies of these definitions. For example [5],

- The Riemann-Liouville derivative does not satisfy $D_a^\alpha 1 = 0$ (Caputo derivative satisfies), if $\alpha$ is not a natural number.
- Not all fractional derivatives satisfy the known formula of the derivative of the product of two functions.

$$D_a^\alpha (fg) = gD_a^\alpha (f) + fD_a^\alpha (g).$$

- Not all fractional derivatives satisfy the known formula of the derivative of the quotient of two functions.

$$D_a^{\alpha}\left(\frac{f}{g}\right)=\frac{fD_a^{\alpha}(f)-gD_a^{\alpha}(g)}{g^2}.$$

- Not all fractional derivatives satisfy the chain rule.

$$D_a^{\alpha}(fog)(t)=f^{\alpha}(g(t))g^{\alpha}(t).$$

- Not all fractional derivatives satisfy $D^{\alpha}D^{\beta}=D^{\alpha+\beta}$ in general.
- In the Caputo definition, it is assumed that the function $f$ is differentiable.

However, in 2014, a new definition of fractional derivative and fractional integral, which satisfy the basic properties of known derivative and integral, are expressed by Khalil *et al.* [5].

**1.1. Definition** Let $f:[0,\infty)\to\mathbb{R}$ be a function. The $\alpha^{th}$ order *"conformable fractional derivative"* of $f$ is defined by,

$$D_{\alpha}(f)(t)=\lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon}$$

for all $t>0,\alpha\in(0,1)$.

**1.2. Definition** If $f$ is $\alpha$-differentiable in some $(0,a),\ a>0$ and $\lim_{t\to 0^+}f^{(\alpha)}(t)$ exists then define $f^{(\alpha)}(0)=\lim_{t\to 0^+}f^{(\alpha)}(t)$. The *"conformable fractional integral"* of a function $f$ starting from $a\geq 0$ is defined as:

$$I_{\alpha}^{a}(f)(t)=\int_a^t f(x)d_{\alpha}x=\int_a^t\frac{f(x)}{x^{1-\alpha}}dx$$

where the integral is the usual Riemann improper integral, and $\alpha\in(0,1]$.

This new definition satisfies the following basic properties and theorems referred in [5,6].

- $D_{\alpha}(cf+dg)=cD_{\alpha}(f)+cD_{\alpha}(g)$ for all $a,b\in\mathbb{R}$.
- $D_{\alpha}(t^{p})=pt^{p-\alpha}$ for all $p\in\mathbb{R}$.
- $D\alpha(\lambda)=0$ for all constant functions $f(t)=\lambda$.
- $D_{\alpha}(fg)=fD_{\alpha}(g)+gD_{\alpha}(f)$.
- $D_{\alpha}\left(\frac{f}{g}\right)=\frac{gD_{\alpha}(f)-fD_{\alpha}(g)}{g^2}$.

- If in addition to $f$ is differentiable, then $D_\alpha(f)(t) = t^{1-\alpha}\dfrac{df}{dt}$.

Integral transforms are very useful tools for obtaining the analytical solutions of both ordinary and partial differential equations. Considered equation can be turned into an algebraic equation by the help of integral transforms so the calculation becomes simpler. In this study we present quadruple Laplace transform including conformable Laplace transform to get the exact solutions of time fractional $(3+1)$ dimensional second order hyperbolic telegraph equation. The basic properties of quadruple Laplace transform is given by Rehman *et al.* [7].

In this study we combine the quadruple Laplace transform with conformable Laplace transform [6,8] to get the exact results for $(3+1)$ dimensional second order hyperbolic telegraph equation. Let us give some basic properties of conformable Laplace transform.

## 2. Conformable Laplace Transform

**2.1. Definition** [6] Let $t_0 \in \mathbb{R}$, $0 < \alpha \le 1$ and $f : [t_0, \infty) \to \mathbb{R}$ real valued function. The conformable fractional Laplace transform of order $\alpha$ for the function $f$ is defined by

$$\mathcal{L}_\alpha^{t_0}[f(t)](s) = \int_{t_0}^{\infty} e^{-s\frac{(t-t_0)^\alpha}{\alpha}} f(t)d_\alpha(t,t_0) = \int_{t_0}^{\infty} e^{-s\frac{(t-t_0)^\alpha}{\alpha}} f(t)(t-t_0)^{\alpha-1}dt. \tag{1.1}$$

**2.2. Example** Here are the fractional Laplace transforms of some elementary functions.

- $\mathcal{L}_\alpha[1] = \dfrac{1}{s}, s > 0$
- $\mathcal{L}_\alpha[t] = \dfrac{\alpha^{1/\alpha}\Gamma\left(1+\dfrac{1}{\alpha}\right)}{s^{\left(1+\frac{1}{\alpha}\right)}}, s > 0$
- $\mathcal{L}_\alpha[t^p] = \dfrac{\alpha^{p/\alpha}\Gamma\left(1+\dfrac{1}{\alpha}\right)}{s^{\left(1+\frac{p}{\alpha}\right)}}, s > 0$
- $\mathcal{L}_\alpha[e^{\frac{t^\alpha}{\alpha}}] = \dfrac{1}{s-1}, s > 1$
- $\mathcal{L}_\alpha[\sin\left(w\dfrac{t^\alpha}{\alpha}\right)] = \dfrac{1}{w^2+s^2}$

- $\mathcal{L}_\alpha[\cos\left(w\frac{t^\alpha}{\alpha}\right)] = \frac{s}{w^2+s^2}$

The relationship between conformable Laplace transform and known Laplace transform can be declares as follows.

**2.3. Lemma** Let $f:[0,\infty)\to\mathbb{R}$ be a function such that $\mathcal{L}_\alpha[f(t)] = F_\alpha(s)$ exists. Then

$$F_\alpha(s) = \mathcal{L}[f(\alpha t)^{1/\alpha}]$$

where $\mathcal{L}[g(t)] = \int_0^\infty g(t)dt$ [6].

Then Özkan [9] expressed the existence theorem, linearity theorem, conformable Laplace transform of the conformable fractional derivative with higher orders and they used conformable Laplace transform for solving integral and integro-differential equations in conformable sense.

**2.4. Theorem** [Linearity Theorem] Let $f_1(t)$ and $f_2(t)$ be two functions whose conformable Laplace transform exist for $s>a_1$ and $s>a_2$ respectively. Then for $s$ greater than the maximum of $a_1$ and $a_2$,

$$\mathcal{L}_\alpha[c_1 f_1(t)+c_2 f_2(t)] = c_1\mathcal{L}_\alpha[f_1(t)] + c_2\mathcal{L}_\alpha[f_2(t)]$$

where $c_1, c_2$ are real constants [9].

**2.5. Theorem** Existence Theorem] Let an exponential order function $f$ be piecewise continuous on the interval $0\le t\le A$ for any positive $A$ and $e^{-a\frac{t^\alpha}{\alpha}}|f(t)|<M$ when $t>t_0$. In this equality $M, a, t_0$ are positive real constants and $0<\alpha\le 1$. Then the conformable Laplace transform defined by Eq. (1.1) exists for any $s>a$ [9].

**2.6. Theorem** [9] Suppose that $f(t)$ is continuous and $D_t^\alpha(f(t))$ is piecewise continuous on any interval $0\le t\le A$. Consider further that there exists constants $M, a$ and $t_0$ such that $|f(t)|\le Me^{a\frac{t^\alpha}{\alpha}}$ for $t\ge t_0$. Then $\mathcal{L}_\alpha[D_t^\alpha(f(t))]$ exists for $s>a$ and moreover

$$\mathcal{L}_\alpha[D_t^\alpha(f(t))] = s\mathcal{L}_\alpha[f(t)] - f(0).$$

**2.7. Corollary** [9] Suppose that $f(t), D_t^\alpha(f(t)), D_t^{(2)\alpha}(f(t)), \ldots, D_t^{(n-1)\alpha}(f(t))$ are continuous and $D_t^{(n)\alpha}(f(t))$ is piecewise continuous on any interval $0\le t\le A$. Consider further that there exists constants

$M, a$ and $t_0$ such that $|f(t)| \leq Me^{a\frac{t^\alpha}{\alpha}}, |D_t^\alpha(f(t))| \leq Me^{a\frac{t^\alpha}{\alpha}}, ..., |D_t^{(n-1)\alpha}(f(t))| \leq Me^{a\frac{t^\alpha}{\alpha}}$ for $t \geq t_0$.Then $\mathcal{L}_\alpha[D_t^{(n)\alpha}(f(t))]$ exists for $s > a$ and obtained as

$$\mathcal{L}_\alpha[D_t^{(n)\alpha}(f(t))] = s^n \mathcal{L}_\alpha[f(t)] - s^{n-1}f(0) - ... - sD_t^{(n-2)\alpha}f(0) - D_t^{(n-1)\alpha}f(0)$$

where $D_t^{(n)\alpha}(f(t))$ means $n$ times conformable derivative of function $f(t)$.

**2.8. Theorem** [9]

If $F_\alpha(s) = \mathcal{L}_\alpha[f(t)]$ exists for $s > a \geq 0$. Then following specialties are satisfied.

- If $c$ is a constant then

$$\mathcal{L}_\alpha[e^{c\frac{t^\alpha}{\alpha}} f(t)] = F_\alpha(s-c).$$

- Let $m$ is a constant

$$\mathcal{L}_\alpha[t^m] = \alpha^{\frac{m}{\alpha}} \frac{\Gamma\left(\frac{m+\alpha}{\alpha}\right)}{s^{\frac{m+\alpha}{\alpha}}}.$$

- If $c, m$ are arbitrary constants

$$\mathcal{L}_\alpha[t^m e^{c\frac{t^\alpha}{\alpha}}] = \alpha^{\frac{m}{\alpha}} \frac{\Gamma\left(\frac{m+\alpha}{\alpha}\right)}{(s-c)^{\frac{m+\alpha}{\alpha}}}.$$

Now let us give the quadruple Laplace transform involving conformable Laplace transform inside its definition.

**2.9. Definition** Let $u(x, y, z, t)$ be a continuous function with $(3+1)$ variables then quadruple Laplace transform involving conformable Laplace transform of function $u(x, y, z, t)$ can be expressed by

$$\mathcal{L}_x\mathcal{L}_y\mathcal{L}_z\mathcal{L}_t^\alpha[u(x, y, z, t)] = U(p, r, s, \omega) = \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty e^{-px-ry-sz-\omega\frac{t^\alpha}{\alpha}} u(x, y, z, t) dxdydzd_\alpha t$$

where $p, r, s, \omega \in \mathbb{C}$, $0 < \omega \leq 1$ and subscripts denotes the variables to be Laplace transform applied.

## 3. Application

### 3.1. Example

Consider the following conformable time fractional (3+1) dimensional second order hyperbolic telegraph equation

$$D_t^{2\alpha}u(x,y,z,t)+2D_t^{\alpha}u(x,y,z,t)+3u(x,y,z,t)=D_x^2u(x,y,z,t)+D_y^2u(x,y,z,t)+D_z^2u(x,y,z,t) \quad (1.2)$$

with the conditions

$$\begin{aligned}
&u(x,y,z,0)=\sinh x\sinh y\sinh z,\\
&D_t^{\alpha}u(x,y,z,0)=-2\sinh x\sinh y\sinh z,\\
&u(0,y,z,t)=0,\\
&u(x,y,0,t)=0,\\
&D_z u(x,y,0,t)=e^{-2\frac{t^\alpha}{\alpha}}\sinh x\sinh y.\\
&D_x u(0,y,z,t)=e^{-2\frac{t^\alpha}{\alpha}}\sinh z,\\
&u(x,0,z,t)=0,\\
&D_y u(x,0,z,t)=e^{-2\frac{t^\alpha}{\alpha}}\sinh x\sinh z,
\end{aligned} \quad (1.3)$$

where $0<\alpha\le 1$, $x>0$, $y>0$, $z>0$, $t>0$, $D_t^{2\alpha}$ denotes the two times sequential $\alpha$-th order conformable fractional derivative of function $u(x,y,z,t)$. First of all applying the quadruple conformable Laplace transform to Eq. (1.2) yields

$$\begin{aligned}
&\omega^2U(p,r,s,\omega)-\omega U(p,r,s,0)-D_t^{\alpha}U(p,r,s,0)+2\left(\omega U(p,r,s,\omega)-U(p,r,s,0)\right)\\
&+3U(p,r,s,\omega)=p^2U(p,r,s,\omega)-pU(0,r,s,\omega)-D_xU(0,r,s,\omega)+r^2U(p,r,s,\omega)\\
&-rU(p,0,s,\omega)-D_yU(p,0,s,\omega)+s^2U(p,r,s,\omega)-sU(p,r,0,\omega)-D_zU(p,r,0,\omega)
\end{aligned} \quad (1.4)$$

where $U(p,r,s,\omega)$ is the quadruple Laplace transformed version of the function $u(x,y,z,t)$. Also calculating the Laplace transformed versions of the conditions (1.3) yields

$$
\begin{aligned}
&U(0,r,s,\omega)=0,\\
&D_xU(0,r,s,\omega)=\frac{1}{(\omega+2)(r^2-1)(s^2-1)},\\
&U(p,r,s,0)=\frac{1}{(p^2-1)(r^2-1)(s^2-1)},\\
&D_t^{\alpha}U(p,r,s,0)=\frac{-2}{(p^2-1)(r^2-1)(s^2-1)},\\
&U(p,0,s,\omega)=0,\\
&D_yU(p,0,s,\omega)=\frac{1}{(\omega+2)(p^2-1)(s^2-1)},\\
&U(p,r,0,\omega)=0,\\
&D_zU(p,r,0,\omega)=\frac{1}{(\omega+2)(p^2-1)(r^2-1)}.
\end{aligned}
\tag{1.5}
$$

Substituting the values of Eqns. (1.5) into Eqn. (1.4) and after some algebraic calculations on Eqn. (1.4) we handle

$$U(p,r,s,\omega)=\frac{1}{\left(p^2-1\right)\left(r^2-1\right)\left(s^2-1\right)(\omega+2)}.$$

So $u(x,y,z,t)$ can be obtained as

$$u(x,t)=e^{-2\frac{t^{\alpha}}{\alpha}}\sinh x\sinh y\sinh z.$$

## 4. Conclusion

In this study, authors expressed the quadruple Laplace transform involving conformable Laplace transform and applied to time fractional (3+1)-dimensional second order hyperbolic telegraph equation. It is understood that this new combined type Laplace transform is a relevant, efficient and applicable transform which can be applied the various problems arising in different branches of science.

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# Approximate Solution Procedure for Conformable Fractional Order (3+1)-Dimensional Telegraph Differential Equation by Residual Power Series Method

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## Abstract

The main target of this study is to obtain the numerical solutions of conformable fractional order (3+1)-dimensional Telegraph differential equation with the aid of an approximate solution procedure namely Residual Power Series Method. Conformable derivative with fractional order is applicable, understandable and reliable definition, which satisfies the basic properties of Newtonian concept derivative.

**Keywords:** Fractional differential equations, Conformable fractional derivative, Numerical solution, Residual power series method

## 1. Introduction

Partial fractional differential equations (PFDEs) have been influenced by many scientists and researchers in recent years and they have been extensively researched and implemented for many real problems modeled in various branches of science and engineering [1-4]. It can be included many applications in science and engineering. Therefore, finding numerical and analytic-approximate solutions of NPFDEs has momentous and special position in these mentioned fields. Till now, classical analytical-approximate methods for fractional differential equations (FDEs) have been developed for Adomian decomposition method (ADM) that was implemented to solve fractional diffusion equations and fractional modified KdV equations [5-6], homotopy perturbation method (HPM) [7-9] that has been used in an enhanced format in [10] for solving FDEs and also it has been improved to solve systems of FDEs [11], variational iteration method (VIM) that has been applied to solve some types of FDEs in [12] and homotopy analysis method (HAM) was used to solve fractional KDV-Burgers-Kuromoto equations [13-15], fractional IVPs [16] and time-fractional PDEs [17,18].

Time-fractional (3+1)-dimensional Telegraph equation is defined as follows:

$$\frac{\partial^{2\alpha}u}{\partial t^{2\alpha}}+2\frac{\partial^{\alpha}u}{\partial t^{\alpha}}+3u=\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}+\frac{\partial^2u}{\partial z^2} \tag{1}$$

The equation describes the voltage and current on an electrical transmission line with distance and time. For the first time, Oliver Heaviside formulated this model. The parameter $\alpha\epsilon(0,1]$ that is the order of the fractional derivative, is considered in the conformable sense. Comprehensive mathematical analysis of the fractional order. Telegraph equations are still under study and it plays an important role in many parts of science and engineering such as telegraph wires and radio frequency conductors.

This study looks over the solutions of time-fractional Telegraph equation by residual power series method (RSPM).

## 2. Basic Definitions

**2.1. Definition** The Riemann –Liouville fractional derivative operator $D^{\alpha}f(x)$ for $\alpha > 0$ and $q-1 < \alpha < q$ defined as [5]:

$$D^{\alpha}f(x) = \frac{d^q}{dx^q}\left[\frac{1}{\Gamma(q-\alpha)}\int_a^x \frac{f(t)}{(x-t)^{\alpha+1-q}}dt\right] \tag{2}$$

**2.2. Definition** The Caputo fractional derivative of order $\boldsymbol{\alpha > 0}$ for $\mathbb{N}$, $\boldsymbol{n-1 < \alpha < n}$, $\boldsymbol{D_*^{\alpha}}$ defined as [5]:

$$D_*^{\alpha}\mathrm{f}(x) = \mathrm{J}^{n-\alpha}D^n f(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x (x-t)^{n-\alpha-1}\left(\frac{d}{dt}\right)^n f(t)dt \tag{3}$$

**2.3. Definition** For a given a function $\boldsymbol{f}: [\boldsymbol{0}, \infty) \to \mathbb{R}$, the conformable fractional derivative of $\boldsymbol{f}$ order $\boldsymbol{\alpha}$ is defined by [19]

$$T_{\boldsymbol{\alpha}}(f)(t) = \lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\boldsymbol{\alpha}}) - f(t)}{\varepsilon}. \tag{4}$$

**2.4. Theorem** Let $\boldsymbol{\alpha\epsilon(0,1]}$ and $\boldsymbol{f, g}$ be $\boldsymbol{\alpha}$ differentiable at a point $\boldsymbol{t > 0}$. Then the following properties hold [*19*,*20*].

i. $T_{\boldsymbol{\alpha}}(mf + ng) = mT_{\boldsymbol{\alpha}}(f) + nT_{\boldsymbol{\alpha}}(g)$ for $m, n \in \mathbb{R}$.
ii. $T_{\boldsymbol{\alpha}}(t^p) = pt^{t-\alpha}$ for all $p \in \mathbb{R}$.
iii. $T_{\boldsymbol{\alpha}}(f.g) = fT_{\boldsymbol{\alpha}}(g) + gT_{\boldsymbol{\alpha}}(f)$
iv. $T_{\boldsymbol{\alpha}}(f/g) = \frac{gT_{\boldsymbol{\alpha}}(f) - fT_{\boldsymbol{\alpha}}(g)}{g^2}$
v. $T_{\boldsymbol{\alpha}}(c) = 0$ when $c$ is a constant.
vi. $f$ is differentiable, then $T_{\boldsymbol{\alpha}}(f)(t) = t^{1-\alpha}\frac{df}{dt}$

**2.5. Definition** Let $\boldsymbol{f}$ be a function with $\boldsymbol{n}$ variables as $\boldsymbol{x_1, x_2, \ldots, x_n}$ and the conformable partial derivative of $\boldsymbol{f}$ order $\boldsymbol{\alpha\epsilon(0,1]}$ in $\boldsymbol{x_i}$ is defined as follows [*21*]

$$\frac{d^{\alpha}}{dx_i^{\alpha}}f(x_1, x_2, \ldots, x_n) = \lim_{\varepsilon\to 0}\frac{f(x_1, x_2, \ldots, x_{i-1}, x_i + \varepsilon x_i^{1-\alpha}, \ldots, x_n) - f(x_1, x_2, \ldots, x_n)}{\varepsilon} \tag{5}$$

## 3. Explanation of Residual Power Series Method

In this section we will introduce some important definitions and theorems about the residual power series.

**3.1. Definition** A power series representation of the form [22]:

$$\sum_{n=0}^{\infty} c_n(t-t_0)^{n\alpha} = c_0 + c_1(t-t_0)^{\alpha} + c_2(t-t_0)^{2\alpha} + \cdots \tag{6}$$

where $0 < m-1 < \alpha \leq m$, $m\epsilon\mathbb{N}$ and $t > t_0$ is called a fractional power series about $t_0$. Here $t$ is a variable and $c_n$'s are the coefficients of the series.

**3.2. Definition** Suppose that f has a FPS representation at $t_0 = 0$ of the form [23]

$$f(t) = \sum_{n=0}^{\infty} c_n(t)^{n\alpha}, \qquad 0 < t < R^{\frac{1}{\alpha}}, \qquad R > 0 \tag{7}$$

and suppose that $f$ is an infinitely conformable $\alpha$ differentiable function, for some $0 < m-1 < \alpha \leq m$, $m\epsilon\mathbb{N}$ in a neighborhood of a point $t_0 = 0$ then the coefficients $c_n$ will take the form $c_n = \frac{f^{(n\alpha)}(0)}{\alpha^n n!}$. Here $f^{(n\alpha)}$ stands for the conformable fractional derivative $n$ times.

**3.3. Theorem** A power series of the form $\sum_{n=0}^{\infty} f_n(x,y,z)t^{\alpha}$ is called a multiple fractional power series about $t_0 = 0$ [23].

**3.4. Definition** Suppose that $u(x,t)$has a multiple FPS representation at $t_0 = 0$ of the form [23]

$$u(x,y,z,t) = \sum_{n=0}^{\infty} f_n(x,y,z)t^{n\alpha}, \qquad 0 < m-1 < \alpha \leq m, \qquad x\epsilon I, \qquad 0 < t < R^{\frac{1}{\alpha}}. \tag{8}$$

If ${u_t}^{(n\alpha)}(x,y,z,t), n = 0,1,2, \dots$ are continuous on $I \times (0, R^{\frac{1}{\alpha}})$, then $f_n(x,y,z) = \frac{{u_t}^{(n\alpha)}(x,y,z0)}{\alpha^n n!}$.

To illustrate the basic idea of RPSM, let's take the following fractional differential equation of the form:

$$T_\alpha u(x,y,z,t) + R[x,y,z]u(x,y,z,t) + N[x,y,z]u(x,y,z,t)g(x,y,z,t), \qquad t > 0, x \in \mathbb{R}, \; n-1 < n\alpha \leq n \tag{9}$$

expressed by the initial condition

$$f_0(x,y,z) = u(x,y,z,0) = f(x,y,z), \tag{10}$$

where$R[x,y,z]$ is a linear operator and $N[x,y,z]$ is a non-linear operator and $g(x,y,z,t)$, are continuous functions.

The RPSM method consists of expressing the solution of the equation given below as the fractional power series expansion around $t = 0$.

$$f_{n-1}(x,y,z) = T_t^{(n-1)\alpha} u(x,y,z,0) = h(x,y,z) \tag{11}$$

The expansion form of the solution is given by:

$$u(x,y,z,t) = \sum_{n=0}^{\infty} f_n(x,y,z)\frac{t^{n\alpha}}{\alpha^n n!} \tag{12}$$

In the next step, the $k$-th truncted series of $u(x,y,z,t)$, that is $u_k(x,y,z,t)$can be written as:

$$u_k(x,y,z,t) = \sum_{n=0}^{k} f_n(x,y,z)\frac{t^{n\alpha}}{\alpha^n n!} \tag{13}$$

Since $u(x,y,z,0) = f(x,y,z)$ and $T_\alpha u(x,y,z,0) = f_1(x,y,z)$ we obtain

$$u_k(x,y,z,t) = u_1(x,y,z,t) + \sum_{n=2}^{j} f_n(x,y,z)\frac{t^{n\alpha}}{\alpha^n n!} = f(x,y,z) + f_1(x,y,z)\frac{t^\alpha}{\alpha} + \sum_{n=2}^{j} f_n(x,y,z)\frac{t^{n\alpha}}{\alpha^n n!} \tag{14}$$

where $u_1(x,y,z,t) = f(x,y,z) + f_1(x,y,z)\frac{t^\alpha}{\alpha}$ is assumed to be the first RPSM approximate solution of $u(x,t)$.

To obtain the rest of the $f_n(x,y,z)$ functions for $n = 2,3,4,\ldots$, we solve $T_t^{(n-2)\alpha} Res_k u(x,y,z,0) = 0$ for $k = 2,3,4,\ldots$, where $Res_k u(x,y,z,t)$ is the $k$-th residual function which is defined by for our case

$$\begin{aligned} \mathrm{Res}_k(x,y,z,t) &= (1-\alpha)t^{1-2\alpha}(u_k)_t(x,y,z,t) + t^{2-2\alpha}(u_k)_{\mathrm{tt}}(x,y,z,t) + 2t^{1-\alpha}(u_k)_t(x,y,z,t) \\ &+ 3u_k(x,y,z,t) - (u_k)_{\mathrm{xx}}(x,y,z,t) - (u_k)_{\mathrm{yy}}(x,y,z,t) - (u_k)_{\mathrm{zz}}(x,y,z,t) \end{aligned} \tag{15}$$

To obtain the $f_2(x,y,z)$ coefficient, we replace the second approximate solution $u_2(x,y,z,t) = f(x,y,z) + f_1(x,y,z)\frac{t^\alpha}{\alpha} + f_2(x,y,z)\frac{t^{2\alpha}}{2\alpha^2}$ into $\mathrm{Res}_2(x,y,z,t)$ and solve $\mathrm{Res}_2(x,y,z,0) = 0$. Similarly, we calculate the $f_n(x,y,z)$

coefficients and $u_k(x,y,z,t)$ unknown functions respectively.

## 4. Approximate Solution of the (3 + 1)-Dimensional Telegraph Equation Using the Residual Power Series

Time-fractional (3+1)-dimensional Telegraph equation is expressed as follows [24]:

$$\frac{\partial^{2\alpha}u}{\partial t^{2\alpha}} + 2\frac{\partial^\alpha u}{\partial t^\alpha} + 3u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \tag{16}$$

where $u = u(x,y,z,t)$. The initial conditions of the equation can be written as

$$f(x,y,z,0) = sinhx sinhy sinhz = -T_t^\alpha u(x,y,z,0) \tag{17}$$

The known exact solution of the equation is

$$f(x,y,z,t) = e^{-2\frac{t^\alpha}{\alpha}} sinhx sinhy sinhz \tag{18}$$

Using the RPSM method, the power series expansion of $u(x,y,z,t)$ is expressed as:

$$u(x,y,z,t) = f(x,y,z) + \sum_{n=0}^{\infty} f_n(x,y,z)\frac{t^{n\alpha}}{\alpha^n n!}, \qquad 0 < \alpha \leq 1, \qquad x \in I, \qquad 0 \leq t < R \tag{19}$$

To find approximate solution of the $(3+1)$-dimensional Telegraph equation, $k$-th truncated series $u_k(x,y,z,t)$;

$$u_k(x,y,z,t) = f(x,y,z) + \sum_{n=0}^{k} f_n(x,y,z)\frac{t^{n\alpha}}{\alpha^n n!}, \qquad 0 < \alpha \le 1, \qquad x \in I, \qquad 0 \le t < R \tag{20}$$

for $u(x,y,z,t)$ is obtained.

To obtain the values of the expression $f_n(x,y,z)$, a series expansion is performed in the equation $u(x,y,z,t)$. Residual function $Res(x,y,z,t)$ of the $(3+1)$-dimensional Telegraph equation is defined as

$$Resu(x,y,z,t) = T_t^{2\alpha}u + 2T_t^{\alpha}u + 3u - u_{xx} + u_{yy} + u_{zz} \tag{21}$$

and $k$-th residual function $Resu_k(x,y,z,t)$ is expressed as:

$$Resu_k(x,y,z,t) = T_t^{2\alpha}u_k + 2T_t^{\alpha}u_k + 3u_k - (u_k)_{xx} + (u_k)_{yy} + (u_k)_{zz} \tag{22}$$

The $Resu_2(x,y,z,t)$ term, which will be solved in the first step of the residual power series algorithm is

$$Resu_2(x,y,z,t) = T_t^{2\alpha}u_2 + 2T_t^{\alpha}u_2 + 3u_2 - (u_2)_{xx} + (u_2)_{yy} + (u_2)_{zz} \tag{23}$$

inserting

$$u_2(x,y,z,t) = f(x,y,z) + f_1(x,y,z)\frac{t^{\alpha}}{\alpha} + f_2(x,y,z)\frac{t^{2\alpha}}{2\alpha^2} \tag{24}$$

to the equation yields

$$\begin{aligned} Resu_2&(x,y,z,t) \\ &= (1-\alpha)t^{1-2\alpha}(u_2)_t(x,y,z,t) + t^{2-2\alpha}(u_2)_{\mathrm{tt}}(x,y,z,t) + 2t^{1-\alpha}(u_2)_t(x,y,z,t) \\ &+ 3u_2(x,y,z,t) - (u_2)_{\mathrm{xx}}(x,y,z,t) - (u_2)_{\mathrm{yy}}(x,y,z,t) - (u_2)_{\mathrm{zz}}(x,y,z,t) \end{aligned} \tag{25}$$

So for $Resu_2(x,y,z,0) = 0$ we obtain the first unknown coefficient as

$$f_2(x,y,z) = -3f(x,y,z) - 2f_1(x,y,z) + f_{\mathrm{xx}}(x,y,z) + f_{\mathrm{xx}}(x,y,z) + f_{\mathrm{zz}}(x,y,z) \tag{26}$$

And the first approximate solution as

$$\begin{aligned} u_2(x,y,z,t) &= f(x,y,z) + \frac{t^{\alpha}f_1(x,y,z)}{\alpha} \\ &+ \frac{t^{2\alpha}(-3f(x,y,z) - 2f_1(x,y,z) + f_{\mathrm{xx}}(x,y,z) + f_{\mathrm{yy}}(x,y,z) + f_{\mathrm{zz}}(x,y,z))}{2\alpha^2} \end{aligned} \tag{27}$$

Similarly, for $k = 3$

$$\begin{aligned} Resu_3&(x,y,z,t) \\ &= (1-\alpha)t^{1-2\alpha}(u_3)_t(x,y,z,t) + t^{2-2\alpha}(u_3)_{\mathrm{tt}}(x,y,z,t) + 2t^{1-\alpha}(u_3)_t(x,y,z,t) \\ &+ 3u_3(x,y,z,t) - (u_3)_{\mathrm{xx}}(x,y,z,t) - (u_3)_{\mathrm{yy}}(x,y,z,t) - (u_3)_{\mathrm{zz}}(x,y,z,t) \end{aligned} \tag{28}$$

is obtained. Replacing

$$u_3(x,y,z,t) = f(x,y,z) + f_1(x,y,z)\frac{t^{\alpha}}{\alpha} + f_2(x,y,z)\frac{t^{2\alpha}}{2\alpha^2} + f_3(x,y,z)\frac{t^{3\alpha}}{6\alpha^3} \tag{29}$$

to the equation gives

$$\begin{aligned} Resu_3(x,y,z,t) &= 3f(x,y,z)+f_2(x,y,z)-f_{xx}(x,y,z)-f_{yy}(x,y,z)-f_{zz}(x,y,z)+\frac{1}{6\alpha^3}(6\alpha^2(3t^\alpha \\ &+2\alpha)f_1(x,y,z)+t^\alpha(3\alpha(3t^\alpha+4\alpha)f_2(x,y,z)+3(t^{2\alpha}+2t^\alpha\alpha+2\alpha^2)f_3(x,y,z) \\ &-6\alpha^2((f_1)_{xx}(x,y,z)-(f_1)_{yy}(x,y,z)-(f_1)_{zz}(x,y,z))-3t^\alpha\alpha((f_2)_{xx}(x,y,z) \\ &-(f_2)_{yy}(x,y,z)-(f_2)_{zz}(x,y,z))-t^{2\alpha}((f_3)_{xx}(x,y,z)-(f_3)_{yy}(x,y,z) \\ &-(f_3)_{zz}(x,y,z)))) \end{aligned} \tag{30}$$

Now applying $T_\alpha$ on both sides of it and equating to 0 for $t=0$ gives

$$f_3(x,y,z)=-3f_1(x,y,z)-2f_2(x,y,z)+(f_1)_{xx}(x,y,z)+(f_1)_{yy}(x,y,z)+(f_1)_{zz}(x,y,z) \tag{31}$$

Therefore, the 2nd RPS approximate solution of Telegraph equation is obtained as

$$\begin{aligned} &u_3(x,y,z,t) \\ &=f(x,y,z)+\frac{t^\alpha f_1(x,y,z)}{\alpha}+\frac{t^{2\alpha}f_2(x,y,z)}{2\alpha^2} \\ &+\frac{t^{3\alpha}(-3f_1(x,y,z)-2f_2(x,y,z)+(f_1)_{xx}(x,y,z)+(f_1)_{yy}(x,y,z)+(f_1)_{zz}(x,y,z))}{6\alpha^3} \end{aligned} \tag{32}$$

In this manner, by applying the same procedure for $k=4,5,\ldots$ we obtain the following results respectively

$$f_4(x,y,z)=-3f_2(x,y,z)-2f_3(x,y,z)+(f_2)_{xx}(x,y,z)+(f_2)_{yy}(x,y,z)+(f_2)_{xzz}(x,y,z) \tag{33}$$

and

$$\begin{aligned} &u_4(x,y,z,t) \\ &=f(x,y,z)+\frac{t^\alpha f_1(x,y,z)}{\alpha}+\frac{t^{2\alpha}f_2(x,y,z)}{2\alpha^2}+\frac{t^{3\alpha}f_3(x,y,z)}{6\alpha^3} \\ &+\frac{t^{4\alpha}(-3f_2(x,y,z)-2f_3(x,y,z)+(f_2)_{xx}(x,y,z)+(f_2)_{yy}(x,y,z)+(f_2)_{xzz}(x,y,z))}{24\alpha^4} \end{aligned} \tag{34}$$

Herewith, we have calculated $f_5(x,y,z)$ and $u_5(x,y,z,t)$ also. Now we present some 3D surface plots and a table to introduce e our numerical results.

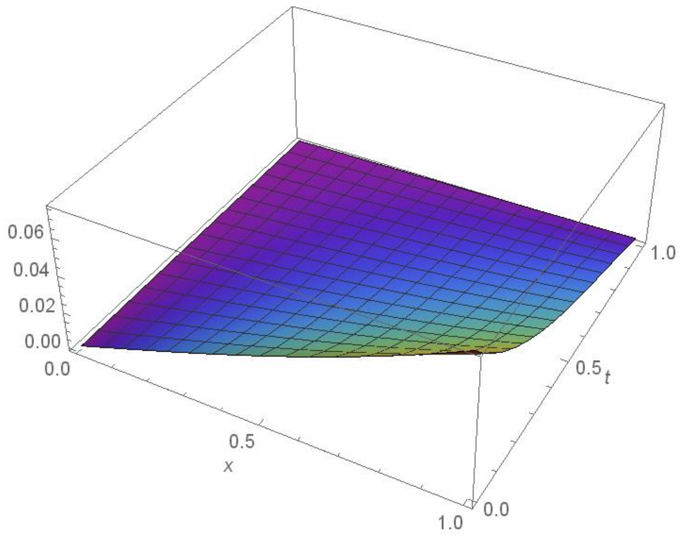


**Figure 1: Surface plot of $u_5(x,y,z,t)$ for $\alpha = 0.95,\ y = z = 0.25$.**

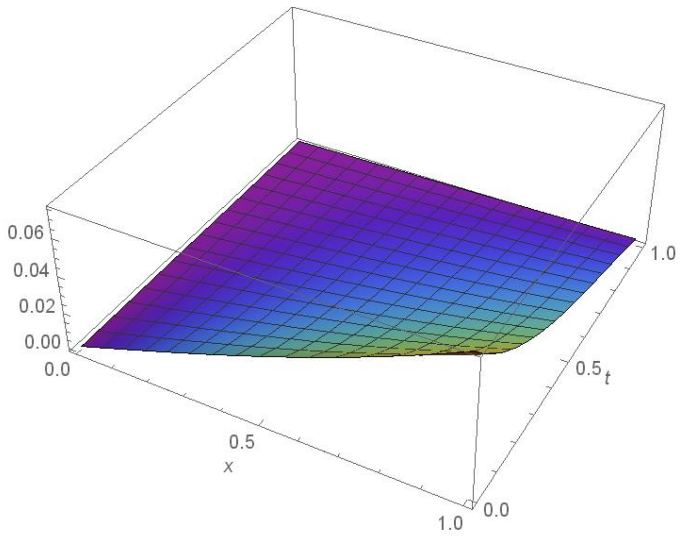


**Figure 2: Surface plot of exact solution for $\alpha = 0.95,\ y = z = 0.25$.**

## 5. Conclusion

In this case study, the residual power series method is implemented to obtain approximate solutions of (3+1)-dimensional time-fractional Telegraph equation. The numerical results are compared with the exact solutions and they reveal the fast convergence rate of the present method even after computing a few iterations. The method does not require any transformations, perturbations or discretization. It is therefore clear that the method is reliable, powerful and easy to implement when compared to other numerical methods. The results prove that the present method could be applied to various fractional linear and nonlinear models occurring in different branches of science and engineering.

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# Sumudu Transform Solution of the Inhomogeneous Burgers Equation

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## Abstract

In this study, we consider Sumudu transform (ST) which is an important integral transformation. It is well-known that using the ST makes the solution method very effective and simple. Moreover, ST can be adapted to many differential equations to obtain their solutions. In this context, this study addresses to obtain the approximate-analytical solution of the inhomogeneous Burgers equation of fractional order by using a method coupled with the ST. Also, by comparing the solutions with respect to the fractional parameter, we conclude that how the method coupled with the SD is effective and accurate.

**Keywords: Fractional differential equation, Sumudu transform, approximate-analytical solution, Caputo fractional derivative**

## 1. Introduction

In the past decades, scientists at various branches have devoted considerable effort to find robust and stable approximate methods for solving fractional differential equations of physical interest. For instance, approximate-analytical methods have included homotopy analysis method (HAM) [1], Adomian decomposition method (ADM) [2, 3], reduced differential transform method (RDTM) [4, 5], homotopy perturbation method (HPM) [6, 7], modified homotopy perturbation method (MHPM) [8, 9]. On the other hand, Sumudu transform techniques have been used successfully by some researchers [10-12]. In this article, the HPM coupled with the Sumudu transform has been considered in order to obtain solutions of the inhomogeneous Burgers equation of fractional order. Caputo derivative operator has been understood as a fractional operator in this paper.

Since there are several types of fractional derivative and integral operators [13-17] the researchers can choose the most comfortable operator to produce the better results of the real-life model problems. This appears in the literature as one of the illustrative advantages of the fractional calculus. For example, some researchers [18-23] have pointed out that which fractional derivative operator is most suitable in modeling. Moreover, in the theory of fractional operators with singular and non-singular kernels and their applications [7, 24-29] and the references stated in these articles.

In this study, we consider the approximate-analytical solutions of fractional nonlinear Burgers equation in the frame of fractional derivative in the Caputo setting. In general case it is difficult to obtain the exact solutions of fractional order nonlinear differential equations, and then numerical solution methods are used to get their solutions. For this reason, theory of numerical and approximate-analytical solution methods play an important role in solving these mentioned problems.

## 2. Preliminaries

In this section, we give some definitions related to the paper.

**Definition 2.1.** The Sumudu transform of a function $f(t)$, defined for all real numbers $t \geq 0$, is the function $\tilde{K}(s)$, defined by [30]

$$\mathcal{S}\{f(t)\} = \tilde{K}(s) = \int_0^\infty \frac{1}{s} \exp\left[-\frac{t}{s}\right] f(t)\, dt. \tag{1}$$

**Definition 2.2.** The Caputo time-fractional derivative is given as [31]:

$${}_0^C D_t^\alpha f(t) = \frac{1}{\Gamma(\rho-\alpha)} \int_0^t (t-k)^{\rho-\alpha-1} f^{(\rho)}(k)\, dk, \quad t > 0, \tag{2}$$

where $f \in L_1(a,b),\ \rho - 1 < \alpha \leq \rho.$

**Definition 2.3.** The Sumudu transform of the Caputo fractional derivative is given by [32]

$$\mathcal{S}\left\{{}_0^C D_t^\alpha f(t)\right\}(s) = s^{-\alpha}\left[\tilde{K}(s) - \sum_{\varpi=1}^{\rho} s^{\varpi-1}\left[D^{\varpi-1} f(t)\right]_{t=0}\right],\ -1 < \rho - 1 < \alpha \leq \rho. \tag{3}$$

### 3. Sumudu Perturbation Approximate-Analytical Method

To investigate the fundamental solution method we take the following general form of fractional nonlinear PDE [6, 33]:

$$ {}_0^C D_t^\alpha u(x,t) + L\left[u(x,t)\right] + N\left[u(x,t)\right] = \theta(x,t), \quad (x,t)\in[0,1]\times[0,T], \quad \rho-1<\alpha\le\rho, \tag{4}$$

with initial condition

$$\frac{\partial^\eta u}{\partial t^\eta}(x,0) = \mu_\eta(x), \qquad \eta = 0,1,\ldots,\rho-1, \tag{5}$$

and the boundary conditions

$$u(0,t) = \gamma_0(t), \qquad u(1,t) = \gamma_1(t), \qquad t\ge 0, \tag{6}$$

where $\mu_\eta, \theta, \gamma_0$ and $\gamma_1$ are known functions. In Eq. (4), we represent the linear part of the equation with $L[.]$, the nonlinear part with $N[.]$ and ${}_0^C D_t^\alpha$ shows the Caputo fractional derivative. We have considered the value of $\rho$ as 1 when constructing the homotopy due to the nature of the problem we used in the study.

We define the recursive approximations for solving the suggested problems (4)-(6). Using the Sumudu transform of the Caputo derivative in Eq. (3), we define the $\boldsymbol{\mathcal{S}}\left\{u(x,t)\right\}(s) = \tilde{K}(x,s)$ for Eq. (4). Then we can obtain the transformed functions for the Caputo fractional derivative

$$\tilde{K}(x,s) = u(x,0) - s^\alpha \boldsymbol{\mathcal{S}}\left\{L\left[u(x,t)\right] + N\left[u(x,t)\right]\right\} + s^\alpha \tilde{\theta}(x,s), \tag{7}$$

where $\boldsymbol{\mathcal{S}}\left\{\theta(x,t)\right\} = \tilde{\theta}(x,s)$. Also considering the Sumudu transforms of the boundary conditions we get

$$\boldsymbol{\mathcal{S}}\left\{\gamma_0(t)\right\} = \tilde{K}(0,s), \quad \boldsymbol{\mathcal{S}}\left\{\gamma_1(t)\right\} = \tilde{K}(1,s), \quad s\ge 0. \tag{8}$$

After that, applying the perturbation method we achieve the solution of Eqs. (4)-(6) as

$$\tilde{K}(x,s) = \sum_{\omega=0}^{\infty} \nabla^\omega \tilde{K}_\omega(x,s), \qquad \omega = 0,1,2,\ldots, \tag{9}$$

The nonlinear part in Eq. (4) can be computed from

$$N\left[u\left(x,t\right)\right]=\sum_{\omega=0}^{\infty}\nabla^{\omega}\mho_{\omega}\left(x,t\right), \tag{10}$$

and the components $\mho_{\omega}\left(x,t\right)$ are given in [34] as

$$\mho_{\omega}\left(u_0,u_1,\ldots,u_{\omega}\right)=\frac{1}{\omega!}\frac{\partial^{\omega}}{\partial\xi^{\omega}}\left[N\left(\sum_{i=0}^{\infty}\xi^{i}u_{i}\right)\right]_{\xi=0},\quad \omega=0,1,2,\ldots \tag{11}$$

Substituting Eqs. (9) and (10) into Eq. (7), we get the recursive relation which gives the solution for the Caputo operator:

$$\sum_{\omega=0}^{\infty}\nabla^{\omega}\tilde{K}_{\omega}\left(x,s\right)=u\left(x,0\right)-\nabla^{\omega}s^{\alpha}\left(\boldsymbol{\mathcal{S}}\left\{L\left[\sum_{\omega=0}^{\infty}\nabla^{\omega}u_{\omega}\left(x,t\right)\right]+\sum_{\omega=0}^{\infty}\nabla^{\omega}\mho_{\omega}\left(x,t\right)\right\}\right)+s^{\alpha}\tilde{\theta}\left(x,s\right). \tag{12}$$

Then, by solving Eqs. (12) with respect to $\nabla,$ we identify the following Caputo homotopies:

$$\begin{aligned}
&\nabla^{0}:\tilde{K}_{0}\left(x,s\right)=u\left(x,0\right)+s^{\alpha}\tilde{\theta}\left(x,s\right), &(13)\\
&\nabla^{1}:\tilde{K}_{1}\left(x,s\right)=-s^{\alpha}\boldsymbol{\mathcal{S}}\left\{L\left[u_0\left(x,t\right)\right]+\mho_0\left(x,t\right)\right\},\\
&\nabla^{2}:\tilde{K}_{2}\left(x,s\right)=-s^{\alpha}\boldsymbol{\mathcal{S}}\left\{L\left[u_1\left(x,t\right)\right]+\mho_1\left(x,t\right)\right\},\\
&\vdots\\
&\nabla^{n+1}:\tilde{K}_{n+1}\left(x,s\right)=-s^{\alpha}\boldsymbol{\mathcal{S}}\left\{L\left[u_n\left(x,t\right)\right]+\mho_n\left(x,t\right)\right\}.
\end{aligned}$$

Considering the case of $\nabla\rightarrow 1,$ we obtain that Eq. (13) shows the approximate solution for problem (12), then the solution is

$$\Im_{n}\left(x,s\right)=\sum_{\sigma=0}^{n}u_{\sigma}\left(x,s\right). \tag{14}$$

Finally, by applying the inverse Sumudu transform of Eq. (14), we obtain the approximate solution of Eq. (4),

$$u\left(x,t\right)\cong u_{n}\left(x,t\right)=\boldsymbol{\mathcal{S}}^{-1}\left\{\Im_{n}\left(x,s\right)\right\}. \tag{15}$$

## 4. Main Results and Discussion

In this part of the study, we obtain a solution of the following Burgers equation:

$$ {}_0^C D_t^{\alpha} u(x,t) + \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + 2x - 2, \quad t>0,\ x \in R,\ 0<\alpha \le 1, \tag{16} $$

with initial condition

$$ u(x,0) = x^2. \tag{17} $$

Using the Sumudu transform of the Caputo derivative in Eq. (3), we can obtain the transformed functions for the Caputo fractional derivative for Eq. (16):

$$ \tilde{K}(x,s) = x^2 - s^{\alpha} \boldsymbol{\mathcal{S}}\left\{ \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} \right\} + s^{\alpha} \boldsymbol{\mathcal{S}}\left\{ \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + 2x - 2 \right\}. \tag{18} $$

We get the recursive relation which gives the solution for the Caputo operator:

$$ \sum_{\omega=0}^{\infty} \nabla^{\omega} \tilde{K}_{\omega}(x,s) = x^2 - \nabla^{\omega} s^{\alpha} \boldsymbol{\mathcal{S}}\left\{ \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} \right\} + s^{\alpha} \boldsymbol{\mathcal{S}}\left\{ \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + 2x - 2 \right\}. \tag{19} $$

Then, by solving Eqs. (12) with respect to $\nabla$, we identify the following Caputo homotopies:

$$ \begin{aligned} &\nabla^0 : \tilde{K}_0(x,s) = x^2 + s^{\alpha} \boldsymbol{\mathcal{S}}\left\{ \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + 2x - 2 \right\} = x^2 + 2s^2 + 2(x-1)s^{\alpha}, \\ &u_0(x,t) = x^2 + t^2 + 2(x-1)\frac{t^{\alpha}}{\Gamma(\alpha+1)}, \\ &\nabla^1 : \tilde{K}_1(x,s) = -s^{\alpha} \boldsymbol{\mathcal{S}}\left\{ \frac{\partial u_0}{\partial x} - \frac{\partial^2 u_0}{\partial x^2} \right\} = -2(x-1)s^{\alpha} - 2s^{2\alpha}, \\ &u_1(x,t) = -2(x-1)\frac{t^{\alpha}}{\Gamma(\alpha+1)} - \frac{2t^{2\alpha}}{\Gamma(2\alpha+1)}, \\ &\nabla^2 : \tilde{K}_2(x,s) = -s^{\alpha} \boldsymbol{\mathcal{S}}\left\{ \frac{\partial u_1}{\partial x} - \frac{\partial^2 u_1}{\partial x^2} \right\} = 2s^{2\alpha}, \\ &u_2(x,t) = \frac{2t^{2\alpha}}{\Gamma(2\alpha+1)}. \end{aligned} \tag{20} $$

Considering the case of $\nabla \to 1,$ we obtain that Eq. (20) shows the approximate solution for problem (16), then the solution is

$$\Im_n(x,s) = \sum_{\sigma=0}^{n} u_\sigma(x,s) = x^2 + 2s^2. \tag{21}$$

Finally, by applying the inverse Sumudu transform of Eq. (14), we obtain the solution of Eq. (16),

$$u(x,t) \cong u_n(x,t) = \boldsymbol{S}^{-1}\{\Im_n(x,s)\} = x^2 + t^2. \tag{22}$$

## 5. Conclusion

In this study, we have obtained the approximate-analytical solution of the nonhomogeneous Burgers equation via Sumudu transformation perturbation method. This method is a good mathematical technique to obtain approximate solutions for obtaining the solutions of fractional PDEs. In addition, this method can gain different meanings depending on the field. Moreover, the mentioned method has given analytical solution of the Burgers equation as seen in Eq. (22).

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# STUDY OF QUASILINEAR PARABOLIC PROBLEMS WITH DATA IN $L^1$

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Abstract. In this paper, we study the existence of entropy solution for quasillinear parabolic problem in bounded open subset $\Omega$ of $\mathbf{R}^N$, with data and $u_0$ in $L^1(\Omega)$. For this we use the Schauder fixed-point method. The results of the problem discussed can be applied to a variety of different fields in applied mathematics for example in elastic mechanics, image processing and electro-rheological fluid dynamics, etc..

Keywords: Quasilinear parabolic equations; fixed point; truncation function; $L^1$ data.

AMS subject classification (2010): 35K59; 37C25.

## 1. Introduction

In this article is devoted to presenting the results of existence of solution for a quasilinear parabolic problem with data in $L^1$, the main difficulty facing one who is interested in such problems is that the classical theories of existence, either using variational methods or compacite methods, are not applicable. Hence the need to use new techniques to prove the existence and uniqueness of solutions for such problems.
In the last years, different methods have been applied to study the existence of the weak solution of elliptic problems with $L^1$ under linear boundary conditions see [4],[6], [11] and [14]. The corresponding parabolic case equations have also been studied by many authors, see for instance [5],[8], [9] and [14].
Besides, partial differential equation (PDE) methods in image processing have proven to be fundamental tools for image diffusion and restoration. We refer the readers to [[1],[2]] and references therein.
The aim of this paper, we treat the existence of solution $u$ for the following quasi-linear parabolic problem of the type

$$(1.1)\qquad \begin{cases} u_t - div(A(u)\nabla u) + \lambda|u|^{p-2}u = f(t,x,u) & \text{in} \quad Q=[0,T]\times\Omega, \\ u=0 & \text{on} \quad \Sigma=[0,T]\times\partial\Omega, \\ u(0,.)=u_0(.) & \text{in} \quad \Omega. \end{cases}$$

In the problem (1.1). Where $\lambda > 0$ and $T > 0$, $\Omega$ is a bounded open spatial domain in $\mathbb{R}^N$ ($N \geq 2$) with a lipschitz boundary denoted by $\partial\Omega$, and $u_0 \in L^1(\Omega)$. The function $\gamma(u) = \lambda|u|^{p-2}u$ such that $\gamma : \mathbb{R} \to \mathbb{R}$ is a continuous increasing function with $\gamma(0) = 0$ and the operator $A : \mathbb{R} \to \mathcal{M}_N(\mathbb{R})$ (or $\mathcal{M}_N(\mathbb{R})$ denotes the set of $N \times N$ matrices with real coefficients), such that satisfies the following assumption for some numbers $0<\alpha<\beta<\infty$:

$$\forall s\in\mathbb{R}, A(s)=(a_{i,j}(s))_{i,j=1,\ldots,N} \text{ where } a_{i,j}\in L^\infty(\mathbb{R})\cap C(\mathbb{R},\mathbb{R}), \tag{1.2}$$

$$\exists\ \alpha>0,\ \text{such that } A(s)\xi.\xi\geq\alpha\,|\xi|^2\,, \quad \forall\xi\in\mathbb{R}^N,\forall s\in\mathbb{R}, \tag{1.3}$$

$$\exists\ \beta>0,\ \text{such that } \|a_{i,j}\|_{L^\infty(\mathbb{R})}\leq\beta, \quad \forall i,j\in\{1,....,N\}\,. \tag{1.4}$$

We will assume that $f:Q\times\mathbb{R}\to\mathbb{R}$ is a Carathèodory function such that the following hypotheses hold

$$|f(t,x,s)|\leq c(t,x)+\sigma|s|, \tag{1.5}$$

$$sf(t,x,s)\geq 0, \tag{1.6}$$

for almost every $(t,x)\in Q$, for every $s\in\mathbb{R}$, where $c$ is a positive function in $L^2(Q)$ and $\sigma>0$.
In this work we are studying the existence of weak solution of the quasilinear parabolic problem (1.1) using the truncation technique and the Schauder fixed point theory see [2],[10].
This result generalizes an analog of this work were made by N. Alaa and all [2] with an increase of $\gamma$ but given $L^1$ and, on the other hand, to extend it to the case $f(t,x,u)$ in $L^1$ data.
To prove our main result, we will proceed by three steps: the first step, we approximate the problem by the fixed point method. In the second step, we estimate on the approximate solution.
In the last step, we study the asymptotic behaviour of the approximate solution as $n$ go to infinity we use the equi-integrable theorem.
The difficulty of this work lies in the fact that the variational method can not be used because $f$ is in $L^1$.

## 2. Main results

Before tackling the main problem, we clearly state our definition of weak solution to the quasilnear parabolic problem.

**Definition 2.1.** *Let $1<p\leq N$ a fixed number with $p>2-\dfrac{1}{N}$. We call $u$ a weak solution of the problem (1.1) in $Q$, if $u\in L^2([0,T]\,,H_0^1(\Omega))\cap C([0,T]\,,L^1(\Omega))$, $u(0,.)=u_0$ for all $\varphi\in C_0^\infty(Q)$ we have*

$$\int\limits_Q -u\varphi_t dxdt+\int\limits_Q A(u)\nabla u\nabla\varphi dxdt+\int\limits_Q \lambda\,|u|^{p-2}\,u\varphi dxdt=\int\limits_Q f(t,x,u)\varphi dxdt, \tag{2.1}$$

*where $f(t,x,u)$ and $\gamma(u)\in L^1(Q)$.*

The main result of this paper is the following theorem:

**Theorem 2.2.** *Under the assumptions (1.2) $-$ (1.6) satisfies, then for all $u_0\in L^1(\Omega)$, there exists a weak solution $u$ of problem (1.1) in the sense defined in (2.1).*

Now we shall prove our main result.

## 3. Proof of the Theorem 2.2

The proof of the theorem consists in the three steps in the first step we solve an approximate problem, in the second step we get estimates on the approximated solutions these estimates allow us and in the third step to go to the limit.

3.1. **First step: solving an approximated problem.** For $n \in \mathbb{N}$ let us define the following approximation of $u_{n,0}$ and $f_n, \gamma_n$. Set

$$f_n(t,x,p) = \begin{cases} f(t,x,p) & \text{if } |f(t,x,p)| \leq n, \\ n \text{ sign}(f(t,x,p)) & \text{if } |f(t,x,p)| > n. \end{cases} \tag{3.1}$$

$$\gamma_n(p) = \begin{cases} \gamma(p) & \text{if } |\gamma(p)| \leq n, \\ 0 & \text{if } |\gamma(p)| > n. \end{cases} \tag{3.2}$$

And $(u_{n,0})_{n\in\mathbb{N}}$ be sequences in $L^2(\Omega)$ such that $(u_{n,0}) \to (u_0)$ in $L^1(\Omega)$.

**Remark**

$$|f_n(t,x)| \leq n \text{ and } |\gamma_n(p)| \leq n,$$
$$\text{so} \quad \gamma_n,\ f_n \in L^\infty(Q) \hookrightarrow L^p(Q),\ p > n \geq 1.$$

We consider the sequence of approximate problems

$$\begin{cases} (u_n)_t - div(A(u_n)\nabla u_n) + \gamma_n(u_n) = f_n(t,x,u_n) & \text{in} \quad Q = [0,T] \times \Omega, \\ u_n = 0 & \text{on} \quad \Sigma = [0,T] \times \partial\Omega, \\ u_n(0,.) = u_{n,0}(.) & \text{in} \quad \Omega. \end{cases} \tag{3.3}$$

We show that for all $n \in \mathbb{N}^*$ and $f_n(t,x,u_n) \in L^2(Q)$, $u_{n,0} \in L^2(\Omega)$ there exists $u_n \in L^2([0,T], H_0^1(\Omega)) \cap C([0,T], L^2(\Omega))$ and $(u_n)_t \in L^2([0,T], H^{-1}(\Omega))$ verify for all $v \in L^2([0,T], H_0^1(\Omega))$, we have

$$\begin{aligned} &\int_0^T \langle (u_n)_t, v\rangle_{H^{-1}(\Omega), H_0^1(\Omega)}\, dt + \int_0^T\int_\Omega A(u_n)\nabla u_n \nabla v dxdt \\ &+ \int_0^T\int_\Omega \gamma_n(u_n) v dxdt = \int_0^T\int_\Omega f_n(t,x,u_n) v dxdt, \end{aligned} \tag{3.4}$$

We will show the existence of a weak solution of the problem (3.3) by the classical Schauder's fixed point theorem. Let us show now that the nonlinear application $F$ defined by

$$\begin{aligned} F:\ L^2([0,T], H_0^1(\Omega)) &\to \quad L^2([0,T], H_0^1(\Omega)) \\ v_n &\mapsto \quad F(v_n) = G \circ F_n(v_n) = v_n, \end{aligned}$$

solution of

$$\begin{aligned} &\int_0^T \langle (v_n)_t, \varphi\rangle_{H^{-1}(\Omega), H_0^1(\Omega)}\, dt + \int_0^T\int_\Omega A(v_n)\nabla v_n \nabla \varphi dxdt \\ &+ \int_0^T\int_\Omega \gamma_n(v_n)\varphi dxdt = \int_0^T\int_\Omega f_n(t,x,v_n)\varphi dxdt, \forall \varphi \in L^2(]0,T[, H_0^1(\Omega)), \end{aligned}$$

is completely continous application of $L^2([0,T], H_0^1(\Omega))$ in $L^2([0,T], H_0^1(\Omega))$.
Where the operator $(F_n)$ is defined by

$$\begin{aligned} F_n:\ L^2([0,T], H_0^1(\Omega)) &\to \quad L^2([0,T], H^{-1}(\Omega)) \\ v_n &\mapsto \quad F_n(v_n) = (v_n)_t + div(A(v_n)\nabla v_n) = f_n(t,x,v_n) - \gamma_n(v_n) \quad = \widetilde{f_n}(v_n), \end{aligned}$$

is continous and compact (natural injection), and $G$ is the Green's operator defined by :

$$\begin{array}{rccc} G: & L^2([0,T], H^{-1}(\Omega)) & \to & L^2([0,T], H_0^1(\Omega)) \\ & \widetilde{f_n}(v_n) = w_n & \mapsto & G(w_n) = v_n, \end{array}$$

is continous because the operator of Green is isomorphism of $L^2([0,T], H^{-1}(\Omega))$ in $L^2([0,T], H_0^1(\Omega))$. Therefore, the operator $F = G \circ F_n$ is completely continous.

The existence of a fixed point of $G \circ F_n$ is an immediate consequence of Schauder's fixed point theorem.

To apply the theorem of Schauder's, you have to choose a closed convex generally suitable a closed ball

$$C = \left\{ v \in L^2([0,T], H_0^1(\Omega)) \text{ such that } \|v\|_{L^2([0,T], H_0^1(\Omega))} \le M \right\},$$

where $M$ is a constant to be determined subsequently, is therefore,

$$\begin{array}{rccc} F: & L^2([0,T], H_0^1(\Omega)) & \to & L^2([0,T], H_0^1(\Omega)) \\ & v_n & \mapsto & F(v_n) = v_n, \end{array}$$

transforms the bounds of $L^2([0,T], H_0^1(\Omega))$ into relatively compact sets in $L^2([0,T], H_0^1(\Omega))$, the set $C$ is a closed convex of $L^2([0,T], H_0^1(\Omega))$ and bounded, so $F$ is relatively compact.

We show that $R(F) = \{F(v_n),\ \forall v_n \in L^2([0,T], H_0^1(\Omega))\}$ is bounded in $L^2([0,T], H_0^1(\Omega))$, as $F(v_n)$ is solution of the variational problem.

$$(3.5) \quad \begin{aligned} &\int_0^T \langle (F(v_n))_t, \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}\, dt + \int_0^T \int_\Omega A(v_n)) \nabla F(v_n) \nabla \varphi dx dt \\ &+ \int_0^T \int_\Omega \gamma_n(v_n) \varphi dx dt = \int_0^T \int_\Omega f_n(t,x,v_n) \varphi dx dt, \forall \varphi \in L^2([0,T], H_0^1(\Omega)). \end{aligned}$$

We choose $F(v_n) = \varphi$ in (3.5), we obtain

$$(3.6) \quad \begin{aligned} &\int_0^T \langle F(v_n)_t, F(v_n) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}\, dt + \int_0^T \int_\Omega A(v_n) \nabla F(v_n) \nabla F(v_n) dx dt \\ &+ \int_0^T \int_\Omega \gamma_n(v_n) F(v_n) dx dt = \int_0^T \int_\Omega f_n(t,x,v_n) F(v_n) dx dt. \end{aligned}$$

By using Cauchy-Schwarz inequality in (3.6), we have

$$\begin{aligned} \frac{1}{2}\|v_n(T)\|_2^2 - \frac{1}{2}\|v_n(0)\|_2^2 + \int_0^T \int_\Omega A(v_n) |\nabla F(v_n)|^2 dx dt \quad \le & \int_Q |\gamma_n(v_n)| |F(v_n)| dx dt \\ & + \int_Q |f_n(t,x,v_n)| |F(v_n)| dx dt, \end{aligned}$$

by, using generalized Young's inequality and the hypothesis (1.3), we get

$$\begin{aligned}
\alpha \int_0^T \int_\Omega |\nabla F(v_n)|^2 \, dxdt &\leq \frac{1}{2} \|v_n(0)\|_2^2 + \|f_n(t,x,v_n)\|_{L^2(Q)} \|F(v_n)\|_{L^2([0,T],H_0^1(\Omega))} \\
&\quad + \|\gamma_n(v_n)\|_{L^2(Q)} \|F(v_n)\|_{L^2(]0,T[,H_0^1(\Omega))} \\
&\leq \frac{1}{2} \|v_n(0)\|_2^2 + \frac{1}{2\varepsilon} \|f_n(t,x,v_n)\|_{L^2(Q)}^2 + \frac{\varepsilon}{2} \|F(v_n)\|_{L^2([0,T],H_0^1(\Omega))}^2 \\
&\quad + \frac{1}{2\varepsilon} \|\gamma_n(v_n)\|_{L^2(Q)}^2 + \frac{\varepsilon}{2} \|F(v_n)\|_{L^2([0,T],H_0^1(\Omega))}^2 .
\end{aligned}$$

We conclude that,

$$(3.7) \qquad (\alpha - \varepsilon) \|F(v_n)\|_{L^2(]0,T[,H_0^1(\Omega))}^2 \leq \frac{1}{2} \|v_n(0)\|_2^2 + \frac{1}{2\varepsilon} \|f_n(t,x,v_n)\|_{L^2(Q)}^2 + \frac{1}{2\varepsilon} \|\gamma_n(v_n)\|_{L^2(Q)}^2 .$$

Therefore the sequence $(F(v_n))_{n\in\mathbb{N}}$ is bounded in $L^2([0,T]\,, H_0^1(\Omega))$. Next we show that $\{(F(v_n)_t)_{n\in\mathbb{N}}, F(v_n) \in R(F)\}$ is bounded in $L^2([0,T], H^{-1}(\Omega))$. We have

$$\int_0^T \langle F(v_n)_t, F(v_n)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \, dt + \int_0^T \int_\Omega A(v_n)\nabla F(v_n) \nabla F(v_n) dxdt$$
$$+ \int_0^T \int_\Omega \gamma_n(v_n) F(v_n) dxdt = \int_0^T \int_\Omega f_n(t,x,v_n) F(v_n) dxdt.$$

By using hypothesis (1.2) and (1.4), we get

$$\begin{aligned}
& \|F(v_n)_t\|_{L^2([0,T],H^{-1}(\Omega))} \|F(v_n)\|_{L^2([0,T],H_0^1(\Omega))} \\
\leq\ & \beta \|F(v_n)\|_{L^2([0,T],H_0^1(\Omega))}^2 + \|f_n(t,x,v_n)\|_{L^2(Q)} \|F(v_n)\|_{L^2([0,T],H_0^1(\Omega))} \\
& + \|\gamma_n(v_n)\|_{L^2(Q)} \|F(v_n)\|_{L^2([0,T],H_0^1(\Omega))} .
\end{aligned}$$

Eventually,

$$\begin{aligned}
& \|F(v_n)_t\|_{L^2([0,T],H^{-1}(\Omega))} \\
\leq\ & \beta \|F(v_n)\|_{L^2([0,T],H_0^1(\Omega))} + \|f_n(t,x,v_n)\|_{L^2(Q)} + \|\gamma_n(v_n)\|_{L^2(Q)} .
\end{aligned}$$

Therefore the sequence $\{(F(v_n)_t)_{n\in\mathbb{N}}, F(v_n) \in R(F)\}$ is bounded in $L^2([0,T]\,, H^{-1}(\Omega))$. As $(F(v_n))_{n\in\mathbb{N}}$ is bounded in $L^2([0,T]\,, H_0^1(\Omega))$ and the sequence $(F(v_n)_t)_{n\in\mathbb{N}}$ is bounded in $L^2([0,T]\,, H^{-1}(\Omega))$ according to the lemma of compactness then gives that $R(F)$ is relatively compact in $L^2([0,T]\,, L^2(\Omega))$, which gives the compactness of $F$. For (3.7), we have $F(C) \subset C$, it is enough to take

$$M = \frac{1}{2(\alpha-\varepsilon)} \|v_{n,0}\|_2^2 + \frac{1}{2(\alpha-\varepsilon)\varepsilon} \|f_n(t,x,v_n)\|_{L^2(Q)}^2 + \frac{1}{2(\alpha-\varepsilon)\varepsilon} \|\gamma_n(v_n)\|_{L^2(Q)}^2 .$$

Therefore the hypotheses of Schauder's fixed point theorem are satisfied consequently there exists at least one solution to the problem in the set $C$.

3.2. **Second step: a priori estimates.** In thise step we proof the estimates of solution $(u_n)_{n\in\mathbb{N}}$ the problem (3.3).
For a given constant $k>0$ we define the truncated function $T_k:\mathbb{R}\to\mathbb{R}$ as

$$T_k(s)=\begin{cases} -k & \text{for } s<-k,\\ s & \text{for } |s|\le k,\\ k & \text{for } s>k.\end{cases}$$

For a function $u=u(x)$, $x\in\Omega$, we define the truncated function $T_ku$ pointwise, i.e., for every $x\in\Omega$ the value of $(T_ku)$ at $x$ is just $T_k(u(x))$. Observe that

$$\lim_{k\to 0}\frac{1}{k}T_k(s)=sign(s)=\begin{cases} 1 & \text{if } s>0,\\ 0 & \text{if } s=0,\\ -1 & \text{if } s<0.\end{cases} \tag{3.8}$$

Let the function $\Phi_k:\mathbb{R}\to\mathbb{R}$ such that, $\Phi_k\ge 0$, $\Phi_k\in L^\infty(\mathbb{R})$ and $|\Phi_k(x)|\le k\,|x|$,

$$\Phi_k(x) = \int_0^x T_k(s)ds.$$

($\Phi_k$ it is the primitive function of $T_k$). We have

$$\langle v_t, T_k(v)\rangle=\frac{d}{dt}\left(\int_\Omega \Phi_k(v)dx\right)\in L^1(Q).$$

What implies that

$$\int_0^T\langle v_t, T_k(v)\rangle=\int_\Omega \Phi_k(v(T))dx-\int_\Omega \Phi_k(v(0))dx,$$

where $\langle .,.\rangle$ denotes the duality between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.
We choose $v=T_k(u_n)$ as test function in (3.4), obtaining

$$\begin{cases} \int_\Omega \Phi_k(u_n(T))dx-\int_\Omega \Phi_k(u_n(0))dx+\int_0^T\int_\Omega A(u_n)\nabla u_n\nabla T_k(u_n)dxdt\\ +\int_Q\gamma_n(u_n)T_k(u_n)dxdt=\int_Q f_n(t,x,u_n)T_k(u_n)dxdt, \forall T_k(u_n)\in L^2([0,T], H_0^1(\Omega)).\end{cases}$$

By using hypothesis (1.3), we obtain

$$\int_0^T\int_\Omega A_n(u_n)\nabla u_n\nabla T_k(u_n)dxdt=\int_0^T\int_\Omega A(u_n)\nabla u_n\nabla u_n T_k^{'}(u_n)dxdt\ge\alpha\int_0^T\int_\Omega |\nabla u_n|^2 T_k^{'}(u_n)dxdt\ge 0,$$

and by $sf(t,x,s)\ge 0$, we have

$$\int_0^T\int_\Omega \gamma_n(u_n)T_k(u_n)dxdt\le\int_\Omega \Phi_k(u_n(0))dx,$$

on the other hand, we have $\gamma_n(u_n)=\lambda|u_n|^{p-2}u_n\geq 0$ because $p>1$ then,

$$\int_0^T\int_\Omega A(u_n)\nabla u_n\nabla T_k(u_n)dxdt\leq\int_\Omega\Phi_k(u_n(0))dx,\forall T_k(u_n)\in L^2([0,T],H_0^1(\Omega)).$$

and,

$$\int_Q f_n(t,x,u_n)T_k(u_n)dxdt\leq\int_\Omega\Phi_k(u_n(0))dx.$$

For all $t\in[0,T]$, we definite the set $Q_T$ by

$$Q_T=\{(t,x)\in Q:u_n>k\}\cup\{(t,x)\in Q:u_n<-k\}\cup\{(t,x)\in Q:-k\leq u_n\leq k\}.$$

By thise definition of $Q_T$, we have

$$\begin{cases}\int\limits_{Q_T} A_n(u_n)\nabla u_n\nabla u_nT_k'(u_n)dxdt=\int\limits_{\{(t,x)\in Q:\ |u_n|\leq k\}} A(u_n)\nabla u_n\nabla u_nT_k'(u_n)dxdt\\ \qquad\leq\int\limits_\Omega\Phi_k(u_n(0))dx,\end{cases}$$

so we have, $\forall k\in\mathbb{R}^+$,

$$\int\limits_{\{(t,x)\in Q:\ |u_n|\leq k\}}|A(u_n)\nabla u_n\nabla u_n|\,dxdt\leq k\int\limits_\Omega|u_{n,0}|\,dx. \tag{3.9}$$

We will now prove that,

$$\int\limits_{\{(t,x)\in Q:\ |u_n|\ \leq k\}} A(u_n)\nabla u_n\nabla u_ndxdt\leq k\,\|u_0\|_{L^1(\Omega)},$$

by hypothesis (1.3), we obtain

$$\alpha\int\limits_{\{(t,x)\in Q:\ |u_n|\ \leq k\}}|\nabla u_n|^2\,dxdt\leq k\,\|u_0\|_{L^1(\Omega)}, \tag{3.10}$$

on the other hand, by (3.8), we obtain:

$$\int\limits_{\{(t,x)\in Q:|u_n|>0\}}|\gamma_n(u_n)|\,dxdt\leq\|u_0\|_{L^1(\Omega)}, \tag{3.11}$$

and,

$$\int\limits_{\{(t,x)\in Q:|u_n|>0\}}|f_n(t,x,u_n)|\,dxdt\leq\|u_0\|_{L^1(\Omega)}. \tag{3.12}$$

New we prove that $(u_n)_{n\in\mathbb{N}}$ is bounded in $C\left([0,T],L^1(\Omega)\right)$.

$$\int_0^T\langle(u_n)_t\,,T_k\,(u_n)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\,dt\leq\int_{Q_T}\gamma_n(u_n)T_k(u_n)dxdt+\int_{Q_T}f_n(t,x,u_n)T_k(u_n)dxdt,$$

we have also, for every $t$ in $[0,T]$

$$\int_\Omega\Phi_k(u_n(t))dx-\int_\Omega\Phi_k(u_n(0))dx\leq k\int\limits_{\{(t,x)\in Q:|u_n|>k\}}|\gamma_n(u_n)|\,dxdt+k\int\limits_{\{(t,x)\in Q:|u_n|>k\}}|f_n(t,x,u_n)|\,dxdt,$$

we now that $\Phi_k(s) \geq |s| - 1$ we deduce that, for every $t$ in $[0,T]$,

$$\begin{aligned}
\int_\Omega |u_n(t)|\, dx &\leq \int_\Omega 1 dx + k \int_{\{(t,x)\in Q:|u_n|>k\}} |\gamma_n(u_n)|\, dxdt + k \int_{\{(t,x)\in Q:|u_n|>k\}} |f_n(t,x,u_n)|\, dxdt + k \|u_{n,0}\|_{L^1(\Omega)} \\
&\leq meas(\Omega) + C \|u_0\|_{L^1(\Omega)},
\end{aligned}$$

which proves that $u_n$ is bounded in $L^2\left([0,T], H_0^1(\Omega)\right)$ and in $C\left([0,T], L^1(\Omega)\right)$, on the other hand, we get

$$\int_\Omega \Phi_k(u_n(T))dx + \alpha \int_{Q_T} |\nabla T_k(u_n)|^2\, dxdt \leq \int_\Omega \Phi_k(u_n(0))dx,$$

$\int_\Omega \Phi_k(u_n(T)) \geq 0$ and for all $s \geq 0, |\Phi_k(s)| \leq k|s|$, we have

$$\alpha \int_{Q_T} |\nabla T_k(u_n)|^2\, dxdt \leq k \|u_0\|_{L^1(\Omega)}. \tag{3.13}$$

That $T_k(u_n)$ is bounded in $L^2\left([0,T], H_0^1(\Omega)\right)$ for every $k > 0$.
Now we prove that $div\,(A(v_n)\nabla u_n)$ is bounded in $L^2\left([0,T], H^{-1}(\Omega)\right)$. By using hypothesis (1.4) and Cauchy-Schwarz inequality, we get

$$\begin{aligned}
|\langle -div\,(A(u_n)\nabla u_n, T_k(u_n)\rangle| &= \left| \int_{Q_T} A(u_n)\nabla u_n \nabla T_k(u_n)\, dxdt \right| \\
&\leq \beta \|\nabla u_n\|_{L^2(Q)} \|\nabla T_k(u_n)\|_{L^2(Q)} \\
&\leq C.
\end{aligned}$$

Since

$$\begin{aligned}
\|-div\,(A(u_n)\nabla u_n)\|^2_{L^2([0,T],H^{-1}(\Omega))} &= \int_0^T \|-div\,(A(u_n)\nabla u_n)\|^2_{H^{-1}(\Omega)}\, dt \\
&= \int_0^T \sup_{\|T_k(u_n)\|_{L^2\left([0,T],H_0^1(\Omega)\right)}\leq 1} |\langle -div\,(A(u_n)\nabla u_n), T_k(u_n)\rangle| \\
&\leq C.
\end{aligned}$$

We know that $div\,(A(u_n)\nabla u_n)$ is bounded in $L^2\left([0,T], H^{-1}(\Omega)\right)$.

Finally, denoting $(u_n)_t = f_n(t,x,u_n) + div\,(A(u_n)\nabla u_n) - \gamma_n(u_n)$ we observe that $f_n + div\,(A(u_n)\nabla u_n) + \gamma_n(u_n)$ is bounded in $L^2\left([0,T], H^{-1}(\Omega)\right) + L^1(Q)$ and by (3.10), $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^2\left([0,T], H_0^1(\Omega)\right)$.

3.3. **Third step: passage to the limit.** We show that $(u_n)_{n\in\mathbb{N}}$ the solution approache the problem (3.4) converges to the solution of the original problem (2.1). By the estimate (3.11) and (3.12), we see that $(\gamma_n(u_n))_{n\in\mathbb{N}}$ is bounded in $L^1(Q)$ and $(f_n(t,x,u_n))_{n\in\mathbb{N}}$ is bounded in $L^1(Q)$. The sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^2([0,T], H_0^1(\Omega))$ and also the sequence $((u_n)_t)_{n\in\mathbb{N}}$ is bounded in $L^2([0,T], H^{-1}(\Omega)) + L^1(Q)$. Therefore, using Aubin-type compactness lemma [16], that $(u_n)_{n\in\mathbb{N}}$ is relatively compact in $L^2([0,T], L^2(\Omega))$, thus we can deduce

$$u_n \to u \text{ in } L^2([0,T], L^2(\Omega)),$$

on the other hand $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^2([0,T]\,, H_0^1(\Omega))$ then, we can extracts a subsequence, still denoted by $(u_n)_{n\in\mathbb{N}}$ such that:

$$u_n \to u \text{ weakly in } L^2([0,T]\,, H_0^1(\Omega)),$$

and

$$\nabla u_n \to \nabla u \text{ weakly in } (L^2([0,T]\,, L^2(\Omega)))^{\mathbb{N}},$$

and $((u_n)_t)_{n\in\mathbb{N}}$ is bounded in $L^2([0,T]\,, H^{-1}(\Omega))$ and in $L^1(Q)$ we can extract a subsequence, still denoted by $((u_n)_t)_{n\in\mathbb{N}}$ such that

$$(u_n)_t \to u_t \text{ weakly in } L^2([0,T]\,, H^{-1}(\Omega)),$$

and either $u_{n,0}$ a sequence of $L^2(\Omega)$ such that

$$\|u_{n,0}\|_{L^1(\Omega)} \leq \|u_0\|_{L^1(\Omega)}\,,$$

and

$$u_{n,0} \longrightarrow u_0 \text{ strongly in } L^1(\Omega). \tag{3.14}$$

We will show that

$$\gamma_n(u_n) \to \gamma(u) \;\text{ strongly in } L^1(Q), \tag{3.15}$$

we have,

$$\begin{aligned}
\|\gamma_n(u_n)\|_{L^1(Q)} &= \int_Q |\gamma_n(u_n)|\, dxdt \\
&\leq \int_{\{(t,x)\in Q:|u_n|>0\}} |\gamma_n(u_n)|\, dxdt \\
&\leq \|u_0\|_{L^1(\Omega)}\,.
\end{aligned}$$

Then,

$$\sup \int_Q \gamma_n(u_n)\, dxdt < +\infty,$$

knowing that,

$$0 \leq \int_Q |\gamma_n(u_n)|\, dxdt, \text{ because } p > 1,$$

for each $(t,x)\in Q$, we pose

$$\lim_{n\to+\infty} \inf \gamma_n(u_n) = \gamma(u),$$

by the Fateau's lemma, we have $\gamma(u)$ in $L^1(Q)$.
As that

$$u_n \to u \text{ weakly in } L^2([0,T]\,, L^2(\Omega)),$$

on the other hand, we have

$$\nabla u_n \to \nabla u \text{ in } (L^2([0,T]\,, L^2(\Omega)))^{\mathbb{N}},$$

we note that

$$\begin{aligned}
&\int\limits_{\{(t,x)\in Q:|\gamma_n(u_n)|\leq n\}} |\gamma_n(u_n)-\gamma(u)|\,dxdt \\
\leq &\int\limits_Q |\gamma_n(u_n)-\gamma(u)|\,dxdt \to 0 \text{ when } n\to+\infty.
\end{aligned}$$

So,

$$\gamma_n(u_n)\to\gamma(u) \text{ when } n\to+\infty \text{ on } \{(t,x)\in Q\ :\ |\gamma_n(u_n)|\leq n\}.$$

For every $n\in\mathbb{N}$, we have

$$\begin{aligned}
meas(\{(x,t)\in Q:|\gamma_n(u_n)|>n\}) &\leq \frac{1}{n}\int\limits_Q |\gamma_n(u_n)|\,dxdt \\
&\leq \frac{1}{n}\|\gamma_n(u_n)\|_{L^1(Q)} \\
&\leq \frac{c}{n}\to 0 \text{ when } n\to+\infty,
\end{aligned}$$

thus $\{(t,x)\in Q:|\gamma_n(u_n)|>n\}$ is the zero measurement set where $(\gamma_n(u_n))_{n\in\mathbb{N}}$ may not converge to $(\gamma(u))$, which shows that

$$\gamma_n(u_n)\to\gamma(u) \text{ almost everywhere in } Q.$$

For proof (3.15) we show that the sequence $(\gamma_n(u_n))_{n\in\mathbb{N}}$ is equi-integrable.
Let $\delta>0$ and $\mathbf{A}$ be a measurable subset belonging to $[0,T]\times\Omega$, we define the following sets,

$$B_\delta = \{(t,x)\in Q:\ |u_n|\leq\delta\}, \tag{3.16}$$
$$F_\delta = \{(t,x)\in Q:\ |u_n|>\delta\}, \tag{3.17}$$

$$\begin{aligned}
\int\limits_{\mathbf{A}} |\gamma_n(u_n)|\,dxdt &= \int\limits_{\mathbf{A}\cap B_\delta} |\gamma_n(u_n)|\,dxdt + \int\limits_{\mathbf{A}\cap F_\delta} |\gamma_n(u_n)|\,dxdt \\
&\leq \int\limits_{\mathbf{A}\cap B_\delta} |\gamma_n(u_n)|\,dxdt + \|u_0\|_{L^1(\Omega)} \\
&\to 0 \text{ when } meas(\mathbf{A})\to 0.
\end{aligned}$$

Using the generalized Hölder's inequality and Poincaré inequality, we get

$$
\begin{aligned}
\int\limits_{\mathbf{A}} |\gamma_n(u_n)|\, dxdt &\leq \left(\int\limits_{\mathbf{A}} |\lambda|^2\, dxdt\right)^{\frac{1}{2}} \left(\int\limits_{B_\delta} |u_n|^{(p-1)2}\, dxdt\right)^{\frac{1}{2}} \\
&\quad + \int\limits_{\mathbf{A}} |\gamma_n(u_n)|\, dxdt \\
&\leq \left(|\lambda|^2\, meas(\mathbf{A})\right)^{\frac{1}{2}} \left(\int\limits_{B_\delta} |\nabla u_n|^2\, dxdt\right)^{(p-1)\frac{1}{2}} \\
&\quad + \int\limits_{\mathbf{A}} |\gamma_n(u_n)|\, dxdt \\
&\leq \left(|\lambda|^2\, meas(\mathbf{A})\right)^{\frac{1}{2}} \left(\frac{k}{\alpha}\left(\|u_0\|_{L^1(\Omega)}\right)\right)^{(p-1)\frac{1}{2}} \\
&\quad + \int\limits_{\mathbf{A}} |\gamma_n(u_n)|\, dxdt \\
&\to 0 \text{ when } meas(\mathbf{A}) \to \mathbf{0}.
\end{aligned}
$$

Which shows that $(\gamma_n(u_n))_{n\in\mathbb{N}}$ is equi-integrable. By using Vitali's theorem, we obtain:

(3.18) $$\gamma_n(u_n) \to \gamma(u) \text{ strongly in } L^1(Q).$$

Now we prove that

$$f_n(t,x,u_n) \to f(t,x,u) \text{ strongly in } L^1(Q),$$

we have,

$$
\begin{aligned}
\|f_n(t,x,u_n)\|_{L^1(Q)} &= \int\limits_Q |f_n(t,x,u_n)|\, dxdt \\
&\leq \int\limits_{\{(t,x)\in Q: |u_n|>0\}} |f_n(t,x,u_n)|\, dxdt \\
&\leq \|u_0\|_{L^1(\Omega)},
\end{aligned}
$$

then,

$$\sup \int\limits_Q f_n(t,x,u_n) dxdt < +\infty.$$

By (1.6) knowing that, $0 \leq f_n(t,x,u_n)$ for each $(t,x) \in Q$, we pose

$$\lim_{n\to+\infty} \inf f_n(t,x,u_n) = f(t,x,u),$$

by the Fateau's lemma, we have $f(t,x,u)$ in $L^1(Q)$. As that

$$u_n \to u \text{ weakly in } L^2([0,T], L^2(\Omega)),$$

on the other hand, we have

$$\nabla u_n \to \nabla u \text{ in } (L^2([0,T]\,, L^2(\Omega)))^{\mathbb{N}},$$

we note that,

$$\begin{aligned}
&\int\limits_{\{(t,x)\in Q : |f_n(t,x,u_n)|\le n\}} |f_n(t,x,u_n) - f(t,x,u)|\,dxdt \\
\le\ &\int\limits_{Q} |f_n(t,x,u_n) - f(t,x,u)|\,dxdt \to 0 \text{ when } n \to +\infty.
\end{aligned}$$

So,

$$f_n(t,x,u_n) \to f(t,x,u) \text{ when } n \to +\infty \text{ on } \{(t,x) \in Q\ :\ |f(t,x,u)| \le n\}\,.$$

For every $n \in \mathbb{N}$, we have

$$\begin{aligned}
meas(\{(x,t) \in Q : |f_n(t,x,u_n)| > n\}) &\le \frac{1}{n} \int\limits_{Q} |f_n(t,x,u_n)|\,dxdt \\
&\le \frac{1}{n} \|f_n(t,x,u_n)\|_{L^1(Q)} \\
&\le \frac{c}{n} \to 0 \text{ when } n \to +\infty,
\end{aligned}$$

thus $\{(t,x) \in Q : |f_n(t,x,u_n)| > n\}$ is the zero measurement set where $(f_n(t,x,u_n))_{n\in\mathbb{N}}$ may not converge to $(f(t,x,u))$, which shows that

$$f_n(t,x,u_n) \to f(t,x,u) \text{ almost everywhere in } Q.$$

For proof (3.15) we show that the sequence $(f_n(t,x,u_n))_{n\in\mathbb{N}}$ is equi-integrable.
By the definitions of the sets (3.16) and (3.17), we get

$$\begin{aligned}
\int\limits_{\mathbf{A}} |f_n(t,x,u_n)|\,dxdt &= \int\limits_{\mathbf{A}\cap B_\delta} |f_n(t,x,u_n)|\,dxdt + \int\limits_{\mathbf{A}\cap F_\delta} |f_n(t,x,u_n)|\,dxdt \\
&\le \int\limits_{\mathbf{A}\cap B_\delta} |f_n(t,x,u_n)|\,dxdt + \|u_0\|_{L^1(\Omega)} \\
&\to\ 0 \text{ when } meas(\mathbf{A}) \to 0.
\end{aligned}$$

Let $\delta > 0$ be large enough. Using the generalized Hölder's inequality and Poincaré inequality, we have

$$\int\limits_{\mathbf{A}} |f_n(t,x,u_n)|\,dxdt = \int\limits_{\mathbf{A}\cap B_\delta} |f_n(t,x,u_n)|\,dxdt + \int\limits_{\mathbf{A}\cap F_\delta} |f_n(t,x,u_n)|\,dxdt,$$

therefore

$$
\begin{aligned}
\int\limits_{\mathbf{A}} |f_n(t,x,u_n)|\, dxdt &\leq \int\limits_{\mathbf{A}\cap B_\delta} (c(x,t)+\sigma |u_n|)\, dxdt + \int\limits_{\mathbf{A}\cap F_\delta} |f_n(t,x,u_n)|\, dxdt \\
&\leq \int\limits_{\mathbf{A}} c(x,t) dxdt + \sigma \int\limits_{Q} |\nabla T_\delta(u_n)|\, dxdt \\
&\quad + \int\limits_{\mathbf{A}\cap F_\delta} |f_n(t,x,u_n)|\, dxdt \\
&\leq \int\limits_{\mathbf{A}} c(x,t) dxdt + \sigma\, (meas(\mathbf{A}))^{\frac{1}{2}} \left( \int\limits_{Q_T} |\nabla T_\delta(u_n)|^2\, dxdt \right)^{\frac{1}{2}} \\
&\quad + \int\limits_{\mathbf{A}\cap F_\delta} |f_n(t,x,u_n)|\, dxdt \\
&\leq K_1 + C_1 \left( \frac{k}{\alpha} \|u_0\|_{L^1(\Omega)} \right)^{\frac{1}{2}} + \int\limits_{\mathbf{A}\cap F_\delta} \frac{1}{|u_n|} |u_n f_n(t,x,u_n)|\, dxdt \\
&\leq K_2 + \int\limits_{\mathbf{A}\cap F_\delta} \frac{1}{\delta} |u_n f_n(t,x,u_n)|\, dxdt \\
&\leq K_2 + \frac{1}{\delta} \left( \int\limits_{\mathbf{A}\cap F_\delta} |u_n|^2\, dxdt \right)^{\frac{1}{2}} \left( \int\limits_{\mathbf{A}\cap F_\delta} |f_n(t,x,u_n)|^2\, dxdt \right)^{\frac{1}{2}} \\
&\to 0 \text{ when } meas(\mathbf{A}) \to \mathbf{0}.
\end{aligned}
$$

Which shows that $(f_n(t,x,u_n))_{n\in\mathbb{N}}$ is equi-integrable. By using Vitali's theorem, we get

$$f_n(t,x,u_n) \to f(t,x,u) \text{ strongly in } L^1(Q). \tag{3.19}$$

Since $u_n \in C([0,T]\,, L^2(\Omega))$, in order to see that $u \in C([0,T]\,, L^1(\Omega))$, we only have to prove that

$$u_n \to u \text{ in } C([0,T]\,, L^1(\Omega)).$$

To do this fix $\tau \in [0,T]$. Choosing $T_k(u_n - u_m)1_{\{[0,\tau[\}}$ as test function in the weak formulation of $u_n$ and $-T_k(u_n - u_m)1_{\{[0,\tau[\}}$ in that of $u_m$ with $\tau \leq T$, we get

$$
\begin{aligned}
& \int_\Omega \Phi_k(u_n(\tau) - u_m(\tau))dx - \int_\Omega \Phi_k(u_n(0) - u_m(0))dx \\
& + \int_0^\tau \int_\Omega A(u_n - u_m)\nabla(u_n - u_m)\nabla T_k(u_n - u_m)dxdt \\
& + \int_0^\tau \int_\Omega \lambda \left[ |u_n|^{p-2} u_n - |u_m|^{p-2} u_m \right] T_k(u_n - u_m)dxdt \\
= & \int_0^\tau \int_\Omega \left( f_n(t,x,u_n) - f_n(t,x,u_m) \right) T_k(u_n - u_m)dxdt,
\end{aligned}
$$

where $\Phi_k$ is the primitive of $T_k$ such that $\Phi_k(0) = 0$,

$$
\begin{aligned}
\int_\Omega \Phi_k(u_n(\tau) - u_m(\tau))dx \quad \leq \quad & \int_0^\tau \int_\Omega \lambda \left| |u_n|^{p-2} u_n - |u_m|^{p-2} u_m \right| dxdt \\
& + k \int_0^\tau \int_\Omega |f_n(t,x,u_n) - f_n(t,x,u_m)| \, dxdt \\
& + k \int_\Omega |u_{n,0} - u_{m,0}| \, dx.
\end{aligned}
$$

Next, we divide this inequality by $k$ and the Monotone convergence theorem and let $k$ go to 0, to obtain

$$
\begin{aligned}
\int_\Omega |u_n(\tau) - u_m(\tau)| \, dx \quad \leq \quad & \int_0^\tau \int_\Omega \lambda \left| |u_n|^{p-2} u_n - |u_m|^{p-2} u_m \right| dxdt \\
& + \int_0^\tau \int_\Omega |f_n(t,x,u_n) - f_n(t,x,u_m)| \, dxdt \\
& + \int_\Omega |u_{n,0} - u_{m,0}| \, dx.
\end{aligned}
$$

Hence,

$$\sup_{\tau\in[0,T]}\int_{\Omega}|u_n(\tau)-u_m(\tau)|\,dx \le \int_0^{\tau}\int_{\Omega}\lambda\left||u_n|^{p-2}u_n-|u_m|^{p-2}u_m\right|dxdt$$
$$+\int_0^{\tau}\int_{\Omega}|f_n(t,x,u_n)-f_n(t,x,u_m)|\,dxdt$$
$$+\int_{\Omega}|u_{n,0}-u_{m,0}|\,dx.$$

Thus, it follows from (3.14), (3.19) and (3.18), that sequence $(u_n)$ is a Cauchy sequence in $C\left([0,T],L^1(\Omega)\right)$ then $u_n\to u$ in $C([0,T],L^1(\Omega))$. Finally,

$$u\in C([0,T],L^1(\Omega)). \tag{3.20}$$

## 4. Conclusion

We conclude by the main purpose from our work. In this article we demonstrated the existence of entropy solution for quasi-linear parabolic problem with $L^1$ data, we also proved that the problem admits a weak solution according to Schauder fixed point theorem. For unbounded nonlinearities satisfying suitable conditions, we established equi-integrablity and we derived a compactness results to be able to pass to the limit to get the desired result.

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**3-5 July 2019, Istanbul, Turkey**

# Some Curvature Properties Of Non-Reductive Pseudo Riemannian Manifolds Of Dimension Four

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**Abstract**

In this paper we considered non-reductive homogeneous pseudo Riemannian manifolds of dimension four and investigated recurrent curvature tensor condition for those curvature tensor. We classified non-reductive homogeneous pseudo Riemannian manifolds with recurrent curvature tensor, and in the other cases, admitting the recurrent curvature tensor condition is equivalent to be locally symmetric or flat curvature tensor and only for one case we have not any 1-form so that satisfy in recurrent curvature tensor. Then we investigated some geometrical concepts for them like Weyl tensor, Einstein property and etc., we obtained some results, for example we show when any non-reductive homogeneous manifold with recurrent curvature tensor is locally conformally flat. We also studied Ricci solitons for these spaces and concluded that any non-reductive homogeneous manifold with non-trivial recurrent curvature tensor is a steady Ricci soliton.

**Keywords: Pseudo Riemannian manifold, Recurrent curvature tensor, Locally Symmetric, Flat, Ricci soliton.**

## 1. Introduction

Let *(M,g)* be homogeneous pseudo-Riemannian manifold then it is said a reductive manifold if in homogeneous space $M=\frac{G}{H}$, the lie algebra $\mathfrak{g}$ can be decompose as sum direct $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{m}$, where $\mathfrak{k}$ is lie algebra of $H$, $\mathfrak{m}$ is lie algebra of $M$ and is Ad($H$)-Invariant subspace of $\mathfrak{g}$. this conditionis equivalent to the algebraic condition $[\mathfrak{k},\mathfrak{m}]\subseteq\mathfrak{m}$. Homogeneous Riemannian manifolds are reductive but there exist homogenous pseudo Riemannain manifolds that are not reductive. If a space is not reductive it is said non-reductive. Non-reductive homogeneous pseudo Riemannian manifolds were classified in term of thier lie algebras in [3] by M.E. Fels and A.G. Renner. In [1] geometry and Ricci solitons on these spaces were studied by G. Calvaruso and A. Fino. after several years they with A. Zaeim obtained an explicit description in coordinates of invariant metrics on these spaces in [2] In this paper we investigate recurrent curvature tensor condition for these spaces and classify non-reductive homogeneous pseudo-Riemannian manifolds of dimension four with recurrent curvature tensor and investigate some geometrical concepts for them.

## 2. Preliminaries

Let $x_1, ..., x_n$ be a coordinate system on pseudo-Riemannian manifold M. The Christoffel symbols for this coordinate system comput by: $\Gamma_{ij}^k = \frac{1}{2}\sum_m g^{mk}\left\{\frac{\partial g_{jm}}{\partial x_i} + \frac{\partial g_{im}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_m}\right\}$. (1)

**Theorem (2.1):**

On a pseudo-Riemannian manifold M there is a unique levi-civita $\nabla$ such that: $\nabla_{\partial_i}^{\partial_j} = \Gamma_{ij}^k \partial_k$. (2)

**Proof**

Refer to [5]

□

**Lemma(2.2):**

Let M be a pseudo-Riemannian manifold with Levi-Civita connection . The tensor $R: \mathfrak{X}^3(M) \to \mathfrak{X}(M)$ given by: $R(X,Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z$. (3)

**Proof**

Refer to [5] □

(0,4)-curvature tensor compute by: **R**(X,Y,Z,W)=$g$($R$(X,Y)Z,W). (4)

Ricci tensor that conclude from contraction on curvature tensor is 2-form and computing by:

$$\rho(X,Y) = \sum_m \varepsilon_m g(R(X,\partial_m)Y,\partial_m). \qquad (5)$$

Scalar curvature is contraction on Ricci form and computing by: $S = g^{ij}\rho_{ij}$. (6)

Weyl tensor for pseudo-Riemannian manifolds with dimention greater than three is as follow:

$$\begin{aligned} W(X,Y,Z,T) = R(X,Y,Z,T) &- \frac{S}{(n-1)(n-2)}\{g(X,T)g(Y,Z) - g(X,Z)g(Y,T)\} \\ &- \frac{1}{(n-2)}\{\rho(X,Z)g(Y,T) + \rho(Y,T)g(X,Z) - \rho(X,T)g(Y,Z) \\ &- \rho(Y,Z)g(X,T)\}. \qquad (7) \end{aligned}$$

Pseudo-Riemannian manifold is conformally flat if and only if $W=0$.

Pseudo-Riemannian manifold is Einstein if Ricci tensor satisfy in: $\rho = \lambda g$. (8)

**Definition(2.3):**

Let *(M,g)* be a pseudo-Riemannian manifold and if there exist vector field $V=\sum_i v_i \partial_i$ Ricci tensor and metric tensor satisfy in following relation then metric $g$ is Ricci Soliton:

$$\rho + \mathcal{L}_V g = \lambda g, \qquad (9)$$

where $\mathcal{L}_V$ is lie derivative in direction *V*. if *V* be gradient vector field then metric $g$ is gradient Ricci soliton and Ricci soliton relation become as: $\rho + \ Hess_f = \lambda g$. (10)

where $Hess_f$ is hessian of potential function *f* and following relation is for compute it in coordinate system : $\frac{\partial^2 f}{\partial x_i \partial x_j} - \ \sum_k \Gamma_{ij}^k \frac{\partial f}{\partial x_k}$. (11)

**Definition(2.4):**

Let *(M,g)* be a pseudo-Riemannian manifold it call with Recurrent curvature tensor if it's curvature tensor satisfy in following relation:

$$\nabla \boldsymbol{R} = \boldsymbol{R} \otimes \omega, \qquad (12)$$

where $\boldsymbol{R}$ (0,4)-curvature tensor and $\omega$ is 1-form.

**Theorem(2.5):**

Let *M* be a non-reductive pseudo-Riemannian homogeneous four-manifold, then it is locally isometric to $\mathbb{R}^4$, equipped with a pseudo-Riemannian metric $g$, which takes the following explicit form (in terms of some real constants a, b, c, q)*:*

$$A1 \quad g = (4bx_2^2 + \ a)dx_1^2 + 4bx_2dx_1dx_2 - (4ax_2x_4 - 4cx_2 + \ a)dx_1dx_3 + 4ax_2dx_1dx_4 + bdx_2^2 - 2(ax_4 - c)dx_2dx_3 + \ \ 2adx_2dx_4 + qdx_3^2,$$

$$A2 \qquad g = -2ae^{2\alpha x_4}dx_1dx_3 + ae^{2\alpha x_4}dx_2^2 + be^{2(\alpha-1)x_4}dx_3^2 + 2ce^{(\alpha-1)x_4}dx_3dx_4 + \ qdx_4^2,$$

$$A3 \quad \varepsilon = 1 \qquad g = 2ae^{2x_3}dx_1dx_4 + ae^{2x_3}\cos(x_4)^2\, dx_2^2 + bdx_3^2 + 2cdx_3dx_4 + \ qdx_4^2,$$

$$A3 \quad \varepsilon = -1 \qquad g = 2ae^{2x_3}dx_1dx_4 + ae^{2x_3}\cosh(x_4)^2\, dx_2^2 + bdx_3^2 + 2cdx_3dx_4 + \ qdx_4^2,$$

$$A4 \qquad g = \left(\frac{a}{2}x_4^2 + 4bx_2^2 + \ a\right)dx_1^2 + 4bx_2dx_1dx_2 + ax_2(4 + \ x_4^2)dx_1dx_3 + ax_4(1 + 2x_2x_3)dx_1dx_4 + bdx_2^2 \ \ + \frac{a}{2}(4 + \ x_4^2)dx_2dx_3 + \ \ ax_3x_4dx_2dx_4 + \frac{a}{2}dx_4^2,$$

$$B1 \qquad g = (q(4x_2x_3x_4 + 4x_2^2x_4^2 + x_3^2) + 4cx_2x_3 + 8cx_2^2x_4 + 2ax_3 + 4bx_2^2)dx_1^2 + 2(q(x_3x_4 + 2x_2x_4^2) + 4cx_2x_4 + cx_3 + 2bx_2)dx_1dx_2 + 2(q(x_3 + 2x_2x_4) + 2cx_2 + a)dx_1dx_3 + 4ax_2dx_1dx_4 + (qx_4^2 + 2cx_4 + b)dx_2^2 + 2(qx_4 + c)dx_2dx_3 + 2adx_2dx_4 + qdx_3^2,$$

$$B2 \quad g = \left(-\frac{a}{2}x_4^2 + 4bx_2^2 + a\right)dx_1^2 + 4bx_2dx_1dx_2 - ax_2(-4 + x_4^2)dx_1dx_3 - ax_4(1 + 2x_2x_3)dx_1dx_4 + bdx_2^2 - \frac{a}{2}(-4 + x_4^2)dx_2dx_3 - ax_3x_4dx_2dx_4 - \frac{a}{2}dx_4^2,$$

$$B3 \quad g = -2ae^{-x_2}x_3dx_1dx_2 + 2ae^{-x_2}dx_1dx_3 + 2(2bx_3^2 - ax_4)dx_2^2 - 4bx_3dx_2dx_3 + 2adx_2dx_4 + bdx_3^2,$$

or it is locally isometric to:

$$(\mathbb{R}^2 - \{0,0\}) \times \mathbb{R}^2$$

with following metric:

$$A5 \qquad g = \frac{-ax_4}{4x_2}dx_1dx_2 + \frac{a}{2}dx_1dx_4 + \frac{a(2 + 2x_1x_4 + x_3^2)}{8x_2^2}dx_2^2 - \frac{ax_3}{4x_2}dx_2dx_3 - \frac{ax_1}{4x_2}dx_2dx_4 + \frac{a}{8}dx_3^2,$$

## 3. Main Results

**Theorem(3.1):**

Let *(M,g)* be non-reductive homogeneous pseudo Riemannian manifold of dimension four then if *M* admit metric *g* that has recurrent curvature tensor then *(M,g)* is isometric to one case of following table:

| case | Condition | | If admit recurrent curvature tensor |
|---|---|---|---|
| A1 | ✓ | | Locally symmetric |
| A2 | 1 | $\alpha = 0$ | $w_1=w_2=w_3=0$, $w_4=-2$ |
| | 2 | b=0 | Locally symmetric |
| | 3 | $\alpha = 0$, b=0 | Flat |
| A3 | × | | × |
| A4 | b=0 | | Locally symmetric |
| A5 | ✓ | | Locally symmetric |
| B1 | 1 | q=0, c=0 | Locally symmetric |
| | 2 | $bq - c^2=0$ | Locally symmetric |
| | 3 | b=0, c=0, q=0 | Flat |
| B2 | b=0 | | Locally symmetric |
| B3 | 1 | ✓ | Ricci flat |
| | 2 | b=0 | Flat |

**Proof:**

For proof of this theorem we studied non-reductive homogeneous pseudo Riemannian manifolds of dimension four case-by-case. For example in case A2 first after computing christoffel symbols by using (1), by using (2) the levi-civita connection are as follow:

$$\nabla_{\partial_3}^{\partial_1}=\frac{c\alpha e^{2(\alpha-1)x_4}}{q}\partial_1+\frac{a\alpha e^{2\alpha x_4}}{q}\partial_4,\ \nabla_{\partial_4}^{\partial_1}=\alpha\partial_1,\ \nabla_{\partial_2}^{\partial_2}=\frac{-c\alpha e^{2(\alpha-1)x_4}}{q}\partial_1-\frac{a\alpha e^{2\alpha x_4}}{q}\partial_4,\ \nabla_{\partial_4}^{\partial_2}=\alpha\partial_2,$$

$$\nabla_{\partial_3}^{\partial_3} = \frac{cb(\alpha-1)e^{2(\alpha-2)x_4}}{aq}\partial_1 + \frac{b(\alpha-1)e^{2(\alpha-1)x_4}}{q}\partial_4,\ \nabla_{\partial_4}^{\partial_3} = \frac{-e^{-2x_4}(-bq+c^2\alpha e^{2(\alpha-1)x_4})}{aq}\partial_1 + \alpha\partial_2 - \frac{c\alpha e^{2(\alpha-1)x_4}}{q}\partial_4,$$

$$\nabla_{\partial_4}^{\partial_4} = \frac{-2c(\alpha-1)e^{-2x_4}}{a}\partial_1\,,$$

Now by using (3) curvature tensor is as follow:

$$R(\partial_1,\partial_2)\partial_2 = \frac{a\alpha^2 e^{2\alpha x_4}}{q}\partial_1,\ R(\partial_1,\partial_2)\partial_3 = \frac{a\alpha^2 e^{2\alpha x_4}}{q}\partial_2,\ R(\partial_1,\partial_3)\partial_1 = \frac{-a\alpha^2 e^{2\alpha x_4}}{q}\partial_1,$$

$$R(\partial_1,\partial_3)\partial_3 = \frac{b\alpha^2 e^{2(\alpha-1)x_4}}{q}\partial_1 + \frac{a\alpha^2 e^{2\alpha x_4}}{q}\partial_3,\ R(\partial_1,\partial_3)\partial_4 = \frac{c\alpha^2 e^{2(\alpha-1)x_4}}{q}\partial_1,\ R(\partial_1,\partial_4)\partial_4 = \alpha^2\partial_1,$$

$$R(\partial_1,\partial_4)\partial_3 = \frac{c\alpha^2 e^{2(\alpha-1)x_4}}{q}\partial_1 + \frac{a\alpha^2 e^{2\alpha x_4}}{q}\partial_4,\ R(\partial_2,\partial_3)\partial_2 = \frac{-b\alpha e^{2(\alpha-1)x_4}}{q}\partial_1 - \frac{a\alpha^2 e^{2\alpha x_4}}{q}\partial_3,$$

$$R(\partial_2,\partial_3)\partial_1 = \frac{-a\alpha^2 e^{2\alpha x_4}}{q}\partial_2,\ R(\partial_2,\partial_3)\partial_3 = \frac{b\alpha(\alpha-1)e^{2(\alpha-1)x_4}}{q}\partial_2,\ R(\partial_2,\partial_3)\partial_4 = \frac{c\alpha^2 e^{2(\alpha-1)x_4}}{q}\partial_2,$$

$$R(\partial_2,\partial_4)\partial_2 = \frac{-a\alpha^2 e^{2\alpha x_4}}{q}\partial_4,\ R(\partial_2,\partial_4)\partial_3 = \frac{c\alpha^2 e^{2(\alpha-1)x_4}}{q}\partial_2,\ R(\partial_2,\partial_4)\partial_4 = \alpha^2\partial_2,$$

$$R(\partial_3,\partial_4)\partial_1 = \frac{a\alpha^2 e^{2\alpha x_4}}{q}\partial_1,\ R(\partial_3,\partial_4)\partial_3 = \frac{2bc(\alpha-1)e^{2(\alpha-2)x_4}}{aq}\partial_1 + \frac{c\alpha^2 e^{2(\alpha-1)x_4}}{q}\partial_3 - \frac{b(\alpha^2-2\alpha+2)e^{2(\alpha-1)x_4}}{q}\partial_4,$$

$$R(\partial_3,\partial_4)\partial_4 = \alpha^2\partial_3 - \frac{c\alpha^2 e^{2(\alpha-1)x_4}}{q}\partial_4,$$

and after computing (0,4)-curvature tensor by (4) by using (12) we investigate recurrent curvature condition that a PDE system conclude that by solving it we obtain results as written in above table.

□

**Theorem(3.2):**

Let *(M,g)* be non-reductive homogeneous pseudo Riemannian manifold with recurrent curvature tensor of dimension four non flat then condition for that *(M,g)* be Einstein and conformally flat and it's scalar curvature tensor are as follow:

| case | | Einstein | Conformally flat | Scalar curvature |
|---|---|---|---|---|
| A1 | | ✗ | ✓ | $\frac{-6}{a}$ |
| A2 | 1 | ✗ | ✗ | 0 |
| | 2 | $\lambda = \frac{-3\alpha^2}{q}$ | ✓ | $\frac{-12\alpha^2}{q}$ |
| A4 | | $\lambda = \frac{-3}{a}$ | ✓ | $\frac{-12}{a}$ |
| A5 | | $\lambda = \frac{-12}{a}$ | ✓ | $\frac{-48}{a}$ |
| B1 | 1 | $\lambda = \frac{3q}{2a^2}$ | ✗ | $\frac{6q}{a^2}$ |
| | 2 | $\lambda = \frac{3c^2}{a^2b}$ | ✗ | $\frac{6c^2}{a^2b}$ |
| B2 | | $\lambda = \frac{-3}{a}$ | ✓ | $\frac{-12}{a}$ |
| B3 | 1 | ✗ | ✗ | 0 |

**Proof**

Proof of this theorem is like previous one. For example consider case A2 again by using (5) Ricci tensor is:

$$\rho_{ij} = \frac{6a\alpha^2 e^{2\alpha x_4} dx_1 dx_3}{q} - \frac{6a\alpha^2 e^{2\alpha x_4} dx_2^2}{q} - \frac{b(3\alpha^2-3\alpha+2)e^{2(\alpha-1)x_4} dx_3^2}{q} - \frac{6c\alpha^2 e^{2(\alpha-1)x_4} dx_3 dx_4}{q} - 3\alpha^2 dx_4^2 ,$$

And now by using (8) we have following equations:

$$\{\frac{ae^{2\alpha x_4}(\lambda q+3\alpha^2)}{q}, \frac{be^{2(\alpha-1)x_4}(\lambda q+3\alpha^2-3\alpha+2)}{q}, \frac{ce^{2(\alpha-1)x_4}(\lambda q+3\alpha^2)}{q}, \lambda q+3\alpha^2\}$$

Therefore for that this case be Einstein, this above equations must be identically zero hence $\lambda = \frac{-3\alpha^2}{q}$ and b=0. This show what written in above table. For that this case be conformally flat by (7) we conclude following equations vanish:

$$\{be^{2(\alpha-1)x_4}(\alpha-2),\ \frac{ba\,e^{2(2\alpha-1)x_4}(\alpha-2)}{q}\}$$

Therefore b=0 or $\alpha = -2$, hence A2.1 is not conformally flat and A2.2 is conformally flat. By using (6) scalar curvature for A2.1 and A2.2 respectively is 0 and $\frac{-12\alpha^2}{q}$.

□

**Theorem(3.3):**

Let *(M,g)* be non-reductive homogeneous pseudo Riemannian manifold with non-trivially recurrent curvature tensor of dimension four then *(M,g)* is steady Ricci Soliton with following vector field

$$v_1(x_1,x_2,x_3,x_4)=c_3x_2 - c_1x_1 + \frac{c(-2+qc_1)e^{-2x_4}}{2aq},\ v_2(x_1,x_2,x_3,x_4)= c_3x_3 + c_4,\ v_3(x_1,x_2,x_3,x_4)= c_1x_3 + c_2,$$

$$v_4(x_1,x_2,x_3,x_4)=\frac{qc_1-1}{q}$$

**Proof**:

If *(M,g)* be non-reductive homogeneous pseudo Riemannian manifold with non-trivially recurrent curvature tensor of dimension four then as mentioned in table of theorem (3.1) if in case A2 of non-reductive homogeneous pseudo Riemannian manifolds $\alpha = 0$ then it’s curvature tensor satisfy in (12) now lie derivative of metric tensor in directional of above vector field is negative of Ricci tensor that this show non-reductive homogeneous pseudo Riemannian manifold with non-trivially recurrent curvature tensor of dimension four is steady Ricci Soliton with above vector field.

□

**Theorem(3.4):**

Let *(M,g)* be non-reductive homogeneous pseudo Riemannian manifold with non-trivially recurrent curvature tensor of dimension four then *(M,g)* is steady gradient Ricci Soliton with following potential function :

$$f(x_1,x_2,x_3,x_4)=\frac{-2}{q}x_1 + c_1x_2 + c_2x_3 + c_3x_4 + c_4$$

**Proof:**

For proof of this theorem exactly like previous theorem with computing hessian of above potential function it will be seen that hessian of $f$ is negative of Ricci tensor that this show non-reductive homogeneous pseudo Riemannian manifold with non-trivially recurrent curvature tensor of dimension four is steady gradient Ricci Soliton with above potential function. $\square$

## 4. Conclusion

Non-reductive homogeneous pseudo Riemannian manifolds with recurrent curvature tensor of dimension four are locally symmetric except case A2.1.

Let *(M,g)* be non-reductive homogeneous pseudo Riemannian manifold with recurrent curvature tensor of dimension four, if it be Einstein with $\lambda \neq 0$, then $\lambda = \frac{S}{4}$.

Non-reductive homogeneous pseudo Riemannian manifolds with non-trivially recurrent curvature tensor of dimension four are steady Ricci soliton and steady gradient ricci soliton.

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**3-5 July 2019, Istanbul, Turkey**

## On Some Metric With Recurrent Curvature Tensor

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**Abstract**

In this paper we studied some curvature properties of special metrics with physical applications. We showed that strictly walker manifolds of dimension four are with recurrent curvature tensor and also we specified the sufficient condition for these manifolds to be locally symmetric. We also showed that oscillator groups, equipped with a one parameter family of left invariant Lorentzian metrics and with recurrent curvature tensor, are locally symmetric. Also recurrent curvature condition for some other metrics which are physically relevant significant were checked, e.g., generalized symmetric pseudo Riemannian manifolds which were showed that are not with recurrent curvature tensor.

**Keywords: Strictly walker manifolds, Pseudo Riemannian manifold, Recurrent curvature tensor, Oscillator group, Generalized symmetric, Locally Symmetric.**

## 1. Introduction

The shape of a manifolds and also the geometry of it mainly depends on the curvature therefore curvature is one of the most important concepts of a manifold. In the other hands one of the most important property of a manifolds is symmetry. Symmetry first time studied by cartan. He classified complete simply connected locally symmetric spaces after that cahen and parker do similar work for non-riemannian case. Later generalization of cartan's notion such as recurrent manifolds and generalized recurrent manifolds were studied. In other hand when analyze the curvature of a manifold, covariant derivative must be computed therefore locally symmetric case reduced to purely algebraic level hence manifolds with recurrent curvature tensor are considered by many authors

Manifolds with Recurrent curvature first time introduce by Ruse [6]. At the first he called these spaces *kappa spaces*. But after while he decided to change this name with name that emphasizing to property that define them so he called them recurrent curvature manifolds. In [7] walker showed manifolds with recurrent curvature tensor admit a parallel vector field but locally symmetric manifolds necessarily don't admit. Garcia, Gilkey and Nikcevic classified locally homogeneous manifolds with recurrent curvature of dimension three in [4] and later Calvaroso and zaeim investigated collineation on these spaces.

## 2. Preliminaries

Walker metrics, those are important metric and also physically relevant significant, first time introdued by walker and a canonical form for a strictly Walker metrics has been obtained by him, showing the existence of suitable coordinates where the metric expresses as:

$$g = 2dx_1dx_3 + 2dx_2dx_4 + f(x_3,x_4)dx_3^2 + 2g(x_3,x_4)dx_3dx_4 + h(x_3,x_4)dx_4^2,$$

Where $f$,$g$ and $h$ are smooth functions. The levi-civita connection, curvature tensor and etc, for this metric is computed in [2].

**Definition(2.1):**

Oscillator group is a four dimensional lie group, whose lie algebra is the one generated by the differential operators, acting on functions of one variable, associated to the harmonic oscillator problem. This group is given by $\mathbb{R} \times \mathbb{C} \times \mathbb{R}$ with the product:

$$(x_1,z_1,y_1).(x_2,z_2,y_2) = \left(x_1 + x_2 + Im(\overline{z_1}e^{iy_1}z_2), z_1 + e^{iy_1}z_2, y_1 + y_2\right)$$

oscillator group with one-parameter family of left-invariant Lorentzian metrics that state as:

$$g_a = adx_1^2 + 2ax_3dx_1dx_2 + 2dx_1dx_4 + (1 + ax_3^2)dx_2^2 + 2x_3dx_2dx_4 + dx_3^2 + adx_4^2,$$

Oscillator group with above metric is an interesting object to study both in differential geometry and in mathematical physics. Also it is one of the most celebrated examples of Lorentzian naturally reductive spaces.

**Definition(2.2):**

Let ($M$,$g$) be pseudo-Riemannian manifolds a $s$-structure on $M$ is a family of isometries $\{s_p \mid p \in M\}$ such that:

$\forall p \in M$: $p$ is isolated fix point of $s_p$.

$\forall p,q \in M$: $s_p \, o \, s_q = s_{s_p(q)} o \, s_p$

The following mapping be smooth:

$$M \times M \to M$$
$$(p,q) \to s_p(q).$$

**Definition(2.3):**

A generalized symmetric space is a connected, pseudo-Riemannian manifold, carrying at least one regular s-structure.

**Theorem(2.4):**

All non-symmetric, simply connected generalized symmetric spaces ($M$, $g$) of dimension 4 are belong, up to isometry, to the following four types:

Type A: ($M$,$g$) is space $\mathbb{R}^4$ $(x_1,x_2,x_3,x_4)$ with pseudo-Riemannian metric:

$$g = \frac{\lambda(1 + x_2^2)dx_1^2 - 2\lambda x_1x_2dx_1dx_2 + \lambda(1 + x_1^2)dx_2^2}{1 + x_1^2 + x_2^2}$$
$$\pm\left(-2x_2^2dx_3dx_4 + \left(-x_1 + \sqrt{1 + x_1^2 + x_2^2}\right)dx_3^2 + \left(x_1 + \sqrt{1 + x_1^2 + x_2^2}\right)dx_4^2\right),$$

possible signature are *(4, 0)* , *(0, 4)* , *(2, 2)*.

Type B: ($M$,$g$) is space $\mathbb{R}^4$ $(x_1,x_2,x_3,x_4)$ with pseudo-Riemannian metric:

$$g = \lambda(dx_1^2 + dx_1dx_2 + dx_2^2) + 2e^{-x_2}dx_1dx_4 + e^{-x_2}dx_2dx_4 + e^{-x_1}dx_1dx_3 + 2e^{-x_1}dx_2dx_3,$$

possible signature is *(2, 2)* .

Type C: ($M$,$g$) is space $\mathbb{R}^4$ $(x_1,x_2,x_3,x_4)$ with pseudo-Riemannian metric:

$$g = \pm(e^{2x_4}dx_1^2 + e^{-2x_4}dx_2^2) + dx_3dx_4,$$

possible signature are *(1, 3)*,*(3,1)*.

Type D: (*M*,*g*) is space $\mathbb{R}^4$ $(x_1, x_2, x_3, x_4)$ with pseudo-Riemannian metric:

$$D \quad g = (\sinh(2x_3) - \cosh(2x_3)\sin(2x_4))dx_1^2 - 2\cosh(2x_3)\cos(2x_4)dx_1dx_2 + (\sinh(2x_3) + \cosh(2x_3)\sin(2x_4))dx_2^2 + \lambda dx_3^2 - \lambda cosh^2(2x_3)dx_4^2.$$

possible signature is *(2, 2)* .

**Proof:** refer to [1]. □

**Definition(2.5):**

Let *(M,g)* be a pseudo-Riemannian manifold it call with Recurrent curvature tensor if it's curvature tensor satisfy in following relation:

$$\nabla \boldsymbol{R} = \boldsymbol{R} \otimes \omega, \qquad (1)$$

where $\boldsymbol{R}$ (0,4)-curvature tensor and $\omega$ is 1-form

## 3. Main Results

**Theorem(3.1):**

Let *(M,g)* be strictly walker manifolds of dimension four then it's metric has recurrent curvature tensor with following 1-form:

$$w_1 = w_2 = 0, \;\; w_3 = \frac{\frac{-2\partial^3 g}{\partial x_4 \partial x_3^2} + \frac{\partial^3 f}{\partial x_4^2 \partial x_3} + \frac{\partial^3 h}{\partial x_3^3}}{\frac{\partial^2 h}{\partial x_3^2} - \frac{2\partial^2 g}{\partial x_4 \partial x_3} + \frac{\partial^2 h}{\partial x_4^2}}, w_4 = \frac{\frac{-2\partial^3 g}{\partial x_4^2 \partial x_3} + \frac{\partial^3 f}{\partial x_4^3} + \frac{\partial^3 h}{\partial x_4 \partial x_3^2}}{\frac{\partial^2 h}{\partial x_3^2} - \frac{2\partial^2 g}{\partial x_4 \partial x_3} + \frac{\partial^2 h}{\partial x_4^2}},$$

**Proof:**

This theorem will prove with straightforward computation. By theorem 5.5 in [2] the only non-zero component of (0,4)-curvature tensor for walker manifold of dimension four is:

$$\boldsymbol{R}(\partial_3, \partial_4, \partial_3, \partial_4) = \frac{-1}{2}\left(\frac{\partial^2 h}{\partial x_3^2} - \frac{2\partial^2 g}{\partial x_4 \partial x_3} + \frac{\partial^2 h}{\partial x_4^2}\right),$$

In the other hand, by using [2] again the levi-civita connection for these manifolds is as follow:

$$\nabla_{\partial_3}^{\partial_3} = \frac{1}{2}\frac{\partial f}{\partial x_3}\partial_1 + \left(\frac{\partial g}{\partial x_3} + \frac{1}{2}\frac{\partial f}{\partial x_4}\right)\partial_2, \nabla_{\partial_4}^{\partial_3} = \frac{1}{2}\frac{\partial f}{\partial x_4}\partial_1 + \frac{1}{2}\frac{\partial h}{\partial x_3}\partial_2, \nabla_{\partial_4}^{\partial_3} = \left(\frac{\partial g}{\partial x_4} - \frac{1}{2}\frac{\partial h}{\partial x_4}\right)\partial_1 + \frac{1}{2}\frac{\partial h}{\partial x_4}\partial_2,$$

Now by straightforward computation, it will be seen that the curvature tensor of strictly walker manifolds satisfy in (1) □

**Corollary(3.2):**

Let *(M,g)* be strictly walker manifolds of dimension four if

$$f(x_3,x_4)=\int(\int(-\frac{\partial^2 h}{\partial x_3^2}+\frac{2\partial^2 g}{\partial x_4\partial x_3})dx_4+\ c_1x_4)dx_4+\ f_1(x_3)x_4+\ f_2(x_3)$$

Then *(M,g)* is locally symmetric. where $f_1$ and $f_2$ are arbitrary function of $x_3$.

In the during proof of theorem (3.1) we saw that if *f* has been above condition then (*M,g*) is locally symmetric.

**Theorem(3.3):**

Let $(M,g_a)$ be lorentzian oscillator group of dimension four if curvature tensor of $g_a$ satisfy in (1) then $(M,g_a)$ is locally symmetric.

**Proof:**

The levi-civita connection and non-zero components of (0,4)-curvature tensor for these manifolds by [3] are as follow:

$$\nabla_{\partial_1}^{\partial_3}=\frac{-1}{2}ax_3\partial_1+\frac{1}{2}a\partial_2,\ \nabla_{\partial_1}^{\partial_2}=\frac{-1}{2}a\partial_3,\ \nabla_{\partial_2}^{\partial_2}=-ax_3\partial_3, \nabla_{\partial_2}^{\partial_3}=(\frac{-1}{2}ax_3^2+\frac{1}{2})\partial_1+\frac{1}{2}ax_3\partial_2, \nabla_{\partial_2}^{\partial_4}=\frac{-1}{2}\partial_3,$$

$$\nabla_{\partial_3}^{\partial_4}=\frac{-1}{2}x_3\partial_1+\frac{1}{2}\partial_2,$$

$$\boldsymbol{R}(\partial_3,\partial_4,\partial_1,\partial_3)=\frac{-1}{4}a, \boldsymbol{R}(\partial_4,\partial_2,\partial_1,\partial_2)=\frac{1}{4}a, \boldsymbol{R}(\partial_3,\partial_1,\partial_1,\partial_3)=\frac{1}{4}a^2, \boldsymbol{R}(\partial_2,\partial_1,\partial_1,\partial_2)=\frac{-1}{4}a^2,$$

$$\boldsymbol{R}(\partial_3,\partial_2,\partial_2,\partial_3)=\frac{-1}{4}a^2x_3^2+\frac{3}{4}a, \boldsymbol{R}(\partial_3,\partial_1,\partial_3,\partial_2)=\frac{1}{4}a^2x_3, \boldsymbol{R}(\partial_4,\partial_3,\partial_3,\partial_2)=\frac{-1}{4}ax_3,$$

$$\boldsymbol{R}(\partial_4,\partial_3,\partial_3,\partial_4)=\frac{-1}{4}, \boldsymbol{R}(\partial_4,\partial_2,\partial_2,\partial_4)=\frac{-1}{4},$$

Now by using (1) for that these manifolds be recurrent curvature tensor, a PDE system conclude that solving it show that *{a}* must be zero and if *a=0* then these manifolds are locally symmetric □

In next theorem in addition generalized symmetric manifold we checked recurrent condition for following pseudo-Riemannian metric that are physically relevant significant.

$$g=\left(1+\frac{2e^{x_1}}{\mu}\right)(dx_1^2+\ dx_2^2+dx_3^2)+dx_4^2, \qquad (2)$$

$$g = \left(1 + \frac{2e^{x_1}}{\mu}\right)(dx_1^2 + \ dx_2^2 + dx_3^2 \ + dx_4^2). \qquad (3)$$

Where $\mu$ is real constant.

**Theorem(3.4):**

Let *(M,g)* be generalized symmetric manifold of dimension four such that it is isometric to one of type A, B, D or is pseudo-Riemannian manifold with metric (3 ,4) then there aren't 1-form such that curvature tensor of this manifolds satisfy in (1), i.e, these manifolds are not with recurrent curvature tensor.

**Proof:**

Exactly like previous theorems that's enough to checking recurrent condition (1) and by considering levi-civita connections and curvature tensor that is brought in [1] and [4] and with straightforward computation will obtain some equations that are not solvable for example for generalized symmetric manifolds of dimension four of type B, must $\frac{1}{3}e^{-x_1}$ be zero that's impossible.

$\square$

**Corollary(3.5):**

When we checked recurrent condition for pseudo-Riemannian manifold with metric (2) we understood that if we change this metric as follow also it don't admit recurrent curvature tensor

$$g = \left(1 + \frac{2e^{x_1}}{\mu}\right)(dx_1^2 - \ dx_2^2 - dx_3^2) - dx_4^2,$$

$$g = \left(1 + \frac{2e^{x_1}}{\mu}\right)(-dx_1^2 + \ dx_2^2 + dx_3^2) + dx_4^2,$$

$$g = \left(1 + \frac{2e^{x_1}}{\mu}\right)(dx_1^2 + \ dx_2^2 + dx_4^2) + dx_3^2,$$

$$g = \left(1 + \frac{2e^{x_1}}{\mu}\right)(dx_1^2 + \ dx_3^2 + dx_4^2) + dx_2^2,$$

## 4. Conclusion

strictly walker manifolds of dimension four are manifolds with recurrent curvature.

lorentzian oscillator group of dimension four with recurrent curvature tensor are locally symmetric.

generalized symmetric manifold with recurrent curvature tensor of dimension four is locally symmetric and isometric to type C .

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# More accurate numerical radius inequalities

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## Abstract

In this paper, first we generalize the definition of Aluthge transform for non-negative continuous functions $f, g$ such that $f(x)g(x)=x\,\,(x\geq0)$. Then, by using this definition, we get some numerical radius inequalities.

**Keywords: Aluthge transform, Numerical radius, Operator matrices, Polar decomposition.**

## 1. Introduction

Let $B(H)$ denote the $C^*$ −algebra of all bounded linear operators on a complex Hilbert space $H$ with an inner product $\langle \ .,. \ \rangle$ and the corresponding norm $||\ .\ ||$. The numerical radius of $T \in B(H)$ is defined by

$$w(T) := sup\{\langle\ Tx, x\ \rangle:\ x\ \in H, ||\ x\ || = 1\}.$$

It is well known that $w(.)$ defines a norm on $B(H)$, which is equivalent to the usual operator norm. In fact $½||T\ || \leq w(T)\ \leq ||\ T\ ||$ for any $T \in B(H)$. The quantity $w(T)$ is useful in studying perturbation, convergence and approximation problems as well as interactive method, etc. The classical Young inequality says that if $0 \leq s\ \leq 1$, then $a^s b^{1-s} \leq\ sa + (1-s)b\ \ (a,b > 0)$. During the last decades several generalizations, reverses, refinements and applications of the Young Inequality.

In this present talk, we refine the numerical radius inequalities and find an upper bound for the functional $w_p$.

## 2. Preliminaries

Let $\mathbb{B}(\mathcal{H})$ denotes the $C^*$-algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ with an inner product $\langle\cdot,\cdot\rangle$ and the corresponding norm $\|\cdot\|$. In the case when $\dim\mathcal{H} = n$, we identify $\mathbb{B}(\mathcal{H})$ with the matrix algebra $\mathbb{M}_n$ of all $n \times n$ matrices with entries in the complex field. For an operator $A \in \mathbb{B}(\mathcal{H})$, let $A = U|A|$ (U is a partial isometry with $kerU = range|A|^{\perp}$) be the polar decomposition of $A$. The Aluthge transform of the operator $A$, denoted by $\widetilde{A}$, is defined as $\widetilde{A} = |A|^{\frac{1}{2}} U |A|^{\frac{1}{2}}$. In [6, 20], a more general notion called t-Aluthge transform has been introduced which has later been studied. This is defined for any

$0 < t \le 1$ by $\widetilde{A}_t = |A|^t U |A|^{1-t}$. Clearly, for $t = \frac{1}{2}$ we obtain the usual Aluthge transform. For the case $t = 1$, the operator $\widetilde{A}_1 = |A|U$ is called the Duggal transform of $A \in \mathbb{B}(\mathbb{H})$. For $A \in \mathbb{B}(\mathbb{H})$, we generalize the Aluthge transform of the operator $A$ to the form

$$\widetilde{A}_{f,g} = f(|A|)Ug(|A|),$$

in which $f, g$ are non-negative continuous functions such that $f(x)g(x) = x \, (x \ge 0)$. The numerical radius of $A \in \mathbb{B}(\mathbb{H})$ is defined by

$$w(A) := \sup\{|\langle Ax, x\rangle| : x \in \mathbb{H}, \|x\| = 1\}.$$

It is well known that $w(\cdot)$ defines a norm on $\mathbb{B}(\mathbb{H})$, which is equivalent to the usual operator norm $\|\cdot\|$. In fact, for any $A \in \mathbb{B}(\mathbb{H})$, $\frac{1}{2}\|A\| \le w(A) \le \|A\|$; see [7]. Let $r(\cdot)$ denote the spectral radius. It is well known that for every operator $A \in \mathbb{B}(\mathbb{H})$, we have $r(A) \le w(A)$. An important inequality for $\omega(A)$ is the power inequality stating that $\omega(A^n) \le \omega(A)^n \, (n = 1, 2, \cdots)$. For further information about the numerical radius we refer the reader to [9, 10, 11] and references therein. The quantity $w(A)$ is useful in studying perturbation, convergence and approximation problems as well as integrative methods, etc. For more information see [3, 5, 8, 12, 13, 14, 16].

Let $A, B, C, D \in \mathbb{B}(\mathbb{H})$. The operator matrices $\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$ and $\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$ are called the diagonal and off-diagonal parts of the operator matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$, respectively.

In [15], it has been shown that if $A$ is an operator in $\mathbb{B}(\mathbb{H})$, then

$$w(A) \le \frac{1}{2}\left(\|A\| + \|A^2\|^{\frac{1}{2}}\right). \tag{1.1}$$

Several refinements and generalizations of inequality (1.1) have been given; see [4, 1, 20, 21, 22, 23]. Yamazaki [21] showed that for $A \in \mathbb{B}(\mathbb{H})$ and $t \in [0,1]$ we have

$$w(A) \le \frac{1}{2}\left(\|A\| + w(\tilde{A}_t)\right). \tag{1.2}$$

Davidson and Power [6] proved that if $A$ and $B$ are positive operators in $\mathbb{B}(\mathbb{H})$, then

$$\|A + B\| \le \max\{\|A\|, \|B\|\} + \|AB\|^{\frac{1}{2}}. \tag{1.3}$$

Inequality (1.3) has been generalized in [2, 19] and improved in [17, 18]. In [19], the author extended this inequality to the form

$$\|A + B^*\| \le \max\{\|A\|, \|B\|\} + \frac{1}{2}\left(\| \, |A|^t |B^*|^{1-t} \| + \| \, |A^*|^{1-t} |B|^t \|\right), \tag{1.4}$$

in which $A, B \in \mathbb{B}(\mathbb{H})$ and $t \in [0,1]$.

## 3. Main Results

We are ready to present our first result. The following theorem shows a generalization of inequality (1.2).

**Theorem 1** *Let* $A \in \mathbb{B}(\mathbb{H})$ *and* $f, g$ *be two non-negative continuous functions on* $[0,\infty)$ *such that* $f(x)g(x) = x \, (x \ge 0)$. *Then, for all non-negative and non-decreasing convex function* $h$ *on* $[0,\infty)$, *we have*

$$h\left(w(A)\right) \le \frac{1}{4}\left\| h\left(g^2(|A|)\right) + h\left(f^2(|A|)\right)\right\| + \frac{1}{2} h\left(w\left(\tilde{A}_{f,g}\right)\right).$$

**Theorem 2** *Let* $A, B \in \mathbb{B}(\mathbb{H})$, $f, g$ *be two non-negative continuous functions on* $[0,\infty)$ *such that* $f(x)g(x) = x \, (x \ge 0)$ *and* $s \ge 1$. *Then*

$$w^s\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \le \frac{1}{4}\max\left(\left\| g^{2s}(|A|) + f^{2s}(|A|)\right\|, \left\| g^{2s}(|B|) + f^{2s}(|B|)\right\|\right)$$

$$+ \frac{1}{4}\left(\|f(|B|)g(|A^*|)\|^s + \|f(|A|)g(|B^*|)\|^s\right).$$

**Theorem 3** *Let* $A \in \mathbb{B}(\mathbb{H})$ *and* $f, g, h$ *be non-negative and non-decreasing continuous functions on* $[0,\infty)$ *such that* $f(x)g(x) = x \, (x \geq 0)$ *and* $h$ *is convex. Then*

$$h(w(A)) \leq \frac{1}{2}\left(h\left(w\left(\tilde{A}_{f,g}\right)\right) + \|h(|A|)\|\right).$$

**Remark 4** *For the special case* $f(x) = x^t$ *and* $g(x) = x^{1-t}$ $(t \in [0,1])$, *we obtain the inequality (1.2)*

$$w(A) \leq \frac{1}{2}\left(w\left(\tilde{A}_t\right) + \|A\|\right),$$

where $A \in \mathbb{B}(\mathbb{H})$.

Let $T = \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$. Using Theorem 2.10, we get the following result.

**Corollary 5** *Let* $A, B \in \mathbb{B}(\mathbb{H})$ *and* $f, g$ *be two non-negative and non-decreasing continuous functions such that* $f(x)g(x) = x \, (x \geq 0)$. *Then*

$$2w^s\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \leq \max\{\|A\|^s, \|B\|^s\} + \frac{1}{2}\left(\left\|f(|B|)g(|A^*|)\right\|^s + \left\|f(|A|)g(|B^*|)\right\|^s\right),$$

where $s \geq 1$.

**Corollary 6** *Let* $A, B \in \mathbb{B}(\mathbb{H})$ *and* $f, g$ *be two non-negative and non-decreasing continuous functions on* $[0,\infty)$ *such that* $f(x)g(x) = x \, (x \geq 0)$. *Then*

$$\|A + B\| \leq \max\{\|A\|, \|B\|\} + \frac{1}{2}\left(\|f(|B|)g(|A|)\| + \|f(|A^*|)g(|B^*|)\|\right).$$

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**3-5 July 2019, Istanbul, Turkey**

# B-Spline wavelets for solving Troesch problem

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**Abstract**

A new and applicable approach based on cubic B-spline wavelets and spectral methods is applied for solving a special case of strongly nonlinear two-point boundary value problems, namely Troesch problem. The purposed method is devoted to application of cubic B-spline wavelets and their operational matrix of derivative via Galerkin and collocation methods to approximate the numerical solution of Troesch equation. Comparison the results of presented method with the results of some other exiting methods for solving this kind of equations, show the high accuracy and efficiency of suggested scheme.

**Keywords:** Troesch problem, cubic B-spline wavelets, spectral methods, operational matrices

## 1. Introduction

Troesch problem arises from a system of nonlinear ordinary differential equations which occur in the theory of gas porous electrodes [1] and investigation of the confinement of a plasma column by radiation pressure [2]. Troesch problem is a special nonlinear two point boundary value problem, defined as

$$\nu''(x) - \lambda \, sinh\big(\lambda \; \nu(x)\big) = 0, \qquad 0 \le x \le 1, \tag{1}$$

$$\nu(0) = 0, \qquad \nu(1) = 1, \tag{2}$$

where Troesch parameter $\lambda$ is a positive constant. Existence of the solution of (1)-(2) is proved in [3] for $\lambda < 5$. The closed form solution to this problem in terms of the Jacobian elliptic function has been given [4] as

$$\nu(x) = \frac{2}{\lambda} \, Sinh^{-1}\left(\frac{\nu'(0)}{2} sc(\lambda\, x | 1 - \frac{1}{4}\nu'(0)^2)\right),$$

where $\nu'(0) = 2\sqrt{1-\mu}$, and $\mu$ satisfies the transcendental $sc(\lambda|\mu) = \frac{Sinh\left(\frac{\lambda}{2}\right)}{\sqrt{1-\mu}}$, where the Jacobian elliptic function $sc(\lambda \mid \mu) = \tan(\alpha)$, and $\alpha$ and $\lambda$ are related through the integral

$$\lambda = \int_0^{\alpha} \frac{1}{Cosh\,\theta - \mu} d\theta.$$



It is clear that $\nu(x)$ has a singularity located at a pole of $sc(\lambda\, x|\mu)$ or approximately at

$x_s = \frac{1}{\lambda} Ln\left(\frac{8}{\nu'(0)}\right)$ [5], which implies that for $\nu'(0) > 8e^{-\lambda}$ the singularity lies within the integration range. Because of difficulty in direct solving of this equation, many researchers have paid considerable attention for numerical solving of this equation; see [4], [6]-[10].

In this paper we apply collocation and Galerkin methods via operational matrix of derivative based on cubic B-spline wavelets for reducing the main equation to some algebraic equation.

## 2. Cubic B-spline wavelets

The basic concepts and preliminaries of the wavelet theory and multi resolution analysis are given in literature [11]-[12]. Cubic B-spline scaling function is defined as

$$\rho_4(x) = \frac{1}{6}\begin{cases} x^3, & x \in [0,1) \\ -3x^3 + 12x^2 - 12x + 4, & x \in [1,2) \\ 3x^3 - 24x^2 + 60x - 44, & x \in [2,3) \\ (4-x)^3, & x \in [3,4) \\ 0, & otherwise \end{cases}$$

### 2.1. Boundary scaling adaptation

We define the boundary near functions at the left boundary by

$$\varrho_{3,k}(x) = \rho_4(8x - k)\chi_{[0,1]}(x), \qquad k = -3,-2,-1, \tag{3}$$

and for other levels of $m$, we have

$$\varrho_{m,k}(x) = \rho_4(2^m x - k)\chi_{[0,1]}(x), \qquad k = -3,-2,-1, \qquad m = 4,5,\dots . \tag{4}$$

For the right end of the interval, by symmetry we have the following relations

$$\varrho_{3,5}(x) = \rho_{3,-1}(1-x),\ \ \varrho_{3,6}(x) = \rho_{3,-2}(1-x),\ \ \varrho_{3,7}(x) = \rho_{3,-3}(1-x), \tag{5}$$

and for other levels of $m$, we have

$$\varrho_{m,2^{m-k-3}}(x) = \rho_{3,k}(2^m x - k),\ \ k = -3,-2,-1,\ \ m = 4,5,\dots. \tag{6}$$

### 2.2. Interior scalings

Interior cubic B-spline scaling functions are chosen as

$$\varrho_{3,k}(x) = \rho_4(8x - k)\chi_{[0,1]}(x), \qquad k = 0,1,2,3,4, \tag{7}$$

and for other levels of $m$, we get

$$\varrho_{m,k}(x) = \rho_4(2^m x - k)\chi_{[0,1]}(x), \qquad k = 0,1,\dots,2^m - 4, \qquad m = 4,5,\dots . \tag{8}$$

Two scale dilation equation for cubic B-spline wavelet is given by

$$\varpi_4(x) = \sum_{k=0}^{10} \frac{(-1)^k}{8} \sum_{l=0}^{4} \binom{4}{l} \rho_8(k - l + 1)\rho_4(2x - k). \tag{9}$$

Other inner and boundary wavelets are constructed similarly as in [13].

## 2.3. Function approximation

We can use cubic B-spline wavelets as basic functions for representing any function $\nu(x) \in L^2(R)$ as

$$\nu(x) = \sum_{i=-3}^{2^m-1} \eta_{m,i}\varrho_{m,i}(x) + \sum_{j=m}^{\infty}\sum_{k=-3}^{2^j-4} \zeta_{j,k}\varpi_{j,k}(x), \tag{10}$$

where $\varrho_{m,i}$ and $\varpi_{j,k}$ are scaling functions and wavelets, respectively. By truncating the infinite series in equation (10) we can rewrite

$\nu($

where $\theta$ and $\Lambda$ are $2^{M+1} + 3$ column vectors given by

$$\theta = \left(\eta_{m,-3},\dots,\eta_{m,2^m-1},\zeta_{m,-3},\dots,\zeta_{M,2^M-4}\right)^T, \tag{12}$$

$$\Lambda = \left(\varrho_{m,-3},\dots,\varrho_{m,2^m-1},\varpi_{m,-3},\dots,\varpi_{M,2^M-4}\right)^T, \tag{13}$$

with

$$\eta_{m,i} = \int_0^1 \nu(x)\,\tilde{\varrho}_{m,i}(x)\,dx, \quad i = -3,\dots,2^{j_0} - 1,$$

$$\zeta_{j,k} = \int_0^1 \nu(x)\tilde{\varpi}_{j,k}(x)dx\,, \quad j = m,\dots,M, \;\; k = -3,\dots,2^M - 4,$$

where $\tilde{\varrho}_{m,i}$ and $\tilde{\varpi}_{j,k}$ are dual functions of $\varrho_{m,i}$ and $\varpi_{j,k}$, respectively, which can be made by linear combinations of $\varrho_{m,i}$ and $\varpi_{j,k}$ [13].

### 2.4. Operational matrices of product and derivative

Put

$$\varrho(x) = \left(\varrho_{m,-3}(x), \varrho_{m,-2}(x), \ldots, \varrho_{m,2^m-4}(x)\right),$$

$$\varpi(x) = \left(\varpi_{m,-3}(x), \ldots, \varpi_{m,2^m-4}(x), \ldots, \varpi_{M,-3}(x), \ldots, \varpi_{M,2^M-4}(x)\right).$$

the product matrices of vectors $\varrho$ and $\varpi$ are defined as

$$\int_0^1 \varrho(x)\varrho^T(x)\,dx = \Pi_1, \qquad \int_0^1 \varpi(x)\varpi^T(x)\,dx = \Pi_2,$$

and the product matrix of B-spline wavelets is defined as

$$\Pi = \begin{pmatrix} \Pi_1 & 0 \\ 0 & \Pi_2 \end{pmatrix}.$$

The derivative of vector $\Lambda$ in equation (13) can be expressed as $\Lambda'(x) = D\,\Lambda(x)$, where $D$ is $(2^{M+1}+3)$-dimensional square operational matrix of derivative for cubic B-spline wavelets on $[0,1]$. The matrix $D$ can be obtained by considering

$$D = \int_0^1 \Lambda'(x)\widetilde{\Lambda}^T(x)\,dx = E\,(\Pi^{-1})^T,$$

where $E$ $(2^{M+1}+3)$-dimensional square matrix defined as follows

$$E = \int_0^1 \Lambda'(x)\Lambda^T(x)\,dx = \begin{pmatrix} E_1 & E_2 \\ E_3 & E_4 \end{pmatrix}$$

$$E_1 = \left(\int_0^1 \varrho'_{m,i}(x)\varrho_{m,r}(x)dx\right)_{i,r}, \qquad E_2 = \left(\int_0^1 \varpi'_{j,k}(x)\varrho_{m,r}(x)dx\right)_{j,k,r},$$

$$E_3 = \left(\int_0^1 \varrho'_{m,i}(x)\varpi_{l,s}(x)dx\right)_{i,l,s}, \qquad E_4 = \left(\int_0^1 \varpi'_{j,k}(x)\varpi_{l,s}(x)dx\right)_{j,k,l,s},$$

and the subscripts $i, r, k, j, l$ and $s$ assume values as given below

$$i, r = -3, -2, \ldots, 2^m - 1, \qquad k, s = -3, \ldots, 2^j - 4, \qquad j, l = m, \ldots, M.$$

## 3. Numerical method

In this section we put $\omega(x) = Sinh(\lambda\, \nu(x))$ and by using cubic B-spline operational matrix of derivative we can write

$$\nu(x) = \theta_\nu^T \Lambda(x), \qquad \nu''(x) = \theta_\nu^T D^2 \Lambda(x), \qquad \omega(x) = \theta_\omega^T \Lambda(x). \tag{16}$$

Substituting equations (16) in the equation (1) we have

$$\theta_\nu^T D^2 \Lambda(x) - \lambda\, \theta_\omega^T \Lambda(x) = 0, \tag{17}$$

now we apply Galerkin approach via cubic B-spline wavelets as weighting functions for solving equation (17)

$$\int_0^0 (\theta_\nu^T D^2 - \lambda\, \theta_\omega^T)\, \Lambda(x)\Lambda^{\mathrm{T}}(x) dx = (\theta_\nu^T D^2 - \lambda\, \theta_\omega^T)\Pi = 0,$$

therefore $(\theta_\nu^T D^2 - \lambda\, \theta_\omega^T) = 0$, in the obtained system there are $2 \times (2^{M+1} + 3)$ unknowns and $(2^{M+1} + 3)$ equations. The difference between numbers of equations and number of unknowns is removable by using the relation between $\nu(x)$ and $\omega(x)$:

$$\omega\big(\theta_\nu^T \Lambda(x)\big) = Sinh\big(\lambda\theta_\nu^T \Lambda(x)\big) = \theta_\omega^T \Lambda(x). \tag{18}$$

By collocating the equation (18) in the points

$$x_j = \frac{j}{2^{M+1} + 3}, \qquad j = 1, 2, \dots, 2^{M+1} + 1,$$

we get

$$Sinh\left(\lambda\theta_\nu^T \Lambda\big(x_j\big)\right) = \theta_\omega^T \Lambda\big(x_j\big), \qquad j = 1,2, \dots, 2^{M+1} + 1,$$

and applying the boundary conditions

$$\theta_\nu^T \Lambda(0) = 0, \qquad \theta_\nu^T \Lambda(1) = 1.$$

So we get a system of $2 \times (2^{M+1} + 3)$ equations with $2 \times (2^{M+1} + 3)$ unknowns, which can be solved easily.

## 4. Test problem

Using purposed method, we solve nonlinear boundary value problem (1)-(2) for start level $m = 3$ and scales $M = 4, 5$ and different values of Troesch parameter $\lambda$ and compare our findings with results of some other exiting methods. The attained results for $\lambda = 0.5$ are given in table 1 and are compared with the solutions of decomposition method (DM) of [4]. Table 2 contains the numerical solutions for scales $M = 4, 5$ and Troesch parameter $\lambda = 1$ and also the results of coupled Laplace Transform and modified decomposition method (LT-MDM) of [8]. In table 3 numerical solution of problem (1)-(2) for $\lambda = 5$ are given in some arbitrary points and compared with the results of Fortran code (FD) [14] and B-spline collocation method (B-SCM) [14].

Table 1. Approximated solutions for $\lambda = 0.5$

| $x$ | $M = 4$ | $M = 5$ | DM [4] | Exact |
|---|---|---|---|---|
| 0 | 0.000000 | 0.000000 | 0.000000 | 0.000000 |
| 0.2 | 0.196057 | 0.186122 | 0.1921352 | 0.1906339 |
| 0.4 | 0.393094 | 0.378081 | 0.3861955 | 0.3835229 |
| 0.6 | 0.592097 | 0.576957 | 0.5841442 | 0.5810020 |
| 0.8 | 0.794063 | 0.783875 | 0.7880234 | 0.7855718 |
| 1 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 |

Table 2. Approximated solutions for $\lambda = 1$

| $x$ | $M = 4$ | $M = 5$ | LT-MDM [8] | Exact |
|---|---|---|---|---|
| 0.1 | 0.0930076 | 0.0852337 | 0.0846631 | 0.0817969 |
| 0.3 | 0.2736864 | 0.2591218 | 0.2573995 | 0.2491673 |
| 0.5 | 0.4565536 | 0.4434094 | 0.4406094 | 0.4283471 |
| 0.7 | 0.6529541 | 0.6454926 | 0.6421421 | 0.6289711 |
| 0.9 | 0.8750744 | 0.8734816 | 0.8713749 | 0.8639700 |

**Table 3. Approximated solutions for $\lambda = 5$**

| $x$ | $M = 4$ | $M = 5$ | FD [14] | B-SCM [14] |
|---|---|---|---|---|
| 0.1 | 0.0076520 | 0.0095523 | -- | -- |
| 0.2 | 0.0110199 | 0.0109043 | 0.01075342 | 0.01002027 |
| 0.3 | 0.0213986 | 0.0250004 | -- | -- |
| 0.4 | 0.0314220 | 0.0328037 | 0.0332005 | 0.0309979 |
| 0.5 | 0.0537343 | 0.0720751 | -- | -- |
| 0.6 | 0.0114229 | 0.9600047 | -- | -- |
| 0.7 | 0.1349407 | 0.1579514 | -- | -- |
| 0.8 | 0.2496878 | 0.2600308 | 0.2582166 | 0.2417049 |
| 0.9 | 0.4369226 | 0.4544960 | 0.4550603 | 0.4246183 |
| 1 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 |

## 5. Conclusion

In this paper a new and applicable method is implemented on a class of strongly nonlinear differential equations. The purposed method is based on cubic B-spline wavelets spectral methods. Common property of the presented method was reducing the nonlinear equation to a system of algebraic equations. Comparison between our results and the findings of some other exiting methods for solving this kind of problems, shows the high accuracy and efficiency of the methods. Because of some significant properties of B-spline wavelets, such as semi orthogonality, having compact support and vanishing moments, the operational matrices are so sparse and consequently relevant required computational time and memory is so low. The presented method is attractive and can be extended for similar high order nonlinear differential equations even fractional order with little additional work.

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# Fractional Scaling Functions for Solving Fractional Pantograph Equation

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**Abstract**

In this paper cubic B-spline scaling fuvctions and their operational matrix of fractional integration are applied for solving a special case of fractional delay differential equations, namely pantograph equation. Comparison the results of presented method with the exact solution of main problem in spacial cases, show the high accuracy and efficiency of suggested scheme.

**Keywords:** delay differential equations, fractional calculus, cubic B-spline scaling functions, operational matrix of fractional integration,

## 1. Introduction

Delay differential equations (DDEs) have numerous applications in mathematical modeling, such as; physiological and pharmaceutical kinetics, chemical kinetics, navigational control of ships and aircrafts, population dynamics, and infectious diseases [1]. They arise when the rate of change of a time-dependent process in its mathematical modeling is not only determined by its present state but also by a certain past state. The pantograph equation is one of the most important kinds of DDEs and plays an important role in explaining many different phenomena [2]. Fractional delay differential equation (FDDE) is a generalization of the delay differential equation to arbitrary non-integer order. FDDEs have also been in center of attention of some authors and been played a significant role in modeling of many real areas of sciences such as physics, engineering, biology, medicine, and economics. FDDEs often cannot be solved analytically so the approximate and numerical methods should be adapted to solve this type of equations. During the last decades, several methods have been used to solve this kind of problems such as; Adams-Bashforth-Moulton algorithm [3], Bernoulli wavelet method [4], Shifted Chebyshev approximations [5], Legendre multiwavelet collocation method [6], Finite difference method [7] and the references therein.

The main aim of this research is evaluating the numerical solution of fractional delay differential equation, well known as pantograph equation, in the following form

$$D^{\beta}\,\xi(t)=\nu(t)\xi(t)+\sum_{j=1}^{m}\omega_j\,(t)D^{\beta_j}\xi\left(\rho_j t\right),\quad 0\le t\le\,T, \tag{1}$$

$$\xi^{(k)}(t) = \lambda_k, \;\; k = 0,1,\dots,n-1, \tag{2}$$

where

$$n-1 < \beta \le n, \quad 0 \le \beta_j < \beta, \quad 0 < \rho_j < 1, \quad j = 1,2,\dots,m.$$

For this aim, cubic B-spline scaling functions and their operational matrix of fractional integration is utilized via collocation method to transform the fractional delay differential equation to some algebraic system.

## 2. Preliminaries of fractional calculus

In this section we briefly present some definitions and results in fractional calculus for our subsequent discussion [8].

**Definition 1.** Fractional integral operator of order $\alpha > 0$ in the means of Riemann Liouville is defined as

$$J^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{u(s)}{(t-s)^{1-\alpha}}\,ds. \tag{3}$$

**Definition 2.** Fractional derivative operator of order $\alpha > 0$ in Caputo means is defined as

$$D^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{u^{(n)}(s)}{(t-s)^{\alpha-n+1}}\,ds, \quad n-1 < \alpha \le n, \quad n \in N. \tag{4}$$

For constants $\alpha$, $\beta$, $\nu$, $\lambda$, $\mu$ and $C$, Riemann Liouville fractional integral operator and Caputo fractional derivative operator are subject to the following conditions:

i. $J^{\alpha}u(t)J^{\beta}u(t) = J^{\alpha+\beta}u(t)$,

ii. $J^{\alpha}x^{\nu} = \frac{\Gamma(\nu+1)}{\Gamma(\alpha+\nu+1)}x^{\alpha+\nu}$ ,

iii. $D^{\alpha}\, x^{\beta} = \begin{cases} 0; & \alpha \in N, \;\; \beta < \alpha \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}x^{\beta-\alpha}; & \beta \ge \alpha, \end{cases}$

iv. $D^{\alpha}J^{\alpha}u(t) = u(t)$,

v. $J^{\alpha}D^{\alpha}u(t) = u(t) - \sum_{j=0}^{n-1} u^{(j)}(0)\frac{t^j}{j!}$.



## 3. Cubic B-Spline scaling functions

Cubic B-spline scaling function $\phi_4(x)$ is given by [9]

$$\phi_4(x) = \left(\frac{1}{6}\sum_{k=0}^{4}\binom{4}{k}(-1)^k(x-k)_+^3\right).\chi_{[0,4]}(x), \tag{5}$$

$$x_+^n = \begin{cases} x^n; & x > 0, \\ 0; & x \le 0\,, \end{cases}$$

and its *two-scale dilation equation* defined as follows

$$\phi_4(x) = \sum_{k=0}^{4}\frac{1}{8}\binom{4}{k}\phi_4(2x-k).$$

Let $\varphi_{j,k}(x) = \phi_4\left(2^j x - k\right),\ \ k, j \in Z$. $\varphi_{j,k}$ are given by

$$\varphi_{j,k}(x) = \frac{2^{3j}}{6}\begin{cases} \gamma(x)^3, & x \in \left[\frac{k}{2^j},\frac{k+1}{2^j}\right) \\ -3\gamma(x)^3 + 12\gamma(x)^2 - 12\gamma(x) + 4, & x \in \left[\frac{k+1}{2^j},\frac{k+2}{2^j}\right) \\ 3\gamma(x)^3 - 24\gamma(x)^2 + 60\gamma(x) - 44, & x \in \left[\frac{k+2}{2^j},\frac{k+3}{2^j}\right) \\ -\left(\gamma(x) - \frac{4}{2^j}\right)^3, & x \in \left[\frac{k+3}{2^j},\frac{k+4}{2^j}\right) \\ 0, & otherwise \end{cases} \tag{6}$$

where $\gamma(x) = x - \frac{k}{2^j}$. Scaling functions can be used to expand any function in $L^2(R)$.

### 3.1. Function approximation

A function $f(x)$ defined over [0,1] may be approximated by cubic B-spline scaling functions as

$$f(x) = \sum_{k=-3}^{2^j-1} c_k\,\varphi_{j,k}(x) = C^T\Phi_{\mathrm{j}}(x), \tag{7}$$

where

$$C = \left(c_{-3},\dots,c_{2^j-1}\right)^T, \qquad \Phi_{\mathrm{j}} = \left(\phi_{j,-3},\dots,\phi_{j,2^j-1}\right)^T,$$

$$c_k = \int_0^1 f(x)\,\tilde{\phi}_{j,k}(x)dx, \quad k = -3,\dots,2^j-1,$$

and $\tilde{\phi}_{j,k}$ are dual functions of $\varphi_{j,k}$ which can be obtained by linear combinations of $\varphi_{j,k}$.

Let

$$\int_0^1 \Phi_j(x)\Phi_j^T(x)dx = \Pi_j, \tag{8}$$

where $\Pi_j$ is $\left(2^j+3\right)\times\,(2^j+3)$ matrix. Suppose $\widetilde{\Phi}_j(x)$ is the dual function of $\Phi_j(x)$, given by

$$\widetilde{\Phi}_j(x) = \left(\tilde{\phi}_{j,-3}(x), \tilde{\phi}_{j,-2}(x), \dots, \tilde{\phi}_{j,2^j-1}(x)\right)^T.$$

Using equation (8) we have

$$\int_0^1 \widetilde{\Phi}_j(x)\Phi_j^T(x)dx = \mathrm{I}_{2^j+3},$$

So we get

$$\widetilde{\Phi}_j = \Pi_j^{-1}\Phi_j. \tag{9}$$

**3.2. Fractional integration of cubic B-spline scaling function**

In this section using Laplace transform, we evaluate the Riemann-Liouville fractional integration of cubic B-spline scaling functions. For this purpose, Suppose $u(x)$ be Heaviside step function, we can rewrite $\varphi_{j,k}$ as follows

$$\varphi_{j,k}(x) = \frac{2^{3j}}{6}\sum_{l=0}^{4}(-1)^l\binom{4}{l}\left(x-\frac{k+l}{2^j}\right)^3 u\left(x-\frac{k+l}{2^j}\right), \tag{10}$$

taking Laplace transform of $\varphi_{j,k}(x)$, we get

$$\mathcal{L}\left(\varphi_{j,k}(x)\right) = \frac{2^{3j}}{s^3}\sum_{l=0}^{4}(-1)^l\binom{4}{l}Exp\left(-\frac{k+l}{2^j}s\right).$$

Now using convolution theorem for Laplace transform and definition of Riemann-Loiuville fractional integration we have

$$\mathcal{L}\left(J^\alpha\varphi_{j,k}(x)\right) = \frac{1}{\Gamma(\alpha)}\mathcal{L}(x^{\alpha-1})\mathcal{L}\left(\varphi_{j,k}(x)\right) = \frac{2^{3j}}{s^{3+\alpha}}\sum_{l=0}^{4}(-1)^l\binom{4}{l}Exp\left(-\frac{k+l}{2^j}s\right). \tag{11}$$

It is clear that by taking inverse Laplace transform of equation (9), we get

$$J^{\alpha}\varphi_{j,k}(x)=\frac{2^{3j}}{\Gamma(3+\alpha)}\sum_{l=0}^{4}(-1)^{l}\binom{4}{l}\left(x-\frac{k+l}{2^{j}}\right)_{+}^{3}.$$

### 3.3. Operational matrix of fractional integration

Suppose that $\boldsymbol{J}^{\alpha}$ be the operational matrix of fractional integration, that is

$$J^{\alpha}\Phi_{\mathrm{j}}(x)=\boldsymbol{J}^{\alpha}\Phi_{\mathrm{j}}(x), \tag{12}$$

multiplying equation (10) in $\widetilde{\Phi}_{j}^{T}(x)$ and integrating from 0 to 1 and using equation (9) we get

$$\boldsymbol{J}^{\alpha}=\left(\int_{0}^{1}J^{\alpha}\Phi_{j}\ (x){\Phi^{T}}_{j}(x)dx\right)\Pi^{-1}=\Theta_{j}^{\alpha}\Pi^{-1}$$

where

$$\Theta_{j}^{\alpha}=\left(\int_{0}^{1}J^{\alpha}\varphi_{j,k}\ (x)\varphi_{j,l}\ (x)dx\right)_{k,l},\qquad k,l=-3,-2,\ldots,3.$$

### 4. Numerical Implementation

In this section, pantograph fractional differential equation is solved using cubic B-spline scaling functions. First we represent the fractional derivative of unknown function by cubic B-spline scaling functions as equation (1)

$$D^{\beta}\xi(t)=C^{T}\Phi_{j}(t), \tag{13}$$

applying property vii of fractional calculus, we get

$$J^{\beta}D^{\beta}\xi(t)=\xi(t)-\sum_{i=0}^{n-1}\frac{\xi^{(i)}\ (0)}{i!}t^{i}, \tag{14}$$

also

$$J^{\beta}\left(C^{T}\Phi_{j}(t)\right)=C^{T}J^{\beta}\left(\Phi_{j}(t)\right)=C^{T}\boldsymbol{J}^{\beta}\left(\Phi_{j}(t)\right), \tag{15}$$

considering equation (14)-(15), we have

$$\xi(t)=C^{T}\boldsymbol{J}^{\beta}\left(\Phi_{j}(t)\right)+\sum_{i=0}^{n-1}\frac{\xi^{(i)}\ (0)}{i!}t^{i}. \tag{16}$$

For evaluating $D^{\beta_\mu}\xi(\rho_\mu t), \mu = 0,1,\dots,m$ in equation (1), first $\xi\big(\rho_\mu t\big)$ is obtained. Using equation (16) we get

$$\xi(\rho_\mu t) = C^T \boldsymbol{J}^\beta\big(\Phi_j(\rho_\mu t)\big) + \sum_{i=0}^{n-1} \frac{\lambda_i \rho_\mu^i}{i!} t^i,$$

thus

$$D^{\beta_\mu}\xi\big(\rho_\mu t\big) = C^T D^{\beta_\mu} \boldsymbol{J}^\alpha\big(\Phi_j(\rho_\mu t\big) + \sum_{i=0}^{n-1} \frac{\lambda_i \rho_\mu^i}{i!} D^{\beta_\mu} t^i,$$

note that

$$D^{\beta_\mu}\boldsymbol{J}^\alpha\big(\Phi_j(t)\big) = D^{\beta_\mu} J^\alpha \Phi_j(t) = J^{\alpha-\beta_\mu}\left(\Phi_j(t)\right) = \boldsymbol{J}^{\alpha-\beta_\mu}\Phi_j(t),$$

therefore we get

$$D^{\beta_\mu}\xi\big(\rho_\mu t\big) = C^T \boldsymbol{J}^{\alpha-\beta_\mu}\Phi_j\big(\rho_\mu t\big) + + \sum_{i=0}^{n-1} \frac{\lambda_i \rho_\mu^i}{i!} D^{\beta_\mu} t^i. \tag{17}$$

Substituting equations (13) and (17) in (1), we have

$$\begin{aligned} C^T\Phi_j(t) &= \nu(t)\left(C^T \boldsymbol{J}^\beta\left(\Phi_j(t)\right) + \sum_{i=0}^{n-1}\frac{\lambda_i}{i!}t^i\right) \\ &+ \sum_{\mu=1}^{m}\omega_\mu(t)\left(C^T\boldsymbol{J}^{\beta-\beta_\mu}\Phi_j\big(\rho_\mu t\big)\right) + \sum_{\mu=1}^{m}\sum_{i=0}^{n-1}\omega_\mu(t)\left(\frac{\lambda_i\rho_\mu^i}{i!}D^{\beta_\mu}t^i\right), \end{aligned} \tag{18}$$

consequently the following algebraic system is obtained

$$C^T\left(\Phi_j(t) - \nu(t)\boldsymbol{J}^\beta\Phi_j(t) - \sum_{\mu=1}^{m}\omega_\mu(t)\left(\boldsymbol{J}^{\beta-\beta_\mu}\Phi_j\big(\rho_\mu t\big)\right)\right) = S(t), \tag{19}$$

where

$$S(t) = \sum_{i=0}^{n-1}\left(\nu(t)\frac{\lambda_i}{i!}t^i + \sum_{\mu=1}^{m}\omega_\mu(t)\left(\frac{\lambda_i\rho_\mu^i}{i!}D^{\beta_\mu}t^i\right)\right).$$

Now for solving the current algebraic system we collocate equation (19) in the following points

$$t_k = \frac{k}{2^j+3}, \quad k = 1,2,\dots,2^j+3.$$

The obtained system can be solved by some iterative method, in this paper we applied Newton method for evaluating the approximated solution.

## 5. Illustrative example

In this section, for showing the applicability and accuracy of the purposed method an illustrative example is solved by introduced method in section 4. Computations have been done using Wolframe Mathematica software.

**Example.** Consider the following pantograph differential equation

$$D^{\beta}\,\xi(t) = 2\xi(t) + 27\xi\left(\frac{t}{3}\right) - 3t^3 + 44t + 6, \quad 1 < \beta \le 3$$

$$\xi(0) = 0, \quad \xi'^{(0)} = -4, \quad \xi''^{(0)} = 0,$$

where the exact solution for $\beta = 3$ is $\xi(t) = t^3 - 4t$. The approximated solution for $\beta = 3$, $j = 4$ and $\alpha = 1, 1.5, 2, 3$ are evaluated and the absolute error are given in Table 1.

Table 1. The absolute errors for $\beta = 3, j = 4$ and $\alpha = 1, 1.5, 2, 3$

| $t$ | $\alpha = 1$ | $\alpha = 1.5$ | $\alpha = 2$ | $\alpha = 3$ |
|---|---|---|---|---|
| 0.0 | $1.05 \times 10^{-3}$ | $2.87 \times 10^{-4}$ | $9.05 \times 10^{-9}$ | $8.49 \times 10^{-16}$ |
| 0.4 | $2.38 \times 10^{-2}$ | $4.68 \times 10^{-4}$ | $4.20 \times 10^{-9}$ | $1.11 \times 10^{-16}$ |
| 0.8 | $5.00 \times 10^{-2}$ | $7.01 \times 10^{-3}$ | $1.03 \times 10^{-8}$ | $5.09 \times 10^{-15}$ |
| 1.2 | $2.94 \times 10^{-2}$ | $3.07 \times 10^{-3}$ | $5.51 \times 10^{-8}$ | $7.72 \times 10^{-15}$ |
| 1.6 | $3.31 \times 10^{-2}$ | $4.93 \times 10^{-2}$ | $4.73 \times 10^{-8}$ | $8.32 \times 10^{-15}$ |
| 2.0 | $6.42 \times 10^{-1}$ | $8.99 \times 10^{-2}$ | $6.95 \times 10^{-8}$ | $6.00 \times 10^{-15}$ |

## 6. Conclusion

In this paper a new and applicable approach based on cubic B-spline scaling functions is implemented for numerically solving a class of fractional delay differential equations. For this purpose fractional cubic B-spline scaling functions are constructed and consequently the operational matrix of fractional integration is made. Then by expanding the unknown functions of main problem by fractional B-spline scaling functions, the main problem is reduced to some algebraic system. Comparison between obtained results and exact solution, shows the high accuracy and efficiency of presented method.

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# $(A,\phi)-$ ALMOST STATISTICAL CONVERGENCE OF ORDER $\gamma$

EKREM SAVAŞ

ABSTRACT. The purpose of this paper is to introduce and study some properties of the almost statistical convergence of order $\gamma$, which is defined using almost convergence and the $\phi$-function. Additional we prove some inclusion theorems.

## 1. Introduction and Background

Let $s$ denote the set of all real and complex sequences $x=(x_k)$. By $l_\infty$ and $c$, we denote the Banach spaces of bounded and convergent sequences $x=(x_k)$ normed by $||x||=\sup_n|x_n|$, respectively.

If $x=(x_k)$ is a sequence and $A=(a_{nk})$ is an infinite matrix, then $Ax$ is the sequence whose $nth$ term is given by $A_n(x)=\sum_{k=0}^{\infty} a_{nk}x_k$. Thus we say that $x$ is $A$-summable to $L$ if $\lim_{n\to\infty} A_n(x)=L$. Let $X$ and $Y$ be two sequence spaces and $A=(a_{nk})$ an infinite matrix. If for each $x\in X$ the series $A_n(x)=\sum_{k=0}^{\infty} a_{nk}x_k$ converges for each $n$ and the sequence $Ax=A_n(x)\in Y$ we say that $A$ maps $X$ into $Y$. By $(X,Y)$ we denote the set of all matrices which maps $X$ into $Y$, and in addition if the limit is preserved then we denote the class of such matrices by $(X,Y)_{reg}$.

A linear functional $L$ on $l_\infty$ is said to be a Banach limit if it has the following properties:

(1) $L(x)\geq 0$ if $n\geq 0$ (i.e. $x_n\geq 0$ for all $n$),
(2) $L(e)=1$ where $e=(1,1,\ldots)$,
(3) $L(Dx)=L(x)$, where the shift operator $D$ is defined by $D(x_n)=\{x_{n+1}\}$.

Let $B$ be the set of all Banach limits on $l_\infty$. A sequence $x\in\ell_\infty$ is said to be almost convergent if all of its Banach limits coincide. Let $\hat{c}$ denote the space of almost convergent sequences .

Lorentz [8] has shown that

$$\hat{c}=\left\{x\in l_\infty : \lim_m t_{m,n}(x) \text{ exists uniformly in } n\right\}$$

where

$$t_{m,n}(x)=\frac{x_n+x_{n+1}+x_{n+2}+\cdots+x_{n+m}}{m+1}.$$

The space $[\hat{c}]$ of strongly almost convergent sequences was introduced by Maddox [9] and also independently by Freedman et al [5] as follows :

$$[\hat{c}]=\left\{x\in l_\infty : \lim_m t_{m,n}(|x-L|)=0, \text{ uniformly in } n, \text{ for some } L\right\}.$$

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2010 *Mathematics Subject Classification.* Primary 40H05; Secondary 40C05.
*Key words and phrases.* $\phi$-function, almost convergence of order $\gamma$, almost statistical convergence of order $\gamma$.

Let $\lambda = (\lambda_i)$ be a non-decreasing sequence of positive numbers tending to $\infty$ such that

$$\lambda_{i+1} \leq \lambda_i + 1, \lambda_1 = 1.$$

Waszak [17] defined the lacunary strong $(A, \phi)-$ convergence with respect to a modulus function.

## 2. Main results

The idea of statistical convergence of a sequence was introduced by Fast [4] (see also Schoenberg [16]) . Statistical convergence was generalized by Buck [2] and studied by other authors, using a regular nonnegative summability matrix $A$ in place of Cesàro matrix.

The idea of convergence of a real sequence was extended to statistical convergence by Fast [4] ( see also Schoenberg [16] ) as follows : If $\mathbb{N}$ denotes the set of natural numbers and $E \subset \mathbb{N}$ then $E(m, n)$ denotes the cardinality of the set $E \cap [m, n]$. The upper and lower natural density of the subset $E$ is defined by

$$\overline{d}(E) = \lim_{n\to\infty} \sup \frac{E(1, n)}{n} \text{ and } \underline{d}(E) = \lim_{n\to\infty} \inf \frac{E(1, n)}{n}.$$

If $\overline{d}(E) = \underline{d}(E)$ then we say that the natural density of $E$ exists and it is denoted simply by $d(E)$. Clearly $d(E) = \lim_{n\to\infty} \frac{E(1, n)}{n}$.

Statistical convergence turned out to be one of the most active areas of research in summability theory after the work of Fridy [6] and Šalát [12]. In another direction, a new type of convergence called $\lambda$- statistical convergence was introduced in [11] as follows: A sequence $(x_k)$ of real numbers is said to be $\lambda$- statistically convergent to $L$ ( or, $S_\lambda$-convergent to $L$ ) if for any $\varepsilon > 0$,

$$\lim_{j\to\infty} \frac{1}{\lambda_j} |\{k \in I_j : |x_k - L| \geq \varepsilon\}| = 0,$$

where $|A|$ denotes the cardinality of $A \subset \mathbb{N}$. In [11] the relation between $\lambda$- statistical convergence and statistical convergence was established among other things.

A sequence $(x_k)$ of real numbers is said to be almost statistically convergent to $L$ if for arbitrary $\varepsilon > 0$, for all $m$, the set

$$E(\varepsilon) = \{k \in \mathbb{N} : |x_{k+m} - L| \geq \varepsilon\}$$

has natural density zero.

Recently E.Savas [13] defined almost $\lambda$-statistical convergence as follows by using the notion of $(V, \lambda)$-summability to generalize the concept of statistical convergence. A sequence $(x_k)$ of real numbers is said to be $\lambda$- almost statistically convergent to $L$ ( or, $S_\lambda$-convergent to $L$ ) if for any $\varepsilon > 0$,

$$\lim_{j\to\infty} \frac{1}{\lambda_j} |\{k \in I_j : |x_{k+m} - L| \geq \varepsilon\}| = 0, \text{ uniformly in m.}$$

If we take $\lambda_j = j$, the definition of $\lambda$- almost statistically convergent reduces to almost statistically convergent. Kolk [7] has given very interesting definition which is $A$-statistical convergence.

Assume that $A$ is a non-negative regular summability matrix. Then the sequence $x = (x_k)$ is called $A$-statistically convergent to $L$ provided that, for every $\varepsilon > 0$,

$$lim_j \sum_{n:|x_k - L| \geq \varepsilon} a_{jn} = 0$$

We denote this by $st_A - lim_n x_k = L$.

Let $A = (a_{nk})$ be the real matrix and the sequence $x = (x_k)$, the $\phi$- function $\phi(u)$ and a positive number $\varepsilon > 0$ be given. We write, for all $m$

$$K_{\lambda_j}((A,\phi),\varepsilon,m) = \{n \in I_j : \sum_{k=1}^{\infty} a_{nk}\phi(|x_{k+m}|) \geq \varepsilon\}.$$

The sequence $x$ is said to be $(A,\phi)$- statistically almost convergent of order $\gamma$, $0 < \gamma \leq 1$, to a number zero if for every $\varepsilon > 0$

$$lim_j \frac{1}{\lambda_j^{\gamma}} \mu(K_{\lambda_j}((A,\phi),\varepsilon,m)) = 0, \text{ uniformly in } m,$$

where $\mu(K^{\gamma}_{\lambda_j}((A,\phi),\varepsilon,m))$ denotes the number of elements belonging to $K^{\gamma}_{\lambda_j}((A,\gamma),\varepsilon,m)$. We denote by $\hat{s}^{\gamma}_{\lambda}((A,\phi))_0$, the set of sequences $x = (x_k)$ which are $(A,\phi)-$ almost statistical convergent to zero.

If we take $A = I$ and $\phi(x) = x$ respectively, then $\hat{s}^{\gamma}_{\lambda}((A,\phi))_0$ reduce to $(\hat{s}^{\gamma}_{\lambda})_0$.

$$(\hat{s}^{\gamma}_{\lambda})_0 = \left\{ x = (x_k) : lim_j \frac{1}{\lambda_j^{\gamma}} |\{k \in I_j : |x_{k+m}| \geq \varepsilon\}| = 0, \text{ uniformly in } m \right\}.$$

If we take $A = I$, $\phi(x) = x$ and $\gamma = 1$ respectively, then we get the following:

$$(\hat{s}_{\lambda})_0 = \left\{ x = (x_k) : lim_j \frac{1}{\lambda_j} |\{k \in I_j : |x_{k+m}| \geq \varepsilon\}| = 0, \text{uniformly in m} \right\}.$$

**Remark 1.** *(i) If*

$$a_{nk} := \{ \begin{array}{lcc} \frac{1}{n}, & if & n \geq k, \\ 0, & & otherwise. \end{array}$$

*then $\hat{s}^{\gamma}_{\lambda}((A,\phi))_0$ reduce to $\hat{s}^{\gamma}_{\lambda}((C,\phi))_0$,i.e., $(C,\phi)-$ almost statistical convergence of order $\gamma$ to zero. (ii) If for all $m$,*

$$a_{nk} := \{ \begin{array}{lcc} \frac{p_k}{P_n}, & if & n \geq k, \\ 0, & & otherwise. \end{array}$$

*then $\hat{s}^{\gamma}_{\lambda}((A,\phi))_0$ reduce to $\hat{sS}^{\gamma}_{\lambda}((N,p),\phi))_0$,i.e., $((N,p),\phi)-$ almost statistical convergence of order $\gamma$ to zero, where $p = p_k$ is a sequence of nonnegative numbers such that $p_0 > 0$ and*

$$P_n = \sum_{k=0}^{n} p_k \to \infty (n \to \infty).$$

We now have

**Theorem 2.1.** *If $\psi \prec \phi$ then $\hat{s}_\lambda^\gamma((A,\psi))_0 \subset \hat{s}_\lambda^\gamma((A,\phi))_0$.*

*Proof.* By assumption we have $\psi(|x_k|) \le b\phi(c|x_k|)$ and we have for all $i$,

$$\sum_{k=1}^{\infty} a_{nk}\psi(|x_{k+m}|) \le b\sum_{k=1}^{\infty} a_{nk}\phi(c|x_{k+m}|) \le L\sum_{k=1}^{\infty} a_{nk}(i)\phi(|x_{k+m}|)$$

for $b, c > 0$, where the constant $L$ is connected with properties of $\phi$. Thus, the condition $\sum_{k=1}^{\infty} a_{nk}\psi(|x_{k+m}|) \ge \varepsilon$ implies the condition $\sum_{k=1}^{\infty} a_{nk}\phi(|x_{k+m}|) \ge \varepsilon$ and in consequence we get

$$\mu(K_{\lambda_j}^{\gamma}((A,\phi),\varepsilon,m)) \subset \mu(K_{\lambda_j}^{\gamma}((A,\psi),\varepsilon,m))$$

and

$$lim_j \frac{1}{{\lambda_j}^{\gamma}}\mu\Big(K_{\lambda_j}^{\gamma}((A,\phi),\varepsilon,m)) \le lim_j \frac{1}{{\lambda_j}^{\gamma}}\mu(K_{\lambda_j}^{\gamma}((A,\psi),\varepsilon,m))\Big).$$

This completes the proof. □

**Theorem 2.2.** *If $0 < \alpha \le \beta \le 1$ then $\hat{s}_\lambda^\gamma(A,\phi)_0 \subset \hat{s}_\lambda^\beta(A,\phi)_0$.*

*Proof.* Let $0 < \gamma \le \beta \le 1$. Then

$$\frac{1}{\lambda_j^{\gamma}}\mu(K(A,\phi),\varepsilon,m) \le \frac{1}{\lambda_j^{\beta}}\mu(K(A,\phi),\varepsilon,m)$$

for every $\varepsilon > 0$ and finally we have that $\hat{s}_\lambda^\gamma(A,\phi)_0 \subset \hat{s}_\lambda^\beta(A,\phi)_0$. This proves the theorem. □

**Theorem 2.3.** *Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ be two sequences in $\Lambda$ such that $\lambda_n \le \mu_n$ for all $n \in \mathbb{N}$ and let $\gamma$ and $\beta$ be fixed real numbers such that $0 < \alpha \le \beta \le 1$, If*

$$\lim_{n\to\infty} \inf \frac{\lambda_n^{\gamma}}{\mu_n^{\beta}} > 0 \tag{2.1}$$

*then $\hat{s}_\mu^\gamma(A,\phi)_0 \subseteq \hat{s}_\lambda^\beta(A,\phi)_0$.*

*Proof.* Suppose that $\lambda_n \le \mu_n$ for all $n \in \mathbb{N}$ and let (2.1) be satisfied. For given $\varepsilon > 0$ we have

$$\left\{n \in J_n : \left|\sum_{k=1}^{\infty} a_{nk}\varphi(|x_{k+m}|)\right| \ge \varepsilon\right\} \supseteq \left\{n \in I_n : \left|\sum_{k=1}^{\infty} a_{nk}\varphi(|x_{k+m}|)\right| \ge \varepsilon\right\}$$

where $I_n = [n - \lambda_n + 1, n]$ and $J_n = [n - \mu_n + 1, n]$. Therefore we can write

$$\frac{1}{\mu_n^{\beta}}\left|\left\{n \in J_n : \left|\sum_{k=1}^{\infty} a_{nk}\varphi(|x_{x_{k+m}}|)\right| \ge \varepsilon\right\}\right| \ge \frac{\lambda_n^{\gamma}}{\mu_n^{\beta}}\frac{1}{\lambda_n^{\gamma}}\left|\left\{n \in I_n : \left|\sum_{k=1}^{\infty} a_{nk}\varphi(|x_{x_{k+m}|)}\right| \ge \varepsilon\right\}\right|$$

Hence $\hat{s}_\mu^\beta(A,\phi)_0 \subseteq \hat{s}_\lambda^\beta(A,\phi)_0$.

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# On the Spectrum of Dissipative Singular Differential Operators of First Order

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**Abstract**

In this paper, firstly all maximally dissipative extensions of the minimal operator generated by first order linear singular differential expression in the weighted Hilbert space of vector-functions on right semi-axis are described. Later on, the structure of spectrum set of these extensions has been researched. Then, the obtained results are supported by application.

**Keywords:** Dissipative operator, Deficiency index, Space of boundary values, Spectrum

## 1. Introduction

Operator theory is important to understand the nature of the spectral properties of an operator associated with a boundary value problem acting on a Hilbert space. To obtain such an information as is well known that the corresponding inner product is useful. A linear closed densely defined operator $T: D(T) \subset X \to X$ in a Hilbert space $X$ is called to be dissipative if and only if

$$Im\,(T\vartheta, \vartheta) \geq 0,$$

where $Im(\cdot,\cdot)$ and $D(T)$ denote the imaginary part of the inner product and the domain of the operator $T$, respectively (see [3]). If a dissipative operator has no any proper dissipative extension, then it is called maximally dissipative [3]. A direct result on dissipative operators is that their spectrum lie in the closed upper half-plane. Therefore open lower half-plane does not belong to the spectrum of $T$ . Maximally dissipative operators play a very important role in mathematics and physics. In physics, there are many interesting applications of the dissipative operators in areas like hydrodynamic, laser and nuclear scattering theories.

Note that the general theory of self-adjoint extensions of linear densely-defined closed symmetric operators in any Hilbert space was mentioned in the well-known work of Neumann [7].

The complete informations of Vishik's and Birman's investigations on the all non-negative selfadjoint extensions of a positive closed symmetric operator have been given by Fischbacher in [1]. Functional model theory of Nagy and Foias [4] is a basic method for investigation the spectral properties of dissipative operators.

The maximal dissipative extensions and their spectral analysis of the minimal operator having equal deficiency indices generated by formally symmetric differential-operator expression in the Hilbert space of vector-functions defined in one finite or infinite interval case have been researched by Gorbachuk, Gorbachuk [3] and Rofe-Beketov, Kholkin [6] in terms of generalized boundary values.

In this work, In Section 3, using the Calkin-Gorbachuk method the representation of all maximally dissipative extensions of the minimal operator generated by the first order linear symmetric differential expression with operator coefficient in the weighted Hilbert spaces of vector-functions defined in the right semi-infinite interval case is given. In section 4, the structure of the spectrum of these type extensions is investigated.

## 2. Statement of the problem

Let $X$ be a separable Hilbert space and $a \in \mathbb{R}$. In the weighted Hilbert space $L^2_{1/\kappa}(X,(a,\infty))$ of vector-functions consider the following linear differential-operator expression for first order in the form

$$l(\nu) = i\kappa(\varsigma)\nu'(\varsigma) + A\nu(\varsigma),$$

where $\kappa \in C(a,\infty)$, $\frac{1}{\kappa} \in L^1(a,\infty)$ for simplicity assumed that $A$ is a linear bounded selfadjoint operator in $X$.

In similar way in [3] the minimal $\Upsilon_0$ and maximal $\Upsilon$ operators associated with differential expression in $l(\cdot)$ in $L^2_{1/\kappa}(X,(a,\infty))$ can be constructed.

The operators $\Upsilon_0$ and $\Upsilon$ in the Hilbert space $L^2_{1/\kappa}(X,(a,\infty))$ are called minimal and maximal operators associated with differential expression $l(\cdot)$, respectively. It is clear that the operator $\Upsilon_0$ is a symmetric and $(\Upsilon_0)^* = \Upsilon$ in $L^2_{1/\kappa}(X,(a,\infty))$ . The minimal operator $\Upsilon_0$ is not maximal. Indeed, differential expression $l(\cdot)$ with boundary condition $\nu(a) = \nu(\infty)$ generates a dissipative extension of $\Upsilon$.

The main goal of this paper is to describe all dissipative extensions of the minimal operator $\Upsilon_0$ in $L^2_{1/\kappa}(X,(a,\infty))$ in terms of boundary values (see Section 3). In section 4, the structure of the spectrum of these extensions will be investigated.

## 3. Description of maximally dissipative extensions

In this section using the Calkin-Gorbachuk method will be investigated the general representation of all maximally dissipative extensions of the minimal operator $\Upsilon_0$ in $L^2_{1/\kappa}(X,(a,\infty))$.

Firstly, let us define the deficiency indices of any symmetric operator in a Hilbert space.

**Definition 3.1** [5] Let $T$ be a symmetric operator, $\lambda$ be an arbitrary non-real number and $X$ be a Hilbert space. We denote by $R_{\bar{\lambda}}$ and $R_\lambda$ the ranges of the operator $(T-\bar{\lambda}I)$ and $(T-\lambda I)$, respectively, where $I$ is identity operator on $X$. Clearly, $R_{\bar{\lambda}}$ and $R_\lambda$ are subspaces of $X$, which need not necessarily be closed. We call $(X - R_{\bar{\lambda}})$ and $(X - R_\lambda)$, which are their orthogonal complements, the deficiency spaces of the operator $T$ and we denote them by $N_{\bar{\lambda}}$ and $N_\lambda$ respectively: thus



$$N_{\bar{\lambda}} = X - R_{\bar{\lambda}}, \quad N_\lambda = X - R_\lambda.$$

The numbers

$$n_{\bar{\lambda}} = dimN_{\bar{\lambda}},\ n_{\lambda} = dimN_{\lambda},$$

are called deficiency indices of the operator $T$.

Firstly, prove the following proposition.

**Lemma 3.2** The deficiency indices of the operator $\Upsilon_0$ are in the form

$$\big(n_+(\Upsilon_0), n_-(\Upsilon_0)\big) = (\dim X, \dim X).$$

*Proof.* For the simplicity of calculations it will be taken $A = 0$. It is clear that the general solutions of the differential equations

$$i\kappa(\varsigma)\nu'_{\pm}(\varsigma) \pm i\nu_{\pm}(\varsigma) = 0, \varsigma > a$$

in $L^2_{1/\kappa}\big(X, (a, \infty)\big)$ are in form

$$\nu_{\pm}(\varsigma) = exp\left(\mp\int_a^{\varsigma}\frac{d\xi}{\kappa(\xi)}\right)f, f \in X,\ \ \varsigma > a\,.$$

From these representations we have

$$\begin{aligned}\|\nu_+\|^2_{L^2_{1/\kappa}(X,(a,\infty))} &= \int_a^{\infty}\frac{1}{\kappa(\varsigma)}\|\nu_+(\varsigma)\|^2_X d\varsigma = \int_a^{\infty}\frac{1}{\kappa(\varsigma)}\left\|exp\left(-\int_a^{\varsigma}\frac{d\xi}{\kappa(\xi)}\right)f\right\|^2_X d\varsigma\\ &= \int_a^{\infty}\frac{1}{\kappa(\varsigma)}exp\left(-2\int_a^{\varsigma}\frac{d\xi}{\kappa(\xi)}\right)d\varsigma\|f\|^2_X\\ &= \frac{1}{2}\left(1 - exp\left(-2\int_a^{\infty}\frac{d\xi}{\kappa(\xi)}\right)\right)\|f\|^2_X < \infty.\end{aligned}$$

Hence $n_+(\Upsilon_0) = \dim\ker(\Upsilon + iE) = \dim X$.

Similarly,

$$\|\nu_-\|^2_{L^2_{1/\kappa}(X,(a,\infty))} = \int_a^{\infty}\frac{1}{\kappa(\varsigma)}\|\nu_-(\varsigma)\|^2_X d\varsigma = \int_a^{\infty}\frac{1}{\kappa(\varsigma)}\left\|exp\left(\int_a^{\varsigma}\frac{d\xi}{\kappa(\xi)}\right)f\right\|^2_X d\varsigma$$

$$= \int_a^\infty \frac{1}{\kappa(\varsigma)} exp\left(2\int_a^\varsigma \frac{d\xi}{\kappa(\xi)}\right) d\varsigma \|f\|_X^2$$

$$= \frac{1}{2}\left(exp\left(2\int_a^\infty \frac{d\xi}{\kappa(\xi)}\right) - 1\right)\|f\|_X^2 < \infty.$$

Hence, $n_-(\Upsilon_0) = \dim\ker(\Upsilon - iE) = \dim X$. This completes the proof.

Consequently, the minimal operator has a maximally dissipative extension (see [3]). In order to describe these extensions we need to obtain the space of boundary values.

**Definition 3.3** [3] Let $\mathcal{X}$ be any Hilbert space and $S: D(S) \subset \mathcal{X} \to \mathcal{X}$ be a closed densely defined symmetric operator in the Hilbert space $\mathcal{X}$ having equal finite or infinite deficiency indices. A triplet $(\mathbb{X}, \beta_1, \beta_2)$ where $\mathbb{X}$ is a Hilbert space, $\beta_1$ and $\beta_2$ are linear mappings from $D(S^*)$ into $\mathbb{X}$, is called a space of boundary values for the operator $S$ if for any $\nu, \vartheta \in D(S^*)$

$$(S^*\nu, \vartheta)_{\mathcal{X}} - (\nu, S^*\vartheta)_{\mathcal{X}} = (\beta_1(\nu), \beta_2(\vartheta))_{\mathbb{X}} - (\beta_2(\nu), \beta_1(\vartheta))_{\mathbb{X}}$$

while for any $F_1, F_2 \in \mathbb{X}$, there exists an element $\nu \in D(S^*)$ such that $\beta_1(\nu) = F_{\,1}$ and

$\beta_2(\nu) = F_2$.

It is known that for any symmetric operator with equal deficiency indexes have at least one space of boundary values (see [3]).

**Lemma 3.4** The triplet $(X, \beta_1, \beta_2)$, where

$$\beta_1: D(\Upsilon) \to X, \beta_1(\nu) = \frac{1}{\sqrt{2}}\big(\nu(\infty) - \nu(a)\big)$$

$$\beta_2: D(\Upsilon) \to X, \beta_2\,(\nu) = \frac{1}{i\sqrt{2}}\big(\nu(\infty)+\nu(a)\big), \nu \in D(\Upsilon)$$

is a space of boundary values of $\Upsilon_0$ in $L^2_{1/\kappa}\big(X,(a,\infty)\big)$ .

*Proof.* For any $\nu,\ \vartheta\ \in D(\Upsilon)$,

$$(\Upsilon\nu, \vartheta)_{L^2_{1/\kappa}(X,(a,\infty))} - (\nu, \Upsilon\vartheta)_{L^2_{1/\kappa}(X,(a,\infty))}$$

$$= (i\kappa(\varsigma)\nu'(\varsigma) + A\nu(\varsigma), \vartheta(\varsigma))_{L^2_{1/\kappa}(X,(a,\infty))} - (\nu(\varsigma), i\kappa(\varsigma)\vartheta'(\varsigma) + A\vartheta(\varsigma))_{L^2_{1/\kappa}(X,(a,\infty))}$$

$$= i\left[(\kappa(\varsigma)\nu'(\varsigma),\vartheta(\varsigma))_{L^2_{1/\kappa}(X,(a,\infty))} + (\nu(\varsigma),\kappa(\varsigma)\vartheta'(\varsigma))_{L^2_{1/\kappa}(X,(a,\infty))}\right]$$

$$= i\int_a^\infty \frac{1}{\kappa(\varsigma)}[(\kappa(\varsigma)\nu'(\varsigma),\vartheta(\varsigma))_X + (\nu(\varsigma),\kappa(\varsigma)\vartheta'(\varsigma))_X]d\varsigma$$

$$= i\int_a^\infty (\nu(\varsigma),\vartheta(\varsigma))'_X d\varsigma$$

$$= i\left[\big(\nu(\infty),\vartheta(\infty)\big)_X - (\nu(a),\vartheta(a))_X\right]$$

$$= (\beta_1(\nu),\beta_2(\vartheta))_{\mathbf{X}} - (\beta_2(\nu),\beta_1(\vartheta))_{\mathbf{X}}$$

Now let $f,g \in X$. Let us find the function $\nu \in D(\Upsilon)$ such that

$$\beta_1(\nu) = \frac{1}{\sqrt{2}}\big(\nu(\infty) - \nu(a)\big) = f, \beta_2(\nu) = \frac{1}{i\sqrt{2}}\big(\nu(\infty)+\nu(a)\big) = g.$$

From this we can obtain

$$\nu(a) = (-f + ig)/\sqrt{2},\quad \nu(\infty) = (f + ig)/\sqrt{2}.$$

If we choose the function $\nu(\cdot)$ as

$$\nu(\varsigma) = e^{a-\varsigma}(-f + ig)/\sqrt{2} + (1 - e^{a-\varsigma})(f + ig)/\sqrt{2},\ \ \varsigma > a,$$

then it is obvious $\nu \in D(\Upsilon)$ and $\beta_1(\nu) = f$, $\beta_2(\nu) = g$. Hence the lemma is proof.

By using the method in [3] it can be established the following result.

**Theorem 3.5** If $\widetilde{\Upsilon}$ is a maximally dissipative extension of $\Upsilon_0$ in $L^2_{1/\kappa}\big(X,(a,\infty)\big)$, then it is generated by the differential-operator expression $l(\cdot)$ and the boundary condition

$$\nu(a) = \Gamma\nu(\infty),$$

where $\Gamma: X \to X$ is a contraction operator. Moreover, the contraction operator $\Gamma$ in $X$ is uniquely determined by the extension $\widetilde{\Upsilon}$, i.e. $\widetilde{\Upsilon} = \Upsilon_\Gamma$ and vice versa.

*Proof.* It is known that each maximally dissipative extension $\widetilde{\Upsilon}$ of the operator $\Upsilon_0$ is described by differential-operator expression $l(\cdot)$ with boundary condition

$$(V - I)\beta_1(\nu) + i(V + I)\beta_2(\nu) = 0,$$

where $V: X \to X$ is a contraction operator and $I$ is identity operator on $X$. Therefore from Lemma 3.4 we obtain

$$(V - E)\big(\nu(\infty) - \nu(a)\big) + (V + E)\big(\nu(\infty) + \nu(a)\big) = 0, \nu \in D(\widetilde{\Upsilon}).$$

From this it is implies that $\nu(a) = -V\nu(\infty)$. Choosing $\Gamma = -V$ in the last boundary condition we have

$$\nu(a) = \Gamma\nu(\infty).$$

## 4. The spectrum of the maximally dissipative extension

In this section the structure of the spectrum of the maximally dissipative extensions of $\Upsilon_0$ in $L^2_{1/\kappa}\big(X, (a, \infty)\big)$ will be investigated.

**Theorem 4.1** The spectrum of any maximally dissipative extension $\Upsilon_\Gamma$ is of the form

$$\sigma(\Upsilon_\Gamma) = \left\{ \lambda \in \mathbb{C}: \lambda = \left( \int_a^\infty \frac{1}{\kappa(\varsigma)} d\varsigma \right)^{-1} (arg\mu + 2n\pi + iln|\mu|^{-1}), \right.$$

$$\left. \mu \in \sigma\left( \Gamma exp\left( iA \int_a^\infty \frac{1}{\kappa(\varsigma)} d\varsigma \right) \right), n \in \mathbb{Z} \right\}.$$

*Proof.* Consider the following problem to spectrum for the extension $\Upsilon_\Gamma$ , i.e.

$$\Upsilon_\Gamma(\nu) = \lambda\nu + f, \qquad f \in L^2_{1/\kappa}\big(X, (a, \infty)\big), \qquad \lambda \in \mathbb{C}, \qquad \lambda_i = Im\lambda \geq 0.$$

Then we have

$$i\kappa(\varsigma)\nu'(\varsigma) + A\nu(\varsigma) = \lambda\nu(\varsigma) + f(\varsigma), \qquad \varsigma > a,$$
$$\nu(a) = \Gamma\nu(\infty).$$

The general solution of the last differential equation is of the form

$$\nu(\varsigma; \lambda) = exp\left( i(A - \lambda I) \int_a^\varsigma \frac{1}{\kappa(\xi)} d\xi \right) f_\lambda$$

$$-i \int_a^\varsigma exp\left( i(A - \lambda I) \int_\xi^\varsigma \frac{1}{\kappa(\tau)} d\tau \right) \frac{f(\xi)}{\kappa(\xi)} d\xi,\ f_\lambda \in X, \varsigma > a.$$

In this case

$$\left\| exp\left( i(A-\lambda I)\int_a^{\varsigma}\frac{1}{\kappa(\xi)}d\xi \right) f_\lambda \right\|^2_{L^2_{1/\kappa}(X,(a,\infty))}$$
$$=\frac{1}{2\lambda_i}\left( exp\left( 2\lambda_i\int_a^{\infty}\frac{1}{\kappa(\xi)}d\xi \right)-1\right)\|f_\lambda\|^2_X<\infty$$

and

$$\left\| -i\int_a^{\varsigma} exp\left( i(A-\lambda I)\int_{\xi}^{\varsigma}\frac{1}{\kappa(\tau)}d\tau \right)\frac{f(\xi)}{\kappa(\xi)}d\xi \right\|^2_{L^2_{1/\kappa}(X,(a,\infty))}$$
$$=\int_a^{\infty}\frac{1}{\kappa(\varsigma)}\left\| \int_a^{\varsigma} exp\left( i(A-\lambda I)\int_{\xi}^{\varsigma}\frac{1}{\kappa(\tau)}d\tau \right)\frac{f(\xi)}{\kappa(\xi)}d\xi \right\|^2_X d\varsigma$$
$$\leq\int_a^{\infty}\frac{1}{\kappa(\varsigma)}\left( \int_a^{\varsigma} exp\left( \lambda_i\int_{\xi}^{\varsigma}\frac{1}{\kappa(\tau)}d\tau \right)\frac{\|f(\xi)\|_X}{\kappa(\xi)}d\xi \right)^2 d\varsigma$$
$$=\int_a^{\infty}\frac{1}{\kappa(\varsigma)}\left( \int_a^{\varsigma}\frac{1}{\sqrt{\kappa(\xi)}} exp\left( \lambda_i\int_{\xi}^{\varsigma}\frac{1}{\kappa(\tau)}d\tau \right)\frac{\|f(\xi)\|_X}{\sqrt{\kappa(\xi)}}d\xi \right)^2 d\varsigma$$
$$\leq\int_a^{\infty}\frac{1}{\kappa(\varsigma)}\left( \int_a^{\infty}\frac{1}{\kappa(\xi)} exp\left( 2\lambda_i\int_{\xi}^{\infty}\frac{1}{\kappa(\tau)}d\tau \right)d\xi \right)\left( \int_a^{\infty}\frac{\|f(\xi)\|^2{}_X}{\kappa(\xi)}d\xi \right)d\varsigma$$
$$=\frac{1}{2\lambda_i}\int_a^{\infty}\frac{1}{\kappa(\varsigma)}d\varsigma\left( exp\left( 2\lambda_i\int_a^{\infty}\frac{1}{\kappa(\tau)}d\tau \right)-1\right)\|f_\lambda\|^2_{L^2_{1/\kappa}(X,(a,\infty))}<\infty.$$

Hence, for $\lambda\in\mathbb{C},\lambda_i=Im\lambda\geq 0,\ v(\cdot,\lambda)\in\ L^2_{1/\kappa}(X,(a,\infty))$ .

From the boundary condition we have

$$\left( \Gamma exp\left( i(A-\lambda E)\int_a^{\infty}\frac{1}{\kappa(\xi)}d\xi \right)-I\right)f_\lambda=i\Gamma\int_a^{\infty} exp\left( i(A-\lambda I)\int_{\varsigma}^{\infty}\frac{1}{\kappa(\tau)}d\tau \right)\frac{f(\xi)}{\alpha(\xi)}d\xi.$$

Therefore in order to $\lambda\in\sigma(\Upsilon_\Gamma)$ the necessary and sufficient condition is

$$exp\left( i\lambda\int_a^{\infty}\frac{1}{\kappa(\xi)}d\xi \right)=\mu\in\sigma\left( \Gamma exp\left( iA\int_a^{\infty}\frac{1}{\kappa(\xi)}d\xi \right)\right).$$



Consequently,

$$\lambda \int_a^\infty \frac{1}{\kappa(\xi)} d\xi = iln|\mu|^{-1} + arg\mu + 2n\pi, n \in \mathbb{Z},$$

that is,

$$\lambda = \left( \int_a^\infty \frac{1}{\kappa(\xi)} d\xi \right)^{-1} (arg\mu + 2n\pi + iln|\mu|^{-1}), \mu \in \sigma \left( \Gamma exp \left( iA \int_a^\infty \frac{1}{\kappa(\xi)} d\xi \right) \right), n \in \mathbb{Z}.$$

This completes proof of theorem.

**Example 4.2** All maximally dissipative extensions $\Upsilon_r$ of the minimal operator $\Upsilon_0$ generated by the following first order linear symmetric singular differential expression

$$l(\nu) = i\varsigma^{\gamma-\delta}\nu'(\varsigma) + a\nu(\varsigma), \gamma, \delta, a \in \mathbb{R} \text{ and } \gamma - \delta - 1 > 0$$

in the Hilbert space $L^2_{\varsigma^{\delta-\gamma}}(1, \infty)$ are described by the boundary condition

$$\nu(1) = r\nu(\infty),$$

where $r \in \mathbb{C}$ and $|r| \leq 1$. Moreover, in case when $r \neq 0$ the spectrum of maximally dissipative extension $\Upsilon_r$ is of the form

$$\sigma(\Upsilon_r) = (\gamma - \delta - 1)\left(\arg(r) + \frac{a}{\gamma - \delta - 1} + 2n\pi + iln|r|^{-1}\right), n \in \mathbb{Z}.$$

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# Compact Inverses of First Order Normal Differential Operators with Lorentz-Schatten Properties

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**Abstract**

In this work, the Lorentz-Schatten properties of the compact inverses of normal extensions of a minimal operator generated by linear differential-operator expression for first order in the weighted Hilbert space of vector-functions on right semi-axis is investigated.

**Keywords:** Differential and normal operators, s-numbers of compact operator, Lorentz-Schatten operator classes

## 1. Introduction

It is known that operator theory plays an exceptionally important role in modern mathematics and physics, quantum mechanics, deformation theory and etc. And also spectral analysis of operators is one of the most important area of modern mathematical physic. In addition the investigation of normal extensions of densely defined closed formal normal operators in any Hilbert space is among the fundamental mathematical problems arising in any physical model. It should be noted that the detail analysis of selfadjoint extensions of any linear closed densely defined having equal deficiency indexes in Hilbert space of vector-functions has been given in [1].

Let us remember that a linear densely defined closed operator $T$ in any Hilbert space $H$ is called formally normal if $D(T) \subset D(T^*)$ and $\|Tx\|_H = \|T^*x\|_H$ for all $x \in D(T)$. If a formally normal operator has no formally normal non-trivial extension, then it is called maximally formally normal operator. If a formally normal operator $T: D(T) \subset H \to H$ satisfies the condition $D(T) = D(T^*)$, then it is called normal operator (see [2]). The general theory of normal extensions of linear unbounded densely (and non-densely) defined formally normal operators has been given in [2]. Some application of this theory to the theory of differential operators in Hilbert space of vector-functions can be found in [3]-[5] ( see references in it).

The general theory of singular numbers and operator ideals was given by A. Pietsch in [6], [7] and the case of linear compact operators was investigated by I. C. Gohberg and M. G. Krein in [8]. However, the first result in this area can be found in the works of E. Schmidt [9], and J. von Neumann, R. Schatten [10] who used these concepts in the theory of non-selfadjoint integral equations.

Later on, the main aim of the mini-workshop hold in Oberwolfach (Germany) was to present and discuss some modern applications of the functional-analytic concepts of $s-$numbers and operator ideals in areas like numerical analysis, theory of function spaces, signal processing, approximation theory, probability of Banach spaces and statistical learning theory (see [11]).

Let $\mathcal{H}$ be a Hilbert space, $S_\infty(\mathcal{H})$ be a class of linear compact operators in $\mathcal{H}$ and $s_n(T)$ be the $n-th$ singular numbers of the operator $T \in S_\infty(\mathcal{H})$. The Lorentz-Schatten operator ideals are defined as

$$S_{p,q}(\mathcal{H})=\left\{T\in S_\infty(\mathcal{H}):\sum_{n=1}^{\infty}n^{\frac{q}{p}-1}s_n^q(T)<\infty\right\},0<p\le\infty,0<q<\infty$$

and

$$S_{p,\infty}(\mathcal{H})=\left\{T\in S_\infty(\mathcal{H}):sup_{n\ge1}n^{\frac{1}{p}}s_n(T)<\infty\right\},0<p\le\infty$$

in [6], [7], [13].

In this work,, the problem of belonging to Lorentz-Schatten operator classes of the inverse (consequently of resolvent operator) of the normal extensions of the minimal operator generated by the differential-operator expression for first order in the weighted Hilbert space of vector-functions on right semi-axis is studied.

**2. On the singular numbers of inverses of normal extensions of the minimal operator**

Let $H$ be a separable Hilbert space, $a\in\mathbb{R}$ and $L_\alpha^2\big(H,(a,\infty)\big)$ be a Hilbert space $H$ −valued vector-functions on $(a,\infty)$. In the $L_\alpha^2\big(H,(a,\infty)\big)$ consider the following differential-operator expression for first order in the form

$$l(u)=(\alpha u)'(t)+Au(t), \qquad (1)$$

where,

1. $\alpha:(a,\infty)\to(0,\infty)$, $\alpha\in C(a,\infty)$ and $\alpha^{-1}\in L^1(a,\infty)$,
2. $A:D(A)\subset H\to H$ and $A^*=A\ge I$ where $I$ is identity operator on $X$.

By standard method the minimal $L_0(L_0^+)$ and maximal $L(L^+)$ operators corresponding to differential expression $l\left(l^+=-\frac{d}{dt}(\alpha)+A^*\right)$ in $L_\alpha^2\big(H,(a,\infty)\big)$ can be easily defined (see [4]). In this case the minimal operator $L_0$ is formally normal, but it is not maximal in $L_\alpha^2\big(H,(a,\infty)\big)$.

Let give auxiliary propositions from [14].

**Theorem 1** Let $A^{1/2}W_{2,\alpha}^1\big(H,(a,\infty)\big)\subset W_2^1\big(H,(a,\infty)\big)$. Each normal extensions $\tilde{L}, L_0\subset\tilde{L}\subset L$ of the minimal operator $L_0$ in $L_\alpha^2\big(H,(a,\infty)\big)$ generated by the differential operator expression $l(\cdot)$ with boundary condition

$$(\alpha u)(\infty)=W(\alpha u)(a),$$

where $W$ and $A^{1/2}WA^{-1/2}$ are unitary operators in $H$. The unitary operator $W$ is uniquely determined by the extension $\tilde{L}$, i.e. $\tilde{L}=L_W$.

On the contrary, the restriction of the maximal operator $L$ to the linear manifold of vector-functions $(\alpha u)\in W_{2,\alpha}^1\big(H,(a,\infty)\big)$ that satisfy mentioned above condition for some unitary operator $W$, where $A^{1/2}WA^{-1/2}$ is also unitary operator in $H$, is a normal extension of the minimal operator $L_0$ in $L^2_\alpha(H,(a,\infty))$.

**Theorem 2** The spectrum of any normal extension $L_W$ in $L^2_\alpha(H,(a,\infty))$ of the minimal operator $L_0$ has a form

$$\sigma(L_W)=\left\{\lambda\in\mathbb{C}:\lambda=\left(\int_a^\infty\frac{ds}{\alpha(s)}\right)^{-1}(ln|\mu|^{-1}+2n\pi i-iarg\mu),n\in\mathbb{Z},\right.$$

$$\left.\mu\in\sigma\left(W^*exp\left(-A\int_a^\infty\frac{ds}{\alpha(s)}\right)\right)\right\}.$$

Let now give auxiliary two propositions from [15].

**Theorem 3** If $A^{-1}\in S_\infty(H)$ and the operator $L_W$ is any normal extension of the minimal operator $L_0$, then $L_W^{-1}\in S_\infty\left(L^2_\alpha(H,(a,\infty))\right)$.

**Theorem 4** If $A^{-1}\in S_\infty(H)$ and $\lambda_n(A)\sim cn^\alpha, 0<c,\alpha<\infty$ as $n\to\infty$, then $L_W^{-1}\in S_\infty\left(L^2_\alpha(H,(a,\infty))\right)$ and $s_n(L_W^{-1})\sim dn^{-\beta}, 0<d<\infty, \beta=\frac{\alpha}{1+\alpha}$ as $n\to\infty$.

**3. Lorentz-Schatten characteristic of inverses of normal extensions of the minimal operator**

Now give the main results of this work in the following theorems.

**Theorem 5** Let $A^{-1}\in S_\infty(H)$, $\lambda_n(A)\sim cn^\alpha, 0<c,\alpha<\infty$ as $n\to\infty$ and $L_W$ be any normal extension of the minimal operator $L_0$. In order to $L_W^{-1}\in S_{p,q}\left(L^2_\alpha(H,(a,\infty))\right), 0<q<\infty$ the necessary and sufficient condition is $p>1+\frac{1}{\alpha}$.

*Proof.* In this case from mentioned above Theorem 4 we have

$$s_n(L_W^{-1})\sim dn^{-\beta}, 0<d<\infty,\ \beta=\frac{\alpha}{1+\alpha}\ \text{as } n\to\infty.$$

Consequently, for the convergence of the series $\sum_{n=1}^\infty n^{\frac{q}{p}-1}s_n^q(L_W^{-1}),\quad 0<p,q<\infty$, i.e., $\sum_{n=1}^\infty n^{\frac{q}{p}-\frac{\alpha q}{1+\alpha}-1}$ the necessary and sufficient condition is $1+\frac{\alpha q}{1+\alpha}-\frac{q}{p}>1$. From this implies that $\frac{1}{p}<\frac{\alpha}{1+\alpha}$. From the last inequality we obtain $p>1+\frac{1}{\alpha}$.

**Theorem 6** If $A^{-1} \in S_\infty(H)$, $\lambda_n(A) \sim cn^\alpha, 0 < c, \alpha < \infty$ as $n \to \infty$ and $L_W$ be any normal extension of the minimal operator $L_0$. In order to $L_W^{-1} \in S_{p,\infty}\left(L_\alpha^2\big(H,(a,\infty)\big)\right), 0 < p \leq \infty$ the necessary and sufficient condition is $p > 1 + \frac{1}{\alpha}$.

*Proof.* In this case from mentioned above Theorem 4 we have

$$s_n(L_W^{-1}) \sim dn^{-\beta}, 0 < d < \infty,\ \beta = \frac{\alpha}{1+\alpha} \text{ as } n \to \infty.$$

Then for the validity of following condition $sup_{n\geq 1} n^{\frac{1}{p}} s_n(L_W^{-1}) < \infty$, that is,

$$sup_{n\geq 1} n^{\frac{1}{p}-\frac{\alpha}{1+\alpha}} < \infty,$$

the necessary and sufficient condition is $\frac{1}{p} < \frac{\alpha}{1+\alpha}$. From this we obtain $p > 1 + \frac{1}{\alpha}$.

**Corollary 7** Under the conditions of above two theorems $L_W^{-1} \notin S_{p,q}\left(L_\alpha^2\big(H,(a,\infty)\big)\right)$ for $0 < p \leq 1,\ 0 < q \leq \infty$.

**Corollary 8** Under the conditions of the Theorem 4, $L_W^{-1} \in S_p\left(L_\alpha^2\big(H,(a,\infty)\big)\right)$ if and only if $1 + \frac{1}{\alpha} < p < \infty$, where $S_p(\cdot)$ denotes a Schatten-von Neumann operators ideal.

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**INTERNATIONAL CONFERENCE ON MATHEMATICS**
**"An Istanbul Meeting for World Mathematicians"**
**3-5 July 2019, Istanbul, Turkey**

# Super decisions for the influenza activity viruses using AHP and ANP methods

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**Abstract**

Nowadays we are very interested about the infections caused by different viruses, to know the most activity spread during years and to make predictions for the future. According to the world health organization WHO online data updated every week, we can evaluate the seasonal influence activity of viruses A, B and their subtypes. We will propose a decision-making model based on two methods AHP and ANP. According to the decision-maker Goal, we can choose the most spread virus by his activity. There are 5 types of activity according to one year of study: no activity, sporadic, local outbreak, widespread outbreak, regional outbreak. The software used is "Super Decision" version 2.10. In fact we can`t agree that one method is better than another because it depends on the purpose of the problem. We will see results in both methods and we will make their comparisons in each case. Generally in the last 10 years, the two methods show that the priority activity in general in each season is no activity, and the more spread virus is $AH_1N_1$.

**Keywords: Super decisions, AHP, ANP, influenza virus activity, pairwise comparisons.**

## 1. Introduction

In 1952 the World Health Organization (WHO) Executive Board decided to have a system for the influenza surveillance in order to collect some data regarding occurrence, epidemiology, viruses etc. The laboratory is called "The Global Influenza Surveillance and Response Systems" (GISRS). It includes 143 institutions and 113 member states, as a network built on voluntary collaboration and real time reporting. In 11 March 2019 GISRS launched the strategy for 2019-2030 in order to protect people from the threat of influenza. The goal of the strategy is to prevent seasonal influenza, in order to prevent the next influenza from animals to humans. Regarding the situation about the predictive modeling, we have used the multi criteria decision making (MCDM) models as: analytic hierarchic process (AHP) and analytic network process (ANP) for the data collected from WHO European Region, United Kingdom of Great Britain and North Ireland from 2010 till nowadays. The report is updated every week, and data are at real time collected. The aim of this study is to compare the two models. T. L. Saaty developed the AHP in 1971- 1975 (University of Pennsylvania, Philadelphia). AHP is used to determine relative priorities on absolute scales from both discrete and continuous cases of the paired comparisons in hierarchic structures (Saaty and Vargas, 1996). The importance measurement has been developed by Saaty (1980, 1996) to represent the relative importance of the criteria, known as Saaty Scale. Pairwise comparisons matrices of these factors provide the means for calculation of importance (Sharma et al., 2008). AHP is a hierarchic decision model with a Goal (Main scope), then next level is Criteria as a cluster of nodes that are being pairwise compared for their importance to the goal, next level of criteria are alternatives also evaluated for the preference to each criterion. In other hand ANP does not have a Goal, it has only Criteria and Alternatives. The ANP method is a mathematical theory for evaluating a network and all kinds of dependence and feedback on it, by priorities as ratio scales of criterion and alternatives. The connection between nodes of each cluster is anyway for the inner and outer dependence. The AHP model is an hierarchic structure that rank the alternatives according to the Goal, while ANP compares the dependence between the nodes of criteria cluster and nodes of alternatives cluster called outer dependence, and the inner dependence between nodes to a cluster [6].

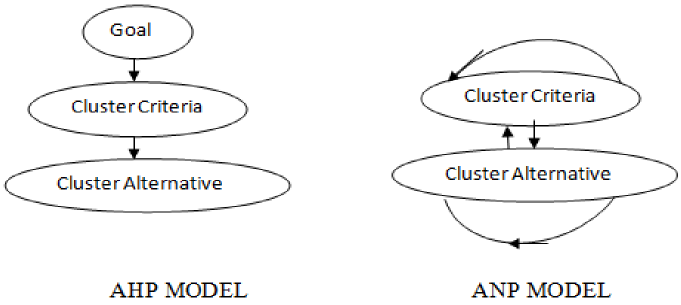



Fig. 1 Structure models

## 2. Materials and Methods

The data used in this paper are from World Health Organization (WHO) from GISRS, Flu-Net functions online data for United Kingdom of Great Britain and North Ireland, from 01.01.2010 to 29.04.2019, week by week all these years. They are organized with type A viruses including subtypes AH1, AH1N1, AH3, A and B viruses that are B Yamangata Lineage, B Victoria Lineage, B Lineage. For every week, we have a column named "ILI activity" for each virus with types: no activity, sporadic, local outbreak, widespread outbreak, regional outbreak. We have formulated a Goal Cluster named "the most spread virus" for the decision-maker. The Goal: Which is the most spread virus over these years for these "ILI activities"? According to the data we will built a hierarchy with AHP method by taking as a first level a cluster that will be called Criteria, and the next level a cluster that will be Alternative. The Criteria cluster will have nodes of five activities, and Alternative cluster will have seven nodes of types of viruses. While the ANP process will be the same hierarchy without the goal, and the clusters will be the same with their nodes as AHP Hierarchy.

### AHP Method

In the literature AHP, has been widely used in solving many decision making problems, in many areas and applications. Kangas et al., 2001, Kajanusa et al., 2004; Arslan and Turan, 2009; Kandakoğlu et al., 2009; Dinçer and Görener, 2011; Lee and Walsh, 2011; Saaty and Vargas, L.G. (1982, 1991, 2000, 2006); Dinçer and Görener, 2011; Lee and Walsh, 2011; Amir Azizi 2014; Naila Jan 2018; Luis G Vargas, H. J. Zoffer 2019. Clusters are connected by a line, we say nodes in them are connected and it means that the criteria must be pairwise compared for their importance with respect to the Goal, similarly Alternatives must be pairwise compared for their importance to Criteria. Clusters are connected by a line, we say nodes in them are connected and it means that the criteria must be pairwise compared for their importance with respect to the Goal, similarly Alternatives must be pairwise compared to Criteria for their importance. In order to determine the relative importance we have used Saaty's scale. Many questionnaire have been formulated to answer by experts in health based on Saaty scale, evaluations are made from mathematicians [5].

**Table 1**

| Relative importance value | Importance | Explanation |
|---|---|---|
| 1 | Equal | Two nodes have equal importance. |
| 3 | Moderate | Experience moderately favors one node over the other. |
| 5 | Strong | Experience strongly favors one node over the other. |
| 7 | Demonstrated | A node is strongly favored and has a demonstrated dominance. |
| 9 | Extreme strong | A node is on the highest possible order domination. |
| 2,4,6,8 | Intermediate values | A node with compromise intermediate value. |

The relative weights were measured using the Super Decision Software. The instructions on how to use the Super Decisions software were prepared by Rozann W. Saaty,wife of Thomas L. Saaty of the Creative Decisions Foundation. The software that implements the Analytic Network Process, Super Decisions, was developed by William J. Adams of Embry Riddle Aeronautical University, Daytona Beach, Florida, working with Rozann W. Saaty. The dictionary of ANP applications, the Encyclicon, included here as an appendix, was compiled from materials by Thomas L. Saaty and his students, Luis Vargas etc [3].

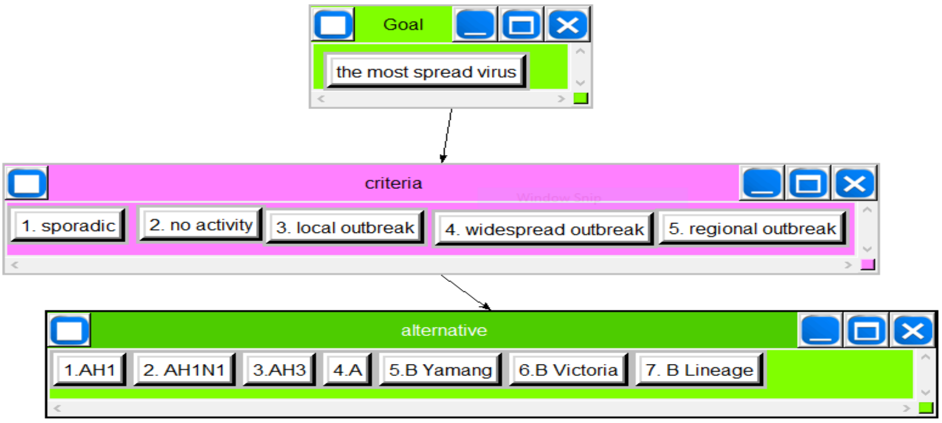



Fig.2 AHP model with Super Decision

The matrix of pairwise comparisons of Criteria cluster is a matrix with elements 1-9 according to the data obtained for the ILI Activity (Figure 2) of the viruses as:

$$A = (a_{ij})_{n \times n} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \text{ (1).}$$

The relative weights are given by the right eigenvector $\omega$, corresponding to the largest eigenvalue $\lambda_{max}$ where $A \cdot \omega = \lambda_{max} \cdot \omega$ [4]. Decision makers can weight the elements at each level using Saaty's scale from 1 to 9 and then calculate the global weights at the bottom level using pairwise comparisons (2).

$$\omega_i = \frac{1}{n}\sum_{j=1}^{n} \frac{a_{ij}}{\sum_{j=1}^{n} a_{ij}} \ , \quad \sum_{i=1}^{n} w_i = 1 \quad (2)$$

The inconsistency index is associated with matrix of the weights $CI = \frac{\lambda_{max}-n}{n-1}$. The consistence ratio is $CR = CI/RI$, where RI is the average of the eigenvalues as shown in the table nr 2 below. In order to improve the consistency of the pairwise comparisons CR, we need to adjust CI, but not larger as the judgment is, and thus the overall inconsistency should be less than 10%.

**Table nr 2**

| Order | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| R.I | 0 | 0 | 0.52 | 0.89 | 1.11 | 1.25 | 1.35 | 1.40 | 1.45 | 1.49 |

**ANP Method**

Everything we decide to do, and the decisions we make, in essence we are all decision-makers. To improve our understanding and judgments is not useful all the information. In some papers authors say that too much information is as bad as little information. The information that we have to use for the judgments is to help us understand occurrences. There are many uses of the ANP model. Dağdeviren and Yüksel (2007) developed an ANP-based personel selection system and weighted personel selection factors.Yang et al. (2009) developed a manufacturing evaluation system model with ANP approach for wafer fabricating industry. Valmohammadi (2010) used the ANP to identify specific resources and capabilities of an Iranian dairy products firm and to develop an evaluation framework of business strategy, Amir Azizi (2014) proposed a paper in comperative study of AHP and ANP on multi automotive suppliers with Multi Criteria, Feibert (2016) used the ANP to assess the distribution of pharmaceuticals in hospitals, Sajad Zare (2018) used the ANP method for prioritizing and weighting shift work disorders among the personnel of hospitals of Kerman University of medical Science. The ANP model does not have the top-bottom form of the AHP hierarchy. ANP structure seems like a network with cycles connecting the criteria level itself and with the alternative level, and vice versa. ANP consist of four steps (Satty, 1996), [8],[9].

**Step 1**. The problem have to be construct like a network with connections and loops. We have criteria cluster with five nodes as subcriteria, and alternative cluster with seven nodes connected respectively to each other and with the loops itself.
**Step 2**. Perform pairwise comparisons on the clusters connected to each others, evaluating their importance respectively to criteria and alternative.
**Step 3**. Compute the limit supermatrix. Synthesize to obtan the limit priority and ideal alternative.
**Step 4**. Create a ratings model and conduct a sensitivity analysis for the final outcame.

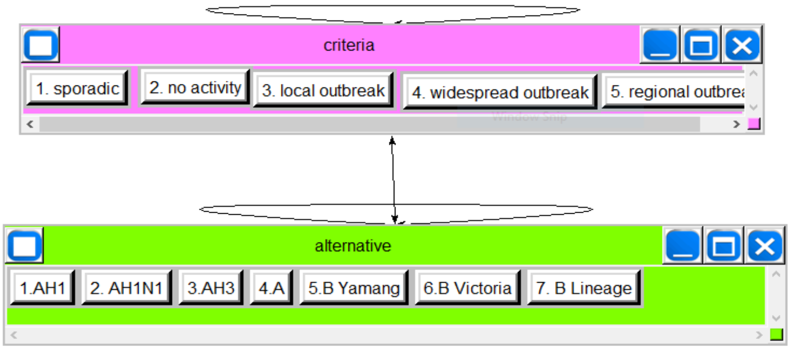



Fig.3 ANP model with Super Decision

The ANP provides a way to judge and measure ratio scales priorities for the distribution of influenza data. In fact the AHP theory is a special case of ANP theory. It’s not true that an ANP model always outputs better results than the AHP. ANP is a theory that extends the AHP to a structure of dependence and feedback and generalizes on the supermatrices approach introduced in Thomas Saaty’s 1980 book. It allows interactions and feedback to all nodes of the cluster as inner dependence and between clusters outer dependence. Similarly as the AHP method the pairs of comparisons for each cluster are being compared respectively to their importance within the nodes of the cluster and between the clusters [7]. A questionnaire was made to the decision makers to respond for the Saaty scale of two comparisons.

## 3. Main Results

**AHP Method.** [9],[10] Firstly we have to construct the A matrix of comparisons to criteria cluster by Saaty scale. Using the super decision software we have these values for our data base [2]:

| | Sporadic | No activity | Local | Widespread | Regional |
|---|---|---|---|---|---|
| Sporadic | 1 | 0.2 | 0.33 | 3.03 | 1.85 |
| No activity | 5 | 1 | 1.11 | 2 | 3 |
| Local | 3 | 0.9 | 1 | 4 | 2 |
| Widespread | 0.33 | 0.5 | 0.25 | 1 | 0.5 |
| Regional | 0.54 | 0.33 | 0.5 | 2 | 1 |

The weights of the $\omega$ vector of pairwise comparisons are:

$\omega = (sporadic, no\ activity,\ local, widespread, regional)$

$\omega_{criteria} = (0.14464; 0.35739; 0.29971; 0.08189; 0.11636)$ with $CI = 0.08714\ < 0.1 = 10\%$,

If $CI$ is larger than 10%, the input data have to be reconsider by Saaty scale to explain better the problem decision making. The next step is the pairwise comparison between each node of criteria cluster to all nodes of alternative cluster. The consistence ratio for cluster of 5 nodes is

$$CR = {}^{CI}/_{RI} = {}^{0.087}/_{1.11} = 0.078$$

The values of a higher CR also depend on the specific decision making problem, the out coming priorities and the required accuracy. The perfectly priorities are being selected well if the number of the criteria is 5-9 nodes. This is because the human limits on our capacity for generating information, published by George A. Miller in 1956, and taken-up Saaty and Ozdemir in 2003. For our data we have to do the pairwise comparisons for each node of Criteria to the alternative cluster.

The $\omega$ normalized vector for ***Sporadic*** node, by computing the comparisons with alternative nodes is:

$\omega_{sporadic} = (AH_1,\ AH_1N_1,\ AH_3, A\ , B\ Yamagata, B\ Victoria, B\ lineage) =$

$(0.092; 0.322; 0.077; 0.207; 0.122; 0.086; 0.091)$ with $CI = 0.085\ < 0.1$.

$$CR = {}^{CI}/_{RI} = {}^{0.085}/_{1.11} = 0.076$$

The $\omega$ normalized vector for ***No Activity*** node, by computing the pairwise comparisons with alternative nodes is: $\omega_{no\ activity} = (AH_1,\ AH_1N_1,\ AH_3, A\ , B\ Yamagata, B\ Victoria, B\ lineage)$

$\omega_{no\ activity} = (0.159; 0.165; 0.067; 0.238; 0.179; 0.08; 0.108)$ with $CI = 0.0852 < 0.1$

$$CR = {}^{CI}/_{RI} = {}^{0.0852}/_{1.11} = 0.077$$

The $\omega$ normalized vector for ***Local outbreak*** node, by computing the pairwise comparisons with alternative nodes is: $\omega_{local\ outbreak} = (AH_1,\ AH_1N_1,\ AH_3, A\ , B\ Yamagata, B\ Victoria, B\ lineage)$

$\omega_{local\ outbreak} = (0.176; 0.343; 0.062; 0.087; 0.086; 0.082; 0.162)$ with $CI = 0.089 < 0.1$

$$CR = {}^{CI}/_{RI} = {}^{0.089}/_{1.11} = 0.08$$

The $\omega$ normalized vector for ***Widespread outbreak*** node, by computing the comparisons with alternative nodes is: $\omega_{widespread\ outbreak} = (AH_1,\ AH_1N_1,\ AH_3, A\ , B\ Yamagata, B\ Victoria, B\ lineage)$

$\omega_{widespread\ outbreak} = (0.130; 0.271; 0.112; 0.083; 0.163; 0.111; 0.127)$ with $CI = 0.0992 < 0.1$

$$CR = {}^{CI}/_{RI} = {}^{0.0992}/_{1.11} = 0.089$$

The $\omega$ normalized vector for ***Regional outbreak*** node, by computing the pairwise comparisons with alternative nodes is:

$\omega_{regional\ outbreak} = (AH_1,\ AH_1N_1,\ AH_3, A\ , B\ Yamagata, B\ Victoria, B\ lineage) =$

$\omega_{regional\ outbreak} = (0.128; 0.267; 0.118; 0.148; 0.121; 0.115; 0.098)$ with $CI = 0.093 < 0.1$

$$CR = {}^{CI}/_{RI} = {}^{0.093}/_{1.11} = 0.083$$

The matrix of the $\omega$ weights normalized for the alternatives have to be multiplicative with the $\omega$ global weights of the criteria cluster.

$$\begin{pmatrix} 0.092 & 0.159 & 0.176 & 0.130 & 0.128 \\ 0.322 & 0.165 & 0.343 & 0.271 & 0.267 \\ 0.077 & 0.067 & 0.062 & 0.112 & 0.118 \\ 0.207 & 0.238 & 0.087 & 0.083 & 0.148 \\ 0.122 & 0.179 & 0.086 & 0.163 & 0.121 \\ 0.086 & 0.08 & 0.082 & 0.111 & 0.115 \\ 0.091 & 0.108 & 0.162 & 0.127 & 0.098 \end{pmatrix} \cdot \begin{pmatrix} 0.14464 \\ 0.35739 \\ 0.29971 \\ 0.08189 \\ 0.11636 \end{pmatrix} = \begin{pmatrix} 0.145 \\ 0.257 \\ 0.104 \\ 0.163 \\ 0.132 \\ 0.0754 \\ 0.12 \end{pmatrix} \quad \text{Rank} \begin{pmatrix} 3 = AH_1 \\ 1 = AH_1N_1 \\ 6 = AH_3 \\ 2 = A \\ 4 = Byam \\ 7 = Bvict \\ 5 = Bline \end{pmatrix}$$

Ranking the most spread virus we find out that the most spread is $AH_1N_1$ about 25.7%, then the second virus is A with its subtypes about 16.3%, then virus $AH_1$ about 14.5%, virus B Yamagata about 13.2%, virus B lineage 12%, virus $AH_3$ 10.4%, the last virus B Victoria 7.54%.

**ANP Method**

ANP method is composed as a network, in which we have to compare the dependences in the same level and between levels [9]. So the calculations have to be double compared to AHP method. Since there are many calculations for ANP, we better share the nodes in the cluster for having the efficient results. There are three supermatrices with the network [1]: The Unweighted Supermatrix contains the priorities from the pairwise comparisons, the Weighted supermatrix obtains the multiplications of all the elements in a component of the unweighted supermatrix by the corresponding cluster weight, and Limit Supermatrix is obtained by raising the weighted supermatrix to powers until the column of numbers is the same for every column, in alphabetical order for their nodes of comparisons [3]. The inner dependence is for the same nodes of the clusters [10]. So we have results for Criteria and Alternatives nodes:

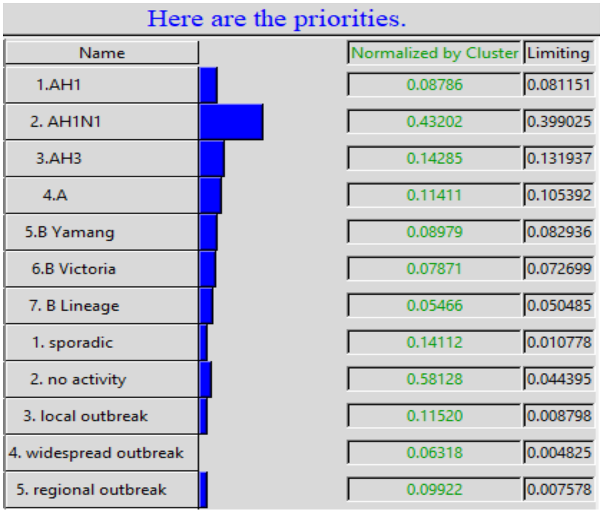
IC=0.08714

| | |
|---|---|
| Local outspread | 0.29971 |
| No activity | 0.35739 |
| Regional | 0.11636 |
| Sporadic | 0.14464 |
| Widespread | 0.08189 |

| | |
|---|---|
| $AH_1$ | 0.09246 |
| $AH_1N_1$ | 0.32266 |
| $AH_3$ | 0.07708 |
| A | 0.20768 |
| B Yamagata | 0.12258 |
| B Victoria | 0.08649 |
| B Lineage | 0.09194 |

IC=0.08533

After comparing the outer dependence and constructing the supermatrices, we have the priorities: [3]



Here are the priorities.

| Name | Normalized by Cluster | Limiting |
|---|---|---|
| 1.AH1 | 0.08786 | 0.081151 |
| 2. AH1N1 | 0.43202 | 0.399025 |
| 3.AH3 | 0.14285 | 0.131937 |
| 4.A | 0.11411 | 0.105392 |
| 5.B Yamang | 0.08979 | 0.082936 |
| 6.B Victoria | 0.07871 | 0.072699 |
| 7. B Lineage | 0.05466 | 0.050485 |
| 1. sporadic | 0.14112 | 0.010778 |
| 2. no activity | 0.58128 | 0.044395 |
| 3. local outbreak | 0.11520 | 0.008798 |
| 4. widespread outbreak | 0.06318 | 0.004825 |
| 5. regional outbreak | 0.09922 | 0.007578 |

Fig. 4 Priorities ANP

For the whole network the most spread virus is $AH_1N_1$ with 0.43=43% priority value for the alternatives, and the best activity node NO Activity with 0.58=58% priority value for criteria cluster. The importance is ranked as follows: $AH_3$=14.28%, A=11.4%, B Yamagata=8.9%, $AH_1$=8.7%, B Victoria=7.8%, B lineage=5.4%. Comparing to AHP we have:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| **Rank AHP** | 1.$AH_1N_1$ | 2.A | 3.$AH_1$ | 4.B Yamagata | 5.B Lineage | 6.$AH_3$ | 7.B Victoria |
| **Rank ANP** | 1.$AH_1N_1$ | 2.$AH_3$ | 3.A | 4. B Yamagata | 5. $AH_1$ | 6.B Victoria | 7. B Lineage |

## 4. Conclusion

The final ranking for the most spread virus during the application of the AHP and ANP methods are significantly the same for the best alternative node $AH_1N_1$, but different for the other nodes. The reason is that AHP is a hierarchy model with a main goal, but ANP a network with inner and outer dependence. Is better using AHP method instead of ANP wherever possible, trying to keep the nodes in a cluster between 5-9 for both methods. Always use AHP as a method to get consolidated results in ranking alternatives and use ANP as a tool to gain deeper inside into a decision problem, evaluated its ranking by decision makers main scopus.

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# Performance evaluation of University Academic Staff in teaching, research, and service using MCDA method and AHP

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**Abstract**
The aim of the education system is the continuous and progressive preparation of students' skills so that they can contribute to the sustainable development of their country. Among the different levels of education, university education plays a very important role in completing basic knowledge and methods of scientific research for their best performance and success in the future. The university education performance, results and success largely depends on the quality of programs, performance of the lectors, students achievements, and academic services. The Albanian universities have completed successfully their accreditation process by the Albanian Quality Assurance Agency for Higher Education (ASCAL), while it is the process of accreditation of bachelor and master programs. Still, Albanian universities need to perform better and have higher expectations. Albanian universities are always under the "watch" of international academic community ranking. And, to our disappointment, Albanian universities are behind universities of Balkan neighboring countries. İt needs a lot to be done. The process of academic staff evaluation should be a daily concern for the higher education Institutions as it is a very useful procedure which will help faculties and universities to have useful information about academic activities, qualifications, and projects, teaching process, achievements, and problems. Performance evaluation includes academic achievement, publications and projects, scientific research, teaching process, student's evaluations, and other activities, inside or outside the university area. Regular evaluation creates a healthy competition among university lectors. But for correct and successful evaluation, it is necessary to develop a performance-based, structured and effective evaluation model. The purpose of this study is to develop some performance indicators and an evaluation mechanism to evaluate the lectors' performance of a math department with respect to those indicators. Criteria and factors will be evaluated through the method of pairwise decision analysis. The Analytical Hierarchy Process (AHP) will be used to determine the weights of those performance indicators and, after evaluating alternatives for each criterion, we will have the rankings of alternatives which is the purpose and the outcome of our study.

Keywords: academic, evaluation, performance, ahp, mcda, university.

## 1. Introduction

The intention of the education system is to educate and improve students' skills and capacity so that they can contribute to the sustainable development of their country. It is the first and principal and factor of human resource development. Education is the key to a better future for any nation. Among the different levels of education, higher university education plays a significant role in the development of students' capability to meet the social, economic and technological needs of the country and to ensure the continuity of economic progress and well-being. The Albanian education system has been through many reforms since the fall of the communist system, many reforms are implemented in the education system, not all of them successful, to our opinion. The need to improve system performance and quality is becoming urgent, in order to prepare capable specialists, teachers, doctors, and future leaders. Among about 20 public universities and more than 20 private colleges and universities and other higher institutions, there are only a few institutions that are performing better in terms of producing globally quality graduates. However, there are still many quality and performance problems in the 9-year and high school institutions that prepare students for admission to Albanian or foreign universities that are related to teacher level, pupil preparation, scientific and educational level books, etc. Yet, the quality of higher education falls short of keeping up with the global level of excellence. Many graduates from high schools fail to pass the first year of Universities, or, their grades are far below the average of high school results. On the other hand, graduate students do not succeed in competing successfully in the labor market because they are not fully equipped. Yet, there is a lack of day-to-day and clear information to help them understand the rules and job-seeking requirements. One of the major causes of this condition is the lack of highly qualified high school teachers and poor performance of them. There is no cooperation and understanding among university professors and high school teachers, the first helping others to get qualified and improved with new knowledge and teaching materials and methods.

In short, there is a great need for creating a performance evaluation mechanism for that matter and there are hopeful signs in the right direction. ASCAL, Albanian Quality Assurance Agency for Higher Education is working to evaluate the whole higher university system, as part of "University pact" in order to understand, find and create methods and procedures to improve the education system. Still, there is no regulated practice of identifying key performance indicators of university academic staff and evaluating them on a regular yearly basis. If no evaluation mechanism in accordance with some key performance indicators exists, then any improvement in the quality and performance of the lectors will be limited in scope and conclusions. Key performance indicators help to identify the areas on which strong focus should be given for achieving better results. Identifying these important indices and evaluating the lectors on these indices regularly will help them to understand their strength and weakness and will stimulate them to overcome the weakness in the future.

## 2. Preliminaries

Regular and periodic performance evaluation of academic staff of higher education institution is important because the quality of teaching depends largely on the performance and academic services of the departments, faculties, and university. It depends on other academic factors, anyway, such as study programs, bachelor, master and doctorate, academic services, projects, students' performance, investment on programs, laboratories, etc. The evaluation result will be a higher and more complete knowledge of the situations and problems and will be helpful to academic staff members in order to improve their performance and the skills needed for better results. The problem of performance evaluation of lecturers is not a simple linear problem because the quality of teaching depends on many factors. The performance evaluation should be treated as a multidimensional problem for higher educational institutions and the state institutions on the basis of numerical and qualitative criteria. Chen et al. (2014) [1] proposed a method for compiling a lector evaluation system using the fuzzy AHP method and fuzzy assessment. In his study, fuzzy AHP was used to determine the weight of key performance factors and the fuzzy technique was used to evaluate the faculty performance. Kumar et al. (2013) [2] used multiple decision- making method (MCDM) to select the best candidate among some choices for academic staff. Some other methods like Simple Additive Weight method (SAW), weighted product method (WPM), AHP, and TOPSIS were used to evaluate the performance of department members. Ghosh (2011) [3] proposed a two-step AHP and TOPSIS method for evaluating the performance of academic staff at the Engineering Faculty. Wu et al. (2009) [4] used the Balanced Scorecard (BSC) to create performance indicators for bank performance evaluation. Lee et al. (2008) [5] used a Balanced Scorecard (BSC) and a Fuzzy Analytic Hierarchic process (AHP) to evaluate the company IT service performance. Bozbura et al. (2006) [6] proposed a model that uses Fuzzy AHP to give priority to factors to measure human capital weight. In this paper, we have built a model for evaluation of a random department and its staff members. The key performance factors were chosen based on literature models and expertise. Here, AHP is used to assign a weight to these factors. Finally, TOPSIS or other tools are used to ranking department staff members by measuring performance and using the quantitative and qualitative data collected by students.

## 3. Main Results

Multiple-criteria decision analysis, making (MCDA/ MCDM) is a powerful tool of operations research which is used for screening, prioritizing, ranking, or selecting a set of alternatives under given criteria, (Hwang & Yoon, 1981) [7]. First of all, selecting the criteria is very important because they influence the decision making process of MCDA/MCDM methods. A combination of AHP and TOPSIS or other tools, ELECTRE, PROMETHEE, etc, has been used in faculty performance evaluation. The evaluation procedure consists of four main steps, figure 1.

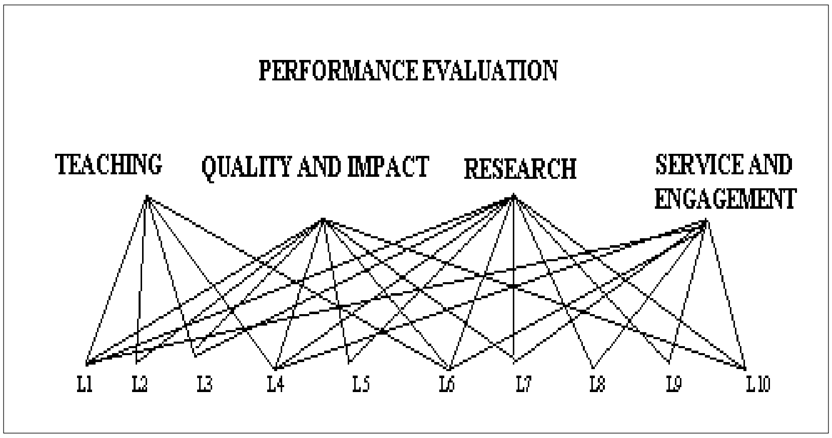


Fig 1. Criteria and alternatives for performance evaluation.

Step1. The identification of evaluation criteria,

Step 2. The construction of hierarchy of the evaluation criteria,

Step 3. Calculation of weights of chosen criteria using AHP method of MCDA,

Step 4. Calculate values for alternatives and get the final ranking results.

### 3.1 Determining the criteria weights. AHP approach

The Analytic Hierarchy Process (AHP) is a very popular MCDA/ MCDM method which was developed by Thomas L. Saaty (1980), [8]. It is widely used for solving complex problems having several attributes. This method converts unstructured problem under study into hierarchical forms of elements which are the main goal of the selected problem, criteria that affect the overall goal, and sub-criteria that influence the main criteria and finally the alternatives available to the problem.

The stepwise procedure to calculate the criteria weights by AHP as follows:

Step 1: Construct the structural hierarchy.

Step 2: Construct the pair-wise comparison matrix.

### Matrix:

Let's assuming $n$ attributes, then the pairwise comparison of any attribute $i$ with any attribute $j$ yields a square matrix $A_{n \times n}$ where the term $a_{i,j}$ denotes the comparative importance of attributes $i$ with respect to attribute $j$.

In the comparison matrice $A_{n \times n}$ we have; $a_{i,j} = 1, for\ i = j$, and $a_{i,j} = \frac{1}{a_{j,i}}, i \neq j$.

$$A_{nxn} = \overbrace{\begin{pmatrix} 1 \\ 2 \\ \dots \\ \dots \\ n \end{pmatrix}}^{Attributes} \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n-1} & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n-1} & a_{2n} \\ \dots & \dots. & \dots. & \dots\dots & \dots.. \\ \dots. & \dots. & \dots. & \dots\dots & \dots\dots \\ a_{n1} & a_n2 & \dots & a_{nn-1} & a_{nn} \end{bmatrix}$$

Step 3. Construct normalized decision matrix,

$$c_{i,j} = \frac{a_{i,j}}{\sum_{j=1}^{n} a_{i,j}}, \ i = \overline{1,n}; \ j = \overline{1,n}. \tag{1}$$

Step 4. Construct the weighted normalized matrix,

$$w_i = \sum_{j=1}^{n} \frac{c_{i,j}}{n}, \ i = \overline{1,n} \tag{2}$$

$$W = \begin{bmatrix} w_1 \\ w_2 \\ \dots \\ \dots \\ w_n \end{bmatrix} \tag{3}$$

Step 5. Calculate eigenvectors and Row matrix

$$E = \frac{N^{th} rootvalue}{\sum N^{th} rootvalue} \tag{4}$$

$$Rowmatrix = \sum_{j=1}^{n} a_{ij} * e_{j1} \tag{5}$$

Step 6. Calculate the largest Eigen value which is called the Principal Eigen value,

$$\lambda_{max} = \frac{Rowmatrix}{E} \tag{6}$$

Step 7. Calculate the consistency index $CI = \frac{(\lambda_{max} - n)}{n-1}$ (7)

where $n$ is the matrix order

and consistent ratio $CR = \frac{CI}{RI}$ (8)

RI is called random Consistency index.

If the value of Consistency Ratio is smaller or equal to 10%, then the inconsistency is acceptable. If the Consistency Ratio is greater than 10%, we need to revise the subjective judgment of the criteria pairwise comparison.

## 3.2 Description of academic staff activities

**Teaching and materials.**

Teaching includes: the amount and scope of teaching; tutorials, classes, online distribution, field, work, studies, and labs, the degree of coordination at the level of the bachelor course, diploma or master program. İt includes supervision and research of high- level students, the orientation, examination and supervision of students in training programs and specialized programs, the level of development of teaching materials, the participation in conferences and other forums, membership in the (internal and external), and the participation in teaching and learning network and forums, integrating innovative pedagogical and technological network practices, etc.

**Teaching quality and impact.**

Teaching quality and impact includes results for independent, University- approved teaching evaluations, numbers of students, the outcomes for students, industry satisfaction on the preparation of students for practice, prizes, and awards.
İt includes effective response to students' feedback and students' outcomes, continuous improvement of curricula, teaching resources, and teaching approaches, peer review of teaching methods, course design, and improvement of teaching materials, courses and programs to improve content, performance and results. İt includes student progression into research, contribution to mentoring and peer review of colleagues in teaching and learning, contribution to tutor training, provision of opportunities for students to engage with key researchers and industry partners, student seminars, laboratory and other research and industry-based projects and tasks, integration of relevant case studies and industry experiences into teaching, etc.

**Research and creative work.**

Research and creative work includes publications: journals, books, chapters, monographs, conference papers, edited books, and special journal issues, conference presentations and organization. İt includes research grants for collaborations and contracts with industry, government, community, etc. Leadership and project management in major projects, editorial and industry board memberships and book series editorships. Significance of research collaborations: interdisciplinary initiatives; major international collaborations, international awards and fellowships, national awards from academies and societies.
Membership government advisory committees, significance of research, collaborations of major industry, government, profession, business, not-for-profit organizations, and community partnerships.

**Service and engagement.**

Service and engagement includes contribution to activities that create benefits and improve performance and productivity, services for the University, Faculty, and Department, offer and contributions to ensure his effective leadership in any possible area of the University's activity, within his expertise and knowledge, contributions to provide effective connection and cooperation between University and outside activities, local community, companies, businesses, social and professional interests, etc.

### 3.3 AHP, criteria, weighting and ranking

The chosen department of the University of Durres has 20 lectors of different ages and experiences of work. We have used the academic activities of lectors during the last 5 years. We have used the data for only 10 random lectors out of 20 lectors, for the purpose of this study.
The four fields of activities mentioned before will be the criteria for evaluating the lectors' performance. For each criterion, a 100 points maximum is selected for the top performance, based on literature recomendation, commission evaluation opinion, lectors' self- evaluation form, and student's feedback. Pair wise comparison is used to measure the relative importance of criteria and calculate the criteria relative weighs. The results of AHP are in tables 1, 2, 3, 4, 5.

Table 1. AHP coeficcients of pair wise comparison.

| Rating Description | |
|---|---|
| 1- Equal | Both alternatives have equal importance. |
| 3- Moderate | One of the criteria is slightly more important than the other |
| 5- Strong. | One of the criteria is strongly more important than the other one |
| 7- Very Strong, | One of the criteria is very strongly important to the other one. |
| 9- Extreme Importance | One of the criteria is strictly superior to the other one. |
| - 2,4,6,8 | Intermediate values. |

Table 2. The resulting weights for the criteria based on pairwise comparisons.

| Criteria | | Priority | Rank | (+) | (-) |
|---|---|---|---|---|---|
| C1 | Teaching and materials | 45.7% | 1 | 11.0% | 11.0% |
| C2 | Teaching quality and impact | 27.1% | 2 | 6.9% | 6.9% |
| C3 | Research and Creative work | 19.3% | 3 | 8.1% | 8.1% |
| C4 | Service and engagement | 8.0% | 4 | 2.6% | 2.6% |

Table 3. Decision Matrix.

| Criteria | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 1 | 2.00 | 3.00 | 4.00 |
| 2 | 0.50 | 1 | 2.00 | 3.00 |
| 3 | 0.33 | 0.50 | 1 | 4.00 |
| 4 | 0.25 | 0.33 | 0.25 | 1 |

Number of comparisons = 6
Consistency Ratio CR = 5.6% < 10%.
Principal Eigenvalue = 4.154
Eigenvector solution: 5 iterations, delta = 4.4E-8

Table 4. Value estimation of lector performance and their ranking.

| Alternatıves, crıterıa estımatıon of performance and their rankıng. | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Lectors | C1 | C2 | C3 | C4 | Poınts | Lectors | Poınt | Nr | Poınts | Rankıng |
| L1 | 70 | 80 | 60 | 60 | 70 | L1 | 70 | 1 | 77.3 | L2 |
| L2 | 80 | 80 | 70 | 70 | 77.3 | L2 | 77.3 | 2 | 71.1 | L5 |
| L3 | 70 | 60 | 70 | 80 | 68.1 | L3 | 68.1 | 3 | 70 | L1 |
| L4 | 60 | 50 | 50 | 70 | 42.7 | L4 | 42.7 | 4 | 69.5 | L7 |
| L5 | 80 | 50 | 80 | 70 | 71.1 | L5 | 71.1 | 5 | 68.9 | L8 |
| L6 | 50 | 40 | 80 | 70 | 54.6 | L6 | 54.6 | 6 | 68.1 | L3 |
| L7 | 70 | 50 | 100 | 60 | 69.5 | L7 | 69.5 | 7 | 67.3 | L9 |
| L8 | 60 | 80 | 70 | 80 | 68.9 | L8 | 68.9 | 8 | 58.1 | L10 |
| L9 | 60 | 70 | 80 | 70 | 67.3 | L9 | 67.3 | 9 | 54.6 | L6 |
| L10 | 40 | 70 | 80 | 70 | 58.1 | L10 | 58.1 | 10 | 42.7 | L4 |

## 4. Conclusion

- The process of University performance evaluation, departments, faculties, and academic staff is a multi-staged decision- analyzing- making problem having both numerical and qualitative criteria.
- Evaluation of department‘s member performance in higher education Institutions is a very important and useful process in order to know each member’s contribution to the department and Institution.
- Performance evaluation helps to improve the cooperation among departments ‘members for the benefit of academic services, better performance, and better results.
- Performance evaluation helps to improve the quality of the education system through determining student’s opinions and academic criteria of qualifications.
- Performance evaluation is still missing in the Albanian academic system as a useful tool of internal evaluation of academic services.
- The proposed method is also effective in group decision studies like service companies, manufacturing industry, mobile companies, hotels, and restaurants, environmental, everywhere there are criteria to fulfill and alternatives to choose.
- The evaluation of performance within any university will help to recognize the overall state of the university, in terms of the importance of the factors, the comparison of work as well as the promotion of the of best lecturers and as a result, the improvement university or Institution performance in the international ranking.

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## On the Spectral Radius and Operator Norm of Finite Upper Triangular Block Operator Matrices

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In this work, a class of finite square upper triangular block operator matrices on the direct sum Hilbert spaces for which spectral radius is strongly less than of operator norm has been found.

**Keywords:** Block operator matrix, spectrum, spectral radius, operator norm

### 1. Introduction

Let $A$ be a bounded linear operator on a Banach space. It is well-known that the only formula for spectral radius is $r(A) = \lim_{n\to\infty}\|A^n\|^{1/n}$ and it has been given by I.M. Gelfand. For a normal bounded linear operator $A$, the spectral radius is the operator norm of $A$. In generally for the operators which are linear bounded but not normal, the spectral radius may not be equal to the norm of the operator. Practically, some operators such as generated by square matrices has been constructed as follows:

1. $A = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} : \mathbb{R}^2 \to \mathbb{R}^2$, $\|A\| = 2$, $\mathrm{r}(\mathrm{A}) = 1$.
2. $V: L_2(0,1) \to L_2(0,1)$, $V(f)(x) = \int_0^x f(t)dt$, $f \in L_2(0,1)$, $r(V) = 0$, $\|V\| = 2/\pi$.
3. $V_0: L_2(-1,1) \to L_2(-1,1)$, $V_0(f)(x) = \int_{-x}^x f(t)dt$, $f \in L_2(-1,1)$, $r(V_0) = 0$ and $\|V_0\| = 4/\pi$. $V_0$ is anti-symmetric and nilpotent of index 2.

In [3], the norm of Hardy operator has been discussed and it has been shown that this operator is compact, quasinilpotent and type of Volterra by M.Gürdal, M.Karaev and S.Saltanın. In [4], R.Withley has investigated the nilpotentness of the Volterra composition operator on $L_p(0,1), 1 \le p \le \infty$. In [5], Y.S.Tong has generalized this result for integral operators defined on the space of continuous functions and Lebesque spaces but the property $r(A) < \|A\|$ has not been studied.

Besides this, for the integral operator $(Af)(t) = L\int_0^t t^\alpha f(t^\beta)\,dt$, $f \in C[0,1]$, $\alpha > 0$,

$\beta > 0$ on $C[0,1]$ the spectral radius has been calculated as r( $A$)= $\frac{L}{\alpha+1} \frac{1}{\sum_{n=1}^{\infty} \beta^n}$. In [6], I. Domanov has given the exact formula of the point spectrum of the composition operator which has the form $\varphi \circ V$ where $\varphi(x) = x^{\alpha}, 0 < \alpha < 1$ and $V: L_2(0,1) \to L_2(0,1)$ the classical Volterra operator. It has been found as $\sigma_p(\varphi \circ V) = \{(1-\alpha)\alpha^{n-1}\}$ and r($\varphi \circ V$)= $1 - \alpha$. In [7], I. Domanov has also studied the composition operator $\varphi \circ V$ for $\varphi(x) > x$ , $x \in (0,1)$. But in [6], [7] the norms of the operator have not been calculated. The details of the spectral theory of such composition operators on $L_p(0,1), 1 \leq p \leq \infty$ can be found in [8].

Furthermore in [9], F. Kittaneh has investigated the relations between spectral radius and spectral norm of the sum, product and commutator operators but the strict inequality has not been touched. In [10], the fine spectra of upper triangular double-band matrices over the sequence spaces $c_0$ and $c$ has been determined and it has been observed that the strict inequality between the spectral radius and spectral norm does not hold.

In this work, we provide a class of finite square upper triangular block operator matrices on direct sum of Hilbert spaces for which spectral radius is strongly less than of operator norm.

Note that this research has been motivated from M. Demuth's open problem which is stated in conference of AIM in 2015 (see [11]).

## 2. Spectrum of Finite Upper Triangular Block Operator Matrix

In the direct sum $H = \bigoplus_{m=1}^{n} H_m$ of Hilbert spaces $H_m$, $1 \leq m \leq n$ consider the following finite square upper triangular block operator matrix

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ & A_{22} & \cdots & A_{2n} \\ 0 & & \ddots & \\ & & & A_{nn} \end{pmatrix},$$

$$A: D(A) \subset H \to H$$

where

$$A_{mj}: D(A_{mj}) \subset H_j \to H_m, \ 1 \leq m \leq j \leq n$$

are linear closed densely defined operators.

First of all we prove the next proposition.

**Theorem 2.1.** Let $A$ be the operator as above. Then

1. $\rho(A) = \bigcap_{m=1}^{n} \rho(A_{mm}),$
2. $\sigma(A) = \bigcup_{m=1}^{n} \rho(A_{mm}).$

**Proof.** 1. For any $\lambda \in \bigcap_{m=1}^{n} \rho(A_{mm})$ from

$$(A - \lambda I)x = y,\ x, y \in H$$

one can obtain that

$$x_n = R_\lambda(A_{nn})y_n$$

$$x_{n-1} = R_\lambda\left(A_{n-1,n-1}\right)y_{n-1} - R_\lambda(A_{n-1,n-1})A_{n-1,n}R_\lambda(A_{nn})y_n$$

$$\vdots$$

$$x_1 = R_\lambda(A_{11})y_1$$

Therefore $\lambda \in \rho(A)$ which implies that $\bigcap_{m=1}^{n} \rho(A_{mm}) \subset \rho(A)$.

Now let us $\lambda \in \rho(A)$. Then the equation

$$(A - \lambda I)x = y$$

is solvable for any $x, y \in H$. In special case this equation is solvable in $0 \oplus 0 \oplus \ldots \oplus 0 \oplus H_n$. This means that $\lambda \in \rho(A_{nn})$. Similarly it is solvable in subspaces $0 \oplus 0 \oplus \ldots \oplus H_{n-1} \oplus H_n$. Consequently the following equations

$$\begin{cases} (A_{n-1,n-1} - \lambda)x_{n-1} + A_{n-1,n}x_n = y_{n-1} \\ (A_{nn} - \lambda)x_n = y_n \end{cases}$$

are solvable for any $y_{n-1} \in H_{n-1}$, $y_n \in H_n$. Since $x_n = R_\lambda(A_{nn})y_n$, then the equation

$$\left(A_{n-1,n-1} - \lambda\right)x_{n-1} = y_{n-1} - A_{n-1,n}R_\lambda(A_{nn})y_n$$

is solvable for any $y_{n-1} \in H_{n-1}$, $y_n \in H$. This means $\lambda \in \rho(A_{n-1,n-1})$. This procedure can be applied repeatedly $(n-1)$ times and this implies

$$\rho(A) \subset \bigcap_{m=1}^{n} \rho(A_{mm}).$$

Hence

$$\rho(A) = \bigcap_{m=1}^{n} \rho(A_{mm}).$$

2. By (1) one can easily obtain

$$\sigma(A) = \bigcup_{m=1}^{n} \rho(A_{mm}).$$

**Theorem 2.2.** If

$\max_{2\le m\le n}\|A_{mm}\| < \|A_{11}\|$ or

$\max_{1\le m\le n-1}\|A_{mm}\| < \|A_{nn}\|$

and $r(A_{11}) < \|A_{11}\|$ or $r(A_{nn}) < \|A_{nn}\|$ respectively, then

$r(A) < \|A\|$.

**Proof.** First of all we prove the following inequality

$\|A_{11}\| \le \|A\|$.

In the subspace

$\{(x_1, 0,0, \dots\dots): x_1 \in H_1\} \subset H$

we have

$\frac{\|A_{11}x_1\|}{\|x_1\|} = \frac{\|Ax\|}{\|x\|}$ for any $x_1 \neq 0,\, x_1 \in H_1$.

Then

$\|A_{11}\| \le \|A\|$.

By similar technique, it can be shown that

$\|A_{nn}\| \le \|A\|$.

Consequently from this and the assumptions of the theorem, we have

$\max_{1\le m\le n}\|A_{mm}\| < \|A_{11}\| \le \|A\|$ or

$\max_{1\le m\le n}\|A_{mm}\| < \|A_{nn}\| \le \|A\|$

respectively.



Then from these inequality and Theorem 2.1 it is obtained

$$r(A) < \|A\|.$$

This completes the proof.

**Corollary 2.3.** If

$$A_{mj} \in L(H_j, H_m),\ 1 \le m \le j \le n,$$

$$A_{mm} = 0,\ m = 1,2, \dots, n \text{ and } A \neq 0,$$

then $r(A) < \|A\|$.

**Remark 2.4.** In similar way, the analogues of this problem can be researched for finite square lower triangular block operator matrices in the direct sum of Hilbert spaces.

**Example 1.** Let $H = \mathbb{R} \oplus L^2(0,1)$,

$$A_{11}x = x,\ x \in \mathbb{R}$$

$$A_{12}: L^2(0,1) \to L^2(0,1),\ A_{12} \in L(L^2(0,1)),$$

$$A_{22}f(t) = \int_0^t f(x)dx,\ f \in L^2(0,1),$$

$$A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} : H \to H.$$

Then one can easily obtain

$$\|A_{11}\| = 1,\ r(A_{11}) = 1$$

$$\|A_{22}\| = \frac{2}{\pi},\ r(A_{22}) = 0.$$

Hence by Theorem 2.3, we have that

$$r(A) = 1 < \|A\|.$$

**Example 2.** Let

$$A_{11} = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix},\ A_{12} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},\ A_{22} = \begin{pmatrix} 4 & 0 \\ 3 & 0 \end{pmatrix},$$

$$A_{11}, A_{12}, A_{22}: \mathbb{C}^2 \to \mathbb{C}^2,$$

$$A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{13} \end{pmatrix} : \mathbb{C}^4 \to \mathbb{C}^4.$$

Then one can easily obtain

$$r(A_{11}) = 1,\ \|A_{11}\| = \sqrt{5},$$

$r(A_{22}) = 4$, $\|A_{22}\| = 5$.

Since $\|A_{11}\| < \|A_{22}\|$, by Theorem 2.3 we have that

$$r(A) = 4 < 5 < \|A\|.$$

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# Dissipative Singular Differential Operators of First Order

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**Abstract**

In this paper, the parametrization of maximally dissipative extensions of the minimal operator generated by first order linear symmetric singular differential-operator expression in the Hilbert space of vector-functions defined at the right semi-axis has been given with the using of Calkin-Gorbachuk method. Finally, the structure of spectrum set of such extensions is researched.

**Keywords:** Dissipative operators, deficiency indices, spectrum

## 1. Introduction

It is known that a linear closed densely defined operator $T: D(T) \subset H \rightarrow H$ in Hilbert space $H$ is said to be dissipative if and only if

$$Im\,(Tx, x) \geq 0$$

for all $x \in D(T)$, i.e. in other words its numerical range is contained in the upper complex plane. Moreover, it is called maximally dissipative if it has no non-trivial dissipative extension [2].

Maximally dissipative operators play a very important role in mathematics and physics. Dissipative operators have many interesting applications in physics like hydrodynamic, laser and nuclear scattering theories.

Note that the study of abstract extension problems for operators in Hilbert spaces goes at least back to J.von Neumann [7] such that in [7] a full characterization of all selfadjoint extensions of a given closed symmetric operator with equal defect indices has been investigated. The characterization of all non-negative selfadjoint extensions of a positive closed symmetric operator has been studied by M. I. Vishik and M. S. Birman in details (see [3]). More general information can also be found in [1]. The class of dissipative operators is an important class of non-selfadjoint operators in the operator theory. Functional model theory of B. Sz.-Nagy and C. Foias [5] is a basic method for investigation the spectral properties of dissipative operators. Note that spectrum set of the dissipative operator lies in closed upper half-pane. The maximal dissipative extensions and their spectral analysis of the minimal operator having equal deficiency indices generated by formally symmetric differential-operator expression in the Hilbert space of vector-functions defined in one finite or infinite interval case have been researched by V. I. Gorbachuk, M. I. Gorbachuk [2] and F. S. Rofe-Beketov, A. M. Kholkin [6] in terms of generalized boundary values.

**2. Statement of the Problem**

Let $H$ be a separable Hilbert space and $a \in \mathbb{R}$. In the Hilbert space $L^2(H,(a,\infty))$ consider the following differential-operator expression in a form

$$l(u) = i\rho u' + \frac{1}{2} i\rho' u + Au,$$

where:

(1) $\rho: (a,\infty) \to (0,\infty)$;

(2) $\rho \in AC_{loc}(a,\infty)$;

(3) $\int_a^\infty \frac{ds}{\rho(s)} < \infty$;

(4) $A^* = A: D(A) \subset H \to H$.

The minimal operator $L_0$ corresponding to differential-operator expression $l(\cdot)$ in $L^2(H,(a,\infty))$ can be defined by standard way (see [3]). The operator $L = (L_0)^*$ is called the maximal operator corresponding to $l(\cdot)$ in $L^2(H,(a,\infty))$ (see [3]).

In this paper, firstly the representation of maximally dissipative extensions of the minimal operator $L_0$ will be described. Then, structure of the spectrum of these extensions will also be investigated.

**3. Description of Maximally Dissipative Extensions**

In this section, the general representation of maximally dissipative extensions of the minimal operator $L_0$ will be investigated by using the Calkin-Gorbachuk method.

Now, we present the following lemma which we need for the proofs of our main results.

**Lemma 3.1.** The deficiency indices of the operator $L_0$ is in form $\big(m(L_0), n(L_0)\big) = (\dim H, \dim H)$.

**Proof.** For the simplicity of calculations it will be taken $A = 0$. It is clear that the general solutions of following differential equations



$$i\rho(t)u_\pm'(t) + \frac{1}{2} i\rho'(t)u_\pm(t) \pm iu_\pm(t) = 0,$$

in the $L^2(H,(a,\infty))$ are in forms

$$u_{\pm}(t) = \exp\left(\mp\int_a^t \frac{2\pm\rho'(s)}{2\rho(s)}ds\right)f,\ f \in H,\ t > a.$$

From these representations, we have

$$\begin{aligned}\|u_+\|^2_{L^2(H,(a,\infty))} &= \int_a^\infty \|u_+(t)\|^2_H dt \\ &= \int_a^\infty \exp\left(-\int_a^t \frac{2+\rho'(s)}{\rho(s)}ds\right)dt\,\|f\|^2_H \\ &= \int_a^\infty \frac{\rho(a)}{\rho(t)}\exp\left(-\int_a^t \frac{2}{\rho(s)}ds\right)dt\,\|f\|^2_H \\ &= \frac{\rho(a)}{2}\left[1-\exp\left(-\int_a^\infty \frac{2}{\rho(s)}ds\right)\right]\|f\|^2_H < \infty.\end{aligned}$$

Consequently $m(L_0) = \dim\ker(L+iE) = \dim H$.

On the other hand it is clear that for any $f \in H$

$$\begin{aligned}\|u_-\|^2_{L^2(H,(a,\infty))} &= \int_a^\infty \|u_-(t)\|^2_H dt \\ &= \int_a^\infty \exp\left(\int_a^t \frac{2-\rho'(s)}{\rho(s)}ds\right)dt\,\|f\|^2_H \\ &= \int_a^\infty \frac{\rho(a)}{\rho(t)}\exp\left(\int_a^t \frac{2}{\rho(s)}ds\right)dt\,\|f\|^2_H \\ &= \frac{\rho(a)}{2}\left[\exp\left(-\int_a^\infty \frac{2}{\rho(s)}ds\right)-1\right]\|f\|^2_H < \infty\end{aligned}$$

holds. It follows from that $n(L_0) = \dim\ker(L-iE) = \dim H$. This completes the proof of theorem.

Consequently, the operator $L_0$ has a maximally dissipative extension (see [2]). In order to describe maximally dissipative extensions of $L_0$, it is necessary to construct a space of boundary values for it.

**Definition 3.2.** [2] Let $\mathcal{H}$ be any Hilbert space and $S: D(S) \subset \mathcal{H} \to \mathcal{H}$ be a closed densely defined symmetric operator in the Hilbert space $\mathcal{H}$ having equal finite or infinite deficiency indices. A triplet $(\boldsymbol{H},\gamma_1,\gamma_2)$, where $\boldsymbol{H}$ is a Hilbert space, $\gamma_1$ and $\gamma_2$ are linear mappings from $D(S^*)$ into $\boldsymbol{H}$, is called a space of boundary values for the operator $S$ if for any $f,g \in D(S^*)$

$$(S^*f,g)_{\mathcal{H}} - (f,S^*g)_{\mathcal{H}} = \big(\gamma_1(f),\gamma_2(g)\big)_{\boldsymbol{H}} - (\gamma_2(f),\gamma_1(g))_{\boldsymbol{H}}$$

while for any $F_1, F_2 \in \boldsymbol{H}$, there exists an element $f \in D(S^*)$ such that $\gamma_1(f) = F_1$ and $\gamma_2(f) = F_2$.

**Lemma 3.3.** The triplet $(H, \gamma_1, \gamma_2)$,

$$\gamma_1 : D(L) \to H,\ \gamma_1(u) = \frac{1}{\sqrt{2}}\Big(\big(\sqrt{\rho}u\big)(\infty) - (\sqrt{\rho}u)(a)\Big),$$

$$\gamma_2 : D(L) \to H,\ \gamma_2(u) = \frac{1}{i\sqrt{2}}\Big(\big(\sqrt{\rho}u\big)(\infty) + (\sqrt{\rho}u)(a)\Big),\ u \in D(L)$$

is a space of boundary values of the minimal operator $L_0$ in $L^2(H, (a, \infty))$.

**Proof.** In this case by direct calculations we have the following for arbitrary $u, v \in D(L)$

$$(Lu, v)_{L^2(H,(a,\infty))} - (u, Lv)_{L^2(H,(a,\infty))} = \Big(i\rho u' + \frac{1}{2} i\rho' u + Au, v\Big)_{L^2(H,(a,\infty))}$$

$$-(u, i\rho v' + \frac{1}{2} i\rho' v + Av)_{L^2(H,(a,\infty))}$$

$$= (i\rho u', v)_{L^2(H,(a,\infty))} + \frac{1}{2}(i\rho' u, v)_{L^2(H,(a,\infty))}$$

$$-(u, i\rho v')_{L^2(H,(a,\infty))} - (u, \frac{1}{2} i\rho' v)_{L^2(H,(a,\infty))}$$

$$= i\Big[(\rho u', v)_{L^2(H,(a,\infty))} + (\rho' u, v)_{L^2(H,(a,\infty))} + (\rho u, v')_{L^2(H,(a,\infty))}\Big]$$

$$= i\Big[((\rho u)', v)_{L^2(H,(a,\infty))} + (\rho u, v')_{L^2(H,(a,\infty))}\Big]$$

$$= i((\rho u, v))'_{L^2(H,(a,\infty))}$$

$$= i\big((\sqrt{\rho}u, \sqrt{\rho}v)\big)'_{L^2(H,(a,\infty))}$$

$$= i\Big[\Big(\big(\sqrt{\rho}u\big)(\infty), \big(\sqrt{\rho}v\big)(\infty)\Big)_H - \Big(\big(\sqrt{\rho}u\big)(a), \big(\sqrt{\rho}v\big)(a)\Big)_H\Big]$$

$$= (\gamma_1(u), \gamma_2(v))_H - (\gamma_2(u), \gamma_1(v))_H.$$

Now for any given elements $f, g \in H$, one can find the function $u \in D(L)$ satisfying

$$\gamma_1(u) = \frac{1}{\sqrt{2}}\Big(\big(\sqrt{\rho}u\big)(\infty) - (\sqrt{\rho}u)(a)\Big) = f \text{ and}$$

$$\gamma_2(u) = \frac{1}{i\sqrt{2}}\Big(\big(\sqrt{\rho}u\big)(\infty) + (\sqrt{\rho}u)(a)\Big) = g.$$

From this

$$\big(\sqrt{\rho}u\big)(\infty) = \frac{ig+f}{\sqrt{2}} \text{ and } \big(\sqrt{\rho}u\big)(a) = \frac{ig-f}{\sqrt{2}}$$

is obtained.

If we choose the function $u$ in the following form

$$u(t) = \frac{1}{\sqrt{\rho(t)}}(1 - e^{a-t})\frac{ig+f}{\sqrt{2}} + \frac{1}{\sqrt{\rho(t)}}e^{a-t}\frac{ig-f}{\sqrt{2}},$$

$u \in D(L)$ we have that $\gamma_1(u) = f$ and $\gamma_2(u) = g$.

Finally, using the method given in [2], we can immediately obtain the following result.

**Theorem 3.4.** If $\tilde{L}$ is maximally dissipative extension of the minimal operator $L_0$ in $L^2(H,(a,\infty))$, then it is generated by the differential-operator expression $l(\cdot)$ and boundary condition

$$\left(\sqrt{\rho}u\right)(a) = W\left(\sqrt{\rho}u\right)(\infty),$$

where $W: H \to H$ is a contraction operator. Moreover, the contraction operator $W$ in $H$ is determined uniquely by the extension $\tilde{L}$, i.e. $\tilde{L} = L_W$ and vice versa.

**Proof.** Each maximally dissipative extensions of the minimal operator $L_0$ are described by differential-operator expression $l(\cdot)$ and the boundary condition

$$(V - E)\gamma_1(u) + i(V + E)\gamma_2(u) = 0,$$

where $V: H \to H$ is a contraction operator. So from Lemma 3.3, we have

$$(V - E)\left(\left(\sqrt{\rho}u\right)(\infty) - \left(\sqrt{\rho}u\right)(a)\right) + (V + E)\left(\left(\sqrt{\rho}u\right)(\infty) + \left(\sqrt{\rho}u\right)(a)\right) = 0.$$

Hence, we obtain

$$\left(\sqrt{\rho}u\right)(a) = -V\left(\sqrt{\rho}u\right)(\infty).$$

Choosing $W = -V$ in last boundary condition, we have

$$\left(\sqrt{\rho}u\right)(a) = W\left(\sqrt{\rho}u\right)(\infty).$$

## 4. The Spectrum of the Maximally Dissipative Extensions

In this section the structure of the spectrum of the maximally dissipative extensions $L_W$ of the minimal operator $L_0$ in $L^2(H,(a,\infty))$ will be investigated.

First of all let us prove the following result.

**Theorem 4.1.** The spectrum of any maximally dissipative extension $L_W$ is in form

$$\sigma(L_W) = \left\{\lambda \in \mathbb{C} : \lambda = \left(\int_a^\infty \frac{ds}{\rho(s)}\right)^{-1} (i \ln|\mu|^{-1} + \arg\mu + 2n\pi),\ \ n \in \mathbb{Z},\ \ \mu \in \sigma\left(W \exp\left(iA \int_a^\infty \frac{ds}{\rho(s)}\right)\right)\right\}.$$

**Proof.** Consider the following problem for spectrum of the extension $L_W$

$$l(u) = \lambda u + f,\ u, f \in L^2(H, (a, \infty)),\ \lambda \in \mathbb{C},\ \lambda_i = \operatorname{Im}\lambda \geq 0,$$

$$(\sqrt{\rho}u)(a) = W(\sqrt{\rho}u)(\infty),$$

that is,

$$i\rho(t)u'(t) + \frac{1}{2}i\rho'(t)u(t) + Au(t) = \lambda u(t) + f(t), t > a,$$

$$(\sqrt{\rho}u)(a) = W(\sqrt{\rho}u)(\infty).$$

The general solution of the last differential equation is in the following form

$$u(t;\lambda) = \frac{1}{\sqrt{\rho(t)}} \exp\left(i(A - \lambda E)\int_a^\infty \frac{ds}{\rho(s)}\right) f_\lambda$$

$$+ \frac{i}{\sqrt{\rho(t)}} \int_t^\infty \exp\left(i(A - \lambda E)\int_s^t \frac{d\tau}{\rho(\tau)}\right) \frac{f(s)}{\sqrt{\rho(s)}} ds,\ \ f_\lambda \in H,\ \ t > a.$$

In this case

$$\left\|\frac{1}{\sqrt{\rho(t)}} \exp\left(i(A - \lambda E)\int_a^t \frac{ds}{\rho(s)}\right) f_\lambda\right\|^2_{L^2(H,(a,\infty))} = \frac{1}{2\lambda_i}\left(\exp\left(2\lambda_i \int_a^\infty \frac{ds}{\rho(s)}\right) - 1\right)\|f_\lambda\|_H^2 < \infty$$

and

$$\left\|\frac{i}{\sqrt{\rho(t)}} \int_t^\infty \exp\left(i(A - \lambda E)\int_s^t \frac{d\tau}{\rho(\tau)}\right)\frac{f(s)}{\sqrt{\rho(s)}} ds\right\|^2_{L^2(H,(a,\infty))}$$

$$= \int_a^\infty \frac{1}{\rho(t)} \left\|\int_t^\infty \exp\left(i(A - \lambda E)\int_s^t \frac{d\tau}{\rho(\tau)}\right)\frac{f(s)}{\sqrt{\rho(s)}} ds\right\|_H^2 dt$$

$$\leq \int_a^\infty \frac{1}{\rho(t)} \left[\int_t^\infty \left\|\exp\left(i(A - \lambda E)\int_s^t \frac{d\tau}{\rho(\tau)}\right)\right\|_H \frac{\|f(s)\|_H}{\sqrt{\rho(s)}} ds\right]^2 dt$$

$$= \int_a^\infty \frac{1}{\rho(t)} \left[\int_t^\infty \exp\left(\lambda_i \int_s^t \frac{d\tau}{\rho(\tau)}\right)\frac{\|f(s)\|_H}{\sqrt{\rho(s)}} ds\right]^2 dt$$

$$\leq \int_a^\infty \frac{1}{\rho(t)} \left(\int_t^\infty \frac{1}{\rho(s)} \exp\left(2\lambda_i \int_s^t \frac{d\tau}{\rho(\tau)}\right) ds\right)\left(\int_t^\infty \|f(s)\|_H^2 ds\right) dt$$

$$\leq \int_a^\infty \frac{1}{\rho(t)} \left(\int_a^\infty \frac{1}{\rho(s)} \exp\left(2\lambda_i \int_s^t \frac{d\tau}{\rho(\tau)}\right) ds\right) dt\, \|f\|^2_{L^2(H,(a,\infty))}$$

$$= \frac{1}{4\lambda_i^2}\left(\exp\left(2\lambda_i \int_a^\infty \frac{d\tau}{\rho(\tau)}\right) + \exp\left(-2\lambda_i \int_a^\infty \frac{d\tau}{\rho(\tau)}\right) - 2\right) \|f\|^2_{L^2(H,(a,\infty))} < \infty.$$

Hence for $u(\cdot,\lambda) \in L^2(H,(a,\infty))$ for $\lambda \in \mathbb{C}$, $\lambda_i = \operatorname{Im}\lambda \geq 0$.

Using this and boundary condition, we have

$$\left(\exp\left(i\lambda \int_a^\infty \frac{ds}{\rho(s)}\right) - W \exp\left(iA \int_a^\infty \frac{ds}{\rho(s)}\right)\right) f_\lambda$$

$$= i \exp\left(i\lambda \int_a^\infty \frac{ds}{\rho(s)}\right) \int_a^\infty \exp\left(i(A-\lambda) \int_s^a \frac{d\tau}{\rho(\tau)}\right) \frac{f(s)}{\sqrt{\rho(s)}} ds.$$

In order to get $\lambda \in \sigma(L_W)$, the necessary and sufficient condition is

$$\exp\left(i\lambda \int_a^\infty \frac{ds}{\rho(s)}\right) = \mu \in \sigma\left(W \exp\left(iA \int_a^\infty \frac{ds}{\rho(s)}\right)\right).$$

Consequently,

$$\lambda = \left(\int_a^\infty \frac{ds}{\rho(s)}\right)^{-1} (i \ln|\mu|^{-1} + \arg\mu + 2n\pi), \;\; n \in \mathbb{Z}.$$

**Remark.** In the special case the representation of selfadjoint extensions of corresponding mentioned above minimal operator and their spectral analysis on right semi-axis have been studied in [4].

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**INTERNATIONAL CONFERENCE ON MATHEMATICS**
**“An Istanbul Meeting for World Mathematicians”**
**3-5 July 2019, Istanbul, Turkey**

# The *(p,q)*-Chebyshev polynomial coefficients for a certain subclass of analytic and bi-univalent functions

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In the present investigation, we consider the *(p,q)*-Chebyshev polynomials to find upper bounds for a certain subclass of analytic and bi-univalent functions. We also determine the Fekete-Szegö functional for functions in this class.

**Keywords:** *(p,q)*-Chebyshev polynomials, coefficient estimates, analytic functions, bi-univalent functions, subordination.

## 1. Introduction

Let $\mathbb{R}=(-\infty,\infty)$ be the set of real numbers, $\mathbb{C}$ be the set of complex numbers and

$$\mathbb{N}:=\{1,2,3,\ldots\}=\mathbb{N}_0\setminus\{0\}$$

be the set of positive integers. Let also $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$ be open unit disc in $\mathbb{C}$.

For any integer $n\geq 2$ and $0<q<p\leq 1$, the *(p,q)*-Chebyshev polynomials of the second kind is defined by the following recurrence relations:

$$U_n(x,s,p,q)=(p^n+q^n)xU_{n-1}(x,s,p,q)+(pq)^{n-1}sU_{n-2}(x,s,p,q)$$

with the initial values $U_0(x,s,p,q)=1$, $U_1(x,s,p,q)=(p+q)x$ and $s$ is a real variable.

In the light of this recurrence relation, we will give the following interesting table:

| $x$ | $s$ | $p$ | $q$ | $U_n(x,s,p,q)$ | *(p,q)*-Chebyshev polynomials of the second kind |
|---|---|---|---|---|---|
| $x/2$ | $s$ | $p$ | $q$ | $F_n(x,s,p,q)$ | *(p,q)*-Fibonacci polynomials |
| $x$ | -1 | 1 | 1 | $U_n(x)$ | Second kind of Chebyshev polynomials |
| $x/2$ | 1 | 1 | 1 | $F_{n+1}(x)$ | Fibonacci polynomials |
| $1/2$ | 1 | 1 | 1 | $F_{n+1}$ | Fibonacci numbers |
| $x$ | 1 | 1 | 1 | $P_{n+1}(x)$ | Pell polynomials |
| 1 | 1 | 1 | 1 | $P_{n+1}$ | Pell numbers |
| $1/2$ | $2y$ | 1 | 1 | $J_{n+1}(y)$ | Jacobsthal polynomials |

| | | | | | |
|---|---|---|---|---|---|
| 1/2 | 2 | 1 | 1 | $J_{n+1}$ | Jacobsthal numbers |

These polynomials defined recursively over the integers share numerous interesting properties and have been extensively studied. They have been also found to be topics of interest in many different areas of pure and applied science, please see the papers Doha 1994, Mason 1967, Filipponi and Horadam 1991, Wang and Zhang 2012 and closely related references therein.

Very recently, Kızılateş et al. (2012) defined *(p,q)*-Chebyshev polynomials of the first and second kind and derived explicit formulas, generating functions and some interesting properties of these polynomials. The generating function of the *(p,q)*-Chebyshev polynomials of the second kind is as follows:

$$G_{p,q}(z)=\frac{1}{1-xpz\tau_p-xqz\tau_q-spqz^2\tau_{p,q}}$$

$$=\sum_{n=0}^{\infty}U_n(x,s,p,q)z^n \quad (z\in\mathbb{U}),$$

where the Fibonacci operator $\tau_p$ was introduced in Mason and Handscomb (2003), by $\tau_q f(z)=f(qz)$. Similarly, $\tau_{p,q}f(z)=f(pqz)$.

Let $A$ be the class of functions $f$ of the form:

$$f(z)=z+a_2z^2+a_3z^3+\cdots, \tag{1}$$

which are analytic in the open unit disc $\mathbb{U}$ and normalized by $f(0)=0,\ f'(0)=1$. Further, by $S$ we shall denote the class of all functions in $A$ which are univalent in $\mathbb{U}$.

For two analytic functions, $f$ and $g$, such that $f(0)=g(0)$, we say that $f$ is subordinate to $g$ in $\mathbb{U}$ and write $f(z)\prec g(z),\ z\in\mathbb{U}$, if there exists a Schwarz function $w(z)$ with $w(0)=0$ and $|w(z)|\le|z|,\ z\in\mathbb{U}$ such that $f(z)=g(w(z)),\ z\in\mathbb{U}$.

The Koebe-One Quarter Theorem (Duren 1983) ensures that the image of $\mathbb{U}$ under every univalent function $f\in A$ contains a disc of radius ¼. Thus every univalent function $f$ has an inverse $f^{-1}$ satisfying $f^{-1}(f(z))=z,\ z\in\mathbb{U}$ and $f(f^{-1}(w))=w,\ (|w|<r_0(f),\ r_0(f)\ge 1/4)$. The inverse function $f^{-1}$ is given by

$$g(w)=f^{-1}(w)=w-a_2w^2+(2a_2^2-a_3)w^3-(5a_2^3-5a_2a_3+a_4)w^3+\cdots. \tag{2}$$

A function $f\in A$ is said to be bi-univalent in $\mathbb{U}$ if both $f$ and $f^{-1}$ are univalent in $\mathbb{U}$. Let $\Sigma$ denote the class of bi-univalent functions defined in $\mathbb{U}$. For a brief history and interesting examples in the class

$\Sigma$, see Srivastava et al. 2010 (see also Altınkaya and Yalçın 2015, Brannan and Clunie 1979, Brannan and Taha 1986, Hussain et al. 2017, Khan et al. 2017, Lewin 1967**)**. However, there are only a few works determining the general coefficient bounds $|a_n|$ for the analytic bi-univalent functions in the literature, that is, the coefficient estimate problem for each of

$$|a_n|, \; n \in \mathbb{N} - \{1,2,3\} \; \left(\mathbb{N} = \{1,2,3,\ldots\}\right)$$

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The classical Fekete-Szegö inequality, presented by means of Loewner's method, for the coefficients of $f \in S$ is

$$\left|a_3 - \rho a_2^2\right| \le 1 + 2\exp(-2\rho/(1-\rho)) \quad \text{for } \rho \in [0,1).$$

As $\rho \to 1$, we have the elementary inequality $\left|a_3 - a_2^2\right| \le 1$. Moreover, the coefficient functional $\Gamma_\rho(f) = a_3 - \rho a_2^2$ on the normalized analytic functions $f$ in the unit disc $\mathbb{U}$ plays an important role in function theory. The problem of maximizing the absolute value of the functional $\Gamma_\rho(f)$ is called the Fekete-Szegö problem (Fekete-Szegö 1933).

Now, we will de.ne the bi-univalent function class $M_\Sigma(\lambda; p,q)$ associated with the *(p,q)*-Chebyshev Chebyshev polynomials.

**Definition 1.2.** A function $f \in \Sigma$ given by (1) is said to belong to the class

$$M_\Sigma(\lambda; p,q) \quad \left(0 \le \lambda < 1, z, w \in \mathbb{U}\right)$$

if the following subordinations are satisfied:

$$\frac{zf'(z)}{(1-\lambda)f(z) + \lambda z f'(z)} \prec G_{p,q}(z)$$

and

$$\frac{wg'(w)}{(1-\lambda)g(w) + \lambda w g'(w)} \prec G_{p,q}(w)$$

where the function $g$ is the extension of $f^{-1}$ to $\mathbb{U}$.

**Remark 1.3.** It should be remarked that the class $M_\Sigma(\lambda; p,q)$ is a generalization of well-known classes consider earlier. For example:

- For $\lambda = 0, p = q = 1$ and $s = -1$, the class $M_\Sigma(\lambda; p,q)$ reduce to the class $S_\Sigma^{\eta,\mu}(t)$ which was introduced recently by Altınkaya and Yalçın (2017).

## 2. Coefficient estimates

In this section, we will derive the *(p,q)*-Chebyshev polynomial bounds for the initial coefficients and determine the Fekete-Szegö functional for $f \in M_\Sigma(\lambda; p,q)$.

**Theorem 2.1.** Let the function $f$ given by (1) be in the class $M_\Sigma(\lambda; p,q)$. Then

$$|a_2| \le \frac{(p+q)x\sqrt{(p+q)x}}{\sqrt{\left|(\lambda^2-2\lambda+1)(p+q)^2x^2-(1-\lambda)^2\left[(p^2+q^2)(p+q)x^2+pqs\right]\right|}},$$

$$|a_3| \le \frac{(p+q)x}{2(1-\lambda)}\left[1+\frac{2(p+q)x}{1-\lambda}\right]$$

and for any real number $\rho$

$$\left|a_3-\rho a_2^2\right| \le$$

$$\begin{cases} \dfrac{(p+q)x}{2(1-\lambda)}, & |1-\rho| \le \dfrac{(1-\lambda)}{2}\left|1-\left(\dfrac{p^2+q^2}{p+q}+\dfrac{pqs}{(p+q)^2x^2}\right)\right| \\ \dfrac{|1-\rho|(p+q)^3x^3}{\left|(\lambda^2-2\lambda+1)(p+q)^2x^2-(1-\lambda)^2\left[(p^2+q^2)(p+q)x^2+pqs\right]\right|}, & |1-\rho| \ge \dfrac{(1-\lambda)}{2}\left|1-\left(\dfrac{p^2+q^2}{p+q}+\dfrac{pqs}{(p+q)^2x^2}\right)\right| \end{cases}.$$

**Proof.** Suppose that $f \in M_\Sigma(\lambda; p,q)$. In view of the definition of subordination, we can write

$$\frac{zf'(z)}{(1-\lambda)f(z)+\lambda zf'(z)} = G_{p,q}(\Phi(z)),$$

$$\frac{zf'(z)}{(1-\lambda)f(z)+\lambda zf'(z)} = G_{p,q}(\psi(w))$$

or,

$$\frac{zf'(z)}{(1-\lambda)f(z)+\lambda zf'(z)} = U_0(x,s,p,q)+U_1(x,s,p,q)\Phi(z)+U_2(x,s,p,q)\Phi^2(z)+\cdots, \tag{4}$$

$$\frac{wg'(w)}{(1-\lambda)g(w)+\lambda wg'(w)} = U_0(x,s,p,q)+U_1(x,s,p,q)\psi(w)+U_2(x,s,p,q)\psi^2(w)+\cdots \tag{5}$$

for some analytic functions $\Phi, \psi$ such that $\Phi(0)=\psi(0)=0$, $|\Phi(z)| = |m_1z+m_2z^2+\cdots| < 1$, $|\psi(w)| = |n_1w+n_2w^2+\cdots| < 1$ and

$$|m_i| \le 1,\ |n_i| \le 1,\ \forall i \in \mathbb{N}. \tag{6}$$

In the light of (4) and (5), we obtain

$$(1-\lambda)a_2 = U_1(x,s,p,q)m_1, \tag{7}$$

$$(\lambda^2-1)a_2^2+2(1-\lambda)a_3 = U_1(x,s,p,q)m_2+U_2(x,s,p,q)m_1^2, \tag{8}$$



$$-(1-\lambda)a_2 = U_1(x,s,p,q)n_1, \tag{9}$$

$$(\lambda^2-4\lambda+3)a_2^2-2(1-\lambda)a_3 = U_1(x,s,p,q)n_2+U_2(x,s,p,q)n_1^2. \tag{10}$$

From (7) and (9), it follows that

$$m_1 = -n_1 \tag{11}$$

and

$$2(1-\lambda)^2 a_2^2 = U_1^2(x,s,p,q)(m_1^2+n_1^2). \tag{12}$$

If we add (8) to (10), we get

$$2(\lambda^2-2\lambda+1)a_2^2 = U_1(x,s,p,q)(m_2+n_2)+U_2(x,s,p,q)(m_1^2+n_1^2). \tag{13}$$

Therefore, by using (6) and (11) in the equality (13), we immediately have

$$|a_2| \le \frac{(p+q)x\sqrt{(p+q)x}}{\sqrt{\left|(\lambda^2-2\lambda+1)(p+q)^2x^2-(1-\lambda)^2\left[(p^2+q^2)(p+q)x^2+pqs\right]\right|}},$$

Next, if we subtract (10) from (8), we have

$$4(1-\lambda)(a_3-a_2^2) = U_1(x,s,p,q)(m_2-n_2). \tag{14}$$

In view of (6) and (12), we readily get the bound on $|a_3|$ as asserted in Theorem 2.1.

From (13) and (14), we have

$$a_3-\rho a_2^2 = \frac{U_1^3(x,s,p,q)(1-\rho)(m_2+n_2)}{2\left\{(\lambda^2-2\lambda+1)U_1^2(x,s,p,q)-(1-\lambda)^2U_2(x,s,p,q)\right\}}$$

$$+\frac{U_1(x,s,p,q)(m_2-n_2)}{4(1-\lambda)}$$

$$= U_1(x,s,p,q)\left[\left(h(\rho)+\frac{1}{4(1-\lambda)}\right)m_2+\left(h(\rho)-\frac{1}{4(1-\lambda)}\right)n_2\right],$$

where

$$h(\rho) = \frac{U_1^2(x,s,p,q)(1-\rho)}{2\left\{(\lambda^2-2\lambda+1)U_1^2(x,s,p,q)-(1-\lambda)^2U_2(x,s,p,q)\right\}}.$$

Along the way, we conclude that

$$|a_3-\rho a_2^2| \le \begin{cases} \dfrac{(p+q)x}{2(1-\lambda)}, & 0 \le |h(\rho)| \le \dfrac{1}{4(1-\lambda)} \\ 2(p+q)x|h(\rho)|, & |h(\rho)| \ge \dfrac{1}{4(1-\lambda)} \end{cases}.$$

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**INTERNATIONAL CONFERENCE ON MATHEMATICS**
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**3-5 July 2019, Istanbul, Turkey**

# Some applications of the *(p,q)*-Lucas polynomials to an unified class of bi-univalent functions

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In the present paper, by using the $L_{p,q,n}(x)$ functions, our methodology intertwine to yield the Theory of Geometric Functions and that of Special Functions, which are usually considered as very different fields. Thus, also making use of the Ruscheweyh derivative operator $D^n$, we aim at introducing a new class of bi-univalent functions defined through the *(p,q)*-Lucas polynomials. Furthermore, we derive coefficient inequalities and obtain Fekete-Szegö problem for this new function class.

**Keywords:** *(p,q)*-Lucas polynomials, coefficient bounds, bi-univalent functions, Ruscheweyh derivative operator.

## 1. Introduction

Let $A$ be the class of functions $f$ of the form:

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots, \tag{1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by $f(0) = 0,\ f'(0) = 1$. Further, by $S$ we shall denote the class of all functions in $A$ which are univalent in $\mathbb{U}$.
For two analytic functions, $f$ and $g$, such that $f(0) = g(0)$, we say that $f$ is subordinate to $g$ in $\mathbb{U}$ and write $f(z) \prec g(z),\ z \in \mathbb{U}$, if there exists a Schwarz function $w(z)$ with $w(0) = 0$ and $|w(z)| \le |z|,\ z \in \mathbb{U}$ such that $f(z) = g(w(z)),\ z \in \mathbb{U}$.

The Koebe-One Quarter Theorem (Duren 1983) ensures that the image of $\mathbb{U}$ under every univalent function $f \in A$ contains a disc of radius ¼. Thus every univalent function $f$ has an inverse $f^{-1}$ satisfying $f^{-1}(f(z)) = z,\ z \in \mathbb{U}$ and $f(f^{-1}(w)) = w,\ (|w| < r_0(f),\ r_0(f) \ge 1/4)$. The inverse function $f^{-1}$ is given by

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^3 + \cdots. \tag{2}$$

A function $f \in A$ is said to be bi-univalent in $\mathbb{U}$ if both $f$ and $f^{-1}$ are univalent in $\mathbb{U}$. Let $\Sigma$ denote the class of bi-univalent functions defined in $\mathbb{U}$. For a brief history and interesting examples in the class $\Sigma$, see Srivastava et al. 2010 (see also Altınkaya and Yalçın 2015, Brannan and Clunie 1979, Brannan and Taha 1986, Lewin 1967, Nehanyahu 1969**)**. However, there are only a few works determining the general coefficient bounds $|a_n|$ for the analytic bi-univalent functions in the literature, that is, the coefficient estimate problem for each of



$$|a_n|,\ n \in \mathbb{N} - \{1,2,3\}\ \left(\mathbb{N} = \{1,2,3,\ldots\}\right)$$

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The classical Fekete-Szegö inequality, presented by means of Loewner's method, for the coefficients of $f \in S$ is

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For $f \in A$, Ruscheweyh (1975) defined the symbol $D^n f(z)$ by

$$D^n f(z) = \frac{z}{n!}\frac{d^n}{dz^n}\left[z^{n-1} f(z)\right] \quad \left(n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\right).$$

This symbol referred to as the the $n^{th}$ order Ruscheweyh derivative of $f$ by Al- Amiri (1980). Given $f$ of the form (1), we notice that

$$D^n f(z) = \frac{z}{(1-z)^{n+1}} * f(z) = z + \sum_{k=2}^{\infty} \binom{n+k-1}{n} a_k z^k$$

for all $z \in \mathbb{U}$, where the operator "*" is the usual Hadamard product of series, that is, if $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$, then $(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k$.

Fibonacci polynomials, Lucas polynomials, Lucas-Lehmer polynomials, Chebychev polynomials, Pell polynomials, Morgan-Voyce polynomials, Orthogonal polynomials and the other special polynomials and their generalizations are of wide spectra in a variety of branches such as Physics, Engineering, Architecture, Nature, Art, Number Theory, Combinatorics and Numerical analysis (see, for example, Lupas 1999, Özkoç and Porsuk 2017, Filipponi and Horadam 1991, Velluci and Bersani 2016).

The well-known *(p,q)*-Lucas polynomials are defined by the following definition:

**Definition 1.1.** (Lee and Aşçı 2012) Let $p(x)$ and $q(x)$ be polynomials with real coefficients. The *(p,q)*-Lucas polynomials $L_{p,q,n}(x)$ are established by the recurrence relation

$$L_{p,q,n}(x) = p(x)L_{p,q,n-1}(x) + q(x)L_{p,q,n-2}(x) \quad (n \geq 2),$$

from which the first few Lucas polynomials can be found as

$$L_{p,q,0}(x) = 2, \quad L_{p,q,1}(x) = p(x), \quad L_{p,q,2}(x) = p^2(x) + 2q(x),$$



$$L_{p,q,3}(x) = p^3(x) + 3p(x)q(x), \ldots. \tag{3}$$

For the special cases of $p(x)$ and $q(x)$, we can get the polynomials given in the following table.

| $p(x)$ | $q(x)$ | $L_{p,q,n}(x)$ |
|---|---|---|
| $x$ | $1$ | Lucas polynomials $L_n(x)$ |
| $2x$ | $1$ | Pell-Lucas polynomials $D_n(x)$ |
| $1$ | $2x$ | Jacobsthal-Lucas polynomials $j_n(x)$ |
| $3x$ | $-2$ | Fermat-Lucas polynomials $f_n(x)$ |
| $2x$ | $-1$ | Chebyshev polynomials first kind $T_n(x)$ |

**Theorem 1.1.** (Lee and Aşçı 2012) Let $G_{\{L_{p,q,n}(x)\}}(z)$ be the generating function of the *(p,q)*-Lucas polynomials sequence $L_{p,q,n}(x)$. Then

$$G_{\{L_{p,q,n}(x)\}}(z)=\sum_{n=0}^{\infty}L_{p,q,n}(x)z^n=\frac{2-p(x)z}{1-p(x)z-q(x)z^2}.$$

**Definition 1.2.** A function $f\in\Sigma$ given by (1) is said to belong to the class

$$T_\Sigma^n(\tau;x)\quad\left(\tau\geq 1,z,w\in\mathbb{U}\right)$$

if the following subordinations are satisfied:

$$\left((1-\tau)\frac{D^n f(z)}{z}+\tau\left[D^n f(z)\right]'\right)\prec G_{\{L_{p,q,n}(x)\}}(z)-1$$

and

$$\left((1-\tau)\frac{D^n g(w)}{w}+\tau\left[D^n g(w)\right]'\right)\prec G_{\{L_{p,q,n}(x)\}}(w)-1,$$

where the function $g$ is the extension of $f^{-1}$ to $\mathbb{U}$.

The remaining case is established by the following remark.

**Remark 1.1.** For $n=0,$ a function $f\in\Sigma$ given by (1) is said to belong to the class

$$T_\Sigma(\tau;x)\quad\left(\tau\geq 1,z,w\in\mathbb{U}\right)$$

if the following subordinations are satisfied:

$$\left((1-\tau)\frac{f(z)}{z}+\tau f'(z)\right)\prec G_{\{L_{p,q,n}(x)\}}(z)-1,$$

$$\left((1-\tau)\frac{g(w)}{w}+\tau g'(w)\right)\prec G_{\{L_{p,q,n}(x)\}}(w)-1.$$

## 2. Coefficient estimates

In this section, In this section, we shall make use of the *(p,q)*-Lucas polynomials to get the estimates on the coefficients $|a_2|$, $|a_3|$ and provide the Fekete-Szegö inequalities for functions in the class $T_\Sigma^n(\tau;x)$ proposed by Definition 1.2.

**Theorem 2.1.** Let $f\in T_\Sigma^n(\tau;x)$. Then

$$|a_2|\le\frac{|p(x)|\sqrt{2|p(x)|}}{\sqrt{(n+1)\left|\left[(n+2)(1+2\tau)-2(n+1)(1+\tau)^2\right]p^2(x)-4(n+1)(1+\tau)^2q(x)\right|}},$$

$$|a_3|\le\frac{p^2(x)}{(n+1)^2(1+\tau)^2}+\frac{2|p(x)|}{(n+1)(n+2)(1+2\tau)}$$

and for $\zeta\in\mathbb{R}$

$$|a_3-\zeta a_2^2|\le$$

$$\begin{cases}\dfrac{2|p(x)|}{(n+1)(n+2)(1+2\tau)}, & |1-\zeta|\le\left|1-\dfrac{2(n+1)(1+\tau)^2}{(n+2)(1+2\tau)}\left(1+\dfrac{2q(x)}{p^2(x)}\right)\right|\\ \dfrac{2|p^3(x)||1-\zeta|}{(n+1)\left|\left[(n+2)(1+2\tau)-2(n+1)(1+\tau)^2\right]p^2(x)-4(n+1)(1+\tau)^2q(x)\right|}, & |1-\zeta|\ge\left|1-\dfrac{2(n+1)(1+\tau)^2}{(n+2)(1+2\tau)}\left(1+\dfrac{2q(x)}{p^2(x)}\right)\right|\end{cases}.$$

**Proof.** Suppose that $f\in T_\Sigma^n(\tau;x)$. In view of the definition of subordination, we can write

$$(1-\tau)\frac{D^nf(z)}{z}+\tau\left[D^nf(z)\right]'=-1+L_{p,q,0}(x)+L_{p,q,1}(x)\Phi(z)+L_{p,q,2}(x)\Phi^2(z)+\cdots,\tag{4}$$

$$(1-\tau)\frac{D^ng(w)}{w}+\tau\left[D^ng(w)\right]'=-1+L_{p,q,0}(x)+L_{p,q,1}(x)\psi(w)+L_{p,q,2}(x)\psi^2(w)+\cdots\tag{5}$$

for some analytic functions $\Phi,\psi$ such that $\Phi(0)=\psi(0)=0,$ $|\Phi(z)|=|t_1z+t_2z^2+\cdots|<1,$ $|\psi(w)|=|s_1w+s_2w^2+\cdots|<1$ and

$$|t_i|\le1,\ |s_i|\le1,\ \forall i\in\mathbb{N}.\tag{6}$$

In the light of (4) and (5), we obtain

$$(n+1)(1+\tau)a_2=L_{p,q,1}(x)t_1,\tag{7}$$

$$\frac{(n+1)(n+2)(1+2\tau)}{2}a_3=L_{p,q,1}(x)t_2+L_{p,q,2}(x)t_1^2,\tag{8}$$

$$-(n+1)(1+\tau)a_2=L_{p,q,1}(x)s_1,\tag{9}$$



$$\frac{(n+1)(n+2)(1+2\tau)}{2}(2a_2^2-a_3)=L_{p,q,1}(x)s_2+L_{p,q,2}(x)s_1^2.\tag{10}$$

From (7) and (9), it follows that

$$t_1 = -s_1 \tag{11}$$

and

$$2(n+1)^2(1+\tau)^2 a_2^2 = L_{p,q,1}^2(x)(t_1^2 + s_1^2). \tag{12}$$

Now, by adding (8) and (10), we obtain

$$(n+1)(n+2)(1+2\tau)a_2^2 = L_{p,q,1}(x)(t_2 + s_2) + L_{p,q,2}(x)(t_1^2 + s_1^2). \tag{13}$$

Therefore, by using (3), (6) and (11) in the equality (13), we immediately have

$$|a_2| \le \frac{|p(x)|\sqrt{2|p(x)|}}{\sqrt{(n+1)\left|\left[(n+2)(1+2\tau) - 2(n+1)(1+\tau)^2\right]p^2(x) - 4(n+1)(1+\tau)^2 q(x)\right|}}.$$

Additionaly, in order to calculate the bound on $|a_3|$, by subtracting (10) from (8), we obtain

$$(n+1)(n+2)(1+2\tau)(a_3 - a_2^2) = L_{p,q,1}(x)(t_2 - s_2). \tag{14}$$

In view of (3), (6) and (12), we readily get the bound on $|a_3|$ as asserted in Theorem 2.1.

From (13) and (14), we have

$$a_3 - \zeta a_2^2 = \frac{L_{p,q,1}^3(x)(1-\zeta)(t_2 + s_2)}{(n+1)\left[(n+2)(1+2\tau)L_{p,q,1}^2(x) - 2(n+1)(1+\tau)^2 L_{p,q,2}(x)\right]}$$

$$+\frac{L_{p,q,1}(x)(t_2 - s_2)}{(n+1)(n+2)(1+2\tau)}$$

$$= L_{p,q,1}(x)\left[\left(K(\zeta;x) + \frac{1}{(n+1)(n+2)(1+2\tau)}\right)t_2 + \left(K(\zeta;x) - \frac{1}{(n+1)(n+2)(1+2\tau)}\right)s_2\right]$$

where

$$K(\zeta;x) = \frac{L_{p,q,1}^2(x)(1-\zeta)}{(n+1)\left[(n+2)(1+2\tau)L_{p,q,1}^2(x) - 2(n+1)(1+\tau)^2 L_{p,q,2}(x)\right]}.$$

Along the way, in view of (3), we conclude that

$$|a_3 - \zeta a_2^2| \le \begin{cases} \dfrac{2|p(x)|}{(n+1)(n+2)(1+2\tau)}, & 0 \le |K(\zeta;x)| \le \dfrac{1}{(n+1)(n+2)(1+2\tau)} \\ 2|p(x)||K(\zeta;x)|, & |K(\zeta;x)| \ge \dfrac{1}{(n+1)(n+2)(1+2\tau)} \end{cases}.$$

**Corollary 2.1.** Let $f \in T_\Sigma(\tau;x)$. Then

$$|a_2| \le \frac{|p(x)|\sqrt{|p(x)|}}{\sqrt{\left|\tau^2 p^2(x) + 2(1+\tau)^2 q(x)\right|}},$$



$$|a_3| \le \frac{p^2(x)}{(1+\tau)^2} + \frac{2|p(x)|}{1+2\tau}$$

and for $\zeta \in \mathbb{R}$

$$\left|a_3-\zeta a_2^2\right| \le \begin{cases} \dfrac{|p(x)|}{1+2\tau}, & |1-\zeta| \le \left|1-\dfrac{(1+\tau)^2}{(1+2\tau)}\left(1+\dfrac{2q(x)}{p^2(x)}\right)\right| \\ \dfrac{|p^3(x)||1-\zeta|}{\left|\tau^2 p^2(x)+2(1+\tau)^2 q(x)\right|}, & |1-\zeta| \ge \left|1-\dfrac{(1+\tau)^2}{(1+2\tau)}\left(1+\dfrac{2q(x)}{p^2(x)}\right)\right| \end{cases}.$$

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**INTERNATIONAL CONFERENCE ON MATHEMATICS**
**"An Istanbul Meeting for World Mathematicians"**
**3-5 July 2019, Istanbul, Turkey**

## Optimal Control of Advection-Diffusion Process with a Bilinear Control

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**Abstract**

The aim of this study is to discuss an optimal control of advection-diffusion process governed by a bilinear control. The problem formulation, uniqueness and controllability of the system are examined. Performance index is defined as a measure of the dynamic response and a penalty term on control energy. The proposed approach uses a reduction of order in the model and the Pontryagin's maximum principle. Using a modal space expansion method the distributed parameter system is transformed into a lumped parameter system. The obtained system corresponds to a bilinear system in the temporal term. Pontryagin's maximum principle is used by means of introducing a suitable Hamiltonian to obtain the optimal control function. By using the Pontryagin's maximum principle optimal control problem is reduced to solve a nonlinear two-point boundary value problem.

**Keywords:** Bilinear control, one-dimensional advection-diffusion equation, Pontryagin's maximum principle.

## 1. Introduction

Bilinear systems are one of the most important subclasses of nonlinear systems due to their various applications in engineering and other fields. Linear models cannot represent many systems in which control is applied in multiplicative ways. These multiplicative controls yield bilinear systems (BLS). The BLS concept was first introduced by a U.S scientist, Mohler in the 1960s [1]. Bilinear systems are specific types of nonlinear systems. In order to approximate and analyze complex nonlinear systems, the BLS are used due to the simplicity of the system. The BLS include products of control and state, namely, they are linear in state and control but not jointly linear in control and state. The terms formed by multiplication of control vector and state vector make these systems nonlinear. Due to the fact that the linearization of nonlinear systems loses real features, bilinear systems are important to understand their natural properties and to improve their performance. Thus, modeling and control of nonlinear systems in a bilinear framework are crucial problems in engineering [2]. In various chemical and biological reactions, bilinear controls are used in modeling reaction diffusion-convection processes controlled via catalysts that can accelerate or decelerate the reaction [3]. Advection-diffusion equation is used to represent the heat transfer, fluid flow, mass transport etc. [4, 5, 6, 7].

In this study, optimal control of advection-diffusion process with a bilinear control is considered. The optimal control problem formulation, uniqueness and controllability of the system are examined. The presented approach is based on reduced order modeling and the Pontryagin's maximum principle. The obtained system corresponds to a bilinear system in the temporal term. Pontryagin's maximum principle is used to obtain the optimal control function that leads to a nonlinear two-point boundary value problem (TPBVP).

## 2. Bilinear-Quadratic Optimal Control Problem Setting

Consider the one-dimensional advection-diffusion equation with a source parameter

$$u_t + Uu_x = Du_{xx} + p(t)u + \phi(x,t) \tag{1}$$

where $u = u(x,t) \in \Omega = \Omega_x \times \Omega_t$ is the displacement of the system at position $x$ and time $t$, $D$ is the diffusion coefficient and $U$ is the constant advection velocity. $p(t)$ is the control function to be determined optimally with the source term $\phi(x,t)$. The set $\Omega_x = [0,\ell]$ is a subset of Euclidean space $\mathbb{R}^1$ and $\Omega_t$ denote a given time interval $(0,t_f)$ where $t_f$ is a predetermined terminal time. Eq. (1) is subjected to the following initial and boundary conditions, respectively;

$$u(x,0) = u_0(x) \tag{2}$$

$$u(0,t) = g_0(t), \quad u(\ell,t) = g_1(t) \tag{3}$$

for given functions $g_0(t)$, $g_1(t)$ and $u_0(x)$. Let the admissible control set be

$$P_{ad} = \{p(t) : p(t) \in L^2(\Omega_t)\}.$$

H (0, $\ell$) = $L^2$ (0, $\ell$) is a Hilbert space such that

$$H(0,\ell) = \{p(t) : (0,t_f) \to \mathbb{R}, \|p(t)\|^2 < \infty\}.$$

The performance index functional $J[p(t)]$ is specified as a weighted quadratic functional of the dynamic response which is to be minimized at the terminal time $t_f$ subject to Eq. $(1)-(3)$,

$$\mathcal{J}[p(t)] = \frac{1}{2}\int_{\Omega_x} [r_1 u^2(x,t_f) + r_2 u_t{}^2(x,t_f)]dx + \frac{1}{2}\int_{\Omega} qu^2(x,t)dxdt + \frac{1}{2}\int_{\Omega_t} sp^2(t)dt \tag{4}$$

where $r_1$, $r_2$, $q$ and $s$ are weighting constants satisfying the condition $r_1, r_2, q \geq 0$ and $s > 0$. The last term on the right hand side of Eq. (4) is a penalty term which limits expending large amounts of control effort.

It is desired to find optimal control function $p^*(t) \in P_{ad}$ that minimizes the performance index

$$\mathcal{J}[p^*(t)] = \min_{p(t) \in P_{ad}} \mathcal{J}[p(t)] \tag{5}$$

providing equations Eqs. $(1) - (3)$.

### 2.1. Uniqueness of the Solution and the Controllability

The existence of the problem Eqs. $(1) - (3)$ are discussed in [8]. The uniqueness of the problem is proved by means of energy methods, because the uniqueness of the solution of the system yields the uniqueness of the control.

**2.1.1. Lemma.** The solution $u(x,t) \in L^2(\mathbb{R}^N)$ of Eq. (1) subject to Eqs. $(2) - (3)$ is unique.

**Proof.** Suppose that $u_1$ and $u_2$ are two solutions to the system given by Eqs. $(1) - (3)$. Then, the difference function

$$\tilde{u} = u_1 - u_2$$

satisfies the following homogeneous equation

$$\tilde{u}_t + U\tilde{u}_x - D\tilde{u}_{xx} - p(t)\tilde{u} = 0$$

with zero initial-boundary conditions

$$\tilde{u}(0,t) = \tilde{u}(\ell,t) = 0, \qquad 0 < t < t_f$$

$$\tilde{u}(x,0) = 0. \quad 0 < x < \ell$$

Now define the following energy integral as follows

$$E[\tilde{u}] = \frac{1}{2}\int_0^\ell |\tilde{u}(x,t)^2| dx \tag{6}$$

Differentiating Eq. (6) with respect to time, and using the equation (1) gives

$$\frac{d}{dt}E = \int_0^\ell \tilde{u}\tilde{u}_t dx = \int_0^\ell \tilde{u}(D\tilde{u}_{xx} + p(t)\tilde{u} - U\tilde{u}_x)dx.$$

Using integrating by parts in the last integral leads to

$$\frac{d}{dt}E[\tilde{u}] = -D\int_0^\ell {\tilde{u}_x}^2 dx + \int_0^\ell p(t)\tilde{u}^2 dx \leq p(t)\|\tilde{u}\|_{L^2}^2$$

Then the following inequality is obtained

$$\frac{d}{dt}\|\tilde{u}\|_{L^2} \leq p(t)\|\tilde{u}\|_{L^2}$$

Applying the Gronwall's lemma gives

$$\|u_1 - u_2\|_{L^2} \leq \|u_1 - u_2\|_{t=0}\exp(\int_0^{t_f} p(r)dr)$$

As an immediate consequence uniqueness of solutions is obtained.

The control function $p(t)$ is unique to preserve the uniqueness of the solution provided by Lemma 2.1.1. The system $(1)-(3)$ is observable because the system has a unique solution and control function. By taking Hilbert Uniqueness into account, the observability is equivalent to the controllability [9], [10]. Briefly, the system is controllable.

## 3. Modal Control Space Problem

In the present section, the optimal control of distributed parameter system $(1)-(5)$ is transformed into the optimal control of lumped parameter system by implementing modal space expansion technique. New system gives rise to a bilinear system in the temporal term. In order to achieve the transformation, first a new parameter $w(x,t)$ is introduced to convert nonhomogeneous boundary conditions to homogeneous boundary conditions.

By letting

$$w(x,t) = u(x,t) - \frac{\ell - x}{\ell}g_0(t) - \frac{x}{\ell}g_1(t) \tag{7}$$

in (1), the following new distributed parameter system is obtained

$$w_t + aw_x - bw_{xx} = \frac{(x-\ell)}{\ell} g_{0t}(t) - \frac{x}{\ell} g_{1t}(t) + a\frac{g_0(t)}{\ell} - a\frac{g_1(t)}{\ell} + p(t)(w + \frac{\ell-x}{\ell} g_0(t) + \frac{x}{\ell} g_1(t)) + \phi(x,t) \quad (8)$$

subject to

$$w(x,0) = u_0(x) - \frac{\ell - x}{\ell} g_0(0) - \frac{x}{\ell} g_1(0) \quad (9)$$

$$w(0,t) = w(\ell,t) = 0 \quad (10)$$

**3.1. Theorem.** Any $w(x,t) \in \mathrm{H}(0,\ell)$ has a unique representation [11]

$$w(x,t) = \sum_{n=1}^{N} \psi_n(x) y_n(t) \quad (11)$$

where $\{\psi_n(x)\}_{n=1}^{\infty}$ is a complete orthonormal basis in $\mathrm{H}(0,\ell)$ and $y_n(t)$ is the temporal term.

Having a modal space expansion gives rise to an infinite-dimensional system theoretically which makes the problem physically insurmountable since there will be a large number of modes to control. Hence, a truncated Fourier series expansion of (11) is taken in the computations hereafter

$$w(x,t) \approx \sum_{n=1}^{N} \psi_n(x) y_n(t)$$

Denoting a complete orthonormal basis as

$$V = \left\{ v | v, \quad \frac{\partial v}{\partial x} \in \mathrm{H}(0,\ell) \; and \; v|_{\partial(0,\ell)} = 0 \right\} \quad (12)$$

and by multiplying both sides of (8) by a basis function $v$, and by integrating by parts, the solution $w(x,t)$ of the system satisfies

$$\int_0^\ell \frac{\partial w}{\partial t} v dx + U \int_0^\ell \frac{\partial w}{\partial x} v dx - D \int_0^\ell \frac{\partial^2 w}{\partial x^2} v dx = \int_0^\ell \left( \frac{(x-\ell)}{\ell} g_{0t}(t) - \frac{x}{\ell} g_{1t}(t) \right) v dx + U \int_0^\ell \left( \frac{g_0}{\ell} - \frac{g_1}{\ell} \right) v dx + \int_0^\ell p \left( w + \frac{(\ell-x)}{\ell} g_0 - \frac{x}{\ell} g_1 \right) v dx + \int_0^\ell \phi(x,t) v dx$$

where $w, v \in V$. If the expression (11) for $w(x,t)$ and $v = \psi m,\ m = 1,2,\ldots$ are used the finite dimensional system is got:

$$\frac{dz}{dt} = Lz + pz + Gp + S \quad (13)$$

where $z(t) = (z_1(t), z_2(t), \ldots, z_n(t))^T \in \mathbb{R}^N$, $M, N, K, L \in \mathbb{R}^{N\times N}$ , $F, G, H, S$ , $S \in \mathbb{R}^N$ and $p = p(t)$ is control function. The vector $z(t)$ is the finite dimensional approximation to the temporal term. The initial values are determined by

$$z_m(0) = (\mathrm{w}(\mathrm{x}, 0), \psi_m), \tag{14}$$

$m = 1,2, \ldots, N.$

In Eq. (13), the following notations are used ($\mathcal{D} \triangleq \partial/\partial \mathrm{x}$):

$$M_{mn} = (\psi_m, \psi_n) = \int_0^\ell \psi_m(x)\psi_n(x)dx = \delta_{mn}, \tag{15}$$

$$\delta_{mn} = \begin{cases} 1, & if\ m = n \\ 0, & otherwise \end{cases}$$

$$N_{mn} = U(\psi_m, \mathcal{D}\psi_n) = U\int_0^\ell \psi_m(x)\frac{\partial(\psi_n(x))}{\partial x}dx, \tag{16}$$

$$K_{mn} = D(\psi_m, \mathcal{D}^2\psi_n) = D\int_0^\ell \psi_m(x)\frac{\partial^2(\psi_n(x))}{\partial x^2}dx, \tag{17}$$

$$F_m = \int_0^\ell \left(\frac{(x-\ell)}{\ell}g_{0t} - \frac{x}{\ell}g_{1t} + U\frac{g_0}{\ell} - U\frac{g_1}{\ell}\right)\psi_m(x)dx, \tag{18}$$

$$G_m = \int_0^\ell \left(\frac{(\ell-x)}{\ell}g_0 + \frac{x}{\ell}g_1\right)\psi_m(x)dx, \tag{19}$$

$$H_m = \int_0^\ell \phi(x,t)\psi_m(x)dx, \tag{20}$$

$$S_m = F_m + H_m, \tag{21}$$

$$L_{mn} = K_{mn} - N_{mn}, \tag{22}$$

## 4. Derivation of the Pontryagin's Maximum Principle for the Bilinear System

**4.1. Theorem.** If the Pontryagin's Maximum Principle is applied to the bilinear system in (13), a canonical optimality condition is obtained,

$$\left.\begin{aligned} \dot{z}(t) &= Lz(t) - s^{-1}\Lambda^T(t)(z(t)+G)^2 + S \\ \dot{\Lambda}(t) &= -Qz(t) - L^T\Lambda(t) - s^{-1}(z(t)+G)^T\Lambda^2(t) \\ \Lambda(t_f) &= R_1 z(t_f) + R_2\dot{z}(t_f) \\ z(t_0) &= z_0 \end{aligned}\right\} \tag{23}$$

which is a nonlinear two-point boundary value problem (TPBVP). In $(23)$, $Q, R_1$ and $R_2$ are positive semidefinite symmetric $n \times n$ matrices, $G, S, L$ are defined in $(19), (21)$ and $(22)$, respectively.

**Proof.**

Consider the optimal control problem of the bilinear system $(13) - (14)$

$$\frac{dz}{dt} = Lz + pz + Gp + S$$

where $z(t)$ is the finite dimensional approximation of $w(x,t)$ and $p(t)$ is the control input. The quadratic cost functional is given by

$$min_p \ \mathcal{J} = \frac{1}{2}[z(t_f)]^T R_1 [z(t_f)] + \frac{1}{2}[\dot{z}(t_f)]^T R_2 [\dot{z}(t_f)] + \frac{1}{2}\int_{t_0}^{t_f} z(t)^T Q z(t) dt + \frac{1}{2}\int_{t_0}^{t_f} sp^2(t) dt$$

where $R_{1mn} = \int_0^\ell r_1 \psi_m(x)\psi_n(x)dx$, $R_{2mn} = \int_0^\ell r_2 \psi_m(x)\psi_n(x)dx$ and $Q_{mn} = \int_0^\ell q \psi_m(x)\psi_n(x)dx$, respectively, for $m, n = 1,2, \ldots, N$.

Using the augmented cost functional, the cost functional is minimized

$$\mathcal{J}^*[z,p,\Lambda] = \int_{t_0}^{t_f} \{\frac{1}{2}(z^T Q z + sp^2) - \Lambda^T(t)(\dot{z}(t) - Lz(t) - p(t)z(t) - Gp(t) - S)\} dt$$
$$+ \ \frac{1}{2}[z(t_f)]^T R_1 [z(t_f)] + \ \frac{1}{2}[\dot{z}(t_f)]^T R_2 [\dot{z}(t_f)].$$

Introducing the so-called Hamiltonian,

$$\mathcal{H}(t,z,p,\Lambda) = \frac{1}{2}(z(t)^T Q z(t) + sp^2(t)) + \Lambda^T(t)(Lz(t) + p(t)z(t) + Gp(t) + S),$$

the augmented functional becomes

$$\mathcal{J}^*[z,p,\Lambda] = \int_{t_0}^{t_f} [\mathcal{H}(t,z,p,\Lambda) - \Lambda^T(t)\dot{z}(t)] dt + \frac{1}{2}[z(t_f)]^T R_1 [z(t_f)] \ + \ \frac{1}{2}[\dot{z}(t_f)]^T R_2 [\dot{z}(t_f)]. \qquad (24)$$

If $(z, p, \Lambda)$ is a minimizer of $\mathcal{J}^*$

$$\delta \mathcal{J}^* = \int_{t_0}^{t_f} [\frac{\partial \mathcal{H}}{\partial z} \delta z + \frac{\partial \mathcal{H}}{\partial p} \delta p + \frac{\partial \mathcal{H}}{\partial \Lambda} \delta \Lambda - \delta (\Lambda^T(t)\dot{z}(t))] dt + \delta [\frac{1}{2} z(t_f)^T R_1 z(t_f) ] +$$
$$\delta \left[\frac{1}{2}\dot{z}(t_f)^T R_2 \dot{z}(t_f)\right] = 0$$

After processing of variation operations and using integrating by parts the followings are obtained firstly,

$$\mathcal{H}_p = 0$$

$$p(t) = -s^{-1}\Lambda^T(t)(z(t) + G)$$

Secondly,

$$\mathcal{H}_z + \dot{\Lambda}^T = 0$$

$$\dot{\Lambda}^T = -Qz(t) - L^T\Lambda(t) - p^T\Lambda(t)$$

Thirdly,

$$\mathcal{H}_\Lambda - \dot{z}(t) = 0$$

$$\dot{z}(t) = Lz(t) - s^{-1}\Lambda^T(t)(z(t) + G)^2 + S$$

Lastly,

$$[z(t_f)]^T R_1 + [\dot{z}(t_f)]^T R_2 - \Lambda^T(t_f) = 0$$

$$R_1 z(t_f) + R_2\dot{z}(t_f) = \Lambda(t_f)$$

In this proof, Pontryagin's maximum principle leads to a nonlinear two-point boundary-value problem that cannot be solved analytically to obtain the control law. The difficulty of solving this optimal control problem is caused by the combination of split boundary values and nonlinear differential equations.

## 4. Conclusion

In this paper, an optimal control problem of the advection-diffusion process with bilinear control by means of Pontryagin's Maximum Principle is studied. Uniqueness of the solution and controllability of the system are considered. Using a modal space expansion method the distributed parameter system is transformed into a lumped parameter system. The obtained system corresponds to a bilinear system in the temporal term. Pontryagin's maximum principle is used to obtain the optimal control function that leads to a nonlinear two-point boundary value problem (TPBVP). The difficulty of solving the TPBVP involving state and costate equations is caused by combination of nonlinear equations and split boundary values. To further this study, iterative numerical techniques are to be implemented to solve the nonlinear TPBVP.

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# New Inequalities of Hermite-Hadamard Type for Differentiable s-Preinvex Functions

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**Abstract**

In this study, some new inequalities of the right-hand side of Hermite-Hadamard type for differentiable s-preinvex functions are established. Also some parallel results are obtained which are based on preincavity.

**Keywords:** Hermite-Hadamard type inequality, Hölder's inequality, s-preinvex function

## 1. Introduction

**Definition 1.** A function $f: \emptyset \neq I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex on $I$ if the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$.

Many inequalities have been established for convex functions but the most famous is the Hermite-Hadamard inequality, which is given as (see [4]) :

Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval $I$ of real numbers and $a, b \in I$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

In recent years, many mathematicians generalized the classical convexity in many ways and some of those are given as follows.

**Definition 2 [5].** Let $K$ be a closed set in $\mathbb{R}^n$. Suppose that $f: K \to \mathbb{R}$ and $\eta: K \times K \to \mathbb{R}$ be continuous functions. Let $u \in K$, then the set $K$ is said to be invex at each $u$ with respect to $\eta(.,.)$, if $u + t\eta(v,u) \in K, \forall u, v \in K, t \in [0,1]$.

$K$ is said to be an invex set with respect to $\eta$, if $K$ is invex at each $u \in K$. The invex set $K$ is also called $\eta$-connected set.

Note that if $\eta(v,u) = v - u$, invexity reduces to convexity. Thus, every convex set is also an invex set with respect to $\eta(v,u) = v - u$, but the converse is not true in general.

**Definition 3 [5].** The function $f$ on the invex set $K$ is said to be preinvex with respect to $\eta$, if the inequality

$$f\big(u + t\eta(v,u)\big) \leq (1-t)f(u) + tf(v)$$

holds for all $u, v \in K$ and $t \in [0,1]$.

The function $f$ is said to be preincave if and only if $-f$ is preinvex.

In [3], Noor has obtained the new form of Hermite-Hadamard inequality for the preinvex functions:

**Theorem 4 [3].** Let $f: K = [a, a + \eta(b,a)] \to (0,\infty)$ be a preinvex function on the interval of the real numbers $K^\circ$ (the interior of $K$) and $a, b \in K^\circ$ with $\eta(b,a) > 0$. Then the following inequalities hold:

$$f\left(a + \frac{1}{2}\eta(b,a)\right) \leq \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx \leq \frac{f(a) + f\big(a + \eta(b,a)\big)}{2}.$$

Recently, Li [2] introduced the notion of $s$-preinvexity and established Hermite-Hadamard type inequalities for this class of functions.

**Definition 5 [2]**. Let $K \subseteq [0,\infty)$ be an invex set with respect to $\eta: K \times K \to \mathbb{R}$. A function $f: K \to \mathbb{R}$ is said to be $s$-preinvex with respect to $\eta$, if for all $u, v \in K$, $t \in [0,1]$ and $s \in (0,1]$, the following inequality holds:

$$f\big(u + t\eta(v,u)\big) \leq (1-t)^s f(u) + t^s f(v).$$

The function $f$ is said to be $s$-preincave if and only if $-f$ is $s$-preinvex.

**Theorem 6 [2].** Let $f: K = [a, a + \eta(b,a)] \subseteq [0,\infty) \to [0,\infty)$ be $s$-preinvex function on the interval of the real numbers $K^\circ$ (the interior of $K$) and $a, b \in K^\circ$ with $\eta(b,a) > 0$. Then the following inequalities hold:

$$2^{s-1} f\left(a + \frac{1}{2}\eta(b,a)\right) \leq \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx \leq \frac{f(a) + f(b)}{s+1}.$$

**Lemma 7 [1].** Let $f: K \to (0,\infty)$ be a differentiable mapping and $a, a + \eta(b,a) \in K$ with $a < a + \eta(b,a)$. If $f' \in L[a, a + \eta(b,a)]$, then

$$\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx - \frac{f(a) + f\big(a + \eta(b,a)\big)}{2} = \frac{\eta(b,a)}{2}\int_0^1 (1-2t)f'\big(a + \eta(b,a)\big)dt.$$

## 2. Main Results

**Theorem 8.** Let $f: I \to \mathbb{R}$ be a differentiable mapping on $I^{\circ}$, $a, a+\eta(b,a) \in I^{\circ}$ with $a < a+\eta(b,a)$ and $f' \in L[a, a+\eta(b,a)]$. If $|f'|^q$ is $s$-preinvex on $[a, a+\eta(b,a)]$ for $p > 1$, $q = \frac{p}{p-1}$, then

$$\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \leq \frac{\eta(b,a)}{2}(p+1)^{-\frac{1}{p}}\left(\frac{|f'(a)|^q + \left|f'\big(a+\eta(b,a)\big)\right|^q}{s+1}\right)^{\frac{1}{q}}.$$

**Proof.** From Lemma 7, the properties of modulus and Hölder's inequality, we have

$$\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right|$$

$$\leq \frac{\eta(b,a)}{2}\int_0^1 |1-2t|\left|f'\big(a+t\eta(b,a)\big)\right|dt$$

$$\leq \frac{\eta(b,a)}{2}\left(\int_0^1 |1-2t|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'\big(a+t\eta(b,a)\big)\right|^q dt\right)^{\frac{1}{q}}. \tag{1}$$

Since $|f'|^q$ is $s$-preinvex on $[a, a+\eta(b,a)]$,then

$$\left|f'\big(a+t\eta(b,a)\big)\right| \leq (1-t)^s|f'(a)| + t^s\left|f'\big(a+\eta(b,a)\big)\right|.$$

Further, we have

$$\int_0^1 |1-2t|^p dt = \frac{1}{p+1} \tag{2}$$

and

$$\int_0^1 \left|f'\big(a+t\eta(b,a)\big)\right|^q dt = \frac{|f'(a)|^q + \left|f'\big(a+\eta(b,a)\big)\right|^q}{s+1}. \tag{3}$$

By (2) and (3), we get (1).

**Theorem 9.** Let the assumptions of Theorem 8 are satisfied with $p > 1$ such that $q = \frac{p}{p-1}$. If the mapping $|f'|^q$ is $s$-preincave on $[a, a+\eta(b,a)]$, then

$$\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \leq \frac{\eta(b,a)}{2(p+1)^{\frac{1}{p}}}.2^{\frac{s-1}{q}}\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|. \tag{4}$$

**Proof.** We proceed similarly as in Theorem 8. By $s$-preincavity of $|f'|^q$, we obtain

$$\int_0^1 \left|f'\big(a+t\eta(b,a)\big)\right|^q dt \le 2^{s-1}\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|.$$

Now the inequality (4) immediately follows from Theorem 6.

**Theorem 10.** Let the assumptions of Theorem 8 are satisfied. If $|f'|^q$ is $s$-preinvex on $[a, a+\eta(b,a)]$, then the following inequality holds:

$$\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \le \frac{\eta(b,a)}{2^{\frac{p+1}{p}}}\left[\left(\frac{s.2^s+1}{2^s}\right)\frac{|f'(a)|^q+\left|f'\big(a+\eta(b,a)\big)\right|^q}{(s+1)(s+2)}\right]^{\frac{1}{q}}$$

for $p>1,\ q=\frac{p}{p-1}$.

**Proof.** Using Lemma 7, the properties of modulus and Hölder's integral inequality for $q>1$, we have

$$\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right|$$

$$\le \frac{\eta(b,a)}{2}\int_0^1 |1-2t|\left|f'\big(a+t\eta(b,a)\big)\right|dt$$

$$=\frac{\eta(b,a)}{2}\int_0^1 |1-2t|^{\frac{1}{p}}|1-2t|^{\frac{1}{q}}\left|f'\big(a+t\eta(b,a)\big)\right|dt.$$

$$\le \frac{\eta(b,a)}{2}\left(\int_0^1 |1-2t|dt\right)^{\frac{1}{p}}\left(\int_0^1 |1-2t|\left|f'\big(a+t\eta(b,a)\big)\right|^q dt\right)^{\frac{1}{q}}. \tag{5}$$

By $s$-preinvexity of $|f'|^q$ on $[a, a+\eta(b,a)]$, the inequality (5) can be written as:

$$\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right|$$

$$\le \frac{\eta(b,a)}{2}\left(\frac{1}{2}\right)^p\left[\int_0^1 \left((1-t)^s|1-2t||f'(a)|+t^s|1-2t|\left|f'\big(a+\eta(b,a)\big)\right|\right)dt\right]^{\frac{1}{q}}$$

$$=\frac{\eta(b,a)}{2^{\frac{p+1}{p}}}\left(|f'(a)|^q\int_0^1 (1-t)^s|1-2t|dt+\left|f'\big(a+\eta(b,a)\big)\right|^q\int_0^1 t^s|1-2t|dt\right)^{\frac{1}{q}},$$

where

$$\int_0^1 t^s|1-2t|dt = \int_0^1 (1-t)^s|1-2t|dt = \frac{2^s.s+1}{2^s(s+1)(s+2)}.$$

This completes the proof of theorem.

**Theorem 11.** Let the assumptions of Theorem 8 are satisfied with $p>1$ such that $q=\frac{p}{p-1}$. If the mapping $|f'|^q$ is $s$-preincave on $[a,a+\eta(b,a)]$, then

$$\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \le \frac{\eta(b,a)}{2^{\frac{p-1}{p}}}\left(\frac{s^{2^s}+1}{2(s+2)}\right)^{\frac{1}{q}}\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|. \qquad (6)$$

**Proof.** We proceed similarly as in Theorem 9.

By $s$-preincavity of $|f'|^q$, we obtain

$$\int_0^1 |1-2t|\left|f'\big(a+t\eta(b,a)\big)\right|^q dt \le \frac{s^{2^s}+1}{2(s+2)}\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|^q.$$

Now the inequality (6) immediately follows from Theorem 6.

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# Some New Hermite-Hadamard Type Inequalities for Differentiable s-Preinvex Functions

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**Abstract**

The object of this study is to establish new inequalities for some differentiable mappings that are connected with the Hermite-Hadamard integral inequality for s-preinvex functions.

**Keywords:** Hermite-Hadamard type inequality, Hölder's inequality, s-preinvex function

## 1. Introduction

**Definition 1 [4].** Let $K$ be a closed set in $\mathbb{R}^n$. Suppose that $f: K \to \mathbb{R}$ and $\eta: K \times K \to \mathbb{R}$ be continuous functions. Let $u \in K$, then the set $K$ is said to be invex at each $u$ with respect to $\eta(.,.)$, if $u + t\eta(v,u) \in K, \forall u, v \in K, t \in [0,1]$.

$K$ is said to be an invex set with respect to $\eta$, if $K$ is invex at each $u \in K$. The invex set $K$ is also called $\eta$-connected set.

Note that if $\eta(v,u) = v - u$, invexity reduces to convexity. Thus, every convex set is also an invex set with respect to $\eta(v,u) = v - u$, but the converse is not true in general.

**Definition 2 [4].** The function $f$ on the invex set $K$ is said to be preinvex with respect to $\eta$, if the inequality

$$f\big(u + t\eta(v,u)\big) \leq (1-t)f(u) + tf(v)$$

holds for all $u, v \in K$ and $t \in [0,1]$.

**Theorem 3 [3].** Let $f: K = [a, a + \eta(b,a)] \to (0, \infty)$ be a preinvex function on the interval of the real numbers $K^\circ$ (the interior of $K$) and $a, b \in K^\circ$ with $\eta(b,a) > 0$. Then the following inequalities hold:

$$f\left(a + \frac{1}{2}\eta(b,a)\right) \leq \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx \leq \frac{f(a) + f\big(a + \eta(b,a)\big)}{2}.$$

**Definition 4 [2]**. Let $K \subseteq [0, \infty)$ be an invex set with respect to $\eta: K \times K \to \mathbb{R}$. A function $f: K \to \mathbb{R}$ is said to be $s$-preinvex with respect to $\eta$, if for all $u, v \in K$, $t \in [0,1]$ and $s \in (0,1]$, the following inequality holds:

$$f\big(u+t\eta(v,u)\big)\leq(1-t)^s f(u)+t^s f(v).$$

**Theorem 5 [2].** Let $f\colon K=[a,a+\eta(b,a)]\subseteq[0,\infty)\to[0,\infty)$ be $s$-preinvex function on the interval of the real numbers $K^\circ$ *(*the interior of $K$*)* and $a,b\in K^\circ$ with $\eta(b,a)>0$. Then the following inequalities hold:

$$2^{s-1}f\left(a+\frac{1}{2}\eta(b,a)\right)\leq\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)dx\leq\frac{f(a)+f(b)}{s+1}.$$

## 2. Main Results

**Lemma 6.** Let $f\colon I\to(0,\infty)$ be a differentiable mapping and $a,a+\eta(b,a)\in I$ with $a<a+\eta(b,a)$. *If* $f'\in L[a,a+\eta(b,a)]$, then

$$\frac{f(a)+f\big(a+\eta(b,a)\big)}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)dx$$

$$=\frac{\eta(b,a)}{2}\int_0^1\int_0^1\big[f'\big(a+t\eta(b,a)\big)-f'\big(a+u\eta(b,a)\big)\big](u-t)dtdu.$$

**Theorem 7.** Let $f\colon I\to\mathbb{R}$ be a differentiable mapping on $I^\circ$, $a,a+\eta(b,a)\in I^\circ$ with $a<a+\eta(b,a)$ and $f'\in L[a,a+\eta(b,a)]$. If $|f'|$ is s-preinvex on $[a,a+\eta(b,a)]$ , then

$$\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)dx\right|\leq\frac{\eta(b,a)}{2}\left[\frac{s^2+3s+4}{(s+1)(s+2)(s+3)}\right]\big[|f'(a)|+\big|f'\big(a+\eta(b,a)\big)\big|\big].$$

**Proof.** From Lemma 6 and the properties of modulus, we have

$$\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)dx\right|$$

$$\leq\frac{\eta(b,a)}{2}\int_0^1\int_0^1\big|f'\big(a+t\eta(b,a)\big)-f'\big(a+u\eta(b,a)\big)\big||u-t|dtdu$$

$$\leq\frac{\eta(b,a)}{2}\int_0^1\int_0^1\big|f'\big(a+t\eta(b,a)\big)\big||u-t|dtdu+\frac{\eta(b,a)}{2}\int_0^1\int_0^1\big|f'\big(a+u\eta(b,a)\big)\big||u-t|dtdu$$

$$=\eta(b,a)\int_0^1\int_0^1\big|f'\big(a+t\eta(b,a)\big)\big||u-t|dtdu.$$

By using $s$-preinvexity of $|f'|$ on $[a,a+\eta(b,a)]$, we get

$$\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right|$$

$$\leq \eta(b,a)\int_0^1\int_0^1 \left[(1-t)^s|u-t||f'(a)|+t^s|u-t|\left|f'\big(a+\eta(b,a)\big)\right|\right]dtdu,$$

where

$$\int_0^1\int_0^1 t^s|u-t|dtdu=\int_0^1\int_0^1 (1-t)^s|u-t|dtdu=\frac{s^2+3s+4}{2(s+1)(s+2)(s+3)}.$$

So, the proof is completed.

**Theorem 8.** Let $f\colon I\to\mathbb{R}$ be a differentiable mapping on $I^\circ$, $a, a+\eta(b,a)\in I^\circ$ with $a<a+\eta(b,a)$ and $f'\in L[a,a+\eta(b,a)]$. If $|f'|^q$ is $s$-preinvex on $[a,a+\eta(b,a)]$ for $q>1$, then

$$\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right|\leq \eta(b,a)\left(\frac{2}{(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q+\left|f'\big(a+\eta(b,a)\big)\right|^q}{s+1}\right)^{\frac{1}{q}}.$$

**Proof.** Using Lemma 6, Hölder inequality and $s$-preinvexity of $|f'|^q$, we have

$$\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right|$$

$$\leq \eta(b,a)\int_0^1\int_0^1 \left|f'\big(a+t\eta(b,a)\big)\right||u-t|dtdu$$

$$\leq \eta(b,a)\left(\int_0^1\int_0^1 \left|f'\big(a+t\eta(b,a)\big)\right|^q dtdu\right)^{\frac{1}{q}}\left(\int_0^1\int_0^1 |u-t|^p dtdu\right)^{\frac{1}{p}}.$$

$$\leq \eta(b,a)\left(\int_0^1\int_0^1 \left[(1-t)^s|u-t||f'(a)|^q+t^s|u-t|\left|f'\big(a+\eta(b,a)\big)\right|^q\right]dtdu\right)^{\frac{1}{q}}\left(\int_0^1\int_0^1 |u-t|^p dtdu\right)^{\frac{1}{p}},$$

where

$$\int_0^1\int_0^1 |u-t|^p dtdu=\int_0^1\left\{\int_0^1 (u-t)^p dt+\int_0^1 (t-u)^p dt\right\}du=\frac{2}{(p+1)(p+2)},$$

and

$$\int_0^1\int_0^1 t^s dtdu=\int_0^1\int_0^1 (1-t)^s dtdu=\frac{1}{s+1}.$$

Thus, the proof is completed.

**Corollary 9.** Let $f: I \to \mathbb{R}$ be a differentiable mapping on $I^\circ$, $a, a+\eta(b,a) \in I^\circ$ with $a < a+\eta(b,a)$ and $f' \in L[a, a+\eta(b,a)]$. *If* $|f'|^q$ is $s$-preinvex on $[a, a+\eta(b,a)]$ for $q > 1$, then

$$\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \leq \eta(b,a)\left(\frac{2}{(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left[|f'(a)| + f'\big(a+\eta(b,a)\big)\right].$$

**Theorem 10.** Let $f: I \to \mathbb{R}$ be a differentiable mapping on $I^\circ$, $a, a+\eta(b,a) \in I^\circ$ with $a < a+\eta(b,a)$ and $f' \in L[a, a+\eta(b,a)]$. If $|f'|^q$ is $s$-preinvex on $[a, a+\eta(b,a)]$ for $q > 1$, then

$$\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right|$$

$$\leq \frac{\eta(b,a)}{3^{\frac{1}{p}}}\left[\frac{s^2+3s+4}{2(s+1)(s+2)(s+3)}\right]^{\frac{1}{q}}\left[|f'(a)|^q + \left|f'\big(a+\eta(b,a)\big)\right|^q\right]^{\frac{1}{q}}.$$

**Proof.** From Lemma 6, properties of modulus and by using Hölder inequality, we have

$$\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right|$$

$$\leq \frac{\eta(b,a)}{2}\int_0^1\int_0^1 \left|f'\big(a+t\eta(b,a)\big) - f'\big(a+u\eta(b,a)\big)\right||u-t|dtdu$$

$$\leq \frac{\eta(b,a)}{2}\int_0^1\int_0^1 \left|f'\big(a+t\eta(b,a)\big)\right||u-t|dtdu + \frac{\eta(b,a)}{2}\int_0^1\int_0^1 \left|f'\big(a+u\eta(b,a)\big)\right||u-t|dtdu$$

$$= \eta(b,a)\int_0^1\int_0^1 \left|f'\big(a+t\eta(b,a)\big)\right||u-t|dtdu.$$

$$\leq \eta(b,a)\left(\int_0^1\int_0^1 |u-t|\left|f'\big(a+t\eta(b,a)\big)\right|^q dtdu\right)^{\frac{1}{q}}\left(\int_0^1\int_0^1 |u-t|dtdu\right)^{\frac{1}{p}}. \tag{5}$$

Here,

$$\int_0^1\int_0^1 |u-t|dtdu = \frac{1}{3} \tag{6}$$

and

$$\int_0^1\int_0^1 |u-t|\left|f'\big(a+t\eta(b,a)\big)\right|^q dtdu \le \int_0^1\int_0^1 |u-t|\left[(1-t)^s|f'(a)|^q + t^s\left|f'\big(a+\eta(b,a)\big)\right|^q\right]dtdu.$$

Since $|f'|^q$ is $s$-preinvex on $[a, a+\eta(b,a)]$, we have

$$\int_0^1\int_0^1 |u-t|\left|f'\big(a+t\eta(b,a)\big)\right|^q dtdu \le \frac{s^2+3s+4}{2(s+1)(s+2)(s+3)}\left[|f'(a)|^q + \left|f'\big(a+\eta(b,a)\big)\right|^q\right]. \qquad (7)$$

By using (5), (6) and (7), the proof is completed.

**Corollary 11.** Let $f: I \to \mathbb{R}$ be a differentiable mapping on $I^\circ$, $a, a+\eta(b,a) \in I^\circ$ with $a < a+\eta(b,a)$ and $f' \in L[a, a+\eta(b,a)]$. If $|f'|^q$ is $s$-preinvex on $[a, a+\eta(b,a)]$ for $q > 1$, then

$$\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\right| \le \frac{\eta(b,a)}{3^{\frac{1}{p}}}\left[\frac{s^2+3s+4}{2(s+1)(s+2)(s+3)}\right]^{\frac{1}{q}}\left[|f'(a)| + \left|f'\big(a+\eta(b,a)\big)\right|\right].$$

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**INTERNATIONAL CONFERENCE ON MATHEMATICS**
**"An Istanbul Meeting for World Mathematicians"**
**3-5 July 2019, Istanbul, Turkey**

## Dynamical Quantum Gates as a Tool for Topological Data Analysis

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**Abstract**

Topological Data Analysis (TDA) is deals with data sets from different branches of natural and social sciences and involves the fundamental concepts of persistent homology to study the basic topological properties of Rips complices based on the $n$ point data clouds. Numerically the results of TDA could be represented via the fractal characteristics (like Hausdorff's dimension of the data cloud) or via the set of barcodes for the Betti numbers vs the radii of the bolls surrounding each data point to form the corresponding Rips complex.

The most efficient classical Giovannetti-Lloyd-Maccone algorithm (2008) involves at least $n^2$ bits to evaluate the Betti numbers of such simplicial complex. The $k$-th Betti number demands the time $O(n^k)$ for its calculation, and the estimation of Betti numbers for all orders to the accuracy $d$ takes time $O(2^n \log(1/d))$. From another side, the same Rips simplicial complex can be mapped onto an $n$-qubit quantum state via the Lloyd-Garnerone-Zanardi approach (2016), with the time cost proportional to $O(n^5/d)$ for the calculation of its persistent homological properties.

In all existing quantum approaches the estimation of the persistent homologic properties for the Rips compleces is performed in a sequence of standard computational operations under the Grover algorithm: the simplicial complex quantum state preparation, the uniform mixture state construction, the phase estimation and, finally, the measurement. The second and third stages of the process are based on the fixed CNOT quantum logical gates.

Here we discuss the opportunity to improve the efficiency of TDA using the concept of externally driven dynamical quantum gates. The gate operator is changing in time itself, and it can be converted from CNOT to other operators and back, depending on computational needs. In this case the sequence of logical operators acting on qubits during the computation can be replaced by the dynamical transformation just of few quantum gates. The stages of quantum state preparation and the mixed state construction could be optimized under our dynamical scenario.

**Keywords: Persistent Homology, Quantum Topological Data Analysis, Quantum Logical Gates, Controlled Qubit**

### 1. Introduction: Classical and quantum algorithms for topological data analysis

Topological Data Analysis (TDA) is a fast developing area dealing with the data sets from different branches of natural sciences [1] and involving the fundamental concepts of persistent homology to study the basic topological properties (like connectivity, number of holes and other parameters related to the Betti numbers of the object) of Rips complexes based on the data clouds in a certain space. Numerically the results of TDA could be represented via the fractal characteristics of the data clouds (like Hausdorff dimension) or via the set of barcodes representing the Betti numbers vs the radiae of the bolls surrounding each data point from the data to form the Rips complex [2].

TDA has been found to be very efficient for the deep analysis of the data structure due to its robustness under the noisy perturbation of the input. The main problem of the practical TDA application comes from the sophisticated numerical calculations, especially in the real-time regime, challenging the capacity of the computational devices [3]. The possible solution may be originated in the sufficient improvement of the computational tools using the elements of quantum algorithms [4-5].

Recently the quantum processor has been sufficiently applied to analyze Rips complexes. The most efficient classical Giovannetti-Lloyd-Maccone algorithm (2008) involves at least $n^2$ bits to evaluate the Betti numbers of such simplicial complex [6]. The $k$-th Betti number demands the time $O(n^k)$ for its calculation, and the estimation of Betti numbers for all orders to the accuracy $d$ takes time $O(2^n\log(1/d))$. From another side, the same Rips simplicial complex can be mapped onto an $n$-qubit quantum state via the Lloyd-Garnerone-Zanardi approach (2016), with the time cost proportional to $O(n^5/d)$ for the calculation of its persistent homological properties [4].

In all existing quantum approaches the estimation of the persistent homologic properties for the Rips complexes is performed in a sequence of standard computational operations under the Grover algorithm: the simplicial complex quantum state preparation, the uniform mixture state construction, the phase estimation and, finally, the measurement. The second and third stages of the process are based on the fixed CNOT quantum logical gates.

## 2. Qubit driven by a dynamical quantum gate

Here we discuss the opportunity to improve the performance of TDA using the concept of externally driven dynamical quantum gates. The gate operator is changing in time itself, and it can be converted from CNOT to other operators and back, depending on computational needs. In this case the sequence of logical operators acting on qubits during the computation can be replaced by the dynamical transformation just of few quantum gates. The stages of quantum state preparation and the mixed state construction could be optimized under our dynamical scenario.

The dynamical quantum gate driving qubit is represented by an external (for instance, optical) field $u$ designed according to a feedback control algorithm in the dimensionless form. As an example, we will discuss here the speed gradient (SG) method [7]. The control goal targeting the qubit towards the desired state is defined as a scalar non-negative differentiable function. In our case the goal is tracking, i.e. the function is time-dependent.

Let’s consider the single qubit preserving its evolution on the surface of the Bloch sphere $x^2(t)+y^2(t)+z^2(t)=1$ with the dynamical system in the form [8]:

$$\begin{aligned} \dot{x} &= u \cdot z; \\ \dot{y} &= z; \\ \dot{z} &= -y - u \cdot x; \end{aligned} \tag{1}$$

with the dimensionless representation of the density matrix elements:

$$\begin{aligned} x &= \rho_{22} - \rho_{11}\ ; \\ y &= \rho_{12}\exp\{i\omega t\} + \rho_{21}\exp\{-i\omega t\}\ ; \\ z &= i[\rho_{12}\exp\{i\omega t\} - \rho_{21}\exp\{-i\omega t\}]\,. \end{aligned} \tag{2}$$

Here $\omega = E_2 - E_1$ is the energy interval between two quantum levels, the Plank constant is chosen to be 1. The inversion $x$ defines the measured state of the qubit due to the equalities $\rho_{11} = (1-x)/2$ and $\rho_{22} = (1+x)/2$.

Due to the normalization constrain, the dynamical system (1) can be re-written in the spherical coordinates as:

$$\begin{aligned} \dot{\varphi} &= -u\sin\theta; \\ \dot{\theta} &= -1 - u\cos\theta\cot\varphi, \end{aligned} \tag{3}$$

where $x=\cos\varphi;\ y=\sin\varphi\cos\theta;\ z=\sin\varphi\sin\theta$.

The goal tracking the given target variables $\varphi_G(t),\ \theta_G(t)$ is defined as:

$$G = \frac{1}{2}[\varphi - \varphi_G(t)]^2 - \frac{1}{2}[\theta - \theta_G(t)]^2 \ , \tag{4}$$

and the control $u$ driving the system toward the minimization of the goal function (4) includes the speed gradient feedback [8] (for the details on SG method see also [7]):

$$u = -\Gamma \frac{\partial \dot{G}}{\partial u} = \Gamma(\varphi - \varphi_G)\sin\theta + \Gamma(\theta - \theta_G)\cos\theta\cot\varphi \ , \tag{5}$$

with a positive constant Γ.

Finally, after the substitution of (5) into the dynamical system (3), one can get:

$$\begin{aligned} \dot{\varphi} &= -\Gamma A(\varphi,\theta)(\varphi - \varphi_G) - \Gamma B(\varphi,\theta)(\theta - \theta_G); \\ \dot{\theta} &= -1 - \Gamma B(\varphi,\theta)(\varphi - \varphi_G) - \Gamma C(\varphi,\theta)(\theta - \theta_G), \end{aligned} \tag{6}$$

with

$$A(\varphi,\theta) = \sin^2\theta,\ B(\varphi,\theta) = \sin\theta\cos\theta\cot\varphi,\ C(\varphi,\theta) = \cos^2\theta\cot^2\varphi. \tag{7}$$

Eqs (6)-(7) are robust, i.e. they drive the qubit towards the dynamical goal $\varphi_G(t),\ \theta_G(t)$ from virtually arbitrary initial condition and under a small noisy perturbation of the system. The characteristic time to achieve the control goal could be evaluated as 1/Γ.

## 3. Rips complexes on the qubit Bloch surface

Let's consider for the simplicity that we form a data cloud on the 2-dimensional plane. Then this set of $n$ points can be mapped onto a surface of the Bloch sphere, for instance, with the stereographic projection. Thus, we will construct the Rips complexes based on the set of qubit spherical coordinates: $\{\varphi_k, \theta_k - t\}$, $k = 1, \ldots, n$. (We shift the second angle variable by $t$, because there is the rotation component with the constant angular velocity in the second Eq. (6).) In the Lloyd-Garnerone-Zanardi (LGZ) algorithm we needed $n$ qubits for this procedure. In our approach, we use only one, but driven by a dynamical gate. It means that the goal function (4) is chosen to be periodical with the period $T$, such that:

$$\varphi_G(t) = \varphi_k \, ; \, \theta_G(t) = \theta_k - t \, , \, \frac{T}{n}(k-1) \le t < \frac{T}{n}k \, , \, k = 1,\ldots,n. \tag{8}$$

Due to the robustness of the SG algorithm, we do not need to know the initial qubit state. The data set will be reproduced on the Bloch surface one by one for all $n$ points. The principal difference in our approach is that we involve only one qubit in the place of $n$, while the tool for coding the data set position is the dynamical control field versus the static qubit operators in the LGZ approach.

Now the same computational procedure as in [4] could be applied to the dynamical object (8) to find out all Betti characteristics for the $n$ point Rips complex. The key question here, of course, is how the new version of the algorithm will change the time cost. As we mentioned in Section 1, for LGZ procedure it is evaluated as $O(n^5/d)$. In our case it takes $T$ times longer (to re-check all $n$ points in the time sequence). Because for each point the error for the SG feedback is evaluated as $1/\Gamma$, one can estimate: $T \propto n/\Gamma$. The ratio for the error in SG is about: $d \propto \Gamma/(\Gamma+\delta)$, where $\delta$ stands for the quantum system decay (for the principles of evaluation for SG error see [7]). If we consider $\delta \to 0$, altogether it implies: $O(n^6/\Gamma)$. The choice of constant $\Gamma$ depends on the physical realization of the computational qubit and the external field $u$.

## 4. Conclusions and discussions

The algorithm described here has pros and cons to compare with the LGZ approach. From one side, it does not need some algorithmic stages, like the preparation of the simplicial complex state. We do not need to know the initial qubit state. Our algorithm involves only one qubit, that can be extremely sufficient from the point of physical realization of quantum computational process. From another side, the algorithm is relatively longer: $O(n^6/\Gamma)$, to compare with the LGZ time cost: $O(n^5/d)$.

The open problem is to optimize the present LGZ algorithm of finding Betti numbers for the single qubit driven by a dynamical gate. The application of dynamical logic operators to other stages of the TDA process (entangled state construction, phase evaluation, measurement) could drastically decrease the time cost and the complexity of quantum computations.

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# Magnitude-Frequency Statistic and Fractal Dimension of Seismicity in and around Van Province of Turkey: Assessing the Annual Probabilities and Recurrence Times

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**Abstract**

In this study, a statistical evaluation on the magnitude-frequency distribution and fractal dimension of earthquakes was achieved for Van province of Turkey. Magnitude completeness of the catalog, recurrence times and annual probabilities of the specific magnitude values were also estimated. A rectangular area covered by the co-ordinates 37.6ºN and 39.6ºN in latitude and the co-ordinates 42.2ºE and 44.6ºE in longitude was selected as the study region. Seismic catalog was compiled from Boğazici University, Kandilli Observatory and Earthquake Research Institute and it is homogeneous for duration magnitude, $M_d$. It covers the years between November 28, 1970 and December 30, 2018, and includes 14.179 earthquakes with magnitude equal to and larger than 1.0. The number of earthquakes shows an exponential decay rate from smaller to larger sizes, and magnitude changes are between 2.0 and 3.0 on average. Hence, magnitude completeness was estimated as $Mc$=2.5 for Van earthquakes. $b$-value of magnitude-frequency distribution was computed as $b$=1.05±0.07 with the maximum likelihood method using $Mc$=2.5. This result shows that $b$-value of Van earthquakes is well represented by the Gutenberg-Richter scaling law. Time variations in $b$-value display a strong tendency of decreasing before strong main shocks, and these fluctuations may be an evidence for the next occurrences. Fractal dimension $Dc$-value was calculated as 1.79±004 with the linear curve fitting technique and 95% confidence limit. The scale invariance in the cumulative statistics of correlation dimension was selected between 4.80 and 43.57 km. This relatively large $Dc$-value shows that earthquake activity in and around Van province is more clustered in smaller regions or at larger scales. Recurrence times of the earthquakes has a value from 3 to 7 years for magnitude ranges of 5.5-6.0, and a value from 7 to 20 years for magnitude ranges of 6.0-6.5. Annual probabilities of the earthquakes between 3.5 and 5.0 magnitude ranges exhibit a value between 1 and 20, and a value lower than 1.0 for magnitude ranges of 5.0-6.5. From this point of view, it can be interpreted that there is a remarkable seismic hazard and risk in this region. As a final result, statistical assessments of different seismicity parameters show that Van province of Turkey may have a significant potential for the strong/large earthquake occurrences in the intermediate term and the long term.

**Keywords:** Van, $b$-value, $Dc$-value, Recurrence Time, Annual Probability

## 1. Introduction

Statistical studies on the size scaling distributions of seismicity are one of the most important process in the evaluation of earthquake potential, and many statistical models for different earthquake occurrences in the world have been presented to the literature. Many authors have used different parameters for seismicity analyses, such as magnitude-frequency *b*-value, fractal dimension *Dc*-value, annual probability, recurrence time, moment and energy releases (eg., Wiemer and Wyss, 2000; Awad, 2005; Roy *et al*., 2011; Öztürk, 2018). In this study, two important seismicity parameters were analyzed as well as the magnitude completeness of the catalog, annual probabilities and recurrence times of specific magnitude ranges. The first one of these parameters is known as *b*-value which describes the power law distribution of seismicity, and the second one can be given as *Dc*-value which means that the number of events larger than a specified level has a power law dependence on the size. The magnitude-frequency distribution, stated as *b*-value of Gutenberg-Richter relation (Gutenberg and Richter, 1944), is one of the most frequently used tool in earthquake statistic. *b*-value reflects the relative numbers of both small and large events, and it is related to the properties of the seismic and tectonic structures and stress distributions in region and time. Seismically active fault regions are complex natural systems and display a scale invariant or fractal correlation. Heterogeneity degree of seismicity in active fault system and some mechanical, geological or structural alterations in heterogeneity are dominant on the fractal dimension *Dc*-value. Thus, the higher order fractal dimension is very sensitive in magnitude distributions (Mandelbort, 1982).

Van province of Turkey was struck many strong and large earthquakes in the near past. For this reason, the aim of this study is to provide some useful information for the evaluation of seismic potential in this part of Turkey and hence, statistical analyses of *b* and *Dc*-values, magnitude completeness, recurrence times and annual probabilities of the earthquakes were analyzed in detailed.

## 2. Preliminaries

The earthquake database used in this work was taken from Boğaziçi University, Kandilli Observatory and Earthquake Research Institute (KOERI). Earthquake catalog is homogeneous for duration magnitude, $M_d$. Main tectonics were modified from Şaroğlu *et al*., (1992) and shown in Figure 1a. A catalog including 14.179 events for the time interval between November 28, 1970 and December 30, 2018 was used and the epicenter distributions of these earthquakes with $M_d \geq 1.0$ and $M_d \geq 5.0$ were presented in Figure 1b.

Magnitude-frequency distribution of earthquakes was defined by Gutenberg-Richter (1944) and gives a size scaling distribution of earthquakes as follow:

$$\log_{10} N(M) = a - bM \tag{1}$$

where $N(M)$ is the expected number of events with magnitudes equal to or larger than $M$. $b$-value describes the slope of the magnitude-frequency distribution, and $a$-value is related to seismic activity rate. $b$-value changes roughly between 0.3 and 2.0 for different parts of the world. Many studies suggest that $b$-value is a scale invariant and related to the distribution of earthquake epicenters and fault segments. A reduction in $b$-value may result from a decrease in the pressure and an increase in shear stress (Scholz, 1968). Many factors lead to differences of $b$-values: an increase in thermal gradient, fracture density, material heterogeneity, the number of small and large earthquakes, fault length, stress and strain conditions (Mogi, 1962).

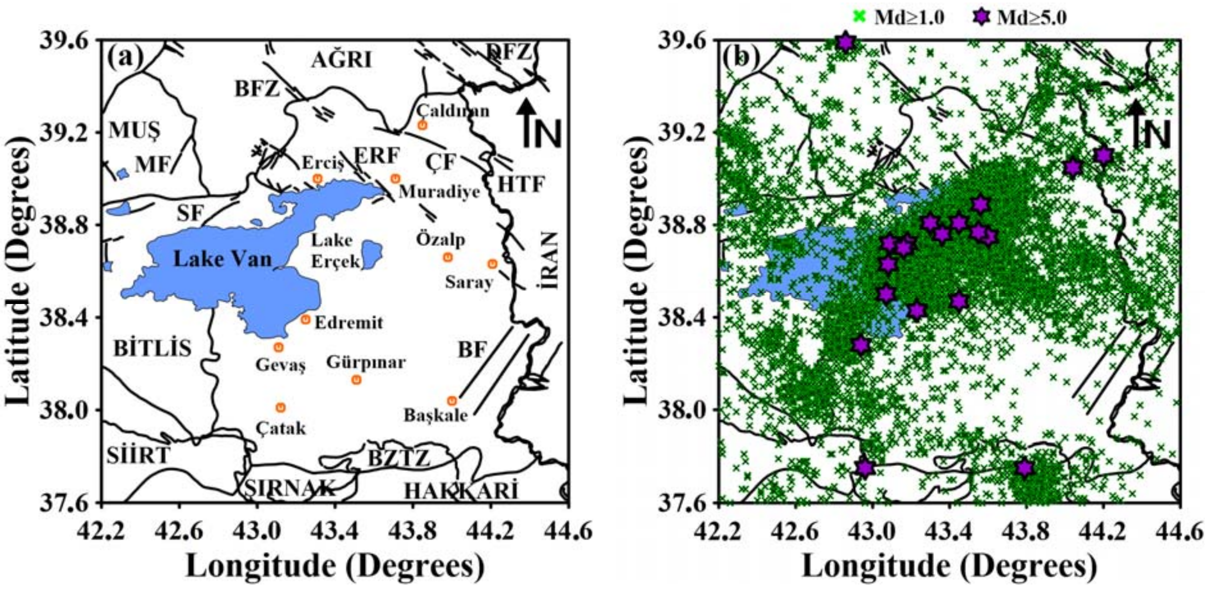


**Figure 1. (a)** Main tectonics in and around Van province. Names of the faults: MF: Malazgirt Fault, SF: Süphan Fault, BFZ: Balıklıgölü Fault Zone, ERF: Erciş Fault, ÇF: Çaldıran Fault, HTF: Hasan-Timur Fault, BF: Başkale Fault, BZTZ: Bitlis-Zagros Thrust Zone. Some significant centers were also given on the figure. **(b)** Epicenter distributions of 14.179 shallow (depth<70 km) earthquakes with $M_d \geq 1.0$ between 1970 and 2019. Stars indicate the strong and large earthquakes with $M_d \geq 5.0$.

It is well known that the usage of the maximum number of earthquakes is important and necessary for high quality and reliable results. For this reason, the identification of magnitude completeness, $Mc$-value, is a very important process. $M$c-value can be defined as the minimum magnitude of complete reporting. In other words, this magnitude level comprises 90% of the earthquake data which can be sampled with a power law fit (Wiemer and Wyss, 2000). A moving time window technique can be used to estimate the changes in *Mc*-value. If magnitude completeness exhibits systematically remarkable fluctuations as a function of time, the estimation of seismicity parameters, especially *b*-value, can be made wrong. Thus, temporal magnitude completeness analysis was achieved with a great care on the catalog.

Fractal analyses are mostly used to describe the clustering features and size scaling properties of earthquake parameters, assuming that earthquake distributions are fractal. A further generalization leads to the correlation dimension *Dc*-value, not based on a covering of the regarded set but based on the distances between pairs of points of the set (Goltz, 1998). Analysis of correlation dimension has been used as a powerful tool in order to quantify the self-similarity of a geometrical object. Correlation dimension *Dc* and the correlation sum *C(r)* was suggested by Grassberger and Procaccia (1983) as follow:

$$Dc = \lim_{r \to 0} \left[ \log C(r) / \log r \right] \tag{2}$$

$$C(r) = 2N_{R<r} / N(N-1) \tag{3}$$

where $C(r)$ is the correlation function, $r$ is the distance between two epicenters and $N$ is the number of earthquakes pairs separated by a distance $R<r$. If the epicenter distribution has a fractal structure, $C(r) \sim r^{Dc}$ is obtained. In this equation, *Dc* is a fractal dimension, more strictly, correlation dimension. Fractal dimension may be estimated to avoid the possible unbroken fields, and these unbroken segments are suggested as potential seismic gaps to be broken in the future (Toksöz *et al*., 1979). The variations in fractal behaviors generally depend on the complexity or quantitative measure of heterogeneity degree of seismicity in the fault systems. Larger *Dc*-value related to lower *b*-value is the dominant structural property in the regions of increased complexity in the active fault system. Thus, it may be resulted from clusters and may be an indicator of stress transfer on fault planes of smaller surface area (Öncel and Wilson, 2002).

## 3. Main Results

In the scope of this study, a statistical assessment on the magnitude-frequency distribution and fractal dimension of seismicity in and around Van province of Turkey was accomplished. In addition, completeness magnitude of the catalog, annual probabilities and recurrence times of specific magnitude sizes were evaluated to supply remarkable results for the next earthquake potential in the intermediate and long terms. *ZMAP* software, introduced by Wiemer (2001) was used for the statistical analyses. Magnitude histogram of earthquake catalog was shown in Figure 2a. Magnitudes of earthquakes in Van region change between 1.0 and 6.6, and earthquake numbers show an exponential decay rate from the smaller to the larger magnitudes. Many of the earthquakes are between 2.0 and 3.0 levels. The number of earthquakes shows a maximum at $M_d$=2.6. Temporal analysis of magnitude completeness was carried out by a moving window approach with the maximum curvature method supplied by *ZMAP* and given in Figure 2b. *Mc*-value was estimated for samples of 150 earthquakes per window using the catalog including all 14.179 earthquakes with $M_d \geq 1.0$. *Mc*-value is relatively large and changes between 2.8 and 3.5 from 1970 to 2012, whereas it fluctuates between 2.0 and 2.5 from 2012 to 2019. Thus, an average of *Mc*=2.5 level for Van region represents the data well for all the statistical evaluations.

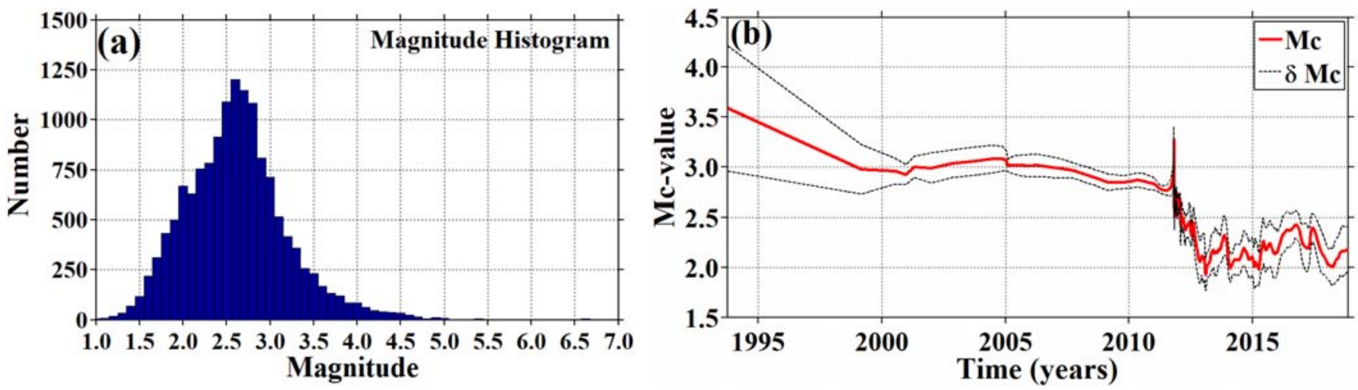


**Figure 2. (a)** Magnitude histogram of earthquake distributions. **(b)** Temporal variation of *Mc*-value. *Mc*-value was estimated with overlapping samples of 150 events/window. $\delta Mc$ indicates standard deviation.

*b*-value of magnitude-frequency distribution was estimated with the maximum likelihood method since it yields a more robust estimation than the last square regression technique (Aki, 1965). Magnitude-frequency relation and correlation integral curve of earthquakes were plotted in Figure 3. Considering *Mc*=2.5 level, *b*-value was calculated as 1.05±0.07 by using whole 14.179 earthquakes. As shown in Figure 3a, *b*-value, its standard deviation, *a*-value and *Mc*-value were given. Average *b*-value is proposed as approximately equal to 1.0 (Frohlich and Davis, 1993), and tectonic earthquakes are represented with a *b*-value between 0.5 and 1.5. Thus, magnitude-frequency distribution of earthquake catalog for Van region matches well with the Gutenberg-Richter scaling law having a characteristic *b*-value close to 1.0.

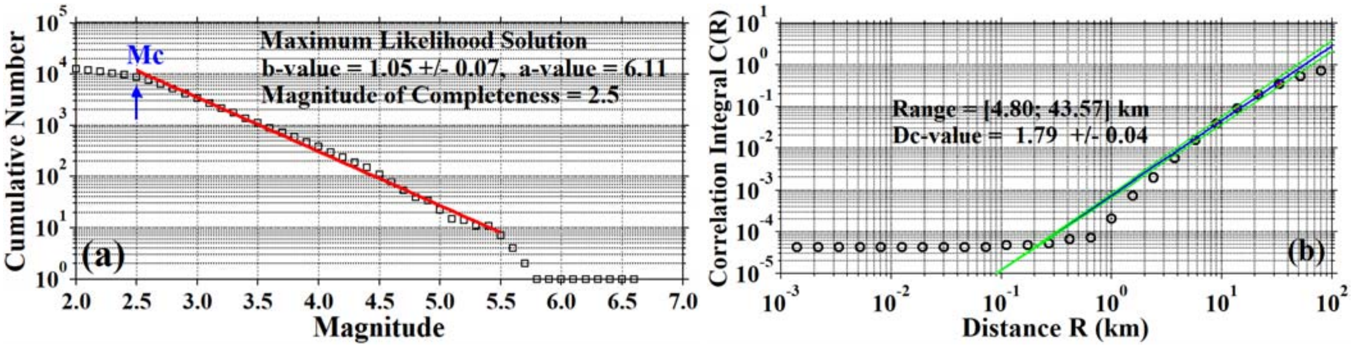


**Figure 3. (a)** Gutenberg-Richter relation and magnitude-frequency distribution of earthquakes in and around Van province. **(b)** Correlation integral curve versus distance. Red dots are the points in the scaling range. The slope of the blue line corresponds to the *Dc*-value and cyan lines represent the standard error.

Fractal properties of earthquake epicenter distributions in Van region were analyzed by fitting a straight line to the curve of the correlation integral, $C(R)$, versus the distance, $R$ (km). *Dc*-value was estimated with 95% confidence limits by linear regression fit and given in Figure 3b. *Dc*-value was computed as 1.79±0.04 for the distribution of 14.179 events and this log-log correlation function exhibits a clear linear range and scale invariance in the cumulative statistics between 4.80 and 43.57 km. As mentioned above, the fractal dimension may define the earthquake distributions since they match the fractal statistics. It is well known that higher *Dc*-values are related to active faults and these fault systems have an increasing complexity, and higher order correlation dimension is increasingly sensitive to the heterogeneity in magnitude distribution (Öncel and Wilson, 2002). This means that earthquake activity is more clustered in smaller areas (or at larger scales) in Van region. Thus, it can be assumed that this higher *Dc*-value is the dominant structural property in this region and may be resulted from the earthquake clusters.

Figure 4a shows the annual probabilities for specific magnitude sizes. Annual probabilities of earthquake occurrences show a value between 1 and 20 for magnitude levels between 3.5 and 5.0, and a value of smaller than 1 for magnitude levels between 4.5 and 6.5. Recurrence times of earthquake occurrences for different magnitude ranges were also plotted in Figure 4b. It is observed quite smaller years (<1.0) for magnitudes from 3.5 to 5.0, and 1-7 years for magnitudes from 5.0 to 6.0. However, the values between 7 and 20 years were estimated for magnitude levels between 6.0 and 6.5, while the values greater than 20 years were estimated for magnitude levels larger than 6.5. There are two large earthquakes in the vicinity of Van region: $M_d$ = 6.6, Tabanlı-Van, October 23, 2011 and $M_d$ = 5.6, Edremit-Van, November 9, 2011 earthquakes. These results of recurrence times and annual probabilities analyses support the existing earthquake potential in the study region. Thus, these types of assessments on the probabilities and recurrence times of earthquake occurrences for specific magnitude levels show that Van and surrounding area has an earthquake risk potential for the possibility of strong earthquake occurrence in the intermediate term and the long term.

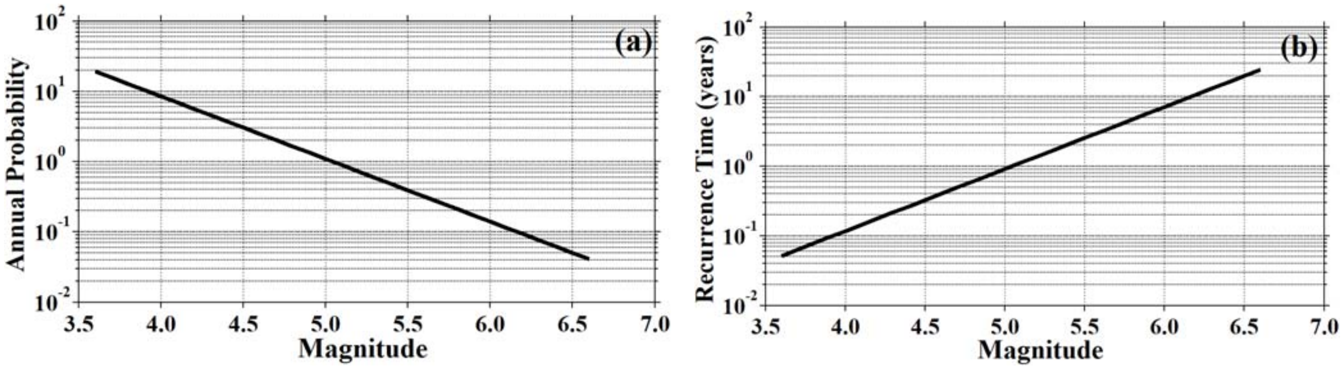


**Figure 4. (a)** Annual probability, **(b)** Recurrence time of specific magnitude levels for Van earthquakes.

Variations in *b*-value as a function of time were plotted in Figure 5. Temporal distribution of *b*-value was estimated for overlapping samples of 400 events per window. There are clear decreases in *b*-value lower than 1.0 before some strong main shocks such as March 14, 2002, February 22, 2011, June 24, 2012, February 18, 2014 and January 23, 2016 earthquakes (arrows on Figure 5). Temporal changes of *b*-value are one of the most important precursors for the future earthquake occurrences. Temporal variations in *b*-value show a tendency to decrease before large earthquake occurrences (Prasad and Singh, 2015; Wang et al., 2016). Prasad and Singh (2015) observed a correlation between small *b*-value for the one-year time interval and the occurrence of large main shocks. They proposed that temporal changes in *b*-value can be used to forecast a major earthquake. Wang *et al*., (2016) stated that seismic observations show abnormal *b*-value changes before the occurrence of some main shocks. It can be pointed out that decreasing trend in *b*-value before the occurrences of some strong main shocks may result from a stress increase. For Van region, the systematic decreasing trend in *b*-value starts in 2017 and continues in 2018. As a remarkable result, this decrease may be significant for earthquake forecasting, and it can be interpreted that these fluctuations may be an indicator of the next earthquake in and around Van province of Turkey.

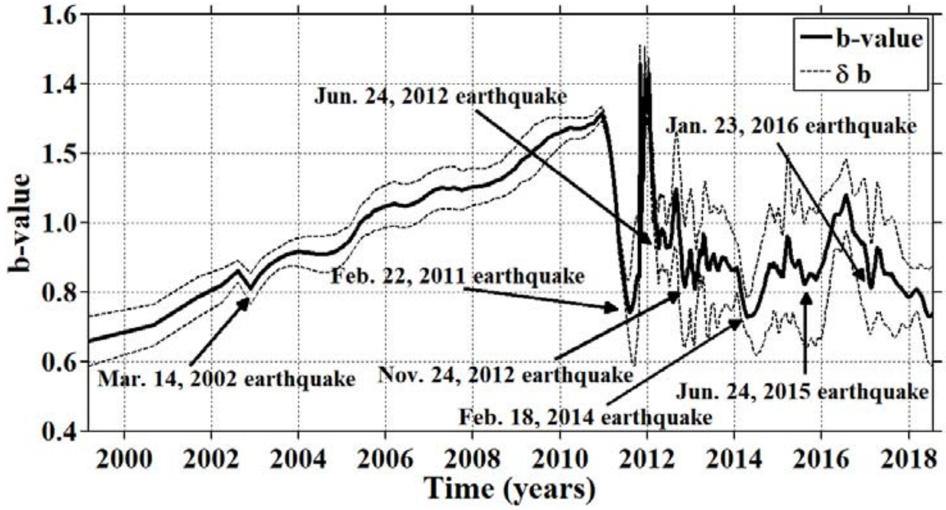


**Figure 5.** Temporal changes in *b*-value. Arrows show large important decreases in *b*-value before strong earthquake occurrences and $\delta b$ indicates standard deviation.

## 4. Conclusions

In the scope of this work, a detailed statistical assessment on the magnitude-frequency distribution, fractal dimension of seismicity, completeness magnitude of the catalog, annual probabilities and recurrence times of earthquakes for Van province of Turkey was made. A homogeneous catalog for duration magnitude, $M_d$, was used and it includes 14.179 earthquakes with $1.0 \leq M_d \leq 6.6$ for shallow earthquakes (depth<70 km) between November 28, 1970 and December 30, 2018. *Mc*-value for Van and surrounding area was calculated as 2.5 and *b*-value was estimated as 1.05±0.07 with this completeness value. This *b*-value is close to 1.0 and well represented with Gutenberg-Richter scaling law. *Dc*-value was calculated as 1.79±0.04, relatively large, and hence, seismicity is more clustered at larger scales (or in smaller areas) in and around Van. This relatively large *Dc*-value means the dominant structural feature and may arise due to clusters. Temporal changes in *b*-value indicate a strong tendency of decreasing before strong main shocks, and these significant decreases may be an evidence for the future events. Analyses of probability and recurrence time of the earthquakes suggest that Van province of Turkey has an earthquake potential for the probability of strong or large earthquake occurrences in the intermediate term and long term.

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# Regional and Temporal Properties of Earthquake Occurrences in Van Province and its vicinity, Turkey: Observations on the Current Seismic Behaviors at the beginning of 2019

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**Abstract**

The aim of this study is to achieve a detailed regional and temporal assessment on the current seismic activity rate changes, *Z*-value, for earthquake occurrences in and around Van province of Turkey. In this context, spatial variations of the most frequently preferred seismicity parameters such as *b*-value of frequency-magnitude distribution and return periods of strong/destructive earthquakes were also analyzed. As the study region, the coordinates limited at 42.2ºE and 44.6ºE in longitude and at 37.6ºN and 39.6ºN in latitude were considered. Earthquake database was provided with the Boğazici University, Kandilli Observatory and Earthquake Research Institute and homogeneous for duration magnitude, $M_d$. Time interval of the catalog is about 48.09-years during from November 28, 1970 and December 30, 2018, and the catalog includes 14.179 shallow events (depth<70 km) with magnitudes greater than or equal to 1.0. Reasenberg's algorithm was used to separate the dependent events from independent ones, and catalog was declustered to image the spatial and temporal variations of precursory seismic quiescence. Completeness magnitude for study region was calculated as 2.5. After declustering procedure and separating the earthquakes with magnitude smaller than 2.5, about 69.18% of total earthquakes was extracted from the catalog and thus, only 4370 earthquakes were used for seismicity rate assessment. Small *b*-values in regional scale were calculated in and around Çaldıran, Muradiye, on Başkale fault, in the north and northeast parts, the middle part of Van between Özalp-Edremit-Gevaş-Çatak-Gürpınar. Four significant anomaly regions indicating precursory quiescence were observed at the beginning of 2019 centered at: (*i*) 38.96ºN-43.06ºE (in and around Erciş), (*ii*) 39.02ºN-43.68ºE (including Muradiye), (*iii*) 38.97ºN-43.97ºE (in the north of Çaldıran fault) and (*iv*) 38.39ºN-43.52ºE (between Lake Erçek-Edremit-Gürpınar). Regional variations of return periods for magnitude level $M_d$=5.0 show that Van province has a significant earthquake potential with regard to occurrences of strong earthquakes in the intermediate and long terms. Regions with small *b*-value, large Z-value and intermediate return periods are significant in terms of the possible earthquake potential, and anomaly regions can be considered to be the most likely place for a strong or large earthquake. Thus, a special attention should be given to these anomaly areas.

**Keywords:** Van, Seismicity Rate Changes, Decluster, *Z*-value, Return Period

## 1. Introduction

Many statistical algorithms have been developed to perform a quantitative temporal and spatial analysis of earthquake occurrences in different parts of the world, and a number of beneficial seismic and tectonic parameters have been used by many researchers (eg., Huang et al., 2001; Polat *et al*., 2008; Öztürk, 2011; Singh, 2016; Ormeni *et al*., 2017). In this scope, statistical assessments on precursors of earthquake occurrences are very important topic and suggest that particular region-time seismicity patterns include the seismic quiescence phenomenon and may be related to the seismic and tectonic process. Thus, there are many different techniques such as Region-Time-Length (RTL), Pattern Informatics (PI), Relative Intensity (RI) or ZMAP ($Z$-value) to measure, map and investigate the possible episodes of seismic quiescence. Wyss and Haberman (1988) defines the seismic quiescence phenomenon as follows: "*The quiescence theorem suggests that some main shocks are preceded by precursory quiescence and this quiescence show a significant decrease in the average seismicity rate, as compared to the preceding declustered background rate in the same crustal depth* (Wyss and Martirosyan, 1988). *Decreasing in the earthquake activity rate may extend up to main shock time or may be separated from main shock by a relatively short period of increasing seismicity rate".* Thus, quiescence hypothesis postulates that the quiet volume overlaps the main shock source volume.

Seismic quiescence can be recognized with a methodology introduced by Wiemer and Wyss (1994) and implemented in *ZMAP* software package (Wiemer, 2001). In this study, it is aimed to detect whether there is a significant quiescence in and around Van, Turkey, with $Z$-value approach at the beginning of 2019 since Van province was struck many strong and large earthquakes in near the past.

## 2. Preliminaries

The data set was compiled from Boğaziçi University, Kandilli Observatory and Earthquake Research Institute (KOERI). It consists of 14.179 earthquakes from November 28, 1970 and December 30, 2018 and is homogeneous for duration magnitude, $M_d$. Active fault systems in the vicinity of Van were modified from Şaroğlu *et al*., (1992) and shown in Figure 1a.

Some occurrences such as foreshocks, aftershocks or swarms generally masks temporal changes of earthquake numbers and thus, eliminating the dependent earthquakes from catalog is a very significant stage in the seismic quiescence assessments. In order to achieve a qualified assessment of seismicity rate changes and to remove the dependent events from the catalog, Reasenberg's (1985) algorithm can be performed. This algorithm "declusters" or decomposes an earthquake catalog into main and secondary events (Arabasz and Hill, 1996). It removes all the dependent earthquakes from each cluster, and defines them as a unique earthquake. Epicenters of the earthquakes for original catalog with $M_d \geq 1.0$ and for declustered catalog with $M_d \geq 2.5$ with strong/large main shocks of $M_d \geq 5.0$ were given in Figure 1b.

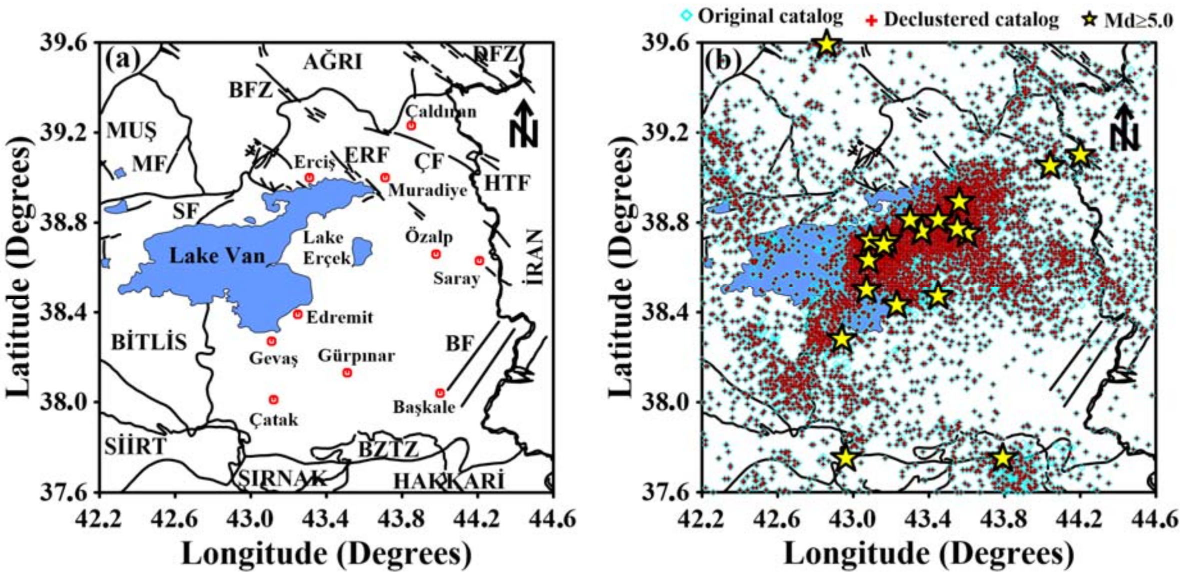


**Figure 1. (a)** Active fault systems in and around Van province. Names of the faults: DFZ: Doğubeyazıt Fault Zone, ÇF: Çaldıran Fault, ERF: Erciş Fault, SF: Süphan Fault, MF: Malazgirt Fault, HTF: Hasan-Timur Fault, BFZ: Balıklıgölü Fault Zone, BF: Başkale Fault, BZTZ: Bitlis-Zagros Thrust Zone. Some significant centers were also shown on the figure. **(b)** Epicenter distributions of 14.179 shallow (depth<70 km) earthquakes with $M_d \geq 1.0$ and $M_d \geq 5.0$ between 1970 and 2019 as well as declustered catalog.

Earthquake catalog was declustered with the Reasenberg's (1985) algorithm to perform a quantitative assessment of seismic quiescence. This technique removed 5510 earthquakes from catalog and 8669 earthquakes remained. Completeness magnitude for Van and vicinity was estimated as 2.5, and the number of earthquakes exceeding this magnitude range is 4299. After declustering and excluding $M_d<2.5$ earthquakes, approximately 69.18% of the earthquake catalog was eliminated and the number of earthquakes for $Z$-test was reduced to 4370. Cumulative number of earthquakes versus time for original catalog, for declustered events and for declustered catalog with $M_d \geq 2.5$ was shown in Figure 2. Earthquake activity does not have any significant changes between 1970 and 2000, and there is a little change between 2000 and 2011. But, there are important changes in seismicity, especially starting after 2011 Van earthquake. Also, the cumulative number of declustered earthquakes with $M_d \geq 2.5$ as a function of time has a smooth slope when compared to original catalog. It can be clearly seen that declustering process removed dependent events from original catalog. After these two process, a more homogeneous, reliable and robust earthquake database is provided for the imaging of seismic quiescence.

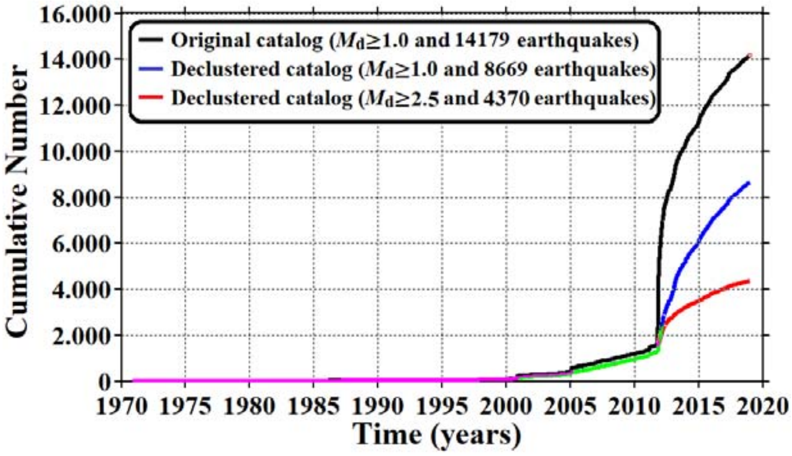


**Figure 2.** Cumulative number of earthquakes against time for original catalog with $M_d$≥1.0 (black line), for declustered (blue line) catalog with $M_d$≥1.0 and for declustered catalog with $M_d$≥2.5 (red line).

The quiescence hypothesis was firstly formulated by Wyss and Habermann (1988). Then, many statistical models for describing and evaluating the seismicity rate changes have been formulated, and most of them focus on the precursory quiescence phenomenon. The standard normal deviate *Z*-test is one of the best known among these statistical models. This evaluation of the regional and temporal changes of seismicity in and around Van province is based on the seismic tool *ZMAP*. This software package is a tool for analysis of seismic quiescence and artificial seismic rate changes. In order to rank the significance of quiescence, the standard deviate *Z*-test was used, generating the *LTA*(*t*) (Log Term Average) function for the statistical evaluation of the confidence level in units of standard deviations:

$$Z = (R_{all} - R_{wl}) / \sqrt{(S_{all}^2 / N_{all}) + (S_{wl}^2 / N_{wl})} \tag{1}$$

where $R_{all}$ is the average seismicity rate in whole period of catalog, $R_{wl}$ is the mean activity rate in the considered time window, $S^2_{all}$ and $S^2_{wl}$ are the standard deviations in these time periods, and $N_{all}$ and $N_{wl}$ the number of samples. The *Z*-value calculated as function of the time, letting the foreground window slide along the time interval of the catalog, is called *LTA* (*t*).

Gutenberg-Richter (1944) proposed an empirical relation for frequency-magnitude distribution of earthquake occurrences as in the following form:

$$\log_{10} N(M) = a - bM \tag{2}$$

where *N(M)* is the expected number of earthquakes with magnitudes greater than or equal to *M*. *b*-value defines the slope of the frequency-magnitude distribution, and *a*-value is related to earthquake activity rate. *b*-value is related to geotectonic and rheological properties of the medium and many factors affect *b*-value: fracture density, thermal gradient, material heterogeneity, fault length, pressure, shear stress etc. (Mogi, 1962). Thus, *b*-value is one of the most important parameters in earthquake statistic.

## 3. Main Results

Regional changes in *b* -value was plotted at every node of the 0.02º grid in longitude and latitude. *b*-value distribution was mapped by using a moving window approach and shown in Figure 3a. Original earthquake catalog with $M_d \geq 1.0$ was included in the estimation of *b*-value with samples of 500 events/window. As seen in Figure 3a, regional distributions in *b*-value are between 0.5 and 1.4. The larger *b*-values (>1.0) were observed in the north and south parts of Lake Van, in and around Erciş, in the north and west parts of Lake Erçek, between Çatak and Başkale, in and around Özalp and Saray, and in the west part of Gevaş. However, the smaller *b*-values (<1.0) were found along the north and northeast parts of Van between Çaldıran and Muradiye, on Başkale fault, in the middle part of Van between Özalp-Edremit-Gevaş-Çatak-Gürpınar. As stated in literature, lower *b*-values may indicate higher stress release, low degree of heterogeneity, high strain due to the subduction tectonics *etc*. Thus, these anomaly regions of small *b*-value may be interpreted as potential earthquake zones for the next earthquake occurrences.

Regional variations in *Z*-value supply the spatial image of seismic quiescence at the beginning of 2019 and were shown in Figure 3b. Study region was divided into a spatial grid of points with 0.02º×0.02º in longitude and latitude. The nearest earthquakes, *N*, at each node were considered as 50 events. Time window length was taken as $T_W$=4.5 years since the quiescence images are better visible for $T_W$=4.5 years. In order to obtain a continuous and dense coverage in time, earthquake distribution was binned into many binning spans of 28 days for each grid point. There are four regions (A, B, C and D) exhibiting seismic quiescence anomalies. The first quiescence anomaly was observed centered at 38.96ºN-43.06ºE (region A, in and around Erciş), the second one was observed centered at 39.02ºN-43.68ºE (region B, including Muradiye), the third one was observed centered at 38.97ºN-43.97ºE (region C, in the north of Çaldıran fault), and the fourth one was observed centered at 38.39ºN-43.52ºE (region D, between Lake

Erçek-Edremit-Gürpınar). The cumulative number curves versus time and correspondent *LTA* (*t*) functions for anomaly regions of *Z*-value were shown in Figure 4. The aim of this process is to describe the beginning year of seismic quiescence. To estimate these beginning times, cumulative number of earthquakes were plotted in a circular area including detected four regions. *Z*-value peaked with $Z_{max}$=4.4 at 2014.26 for a circle of 19.38 km radius centered for region A, $Z_{max}$=3.5 at 2015.11 for a circle of 3.32 km radius centered for region B, $Z_{max}$=4.8 at 2016.81 for a circle of 13.96 km radius centered for region C and $Z_{max}$=3.2 at 2013.66 for a circle of 14.57 km radius centered for region D.

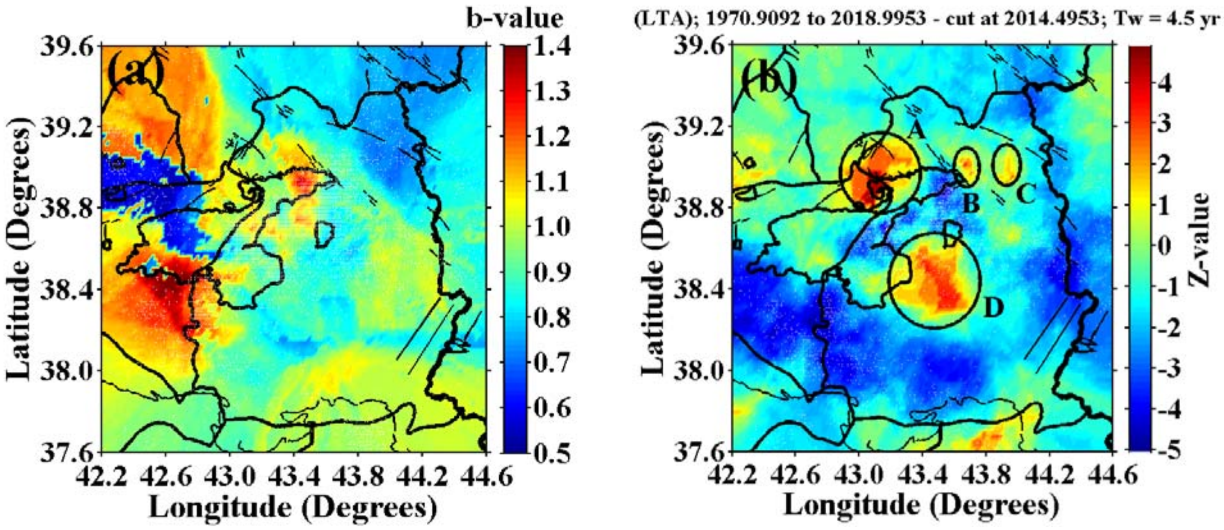


**Figure 3.** Regional variations at the beginning of 2019 for **(a)** *b*-value, **(b)** Z-value.

The starting time was used as 1970 and the same time window lengths as $T_W$=4.5 years was used in both *Z*-value map (Figure 3b) and in cumulative numbers curves (Figure 4). The average duration of seismic quiescence before the occurrence of a strong/large earthquake in the eastern part of Turkey is given as $4.9 \pm 1.5$ years for the earthquakes which occurred after 2000 (Öztürk, 2009). If one considers the fact that the beginning of quiescence is started in time interval between 2013.66 and 2016.81 for study region and the duration of quiescence before the occurrence of an earthquake is described as average 5 years for this area, it can be interpreted that the regions where the quiescence anomalies were observed may have an earthquake risk. Hence, the estimated time for the next earthquake can be expected between 2019 and 2021. However, because the standard deviation of average seismic quiescence is ±1.5 years, it can be interpreted that the upper limit of the next earthquake occurrence may reach to 2022.

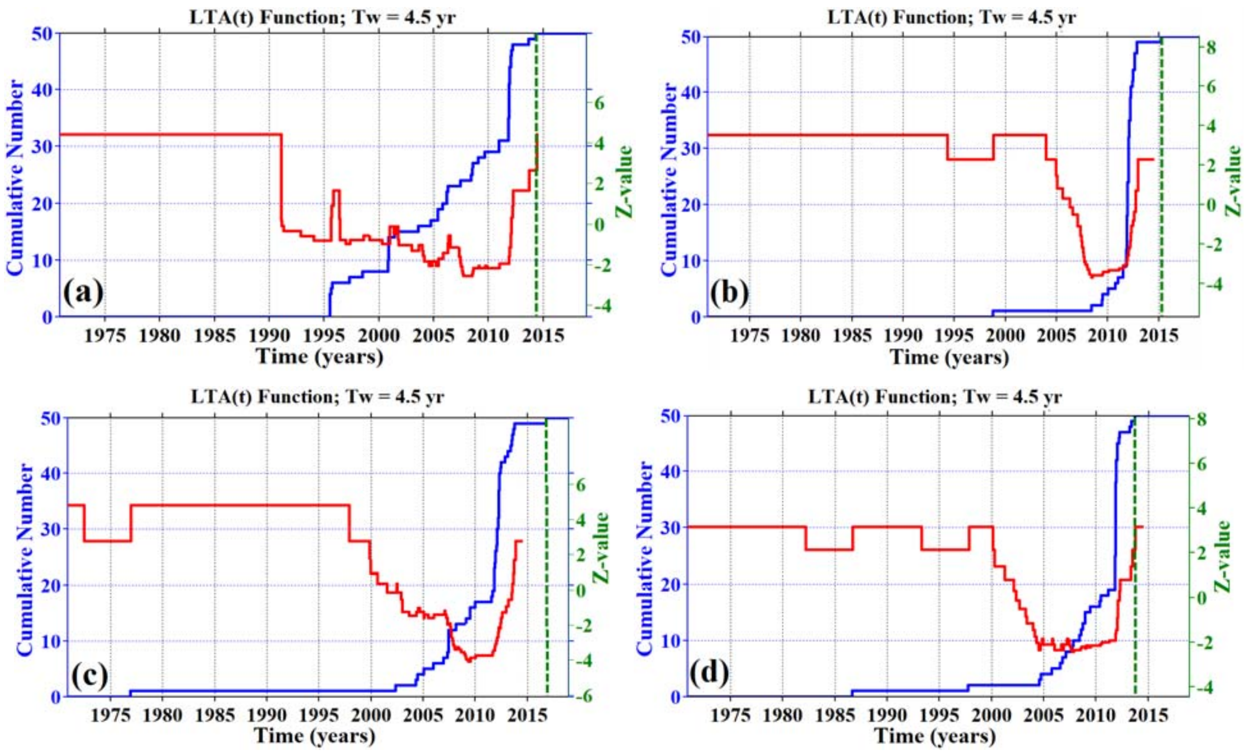


**Figure 4.** Cumulative numbers of earthquakes from declustered catalog between 1970 and 2019 (blue line) for the anomaly regions observed in Figure 3b as a function of time with LTA (*t*) function (red line) for **(a)** region A, **(b)** region B, **(c)** region C and **(d)** region D. Dashed green lines show the Z-value scale and the beginnings of quiescence times.

As in *b*-value map, regional distributions of return period for different magnitude levels were also plotted at every node of 0.02º grid and given in Figure 5. Original earthquake catalog with $M_d \geq 1.0$ was considered in the imaging of recurrence time maps. As shown in Figure 5a, return periods were found to be smaller for magnitude size $M_d=5.0$, changing between 5 and 25 years. The lower return periods (<10 years) were estimated in the northeast part of Van between Çaldıran and Muradiye, between Gevaş-Çatak-Gürpınar, and around Başkale and Süphan faults. The other regions have a return period larger than 20 years. Figure 5b shows also return periods for magnitude level $M_d=6.0$, and return periods for this magnitude level are generally higher than 60 years. The results especially for strong earthqauke occurrences indicate the existing earthquake potential in and around Van region in the intermediate term. Thus, these types of assessments can eventually contribute to forecast the impending main shocks in the future circumstances.

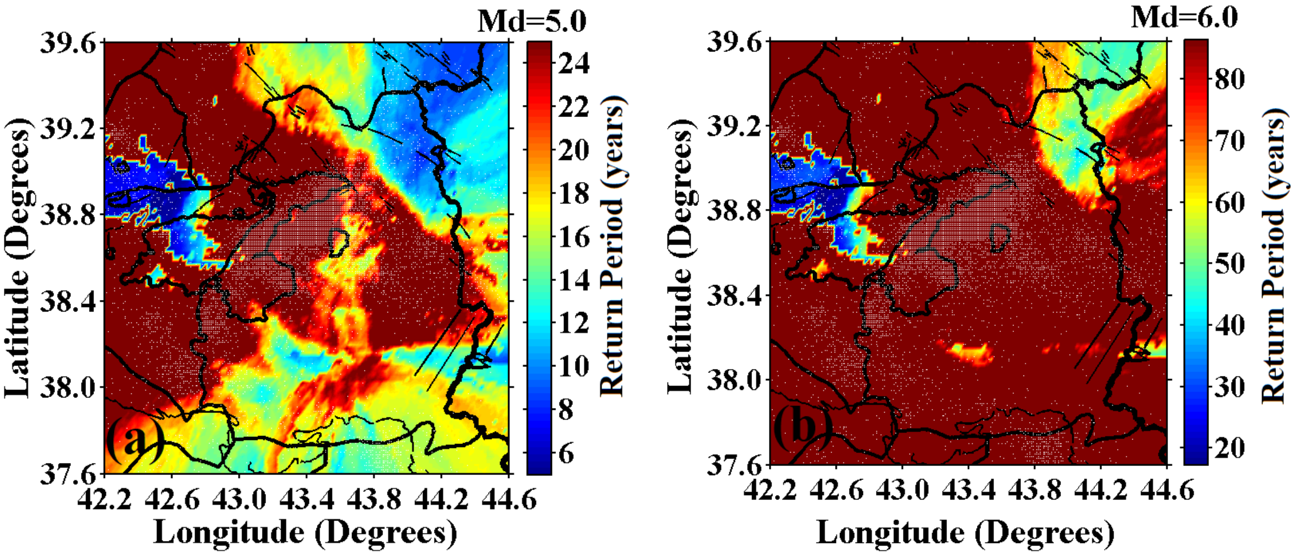


**Figure 5.** Regional variations of return periods for specific magnitude sizes: **(a)** $M_d$=5.0, **(b)** $M_d$=6.0.

It is well known that Turkey, especially the Eastern Anatolian region including Van and its vicinity, is a seismically and tectonically very active region and so, Van province was struck with strong/destructive earthquakes in the past and recent years. Therefore, forecasting of the future earthquake locations in this area would be useful. Such an evaluation must be relied on the observation of phenomena related to *b*-value, precursory quiescence and return periods. Thus, these types of applications to regional and temporal behaviors of earthquake occurrences may provide valuable contributions for earthquake hazard.

## 4. Conclusions

In this study, regional and temporal properties of magnitude-frequency distribution, precursory seismic quiescence and recurrence times of specific magnitude levels for Van, Turkey, earthquakes were evaluated for the beginning of 2019. *b*-values lower than 1.0 were observed along the north and northeast parts of Van between Çaldıran and Muradiye, on Başkale fault, in the middle part of Van between Özalp-Edremit-Gevaş-Çatak-Gürpınar. Seismic quiescence anomalies at the beginning of 2019 were observed centered at 38.96ºN-43.06ºE (in and around Erciş), at 39.02ºN-43.68ºE (including Muradiye), at 38.97ºN-43.97ºE (in the north of Çaldıran fault) and at 38.39ºN-43.52ºE (between Lake Erçek-Edremit-Gürpınar). The regions with smaller *b*-values and larger *Z*-values may be interpreted as the possible regions for the future strong/large earthquakes. The beginning of quiescence times in these areas starts in time interval between 2013.66 and 2016.81, and this result shows that the estimated time for the next earthquake can be expected between 2019 and 2021. Return periods smaller than 10 years for strong earthquakes were estimated in the northeast part of Van between Çaldıran and Muradiye, between Gevaş-Çatak-Gürpınar, and around Başkale and Süphan faults. Thus, these changes can give important clues for strong or large earthquake occurrences in and around Van province in the intermediate and long terms.

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**INTERNATIONAL CONFERENCE ON MATHEMATICS**
**"An Istanbul Meeting for World Mathematicians"**
**3-5 July 2019, Istanbul, Turkey**

# An Application of q-Derivative to a Subclass of Harmonic Univalent Functions

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**Abstract**

We introduce and investigate a special class of harmonic univalent functions by using Al-Oboudi q-differential operator. We first obtained a coefficient characterization of these functions. Using this coefficient estimates, distortion and covering theorems, and some properties were also obtained.

**Keywords:** Al-Oboudi differential operator, Salagean differential operator, harmonic univalent function, q-calculus, starlike and convex functions, analytic functions

## 1. Introduction

Let $H$ denote the family of continuous complex-valued harmonic functions which are harmonic in the open unit disk $U=\{\ z:z\in C$ and $|z|<1\}$ and let $A$ be the subclass of $H$ consisting of functions which are analytic in $U$. A function harmonic in $U$ may be written as $f=h+\overline{g}$, where $h$ and $g$ are members of $A$. We call $h$ the analytic part and $g$ co-analytic part of f. A necessary and sufficient condition for $f$ to be locally univalent and sense-preserving in $U$ is that $|h'(z)|>|g'(z)|$ (see [2] and [7]). To this end, without loss of generality, we may write

$$h(z)=z+\sum_{n=2}^{\infty}a_n z^n \text{ and } g(z)=\sum_{n=1}^{\infty}b_n z^n \qquad (1)$$

Let $SH$ denote the family of functions $f=h+\overline{g}$ which are harmonic, univalent, and sense-preserving in $U$ for which $f(0)=f_z(0)-1=0$. One shows easily that the sense-preserving property implies that $|b_1|<1$.

In 1984, Clunie and Sheil-Small [2] investigated the class $SH$ as well as its geometric subclasses and obtained some coefficient bounds. Since then, there have been several related papers on $SH$ and its subclasses. Also note that *SH* reduces to the class *S* of normalized analytic univalent functions in *U*, if the co-analytic part of f is identically zero.

We recollect here the *q*-difference operator that was used in geometric function theory and in several areas of science. We give basic definitions and properties about the *q*-difference operator that are used in this study (for details see [3], [4] and [5] ). For $0<q<1$, we defined the *q*-integer $[n]_q$ by

$$[n]_q=\frac{1-q^n}{1-q},\ (n=1,2,3,...).$$

Notice that if $q\to 1^-$ then $[n]_q\to n$. The *q*-derivative of a function $h\in A$ is, by definition, given as follows [3]

$$\partial_q h(z)=\begin{cases}\dfrac{h(z)-h(qz)}{(1-q)z} & \text{for } z\neq 0\\ h'(0) & \text{for } z=0\end{cases}. \quad (2)$$

Thus, for the function *h* of the form (1), we have

$$\partial_q h(z)=1+\sum_{n=2}^{\infty}[n]_q a_n z^{n-1}.$$

For $h\in A$, Salagean *q*-difference operator of *h*, denote by $D_q^k h(z)$ is defined by

$$\mathrm{D}_q^0 h(z)=h(z),\ \mathrm{D}_q^1 h(z)=z\partial_q h(z),\ \ldots,\ \mathrm{D}_q^k h(z)=\mathrm{D}_q(\mathrm{D}_q^{k-1}h(z)),$$

$$D_q^k h(z)=z+\sum_{n=2}^{\infty}[n]_q^k a_n z^n \quad (k\in\mathrm{N}_0, z\in\mathrm{U}). \quad (3)$$

Therefore, if $q\to 1^-$ then (1),

$$D^k h(z)=z+\sum_{n=2}^{\infty}n^k a_n z^n \quad (k\in\mathrm{N}_0,\ z\in\mathrm{U})$$

the familiar Salagean derivative [6]. For $h\in A$ given by (1), we define Al-Oboudi *q*-difference operator of *h*:

$$\mathrm{D}^0_{\lambda,q}h(z)=h(z),$$

$$\mathrm{D}^1_{\lambda,q}h(z)=(1-\lambda)D^0_q h(z)+\lambda D^1_q h(z),\ \lambda\geq 0,$$

$$\vdots$$

$$\mathrm{D}^k_{\lambda,q}h(z)=D_{\lambda,q}\left(\mathrm{D}^{k-1}_{\lambda,q}h(z)\right),\ k=0,1,2,...,$$

and

$$D^k_{\lambda,q}h(z)=z+\sum_{n=2}^{\infty}\left[\lambda([n]_q-1)+1\right]^k a_n z^n \quad (k\in \mathrm{N}_0, \mathrm{z}\in \mathrm{U}). \tag{4}$$

When $\lambda=1$, we get Salagean $q$-difference operator. Notice that if $q\to 1^-$ then $\mathrm{D}^k_{\lambda,q}$ reduces to the Al-Oboudi operator [1].

For functions $f_1$ and $f_2\in H$ of the form

$$f_m(z)=z+\sum_{n=2}^{\infty}a_{m,n}z^n+\sum_{n=1}^{\infty}\overline{b_{m,n}z^n},\ (m=1,2),$$

We define the Hadamard product of $f_1$ and $f_2$ by

$$(f_1*f_2)(z)=z+\sum_{n=2}^{\infty}a_{1,n}a_{2,n}z^n+\sum_{n=1}^{\infty}\overline{b_{1,n}b_{2,n}z^n}.$$

## 2. Preliminaries

For $0\leq\alpha<1,\ k\in N_0,\ 0<q<1$ and $z\in U$, we let $SH^k_q(\lambda,\alpha)$ denote by the subclass of $H$ consisting of functions $f$ of the form (1) that satisfy the condition

$$\mathrm{Re}\left\{\frac{z(D^k_{\lambda,q}h(z))'-\overline{z(D^k_{\lambda,q}g(z))'}}{D^k_{\lambda,q}h(z)+\overline{D^k_{\lambda,q}g(z)}}\right\}>\alpha \tag{5}$$

where is $\mathrm{D}^k_{\lambda,q}h(z)$ and $\mathrm{D}^k_{\lambda,q}g(z)$ are defined by (4). We also let the subclass $TSH^k_q(\lambda,\alpha)$ consist of harmonic functions $f=h+\overline{g}$ in $SH^k_q(\lambda,\alpha)$ so that $h$ and $g$ are of the form

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n \text{ and } g(z) = \sum_{n=1}^{\infty} |b_n| z^n . \tag{6}$$

In this paper, we find necessary and sufficient coefficient conditions, distortion bounds, extreme points for the above defined class $TSH_q^k(\lambda,\alpha)$.

## 3. Main Results

The first theorem we introduce a sufficient coefficient bound for harmonic functions in $SH_q^k(\lambda,\alpha)$.

**Theorem 1.** Let $f = h + \overline{g}$ be given by (1). If

$$\sum_{n=1}^{\infty} \frac{\left[\lambda([n]_q - 1) + 1\right]^k}{2(1-\alpha)} \left[(n-\alpha)|a_n| + (n+\alpha)|b_n|\right] \le 1, \tag{7}$$

where $a_1 = 1,\ 0 < q < 1,\ k \in N_0,\ 0 \le \alpha < 1$ and $\lambda \ge 0$, then *f* is sense preserving, harmonic univalent in *U*, and $f \in SH_q^k(\lambda,\alpha)$.

The harmonic function

$$f(z) = z + \sum_{n=2}^{\infty} \frac{(1-\alpha)x_n}{\left[\lambda([n]_q - 1) + 1\right]^k (n-\alpha)} z^n + \sum_{n=1}^{\infty} \frac{(1-\alpha)\overline{y_n}}{\left[\lambda([n]_q - 1) + 1\right]^k (n+\alpha)} \overline{z^n}$$

where

$$\sum_{n=2}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| = 1$$

shows that the coefficient bound given by (7) is sharp.

**Theorem 2.** Let $f = h + \overline{g}$ be given by (6). $f \in TSH_q^k(\lambda,\alpha)$ if and only if the inequality (7) holds.

Our next theorem is on the extreme points of $TSH_q^k(\lambda,\alpha)$.

**Theorem 3.** Let $f = h + \overline{g}$ be given by (6). Then $f \in TSH_q^k(\lambda,\alpha)$ if and only if

$$f(z) = \sum_{n=1}^{\infty} \left(X_n h_n(z) + Y_n g_n(z)\right)$$

where

$$h_1(z)=z,\ h_n(z)=z-\frac{1-\alpha}{\left[\lambda([n]_q-1)+1\right]^k(n-\alpha)}z^n\quad (n=2,3,...),$$

$$g_n(z)=z+\frac{1-\alpha}{\left[\lambda([n]_q-1)+1\right]^k(n+\alpha)}\overline{z^n}\quad (n=1,2,3,...),$$

$$\sum_{n=1}^{\infty}\left(X_n+Y_n\right)=1,\ X_n\geq 0,\ Y_n\geq 0.$$

In particular, the extreme points of $f\in TSH_q^k(\lambda,\alpha)$ are $\{h_n\}$ and $\{g_n\}$.

Finally, we give the distortion bounds for functions in $TSH_q^k(\lambda,\alpha)$ which yields a covering result for this class.

**Theorem 4.** Let $f\in TSH_q^k(\lambda,\alpha)$. Then for $|z|=r<1$ and $|b_1|<\dfrac{1-\alpha}{1+\alpha}$, we have

$$|f(z)|\leq\left(1+|b_1|\right)r+\frac{1-\alpha}{(\lambda q+1)^k(2-\alpha)}\left(1-\frac{1+\alpha}{1-\alpha}|b_1|\right)r^2$$

and

$$|f(z)|\geq\left(1-|b_1|\right)r-\frac{1-\alpha}{(\lambda q+1)^k(2-\alpha)}\left(1-\frac{1+\alpha}{1-\alpha}|b_1|\right)r^2.$$

**Corollary 5.** Let $f=h+\bar{g}$ with $h$ and $g$ are of the form (6). If $f\in TSH_q^k(\lambda,\alpha)$ then

$$\left\{w:|w|<\frac{(\lambda q+1)^k(2-\alpha)-1+\alpha-\left[(\lambda q+1)^k(2-\alpha)-1-\alpha\right]|b_1|}{(\lambda q+1)^k(2-\alpha)}\right\}\subset f(U).$$

**Theorem 6.** The family $TSH_q^k(\lambda,\alpha)$ is closed under convex combination.

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**INTERNATIONAL CONFERENCE ON MATHEMATICS**
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## Mathieu Equation Stability Analysis of a Repulsive Magnetic Bearing Flywheel System

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**Abstract**

In general, if a system has negative real parts of all poles for its transfer function, the system stability is achieved. This condition does not apply to the Mathieu equation. The stability condition of the Mathieu equation depends on the parameters. There are some engineering systems that show Mathieu equation form. In this research study, a flywheel system supported by repulsive magnetic bearings both in the axial and radial directions is proposed to have an energy-free noncontact magnetic bearing flywheel. In the proposed bearing structure, repulsive magnetic bearings do not provide a stable magnetic levitation alone but it is possible to maintain the dynamic stability of the flywheel by controlling the rotor axially. The mathematical model of the flywheel repulsive magnetic bearing system can be transformed to a Mathieu equation form. The stiffness factors generated by repulsive bearings are similar to the stiffness form of Mathieu equation.

In the research work, the stability dynamics of the repulsive magnetic bearing flywheel system is presented using the Mathieu equation approach. The stability diagram of the flywheel system is obtained using the solution of the Mathieu equation.

**Keywords:** Stability analysis; Mathieu equation; Flywheel system, Repulsive magnetic bearing.

## 1. Introduction

Flywheel systems are mainly used in reaction wheels, energy storage system and spacecraft attitude control actuator [1 - 3]. In most conventional systems, flywheel and rotor are supported by ball bearings and such bearings require lubrication and generate problems due to frictions, therefore, more maintenance issues are faced. Magnetic bearing technology provides frictionless mechanical motion for rotating machine elements with a very high precision [4, 5]. This technology has two types of bearings such as active magnetic bearings (AMB) which needs actively feedback control and repulsive magnetic bearings (RMB) which do not require the feedback control.

Active magnetic bearings have distinct advantages over conventional mechanical bearings such as adjustment of stiffness and damping values of the bearing but they consume energy and need feedback control in operation. Repulsive magnetic bearings use magnetic force generated by permanent magnets (PMs) for magnetic levitation [6 - 9]. From Earnshaw theorem, a stable magnetic levitation is impossible in a system composed solely of permanent magnets. A repulsive magnetic bearing is usually combined with a mechanical guidance or actively controlled electromagnets for stable levitation due to instability of at least one direction. Also, this type of bearing has poor damping in the levitation directions. It is an advantage that no energy is required to generate levitation force and larger rotor-stator gaps are possible in this type of bearings [10, 11].

In addition to stable levitation of the RMB systems, the proof of the possible stable dynamics is important. The simple permanent magnet (PM) levitation system can be modeled as a mass spring element in the axial direction. But in radial dynamics axial stiffness factor have to be considered. Mizuno's [12] and Bassani's works [13] are shown that the radial dynamics of the RMB is very similar to the Mathieu's equation. The radial dynamics of the repulsive magnetic bearing system has to satisfying the condition of the stability of Mathieu equation.

## 2. Flywheel system structure

The proposed flywheel system structure is schematically shown as a cross section in Figure 1. Basically, two sets of repulsive magnetic bearings are located at the upper and lower sides of the rotor. The repulsive type permanent magnetic bearings are passive and limit the radial and axial displacements of the rotor. The proposed flywheel system has a complete symmetry inside the housing.

The upper and lower magnetic bearings operating in the repulsion mode has a special structure that creates always opposite forces along the rotor axis. As seen in Figure 1, while A1 (ring PMs) and A2 (disk PMs) pairs limit axial movements of the rotor, R1 and R2 ring permanent magnet pairs limit radial movements of the rotor. Actually, it is aimed to provide the levitation of the rotor in the radial axis only using R1 and R2 ring permanent magnet pairs.  For these bearings, a screw mechanism for the outer ring that move the outer permanent magnets to the center of inner permanent magnet center position is used to set the initial position and eliminate the misalignment issue. In the proposed design, axially magnetized two pairs of permanent magnets are selected for the inner and outer rings (R1 and R2). The poles of the permanent magnets are oriented to repel each other inside the inner and the outer rings. This type of structure is discussed in many different research works [14, 15].

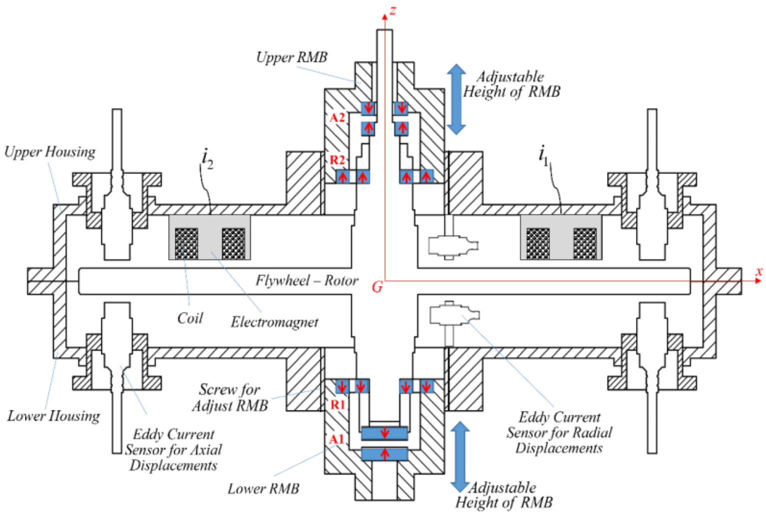


Figure 1. Cross section view of the flywheel system.

In this flywheel system, the stabilization of the flywheel system in the axial direction is performed by using four electromagnetic actuators. On the other hand, there is no active control effect in the radial directions. The stabilization of the radial direction maintained by the repulsive forces of the R1 and R2 bearings. The only criteria for the radial stability is the axial position of the rotor. Displacement sensors are assembled to the system for the feedback control of the actuators. Note that in the content of this study the feedback control design is not presented and only the radial stability is analysed using the Mathieu equation.

## 3. The Mathieu Equation-Stability

Mathieu equation is frequently written as

$$\ddot{\phi}+\left(\delta+\varepsilon\cos 2t\right)\phi=0 \tag{1}$$

The Mathieu equation is commonly used in nonlinear vibration problems. It used in stability studies of the periodic motions in nonlinear systems. In general, if a system has negative real parts of all poles for its transfer function, the system stability is achieved. This condition does not apply to the Mathieu equation. The stability condition of the Mathieu equation depends on the parameters. This parameter dependencies are shown in Figure 2 known as Strutt-Ince diagram.

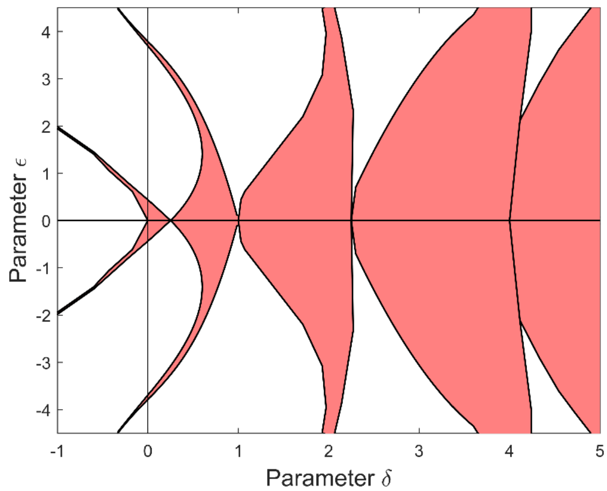


Figure 2. The Strutt-Ince diagram, the ε-δ plane

In this figure the horizontal axis represents the parameter δ, and the vertical axis, the parameter ε. The ε-δ plane is consist of two different regions. The filled areas on the Figure represent stable regions on the other hand white regions are correspond to unstable solutions. The diagram is symmetric about the δ axis. In addition, it can be understood from the stability diagram that stability increases for the smaller values of the ε and larger values of the δ. The stability diagram is also symmetric about the δ axis. To solve Mathieu equation, the equation (1) can be considered as

$$\ddot{\varphi}_i + \left(\sigma_i^2 + \varepsilon_i \cos \omega t\right)\varphi_i = 0, \qquad i = 1,2,3,4. \tag{2}$$

With the transformation of the ωt = $2t_1$ states of the Mathieu equation can be written as;

$$\dot{\varphi}_i = \frac{d\varphi_i}{dt_1}\frac{dt_1}{dt} = \frac{\omega}{2}\frac{d\varphi_i}{dt_1}, \qquad i = 1,2,3,4. \tag{3}$$

$$\ddot{\varphi}_i = \frac{d\dot{\varphi}_i}{dt_1}\frac{dt_1}{dt} = \frac{\omega^2}{4}\frac{d^2\varphi_i}{dt_1^2}, \qquad i = 1,2,3,4. \tag{4}$$

Substituting Eq. (3) and (4) into Eq. (2) gives a Mathieu equation in the standard form which is defined in Eq. (1)

$$\frac{d^2}{dt^2}\varphi_i\left(t\right) + \left(\psi_i + \gamma_i \cos 2t_1\right)\varphi_i\left(t\right) = 0, \qquad i = 1,2,3,4. \tag{5}$$

Where

$$\psi_i = \frac{4\sigma_i^2}{\omega^2} \qquad \gamma_i = -\frac{4\varepsilon_i}{\omega^2} \tag{6}$$

The general aspect of the $\psi$ and $\gamma$ plane is shown in Figure 3.

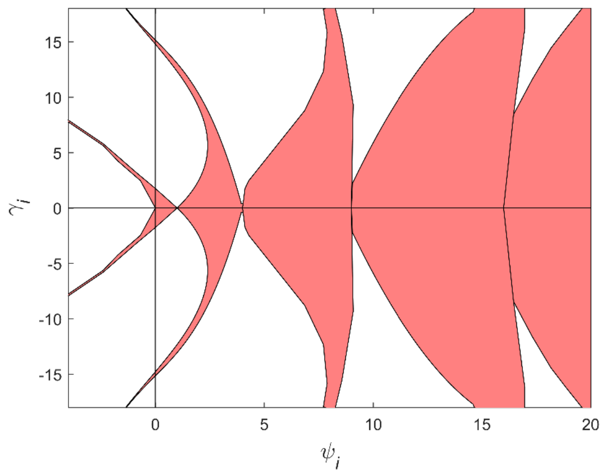


Figure 3. The Strutt-Ince diagram, the $\gamma_i$ - $\psi_i$ plane

## 4. Mathematical model of the system

### 4.1. Axial dynamics of the repulsive magnetic bearing

Consider a permanent magnet axial bearing with equal rings of mass $m_1$ and with opposite fields $B_r$. The upper PM ring is levitated over the fixed PM ring by the axial stable force $F_z$ at a height $z_G$ and it is moved laterally by the radial unstable force $F_r$. The equation of motion in the axial direction for the levitated ring PM is given by

$$m_1\ddot{z} + F_z = 0 \tag{7}$$

where the axial stable force is a vibrating characteristic such as $F_z = F_{zG}\cos\Omega_z t$ and the natural frequency is $\Omega_z = \sqrt{K_z / m_1}$. Since levitation force is vibrating, the axial displacement is $z = z_G \cos\Omega_z t$. For the same axial repulsive magnetic bearing, the radial equation of motion of the ring PM is given by

$$m_1\ddot{r} + f_r - f_{zr} = 0 \tag{8}$$

where $f_{zr} = F_z \sin\theta$, $f_r = (-K_z / 2)r$. With the conversion of the $\sin\theta \approx \theta$, $\theta = \dfrac{r}{z}$, $F_z = F_{zG}\cos\Omega_z t$ and $K_z = F_{zG}/z$.

$$m_1\ddot{r} + \left(-\frac{1}{2}K_z - K_z\cos\Omega_z t\right)r = 0 \tag{9}$$

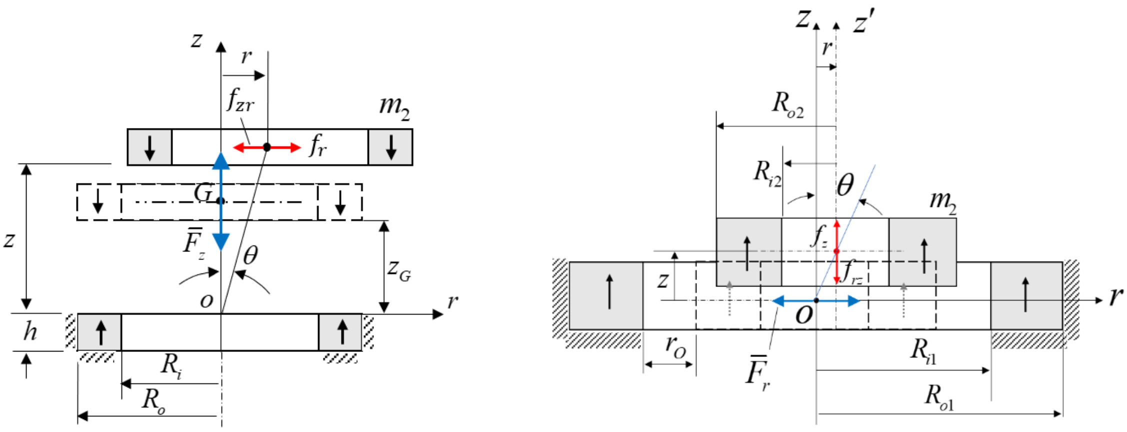


Figure 4. Axial and radial repulsive bearing dynamics

### 4.2. Radial Repulsive Magnetic Bearing

The equation of motion in the radial direction for the levitated inner ring PM is given by

$$m_2\ddot{r} + F_r = 0 \tag{10}$$

where the radial stable force and the natural frequency are $F_r = F_{rO}\cos\Omega_r t$ and $\Omega_r = \sqrt{K_r / m_2}$. Here $K_r$ is the radial stiffness. Similarly, the unstable axial dynamics of the inner ring PM is given by

$$m_2\ddot{z} + f_z - f_{rz} = 0 \tag{11}$$

where $f_z = (-K_r / 2)z$. With the similar conversation;

$$m_2\ddot{z} + \left(-\frac{K_r}{2} - K_r\cos\Omega_r t\right)z = 0 \tag{12}$$

### 4.3. Equation of motion of the flywheel system

The structure of the flywheel system with the repulsive magnetic bearing for modelling is depicted schematically in *xGz* plane in Figure 1. Since the system is completely symmetrical, *yGz* plane has also the same structure. The rotor flywheel system is assumed to be rigid and the center of mass G is known. The flywheel has translational motions in *x*, *y* and *z* directions and has angular motions $\eta_x$ and $\eta_y$ around *x* and *y* axes, respectively. The radial dynamics of the flywheel is written as

$$m_f \ddot{x}_g + F_{x2} + F_{x3} + f_{x1} - f_{zx1} + f_{x4} - f_{zx4} = 0 \tag{13}$$

From equation (15), the following approximation can be done

$$m_f \ddot{x}_g + (K_{x2} + K_{x3})x_g - \left( (\frac{K_{z1}}{2} + K_{z1}\cos\Omega_{z1}t) + (\frac{K_{z4}}{2} + K_{z4}\cos\Omega_{z4}t) \right) x_g = 0 \tag{14}$$

The equation can be written in form of the Methieu equation given in equation (2) as below

$$\ddot{x}_g + \left[ \underbrace{\frac{(K_{x2}+K_{x3})}{m_f} - \frac{(K_{z1}+K_{z4})}{2m_f}}_{\sigma_1^2} \right] x_g + \left( \underbrace{\frac{-(K_{z1}+K_{z4})}{m_f}}_{\varepsilon_1} \cos\Omega_{z_g} t \right) x_g = 0 \tag{15}$$

Since the system is completely symmetrical, same approximation is valid for the radial dynamics $y_g$ which Mathieu equation parameters are $\sigma_1^2 = \sigma_2^2$ and $\varepsilon_1 = \varepsilon_2$ . The angular dynamics of the flywheel-repulsive magnetic bearing system is obtained as with same approximation

$$J_t \ddot{\eta}_y - J_p \omega \dot{\eta}_x + F_{x2}L_m + F_{x3}L_m + f_{x1}L_p - f_{zx1}L_p + f_{x4}L_p - f_{zx4}L_p = f_{a1}L_f - f_{a2}L_f \tag{16}$$

$$J_f \ddot{\eta}_y - J_p \omega \dot{\eta}_x + (K_{x2}+K_{x3})L_m^2 \eta_y + \left( -\frac{(K_{z1}+K_{z4})L_p^2}{2} + (K_{z1}+K_{z4})L_p^2 \cos\Omega_z t \right) \eta_y = f_{a1}L_f - f_{a2}L_f \tag{17}$$

For a homogeneous solution, we take the right side of the equation to zero and make the necessary arrangement, we obtain Mathieu equation form of the angular dynamic equation.

$$\ddot{\eta}_y + \left[ \underbrace{\frac{(K_{x2}+K_{x3})L_m^2}{J_f} - \frac{(K_{z1}+K_{z4})L_p^2}{2J_f}}_{\sigma_3^2} \right] \eta_y + \left( \underbrace{\frac{-(K_{z1}+K_{z4})L_p^2}{J_f}}_{\varepsilon_3} \cos\Omega_z t \right) \eta_y = 0 \tag{18}$$

Same approximation is valid for the angular dynamics $\eta_x$ which parameters are $\sigma_3^2 = \sigma_4^2$ and $\varepsilon_3 = \varepsilon_4$ .

## 5. Repulsive magnetic bearing design and force characteristics

Repulsive magnetic bearings produce magnetic forces that act radially and axially on the rotor. To see the generating magnetic stiffness of the RMB in the flywheel system, a magneto static finite element analysis was performed. The total stiffness acting on the flywheel in the axial direction is given in Figure 5 (a) for different radial movement of the rotor. Similarly, the total stiffness in the radial direction is shown in Figure 5 (b) for different axial movement of the rotor.

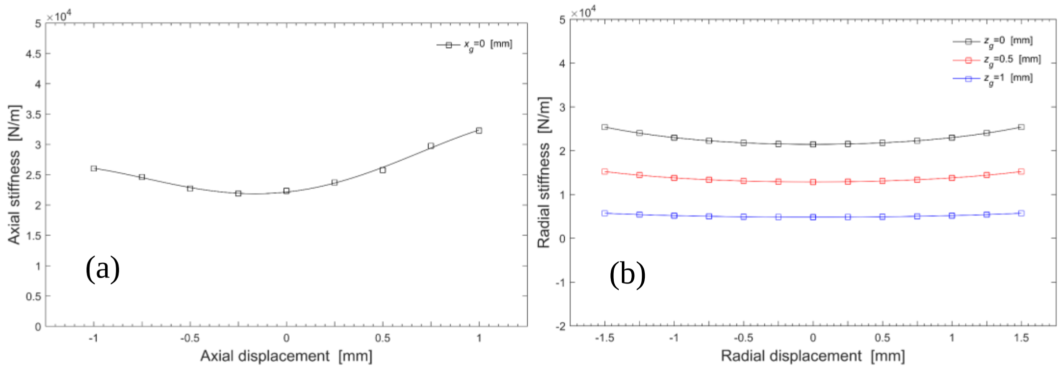


Figure 5. Repulsive magnetic bearing stiffness characteristics

## 6. Homogenous solution

The homogenous solution of the Mathieu Equation is obtained from Equation (4) to (8) using the differential equation solver ode45 in MATLAB software. The stability diagram of the flywheel system for homogenous solution is illustrated in Figure 6.

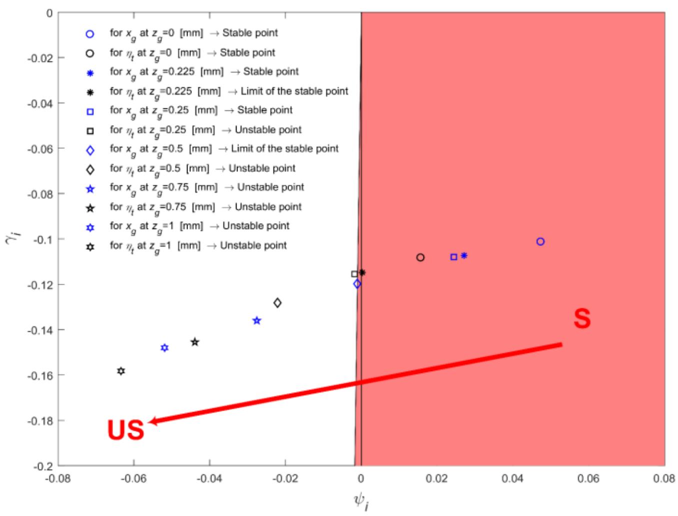


Figure 6. Stability diagram for repulsive magnetic bearing flywheel system

This diagram shows a narrow region of the The Strutt-Ince diagram given in Figure 3. Based on the stability chart different test points are selected to show the stability condition for the states of the flywheel system. Since the flywheel system is symmetric according to $z$ axis, the stable point for system states $x_g$ and

$y_g$ are equal. The same condition is valid for the system states $\eta_y$ and $\eta_x$. Since the stability is determined by the rotor axial movement $z_g$, the stability condition of the states $x_g$, $y_g$, $\eta_y$, $\eta_x$ should be tested for every $z_g$ value. Solution interval of these states can be in either stable or instable regions depending on the value of $z_g$.

## 7. Simulation results

The simulation starts by selecting the axial movement of the flywheel mass center $z_g$ and select the corresponding stiffness value. Then the state space model is solved with selected initial states such as $[x_g(0)\ y_g(0)\ \eta_y(0)\ \eta_x(0)]^T$. Using the stability diagram, some certain points shown in Figure 6 are selected depending on the mass center movement $z_g$ and time domain responses of the flywheel system states for these points are obtained to see the stability condition. When the axial displacement of the rotor is $z_g = 0$ [mm], all states are in the stability regions as seen in Figure 7. The time domain responses of the radial dynamics $x_g$ is illustrated in Figure 7(a). Since the flywheel system is symmetric according to $z$ axis, similar conclusion is available for $y_g$ at $z_g = 0$ [mm]. The responses of the angular dynamics at $z_g = 0$ mm are given in Figure 7 (b). Since $z_g = 0$ [mm] is at the stable region, $x_g$, $y_g$, $\eta_y$, $\eta_x$ states have stable behaviour. The amplitude of these states depends only the initial values of the states.

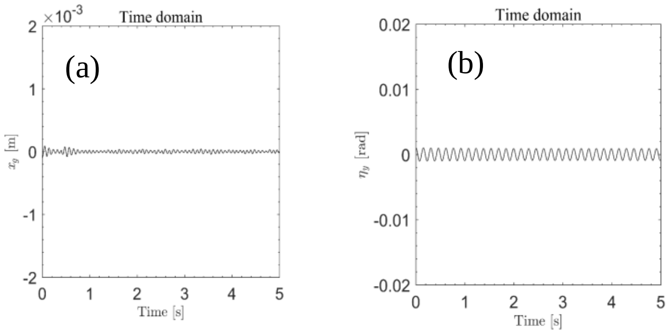


Figure 7. Simulation of the translation and angular dynamics at $z_g = 0$ [mm] (a) Variation of $x_g$ (b) Variation of $\eta_y$.

State behavior in the stability region limit point is checked for every state. Simulation results for the translational and angular dynamics at the stability limit points are given in Figure 8 and 9. The limit points are different for both dynamics and the states are stable at these limit regions. Although the states are stable in the limit points, the time responses increase very slowly.

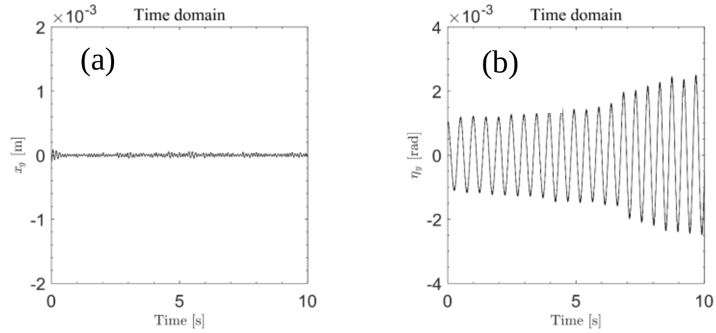


Figure 8. Simulation of the translation and angular dynamics at $z_g$ = 0.225 [mm] (a) Variation of $x_g$ (b) Variation of $\eta_y$.

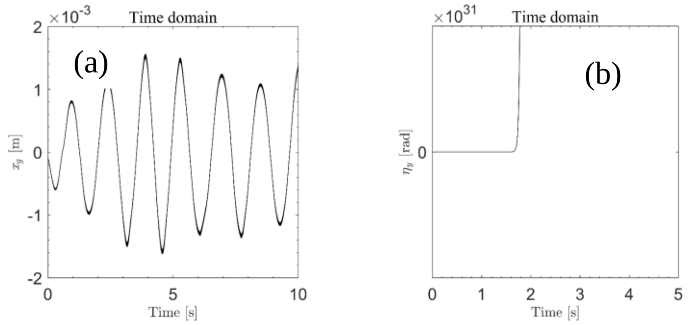


Figure 9. Simulation of the translation and angular dynamics at $z_g$ = 0.5 [mm] (a) Variation of $x_g$ (b) Variation of $\eta_y$.

When the axial distance exceeds the limit of stability point the system states have unstable behavior. Finally, time domain responses are obtained for a different unstable axial point at $z_g$ = 0.75 mm in Figure 10.

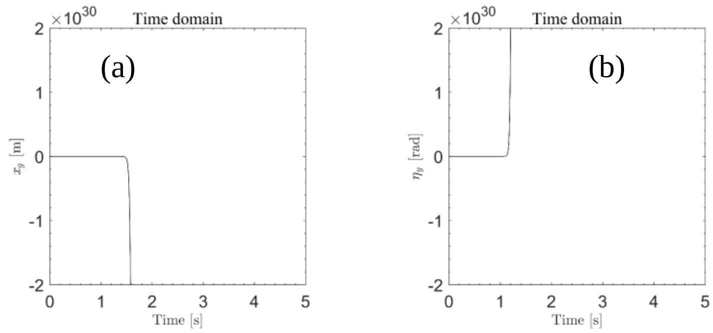


Figure 10. Simulation of the translation and angular dynamics at $z_g$ = 0.75 [mm] (a) Variation of $x_g$ (b) Variation of $\eta_y$.

## 8. Conclusions

In this research work, the stability dynamics of the repulsive magnetic bearing flywheel system is presented using the Mathieu equation approach. The stability diagram of the flywheel system is obtained using the solution of the Mathieu equation. The time domain responses of the system states are depicted for different rotor axial displacement according to stability diagram. In the proposed bearing structure, the stability of the radial repulsive magnetic bearing is determined by the axial movement of the rotor.

**Acknowledgement**

This research study is supported by The Scientific and Technological Research Council of Turkey under the support program of 1001 with the project number 214M284.

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# STUDY OF QUASILINEAR PARABOLIC PROBLEMS WITH DATA IN $L^1$

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Abstract. In this paper, we study the existence of entropy solution for quasillinear parabolic problem in bounded open subset $\Omega$ of $\mathbf{R}^N$, with data and $u_0$ in $L^1(\Omega)$. For this we use the Schauder fixed-point method. The results of the problem discussed can be applied to a variety of different fields in applied mathematics for example in elastic mechanics, image processing and electro-rheological fluid dynamics, etc..

Keywords: Quasilinear parabolic equations; fixed point; truncation function; $L^1$ data.

AMS subject classification (2010): 35K59; 37C25.

## 1. Introduction

In this article is devoted to presenting the results of existence of solution for a quasilinear parabolic problem with data in $L^1$, the main difficulty facing one who is interested in such problems is that the classical theories of existence, either using variational methods or compacite methods, are not applicable. Hence the need to use new techniques to prove the existence and uniqueness of solutions for such problems.
In the last years, different methods have been applied to study the existence of the weak solution of elliptic problems with $L^1$ under linear boundary conditions see [4],[6], [11] and [14]. The corresponding parabolic case equations have also been studied by many authors, see for instance [5],[8], [9] and [14].
Besides, partial differential equation (PDE) methods in image processing have proven to be fundamental tools for image diffusion and restoration. We refer the readers to [[1],[2]] and references therein.
The aim of this paper, we treat the existence of solution $u$ for the following quasi-linear parabolic problem of the type

$$(1.1) \qquad \begin{cases} u_t - div(A(u)\nabla u) + \lambda|u|^{p-2}u = f(t,x,u) & \text{in} \quad Q = [0,T]\times\Omega, \\ u = 0 & \text{on} \quad \Sigma = [0,T]\times\partial\Omega, \\ u(0,.) = u_0(.) & \text{in} \quad \Omega. \end{cases}$$

In the problem (1.1). Where $\lambda > 0$ and $T > 0$, $\Omega$ is a bounded open spatial domain in $\mathbb{R}^N$ ($N \geq 2$) with a lipschitz boundary denoted by $\partial\Omega$, and $u_0 \in L^1(\Omega)$. The function $\gamma(u) = \lambda|u|^{p-2}u$ such that $\gamma : \mathbb{R} \to \mathbb{R}$ is a continuous increasing function with $\gamma(0) = 0$ and the operator $A : \mathbb{R} \to \mathcal{M}_N(\mathbb{R})$ (or $\mathcal{M}_N(\mathbb{R})$ denotes the set of $N \times N$ matrices with real coefficients), such that satisfies the following assumption for some numbers $0 < \alpha < \beta < \infty$:

$$\forall s \in \mathbb{R}, A(s) = (a_{i,j}(s))_{i,j=1,\ldots,N} \text{ where } a_{i,j} \in L^\infty(\mathbb{R}) \cap C(\mathbb{R}, \mathbb{R}), \tag{1.2}$$

$$\exists\ \alpha > 0, \text{ such that } A(s)\xi.\xi \geq \alpha\, |\xi|^2\,, \quad \forall \xi \in \mathbb{R}^N, \forall s \in \mathbb{R}, \tag{1.3}$$

$$\exists\ \beta > 0, \text{ such that } \|a_{i,j}\|_{L^\infty(\mathbb{R})} \leq \beta, \quad \forall i,j \in \{1, ...., N\}\,. \tag{1.4}$$

We will assume that $f : Q \times \mathbb{R} \to \mathbb{R}$ is a Carathèodory function such that the following hypotheses hold

$$|f(t,x,s)| \leq c(t,x) + \sigma |s|, \tag{1.5}$$

$$sf(t,x,s) \geq 0, \tag{1.6}$$

for almost every $(t,x) \in Q$, for every $s \in \mathbb{R}$, where $c$ is a positive function in $L^2(Q)$ and $\sigma > 0$.
In this work we are studying the existence of weak solution of the quasilinear parabolic problem (1.1) using the truncation technique and the Schauder fixed point theory see [2],[10].
This result generalizes an analog of this work were made by N. Alaa and all [2] with an increase of $\gamma$ but given $L^1$ and, on the other hand, to extend it to the case $f(t,x,u)$ in $L^1$ data.
To prove our main result, we will proceed by three steps: the first step, we approximate the problem by the fixed point method. In the second step, we estimate on the approximate solution.
In the last step, we study the asymptotic behaviour of the approximate solution as $n$ go to infinity we use the equi-integrable theorem.
The difficulty of this work lies in the fact that the variational method can not be used because $f$ is in $L^1$.

## 2. Main results

Before tackling the main problem, we clearly state our definition of weak solution to the quasilnear parabolic problem.

**Definition 2.1.** *Let $1 < p \leq N$ a fixed number with $p > 2 - \dfrac{1}{N}$. We call $u$ a weak solution of the problem (1.1) in $Q$, if $u \in L^2([0,T], H_0^1(\Omega)) \cap C([0,T], L^1(\Omega))$, $u(0,.) = u_0$ for all $\varphi \in C_0^\infty(Q)$ we have*

$$\int_Q -u\varphi_t dxdt + \int_Q A(u)\nabla u \nabla\varphi dxdt + \int_Q \lambda\, |u|^{p-2}\, u\varphi dxdt = \int_Q f(t,x,u)\varphi dxdt, \tag{2.1}$$

*where $f(t,x,u)$ and $\gamma(u) \in L^1(Q)$.*

The main result of this paper is the following theorem:

**Theorem 2.2.** *Under the assumptions (1.2) − (1.6) satisfies, then for all $u_0 \in L^1(\Omega)$, there exists a weak solution $u$ of problem (1.1) in the sense defined in (2.1).*

Now we shall prove our main result.

## 3. Proof of the Theorem 2.2

The proof of the theorem consists in the three steps in the first step we solve an approximate problem, in the second step we get estimates on the approximated solutions these estimates allow us and in the third step to go to the limit.

3.1. **First step: solving an approximated problem.** For $n \in \mathbb{N}$ let us define the following approximation of $u_{n,0}$ and $f_n, \gamma_n$. Set

$$f_n(t,x,p) = \begin{cases} f(t,x,p) & \text{if } |f(t,x,p)| \le n, \\ n\ \mathrm{sign}(f(t,x,p)) & \text{if } |f(t,x,p)| > n. \end{cases} \tag{3.1}$$

$$\gamma_n(p) = \begin{cases} \gamma(p) & \text{if } |\gamma(p)| \le n, \\ 0 & \text{if } |\gamma(p)| > n. \end{cases} \tag{3.2}$$

And $(u_{n,0})_{n\in\mathbb{N}}$ be sequences in $L^2(\Omega)$ such that $(u_{n,0}) \to (u_0)$ in $L^1(\Omega)$.

**Remark**

$$|f_n(t,x)| \le n \text{ and } |\gamma_n(p)| \le n,$$
$$\text{so} \quad \gamma_n,\ f_n \in L^\infty(Q) \hookrightarrow L^p(Q),\ p > n \ge 1.$$

We consider the sequence of approximate problems

$$\begin{cases} (u_n)_t - div(A(u_n)\nabla u_n) + \gamma_n(u_n) = f_n(t,x,u_n) & \text{in} \quad Q = [0,T] \times \Omega, \\ u_n = 0 & \text{on} \quad \Sigma = [0,T] \times \partial\Omega, \\ u_n(0,.) = u_{n,0}(.) & \text{in} \quad \Omega. \end{cases} \tag{3.3}$$

We show that for all $n \in \mathbb{N}^*$ and $f_n(t,x,u_n) \in L^2(Q)$, $u_{n,0} \in L^2(\Omega)$ there exists $u_n \in L^2([0,T], H_0^1(\Omega)) \cap C([0,T], L^2(\Omega))$ and $(u_n)_t \in L^2([0,T], H^{-1}(\Omega))$ verify for all $v \in L^2([0,T], H_0^1(\Omega))$, we have

$$\begin{aligned} &\int_0^T \langle (u_n)_t, v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\, dt + \int_0^T\int_\Omega A(u_n)\nabla u_n \nabla v dxdt \\ &+ \int_0^T\int_\Omega \gamma_n(u_n) v dxdt = \int_0^T\int_\Omega f_n(t,x,u_n) v dxdt, \end{aligned} \tag{3.4}$$

We will show the existence of a weak solution of the problem (3.3) by the classical Schauder's fixed point theorem. Let us show now that the nonlinear application $F$ defined by

$$\begin{aligned} F: \quad L^2([0,T], H_0^1(\Omega)) &\to \quad L^2([0,T], H_0^1(\Omega)) \\ v_n &\mapsto \quad F(v_n) = G \circ F_n(v_n) = v_n, \end{aligned}$$

solution of

$$\begin{aligned} &\int_0^T \langle (v_n)_t, \varphi\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\, dt + \int_0^T\int_\Omega A(v_n)\nabla v_n \nabla \varphi dxdt \\ &+ \int_0^T\int_\Omega \gamma_n(v_n)\varphi dxdt = \int_0^T\int_\Omega f_n(t,x,v_n)\varphi dxdt, \forall \varphi \in L^2(]0,T[, H_0^1(\Omega)), \end{aligned}$$

is completely continous application of $L^2([0,T], H_0^1(\Omega))$ in $L^2([0,T], H_0^1(\Omega))$.
Where the operator $(F_n)$ is defined by

$$\begin{aligned} F_n: \quad L^2([0,T], H_0^1(\Omega)) &\to \quad L^2([0,T], H^{-1}(\Omega)) \\ v_n &\mapsto \quad F_n(v_n) = (v_n)_t + div(A(v_n)\nabla v_n) = f_n(t,x,v_n) - \gamma_n(v_n) \quad = \widetilde{f_n}(v_n), \end{aligned}$$

is continous and compact (natural injection), and $G$ is the Green's operator defined by :

$$\begin{array}{rccc} G: & L^2([0,T], H^{-1}(\Omega)) & \to & L^2([0,T], H_0^1(\Omega)) \\ & \widetilde{f_n}(v_n) = w_n & \mapsto & G(w_n) = v_n, \end{array}$$

is continous because the operator of Green is isomorphism of $L^2([0,T], H^{-1}(\Omega))$ in $L^2([0,T], H_0^1(\Omega))$. Therefore, the operator $F = G \circ F_n$ is completely continous.
The existence of a fixed point of $G \circ F_n$ is an immediate consequence of Schauder's fixed point theorem.
To apply the theorem of Schauder's, you have to choose a closed convex generally suitable a closed ball

$$C = \left\{ v \in L^2([0,T], H_0^1(\Omega)) \text{ such that } \|v\|_{L^2([0,T], H_0^1(\Omega))} \leq M \right\},$$

where $M$ is a constant to be determined subsequently, is therefore,

$$\begin{array}{rccc} F: & L^2([0,T], H_0^1(\Omega)) & \to & L^2([0,T], H_0^1(\Omega)) \\ & v_n & \mapsto & F(v_n) = v_n, \end{array}$$

transforms the bounds of $L^2([0,T], H_0^1(\Omega))$ into relatively compact sets in $L^2([0,T], H_0^1(\Omega))$, the set $C$ is a closed convex of $L^2([0,T], H_0^1(\Omega))$ and bounded, so $F$ is relatively compact.
We show that $R(F) = \{F(v_n),\ \forall v_n \in L^2([0,T], H_0^1(\Omega))\}$ is bounded in $L^2([0,T], H_0^1(\Omega))$, as $F(v_n)$ is solution of the variational problem.

$$\begin{aligned} & \int_0^T \langle (F(v_n))_t, \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \, dt + \int_0^T \int_\Omega A(v_n)) \nabla F(v_n) \nabla \varphi dx dt \\ & + \int_0^T \int_\Omega \gamma_n(v_n) \varphi dx dt = \int_0^T \int_\Omega f_n(t,x,v_n) \varphi dx dt, \forall \varphi \in L^2([0,T], H_0^1(\Omega)). \end{aligned} \tag{3.5}$$

We choose $F(v_n) = \varphi$ in (3.5), we obtain

$$\begin{aligned} & \int_0^T \langle F(v_n)_t, F(v_n) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \, dt + \int_0^T \int_\Omega A(v_n) \nabla F(v_n) \nabla F(v_n) dx dt \\ & + \int_0^T \int_\Omega \gamma_n(v_n) F(v_n) dx dt = \int_0^T \int_\Omega f_n(t,x,v_n) F(v_n) dx dt. \end{aligned} \tag{3.6}$$

By using Cauchy-Schwarz inequality in (3.6), we have

$$\begin{aligned} \frac{1}{2} \|v_n(T)\|_2^2 - \frac{1}{2} \|v_n(0)\|_2^2 + \int_0^T \int_\Omega A(v_n) |\nabla F(v_n)|^2 \, dx dt \quad \leq \quad & \int_Q |\gamma_n(v_n)| \, |F(v_n)| \, dx dt \\ & + \int_Q |f_n(t,x,v_n)| \, |F(v_n)| \, dx dt, \end{aligned}$$

by, using generalized Young's inequality and the hypothesis (1.3), we get

$$\begin{aligned}
\alpha \int_0^T \int_\Omega |\nabla F(v_n)|^2 \, dxdt &\leq \frac{1}{2} \|v_n(0)\|_2^2 + \|f_n(t,x,v_n)\|_{L^2(Q)} \|F(v_n)\|_{L^2([0,T],H_0^1(\Omega))} \\
&\quad + \|\gamma_n(v_n)\|_{L^2(Q)} \|F(v_n)\|_{L^2(]0,T[,H_0^1(\Omega))} \\
&\leq \frac{1}{2} \|v_n(0)\|_2^2 + \frac{1}{2\varepsilon} \|f_n(t,x,v_n)\|_{L^2(Q)}^2 + \frac{\varepsilon}{2} \|F(v_n)\|_{L^2([0,T],H_0^1(\Omega))}^2 \\
&\quad + \frac{1}{2\varepsilon} \|\gamma_n(v_n)\|_{L^2(Q)}^2 + \frac{\varepsilon}{2} \|F(v_n)\|_{L^2([0,T],H_0^1(\Omega))}^2 .
\end{aligned}$$

We conclude that,

$$(\alpha - \varepsilon) \|F(v_n)\|_{L^2(]0,T[,H_0^1(\Omega))}^2 \leq \frac{1}{2} \|v_n(0)\|_2^2 + \frac{1}{2\varepsilon} \|f_n(t,x,v_n)\|_{L^2(Q)}^2 + \frac{1}{2\varepsilon} \|\gamma_n(v_n)\|_{L^2(Q)}^2 . \tag{3.7}$$

Therefore the sequence $(F(v_n))_{n\in\mathbb{N}}$ is bounded in $L^2([0,T], H_0^1(\Omega))$. Next we show that $\{(F(v_n)_t)_{n\in\mathbb{N}}, F(v_n) \in R(F)\}$ is bounded in $L^2([0,T], H^{-1}(\Omega))$. We have

$$\begin{aligned}
&\int_0^T \langle F(v_n)_t, F(v_n) \rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \, dt + \int_0^T \int_\Omega A(v_n) \nabla F(v_n) \nabla F(v_n) dxdt \\
&+ \int_0^T \int_\Omega \gamma_n(v_n) F(v_n) dxdt = \int_0^T \int_\Omega f_n(t,x,v_n) F(v_n) dxdt.
\end{aligned}$$

By using hypothesis (1.2) and (1.4), we get

$$\begin{aligned}
& \|F(v_n)_t\|_{L^2([0,T],H^{-1}(\Omega))} \|F(v_n)\|_{L^2([0,T],H_0^1(\Omega))} \\
\leq\; & \beta \|F(v_n)\|_{L^2([0,T],H_0^1(\Omega))}^2 + \|f_n(t,x,v_n)\|_{L^2(Q)} \|F(v_n)\|_{L^2([0,T],H_0^1(\Omega))} \\
& + \|\gamma_n(v_n)\|_{L^2(Q)} \|F(v_n)\|_{L^2([0,T],H_0^1(\Omega))} .
\end{aligned}$$

Eventually,

$$\begin{aligned}
& \|F(v_n)_t\|_{L^2([0,T],H^{-1}(\Omega))} \\
\leq\; & \beta \|F(v_n)\|_{L^2([0,T],H_0^1(\Omega))} + \|f_n(t,x,v_n)\|_{L^2(Q)} + \|\gamma_n(v_n)\|_{L^2(Q)} .
\end{aligned}$$

Therefore the sequence $\{(F(v_n)_t)_{n\in\mathbb{N}}, F(v_n) \in R(F)\}$ is bounded in $L^2([0,T], H^{-1}(\Omega))$. As $(F(v_n))_{n\in\mathbb{N}}$ is bounded in $L^2([0,T], H_0^1(\Omega))$ and the sequence $(F(v_n)_t)_{n\in\mathbb{N}}$ is bounded in $L^2([0,T], H^{-1}(\Omega))$ according to the lemma of compactness then gives that $R(F)$ is relatively compact in $L^2([0,T], L^2(\Omega))$, which gives the compactness of $F$. For (3.7), we have $F(C) \subset C$, it is enough to take

$$M = \frac{1}{2(\alpha-\varepsilon)} \|v_{n,0}\|_2^2 + \frac{1}{2(\alpha-\varepsilon)\varepsilon} \|f_n(t,x,v_n)\|_{L^2(Q)}^2 + \frac{1}{2(\alpha-\varepsilon)\varepsilon} \|\gamma_n(v_n)\|_{L^2(Q)}^2 .$$

Therefore the hypotheses of Schauder's fixed point theorem are satisfied consequently there exists at least one solution to the problem in the set $C$.

3.2. **Second step: a priori estimates.** In thise step we proof the estimates of solution $(u_n)_{n\in\mathbb{N}}$ the problem (3.3).
For a given constant $k>0$ we define the truncated function $T_k:\mathbb{R}\to\mathbb{R}$ as

$$T_k(s)=\begin{cases} -k & \text{for } s<-k,\\ s & \text{for } |s|\le k,\\ k & \text{for } s>k.\end{cases}$$

For a function $u=u(x)$, $x\in\Omega$, we define the truncated function $T_k u$ pointwise, i.e., for every $x\in\Omega$ the value of $(T_k u)$ at $x$ is just $T_k(u(x))$. Observe that

$$\lim_{k\to 0}\frac{1}{k}T_k(s)=sign(s)=\begin{cases} 1 & \text{if } s>0,\\ 0 & \text{if } s=0,\\ -1 & \text{if } s<0.\end{cases} \tag{3.8}$$

Let the function $\Phi_k:\mathbb{R}\to\mathbb{R}$ such that, $\Phi_k\ge 0$, $\Phi_k\in L^\infty(\mathbb{R})$ and $|\Phi_k(x)|\le k\,|x|$,

$$\Phi_k(x) = \int_0^x T_k(s)ds.$$

($\Phi_k$ it is the primitive function of $T_k$). We have

$$\langle v_t,T_k(v)\rangle=\frac{d}{dt}\left(\int_\Omega \Phi_k(v)dx\right)\in L^1(Q).$$

What implies that

$$\int_0^T\langle v_t,T_k(v)\rangle=\int_\Omega\Phi_k(v(T))dx-\int_\Omega\Phi_k(v(0))dx,$$

where $\langle .,.\rangle$ denotes the duality between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.
We choose $v=T_k(u_n)$ as test function in (3.4), obtaining

$$\begin{cases} \int_\Omega\Phi_k(u_n(T))dx-\int_\Omega\Phi_k(u_n(0))dx+\int_0^T\int_\Omega A(u_n)\nabla u_n\nabla T_k(u_n)dxdt\\ +\int_Q\gamma_n(u_n)T_k(u_n)dxdt=\int_Q f_n(t,x,u_n)T_k(u_n)dxdt, \forall T_k(u_n)\in L^2([0,T],H_0^1(\Omega)).\end{cases}$$

By using hypothesis (1.3), we obtain

$$\int_0^T\int_\Omega A_n(u_n)\nabla u_n\nabla T_k(u_n)dxdt=\int_0^T\int_\Omega A(u_n)\nabla u_n\nabla u_n T_k'(u_n)dxdt\ge\alpha\int_0^T\int_\Omega|\nabla u_n|^2\,T_k'(u_n)dxdt\ge 0,$$

and by $sf(t,x,s)\ge 0$, we have

$$\int_0^T\int_\Omega\gamma_n(u_n)T_k(u_n)dxdt\le\int_\Omega\Phi_k(u_n(0))dx,$$

on the other hand, we have $\gamma_n(u_n) = \lambda|u_n|^{p-2}u_n \geq 0$ because $p > 1$ then,

$$\int_0^T\int_\Omega A(u_n)\nabla u_n \nabla T_k(u_n)dxdt \leq \int_\Omega \Phi_k(u_n(0))dx, \forall T_k(u_n) \in L^2([0,T]\,, H_0^1(\Omega)).$$

and,

$$\int_Q f_n(t,x,u_n)T_k(u_n)dxdt \leq \int_\Omega \Phi_k(u_n(0))dx.$$

For all $t \in [0,T]$, we definite the set $Q_T$ by

$$Q_T = \{(t,x) \in Q : u_n > k\} \cup \{(t,x) \in Q : u_n < -k\} \cup \{(t,x) \in Q : -k \leq u_n \leq k\}.$$

By thise definition of $Q_T$, we have

$$\begin{cases} \int\limits_{Q_T} A_n(u_n)\nabla u_n \nabla u_n T_k'(u_n)dxdt = \int\limits_{\{(t,x)\in Q\,:\ |u_n|\leq k\}} A(u_n)\nabla u_n \nabla u_n T_k'(u_n)dxdt \\ \qquad\qquad \leq \int\limits_\Omega \Phi_k(u_n(0))dx, \end{cases}$$

so we have, $\forall k \in \mathbb{R}^+$,

$$\int\limits_{\{(t,x)\in Q\,:\,|u_n|\leq k\}} |A(u_n)\nabla u_n \nabla u_n|\, dxdt \leq k\int_\Omega |u_{n,0}|\, dx. \tag{3.9}$$

We will now prove that,

$$\int\limits_{\{(t,x)\in Q\,:\ |u_n|\,\leq k\}} A(u_n)\nabla u_n \nabla u_n dxdt \leq k\, \|u_0\|_{L^1(\Omega)}\,,$$

by hypothesis (1.3), we obtain

$$\alpha \int\limits_{\{(t,x)\in Q\,:\ |u_n|\,\leq k\}} |\nabla u_n|^2\, dxdt \leq k\, \|u_0\|_{L^1(\Omega)}\,, \tag{3.10}$$

on the other hand, by (3.8), we obtain:

$$\int\limits_{\{(t,x)\in Q:|u_n|>0\}} |\gamma_n(u_n)|\, dxdt \leq \|u_0\|_{L^1(\Omega)}\,, \tag{3.11}$$

and,

$$\int\limits_{\{(t,x)\in Q:|u_n|>0\}} |f_n(t,x,u_n)|\, dxdt \leq \|u_0\|_{L^1(\Omega)}\,. \tag{3.12}$$

New we prove that $(u_n)_{n\in\mathbb{N}}$ is bounded in $C\left([0,T]\,, L^1(\Omega)\right)$.

$$\int_0^T \left\langle (u_n)_t\,, T_k\,(u_n)\right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} dt \leq \int_{Q_T} \gamma_n(u_n)T_k(u_n)dxdt + \int_{Q_T} f_n(t,x,u_n)T_k(u_n)dxdt,$$

we have also, for every $t$ in $[0,T]$

$$\int_\Omega \Phi_k(u_n(t))dx - \int_\Omega \Phi_k(u_n(0))dx \leq k\int\limits_{\{(t,x)\in Q:|u_n|>k\}} |\gamma_n(u_n)|\, dxdt + k\int\limits_{\{(t,x)\in Q:|u_n|>k\}} |f_n(t,x,u_n)|\, dxdt,$$

we now that $\Phi_k(s) \geq |s| - 1$ we deduce that, for every $t$ in $[0,T]$,

$$\begin{aligned}\int_\Omega |u_n(t)|\,dx &\leq \int_\Omega 1dx + k\int_{\{(t,x)\in Q:|u_n|>k\}} |\gamma_n(u_n)|\,dxdt + k\int_{\{(t,x)\in Q:|u_n|>k\}} |f_n(t,x,u_n)|\,dxdt + k\,\|u_{n,0}\|_{L^1(\Omega)} \\ &\leq meas(\Omega) + C\,\|u_0\|_{L^1(\Omega)}\,,\end{aligned}$$

which proves that $u_n$ is bounded in $L^2\left([0,T], H_0^1(\Omega)\right)$ and in $C\left([0,T], L^1(\Omega)\right)$, on the other hand, we get

$$\int_\Omega \Phi_k(u_n(T))dx + \alpha\int_{Q_T} |\nabla T_k(u_n)|^2\,dxdt \leq \int_\Omega \Phi_k(u_n(0))dx,$$

$\int_\Omega \Phi_k(u_n(T)) \geq 0$ and for all $s \geq 0, |\Phi_k(s)| \leq k\,|s|$, we have

$$\alpha\int_{Q_T} |\nabla T_k(u_n)|^2\,dxdt \leq k\,\|u_0\|_{L^1(\Omega)}\,. \tag{3.13}$$

That $T_k(u_n)$ is bounded in $L^2\left([0,T], H_0^1(\Omega)\right)$ for every $k > 0$.
Now we prove that $div\,(A(v_n)\nabla u_n)$ is bounded in $L^2\left([0,T], H^{-1}(\Omega)\right)$. By using hypothesis (1.4) and Cauchy-Schwarz inequality, we get

$$\begin{aligned}|\langle -div\,(A(u_n)\,\nabla u_n, T_k(u_n)\rangle| &= \left|\int_{Q_T} A(u_n)\nabla u_n \nabla T_k(u_n)\,dxdt\right| \\ &\leq \beta\,\|\nabla u_n\|_{L^2(Q)}\,\|\nabla T_k(u_n)\|_{L^2(Q)} \\ &\leq C.\end{aligned}$$

Since

$$\begin{aligned}\|-div\,(A(u_n)\nabla u_n)\|^2_{L^2([0,T],H^{-1}(\Omega))} &= \int_0^T \|-div\,(A(u_n)\nabla u_n)\|^2_{H^{-1}(\Omega)}\,dt \\ &= \int_0^T \sup_{\|T_k(u_n)\|_{L^2\left([0,T],H_0^1(\Omega)\right)}\leq 1} |\langle -div\,(A(u_n)\nabla u_n)\,, T_k(u_n)\rangle| \\ &\leq C.\end{aligned}$$

We know that $div\,(A(u_n)\nabla u_n)$ is bounded in $L^2\left([0,T], H^{-1}(\Omega)\right)$.

Finally, denoting $(u_n)_t = f_n(t,x,u_n) + div\,(A(u_n)\nabla u_n) - \gamma_n(u_n)$ we observe that $f_n + div\,(A(u_n)\nabla u_n) + \gamma_n(u_n)$ is bounded in $L^2\left([0,T], H^{-1}(\Omega)\right) + L^1(Q)$ and by (3.10), $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^2\left([0,T], H_0^1(\Omega)\right)$.

3.3. **Third step: passage to the limit.** We show that $(u_n)_{n\in\mathbb{N}}$ the solution approache the problem (3.4) converges to the solution of the original problem (2.1). By the estimate (3.11) and (3.12), we see that $(\gamma_n(u_n))_{n\in\mathbb{N}}$ is bounded in $L^1(Q)$ and $(f_n(t,x,u_n))_{n\in\mathbb{N}}$ is bounded in $L^1(Q)$. The sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^2([0,T], H_0^1(\Omega))$ and also the sequence $((u_n)_t)_{n\in\mathbb{N}}$ is bounded in $L^2([0,T], H^{-1}(\Omega)) + L^1(Q)$. Therefore, using Aubin-type compactness lemma [16], that $(u_n)_{n\in\mathbb{N}}$ is relatively compact in $L^2([0,T], L^2(\Omega))$, thus we can deduce

$$u_n \to u \text{ in } L^2([0,T], L^2(\Omega)),$$

on the other hand $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^2([0,T],H_0^1(\Omega))$ then, we can extracts a subsequence, still denoted by $(u_n)_{n\in\mathbb{N}}$ such that:

$$u_n \to u \text{ weakly in } L^2([0,T],H_0^1(\Omega)),$$

and

$$\nabla u_n \to \nabla u \text{ weakly in } (L^2([0,T],L^2(\Omega)))^{\mathbb{N}},$$

and $((u_n)_t)_{n\in\mathbb{N}}$ is bounded in $L^2([0,T],H^{-1}(\Omega))$ and in $L^1(Q)$ we can extract a subsequence, still denoted by $((u_n)_t)_{n\in\mathbb{N}}$ such that

$$(u_n)_t \to u_t \text{ weakly in } L^2([0,T],H^{-1}(\Omega)),$$

and either $u_{n,0}$ a sequence of $L^2(\Omega)$ such that

$$\|u_{n,0}\|_{L^1(\Omega)} \le \|u_0\|_{L^1(\Omega)},$$

and

$$u_{n,0} \longrightarrow u_0 \text{ strongly in } L^1(\Omega). \tag{3.14}$$

We will show that

$$\gamma_n(u_n) \to \gamma(u) \;\text{ strongly in } L^1(Q), \tag{3.15}$$

we have,

$$\begin{aligned}\|\gamma_n(u_n)\|_{L^1(Q)} &= \int_Q |\gamma_n(u_n)|\,dxdt \\ &\le \int_{\{(t,x)\in Q:|u_n|>0\}} |\gamma_n(u_n)|\,dxdt \\ &\le \|u_0\|_{L^1(\Omega)}.\end{aligned}$$

Then,

$$\sup \int_Q \gamma_n(u_n)\,dxdt < +\infty,$$

knowing that,

$$0 \le \int_Q |\gamma_n(u_n)|\,dxdt, \text{ because } p>1,$$

for each $(t,x)\in Q$, we pose

$$\lim_{n\to+\infty} \inf \gamma_n(u_n) = \gamma(u),$$

by the Fateau's lemma, we have $\gamma(u)$ in $L^1(Q)$.
As that

$$u_n \to u \text{ weakly in } L^2([0,T],L^2(\Omega)),$$

on the other hand, we have

$$\nabla u_n \to \nabla u \text{ in } (L^2([0,T],L^2(\Omega)))^{\mathbb{N}},$$

we note that

$$\begin{aligned}
&\int\limits_{\{(t,x)\in Q:|\gamma_n(u_n)|\le n\}} |\gamma_n(u_n)-\gamma(u)|\,dxdt \\
&\le \int\limits_Q |\gamma_n(u_n)-\gamma(u)|\,dxdt \to 0 \text{ when } n\to+\infty.
\end{aligned}$$

So,

$$\gamma_n(u_n)\to\gamma(u) \text{ when } n\to+\infty \text{ on } \{(t,x)\in Q \;:\; |\gamma_n(u_n)|\le n\}.$$

For every $n\in\mathbb{N}$, we have

$$\begin{aligned}
meas(\{(x,t)\in Q:|\gamma_n(u_n)|>n\}) &\le \frac{1}{n}\int\limits_Q |\gamma_n(u_n)|\,dxdt \\
&\le \frac{1}{n}\|\gamma_n(u_n)\|_{L^1(Q)} \\
&\le \frac{c}{n}\to 0 \text{ when } n\to+\infty,
\end{aligned}$$

thus $\{(t,x)\in Q:|\gamma_n(u_n)|>n\}$ is the zero measurement set where $(\gamma_n(u_n))_{n\in\mathbb{N}}$ may not converge to $(\gamma(u))$, which shows that

$$\gamma_n(u_n)\to\gamma(u) \text{ almost everywhere in } Q.$$

For proof (3.15) we show that the sequence $(\gamma_n(u_n))_{n\in\mathbb{N}}$ is equi-integrable.
Let $\delta>0$ and $\mathbf{A}$ be a measurable subset belonging to $[0,T]\times\Omega$, we define the following sets,

$$B_\delta = \{(t,x)\in Q:\; |u_n|\le\delta\}, \tag{3.16}$$
$$F_\delta = \{(t,x)\in Q:\; |u_n|>\delta\}, \tag{3.17}$$

$$\begin{aligned}
\int\limits_{\mathbf{A}} |\gamma_n(u_n)|\,dxdt &= \int\limits_{\mathbf{A}\cap B_\delta} |\gamma_n(u_n)|\,dxdt + \int\limits_{\mathbf{A}\cap F_\delta} |\gamma_n(u_n)|\,dxdt \\
&\le \int\limits_{\mathbf{A}\cap B_\delta} |\gamma_n(u_n)|\,dxdt + \|u_0\|_{L^1(\Omega)} \\
&\to 0 \text{ when } meas(\mathbf{A})\to 0.
\end{aligned}$$

Using the generalized Hölder's inequality and Poincaré inequality, we get

$$\begin{aligned}
\int\limits_{\mathbf{A}} |\gamma_n(u_n)|\, dxdt &\leq \left(\int\limits_{\mathbf{A}} |\lambda|^2\, dxdt\right)^{\frac{1}{2}} \left(\int\limits_{B_\delta} |u_n|^{(p-1)2}\, dxdt\right)^{\frac{1}{2}} \\
&\quad + \int\limits_{\mathbf{A}} |\gamma_n(u_n)|\, dxdt \\
&\leq \left(|\lambda|^2\, meas(\mathbf{A})\right)^{\frac{1}{2}} \left(\int\limits_{B_\delta} |\nabla u_n|^2\, dxdt\right)^{(p-1)\frac{1}{2}} \\
&\quad + \int\limits_{\mathbf{A}} |\gamma_n(u_n)|\, dxdt \\
&\leq \left(|\lambda|^2\, meas(\mathbf{A})\right)^{\frac{1}{2}} \left(\frac{k}{\alpha}\left(\|u_0\|_{L^1(\Omega)}\right)\right)^{(p-1)\frac{1}{2}} \\
&\quad + \int\limits_{\mathbf{A}} |\gamma_n(u_n)|\, dxdt \\
&\to 0 \text{ when } meas(\mathbf{A}) \to \mathbf{0}.
\end{aligned}$$

Which shows that $(\gamma_n(u_n))_{n\in\mathbb{N}}$ is equi-integrable. By using Vitali's theorem, we obtain:

(3.18) $$\gamma_n(u_n) \to \gamma(u) \text{ strongly in } L^1(Q).$$

Now we prove that

$$f_n(t,x,u_n) \to f(t,x,u) \text{ strongly in } L^1(Q),$$

we have,

$$\begin{aligned}
\|f_n(t,x,u_n)\|_{L^1(Q)} &= \int\limits_Q |f_n(t,x,u_n)|\, dxdt \\
&\leq \int\limits_{\{(t,x)\in Q: |u_n|>0\}} |f_n(t,x,u_n)|\, dxdt \\
&\leq \|u_0\|_{L^1(\Omega)},
\end{aligned}$$

then,

$$\sup \int\limits_Q f_n(t,x,u_n) dxdt < +\infty.$$

By (1.6) knowing that, $0 \leq f_n(t,x,u_n)$ for each $(t,x) \in Q$, we pose

$$\lim_{n\to+\infty} \inf f_n(t,x,u_n) = f(t,x,u),$$

by the Fateau's lemma, we have $f(t,x,u)$ in $L^1(Q)$. As that

$$u_n \to u \text{ weakly in } L^2([0,T], L^2(\Omega)),$$

on the other hand, we have

$$\nabla u_n \to \nabla u \text{ in } (L^2([0,T]\,, L^2(\Omega)))^{\mathbb{N}},$$

we note that,

$$\begin{aligned}
&\int\limits_{\{(t,x)\in Q:|f_n(t,x,u_n)|\le n\}} |f_n(t,x,u_n) - f(t,x,u)|\,dxdt \\
&\le \int\limits_{Q} |f_n(t,x,u_n) - f(t,x,u)|\,dxdt \to 0 \text{ when } n \to +\infty.
\end{aligned}$$

So,

$$f_n(t,x,u_n) \to f(t,x,u) \text{ when } n \to +\infty \text{ on } \{(t,x) \in Q \;:\; |f(t,x,u)| \le n\}\,.$$

For every $n \in \mathbb{N}$, we have

$$\begin{aligned}
meas(\{(x,t) \in Q : |f_n(t,x,u_n)| > n\}) &\le \frac{1}{n}\int\limits_{Q} |f_n(t,x,u_n)|\,dxdt \\
&\le \frac{1}{n}\,\|f_n(t,x,u_n)\|_{L^1(Q)} \\
&\le \frac{c}{n} \to 0 \text{ when } n \to +\infty,
\end{aligned}$$

thus $\{(t,x) \in Q : |f_n(t,x,u_n)| > n\}$ is the zero measurement set where $(f_n(t,x,u_n))_{n\in\mathbb{N}}$ may not converge to $(f(t,x,u))$, which shows that

$$f_n(t,x,u_n) \to f(t,x,u) \text{ almost everywhere in } Q.$$

For proof (3.15) we show that the sequence $(f_n(t,x,u_n))_{n\in\mathbb{N}}$ is equi-integrable.
By the definitions of the sets (3.16) and (3.17), we get

$$\begin{aligned}
\int\limits_{\mathbf{A}} |f_n(t,x,u_n)|\,dxdt &= \int\limits_{\mathbf{A}\cap B_\delta} |f_n(t,x,u_n)|\,dxdt + \int\limits_{\mathbf{A}\cap F_\delta} |f_n(t,x,u_n)|\,dxdt \\
&\le \int\limits_{\mathbf{A}\cap B_\delta} |f_n(t,x,u_n)|\,dxdt + \|u_0\|_{L^1(\Omega)} \\
&\to 0 \text{ when } meas(\mathbf{A}) \to 0.
\end{aligned}$$

Let $\delta > 0$ be large enough. Using the generalized Hölder's inequality and Poincaré inequality, we have

$$\int\limits_{\mathbf{A}} |f_n(t,x,u_n)|\,dxdt = \int\limits_{\mathbf{A}\cap B_\delta} |f_n(t,x,u_n)|\,dxdt + \int\limits_{\mathbf{A}\cap F_\delta} |f_n(t,x,u_n)|\,dxdt,$$

therefore

$$\begin{aligned}
\int_{\mathbf{A}} |f_n(t,x,u_n)|\,dxdt &\leq \int_{\mathbf{A}\cap B_\delta} (c(x,t)+\sigma\,|u_n|)\,dxdt + \int_{\mathbf{A}\cap F_\delta} |f_n(t,x,u_n)|\,dxdt \\
&\leq \int_{\mathbf{A}} c(x,t)dxdt + \sigma \int_{Q} |\nabla T_\delta(u_n)|\,dxdt \\
&\quad + \int_{\mathbf{A}\cap F_\delta} |f_n(t,x,u_n)|\,dxdt \\
&\leq \int_{\mathbf{A}} c(x,t)dxdt + \sigma\,(meas(\mathbf{A}))^{\frac{1}{2}} \left( \int_{Q_T} |\nabla T_\delta(u_n)|^2\,dxdt \right)^{\frac{1}{2}} \\
&\quad + \int_{\mathbf{A}\cap F_\delta} |f_n(t,x,u_n)|\,dxdt \\
&\leq K_1 + C_1 \left( \frac{k}{\alpha} \|u_0\|_{L^1(\Omega)} \right)^{\frac{1}{2}} + \int_{\mathbf{A}\cap F_\delta} \frac{1}{|u_n|}\,|u_n f_n(t,x,u_n)|\,dxdt \\
&\leq K_2 + \int_{\mathbf{A}\cap F_\delta} \frac{1}{\delta}\,|u_n f_n(t,x,u_n)|\,dxdt \\
&\leq K_2 + \frac{1}{\delta} \left( \int_{\mathbf{A}\cap F_\delta} |u_n|^2\,dxdt \right)^{\frac{1}{2}} \left( \int_{\mathbf{A}\cap F_\delta} |f_n(t,x,u_n)|^2\,dxdt \right)^{\frac{1}{2}} \\
&\to 0 \text{ when } meas(\mathbf{A}) \to \mathbf{0}.
\end{aligned}$$

Which shows that $(f_n(t,x,u_n))_{n\in\mathbb{N}}$ is equi-integrable. By using Vitali's theorem, we get

$$f_n(t,x,u_n) \to f(t,x,u) \text{ strongly in } L^1(Q). \tag{3.19}$$

Since $u_n \in C([0,T], L^2(\Omega))$, in order to see that $u \in C([0,T], L^1(\Omega))$, we only have to prove that

$$u_n \to u \text{ in } C([0,T], L^1(\Omega)).$$

To do this fix $\tau \in [0,T]$. Choosing $T_k(u_n - u_m)1_{\{[0,\tau[\}}$ as test function in the weak formulation of $u_n$ and $-T_k(u_n - u_m)1_{\{[0,\tau[\}}$ in that of $u_m$ with $\tau \le T$, we get

$$\begin{aligned}
&\int_\Omega \Phi_k(u_n(\tau) - u_m(\tau))dx - \int_\Omega \Phi_k(u_n(0) - u_m(0))dx \\
&+\int_0^\tau\int_\Omega A(u_n - u_m)\nabla(u_n - u_m)\nabla T_k(u_n - u_m)dxdt \\
&+\int_0^\tau\int_\Omega \lambda\left[|u_n|^{p-2}u_n - |u_m|^{p-2}u_m\right]T_k(u_n - u_m)dxdt \\
= &\int_0^\tau\int_\Omega \left(f_n(t,x,u_n) - f_n(t,x,u_m)\right)T_k(u_n - u_m)dxdt,
\end{aligned}$$

where $\Phi_k$ is the primitive of $T_k$ such that $\Phi_k(0) = 0$,

$$\begin{aligned}
\int_\Omega \Phi_k(u_n(\tau) - u_m(\tau))dx \quad \le \quad &\int_0^\tau\int_\Omega \lambda\left||u_n|^{p-2}u_n - |u_m|^{p-2}u_m\right|dxdt \\
&+k\int_0^\tau\int_\Omega |f_n(t,x,u_n) - f_n(t,x,u_m)|\,dxdt \\
&+k\int_\Omega |u_{n,0} - u_{m,0}|\,dx.
\end{aligned}$$

Next, we divide this inequality by $k$ and the Monotone convergence theorem and let $k$ go to 0, to obtain

$$\begin{aligned}
\int_\Omega |u_n(\tau) - u_m(\tau)|\,dx \quad \le \quad &\int_0^\tau\int_\Omega \lambda\left||u_n|^{p-2}u_n - |u_m|^{p-2}u_m\right|dxdt \\
&+\int_0^\tau\int_\Omega |f_n(t,x,u_n) - f_n(t,x,u_m)|\,dxdt \\
&+\int_\Omega |u_{n,0} - u_{m,0}|\,dx.
\end{aligned}$$

Hence,

$$\sup_{\tau\in[0,T]}\int\limits_{\Omega}|u_n(\tau)-u_m(\tau)|\,dx \leq \int\limits_0^{\tau}\int\limits_{\Omega}\lambda\left||u_n|^{p-2}u_n-|u_m|^{p-2}u_m\right|dxdt$$
$$+\int\limits_0^{\tau}\int\limits_{\Omega}|f_n(t,x,u_n)-f_n(t,x,u_m)|\,dxdt$$
$$+\int\limits_{\Omega}|u_{n,0}-u_{m,0}|\,dx.$$

Thus, it follows from (3.14), (3.19) and (3.18), that sequence $(u_n)$ is a Cauchy sequence in $C\left([0,T],L^1(\Omega)\right)$ then $u_n\to u$ in $C([0,T],L^1(\Omega))$. Finally,

$$u\in C([0,T],L^1(\Omega)). \tag{3.20}$$

## 4. Conclusion

We conclude by the main purpose from our work. In this article we demonstrated the existence of entropy solution for quasi-linear parabolic problem with $L^1$ data, we also proved that the problem admits a weak solution according to Schauder fixed point theorem. For unbounded nonlinearities satisfying suitable conditions, we established equi-integrablity and we derived a compactness results to be able to pass to the limit to get the desired result.

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# The Relation Between Chebyshev Polynomials and Jacobsthal and Jacobsthal Lucas Sequences

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**Abstract**
The main purpose of this paper is to establish some new properties of Jacobsthal, Jacobsthal Lucas sequences in terms of Chebyshev polynomials. Moreever, some connections among Jacobsthal, Jacobsthal Lucas sequences are revealed by using the power of some special matrices. And also the properties of Jacobsthal, Jacobsthal Lucas numbers are obtained by using the identities of Chebyshev polynomials
**Keywords:** Chebhshev polynomials Jacobsthal and Jacobsthal Lucas Sequences.

## 1. Introduction

For any $n \geq 2$ integers, a, b, p; q are integers, Horadam sequence was defined by Horadam in 1965, denoted by $\{W_n\}_{n=0}^{\infty}$ , by the following recursive relation

$$W_0 = a\,, W_1 = b\,, W_n(a,b,p,q) = W_n = \ p\, W_{n-1} - q\, W_{n-2}$$

where $p^2 - 4q \neq 0$ . For special choices of *a; b, p; q* special integer sequences are obtained as
$W_n(0,1,1,-1) = F_n$, classic Fibonacci sequence

$W_n(2,1,1,-1) = L_n$, classic Lucas sequence

$W_n(0,1,p,-q) = \mathcal{F}_n$, generalized Fibonacci sequence

$W_n(0,1,1,-2) = j_n$, classic Jacobsthal sequence

$W_n(a,b,1,-2) = J_n$, generalized Jacobsthal sequence

$W_n(2,1,1,-2) = c_n$, classic Jacobsthal Lucas sequence

$W_n(0,1,2,-1) = P_n$, classic Pell sequence

$W_n(2,2,2,-1) = Q_n$, classic Pell Lucas sequence

$W_n(1,x,2x,1) = T_n$, first kind Chebyshev polynomials

$W_n(1,2x,2x,1) = U_n$, second kind Chebyshev polynomials

The humankind encountered special integer sequences with Fibonacci in 1202. The importance of Fibonacci sequence was not understood in that century. But now, for rich applications of special sequences, there are many studies on it. For example, the Golden Ratio, the ratio of two consecutive Fibonacci numbers is used in Physics, Art, Architecture, Engineering. We can also encounter Golden Ratio so many areas in nature. One of these special integer sequences is Horadam sequence. It is very important since we can obtain almost all of other special integer sequences by using Horadam sequence. Horadam sequence was studied by Horadam, Carlitz, Riordan and other some mathematicians since 1960. Horadam intended to write the first paper which contains the properties of Horadam sequences in [1,2]. In 1969, the relations between Chebyshev fuctions and Horadam sequences were investigated in [3]. In [6], Udrea found relations with Horadam sequence and Chebyshev polynomials. In [7], Mansour found a formula for the generating functions of powers of Horadam sequence. Horzum and Koçer studied the properties of Horadam polynomial sequences in [8]. The authors established identities involving sums of products of binomial coe¢ cients and coe¢ cients that satisfy the general second order linear recurrence in [9]. In [10], the authors obtained Horadam numbers with positive and negative indices by usind determinants of some special tridiagonal matrices. In [11], the authors established formulas for odd and even sums of generalized Fibonacci numbers by matrix methods. In [12], some properties of the generalized Fibonacci sequence were obained by matrix methods.

One of important special integer sequences is Jacobsthal sequence because of its application in computer science. In [4,13,14,15], you can find some properties and generalizations of Jacobsthal sequence.

## 2. Preliminaries

**Definition 1** For any $n \geq 2$ integers, the Jacobsthal $\{j_n\}_{n=0}^{\infty}$, the Jacobsthal Lucas $\{c_n\}_{n=0}^{\infty}$ and generalized Jacobsthal $\{J_n\}_{n=0}^{\infty}$ sequences are defined by

$$j_n = j_{n-1} + 2j_{n-2} \qquad j_0 = 0\,, j_1 = 1$$

$$c_n = c_{n-1} + 2c_{n-2} \qquad c_0 = 2\,, c_1 = 1$$

$$J_n = J_{n-1} + 2J_{n-2} \qquad J_0 = a\,, J_1 = 1$$

respectively.

**Definition 2** For any $n \geq 2$ integers, the first kind $\{T_n\}_{n=0}^{\infty}$ and second kind $\{U\}_{n=0}^{\infty}$ Chebyshev polynomials are defined by the following recurrence relation

$$T_0 = 1\,, T_1 = x\,, T_n = 2x\,T_{n-1} - T_{n-2}$$

$$U_0 = 1\,, U_1 = 2x\,, U_n = 2x\,U_{n-1} - U_{n-2}$$

respectively.

The Binet formula for the Horadam sequence is given by

$$W_n = \frac{X\alpha^n - Y\beta^n}{\alpha - \beta}$$

where X = b- $\alpha\beta$ , Y = b- $a\alpha$ ; $\alpha$ ,$\beta$ being the roots of the associated characteristic equation of the Horadam sequence $\{W_n\}_{n=0}^{\infty}$ . From the definition of Horadam sequence we obtain the quadratic characteristic equation for $\{W_n\}_{n=0}^{\infty}$ as $x^2 - px + q = 0$ , with roots $\alpha$ and $\beta$ defined by

$$\alpha = \frac{p + \sqrt{p^2 - 4q}}{2}, \qquad \beta = \frac{p - \sqrt{p^2 - 4q}}{2}$$

The summation, diference and product of the roots are given as

$$\alpha + \beta = p\,, \qquad \alpha - \beta = \sqrt{p^2 - 4q}, \qquad \alpha\beta = q$$

The Binet formula for the Jacobsthal, Jacobsthal Lucas and generalized Jacobsthal sequences are given by respectively

$$j_n = \frac{2^n - (-1)^n}{3}$$

$$c_n = 2^n + (-1)^n$$

$$J_n = \frac{X\,2^n - Y(-1)^n}{3}$$

where $X = b+a$ , $Y = b - 2a$.

We define $E_n = -XY = (b - \alpha\beta)(b - a\alpha) = pab - q\,a^2 - b$ for the Horadam sequence $W_n$ . Similarly, for the first kind Chebyshev polynomials $E_n = -XY = x^2 - 1$ , and for the second kind Chebyshev polynomials $E_u = -1$. We know that

$$T_n(\cos\theta) = \cos n\theta, \quad U_{n-1}(\cos\theta) = \frac{\sin n\theta}{\sin\theta}\ , \qquad n\epsilon\,\mathrm{N}, \sin\theta \neq 0$$

## 3. Main Results

**Proposition 3** Jacobsthal, Jacobsthal Lucas, generalized Jacobsthal numbers are obtained by using Chebyshev polynomials as

$$j_n = (2i^2)^{\frac{n-1}{2}}\ U_{n-1}\left(\frac{1}{2\sqrt{2i}}\right)$$

$$c_n = 2\ (2i^2)^{\frac{n}{2}}\ T_n\left(\frac{1}{2\sqrt{2i}}\right)$$

$$J_n = \frac{a\ (2i^2)^{\frac{n}{2}}}{3}\quad T_n\left(\frac{1}{2\sqrt{2i}}\right) + \frac{(2b-a\ )(2i^2)^{\frac{n-1}{2}}}{2}\quad U_{n-1}\left(\frac{1}{2\sqrt{2i}}\right)$$

$$= (2i^2)^{\frac{n}{2}}\ \left[\frac{b}{\sqrt{2i}}\quad U_{n-1}\left(\frac{1}{2\sqrt{2i}}\right) - a\ \ U_{n-1}\left(\frac{1}{2\sqrt{2i}}\right)\right]$$

**Proof.** The roots of characteristic equation for Horadam sequence are $\alpha, \beta = \frac{p \pm \sqrt{p^2-4q}}{2}$ are demonstrated by

$$\alpha,\beta = \sqrt{q}\left(\frac{p}{2\sqrt{q}} \pm \sqrt{\left(\frac{p}{2\sqrt{q}}\right)^2 - 1}\right) = \sqrt{q}\,(\cos\theta \pm i\,\sin\theta)$$

where $\cos\theta = \frac{p}{2\sqrt{q}}$. By De Moivre formula it is written that

$$\alpha^n = q^{\frac{n}{2}}(\cos n\theta + i\sin n\theta\,)$$

$$\beta^n = q^{\frac{n}{2}}(\cos n\theta - i\sin n\theta\,).$$

We know that for $p = 1, q = -2$, Jacobsthal and Jacobsthal Lucas numbers are obtained by Horadam sequence. Then

$$j_n = W_n(0,1,1,-2) = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

$$= \frac{\sqrt{q}^2[(\cos\theta + i\sin\theta)^2 - (\cos\theta - i\sin\theta)^2]}{2\sqrt{q}\,i\ \sin\theta}$$

$$= \sqrt{2i^2}^{\,n-1}\frac{\sin n\theta}{\sin\theta}$$

$$= (2i^2)^{\frac{n-1}{2}}\,U_{n-1}(\cos\theta) = (2i^2)^{\frac{n-1}{2}}\,U_{n-1}\left(\frac{1}{2\sqrt{2i}}\right)$$

$$c_n = W_n(2,1,1,-2) = \alpha^n + \beta^n$$

$$= (2i^2)^{\frac{n}{2}}\, 2(\cos n\theta) = (2i^2)^{\frac{n}{2}}\ T_n(\cos n\theta)$$

$$= (2i^2)^{\frac{n}{2}}\ T_n\left(\frac{1}{2\sqrt{2i}}\right)$$

$$J_n = W_n(a,b,1,-2) = \frac{X\alpha^n - Y\beta^n}{\alpha - \beta}$$

$$= \frac{X\sqrt{q}^{\,n}\left[(\cos\theta + i\sin\theta)^n - Y\sqrt{q}^{\,n}\,(\cos\theta - i\sin\theta)^n\right]}{2\sqrt{q}\ i\ \sin\theta}$$

$$= \frac{\sqrt{q}^{\,n}[\cos n\theta\,(X-Y) + i\,(X+Y)\,i\sin\theta]}{2\sqrt{q}\ i\ \sin\theta}$$

$$= \frac{\sqrt{q}^{\,n}\cos n\theta\,(X-Y)}{3} + \frac{\sqrt{q}^{\,n-1}(X+Y)\ \sin\theta}{2\ \sin\theta}$$

$$= \frac{a\,(2i^2)^{\frac{n}{2}}\ T_n\,(\cos\theta\,)}{3} + \frac{(2b-a\,)\,(2i^2)^{\frac{n-1}{2}}\ U_n\,(\cos\theta\,)}{2}$$

$$= \frac{a\,(2i^2)^{\frac{n}{2}}}{3}\ T_n\left(\frac{1}{2\sqrt{2i}}\right) + \frac{(2b-a\,)\,(2i^2)^{\frac{n-1}{2}}}{2}\ U_{n-1}\left(\frac{1}{2\sqrt{2i}}\right)$$

By using the well- known property of Chebyshev polynomials as $T_n(x) = x\,U_{n-1}(x) - U_{n-2}(x)$ it is easily seen that

$$J_n = (2i^2)^{\frac{n}{2}}\left[a\frac{1}{2\sqrt{2i}}\ U_{n-1}\left(\frac{1}{2\sqrt{2i}}\right) - a\ U_{n-2}\left(\frac{1}{2\sqrt{2i}}\right) + \frac{2b-a}{2\sqrt{2i}}\ U_{n-1}\left(\frac{1}{2\sqrt{2i}}\right)\right]$$

$$= (2i^2)^{\frac{n}{2}}\left[\frac{b}{\sqrt{2i}}\ U_{n-1}\left(\frac{1}{2\sqrt{2i}}\right) - a\,U_{n-2}\left(\frac{1}{2\sqrt{2i}}\right)\right]$$

**Corollary 4:** Jacobsthal, Jacobsthal Lucas, generalized Jacobsthal numbers can also be demonstrated by using Chebyshev polynomials as

$$j_{n+1} = (2)^{\frac{n}{2}}\, i^{3n}\, U_n\left(\frac{i}{2\sqrt{2}}\right)$$

$$c_n = (2)^{\frac{n+2}{2}}\, i^{3n}\, T_n\left(\frac{i}{2\sqrt{2}}\right)$$

$$J_n = i^n\left[\frac{a\,(2)^{\frac{n}{2}}}{3}\, T_n\left(\frac{-i}{2\sqrt{2}}\right) + \frac{(2b-a)\,(2)^{\frac{n-1}{2}}}{2i}\; U_{n-1}\left(\frac{i}{2\sqrt{2}}\right)\right]$$

**Proof.**

$$j_n = (2i^2)^{\frac{n-1}{2}}\; U_{n-1}\left(\frac{1}{2\sqrt{2i}}\right) = 2^{\frac{n-1}{2}} i^{n-1}\; U_{n-1}\left(\frac{-i}{2\sqrt{2}}\right)$$

$$= 2^{\frac{n-1}{2}} i^{n-1}(-1)^{n-1}\; U_{n-1}\left(\frac{i}{2\sqrt{2}}\right)$$

$$= 2^{\frac{n-1}{2}}\, i^{3n-3}\; U_{n-1}\left(\frac{i}{2\sqrt{2}}\right)$$

$$c_n = 2\,(2i^2)^{\frac{n}{2}}\;\; T_n\left(\frac{1}{2\sqrt{2i}}\right) = 2^{\frac{n+2}{2}}\, i^n\; T_n\left(\frac{-i}{2\sqrt{2}}\right)$$

$$= 2^{\frac{n+2}{2}}\, i^n\,(-1)^n\, T_n\left(\frac{i}{2\sqrt{2}}\right)$$

$$= 2^{\frac{n+2}{2}}\, i^{3n}\; T_n\left(\frac{i}{2\sqrt{2}}\right)$$

$$J_n = i^n\left[\frac{a\,(2)^{\frac{n}{2}}}{3}\, T_n\left(\frac{1}{2\sqrt{2i}}\right) + \frac{(2b-a)\,(2)^{\frac{n-1}{2}}}{2i}\; U_{n-1}\left(\frac{1}{2\sqrt{2i}}\right)\right]$$

$$= i^n \left[ \frac{a\ (2)^{\frac{n}{2}}(-1)^n}{3}\ T_n\left(\frac{i}{2\sqrt{2}}\right) + \frac{(2b-a\ )\ (2)^{\frac{n-1}{2}}(-1)^n}{2i}\ U_{n-1}\left(\frac{i}{2\sqrt{2}}\right) \right]$$

$$= i^n \left[ \frac{a\ (2)^{\frac{n}{2}}}{3}\ T_n\left(\frac{-i}{2\sqrt{2}}\right) + \frac{(2b-a\ )\ (2)^{\frac{n-1}{2}}}{2i}\ U_{n-1}\left(\frac{-i}{2\sqrt{2}}\right) \right]$$

**Theorem 5** Generalized Jacobsthal numbers are denoted by using the first kind Chebyshev polynomials as

$$J_n = \frac{(2i^2)^{\frac{n}{2}}\ 2\sqrt{E}}{3}\ T_n\left(\theta - \frac{\phi}{n}\right)$$

where $\cos\phi = \frac{X-Y}{2\sqrt{E}}$ .

**Proof.** It is easily seen that $\sqrt{(X-Y)^2 + (i(X-Y))^2} = 2\sqrt{E}$ . By using this equality and the third part of the proof of Proposition 3, it is obtained that

$$J_n = \frac{\left(2i^2\right)^{n/2}}{\alpha-\beta}\left[\left(X-Y\right)\cos n\theta + i(X+Y)\sin n\theta\right]$$

$$= \frac{\left(2i^2\right)^{n/2}}{3}\left[\frac{(X-Y)\cos n\theta}{\sqrt{\left(X-Y\right)^2+\left(i\left(X+Y\right)\right)^2}} + \frac{i(X+Y)\sin n\theta}{\sqrt{\left(X-Y\right)^2+\left(i\left(X+Y\right)\right)^2}}\right]\sqrt{\left(X-Y\right)^2+\left(i\left(X+Y\right)\right)^2}$$

$$= \frac{\left(2i^2\right)^{n/2}\ 2\sqrt{E}}{3}\left[\frac{(X-Y)\cos n\theta}{2\sqrt{E}} + \frac{i(X+Y)\sin n\theta}{2\sqrt{E}}\right]$$

$$= \frac{\left(2i^2\right)^{n/2}\ 2\sqrt{E}}{3}\left[\cos\phi\cos n\theta + \sin\phi\sin n\theta\right]$$

$$= \frac{\left(2i^2\right)^{n/2}\ 2\sqrt{E}}{3}\cos(n\theta-\phi)$$

$$= \frac{\left(2i^2\right)^{n/2}\ 2\sqrt{E}}{3}T_n(\theta-\frac{\phi}{n})$$

**Theorem 6:** The following relation between generalized Jacobsthal numbers and Jacobsthal numbers is satisfied

$$J_n J_{n+r+s} J_{n+r} J_{n+s} = E(2i^2)^n j_r j_s.$$

**Proof.** By using Theorem 5, it is written that

$$J_n \ \ J_{n+r+s} = \frac{2\sqrt{E}\ (2i^2)^{\frac{n}{2}}}{3}\cos(n\theta - \phi\ )\frac{2\sqrt{E}\ (2i^2)^{\frac{(n+r+s)}{2}}}{3}\cos((n+r+s)\theta - \phi\ )$$

$$= \frac{4E\ (2i^2)^{\frac{(2n+r+s)}{2}}}{9}\cos(n\theta - \phi\ )\cos((n+r+s)\theta - \phi\ )$$

$$J_{n+r}\ J_{n+s} = \frac{4E\ (2i^2)^{\frac{(2n+r+s)}{2}}}{9}\cos((n+r)\theta - \phi\ )\cos((n+s)\theta - \phi\ )$$

By substracting the equalities,

$$J_n \ \ J_{n+r+s} - \quad J_{n+r}\ J_{n+s} = \frac{4E\ (2i^2)^{\frac{(2n+r+s)}{2}}}{9}\left[\begin{array}{c}\dfrac{\cos\big((2n+r+s)\theta - 2\phi\ \big) + \cos(r+s)\theta}{2} \\ -\dfrac{\cos\big((2n+r+s)\theta - 2\phi\ \big) + \cos(r-s)\theta}{2}\end{array}\right]$$

$$= \frac{4E\ (2i^2)^{\frac{(2n+r+s)}{2}}}{9}\left[\frac{\cos(\ (r+s) - \cos(r-s)\theta)}{2}\right]$$

$$= -\frac{4E\ (2i^2)^{\frac{(2n+r+s)}{2}}}{9}\ \sin r\theta \sin s\theta$$

$$= -\sin^2\theta\ \frac{4E\ (2i^2)^{\frac{(2n+r+s)}{2}}}{9}\ \frac{\sin r\theta}{\sin\theta}\ \frac{\sin s\theta}{\sin\theta}$$

$$= \ (\cos^2\theta - 1)\ \frac{4E\ (2i^2)^{\frac{(2n+r+s)}{2}}}{9}\ \ U_{r-1}(\cos\theta)\ \ U_{s-1}(\cos\theta)$$

$$= \left(\left(\frac{1}{2\sqrt{2i}}\right)^2 - 1\right)\frac{4E\ (2i^2)^{\frac{(2n+r+s)}{2}}}{9}\ \ U_{r-1}\left(\frac{1}{2\sqrt{2i}}\right)\ U_{s-1}\left(\frac{1}{2\sqrt{2i}}\right)$$

$$= E\ (2i^2)^{\frac{(2n+r+s)}{2}}\ \ U_{r-1}\left(\frac{1}{2\sqrt{2i}}\right)\ \ U_{s-1}\left(\frac{1}{2\sqrt{2i}}\right)$$

For the other side of the equality, it is obtained that

$$E\,(2i^2)^n \quad J_r\ J_s = E\,(2i^2)^n \ \ (2i^2)^{\frac{r-1}{2}} \ \ U_{r-1}\left(\frac{1}{2\sqrt{2i}}\right)\ (2i^2)^{\frac{r-1}{2}}\, U_{s-1}\left(\frac{1}{2\sqrt{2i}}\right)$$

The equality of the results is proved the theorem.

The applications of the theorem for the Jacobsthal sequence is

$$j_n \ \ j_{n+r+s} - \ \ j_{n+r}\ \ j_{n+s} = -\,(-2)^{\frac{(2n+r+s-2)}{2}}\ \ U_{r-1}\left(\frac{1}{2\sqrt{2i}}\right)\ \ U_{s-1}\left(\frac{1}{2\sqrt{2i}}\right)$$

$$= -\,(-2)^n\ \ j_r j_s$$

The applications of the theorem for the Jacobsthal Lucas sequence

$$c_n \ \ c_{n+r+s} - \ \ c_{n+r}\ \ c_{n+s} = 9\,(-2)^{\frac{(2n+r+s-2)}{2}}\ \ U_{r-1}\left(\frac{1}{2\sqrt{2i}}\right)\ \ U_{s-1}\left(\frac{1}{2\sqrt{2i}}\right)$$

$$= 9\,(-2)^n\ \ j_r j_s$$

**Lemma 7:** It is well-known that if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$A^2 = \begin{cases} \dfrac{r_1{}^n - r_2{}^n}{r_1 - r_2} A - \dfrac{r_1{}^{n-1} - r_2{}^{n-1}}{r_1 - r_2} I_2 & r_1 \neq r_2 \\ n\ r_1{}^{n-1} A - (n-1)\det(A)\ \ r_1{}^{n-2} I_2 & r_1 = r_2 \end{cases}$$

$r_1,\ r_2$being the roots of the associated characteristic equation of the matrix A

$$r^2 - (a+d)r + \det(A) = 0$$

**Corollary 8:** If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a 2x2 square matrix is chosen whose trace is *a + d = 1* and determinant is *det(A) = -2*, then

$$A^2 = j_n A - j_{n-1} I$$

**Proof**. We know that the quadratic characteristic equation for the Jacobsthal sequence is $r^2 - r - 2 = 0$ with roots $r_1$ = 2 and $r_2$ = -1. If a 2x2 square matrix is chosen whose trace is *a+d = 1* and determinant is *det(A) = -2*, then we will get

$$
\begin{aligned}
A^2 &= \frac{{r_1}^n - {r_2}^n}{r_1 - r_2} A - \frac{{r_1}^{n-1} - {r_2}^{n-1}}{r_1 - r_2} I_2 \\
&= j_n A - j_{n-1} I_2
\end{aligned}
$$

**Theorem 9** If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a 2x2 square matrix whose trace $a+d = 1$ and determinant $det(A) = -2$, then another relation with Jacobsthal sequence and Chebyshev polynomials is established by using the matrix of $A$ as

$$
\begin{aligned}
j_n &= \left(\sqrt{2i^2}\right)^{n-1} \left[ U_{n-1}\left(\frac{1}{2\sqrt{2i^2}}\right) A - \frac{1}{\sqrt{2i^2}} \; U_{n-2}\left(\frac{1}{2\sqrt{2i^2}}\right) I_2 \right] \\
&= \left(\sqrt{2i^2}\right)^{n-1} \left[ U_{n-1}\left(\frac{1}{2\sqrt{2i^2}}\right)\left(A - \frac{1}{4i^2} I_2\right) + \frac{1}{\sqrt{2i^2}} \; T_n\left(\frac{1}{2\sqrt{2i^2}}\right) I_2 \right]
\end{aligned}
$$

**Proof.** We know that

$$
j_n = W_n(0,1,1,-2) = \frac{\alpha^n - \beta^n}{\alpha - \beta} = (2i^2)^{\frac{n-1}{2}} \; U_{n-1}\left(\frac{1}{2\sqrt{2i^2}}\right)
$$

By Corollary 8 , we get

$$
\begin{aligned}
A^2 &= j_n A - j_{n-1} I_2 \\
&= \left(\sqrt{2i^2}\right)^{n-1} \left[ U_{n-1}\left(\frac{1}{2\sqrt{2i^2}}\right) A - \frac{1}{\sqrt{2i^2}} \; U_{n-2}\left(\frac{1}{2\sqrt{2i^2}}\right) I_2 \right]
\end{aligned}
$$

By using the property between Chebyshev polynomials $U_{n-2} = x \;\; U_{n-1} - \;\; T_n$. it is obtained that

$$
A^n = \left(\sqrt{2i^2}\right)^{n-1} \left[ \begin{array}{c} U_{n-1}\left(\frac{1}{2\sqrt{2i^2}}\right) A - \\ \frac{1}{\sqrt{2i^2}} \left(\frac{1}{2\sqrt{2i^2}} \; U_{n-1}\left(\frac{1}{2\sqrt{2i^2}}\right) I_2 - \; T_n\left(\frac{1}{2\sqrt{2i^2}}\right) I_2\right) \end{array} \right]
$$

$$
= \left(\sqrt{2i^2}\right)^{n-1} \left[ U_{n-1}\left(\frac{1}{2\sqrt{2i^2}}\right)\left(A + \frac{1}{4} I_2\right) + \frac{1}{\sqrt{2i^2}} \; T_n\left(\frac{1}{2\sqrt{2i^2}}\right) I_2 \right]
$$

$$
= \left[ U_{n-1}\left(\frac{1}{2\sqrt{2i}}\right)\left(A + \frac{1}{4} I_2\right) + \frac{1}{\sqrt{2i}} \; T_n\left(\frac{1}{2\sqrt{2i^2}}\right) I_2 \right]
$$

**Example 10:** Let $A = \begin{bmatrix} 1/2 & 1/3 \\ 3/2 & 1/2 \end{bmatrix}$, then

$$\begin{bmatrix} 1/2 & 1/3 \\ 3/2 & 1/2 \end{bmatrix}^n = j_n \begin{bmatrix} 1/2 & 1/3 \\ 3/2 & 1/2 \end{bmatrix} - j_{n-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} j_n/2 - j_{n-1} & 3j_n/2 \\ 3j_n/2 & j_n/2 - j_{n-1} \end{bmatrix}$$

By the equality of the determinant of matrices, we get

$$(-2)^n = -2\, j_n{}^2 + j_{n-1}{}^2 - j_n j_{n-1}$$

**Example 11:** Let $A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$, then $\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}^n = j_n \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} - j_{n-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -j_{n-1} & j_n \\ 2j_n & j_{n-2} \end{bmatrix}$

By the equality of the determinant of matrices, we get

$$(-2)^{n-1} = j_{n-2}\, j_{n-1} + j_n^2 .$$

**Example 12:** Let $A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$, then $\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}^n = j_n \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} - j_{n-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2j_{n-2} & j_n \\ 2j_n & -j_{n-1} \end{bmatrix}$

By the equality of the determinant of matrices, we get the same result with the previous example.

**Theorem 13** By using the properties of Chebyshev polynomials in [16], we get some properties of Jacobsthal and Jacobsthal Lucas sequences as

*a)* $c_{m+n} + (-2)^n\, c_{m-n} = c_m\, c_n$

*b)* $j_{n+1}\;\; j_{n+2r+1} + (-2)^{n+\frac{r-1}{2}}\, j_r{}^2 = (-2)^{r+\frac{n-1}{2}}\, j_{n+r+1}{}^2$

*c)* $c_n\, c_{n+2r} = (-2)^r \left[\, c_{n+r}{}^2 - 9\; j_r{}^2 \right]$

*d)* $\dfrac{j_{nk}}{j_k(-2)^{k(n-1)/2}} = \dfrac{\sin n\left(\cos^{-1}\frac{c_n}{(-2)^{n/2}}\right)}{\sin\left(\cos^{-1}\frac{c_n}{(-2)^{n/2}}\right)}$

*e)* $c_n{}^2 = 2\,(-2)^n + c_{2n}$

*f)* $c_n{}^2 - c_{n+1}\, c_{n-1} = 9\,(-2)^{n-1}$

*g)* $c_n{}^2 - 9\; j_n{}^2 = (-2)^{n+2}$

**Proof.**

a) Let $x = \frac{1}{2\sqrt{2i}}$. By using this property $c_n = 2\,(2i^2)^{n/2}\; T_n\left(\frac{1}{2\sqrt{2i}}\right)$, $\quad j_n = 2\,(2i^2)^{n-1/2}\; U_{n-1}\left(\frac{1}{2\sqrt{2i}}\right)$ it is obtained that

$$T_{m+n}\ T_{m-n} = \ 2\,T_m\,T_n$$

$$\frac{c_{m+n}}{2\,(2i^2)^{m+n/2}} + \frac{c_{m-n}}{2\,(2i^2)^{m-n/2}} = 2\,\frac{c_m}{2\,(2i^2)^{m/2}}\frac{c_n}{2\,(2i^2)^{n/2}}$$

$$\frac{(2i^2)^{(m-n)/2}\,c_{m+n} + (2i^2)^{(m+n)/2}\ \ c_{m-n}}{2\,(2i^2)^m} = \frac{c_m\,c_n}{2\,(2i^2)^{(m+n)/2}}$$

$$c_{m+n} + (-2)^n\ \ c_{m-n} = c_m\,c_n$$

b) Similarly

$$U_n\,U_{n+2r}\ +\ {U_{r-1}}^2\ =\ \ {U_{n+r}}^2$$

$$\frac{j_{n+1}}{(-2)^{(n-1)/2}} + \frac{j_{n+2r+1}}{(-2)^{(n+2r-1)/2}} + \frac{{j_r}^2}{(-2)^{(n-1)/2}} = \frac{j^2_{n+r+1}}{(-2)^{(n+r)/2}}$$

$$j_{n+1}j_{n+2r+1}+(-2)^{n+\frac{r-1}{2}}\,j_r^2=(-2)^{\frac{n+r}{2}-1}\,j^2_{n+r+1}\,.$$

c)

$$T_n\ \ T_{n+2r} - (x^2-1)\,{U_{r-1}}^2 = {T_{n+r}}^2$$

$$T_n\left(\frac{1}{2\sqrt{2i}}\right)\ \ T_{n+2r}\left(\frac{1}{2\sqrt{2i}}\right) - \left(\frac{1}{-8}-1\right)\ \ {U_{r-1}}^2\left(\frac{1}{2\sqrt{2i}}\right) = \ {T_{n+r}}^2$$

$$\frac{c_n\ \ c_{n+2r}}{2\,(2i^2)^{n/2}\,2\,(2i^2)^{\frac{n}{2}+r}} + \frac{9\,{j_r}^2}{8(2i^2)^{(n-1)}} = \frac{{c_{n+r}}^2}{2(2i^2)^n}$$

$$\frac{c_n\ \ c_{n+2r}}{4\,(2i^2)^{n+r}} + \frac{9\,{j_r}^2}{8(2i^2)^{(n-1)}} = \frac{c^2_{n+r}}{2(2i^2)^n}$$

d) By using this property $U_{r-1}(\,T_k(x)) = \frac{U_{nk-1}\,(x)}{U_{k-1}\,(x)}$ and the equality of the results we prove the statement.

$$U_{n-1}(T_k(x)) = U_{n-1}\left(\frac{c_n}{2\,(-2)^{\frac{n}{2}}}\right) = \frac{\sin n\left(\cos^{-1}\frac{c_n}{2(-2)^{\frac{n}{2}}}\right)}{\sin\left(\cos^{-1}\frac{c_n}{2(-2)^{\frac{n}{2}}}\right)}$$

and

$$\frac{U_{nk-1}\,(x)}{U_{k-1}\,(x)} = \left(\frac{j_{nk}}{(-2)^{\frac{nk-1}{2}}}\right) / \left(\frac{j_k}{(-2)^{(k-1)/2}}\right)$$

$$= \frac{j_{nk}}{(-2)^{k(n-1)/2}\;\; j_k}$$

e)

$$2\,{T_n}^2 = 1 + T_{2n}$$

$$2\left(\frac{c_n}{2\,(-2)^{\frac{n}{2}}}\right)^2 = 1 + \frac{c_n}{2\,(-2)^n}$$

$${c_n}^2 = 2\,(-2)^n + \;\; c_{2n}$$

f)

$${T_n}^2 - \;\; T_{n+1}\;\; T_{n-1} = 1 - x^2$$

$$\left(\frac{c_n}{2\,(-2)^{\frac{n}{2}}}\right)^2 - \frac{c_{n+1}}{2\,(-2)^{\frac{n+1}{2}}}\,\frac{c_{n-1}}{2\,(-2)^{\frac{n-1}{2}}} = 1 - \left(\frac{-1}{8}\right)$$

$${c_n}^2 - \;\; c_{n+1}\;\; c_{n-1} = 9\,(-2)^{n-1}$$

g)

$${T_n}^2 - (x^2\; - 1)U_{n-1}^2 = 1$$

$$\left(\frac{c_n}{2\,(-2)^{\frac{n}{2}}}\right)^2 - \left(\frac{1}{-8} - 1\right)\frac{{j_n}^2}{(2i^2)^{n-1}} = 1$$

## 4. Conclusion

In this paper we find some properties of Jacobsthal and Jacobsthal Lucas sequences by using the properties of Chebyshev polynomials. And some relationships between Jacobsthal and Jacobsthal Lucas sequences and Chebyshev polynomials are obtained.

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# On the Generating Tridiagonal Matrices with Generalized Jacobsthal and Jacobsthal Lucas Polynomial Sequences

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## Abstract

In this study, we define some tridigional matrices with elements depending on a polynomial. By using the determinant of these matrices, the elements of *p(x)*-Jacobsthal and *p(x)*-Jacobsthal Lucas polynomial sequences with even or odd indices are generated. Then we construct the inverse matrices of these tridigional matrices. So, some properties of *p(x)*-Jacobsthal and *p(x)*-Jacobsthal Lucas polynomial sequences are proved by a new way. We also investigate eigenvalues of these matrices.
**Keywords:** *p(x)*-Jacobsthal sequence, *p(x)*-Jacobsthal Lucas sequence, Eigenvalues.

## 1. Introduction

In this study, we define some tridigional matrices with elements depending on a polynomial. By using the determinant of these matrices, the elements of *p(x)*-Jacobsthal and *p(x)*-Jacobsthal Lucas polynomial sequences with even or odd indices are generated. Then we construct the inverse matrices of these tridigional matrices. So, some properties of *p(x)*-Jacobsthal and *p(x)*-Jacobsthal Lucas polynomial sequences are proved by a new way. We also investigate eigenvalues of these matrices.

Special integer sequences are encountered in different branches of science, art, nature, the structure of our body. So it is a popular topic in applied mathematics. One of the special integer sequences are the Jacobsthal sequence. By changing the initial conditions but preserving the recurrence relation the Jacobsthal Lucas sequence is obtained. The recurrence relations for Jacobsthal and Jacobsthal Lucas sequences are $j_n = j_{n-1} + 2j_{n-2}$ ; $j_0 = 0,\ j_1 = 1$ and $c_n = c_{n-1} + 2c_{n-2}$ ; $c_0 = 2,\ c_1 = 1$ for $n \geq 2$; respectively in [1]. There are some generalizations of these integer sequences. For example, a generalization for Jacosthal and Jacobsthal Lucas sequences is given by

$$j_n(s,t) = j_{n-1}(s,t) + 2j_{n-2}(s,t),\ j_0(s,t) = 0,\ j_1(s,t) = 1$$
$$c_n(s,t) = c_{n-1}(s,t) + 2c_{n-2}(s,t),\ c_0(s,t) = 2,\ c_1(s,t) = s$$

for $n \geq 2$ in [2]. In this paper, a new generalization of Jacobsthal and Jacobsthal Lucas sequences by using polynomials as called *p(x)*-Jacobsthal polynomial $J_{p.n}(x)$ and *p(x)*-Jacobsthal Lucas polynomial $C_{p.n}(x)$ are used [3]. There is a long tradition of using matrices and determinants to study special integer sequences. For example Cahill et. al. [6] found some types of the tridiagonal matrices whose determinants are equal to Fibonacci numbers. There are many known connections between determinants of tridiagonal matrices and the Fibonacci and Lucas numbers. The authors constructed the symmetric tridiagonal family of matrices whose determinants form any linear subsequence of the Fibonacci and numbers Lucas numbers in [7]. Feng, [8] found some Fibonacci identities via determinant of tridiagonal matrix. Seibert et. al [9] gave new results about factorization of Fibonacci and Lucas numbers. J. J´ına, P. Trojovsk´ used the determinants of some tridiagonal matrices with Fibonacci numbers in [10]. Falcon, in [11] used determinants of some tridiagonal generating matrices to obtain results about *k*-Fibonacci numbers. In [12], the authors studied on certain matrices whose entries are Pell, Pell-Lucas, *k*-Pell and k-Pell-Lucas quaternions. In [13], Catarino constructed certain matrices with *h(x)*-Fibonacci polynomials. In [14], the authors studied on generalized Jacobsthal and Jacobsthal-Lucas polynomials by using certain matrices.

## 2. Preliminaries

**Definition 1** Assume that *p(x)* is a polynomial with real coefficeients and $n \geq 2$ any integer. The *p(x)*-Jacobsthal polynomial $\{J_{p.n}(x)\}_{n\in\mathbb{N}}$ sequences are described by using the following recurrence relation

$$j_{p,n}(x) = j_{p,n-1}(x) + 2j_{p,n-2}(x), \tag{1}$$

with initial conditions are $j_{p,0}(x)=0,\ j_{p,1}(x)=1$;and the *p(x)*-Jacobsthal Lucas polynomial $\{C_{p.n}(x)\}_{n\in\mathbb{N}}$ sequences are described by using the following recurrence relation

$$c_{p,n}(x) = c_{p,n-1}(x) + 2c_{p,n-2}(x), \tag{2}$$

with initial conditions are $c_{p,0}(x)=2,\ c_{p,1}(x)=p(x)$ Some of the first *p(x)*-Jacobsthal polynomial sequences and *p(x)*-Jacobsthal Lucas polynomial sequences are given in the following tables

| $n$ | $p(x)-Jacobsthal\ polynomials$ |
|---|---|
| $1$ | $1$ |
| $2$ | $p(x)$ |
| $3$ | $p^2(x)+2$ |
| $4$ | $p^3(x)+4p(x)$ |
| $5$ | $p^4(x)+6p^2(x)+4$ |
| $6$ | $p^5(x)+8p^3(x)+12p(x)$ |
| $7$ | $p^6(x)+10p^4(x)+24p^2(x)+8$ |
| $8$ | $p^7(x)+12p^5(x)+40p^3(x)+32p(x)$ |

| $n$ | $p(x)-Jacobsthal\ Lucas\ polynomials$ |
|---|---|
| $1$ | $p(x)$ |
| $2$ | $p^2(x)+4$ |
| $3$ | $p^3(x)+6p(x)$ |
| $4$ | $p^4(x)+8p^2(x)+8$ |
| $5$ | $p^5(x)+10p^3(x)+20p(x)$ |
| $6$ | $p^6(x)+12p^4(x)+36p^2(x)+16$ |
| $7$ | $p^7(x)+14p^5(x)+56p^3(x)+56p(x)$ |
| $8$ | $p^8(x)+16p^6(x)+80p^4(x)+148p^2(x)+32$ |

Special integer sequences are obtained with special numerical choices for *p(x)*-Jacobsthal polynomial and *p(x)*-Jacobsthal Lucas polynomial sequences. For example, if $p(x) = 1$; then we get classic Jacobsthal and Jacobsthal Lucas sequences. If $p(x) = k$; then we get classic *k*-Jacobsthal and *k*-Jacobsthal Lucas sequences.

Let us consider a tridigional matrix as

$$A_n = \begin{bmatrix} a & b & & & & & \\ c & d & e & & & & \\ & c & d & e & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & & & & \\ & & & & c & d & e \\ & & & & & c & d \end{bmatrix}$$

Then

$$\det A_1 = a$$
$$\det A_2 = d \det A_1 - bc$$
$$\det A_3 = d \det A_2 - ce \det A_1.$$

By continuing this prodecure, it is computed that

$$\det A_n = d \det A_{n-1} - ce \det A_{n-2} \qquad (3)$$

The inverse of a matrix $A$ can be obtained by the formula $A^{-1} = \frac{(cof(A))^T}{\det A}$ where $(cof(A))^T$ is the transpose of the cofactor matrix $A$ or adjugate matrix of $A$ [4]. Let $T$ a nonsingular tridigional matrix as

$$T = \begin{bmatrix} a_1 & b_1 & & & & & \\ c_1 & a_2 & b_2 & & & & \\ & c_2 & a_3 & b_3 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & & & & \\ & & & & c_{n-2} & a_{n-1} & b_{n-1} \\ & & & & & c_{n-1} & a_n \end{bmatrix}$$

Usmani [5] gave a formula for the inverse of this matrix $T^{-1} = (t_{i,j})$ as

$$t_{i,j}=\begin{cases}(-1)^{i+j}\dfrac{1}{\theta_n}b_i\dots b_{j-1}\theta_{i-1}\phi_{j+1} & if \quad i\leq j\\ (-1)^{i+j}\dfrac{1}{\theta_n}c_j\dots b_{i-1}\theta_{j-1}\phi_{i+1} & if \quad i\succ j\end{cases} \tag{4}$$

where $\theta_i$ verify the recurrence relation $\theta_i = a_i\theta_{i-1} - b_{i-1}c_{i-1}\theta_{i-2}$ for $i=2,\dots,n$ with the initial conditions $\theta_0 = 1,\ \theta_1 = a_1$. Observe that $\theta_n = det(T)$

$\phi_i$ verify the recurrence relation $\phi_i = a_i\phi_{i+1} - b_i c_i\phi_{i+2}$ for $i=n-1,\dots,1$ with the initial conditions $\phi_{n+1} = 1,\ \phi_n = a_n$.

If the tridigional matrix is given in the following form

$$A_n=\begin{bmatrix} a & b & & & & & \\ c & a & b & & & & \\ & c & a & b & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & & & & \\ & & & & c & a & b \\ & & & & & c & a \end{bmatrix}$$

then the eigenvalues of this matrix are

$$\lambda_r = a + 2\sqrt{bc}\,cos(\frac{r\pi}{n+1}), \quad r = 1,2,\dots,n. \tag{5}$$

## 3. Main Results

### 3.1 Some properties of *p(x)*-Jacobsthal polynomials by tridiagonal matrices *Aj,n(p)*

**Theorem 2** Assume that $A_{j,n}(p)$ is a nxn tridiagonal matrix defined as

$$A_{j,n}(p)=\begin{bmatrix} p(x) & 2 & & & & \\ -1 & p(x) & 2 & & & \\ & -1 & \ddots & \ddots & & \\ & & \ddots & & & \\ & & & & & 2 \\ & & & & -1 & p(x) \end{bmatrix} \tag{6}$$

Then the determinant of $A_{j,n}(p)$

$$det(A_{j,n}(p)) = J_{p,n+1}(x). \tag{7}$$

**Proof.** The proof is made by mathematical induction method applied on $n$. For $n = 1$, we have $det(A_{j,1}(p)) = J_{p,2}(x) = p(x)$. Assume that $det(A_{j,n-1}(p)) = J_{p,n}(x)$, and $det(A_{j,n}(p)) = J_{p,n+1}(x)$ for $n > 2$. Then

$$\begin{aligned} det(A_{j,n+1}(p)) &= p(x)det(A_{j,n}(p)) - 2(-1)det(A_{j,n-1}(p)) \\ &= p(x)J_{p,n+1}(x) + 2J_{p,n}(x) = J_{p,n+2}(x). \end{aligned}$$

$p(x)$-Jacobsthal polynomials are also obtained by using the following tridiagonal matrix with complex entries. Assume that $A_{j,n}(p)$ is a $nxn$ matrix defined as

$$A_{j,n}(p) = \begin{bmatrix} p(x) & 2i & & & & \\ i & p(x) & 2i & & & \\ & i & \ddots & \ddots & & \\ & & \ddots & & & \\ & & & & & 2i \\ & & & & i & p(x) \end{bmatrix}$$

Then it is easily seen that the determinant of $A_{j,n}(p)$ is also $(n+1)$th $p(x)$-Jacobsthal polynomial.

$$det(A_{j,n}(p)) = J_{p,n+1}(x).$$

For the inverse of $A_{j,n}(p)$; by using (4), it is obtained that

$$a_i = p(x), \ b_i = 2, \ c_i = -1,$$

$$\theta_i = J_{p,i+1}, \quad \phi_j = \frac{1}{p(x)} J_{p,(n-j+2)}$$

Therefore the inverse of $A_{j,n}(p)$

$$(A_{j,n}^{-1}(p))_{(i,j)} = \begin{cases} (-1)^{i+j} 2^{j-i} J_{p,i}(x) J_{p,n-j+1}(x) \dfrac{1}{J_{p,n+1}(x)}, & if \quad i \leq j \\ J_{p,j}(x) J_{p,n-i+1}(x) \dfrac{1}{J_{p,n+1}(x)}, & if \quad i \succ j \end{cases}$$

The elements of the cofactor matrix are given as

$$cof(A_{j,n}(p))_{(i,j)} = \begin{cases} J_{p,i}(x) J_{p,n-j+1}(x), & if \quad i \prec j \\ (-1)^{i+j} 2^{i-j} J_{p,j}(x) J_{p,n-i+1}(x), & if \quad i \geq j \end{cases}$$

It is well-known that $\left|cof(A_{j,n}(p))\right| = \left|adj(A_{j,n}(p))\right| = \left|(A_{j,n}(p))\right|^{n-1}$. So $\left|cof(A_{j,n}(p))\right| = J_{p,n+1}^{n-1}(x)$. By using cofactor matrix, we get some properties of $p(x)$-Jacobsthal polynomials.

For $n = 2$; we get

$$\left|cof(A_{j,2}(p))\right| = \begin{vmatrix} J_{p,1}(x)J_{p,2}(x) & J_{p,1}(x)J_{p,1}(x) \\ -2J_{p,1}(x)J_{p,1}(x) & J_{p,2}(x)J_{p,1}(x) \end{vmatrix}$$

$$2J_{p,1}^2(x) + J_{p,2}^2(x) = J_{p,3}(x)$$

For $n = 3$; we get

$$\left|cof(A_{j,3}(p))\right| = \begin{vmatrix} J_{p,1}(x)J_{p,3}(x) & J_{p,1}(x)J_{p,2}(x) & J_{p,1}(x)J_{p,1}(x) \\ -2J_{p,1}(x)J_{p,2}(x) & J_{p,2}(x)J_{p,2}(x) & J_{p,1}(x)J_{p,2}(x) \\ 4J_{p,1}(x)J_{p,1}(x) & -2J_{p,1}(x)J_{p,2}(x) & J_{p,1}(x)J_{p,3}(x) \end{vmatrix}$$

$$J_{p,2}^2(x)(J_{p,3}(x) + 2J_{p,1}(x))^2 = J_{p,4}^2(x)$$

$$\left(\frac{J_{p,3}(x) - 2J_{p,1}(x)}{p(x)}\right)^2 (J_{p,3}(x) + 2J_{p,1}(x))^2 = J_{p,4}^2(x)$$

$$J_{p,3}^2(x) - 4J_{p,1}^2(x) = p(x)J_{p,4}(x)$$

For $n = 4$; we get

$$\left|cof(A_{j,4}(p))\right| = \begin{vmatrix} J_{p,4}(x) & J_{p,3}(x) & J_{p,2}(x) & 1 \\ -2J_{p,3}(x) & J_{p,2}(x)J_{p,3}(x) & J_{p,2}^2(x) & J_{p,2}(x) \\ 4J_{p,2}(x) & -2J_{p,2}^2(x) & J_{p,2}(x)J_{p,3}(x) & J_{p,3}(x) \\ -8 & 4J_{p,2}(x) & -2J_{p,3}(x) & J_{p,4}(x) \end{vmatrix}$$

$$\begin{aligned} &2J_{p,2}^4(x)J_{p,4}^2(x) + 8J_{p,2}^3(x)J_{p,3}(x)J_{p,4}(x) + J_{p,2}^2(x)J_{p,3}^2(x)J_{p,4}^2(x) + 8J_{p,2}^2(x)J_{p,3}^2(x) \\ &+ 4J_{p,2}(x)J_{p,3}^3(x)J_{p,4}(x) + 4J_{p,3}^4(x) = J_{p,5}^3(x) \end{aligned}$$

Generalizing the results, it is obtained that $J_{p,n+1}^2(x) + 2J_{p,n}^2(x) = J_{p,2n+1}(x)$ for even integer $n$ and $J_{p,n+1}^2(x) - 4J_{p,n-1}^2(x) = p(x)J_{p,2n}(x)$ for odd integer $n$ .

Eigenvalues of the matrices $A_{j,n}(p)$ construct the spectra of the $A_{j,n}(p)$. By using the property (5), the sequence of the spectra of $A_{j,n}(p)$ for $n = 1,2,3,4,5$ is

$$n=1 \Rightarrow \lambda_r = \{p(x)\}$$
$$n=2 \Rightarrow \lambda_r = \{p(x)+\sqrt{2}i,\ p(x)-\sqrt{2}i\}$$
$$n=3 \Rightarrow \lambda_r = \{p(x)+2i,\ p(x),\ p(x)-2i\}$$
$$n=5 \Rightarrow \lambda_r = \{p(x)+\sqrt{6}i,\ p(x)+\sqrt{2}i,\ p(x),\ p(x)-\sqrt{6}i,\ p(x)-\sqrt{2}i\}$$

The sequence of the spectra of the matrices $A_{j,n}(p)=A_{j,n}(1)$ for $n = 2,3,4,5,6$ is computed by using Matlab Programme as

$$s_2 = \{1 + 1.4142135623730951i;\ 1 - 1.4142135623730951i\}$$
$$s_3 = \{1 + 2i;\ 1;\ 1 - 2i\}$$
$$s_4 = \begin{Bmatrix} 1 + 2.288245611270738i;\ 1 + 2.288245611270738i \\ ;\ 1 + 0.874032048897642i;\ 1 - 0.874032048897642i \end{Bmatrix}$$
$$s_5 = \begin{Bmatrix} 1 + 2.449489742783177i;\ 1- 2.449489742783177i;\ 1 \\ ;\ 1 + 1.4142135623730951i;\ 1 + 1.4142135623730951i \end{Bmatrix}$$
$$s_6 = \begin{Bmatrix} 1 + 2.548324784527070i;\ 1 - 2.548324784527070i;\ 1 + 0.6293842454258 96i \\ ;\ 1 - 0.629384245425896i;\ 1 + 1.763495467579869i;\ 1 - 1.763495467579869i \end{Bmatrix}$$

Evidently, the product of eigenvalues is the determinant of the matrix and the sum of eigenvalues is the trace of the matrix. Therefore

$$\sum_{i=1}^{n} \lambda_i = tr(A_{j,n}(p)) = np(x), \quad \prod \lambda_i = det(A_{j,n}(p)) = J_{p,n+1}(x)$$

and by using (5) this equality is satisfied.

$$J_{p,n+1}(x) = \prod_{j=1}^{n} (p(x) + 2\sqrt{2}i\, cos(\frac{\pi j}{n+1}))$$

**3.2 Some properties of even p(x)-Jacobsthal polynomials by tridiagonal matrices $E_{j,n}(p)$**

Assume that $E_{j,n}(p)$ is a *nxn* tridiagonal matrix defined as

$$E_{j,n}(p) = \begin{bmatrix} p(x) & 0 & & & & \\ 2 & p^2(x)+4 & 2 & & & \\ & 2 & p^2(x)+4 & \ddots & & \\ & & \ddots & \ddots & & \\ & & & & & 2 \\ & & & & 2 & p^2(x)+4 \end{bmatrix}$$

Then the determinant of $E_{j,n}(p)$ is computed by (3) as

$$det(E_{j,n}(p)) = J_{p,2n}(x).$$

For the inverse of $E_{j,n}(p)$ the values are computed as

$$\begin{aligned}
&a_1 = p(x), \ \ a_i = p^2(x)+4, \quad i \geq 2 \\
&b_1 = 0, \ \ c_1 = 2, \\
&b_i = c_i = 2, \quad i \succ 1 \\
&\theta_i = J_{p,2i}, \quad \phi_j = \frac{1}{p(x)} J_{p,2(n-j+2)}, \quad j \geq 1
\end{aligned}$$

Therefore the inverse of $E_{j,n}(p)$

$$(E^{-1}_{j,n}(p))_{(i,j)} = \begin{cases} 0, & if \quad i = 1 \\ (-1)^{i+j} 2^{j-i} J_{p,2i-2}(x) J_{p,2(n-j+1)}(x) \dfrac{1}{p(x) J_{p,2n}(x)}, & if \quad i \leq j \\ 2^{i-j} J_{p,2j-2}(x) J_{p,2(n-i+1)}(x) \dfrac{1}{p(x) J_{p,2n}(x)}, & if \quad i \succ j \end{cases}$$

If all entries of the matrix are real and nonnegative, then the matrix is called positive. All eigenvalues are real if the matrix positive and tridiagonal [4]. Therefore all eigenvalues of $E_{j,n}(p)$ are real if $p(x) \geq 0$. If we choose $p(x) = 1$; then the sequence of the spectra of the matrix $E_{j,n}(p)$ for $n = 2, 3, 4, 5, 6$ is given in the following result with the help of Matlab programme

$$\begin{aligned}
&s_2 = \{1 ;5 \quad \} \\
&s_3 = \{1; 3; \ 7\} \\
&s_4 = \{1; 2.171572875253810; 5 ; 7.828427124746191\} \\
&s_5 = \left\{ \begin{matrix} 1; 1.763932022500209; 3.763932022500209; 6.236067977499790 \\ ; 8.236067977499776 \end{matrix} \right\} \\
&s_6 = \{1; 1{:}535898384862246; 2{:}999999999999994; 5; 7; 8{:}464101615137750\}
\end{aligned}$$

Evidently

$$\sum_{i=1}^{n} \lambda_i = tr(E_{j,n}(p)) = (n-1)(p^2(x)+4) + p(x), \quad \prod \lambda_i = det(E_{j,n}(p)) = J_{p,2n}(x).$$

If we take care of the spectra, one of the eigenvalues is $p(x) = 1$ for all positive integer $n$. And the minimum eigenvalue of spectra converges to $p(x) = 1$; the maximum eigenvalue of spectra converges to $p^2(x)+8$.

**3.3 Some properties of odd *p(x)*-Jacobsthal polynomials by tridiagonal matrices** $O_{j,n}(p)$

Assume that $O_{j,n}(p)$ is a *nxn* tridiagonal matrix defined as

$$O_{j,n}(p)=\begin{bmatrix} p^2(x)+2 & 2 & & & & \\ 2 & p^2(x)+4 & 2 & & & \\ & 2 & p^2(x)+4 & \ddots & & \\ & & \ddots & \ddots & & \\ & & & & & 2 \\ & & & & 2 & p^2(x)+4 \end{bmatrix}.$$

Then the determinant of $O_{j,n}(p)$ is given by (3) as

$$det(O_{j,n}(p)) = J_{p,2n+1}(x).$$

For the inverse of $O_{j,n}(p)$ the values are computed as

$$\begin{aligned} &a_1 = p^2(x)+2, \ \ a_i = p^2(x)+4, \quad i \geq 2 \\ &b_i = c_i = 2, \quad i \geq 1 \\ &\theta_i = J_{p,2i+1}, \quad i \geq 1 \\ &\phi_j = \frac{1}{p(x)} J_{p,2(n-j+2)}, \quad j \geq 1 \end{aligned}$$

Therefore the inverse of $O_{j,n}(p)$

$$(O^{-1}_{j,n}(p))_{(i,j)} = \begin{cases} (-1)^{i+j} 2^{j-i} J_{p,2i-1}(x) J_{p,2(n-j+1)}(x) \dfrac{1}{p(x) J_{p,2n+1}(x)}, & if \quad i \leq j \\ (-1)^{i+j} 2^{i-j} J_{p,2j-1}(x) J_{p,2(n-i+1)}(x) \dfrac{1}{p(x) J_{p,2n+1}(x)}, & if \quad i \succ j \end{cases}$$

Matrices $O_{j,n}(p)$ are symmetric so the eigenvalues are real. The sequence of the spectra of the matrices $O_{j,n}(p)$ for $n = 2, 3, 4, 5, 6$ is given in the following result with the help of Matlab programme

$$s_2=\{1.76393202250021;\ 6.236067977499790\}$$
$$s_3=\{1.396124528390323;\ 4.109916264174743;\ 7.493959207434934g\}$$
$$s_4=\{1.241229516856366;\ 3;\ 5.694592710667723;\ 8.064177772475910g\}$$
$$s_5=\left\{\begin{array}{l}1.162028105542011;\ 2.380557064218860;\ 4.430740646906859\\ ;\ 6.661660052007548;\ 8.365014131324726\end{array}\right\}$$
$$s_6=\left\{\begin{array}{l}1.116232730295792;\ 2.005957007315595;\ 3.581580451829856\\ ;\ 5.482146721021292;\ 7.272258986924621;\ 8.541824102612837\end{array}\right\}$$

Evidently

$$\sum_{i=1}^{n}\lambda_i=tr(O_{j,n}(p))=(n-1)(p^2(x)+4)+(p^2(x)+2),\quad \prod\lambda_i=det(O_{j,n}(p))=J_{p,2n+1}(x).$$

If we take care of the spectra, minimum eigenvalue converges to $p(x)=1$. The maximum eigenvalue of spectra converges to $p^2(x)+8$.

**Theorem 3** If $\lambda_i$ is an eigenvalue of the matrix $O_{j,n}(p)$, then $\lambda_i+2p(x)+1$ is an eigenvalue of $O_{j,n}(p+1)$.

**Proof.** $\lambda_i$ is an eigenvalue of $O_{j,n}(p)$, so

$$\left|O_{j,n}(p)-\lambda_i I\right|=$$
$$\begin{vmatrix}(p+1)^2+2-(\lambda_i+2p+1) & 2 & & & \\ 2 & 2 & & & \\ & 2 & \ddots & & \\ & & \ddots & \ddots & \\ & & & & 2 \\ & & & 2 & (p+1)^2+2-(\lambda_i+2p+1)\end{vmatrix}$$

We substitute $p$ for $p(x)$.

### 3.4 Some properties of *p(x)*-Jacobsthal Lucas polynomials by tridiagonal matrices $A_{c,n}(p)$

Assume that $A_{c,n}(p)$ is a *nxn* tridiagonal matrix defined as

$$A_{c,n}(p)=\begin{bmatrix} p(x) & 4 & & & & \\ -1 & p(x) & 2 & & & \\ & -1 & \ddots & \ddots & & \\ & & \ddots & & & \\ & & & & & 2 \\ & & & & -1 & p(x) \end{bmatrix}$$

Then the determinant of $A_{c,n}(p)$

$$det(A_{c,n}(p))=C_{p,n}(x).$$

*p(x)*-Jacobsthal Lucas polynomials are also obtained by using the following symmetric matrix with complex entries. Assume that $A_{c,n}(p)$ is a *nxn* tridiagonal matrix defined as

$$A_{c,n}(p)=\begin{bmatrix} p(x) & 4i & & & & \\ i & p(x) & 2i & & & \\ & i & \ddots & \ddots & & \\ & & \ddots & & & \\ & & & & & 2i \\ & & & & i & p(x) \end{bmatrix}$$

Then it is easily seen that the determinant of $A_{c,n}(p)$ is also *n*.th *p(x)*-Jacobsthal Lucas polynomial.

$$det(A_{c,n}(p))=C_{p,n}(x).$$

The sequence of the spectra of the matrices $A_{c,n}(p)=A_{c,n}(1)$ for $n = 2,3,4,5,6$ is computed by using Matlab Programme as

$$s_2=\{-1;\ 3\}$$
$$s_3=\{1.449489742783179;\ 1;\ 3.449489742783181\}$$
$$s_4=\left\{\begin{matrix}1.613125929752755;\ \ 0.082392200292394 \\ ;\ 2.082392200292394;\ 3.613125929752755\end{matrix}\right\}$$
$$s_5=\left\{\begin{matrix}1.689994047855828;\ 0.662507751109815;\ 1 \\ ;\ 2.662507751109814;\ 3.689994047855830\end{matrix}\right\}$$
$$s_6=\left\{\begin{matrix}1.732050807568877;\ \ 1;\ 0.267949192431123 \\ ;\ 1.732050807568878;\ 3;\ 3.732050807568875\end{matrix}\right\}$$

Evidently,

$$\sum_{i=1}^{n}\lambda_i=tr(A_{c,n}(p))=np(x),\quad \prod\lambda_i=det(A_{c,n}(p))=C_{p,n}(x)$$

For the inverse of $A_{j,n}(p)$; by using (4), it is obtained that

$$\begin{aligned}
&a_i = p(x), \quad i \geq 1\\
&b_1 = 4, \; b_i = 2, \quad i \geq 2\\
&c_i = -1, \quad i \geq 1\\
&\theta_0 = 1, \quad \theta_i = C_{p,i}, \quad i \geq 2\\
&\phi_j = J_{p,(n-j+2)} \qquad j \geq 1
\end{aligned}$$

Therefore the inverse of $A_{c,n}(p)$ is the following matrix

$$(A_{c,n}^{-1}(p))_{(i,j)} = \begin{cases} (-1)^{i+j} 2^{j-i+1} C_{p,i-1}(x) J_{p,n-j+1}(x) \dfrac{1}{C_{p,n}(x)}, & if \quad i \prec j \\ (-1)^{1+j} 2^{j} J_{p,n-j+1}(x) \dfrac{1}{C_{p,n}(x)}, & if \quad i = 1, \; j \geq 2 \\ J_{p,n}(x) \dfrac{1}{C_{p,n}(x)}, & if \quad i = j = 1 \\ J_{p,n-i+1}(x) \dfrac{1}{C_{p,n}(x)}, & if \quad j = 1, \; i \geq 2 \\ C_{p,j-1}(x) J_{p,n-i+1}(x) \dfrac{1}{C_{p,n}(x)}, & if \quad i \succ j \end{cases}$$

The elements of the cofactor matrix are given as

$$cof(A_{c,n}(p))_{(i,j)} = \begin{cases} (-1)^{i+j} 2^{i-j+1} C_{p,j-1}(x) J_{p,n-i+1}(x), & if \quad i \succ j \\ (-1)^{1+i} 2^{i} J_{p,n-i+1}(x), & if \quad j = 1, \; i \geq 2 \\ J_{p,n}(x), & if \quad i = j = 1 \\ J_{p,n-j+1}(x), & if \quad i = 1, \; j \geq 2 \\ C_{p,i-1}(x) J_{p,n-j+1}(x), & if \quad i \prec j \end{cases}$$

For $i = j = 1$; we must take care of $\theta_0 = 1$ when constructing the matrix. By using cofactor matrix, we get some properties of *p(x)*-Jacobsthal and *p(x)*-Jacobsthal Lucas polynomials.

For $n = 2$; we get

$$\left|cof(A_{c,2}(p))\right| = \begin{vmatrix} J_{p,2}(x) & J_{p,1}(x) \\ -4J_{p,1}(x) & C_{p,1}(x)J_{p,1}(x) \end{vmatrix}$$

$$C_{p,1}(x)J_{p,2}(x)J_{p,1}(x)+4J_{p,1}^2(x)=J_{p,2}^2(x)+4J_{p,1}^2(x)=C_{p,2}(x).$$

For $n = 3$; we get

$$\left|cof(A_{c,3}(p))\right| = \begin{vmatrix} J_{p,3}(x) & J_{p,2}(x) & J_{p,1}(x) \\ -4J_{p,2}(x) & C_{p,1}(x)J_{p,2}(x) & C_{p,1}(x)J_{p,1}(x) \\ 8J_{p,1}(x) & -2C_{p,1}(x)J_{p,1}(x) & C_{p,2}(x)J_{p,1}(x) \end{vmatrix}$$

$$\begin{aligned}
&= 2J_{p,3}(x)C_{p,1}^2(x)J_{p,1}^2(x)+C_{p,2}(x)J_{p,3}(x)C_{p,1}(x)J_{p,1}(x)J_{p,2}(x) \\
&+8C_{p,1}(x)J_{p,1}^2(x)J_{p,2}(x)+4C_{p,2}(x)J_{p,1}(x)J_{p,2}^2(x) \\
&= 2J_{p,3}(x)C_{p,1}^2(x)+C_{p,2}(x)J_{p,3}(x)C_{p,1}(x)J_{p,2}(x) \\
&+8C_{p,1}(x)J_{p,2}(x)+4C_{p,2}(x)J_{p,2}^2(x) \\
&= p^2(x)\left(2J_{p,3}(x)+C_{p,2}(x)J_{p,3}(x)+8+4C_{p,2}(x)\right)=C_{p,3}^2(x)
\end{aligned}$$

For $n = 4$; we get

$$\left|cof(A_{c,4}(p))\right| = \begin{vmatrix} J_{p,4}(x) & J_{p,3}(x) & J_{p,2}(x) & J_{p,1}(x) \\ -4J_{p,3}(x) & C_{p,1}(x)J_{p,3}(x) & C_{p,1}(x)J_{p,2}(x) & C_{p,1}(x)J_{p,1}(x) \\ 8J_{p,2}(x) & -2C_{p,1}(x)J_{p,2}(x) & C_{p,2}(x)J_{p,2}(x) & C_{p,2}(x)J_{p,1}(x) \\ -16 & 4C_{p,1}(x)J_{p,1}(x) & -2C_{p,2}(x)J_{p,1}(x) & C_{p,3}(x)J_{p,1}(x) \end{vmatrix}$$

$$\begin{aligned}
&=4C_{p,2}^2(x)J_{p,3}^2(x)+2p^4C_{p,3}(x)J_{p,4}(x)+8p^2C_{p,2}(x)J_{p,3}(x) \\
&+2p^3C_{p,2}(x)J_{p,4}(x)+8p^3C_{p,3}(x)J_{p,3}(x)+pC_{p,2}^2(x)J_{p,3}(x)J_{p,4}(x) \\
&+4pC_{p,2}(x)C_{p,3}(x)J_{p,3}^2(x)+pC_{p,2}(x)C_{p,3}(x)J_{p,3}(x)J_{p,4}(x) \\
&=(p^4 + 8p^2 + 8)^3 \\
&= p^{12} + 24p^{10} + 216p^8 + 896p^6 + 1728p^4 + 1536p^2 + 512.
\end{aligned}$$

**3.5 Some properties of even *p(x)*-Jacobsthal Lucas polynomials by tridiagonal matrices** $E_{c,n}(p)$

Assume that $E_{c,n}(p)$ is a *nxn* tridiagonal matrix defined as

$$E_{c,n}(p)=\begin{bmatrix} p^2(x)+4 & 4 & & & & \\ 2 & p^2(x)+4 & 2 & & & \\ & 2 & p^2(x)+4 & \ddots & & \\ & & \ddots & \ddots & & \\ & & & & & 2 \\ & & & & 2 & p^2(x)+4 \end{bmatrix}$$

Then the determinant of $E_{c,n}(p)$ is computed by (3) as

$$det(E_{c,n}(p))=C_{p,2n}(x).$$

For the inverse of $E_{c,n}(p)$ the values are computed as

$$\begin{aligned} &a_i = p^2(x)+4, \quad i\geq 1 \\ &b_1=4, \ c_1=2, \\ &b_i = c_i=2, \quad i\geq 1 \\ &\theta_0=1, \qquad \theta_i=C_{p,2i}, \qquad i\geq 1 \\ &\phi_j=\frac{1}{p(x)}J_{p,2(n-j+2)}, \qquad j\geq 1 \end{aligned}$$

The sequence of the spectra of the matrix $E_{j,n}(p)$ for $n = 2, 3, 4, 5, 6$ is given in the following result with the help of Matlab programme

$$\begin{aligned} &s_2=\{5\ ;5\} \\ &s_3=\{1.535898384862244;\ 5;\ 8.464101615137755\} \\ &s_4=\left\{\begin{matrix}1.304481869954852;\ 3.469266270539637 \\ ;\ 6.530733729460359;\ 8.695518130045146\end{matrix}\right\} \\ &s_5=\left\{\begin{matrix}1.195773934819386;\ 2.648858990830108;\ 2.648858990830108 \\ ;\ 5.000000000000002;\ 7.351141009169893;\ 8.804226065180611\end{matrix}\right\} \\ &s_6=\left\{\begin{matrix}1.136296694843728;\ 2.171572875253808;\ 3.964723819589911 \\ ;\ 6.035276180410079;\ 7.828427124746194;\ 8.863703305156262\end{matrix}\right\} \end{aligned}$$

Evidently

$$\sum_{i=1}^{n}\lambda_i=tr(E_{c,n}(p))=n(p^2(x)+4), \quad \prod\lambda_i=det(E_{c,n}(p))=C_{p,2n}(x).$$

**Theorem 4** If $\lambda_i$ is an eigenvalue of the matrix $O_{c,n}(p)$, then $\lambda_i+2p(x)+1$ is an eigenvalue of $O_{c,n}(p+1)$.

**3.6 Some properties of odd *p(x)*-Jacobsthal Lucas polynomials by tridiagonal matrices** $O_{c,n}(p)$

Assume that $O_{c,n}(p)$ is a *nxn* tridiagonal matrix defined as

$$O_{c,n}(p)=\begin{bmatrix} p(x) & -p(x) & & & & \\ 2 & p^2(x)+4 & 2 & & & \\ & 2 & p^2(x)+4 & \ddots & & \\ & & \ddots & \ddots & & \\ & & & & & 2 \\ & & & & 2 & p^2(x)+4 \end{bmatrix}$$

Then the determinant of $O_{c,n}(p)$ is given by (3) as

$$det(O_{c,n}(p))=C_{p,2n-1}(x).$$

For the inverse of $O_{j,n}(p)$ the values are computed as

$$\begin{aligned} &a_1 = p(x),\ \ a_i = p^2(x)+4,\quad i\geq 2 \\ &b_1 = 2,\quad c_1 = -p(x), \\ &b_i = c_i = 2,\quad i\geq 2 \\ &\theta_0 = 1,\quad \theta_i = C_{p,2i-1},\quad i\geq 1 \\ &\phi_j = \frac{1}{p(x)} J_{p,2(n-j+2)},\quad j\geq 1 \end{aligned}$$

The sequence of the spectra of the matrices $O_{c,n}(p)$ for $n$ = 2, 3, 4, 5, 6 is given in the following result with the help of Matlab programme

$$\begin{aligned} &s_2 = \{0.550510257216821;\ 5.449489742783178\} \\ &s_3 = \{0.454392291525188;\ 3.376939054880550;\ 7.168668653594259\} \\ &s_4 = \left\{\begin{matrix} 0.427137177276751;\ 2.432687766807676; \\ 5.234519519401236;\ 7.905655536514336 \end{matrix}\right\} \\ &s_5 = \left\{\begin{matrix} 0.418527169600049;\ 1.942058836757595;\ 3.98664701329108 \\ ;\ 6.375568580727081;\ 8.277198399624194 \end{matrix}\right\} \\ &s_6 = \left\{\begin{matrix} 0.415676628845785;\ 1.659356259675156;\ 3.190740313743504; \\ 5.158619116030256;\ 7.087162176915573;\ 8.488445504789716 \end{matrix}\right\} \end{aligned}$$

Evidently

$$\sum_{i=1}^{n}\lambda_i = tr(O_{j,n}(p)) = (n-1)(p^2(x)+4)+p(x),\quad \prod\lambda_i = det(O_{c,n}(p)) = C_{p,2n-1}(x).$$

The maximum eigenvalue of spectra converges to $p^2(x)+8$.

## 4. Conclusion

In this study we define six different tridigional matrices whose determinants are *p(x)*-Jacobsthal, even, odd Jacobsthal, Jacobsthal Lucas polynomials. By using the finding inverse matrices property of tridigional matrices, we obtain some property of these sequences.

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# Two-Parameter Analysis of Rician Data as an Efficient Tool of Stochastic Signals' Processing

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**Abstract**

The papers considers new theoretical methods of the Rician data analysis in comparison with the traditional filtration techniques at solving the signal's analysis and noise suppression tasks. These so-called two-parameter methods provide the joint calculation of both the required signal value and the noise dispersion value. The joint computing of the Rice distribution's parameters allows efficient reconstruction of the informative component of the signal against the noise background. One of the main advantages of the proposed approach consists in the absence of restrictions connected with any *a priori* suppositions inherent to the traditional techniques of data analysis based on the so-called one-parameter approximation implying that the noise parameter of the signal to be investigated is known *a priori*. The developed two-parameter approach to data analysis is efficiently applicable to a wide spectrum of scientific and applied tasks, in which the signal to be analyzed is described by the Rice statistical model.

**Keywords:** Rice distribution, Probability density, Likelihood function, Two-parameter analysis; Signal-to-noise ratio .

## 1. Introduction

At the random signals' processing, in particular, at handling the problem of noise suppression, recently an approach is being widely developed based on the statistical methods such the method of moments, the maximum likelihood method, etc. Obviously, at applying such an approach the peculiarities of the statistical distribution of the data being analyzed have a substantial significance for the possibility of the task solution. The Rice distribution describes a wide range of information processing problems when the output signal is composed as a sum of the required initial signal and a random noise generated by many independent normally-distributed summands of zero mean value. The variable to be measured and analyzed is an amplitude, or an envelope of the resulting signal which is known to obey to the Rice distribution, [1].

The so-called two-parameter approach to the Rician signals' analysis consists in solving the task of joint determination of both parameters of the Rice distribution. In contrast to the traditional one-parameter approximation this approach is free of limitations that are inherent to the one-parametric approximation based upon the supposition that one of the task statistical parameters – the noise dispersion – is known a priori [2, 3]. That's why the technique of the two-parametric task solution that have become a subject of the present paper ensures much more correct estimation of the required values.

## 2. Preliminaries

A significant interest to solving a task of joint estimation of both parameters of the Rice distribution has appeared in 60-th years of the 20$^{th}$ century because of the understanding that in conditions of Rice distribution only the knowledge of both Rician parameters allows efficient reconstructing the initial required signal against the noise background. In paper [2] there was first formulated the significance of solving the two-parameter task applicably to radar signals' analysis. However this task is connected with finding the solution of a system of two essentially nonlinear equations what is conjugated with considerable difficulties of both the theoretical and the computational character. Partly due to this reason in [2] the mathematical consideration of the task is limited by the determination of the lower bounds for the standard deviation of these parameters' estimations on the basis of the Cramer-Rao inequality.

Later the simplified methods of the Rician data analysis have been elaborated in the conditions of the so-called one-parameter approximation consisting in estimating only one of the two unknown parameters – the signal value, in supposition that the second parameter – the noise dispersion – is known *a priori.*

The fundamental papers considering the problem within the one-parameter approximation are the papers [3] and [4], in which the required signal parameter is being estimated on the basis of the method of moments and the maximum likelihood techniques, respectively.

However in practice the condition when the Gaussian noise dispersion is known *a priori* never takes place and so is a severe restriction of the one-parameter approach what in accepted by practically all the authors working in this area.

Therefore the theoretical problem of joint estimation of both Rician parameters without any *a priori* conditions has remained unsolved for a few decades, since the 60-th years of the 20$^{th}$ century.

In [5-7] the authors of the present paper first develop an accurate theory of Rician signals statistical processing: new mathematical methods have elaborated and strictly substantiated for the so-called two-parameter approach to Rician data analysis. This approach implies solving the task of Rician data analysis be means of joint signal and noise estimation.

The present paper provides a study of the two-parameters statistical methods' efficiency in comparison with the traditional techniques.

**3. Comparative analysis of the two-parameter approach to Rician data processing and the traditional techniques**

In the tasks of the Rician signal analysis the value to be measured is an amplitude $x=\sqrt{x_{\mathrm{Re}}^2+x_{\mathrm{Im}}^2}$ of the complex variable with the real $x_{\mathrm{Re}}$ and the imaginary $x_{\mathrm{Im}}$ components characterized by their mean value $\nu$ and distorted by the normally distributed Gaussian noise with the dispersion $\sigma^2$. These conditions characterize many tasks of processing the signals of various physical nature. The amplitude $x=\sqrt{x_{\mathrm{Re}}^2+x_{\mathrm{Im}}^2}$ obeys to the Rice distribution with the probability density function:

$$P\left(x|\nu,\sigma^2\right)=\frac{x}{\sigma^2}\cdot\exp\left(-\frac{x^2+\nu^2}{2\sigma^2}\right)\cdot I_0\left(\frac{x\nu}{\sigma^2}\right), \tag{1}$$

where $I_\alpha(z)$ is the modified Bessel function of the first type of the order $\alpha$. The task to be solved consists in determining the unknown parameters $\nu$ and $\sigma^2$ on the basis of data measured in the samples.
In virtue of the specific peculiarities of the Rice statistical distribution the Rician data analysis demands a development of the particular methods and the corresponding mathematical apparatus.
As it is known at processing the Gaussian data an efficient and traditional filtration tool is the data averaging. However, as it has been noticed above, in contrast to the case of the Gaussian distribution an average value of the Rician signal $\bar{x}$ does not coincide to the requires useful signal's value $\nu$. This is illustrated in Fig.1 where the average Rician signal's value $\bar{x}$ that is depicted by a curved line while the useful signal's value $\nu$ depends is depicted by a straight line going from the coordinates' origin. The average Rician signal's value $\bar{x}$ as a function of the Rician parameters $\nu$ and $\sigma$ is expressed by the following formula:

$$\bar{x}=\sigma\cdot\sqrt{\pi/2}\cdot L_{1/2}\left(-\nu^2/2\sigma^2\right) \tag{2}$$

In (2) $L_{1/2}(z)$ is a Laguerre polynomial.

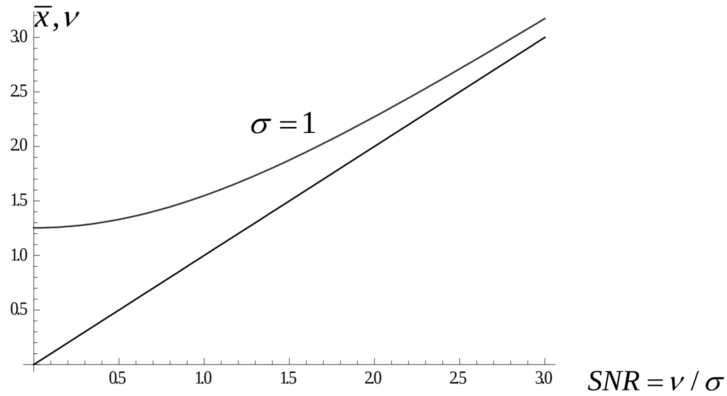



**Fig. 1.** An illustration of the noncoincidence of the Rician signal's average value $\bar{x}$ and the Rician parameter $\nu$, shown in dependency on the signal-to-noise ratio $SNR = \nu / \sigma$

The plots in Fig.1 correspond to the fixed values of parameter $\sigma$: $\sigma = 1$, so the values at the abscissa axis correspond to the signal-to-noise ratio $SNR = \nu / \sigma$.
Thereby if one applies the traditional filtration methods by averaging to the Rician data then in a range of small values of signal-to-noise ratio just the smoothing of the true values of the signal takes place.

## 4. Theoretical aspects and numerical testing results

The particular theoretical methods having been developed within the two-parameter analysis of the Rician signal in [5-7] differ in underlying statistical principles they are based upon. These methods include the method of moments based on the measured data for the random value's 1-st and 2-nd moments, designated as MM12; the method of moments based on measurements of the 2-nd and the 4-th moments, designated as MM12; the two-parametric maximum likelihood method, designated as ML. Each of these methods is based on solving the corresponding equations' system for the sought for parameters $\nu$ and $\sigma$, [7]. The system of equation for method MM12 looks as follows:

$$\begin{cases} \sigma \cdot \sqrt{\pi / 2} \cdot e^{-\frac{\nu^2}{4\sigma^2}} \left[ \left(1 + \frac{\nu^2}{2\sigma^2}\right) I_0\left(\frac{\nu^2}{4\sigma^2}\right) + \frac{\nu^2}{2\sigma^2} I_1\left(\frac{\nu^2}{4\sigma^2}\right) \right] = \bar{x}, \\ 2\sigma^2 + \nu^2 = \overline{x^2}. \end{cases} \qquad (3)$$

Method MM24 is rather an original and simple in its realization with equations' system for method MM24 as:

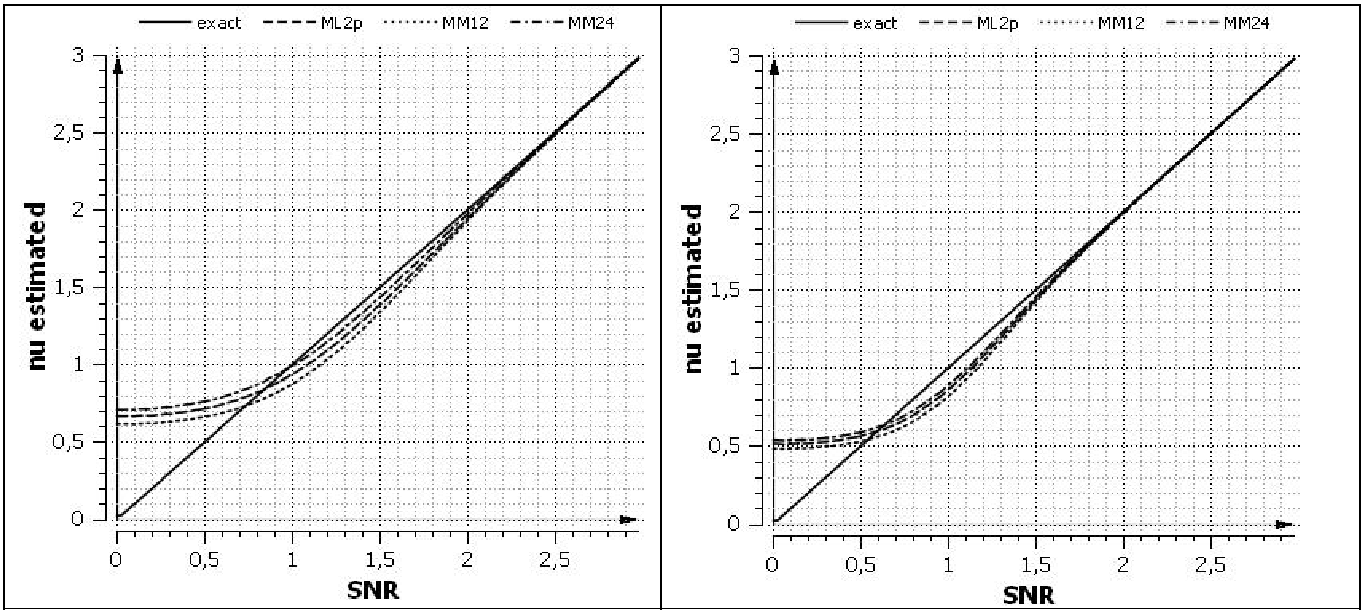
$$\begin{cases} \overline{x^2} = 2 \cdot \sigma^2 + \nu^2 \\ \overline{x^4} = 8 \cdot \sigma^4 + 8 \cdot \sigma^2 \cdot \nu^2 + \nu^4 \end{cases} \tag{4}$$

For ML method we have the following system of equations:

$$\begin{cases} \nu = \dfrac{1}{n} \sum\limits_{i=1}^{n} x_i \cdot I_1\left(x_i \cdot \nu / \sigma^2\right) / I_0\left(x_i \cdot \nu / \sigma^2\right) \\ \sigma^2 = \left(\left\langle x^2 \right\rangle - \nu^2\right) / 2. \end{cases} \tag{5}$$

The existence and the uniqueness of the systems (3)-(5) solutions have been strictly proved. An important theoretical result consists in the fact that for each above mentioned two-parameter method the corresponding system of two nonlinear equations for two variables $\nu$ and $\sigma$ can be reduced to one equation for just one variable. This allows an essential decreasing of the computational resources needed for the task solving, cutting them down to the level requires just at one-parameter approximation.

In Fig.2 there are provided some results of computer simulation of the task of the Rician signal's two-parametric analysis by means of three above mentioned methods.




| n=16 | n=64 |
|---|---|

**Fig.2**. The graphs of the calculated values of the informative signal's component $\nu$ reconstructed against the noise background.

In Fig. 2 the plots are presented that characterize the precisions of calculations of parameter $\nu$ by methods MM12, MM24 and ML. The obtained data were averaged by $10^5$ measurements. The numerical experiments were conducted for various values of the sample length $n$: $n = 16$, 64. The deviations of the curved lines from the straight line in Fig.2. characterize the precision of the methods being compared.

The results of the numerical experiments illustrate the following expected conclusions: the precision of the calculated sought-for parameters noticeably decrease with the increase of the signal-to-noise ratio and with enlarging the sample length $n$.

## 5. Conclusion

New two-parameter statistical signal processing methods have been investigated for the Rice distribution. The two-parameter task by each of the methods: MM12, MM24, ML, has been mathematically reduced to solving just one equation for one unknown variable, what essentially decreases the necessary calculating resources. The comparison of the elaborated two-parameter approach with the traditional ones have proved a high efficiency of the new technique. The numerical results confirm the possibility of solving the problem of the Rician signals' analysis by the developed methods ensuring a high precision in a wide range of the signal-to-noise ratio's values.

**Acknowledgement**: The work has been supported by RFBR, project N17-07-00064 within the Fundamental Research program.

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# An Application of the Rice Statistical Model in Tasks of Optical Metrology

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The paper presents a new approach to solving the tasks in optics and photonics that is based upon analysis and processing of the optical signal's amplitude as a random value obeying the Rice statistical distribution. Recently a new concept of the so-called two-parameter analysis has been developed and mathematically substantiated for Rician signals providing an accurate joint estimation of both the signal and the noise values without any a-priory assumptions concerning the process. One of the proposed applications of such an approach concerns the tasks of measuring the optical properties of a medium, such as an electro-optical coefficient on the basis of analyzing the statistical characteristics of the modulated reflected optical wave. Another perspective application of the developed technique concerns the phase shift measuring at quasi-harmonic signals' interferometry in optical metrology.

**Keyword(s):** Rice distribution, signal processing, quasiharmonic signal, phase shift, electro-optical coefficient, two-parameter analysis.
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## 1. Introduction

The statistical processing of Rician signals has recently become a subject of increasing scientific interest because of a wide circle of tasks which are adequately described by the Rice statistical model, [1]. In particular, these tasks include the high precision measurements in optics and photonics which are in the use in optical metrology, at distance measurements, in ranging systems, at determining the object's geometrical parameters, at non-destructive control and in many other applied tasks, [2-4].

The accurate measuring of two signals' phase difference is known to be one of the most important problems in such fields as radio-physics, optics, radiolocation, radio-navigation. This problem has been investigated for a long time and many various methods for its solving have been elaborated. The traditional methods of measuring the phase difference include the phase compensation technique, the transformation of the time interval into the voltage, the digital technique of accounting the number of pulses [2], the phase measuring method accompanied by the frequency transform, the correlation methods [2, 3], the Fourier transformation technique with the further extraction of the phase component [2, 3].

A number of existing phase measuring methods a-priori use a harmonic signal model, i.e. imply the constant amplitude’s value, what does not correspond to the real circumstances. In practice we normally have the so-called quasi-harmonic signal that is characterized by the random variations of the signal’s amplitude due to the Gaussian noise. Such a random character of a signal’s amplitude value is a serious obstacle for the accurate phase measuring [4]. The original method of the signals’ phase difference measuring that is considered in the present paper differs in principle from the methods of the prior art as it is based entirely upon measuring and processing the amplitude values only.

As for the problem of measuring the optical properties of a medium, such as the electro-optical (EO) coefficient, the traditional approach to calculating the EO coefficient value is based on the light reflection modulation. This modulation is caused by the variation of the refraction index under the influence of alternating electric field due to EO effect [6]. Traditionally a linear regression technique is being used for processing the results of such a modulation. Such an approach implies the evaluation of the combined effect of both the EO qualities of a medium and the inevitable Gaussian noise, which may significantly decrease the accuracy of EO coefficient estimation.

A principal distinguish of the approach proposed in the present paper consists in the increase the accuracy of measurements due to the possibility to evaluate the signal’s noise dispersion value and thus to compensate the inevitable Gaussian noise influence.

## 2. Preliminaries

The amplitude, or the envelope of a signal being formed from the initially determined component under the inevitable influence of Gaussian noise obeys the Rice statistical distribution, first formulated by S. Rice in 1944 as an extension of the classical Rayleigh distribution. So, the Rician distribution describes the amplitude of the random variable, formed by summing an initially determined complex signal and the Gaussian noise distorting this signal.

The Rice statistical model is known to adequately describe a wide range of the signal processing problems in the tasks when the output signal is composed as a sum of the sough-for initial signal and a random noise generated by many independent normally-distributed summands, what always takes place at the optical signal propagation in a medium.

Let $A$ be a determined value that characterizes the physical process to be considered. This value is inevitably distorted by the Gaussian noise created by a great number of independent noise components, while the measured and analyzed value is the amplitude, or the envelope of the resulting signal. The Gaussian noise distorting the initial determined signal is characterized by a zero mean value and a dispersion $\sigma^2$. The signal's amplitude $x=\sqrt{x_{\mathrm{Re}}^2+x_{\mathrm{Im}}^2}$ obeys the Rice statistical distribution, while the real $x_{\mathrm{Re}}$ and imaginary $x_{\mathrm{Im}}$ parts of the complex signal with amplitude $x$ are random Gaussian values with mathematical expectations $\overline{x_{\mathrm{Re}}}$ and $\overline{x_{\mathrm{Im}}}$, satisfying the condition $\overline{x_{\mathrm{Re}}}^2+\overline{x_{\mathrm{Im}}}^2=A^2$, and dispersion $\sigma^2$. The values $\nu=A$ and $\sigma$ are the Rice distribution's parameters for random variable $x$. Obviously, the value of $x$ belongs to the subset of the not-negative real numbers: $x\in(0,\infty)$. The ratio of the Rician parameters $SNR=\nu/\sigma$ characterises the signal-to-noise ratio.

So, the Rician random variable $x$ represents the amplitude of the signal with the Gaussian real and imaginary parts. The Rician probability density function is given by the following formula:

$$P\left(x|\nu,\sigma^2\right)=\frac{x}{\sigma^2}\cdot\exp\left(-\frac{x^2+\nu^2}{2\sigma^2}\right)\cdot I_0\left(\frac{x\nu}{\sigma^2}\right), \tag{1}$$

where $I_0$ is the modified Bessel function of the first kind of order zero. Here and below we'll use the following denotations: $I_\alpha\left(z\right)$ is the modified Bessel function of the first kind (or the Infeld function) of the order $\alpha$; $x_i$ is the signal's value measured as the $i$-th element of a sample; $n$ is the quantity of elements in a sample, called also a sample's length.

The final purpose of the Rician data processing is evidently the evaluation of value $A$ that characterizes the process under the study and coincides with parameter $\nu$ of the Rice distribution.

## 3. Application of Rician data analysis as a tool of EO coefficient calculation

The situation when the resulting signal's envelope is formed by the initially determined component under the inevitable Gaussian noise influence is rather common and takes place in many tasks, in particular, at the optical signal propagation in a medium, at implementation of measurements in optics and photonics. The Rice distribution is known to describe a wide range of the signal processing problems in the tasks when the output signal is composed as a sum of the sough-for initial informative signal and a random noise. The recently developed theoretical methods of the Rician data analysis by means of the Rician parameters computing [6, 7] provide possibility to decompose the Rician signal into its informative and noise components and thereby ensure an efficient reconstruction of the useful signal against the noise background.

The proposed application of the developed technique of Rician signals' analysis to solving the tasks in optics allows rather an accurate estimation of the medium's parameters, such as electro-optical (EO) coefficient estimation. The proposed method for measuring EO coefficient is based on the analysis of the reflected optical signal as a Rician random value: indeed, the inevitable presence of speckle-noise in the reflected light wave causes the stochastic character of the process. The efficient EO coefficient has been shown to be a random value that obeys to the Rice distribution what proves the applicability of the method of Rician signals' analysis to the task of the EO coefficient estimation.

As for the traditional approach to the EO coefficient evaluation, it demands conducting a series of experimental measuring the refractive coefficient while the electric field applied to EO sample is being modulated [5]. The stochastic data to be analyzed in the task is formed by the amplitude of the light wave, reflected from the EO medium, while the reflection coefficient is being periodically modulated under the influence of the controlling electric field in virtue of EO effect. Then the EO coefficient was calculated by means of the application of the least squares method using the straight-line regression for the value of the reflection coefficient variation for each pair of the magnitudes of the modulating periodic electric field and the corresponding refractive coefficient.

Obviously, the reflected optical signal is a sum of the fixed component being determined by the EO coefficient of the medium, and a noise component being formed by the Gaussian speckle noise. So, the reflected optical signal is distorted if compared with the controlling voltage because of the influence of a Gaussian noise. As it has been indicated above, the developed techniques of the Rician signals analysis allow joint calculating of both the informative and the noise components by means of measuring the resultant reflected signal.

The electro-optical effect is known to cause the change of the reflection coefficient of EO material characterized by the normal reflection coefficient: $\rho = (n - 1)^2/(n + 1)^2$, where $n$ is the refractive index for ordinary polarized beam or for non-ordinary polarized beam, depending upon the polarization of the incident beam respectively to the crystal's optical axis. The variation of the medium's reflection coefficient caused by the EO effect is expressed by the following formula:

$$\delta\rho = \frac{\delta I}{I_{inc}} = \frac{\partial \rho}{\partial n}\delta n = \frac{4(n-1)}{(n+1)^3}\delta n \tag{1}$$

In (1) $I_{inc}$ is the intensity of the incident light, $\delta I$ – the change of the reflected light intensity, $\delta n = \frac{1}{2} n_0^3 k_{ef} E$ – the change of the refractive index due to the EO effect, $k_{ef}$ – an efficient EO coefficient, $n_0$ - the refractive index of the medium in the absence of the electric field.

Solving the task of two-parameter analysis of the Rician signal by the so-called MM24 method is based on the known formulas for the 2-nd and the 4-th initial moments for the random Rician value $R$, [6]:

$$\begin{aligned} \overline{R^2} &= 2\sigma^2 + A^2 \\ \overline{R^4} &= 8\sigma^4 + 8\sigma^2 A^2 + A^4 \end{aligned} \tag{2}$$

Considering formulas (2) as a system of two equations for two unknown variables $A$ and $\sigma^2$, one can calculate the sought-for values $A$ and $\sigma^2$ on the basis of data for the second $\overline{R^2}$ and the fourth $\overline{R^4}$ moments of value $R$, having been computed from the sampled measurements.

The principle thing in the proposed technique is that the sought–for EO coefficient measured in an experiment is a Rician value and its real (undistorted) magnitude can be reconstructed against the noise background by the Rician analysis technique, while a traditional approach implies the evaluation of the combined effect of both the EO qualities of a medium and the inevitable Gaussian noise, which may significantly decrease the accuracy of EO coefficient estimation by the traditional method.

Fig. 1 presents a typical histogram of the efficient EO coefficients of a crystallized quartz sample: on the left there are presented the both the EO coefficient values and the standard deviation value σ characterizing the Gaussian speckle noise; on the right there are provided the results of the EO coefficient calculation by the linear regression technique, based upon evaluating the inclination of the averaged straight line displaying the dependence of the change in the refractive coefficient on the electric field magnitude.

An advantage of the proposed Rician analysis application for the EO coefficient measurement, important from the view point of its practical realization, consists in the following: for calculating the EO coefficient by this technique it is sufficient to conduct the measurements of the reflection at any only one value of the electric field, without the necessity to modulate this value, what significantly simplifies the experimental setup and decreases the number of measurements.

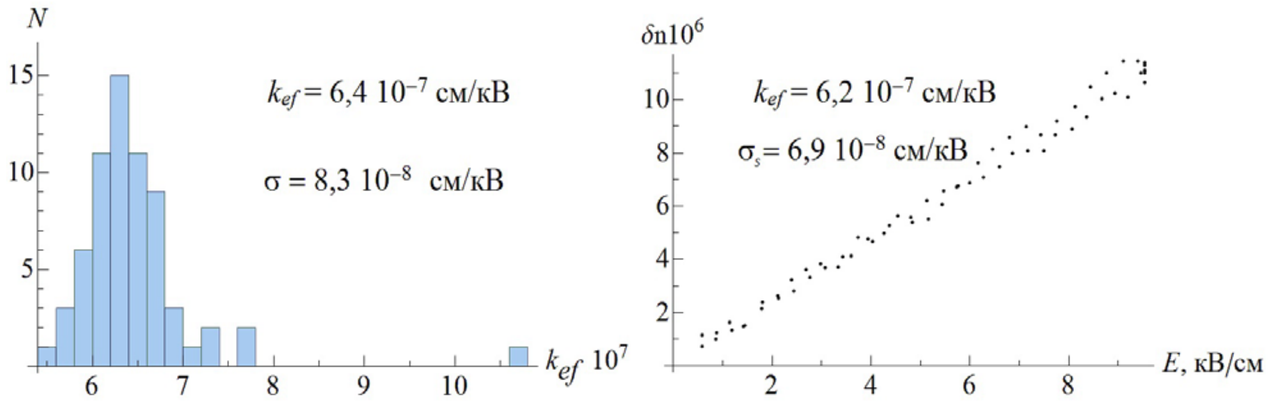


Fig. 1. Results of estimation of the efficient EO coefficient and its standard deviation by means of Rician parameters computing (on the left) and by traditional method (on the right).

### 4. Rician signals' amplitude analysis as a tool for the phase shift measuring

Another perspective application of the developed technique concerns the phase shift measuring at quasi-harmonic signals' interferometry in optical metrology.

The statement of the problem being considered here as a specific application of the Rician data two-parameter analysis is as follows: two initially sine-shaped optical signals of the same frequency propagate through the different channels thus accumulating the phase shift to be measured. In practice the propagation of the harmonic signal in a medium is inevitably accompanied by the noise influence what results in the random variations of the signal's amplitude. Therefore, instead of a sine-shaped signal one has to consider just the quasi-harmonic, or quasi-sinusoidal signal. According to the above the amplitude of such a signal is a random value that satisfies to the Rice statistical distribution.

The time dependence of any quasi-harmonic signal $S(t)$ can be presented as a complex value as:

$$S(t) = R(t)\cdot \exp\left[i\left(\omega t + \varphi(t)\right)\right] = s(t)\cdot \exp(i\omega t) \tag{3}$$

where $\omega$ is the frequency, $R(t)$ is the signal's amplitude, or envelope that randomly varies in time $t$ due to the inevitable Gaussian noise influence, and $\varphi(t)$ is the phase shift that also changes randomly in time due to the so-called amplitude-phase modulation. For measuring the signals' phase values we'll analyze the "slow" signal's component $s(t) = R(t)\cdot\exp[i\varphi(t)]$.

The essence of the proposed phase shift measuring technique by means of the Rician data analysis consists in the following. Let us consider two quasi-harmonic signals propagating in different channels. These signals' phase difference is a characteristic of the object or the process to be studied. We can present these signals as the following vectors: $\overrightarrow{R_1}(R_1,\varphi_1), \overrightarrow{R_2}(R_2,\varphi_2)$, as illustrated in Fig.2.

The quasi-harmonic signals' amplitudes $R_1$ and $R_2$ obey the Rice distribution with parameters $(A_1,\sigma^2)$ and $(A_2,\sigma^2)$, correspondingly, where $A_1$ and $A_2$ are the initial, undistorted signals' amplitudes, $\sigma^2$ is the Gaussian noise dispersion.

The noised signals to be measured can be put down as follows: $\overrightarrow{R_1} = \overrightarrow{A_1} + \overrightarrow{r_1}$ , $\overrightarrow{R_2} = \overrightarrow{A_2} + \overrightarrow{r_2}$, where vectors $\overrightarrow{A_1}$ and $\overrightarrow{A_2}$ denote the two initial, undistorted signals, $\overrightarrow{r_1}, \overrightarrow{r_2}$ - the noise vectors, each of them being characteristic for a corresponding channel of the signal propagation. Let us introduce the third vector that is equal to the sum of the two signals being analyzed: $\overrightarrow{R_3} = \overrightarrow{A_3} + \overrightarrow{r_3}$ . In Fig. 2 the noised signals $\overrightarrow{R_i}\,(i=1,2,3)$ are shown by the dashed lines while the initial, undistorted signals $\overrightarrow{A_1}, \overrightarrow{A_2}$ and their sum $\overrightarrow{A_3}$ are shown by solid lines.

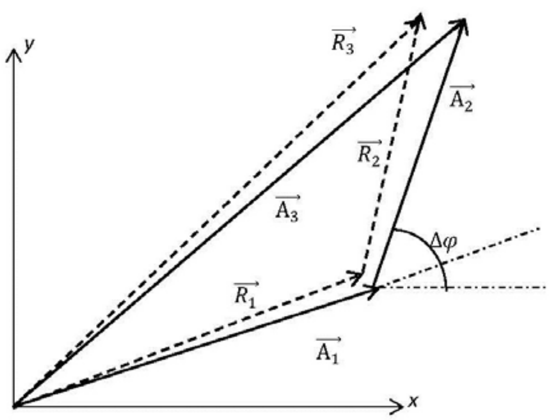


Fig. 2. The vector representation of signals being analyzed at calculating the phase shift value $\Delta\varphi$

The phase difference $\Delta\varphi$ between the two signals is equal to an angle between the corresponding vectors. Vectors $\overrightarrow{R_1}, \overrightarrow{R_2}$ and $\overrightarrow{R_3}$ form a triangle and the phase difference between the two signals can be determined based on the geometrical consideration of this triangle, namely - by calculating the triangle sides' values, i.e. the signals amplitudes' values. The noise distorts each vector independently and the amplitudes measured in each moment of time would provide a false, distorted value for the sought for phase shift. Obviously, the sought for phase difference $\Delta\varphi$ could be correctly found only from the triangle formed by the initial, undistorted amplitudes: $A_1, A_2, A_3$.

As it has been shown above the signals' amplitudes obey to the Rice distribution with the Rician parameters $\left(A_i, \sigma^2\right), i = 1, 2$. As for the third signal $\overrightarrow{R_3} = \overrightarrow{A_3} + \overrightarrow{r_3}$, its amplitude obeys the Rice distribution as well due to the stable character of the Rice statistical distribution [7]. The parameters of the Rice distribution for amplitude of the sum signal are: $\left(A_3, 2\sigma^2\right)$. The so-called two parameter methods elaborated for Rician signals' analysis, [6, 7] allow an accurate estimating of both the signal ($A_i, i = 1, 2, 3$) and the noise ($\sigma^2$) parameters based upon the sampled measurements.

By calculating the initial, undistorted values of the three signals' amplitudes $A_i, i = 1, 2, 3$ we are able to "freeze" the picture as a noise-free one and thus calculate the needed phase difference value just on the basis of geometrical considerations by the formula:

$$\Delta\varphi = \arccos\left(\frac{A_3^2 - A_1^2 - A_2^2}{2A_1A_2}\right) \tag{4}$$

The proposed technique of the signals' phase shift measuring differs in principle from other methods as it is based entirely upon measuring and processing the amplitude values only.

## 5. Conclusion

The paper is devoted to the consideration of a new approach to solving the tasks of high precision measurements in optics and photonics. The approach is based upon analysis of the optical signal's amplitude value within the Rice statistical model. The proposed technique's applications having been demonstrated in the paper, relate the optical metrology problems. In particular, concerning the task of the optical medium's EO coefficient estimation it has been shown that the Rician data analysis provides an efficient reconstruction of the useful signal component against the speckle noise background, thus ensuring the more correct evaluation of the EO coefficient than provided by the traditional linear regression technique, based upon measuring the total, noise-contaminated reflected signal. Besides, the application of the two-parameter technique significantly simplifies the experimental setup and decreases the required number of measurements.

Another example of the application of the proposed approach relates to the problem of accurate measuring the phase shift between two quasi-harmonic signals. An important peculiarity of the proposed technique for solving this task consists in the fact that the phase data are obtained as a result of the amplitude measurements only what significantly decreases the demands to the measuring equipment. The amplitudes of the three signals to be analyzed are shown to obey the Rice statistical distribution.

**Acknowledgement**: The work has been supported by RFBR, project N17-07-00064 within the Fundamental Research program.

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# The use of partial least squares structural equation modeling approach for analysis of the dimensions of poverty, a case study of Albania.

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**Abstract:** Partial Least Squares Structural Equation Modeling (PLS-SEM) is a multivariate analysis technique for modeling the relations in several fields of knowledge including dimensions of poverty. The purpose of this article is to operationalize living conditions, social inclusion, education, expenditures as poverty dimensions with a view to understanding the links between them. The data is derived from Living Standards Measurement Survey (LSMS) 2012. The results show that education has a positive impact on expenditures and social inclusion. Relationships in structural models between latent dimensions are significant. Measurement models indicate allowed values for internal consistency, reliability, validity. Our findings support as instruction for PLS-SEM implementation in multidimensional poverty analysis.
**Keywords: partial least squares structural equation modeling, LSMS, latent dimension, measurement models.**

## 1. Introduction

Nowadays it is necessary to study the phenomenon of poverty, well-being and the factors that cause them as the most important goal of development policies. Researchers studying the causes of poverty are mindful of the fact that the concept of poverty above all is a complex and multidimensional concept, has different meanings, multiple causes that cannot easily be distinguished [1], [2] . Poverty is a complex concept that we need to understand the ties between its dimensions. Partial least squares structural equation modeling (PLS-SEM) is a widely used method to analyze interaction between dimensions or constructs. In his study of Nepal, Wagle 2008 [3] explores the relationship between latent poverty dimensions which in this case are considered well-being, capability, social inclusion, and set of observed indicators for each dimension. By PLS-SEM it is shown how these dimensions are interconnected with each other so that the results obtained can be used by policy makers.

## 2. Materials and Methods

Structural equation modeling (SEM) is a multivariate analysis technique for operationalizing latent variables, and describes the relationship between latent variables (dimensions) and its indicators. SEM includes two models: the measurement model or external model and structural model or inner model. Among the different approaches to estimating the model parameters in SEM are the covariance-based model and the variance-based model or partial least square (PLS) path model for which there has been a growing interest in recent decades [4]. The PLS-SEM ability is that it does not make assumptions about the distribution of data, it is used when distributions are highly skewed used for metric data, nominal, ordinal data [5], is used in small samples and finally builds more complex models with many latent variables, indicators.

In this study, data is derived from Living Standards Measurement Survey 2012 (LSMS 2012) which includes 2000 households. Partial Least Squares Structural Equation Modeling has been used through the SmartPLS3 program. So, in our study we have used the multidimensional approach of poverty it is necessary to quickly determine the dimensions to be taken into account and their corresponding variables. For the dimensions we have taken in the study we are based on available data, expert knowledge and, on the review of the literature on multidimensional poverty, where the latter includes Multidimensional Poverty Index [6].

**2.1 The variables selected in the study are:**
**Educational Level, ED**
The father's educational level is ordinal variable, the values it receives are from 1 to 5 (four-year-old school, four-year high school, high school, some high school, university), $\mathbf{Ed_1}$
The mother's educational level is ordinal variable, the values it takes are from 1 to 5 (four-year-old school, primary school, high school, some high school, university), $\mathbf{Ed_2}$

**Expenditures Household, EX**

Family expenses are taken into account.

**Social Inclusion, SI**
Cinema is the ordinal variable, the values it receives are from 1 to 5 (never, 1 to 6 times, more than 6 times, every month, every week), $\mathbf{SI_1}$
Live is the ordinal variable, the values it receives are from 1 to 5 (never, 1 to 6 times, more than 6 times, every month, every week), $\mathbf{SI_2}$
Cultural Sites is the ordinal variable, the values it receives are from 1 to 5 (never, 1 to 6 times, more than 6 times, every month, every week), $\mathbf{SI_3}$.

**Living Condition, LC**
The condition of dwelling type is variable ordinal, the values it receives are from 1 to 3 (inappropriate for living, suitable for living, very good condition), $\mathbf{LC_1}$
Outside apparence of building is the ordinal variable, the values that are taken are from 1 to 3 (plastered, partially plastered, not plastered), $\mathbf{LC_2}$

**2.2 Conceptual Model**
The proposed model for our work includes four latent dimensions which include: **ED, EX, SI, LC**. Below are represented casual relationships between dimensions.

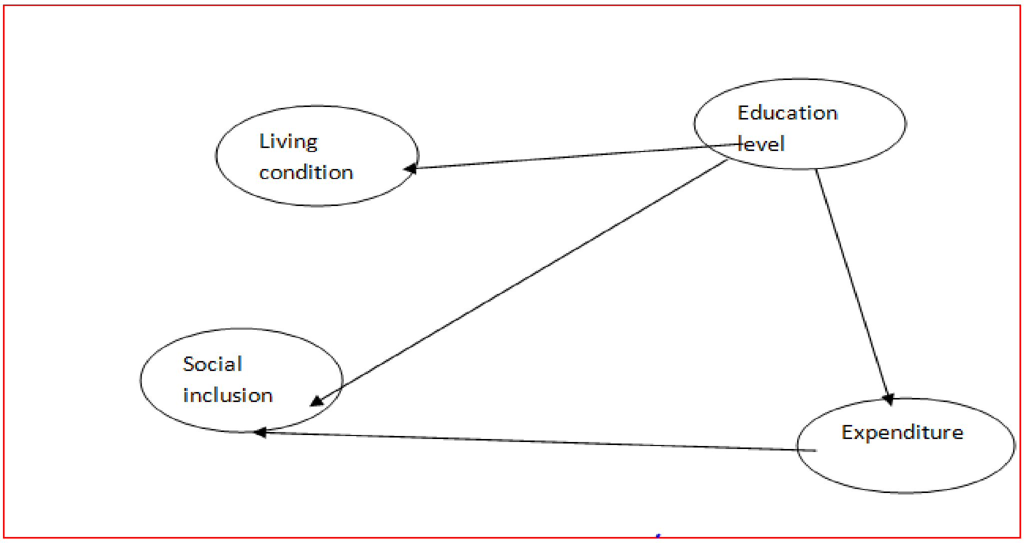


**Figure 1. Conceptual Model**

## 3. Main Results

### 3.1 Assessment of Measurement Model

Our reflective dimension is used in our model. Reflective measurement models are evaluated based on the internal consistency reliability that includes the composite reliability statistic. The composite reliability values (**CR**) should be between 0.7 and 0.9 because values above this limit are problematic due to excessive indicators [4], [7]. Validity, that includes the convergent validity indicator, and, discriminant validity [5]. To estimate convergent validity, we should consider the indicator load and the average variance extracted (AVE), each having at least the value of 0.7 and, 0.5 respectively [5]. To study discriminant validity, consider the Fornell and Larcker’s criterion [8].

**Table 1. Reflective measurement model**

| **Dimensions and indicators** | **Loadings** | **CR** | **AVE** |
|---|---|---|---|
| **Educational level** | | 0.868 | 0.767 |
| Father educational level | 0.893 | | |
| Mother educational level | 0.857 | | |
| **Expenditures** | 1 | 1 | 1 |
| **Social Inclusion** | | 0.881 | 0.713 |
| Cinema | 0.875 | | |
| Live | 0.886 | | |
| Cultural sites | 0.776 | | |
| **Living condition** | | 0.918 | 0.849 |
| Condition of dwelling type | 0.907 | | |
| Outside apparence of building | 0.935 | | |

By the results reflected in Table 1, it is shown that all dimensions of this study are within the permissible parameters for Loading, CR, AVE. To evaluate discriminant validity we have used the Forner and Larcker's criterion that requires the condition to be met:
which requires all the square root of AVE to be higher than their inter-correlations dimensions. The values that are placed in the diagonal of table 2 indicate the AVE square root, and the other values inter-correlations between dimensions.

**Table 2. Forner and Larcker criterion**

| Dimensions | Expenditure | Living Condition | Educational level | Social Inclusion |
|---|---|---|---|---|
| Expenditure | 1 | | | |
| Living Condition | 0.098 | 0.921 | | |
| Educational level | 0.156 | 0.578 | 0.876 | |
| Social Inclusion | 0.185 | 0.180 | 0.238 | 0.844 |

It is noted that all values outside the diagonal are smaller than those in the main diagonal, therefore Forner and Larcker criteria are met.

### 3.2 Assessment of structural model

Assessment of structural model includes the significance of the structural relations, the coefficient of determination $R^2$. Table 3 shows the path coefficients, p-value, t-statistics, significance level for all paths. The analysis shows that the educational level has a significantly positively correlated impact on expenditures, also has a significant positive impact on social inclusion. Household expenditures have a positive impact on social inclusion. Educational level has a positive impact on living condition. Ultimately, all path coefficients are significant.

**Table 3.** Path Coefficients of the Structural Model

| Path | Path Coefficient | t-statistics | p-value |
|---|---|---|---|
| ED→EX | 0.156 | 12.378 | 0.000 |
| ED→ SI | 0.214 | 10.859 | 0.000 |
| EX→ SI | 0.152 | 12.734 | 0.000 |
| ED→ LC | 0.578 | 32.187 | 0.000 |

For $R^2$ values, it is difficult to set a lower limit of its values because it depends on the complexity of the model and field of study [5]. Based on the study of Falk and Miller (1992) [9] it is considered as a criterion that the value of $R^2$ should not be less than 0.1.

**Table 4. Coefficient of determination for dimension, $R^2$**

| Dimension | $\mathbf{R^2}$ |
|---|---|
| Expenditure | 0.03 |
| Living Condition | 0.33 |
| Social Inclusion | 0.09 |

Another size that is used for structural model estimation is effect size $f^2$. The latter shows the effect of removing a dimension in the value of $R^2$. As a rule, the values of $f^2$ 0.02, 0.14, 0.35 are respectively considered small, medium and large [10]. Table 5 shows the effect size values for each dimensional connection. The level of education has a substantial effect on living condition and a small size effect on expenditure and social inclusion.

**Table 5. Effect size**

| Dimension | Expenditure | Living Condition | Level Education | Social Inclusion |
|---|---|---|---|---|
| Expenditure | | | | 0.024 |
| Living Condition | | | | |
| Level Education | 0.025 | 0.501 | | 0.049 |
| Social Inclusion | | | | |

## 4. Conclusions

Using PLS-SEM helps in analyzing the dimensions of poverty by understanding how these dimensions are related to one another. From the results of the model, we draw conclusions about the impact that have dimensions with each other that serve to improve social policies. Specifically, our study confirmed the positive impact of education on social inclusion, expenditures and living conditions. In further studies it is thought that the model will expand and with other latent dimensions.

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# Descriptive Features of Spousal Relation in Turkey

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**Abstract**

The spousal relation is one of the leading problems in today's conditions. Due to the certain economic, social and cultural reasons, the couple divorces after a short time of period and prefers to lives single with their children, parents or alone. In order to determine the major effects which cause the divorce and understand the spousal relations, a very comprehensive survey has been conducting in Turkey in every five-year. This survey is called Turkey Family Structure Research (TAYA) 2016 and executed by the Ministry of Family and Social Policies. In this study, by a special permission from Turkish Statistical Institute, we detect the descriptive features of this survey, by analyzing certain questions in details such as which type of demographic properties are generally seen in couples? How do they choose their couples? Is the consanguineous marriage still common? How do they meet with their spouses? In our analyses, we aim to define these sociological properties via statistical analyses and find the common features in spousal relations at a first glance.

**Keywords:** Spousal relations in Turkey, Statistical Analyses, Extracting Common Features
**Acknowledgement**: The authors thank to Turkish Statistical Institute for their permission to analyze this dataset.

## 1. Introduction

Family is the first type of the community that begins along with the human life. The structure of the family highly depends on the social, economic and cultural developments that occur in the community. Accordingly, marriage is the base organization for starting a family and it is a legal union of spouses to establish a life partnership. The ending of a marriage based on one statutory reason except death is called as divorce. The reasons for divorce are investigated from different aspects by sociology and psychology sciences. The sociological studies consider age, gender, socioeconomic status, social structure and the age of marriage of the individuals as probable reasons while psychology evaluates the situation in terms of the communications of the individuals during the marriage process and the personal characteristic of the individuals (TBNA, 2014). In order to have knowledge about the situation of spousal relations, Turkish family structure and reasons for divorce in Turkey, the study executed by the Ministry of Family and Social Policies can be investigated. The study is called as TBNA (2014) and presents the concept about spousal relation and a detailed research on reasons for divorce in Turkey from sociological and psychological aspects. The study also presents evaluations according to the survey and offers recommendations for the problems. Another extensive study about marriage, relationship with the spouse and divorce is TAYA (2013). The study is implemented also by the Ministry of Family and Social Policies. Indeed, the study is about research on family structure in Turkey and has titles for marriage including relations with spouse and divorce. Yet another resource for the studies about spousal relation, divorce and related topics in Turkey is academic literature. There are various researches from different aspects about the marriage, divorce, relation with the spouse and violence against women which are related topics. Hereby, the main aim of this study is to desribe the major property of spousal relations in the Turkish family structure using the data updated for 2016 in order to better understand all these listed sociological consequences. So, as the first step we obtain the percentages for the answers. Then, we construct tables integrating the questions related with marriage and lastly, we statistically test the significance of the results. In the following part, we describe the data and in the Results part, we present the associated outcomes. Finally, in the Conclusion part, we summarize the findings and discuss the future work.

## 2. Data Description

Accordingly, as stated previously, the ministry of family and social policies of Turkey makes a survey on the Turkish family structure. The study is called the Research on Family Structure and repeated every five years. Previous surveys were carried out in 2006 and 2011. The current report for this study is published in 2013 and it includes the analyses of the data from the surveys in 2006 and 2011. Demographic characteristics such as gender, age, education, marital status, type of family, number of children, socioeconomic status and religious belief are used as independent variables in the current report. On the other hand, dependent variables in this report are listed under three titles (TAYA, 2013). These dependent variables are:

- Attitudes towards marriage: Marital status, age at first marriage, how did s/he got married?, marriage solemnization, how many times did s/he get married?, what kind of ceremonies?, bride price, which social circle did s/he get married from?

- Relations with the spouse: Ideals about marriage, ideal marriage age.

- Relations with the spouse: Level of relationship with the spouse, three problems experienced with the spouse, reactions given to problem.

The study in which data are obtained executed with a protocol between the Ministry of Family and Turkish Statistical Institute in 2016. Based on this protocol, the survey covers 17239 households and 35475 individuals over 15 years old are interviewed in these households. Hereby, in this study, we represent certain descriptive statistics from this survey whose results are presented in the next section. Most of our selected variables in our anlyses have ordinal scale.

## 2. Results

In the survey, there are 35475 individuals over 15 years of age. 15774 of them are men while 19701 are women. The average age by gender and related descriptive statistics are represented in Table 1. The statistical analyses show that there is no significant difference between the average age of men and women ($t=1.137$, $p>0.05$, $df=35473$).

**Table 1:** Age by gender.

| Gender | Number | Percentage | Mean | Standard Deviation |
|---|---|---|---|---|
| Male | 15774 | 44.5 % | 43.11 | 17.448 |
| Female | 19701 | 55.5 % | 42.90 | 17.472 |

On the other hand, the percentages of marital status by gender for 2016 are shown in Table 2. The percentages of never married men and women are 25% and 19%, respectively. In total, the percentage for married with civil ceremony people is 68%, the percentage for never married people is 21%, the percentage for divorced people is 3% and the percentage for widowed people is 7%. Herein, the chi-square test is performed to see whether there is an association between marital status and gender. According to the test result ($\chi^2$=1116.262, p<0.05, df=6), it is found that there is a significant association between marital status and gender. Furthermore, the Cramer's V measure which shows the strength of the association for nominal variables that have more than two categories is also calculated and the value of the measure (0.177) indicates a weak association between the marital status and gender.

**Table 2:** 15+ marital status by gender (%).

| Marital Status | Male | Female | Total |
|---|---|---|---|
| Never married | 24.7 | 18.6 | 21.3 |
| Divorced | 2.0 | 3.5 | 2.9 |
| Widowed | 2.1 | 10.1 | 6.6 |
| Married, civil ceremony | 70.1 | 66.3 | 68 |
| Married, religious ceremony | 0.6 | 0.7 | 0.6 |
| Separated | 0.4 | 0.6 | 0.5 |
| Shacked | 0.1 | 0.1 | 0.1 |

In Table 3, frequencies for the number of marriages are shown. Among the respondents married at least one, 94% of them get married once, 5% of them get married for the second time and about 1% of them get married more than two times.

**Table 3:** Frequency of the number of marriages.

| Number of Marriage | Frequency | Percent | Valid Percent |
|---|---|---|---|
| 1 | 26333 | 74.2 | 94.4 |
| 2 | 1457 | 4.1 | 5.2 |
| 3 | 82 | 0.2 | 0.3 |
| 4 | 22 | 0.1 | 0.1 |
| 5 | 7 | 0.0 | 0.0 |
| 6 | 1 | 0.0 | 0.0 |
| 10 | 1 | 0.0 | 0.0 |

On the other side, the mean of the age at first marriage for men is 24 while it is 20 for women as shown in Table 4. Moreover, we observe that there is a significant difference between the age of men and women for the age at first marriage (t=71.707, p<0.05, df=25353.910).

**Table 4:** Age at first marriage by gender.

| Gender | Mean | Standard Deviation | Minimum | Maximum |
|---|---|---|---|---|
| Male | 24.03 | 4.405 | 12 | 71 |
| Female | 20.23 | 4.337 | 12 | 60 |

To see how individuals get married, it is asked "how did you marry your spouse?" with multiple choice options. Table 5 shows percentages by gender. In total, majority of the answers are under two choices: The percentage of people who said "my decision, with the approval of my family" is 29% and the percentage of people who said "arranged marriage, my decision" is 48%. Later, the chi-square test is used to see whether there is an association between decision of marriage and gender. According to the test result ( $\chi^2$ =310.020, p<0.05, df=6), we see that there is a significant association between decision of marriage and gender. Whereas, the Cramer's V measure (0.105) shows a weak association between the decision of marriage and gender.

**Table 5:** Decision of marriage by gender (%).

| Decision | Male | Female | Total |
|---|---|---|---|
| Arranged marriage with the decision of my family, without taking my opinion | 8.8 | 15.1 | 12.4 |
| Arranged marriage, my decision | 48.9 | 47.8 | 48.3 |
| My own decision, without my family's consent | 2.5 | 2.6 | 2.5 |
| My decision, with the approval of my family | 32.9 | 26.9 | 29.4 |
| Eloping / being eloped | 6.7 | 7.3 | 7.0 |
| Bride exchange | 0.2 | 0.3 | 0.3 |
| Other | 0.0 | 0.0 | 0.0 |

Additonally, in Table 6, the percentages for the question "how was the form of solemnization" by gender are represented. 97% of the married men and women have both civil and religious ceremonies. In order to see that whether there is an association between form of solemnization and gender, the chi-square test is performed. According to the test result ( $\chi^2$ =23.354, p<0.05, df=2), it is found that there is a significant association between form of solemnization and gender. As the Cramer's V measure (0.029) is quite small, it can be interpreted that the association between form of solemnization and gender is very weak.

**Table 6:** Consanguineous marriage (%).

| Answer | Male | Female | Total |
|---|---|---|---|
| Yes | 21.4 | 22.4 | 21.9 |
| No | 78.6 | 77.6 | 78.1 |

Then, the next question of interest is about the degree of kinship for those people who has a consanguineous marriage. The percentages are listed in Table 7 based on the degree types of the kinship. In the table, the degree of kinship is considered under two categories, namely, relative from the father's side and relative from the mother's side. By mergingthe paternal and maternal relatives, it is seen that the degree of kinship from the father's side (58.8%) has a higher frequency from the degree of kinship from the mother's side (41.1%).

**Table 7:** Degree of kinship between spouses (%).

| Degree of Kinship | Percent |
|---|---|
| Son / daughter of paternal uncle | 15.6 |
| Son / daughter of paternal aunt | 10.4 |
| Son / daughter of maternal uncle | 10.6 |
| Son / daughter of maternal aunt | 11.4 |
| Other relative from the father's side | 32.8 |

| | |
|---|---|
| Other relative from the mother's side | 19.1 |
| Other | 0.1 |

On the oher hand, Table 8 shows the percentages for different ways of meeting with the spouse by gender. The most common way to meet their spouses is family and relative network for men and women with percentages 47% and 50%, respectively. After testing via the chi-square test, it is found that there is a significant association between way of meeting the spouse and gender ($\chi^2$ =37.990, p<0.05, df=6). Also by 0.037 Cramer's V measure, it can be stated that the strength of the association is quite weak.

**Table 8:** Way of meeting the spouse by gender (%).

| Way of Meeting | Male | Female | Total |
|---|---|---|---|
| Family / relative network | 46.8 | 50.2 | 48.8 |
| Neighbor network | 30.3 | 28.8 | 29.5 |
| School / course network | 4.0 | 3.7 | 3.9 |
| Work network | 7.7 | 6.5 | 7.0 |
| Friends network (outside school and work) | 10.4 | 10.1 | 10.2 |
| Internet | 0.4 | 0.3 | 0.3 |
| Other | 0.3 | 0.3 | 0.3 |

The opinion about the consanguineous marriage was also asked. Table 9 shows percentages for the answers by gender. In total, for the 14% of the respondents this type of marriage is proper while for the 86% of the them this type of marriage is not proper. According to the Chi-Square test ($\chi^2$ =7.231, p<0.05, df=1) , there is a significant gender effect for the opinion about the consanguineous marriage. The association between the opinion about the consanguineous marriage and gender is very weak as the Cramer's V measure is quite small (0.014).

**Table 9:** Idea for consanguineous marriage by gender (%).

| Idea | Male | Female | Total |
|---|---|---|---|
| Proper | 14.5 | 13.5 | 13.9 |
| Not proper | 85.5 | 86.5 | 86.1 |

Then, if the individuals thought that consanguineous marriage was proper, they are asked "what is the reason for this type of marriage?" and given multiple choices. Table 10 shows percentages for different answers by gender. The most frequent answer is "know and protect of family origins" with 49% percentage. From the ch-square test, it is seen that there is a significant association between the reasons for consanguineous marriage and gender ($\chi^2$ =41.660, p<0.05, df=5) and the reason for this type of marriage changes by gender. The strength of the association is weak (Cramer's V measure=0.092).

**Table 10:** Reason for consanguineous marriage by gender (%)

| Reason | Male | Female | Total |
|---|---|---|---|
| Non-division of assets | 2.1 | 1.5 | 1.8 |
| Know/protection of family origins | 48.7 | 49.5 | 49.1 |
| Better understanding of relative children | 20.7 | 26.5 | 23.8 |
| More respect for family elders in consanguineous marriages | 10.3 | 9.1 | 9.7 |
| Protection of traditions and customs | 14.6 | 10.9 | 12.6 |
| Other | 3.6 | 2.3 | 2.9 |

Furthermore, Table 11 shows the percentages for different ways of meeting with the spouse by gender. The most common way to meet their spouses is family and relative network for men and women with percentages 47% and 50%, respectively, and other choices are presented in the table. After testing via the chi-square test, it is found that there is a significant association between way of meeting the spouse and gender ( $\chi^2$ =37.990, p<0.05, df=6) and due to the small Cramer's V measure (0.037), it can be stated that the strength of the association is quite weak.

**Table 11:** Way of meeting the spouse by gender (%).

| Way of Meeting | Male | Female | Total |
|---|---|---|---|
| Family / relative network | 46.8 | 50.2 | 48.8 |
| Neighbor network | 30.3 | 28.8 | 29.5 |
| School / course network | 4.0 | 3.7 | 3.9 |
| Work network | 7.7 | 6.5 | 7.0 |
| Friends network (outside school and work) | 10.4 | 10.1 | 10.2 |
| Internet | 0.4 | 0.3 | 0.3 |
| Other | 0.3 | 0.3 | 0.3 |

When the social qualities for the future spouse are investigated, the percentages in Tables 12 and 13 are obtained. Hereby, regarding the tabulated values, it is seen that for women the most important feature is having a job with 92% percentage. The other important qualities for women are "similarity of family structures" with 88% percentage, "not being married before" with 84% percentage, "being religious" with 83% percentage and "good education" with 76% percentage. On the other hand, from the answers of men (Table 13), "not being married before" is highly important with 86% (the sum of important and very important options) percentage. "Similarity of family structures", and "being religious" are the other two most important qualities for men with percentages 84% and 76%.

**Table 12:** Social qualities sought in a future spouse for women (%).

| Social Qualities | Not at All Important | Not Important | Moderately Important | Important | Very Important |
|---|---|---|---|---|---|
| Good education | 0.8 | 11.2 | 12.1 | 58.7 | 17.1 |
| High income | 1.9 | 26.8 | 25.4 | 40.7 | 5.2 |
| Has a job | 0.2 | 3.0 | 4.7 | 68.8 | 23.3 |
| Works shorter hours | 1.4 | 15.4 | 18.2 | 57.7 | 7.4 |
| Not being married before | 1.3 | 9.8 | 4.9 | 63.7 | 20.3 |
| Similarity of family structures | 0.5 | 5.6 | 6.0 | 72.5 | 15.3 |
| Being religious | 1.1 | 7.0 | 9.0 | 65.6 | 17.3 |
| From the same religious sect | 2.2 | 14.8 | 6.9 | 63.9 | 12.2 |
| From the same hometown | 7.7 | 42.7 | 8.6 | 36.7 | 4.4 |
| From the same social circle | 3.4 | 25.6 | 12.3 | 53.5 | 5.2 |
| From the same ethnic origin | 3.6 | 24.2 | 10.3 | 55.9 | 6.0 |
| Has similar political view | 6.1 | 33.6 | 11.0 | 44.5 | 4.8 |

Thereby, in order to see whether there is an association between gender and importance level of social qualities sought in a future spouse from the results of Table 11 and 12, we compute the chi-square test for each social quality and gender. The results are represented in Table13 with 4 degrees of freedom. The tabulated values show that an association occurs between gender and each social quality sought in a future spouse. The importance level of each social quality sought in a future spouse differs with respect to gender of the respondent.

**Table 13:** Comparison of social qualities sought in a future spouse with respect to the gender.

| Social Qualities | $\chi^2$ statistic | *p*-value |
|---|---|---|
| Good education | 366.054 | .000* |
| High income | 2235.548 | .000* |
| Has a job | 7627.942 | .000* |
| Works shorter hours | 872.816 | .000* |
| Not being married before | 154.276 | .000* |
| Similarity of family structures | 180.735 | .000* |
| Being religious | 316.318 | .000* |
| From the same religious sect | 562.916 | .000* |
| From the same hometown | 307.389 | .000* |
| From the same social circle | 216.546 | .000* |
| From the same ethnic origin | 393.104 | .000* |
| Has similar political view | 678.634 | .000* |

Then, to detect the association between social qualities sought in a future response, we calculate the Kendall's Tau-B measure. This measure is a nonparametric measure of correlation for ordinal variables. The computed associations are represented in Table 14. From the outcomes, it is observed that there are significant associations between all social qualities sought in a future response at 0.01 significance level except the association between "from the same hometown" and "good education". As it is seen in the Table 14, three highest associations are between "from the same ethnic origin" and "from the same social circle" (0.527), "from the same hometown" and "from the same social circle" (0.508), "being religious" and "from the same religious sect" (0.490). According to these associations, it can be said that having same religious opinion and similar life style are associated characteristics for people and their idea about potential spouses.

**Table 14:** Association between social qualities sought in a future spouse.

| | Good educa-tion | High income | Has a job | Works shorter hours | Not being married before | Similarity of family structures | Being religious | From the same religious sect | From the same hometown | From the same social circle | From the same ethnic origin | Has similar political view |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Good education | 1.000 | .283* | .351* | .240* | .127* | .185* | .078* | .049* | -.002 | .086* | .044* | .067* |
| High income | .283* | 1.000 | .364* | .286* | .046* | .096* | .074* | .118* | .178* | .149* | .136* | .162* |
| Has a job | .351* | .364* | 1.000 | .334* | .101* | .200* | .096* | .112* | .027* | .095* | .083* | .092* |
| Works shorter hours | .240* | .286* | .334* | 1.000 | .089* | .145* | .064* | .079* | .067* | .107* | .081* | .117* |
| Not being married before | .127* | .046* | .101* | .089* | 1.000 | .308* | .344* | .246* | .069* | .102* | .150* | .057* |

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Similarity of family structures | .185* | .096* | .200* | .145* | .308* | 1.000 | .331* | .279* | .093* | .227* | .208* | .129* |
| Being religious | .078* | .074* | .096* | .064* | .344* | .331* | 1.000 | **.490*** | .175* | .194* | .278* | .137* |
| From the same religious sect | .049* | .118* | .112* | .079* | .246* | .279* | **.490*** | 1.000 | .314* | .306* | .429* | .235* |
| From the same hometown | -.002 | .178* | .027* | .067* | .069* | .093* | .175* | .314* | 1.000 | **.508*** | .447* | .357* |
| From the same social circle | .086* | .149* | .095* | .107* | .102* | .227* | .194* | .306* | **.508*** | 1.000 | **.527*** | .365* |
| From the same ethnic origin | .044* | .136* | .083* | .081* | .150* | .208* | .278* | .429* | .447* | **.527*** | 1.000 | .366* |
| Has similar political view | .067* | .162* | .092* | .117* | .057* | .129* | .137* | .235* | .357* | .365* | .366* | 1.000 |

Moreover, Table 15 and Table 16 indicare the percentage of acceptability of some statements about marriage and social quality of spouse for men and women. In this part, the individuals evaluate their agreements for each statement and results are obtained as in Table 15 and Table 16. From the findings it is seen that the men are disagreeing for "live together without getting married" with 89% percentages (the sum of strongly disagree and disagree) and "having child out of wedlock" with 94% percentages. The men are also disagreeing for "marriage with a person from the internet" with 77% percentage.

**Table 15:** Acceptability of some statements about marriage for men (%).

| Acceptability of Some Statements | Strongly Disagree | Disagree | Slightly Agree | Agree | Strongly Agree |
|---|---|---|---|---|---|
| Live together without getting married | 43.9 | 45.4 | 3.0 | 5.9 | 1.8 |
| Marriage of men with a woman of different religions and nationalities | 17.6 | 34.6 | 10.1 | 35.5 | 2.3 |
| Marriage of woman with a men of different religions and nationalities | 20.6 | 39.9 | 8.7 | 29.0 | 1.8 |
| Having child out of wedlock | 44.8 | 48.5 | 1.6 | 3.8 | 1.2 |
| Marriage with a person from the internet | 28.2 | 49.2 | 10.6 | 11.0 | 0.9 |
| Marriage with a person from different religious sect | 12.1 | 32.0 | 14.4 | 38.9 | 2.5 |

On the other side, according to Table 16, the women are disagreeing for "live together without getting married" with 93% percentages (the sum of strongly disagree and disagree) and "having child out of wedlock" with 95% percentages. The women are also disagreeing for "marriage with a person from the internet" with 84% percentage.

Table 17 and Table 18 present the acceptability of some statements about marriage for men and women, respectively. In order to see whether there is an association between gender and acceptability of given statements, the chi-square test is performed for each statement and gender. The results are represented in Table 17 with 4 degrees of freedom. The Chi-Square tests are significant for all of the statements, it means there is an association between gender and each statements about marriage. The acceptability level of each statement about marriage differs with respect to gender of the respondent.

**Table 16:** Acceptability of some statements about marriage for women (%).

| Acceptability of Some Statements | Strongly Disagree | Disagree | Slightly Agree | Agree | Strongly Agree |
|---|---|---|---|---|---|
| Live together without getting married | 48.5 | 43.6 | 2.4 | 4.0 | 1.5 |
| Marriage of men with a woman of different religions and nationalities | 23.4 | 40.2 | 9.6 | 25.3 | 1.6 |
| Marriage of woman with a men of different religions and nationalities | 25.3 | 44.0 | 8.5 | 20.7 | 1.4 |
| Having child out of wedlock | 49.2 | 45.9 | 1.4 | 2.4 | 1.1 |
| Marriage with a person from the internet | 33.6 | 50.3 | 8.0 | 7.3 | 0.9 |
| Marriage with a person from different religious sect | 17.3 | 38.6 | 14.2 | 28.1 | 1.7 |

**Table 17:** Comparison of acceptability of some statements about marriage with respect to gender.

| Acceptability of Some Statements | $\chi^2$ statistic | *p*-value |
|---|---|---|
| Live together without getting married | 135.046 | .000* |
| Marriage of men with a woman of different religions and nationalities | 547.129 | .000* |
| Marriage of woman with a men of different religions and nationalities | 379.611 | .000* |
| Having child out of wedlock | 104.646 | .000* |
| Marriage with a person from the internet | 289.609 | .000* |
| Marriage with a person from different religious sect | 601.677 | .000* |

In order to see the association between acceptability level of statements about marriage, the Kendall’s Tau-B measure is calculated. The significant associations between acceptability level of statements about marriage and the strength of the significant associations are represented in Table 18. As seen in the tabulated values, there are significant associations between all statements about marriage at 0.01 significance level and the three highest associations are related with the religious opinion (0.821 and 0.466) and the life style related with marriage and having children.

**Table 18:** Association between agreement level of statements about marriage.

| | Live together without getting married | Marriage of men with a woman of different religions and nationalities | Marriage of woman with a men of different religions and nationalities | Having child out of wedlock | Marriage with a person from the internet | Marriage with a person from different religious sect |
|---|---|---|---|---|---|---|
| Live together without getting married | 1.000 | .342* | .400* | **.713*** | .404* | .222* |
| Marriage of men with a woman of different religions and nationalities | .342* | 1.000 | **.821*** | .297* | .298* | .466* |

| | | | | | | |
|---|---|---|---|---|---|---|
| Marriage of woman with a men of different religions and nationalities | .400* | **.821*** | 1.000 | .366* | .327* | .443* |
| Having child out of wedlock | **.713*** | .297* | .366* | 1.000 | .459* | .212* |
| Marriage with a person from the internet | .404* | .298* | .327* | .459* | 1.000 | .298* |
| Marriage with a person from different religious sect | .222* | **.466*** | .443* | .212* | .298* | 1.000 |

Additionally, in order to understand about the qualities sought in a future spouse, individuals were asked "which personal qualities below are important for you in a future spouse and how important are they?"

In order to test association between gender and importance of personal qualities with two levels, the frequencies of first two and last three importance level are summed to have two answers as "not important" and "important" and then, the chi-square test is performed again for each personal quality and gender. The results are represented in Table 19. From the tabluted values, it is shown that there is no significant association between gender and personal qualities sought in a future spouse listed as taking care of herself/himself, fidelity to partner, being reliable/not lying, giving importance to partner's feelings, being thrifty, patience/tolerance and behaving properly in society. Moreover, almost all of the personal qualities are considered as important with very high frequencies (about 99%) except being beautiful/handsome for both gender. Being beautiful/handsome is found as not important by males with 32% and females with 38%.

**Table 19:** Comparison of personal qualities sought in a future spouse with two levels by gender

| | Male (%) | | Female (%) | | $\chi^2$ statistic | *p*-value |
|---|---|---|---|---|---|---|
| | Not Important | Important | Not Important | Important | | |
| In love | 12.9 | 87.1 | 14.6 | 85.4 | 21.620 | .000* |
| Beautiful / Handsome | 32.4 | 67.6 | 38.3 | 61.7 | 132.193 | .000* |
| Taking care of herself/himself (paying attention to his/her personal hygiene) | 2.1 | 97.9 | 2.1 | 97.9 | .088 | .767 |
| Fidelity to partner | 0.7 | 99.3 | 0.7 | 99.3 | .002 | .961 |

| | | | | | | |
|---|---|---|---|---|---|---|
| Enjoying the spend time with family | 1.2 | 98.8 | 0.9 | 99.1 | 7.970 | .005* |
| Being reliable / not lying | 0.6 | 99.4 | 0.5 | 99.5 | .002 | .966 |
| Giving importance to his/her partner’s feelings | 0.6 | 99.4 | 0.6 | 99.4 | .000 | .992 |
| Being thrifty | 1.2 | 98.8 | 1.2 | 98.8 | .209 | .647 |
| Generosity | 1.9 | 98.1 | 1.1 | 98.9 | 39.850 | .000* |
| Patience and tolerance | 0.4 | 99.6 | 0.5 | 99.5 | 1.178 | .278 |
| Protecting the partner against his/her own family | 1.1 | 98.9 | 0.6 | 99.4 | 22.499 | .000* |
| Behaving properly in society | 0.5 | 99.5 | 0.5 | 99.5 | .864 | .353 |

Finally, in order to see the association between personal qualities sought in a future response, the Kendall’s Tau-B is calculated. The significant associations between personal qualities sought in a future spouse and the strength of the significant associations are presented in Table 20. As seen in the table, there are significant associations between all personal qualities sought in a future response and gender at 0.01 significance level and three highest associations are between “fidelity to partner” and “being reliable / not lying” (0.798), “giving importance to his/her partner’s feelings” and “being reliable / not lying” (0.794), “protecting the partner against his/her own family” and “behaving properly in society” (0.745).

**Table 20:** Association between personal qualities sought in a future spouse.

| | In love | Beautiful / Handsome | Taking care of herself/himself | Fidelity to partner | Enjoying the spend time with family | Being reliable / not lying | Giving importance to his/her partner’s feelings | Being thrifty | Generosity | Patience and tolerance | Protecting the partner against his/her own family | Behaving properly in society |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| In love | 1.000 | .270* | 0296 * | .254* | .236* | .234* | .253* | .191* | .189* | .220* | .221* | .233* |
| Beautiful / Handsome | .270* | 1.000 | .145* | .061* | .073* | .049* | .067* | .070* | .076* | .061* | .058* | .058* |

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Taking care of herself/himself | .296* | .145* | 1.000 | .596* | .558* | .550* | .548* | .444* | .433* | .519* | .487* | .517* |
| Fidelity to partner | .254* | .061* | .596* | 1.000 | .699* | **.798*** | .718* | .528* | .480* | .671* | .592* | .653* |
| Enjoying the spend time with family | .236* | .073* | .558* | .699* | 1.000 | .708* | .735* | .582* | .565* | .654* | .611* | .629* |
| Being reliable / not lying | .234* | .049* | .550* | **.798*** | .708* | 1.000 | **.794*** | .567* | .522* | .727* | .639* | .704* |
| Giving importance to his/her partner’s feelings | .253* | .067* | .548* | .718* | .735* | **.794*** | 1.000 | .632* | .609* | .740* | .675* | .716* |
| Being thrifty | .191* | .070* | .444* | .528* | .582* | .567* | .632* | 1.000 | .717* | .664* | .581* | .590* |
| Generosity | .189* | .076* | .433* | .480* | .565* | .522* | .609* | .717* | 1.000 | .659* | .595* | .577* |
| Patience and tolerance | .220* | .061* | .519* | .671* | .654* | .727* | .740* | .664* | .659* | 1.000 | .717* | .753* |
| Protecting the partner against his/her own family | .221* | .058* | .487* | .592* | .611* | .639* | .675* | .581* | .595* | .717* | 1.000 | **.745*** |
| Behaving properly in society | .233* | .058* | .517* | .653* | .629* | .704* | .716* | .590* | .577* | .753* | **.745*** | 1.000 |

## 4. Conclusion

From the survey analyses, we have obtained certain interesting results. The summary of the results can be listed as below:

- As most of the variables are in nominal or ordinal scale we have obtained frequency distributions, checked the statistical significance of their answers in gender and correlations.
- We have seen that majority of Turkish people has single marriage and prefers both civil and religious ceremonies.
- The consanguineous marriage is not very common and still has around 22% with mainly relatives of father side.
- The spouse is chosen from family and neighborhood networks and taking into account similarities in family structures (both gender), not married before (for men) and having job and being religious (for women).
- Both gender say that taking care of herself/himself (paying attention to his/her personal hygiene), enjoying the spend time with family and being thrifty are the most important characters for the selection of spouses, and beauty as well as love are the least important features for both gender.

As the extension of this study, we aim to compare the counts performing statistical testing procedures also considering other categorical variables such as educational status, socioeconomic status and territories. Also, we aim to continue with the modeling of this dataset (Hosmer et al., 2013; O’Connell, 2006) and investigate the effects of these preferences on divorce occurrence in Turkey.

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# Assessment of Low Achievers in Solving 'Guessing Game' Combine Problems

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**Abstract**

All teachers encounter children with different abilities. It is a constant challenge and teachers do their best to identify the poor performers and understand the reasons for their poor performance.This study arises out of research in a school where the majority of students, aged between seven and nine, had experienced substantial difficulty in problem-solving in mathematics. The study was designed to discover why such a large body of students had so many difficulties in solving what, to their peers elsewhere, might be seen as simple arithmetic problems well within their capabilities. The study was designed with guessing games to discover the extent of the students' difficulties and the reasons for them. 27 students were studied over a a period of many months. Each was interviewed face-to-face. All interviews were videotaped and the recordings were studied and analysed and relevant data extracted.

The guessing game was found to be particularly illuminating as regards a child's ability to understand part-whole relation. The child was asked to guess what could be the possible combinations of numbers within boxes. In this game, through the different guesses that the child freely makes, the researcher can find out how different numbers relate in their conception of number.

The findings disclosed a number of problems in both procedural and conceptual knowledge, the details of which, it was felt, could help the teachers understand the symptoms of the poor performance and the reasons for it so that they could then design a suitable remedial programme.

**Keywords:** problem-solving, part-whole relation, guessing game, procedural knowledge, conceptual knowledge

## 1. Introduction

One of the most problematic areas of the mathematics curriculum involves solving word problems. Many students experience considerable difficulty with simple word problems. Previous studies in mathematics education emphasize the effectiveness of instruction, focused on teaching strategies, to overcome the students' difficulties (Jordan, Kaplan, Olah & Locuniak, 2006; Henjes, 2007).

Carpenter and Moser (1982) stated that children need to have certain prerequisite conceptual knowledge, such as part-whole relationship, in order to understand and solve simple word problems. Riley, Greeno and Heller (1983) also found that, 'improvement in performance results mainly from improved understanding of certain conceptual relationships." (p.154). My research method was to reveal the conceptual knowledge and procedural knowledge of the children, these being the main areas of their mathematical ability (e.g. Kilpatrick, Swaffords and Findell, 2001; Rittle- Johnson and Star, 2007; Bottage, Rueda, LaRoque; Serlin Kwon, 2007). I did this using carefully chosen problems based, to an extent, on Neuman's work (1987) –guessing game, but with significant differences to probe into the children's abilities in each problem type in greater depth. Neuman (1987) used her number problems to ascertain children's conceptions of numbers and, I used similar problems to examine children's difficulties in mathematics.

My study was to examine what those difficulties were and why they had them – then it would better help the teachers establish how to address those difficulties. In other words, the better we understand the symptoms of poor performance in detail the better we are able to prescribe remedial action. The teachers can then plan their future teaching with a particular eye on the areas where the weaknesses may appear and hence be addressed.

## 2. Preliminaries

The guessing game was the most powerful problem devised by Neuman.The guessing game was found to be particularly illuminating as regards a child's ability to understand part-whole relation. 9 coins were separated into 2 groups unknown to the child and put into two boxes. The child was asked to guess what could be the possible combinations of numbers of the buttons within 2 boxes. Although children may guess any number of combinations the goal is for the child to see whether the child can attend to the one hidden invariable: the whole, namely the number of coins distributed between the two boxes (e.g., 9), yet at the same time, to the parts which can vary with the certain inter- dependence relation between them, so that together they made 9. In this game, through the different guesses that the child freely makes, the researcher can find out how different numbers relate in their conception of number. This is how I see the beauty of the guessing game.

In my study, I did not set out list and test all the fundamental skills which students require in order to study mathematics, rather, I observed closely my students' performance in attempting to solve the carefully-chosen problems I presented to them. In this way I was able to identify problems that were specific to each child and which seemed to be common to the vast majority of them. Thus, I do not pretend to have revealed all the fundamental issues that my students might have faced but, what I do say, is that my study focused on some of fundamental problems inhibiting the learning of mathematics. I focused on the extent to which my students displayed difficulties in procedural and conceptual knowledge. Fundamental to the learning of mathematics is having a wide range of procedures available to a child, as is the ability to use those procedures well. As important, is the ability to choose the appropriate procedure in the circumstances. I also consider it to be of fundamental importance that students have conceptual knowledge – in particular as regards an understanding of the part-whole relationship – both in theory (at school) and in practice (in the students' day-to-day lives).

My study concentrates on the personal think-aloud interviews with 27 students to find out how difficulties arise when the children solve simple mathematics word problems. By adopting this approach, it was possible to explore whether they lacked conceptual or procedural knowledge, or both and whether they did not know enough procedures or how to operate those procedures. In short, the purpose of my study is to get to the bottom of their problems on the basis that there is no point in prescribing a remedy until you know what you are treating. Although I do consider implications, I do not put these forward as solutions nor an intervention plan - that, I felt is best left to their teachers.

Research question:

To explore the difficulties encountered by students, the study was designed to address the following question:

How did the students discern the possible parts in a given number in guessing game combination problem?

## 3. Main Results

Different ways of handling the guessing games by students were noticed. Some children just guessed – on some occasions, guessing correctly. Others didn't guess at all but used other methods to try and solve the problem. In general, the strategies can be split into the six categories below.

Commutative pairs

This indicates whether the student's answer suggests a possibility of their knowledge of the principle of commutativity (e.g., if 4+5 is one answer then 5+4 should also be there).

Unique matching number

Some students did not understand the uniqueness in the combination of numbers. They tried with the same number as one part several times with different numbers as the second part (such as 4, 4 and 4, 3). This indicates whether a child was able to understand that it should be a unique answer for the second part.

Number facts

These are basic subtraction and addition facts that children should have learnt to recall immediately without having to work the problems out. In short, they should know them off by heart. In the context of the guessing game, this relates to a student's knowing, automatically, for example, that a certain combination would produce a certain total (for example 6+3 = 9). Among the different combination, the half-half combination (such as, 3, 3 for the total 6 and 4, 4 for the total 8) seemed to be most easy to recall.

Graphical representation/Assisting ways

This relates to the child's use of graphics or objects (in one form or another) to help. It might be, for example, drawing circles or lines. The way that they used their graphics was most instructive.

a. Most students would draw circles or lines and according to their guessed parts, then use counting to see if, when combined, these two sets equaled the hidden total number.

b. The other way was to draw the total number of coins and try to group them into two parts in different ways.

'Pure' guess

a. By observing carefully a student's demeanor, way of speaking and the speed with which they made their 'guesses' I felt able to reach a conclusion as to their thought processes. For example. Some of them stared at the box and tried to imagine how many button were inside.

b. Sometimes, a child gave me a 'guess' that was wildly inaccurate suggests that no real thought had gone into it. For example, when the total number of coins divided between the two boxes was 9, if a student's guess was Box 1 had 6 and Box 2 had 19 without any apparent attempt to estimate or calculate, then I saw this as a 'pure' guess. Some did not add up the total number which was equal to the original number of coins that I have given at the beginning, they just give any random number. Some students even seemed to have no idea at all how to find out the numbers of the two boxes. They just randomly guessed any number, it seemed.

Systematic variation

By 'systematic variation' I do not refer to my changing the numbers of coins to be used in the guessing game problem. I am talking about the way in which some children, themselves, used the strategy of varying combinations of numbers in a systematic way by, for example starting with a correct combination then reducing one from the first addend and adding one to the second addend. This could be seen in certain combinations (4, 5; 5, 4; 6, 3; 7, 2 and 8, 1).

## 4. Conclusion

It might be thought that the scenario posed by the guessing game would be quite unfamiliar to students and would, therefore, cause them difficulties. That proved to be true but, in unfamiliar situations a teacher is able to see how students' think and solve problems. It was discovered that few students used mathematical concepts as number sentences to help to find the answer, instead, they use concepts which were more native to them.

As the guessing game problem is posed in an open way, it stimulates students' thinking and encourages the children to try and make sense of what is an open situation and turn it into mathematical knowledge and translate that into an appropriate number sentence. It also makes them think what would be the best procedure for solving the problem. It allows the teacher to see what kind of part-whole relationship the students thought of. The teacher can see how student construct the part-whole relationship through the guessing process. From the process, the teacher can discover if the student can think from certain pairs of correct combinations to other combinations. For other students, the different combinations might be well linked together. Some know how to adjust the parts without changing the whole. I noted that some could guess one pair of combinations only, believing this to be the only answer for two boxes. Some could guess one pair of combinations and then decrease one part and increase another part to make the same whole. Or, if they got a wrong pair of combinations, some students would know how to adjust and make another correct pair of combinations. From all these different approaches, I could gain insight into how students think of the partwhole relationship.

The guessing game is undoubtedly most useful for promoting an understanding of important concepts such as part-whole. It encourages the children to make the connections necessary to enhance their conceptual and procedural knowledge. This, in my view, encourages a child to think more widely than when they are faced with a closed question.

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**INTERNATIONAL CONFERENCE ON MATHEMATICS**
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**Parafree Metabelian Nilpotent Lie Algebras**

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**Abstract**

In this work, we define parafree metabelian nilpotent Lie algebras. Moreover using direct limit of free Lie algebras, we prove that under some conditions a parafree metabelian nilpotent Lie algebra is isomorphic to a free metabelian nilpotent Lie algebra.

**Keywords:** Parafree, Metabelian, Nilpotent, Directed system.

## 1.Introduction

Baumslag has introduced the notion of parafree groups and has got some important results about parafree groups [1,2,3,4]. Baumslag has taken his results [5,6] on one relator groups. Because of the close relationship between groups and Lie algebras, one would expect that parafree Lie algebras enjoy properties that are analogous to those of parafree groups. We have taken this opportunity to obtain some results about parafree Lie algebras. Parafree Lie algebras firstly arise in the works of Baur [7,8], Knus and Stammbach [9]. They have obtained basic in the structure of parafree Lie algebras. We carry the formal arguments used in [2] over to parafree Lie algebras. More exactly we prove that under some conditions a parafree metabelian nilpotent Lie algebra is isomorphic to a free metabelian nilpotent Lie algebra.

## 2. Preliminaries

Let L be a Lie algebra over a field k of characteristic zero. The lower central series of L

$$L = \gamma_1(L) \supseteq \gamma_2(L) \supseteq \cdots \gamma_n(L) \supseteq \cdots$$

is defined inductively by

$$\gamma_2(L) = [L, L],\ \gamma_{n+1}(L) = [\gamma_n(L), L],\ n\geq 1.$$

If n is the smallest integer satisfying $\gamma_n(L)= \{0\}$, then L is called nilpotent of class n.

The second term $\gamma_2(L)$ is called the derived subalgebra of L and it is denoted by $L'$. The derived subalgebra of $L'$ is denoted by $L''$. L is called metabelian, if $L'' = \{0\}$.

**Definition 2.1.** A Lie algebra L is called residually nilpotent if

$$\bigcap_{n=1}^{\infty} \gamma_n(L) = \{0\}.$$

We associate the lower central series of L with its lower central sequence: $L/_{\gamma_2(L)}, L/_{\gamma_3(L)}, \ldots$. We say that two Lie algebras L and H have the same lower central sequence if $L/_{\gamma_n(L)} \cong H/_{\gamma_n(H)}$ for every $n \geq 1$.

**Definition 2.2.** The Lie algebra L is called parafree over a set X if,

i) L is residually nilpotent, and
ii) L has the same lower central sequence as a free Lie algebra generated by the set X.

The cardinality of X is called the rank of L.

One of the crucial definition in this work is that the direct limit of Lie algebras.

**Definition 2.3.** Let $(I, \leq)$ be a partially ordered set. Then $(I, \leq)$ is a directed set if for any elements i,j $\in$I, there exists an element k$\in$I such that i$\leq$k ve j$\leq$k.

**Definition 2.4.** Let $(I, \leq)$ be a directed set, and let $\{A_i\}_{i \in I}$ be a collection of Lie algebras indexed by I and $\varphi_{ij}: A_i \rightarrow A_j$ be a homomorphism for all i,j $\in$I such that i$\leq$j with the following properties:

i) $\varphi_{ii}$ is the identity of $A_i$, and
ii) $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ for all i,j,k $\in$I such that i$\leq$j$\leq$k.

Then the pair $\left(\{A_i\}_{i \in I}, \{\varphi_{ij}\}_{i<j}\right)$ is called a directed system over I.

**Definition 2.5.** Let $\left(\{A_i\}_{i \in I}, \{\varphi_{ij}\}_{i<j}\right)$ be a directed system over I. The direct limit $\lim_{\rightarrow} A_i$ is a Lie algebra L such that it is unique up to isomorphism and satisfies the following universal mapping property:

i) For all i,j $\in$I and i $\leq$j there are mappings $\varphi_i: A_i \rightarrow L$ such that $\varphi_i = \varphi_j \circ \varphi_{ij}$.

ii) If there is a Lie algebra C together with maps $\tau_i : A_i \to C$ such that $\tau_i = \tau_j \circ \varphi_{ij}$, for each $i \leq j$ then there exists a unique Lie algebra homomorphism $\tau : L \to C$ such that $\tau_i = \tau \circ \varphi_i$.

We give an alternative definition of direct limit:

Let $\left(\{A_i\}_{i\in I}, \{\varphi_{ij}\}_{i<j}\right)$ be a directed system. The direct limit of this system is defined as the disjoint union of the $A_i$ is modülo a certain equivalence relation $'\sim'$. Denote the set of equivalence classes by $\bigcup_n A_n \,/\!\sim$ .

Here, if $a_i \epsilon A_i$ and $a_j \epsilon A_j$, $a_i \sim a_j$ if there is some k $\epsilon$I such that $\varphi_{ik}(a_i) = \varphi_{jk}(a_j)$. Clearly $\sim$ is an equivalence relation. We will write $\bar{a}_i$ for the equivalence class for an element $a_i \epsilon A_i$. The set of equivalence classes is a Lie algebra with the operation defined by

$$[\bar{a}_\imath, \bar{a}_j] = \left[\overline{\varphi_{\imath k}(a_\imath), \varphi_{\jmath k}(a_\jmath)}\right]$$

This Lie algebra has the same mapping property as does the direct limit. Hence

$$\lim_{\to} A_i = \bigcup_i A_i \,/\!\sim \qquad (1).$$

The proof of (1) is the same as in the group case (see [10]). A routine exercise involving universal mapping properties shows that the direct limit of a Lie algebra if it exists, is unique up to isomorphism.

By the following theorem, one can investigate the direct limit of free Lie algebras and see that direct limit of free Lie algebras is parafree.

**Theorem 2.1.** Let $F_\alpha$ be the free Lie algebra generated by free generators $a_\alpha$ and $b_\alpha$ for $\alpha \geq 1$. Then P=$\bigcup_{\alpha\epsilon I} F_\alpha$ is parafree.

Indeed, let I be a directed set and $F_\alpha$ be the free Lie algebra generated by free generators $a_\alpha$ and $b_\alpha$ for $\alpha \in I$. Consider the homomorphism $\Phi_{\alpha\beta}$ from $F_\alpha$ into $F_\beta$ where $\alpha, \beta \in I, \alpha < \beta$. Then

$$\left(\{\mathrm{F}_\alpha\}_{\alpha\in \mathrm{I}}, \{\Phi_{\alpha\beta}\}_{\alpha<\beta}\right)$$

is a directed system. Let P be the direct limit of this system. Now we consider the equivalence relation "$\sim$"on $\bigcup_{\alpha\in\mathrm{I}} \mathrm{F}_\alpha$, which is defined in the definition of the direct limit. A short calculation shows that the set of the equivalence classes $\bigcup_{\alpha\in\mathrm{I}} \mathrm{F}_\alpha/_\sim$ is equal to $\bigcup_{\alpha\in\mathrm{I}} \mathrm{F}_\alpha$.

Therefore by (1), we obtain

$$\lim_{\rightarrow} F_\alpha = \bigcup_{\alpha\in\mathrm{I}} \mathrm{F}_\alpha.$$

Hence P may be viewed as the union of its subalgebras $\mathrm{F}_\alpha$. So P=$\bigcup_{\alpha\in\mathrm{I}} \mathrm{F}_\alpha$. The parafreeness of P and the details of the proof can be found in [12].

## 3. Main Results

**Definition 3.1** Let L be a Lie algebra over a field k of characteristic zero. A Lie algebra $\mathrm{L}/_{L''} + \gamma_{\mathrm{n}}(\mathrm{L})$ is called metabelian nilpotent of class n.

**Theorem 3.1.** A quotient algebra of a parafree Lie algebra is parafree.

**Proof.** The proof can be found in [11].

**Corollary 3.1.** Let P be a parafree Lie algebra. Then the algebra $\mathrm{P}/_{P''} + \gamma_{\mathrm{n}}(\mathrm{P})$ is parafree.

Therefore one can call this algebra as parafree metabelian nilpotent of class n Lie algebra.

**Theorem 3.2.** Let F be a free Lie algebra freely generated by the set $\{\mathrm{a},\mathrm{b}\}$. Then there exists a parafree Lie algebra P of rank two with the following properties:

i) P is the union of Lie algebras of rank two.
ii) $\mathrm{P}/_{P''} + \gamma_{\mathrm{n}}(\mathrm{P}) \cong \mathrm{F}/_{F''} + \gamma_{\mathrm{n}}(\mathrm{F})$,
iii) P is not free.
iv) Every finitely generated quotient algebra of P can be generated by two elements.

To prove the Theorem 3.2, we need the following lemma.

**Lemma 3.1.** Let F be the free metabelian Lie algebra freely generated by $a$ and $b$ over a field k of characteristic zero. If $u \in$ F then F can be generated by the set $\{a+u, b\}$ if and only if $u = \alpha b$ for some $\alpha \in k$.

**Proof**. The proof can be found in [11].

**Proof of Theorem 3.2:**

**i)** It is clear by Theorem 2.1.

**ii)** Let I be a directed set and for $\alpha \in I$, $\mathrm{F}_\alpha$ be the free Lie algebra freely generated by $a_\alpha$ and $b_\alpha$. Consider the homomorphism defined by

$$\begin{aligned} \phi_{\alpha\beta}\colon \mathrm{F}_\alpha &\to \mathrm{F}_\beta \\ a_\alpha &\to a_\beta + u_\beta \\ b_\alpha &\to b_\beta \end{aligned}$$

where $\alpha, \beta \in I, u_\beta \in \gamma_2(F_\beta)$ and $\alpha < \beta$ It is well known that if $\mathrm{u}_\beta \neq 0$, then $\phi_{\alpha\beta}(\mathrm{F}_\alpha) = \langle \mathrm{a}_\beta + \mathrm{u}_\beta, \mathrm{b}_\beta \rangle$ is a proper subalgebra of $\mathrm{F}_\beta$. We choose

$$\mathrm{u}_\beta \neq 0 \text{ if } \alpha < \beta \text{ and } \mathrm{u}_\beta = 0 \text{ if } \alpha = \beta.$$

Therefore $\phi_{\alpha\alpha} = \mathrm{Id}_{\mathrm{F}_\alpha}$. For $a_\alpha \in A_\alpha$ and $\alpha < \beta < \gamma$, we have

$\phi_{\alpha\gamma}(a_\alpha) = a_\gamma + u_\gamma,\ u_\gamma \in \gamma_2(F_\gamma)$
$\phi_{\beta\gamma} \circ \phi_{\alpha\beta}(a_\alpha) = \phi_{\beta\gamma}(a_\beta + u_\beta), u_\beta \in \gamma_2(F_\beta)$
$\phi_{\beta\gamma}(a_\beta) + \phi_{\beta\gamma}(u_\beta) = a_\gamma + w_\gamma + u_\gamma + w'_\gamma = a_\gamma + u_\gamma,\ w_\gamma, w'_\gamma \in \gamma_2(F_\gamma)$.
By choosing $w_\gamma + w'_\gamma = 0$, we have $\phi_{\beta\gamma} \circ \phi_{\alpha\beta} = \phi_{\alpha\gamma}$. Hence $\phi_{\alpha\alpha} = \mathrm{id}_{\mathrm{F}_\alpha}$ and for all $\alpha < \beta < \gamma$,

$\phi_{\beta\gamma}\phi_{\alpha\beta} = \phi_{\alpha\gamma}$. This provides us

$$\left(\{\mathrm{F}_\alpha\}_{\alpha\in \mathrm{I}}, \{\phi_{\alpha\beta}\}_{\alpha<\beta}\right)$$

is a directed system. Let P be the direct limit of this system. Therefore by the Theorem 2.1, P is parafree and $P = \bigcup_{\alpha\in I} F_\alpha$. Now for $\beta \in$ I and $\alpha < \beta$, we define a homomorphism

$$\varphi_{\alpha\beta}\colon \mathrm{F}_\alpha'' + \gamma_n(\mathrm{F}_\alpha) \to \mathrm{F}_\beta'' + \gamma_n(\mathrm{F}_\beta)$$

such that the homomorphism $\varphi_{\alpha\beta}$ is the restriction of the homomorphism $\phi_{\alpha\beta}$ to $F_\alpha{}'' + \gamma_n(F_\alpha)$. For $\alpha < \beta < \gamma$, $\varphi_{\alpha\alpha} = \mathrm{Id}_{F_\alpha{}''+\gamma_n(F_\alpha)}$ and $\varphi_{\beta\gamma} \circ \varphi_{\alpha\beta} = \varphi_{\alpha\gamma}$, then the system

$$\left\{\left(F_\alpha{}'' + \gamma_n(F_\alpha)\right)_{\alpha\epsilon I}, \{\varphi_{\alpha\beta}\}_{\alpha<\beta}\right\}$$

becomes a directed system. Now we want to show that

$$P'' + \gamma_n(P) = \cup_{m=1}^{\infty}\left(F_m{}'' + \gamma_n(F_m)\right).$$

By the definition we have

$$\cup_{m=1}^{\infty}(F_m{}'' + \gamma_n(F_m)) = \left\{\bar{u}/u \in \cup_{m=1}^{\infty}\left(F_m{}'' + \gamma_n(F_m)\right)\right\}.$$

Hence

$$\lim_{\rightarrow}(F_m{}'' + \gamma_n(F_m)) = \cup_{m=1}^{\infty}(F_m{}'' + \gamma_n(F_m)).$$

On the other hand there exists $l \in I$ such that

$$[\bar{a}_\alpha, \bar{a}_\beta] = \underbrace{\overline{[\phi_{\alpha l}(a_\alpha), \phi_{\beta l}(a_\beta)]}}_{\in\gamma_2(F_l)} \in (F_l{}'' + \gamma_n(F_l)),$$

where $a_\alpha \in F_\alpha$ and $a_\beta \in F_\beta$. Therefore

$$\lim_{\rightarrow}(F_m{}'' + \gamma_n(F_m)) = \bigcup_{m=1}^{\infty}\left(F_m{}'' + \gamma_n(F_m)\right)$$

and so by (1)

$$P'' + \gamma_n(P) = \bigcup_{m=1}^{\infty} F_m{}'' + \gamma_n(F_m).$$

Now let $F$ be the free Lie algebra freely generated by $a$ and $b$. Consider the homomorphism defined by

$$\psi_\alpha : F_\alpha \rightarrow {}^{F}\!/_{F''} + \gamma_n(F)$$
$$a_\alpha \rightarrow a + F'' + \gamma_n(F)$$
$$b_\alpha \rightarrow b + F'' + \gamma_n(F).$$

It is clear that $\psi_\alpha = \psi_\beta \circ \phi_{\alpha\beta}$. By the universal property of the direct limit, there is a unique homomorphism

$$\psi: P \rightarrow {}^{F}\!/_{F''} + \gamma_n(F)$$

with $\psi \circ \phi_\alpha = \psi_\alpha$. Now we compute the kernel of $\psi$. Let $x \in \mathrm{Ker}\,\psi$ so $x \in P$, therefore there exists $\alpha \in I$ such that $x \in F_\alpha$. Consider the map

$$\phi_\alpha : F_\alpha \to P\ ,\ \phi_\alpha(x) = x.$$

Since $\psi \circ \phi_\alpha = \psi_\alpha$ , then

$$\psi_\alpha\,(x) = \psi\big(\phi_\alpha(x)\big) = \psi(x) = \bar{0}.$$

Hence x $\in \mathrm{Ker}\psi_\alpha$ . So for x $\in$Ker $\psi$ there exists $\alpha \in$ I such that x $\in \mathrm{Ker}\psi_\alpha$ . Therefore

$$\mathrm{Ker}\,\psi = \cup_{\alpha\epsilon \mathrm{I}} \mathrm{Ker}\psi_\alpha\,.$$

On the other hand since $\mathrm{Ker}\psi_\alpha = \{x \in F_\alpha / \psi_\alpha(x) = \bar{0}\}$ , then $\psi_\alpha(x) = F'' + \gamma_n(F)$. Therefore for each $\alpha \in$ I, we have $\mathrm{Ker}\psi_\alpha = F_\alpha{}'' + \gamma_n(F_\alpha)$. It is well known that verbal ideals are invariant under homomorphisms so $\psi_\alpha(x) \in F_\alpha{}'' + \gamma_n(F_\alpha)$. Then

$$\mathrm{Ker}\psi_\alpha = F_\alpha{}'' + \gamma_n(F_\alpha).$$

Hence

$$\mathrm{Ker}\,\psi = \cup_{\alpha\epsilon \mathrm{I}} \mathrm{Ker}\psi_\alpha = \cup_{\alpha\epsilon \mathrm{I}}(F_\alpha{}'' + \gamma_n(F_\alpha)) = \mathrm{P}'' + \gamma_\mathrm{n}(\mathrm{P})$$

By the isomorphism theorems, we get

$$\mathrm{P}\big/_{P'' + \gamma_n(P)} \cong \mathrm{F}\big/_{F'' + \gamma_n(F)}.$$

**iii)** By (i), it is clear that the rank of $P$ is two. Furthermore since $P = \cup_n F_n$, then it is not finitely generated. But there is no a free Lie algebra which has a finite rank and is not finitely generated. Therefore $P$ is not free.

**iv)** Let $F_i$ be the free Lie algebra freely generated by $\{a_i, b_i\}$ for $i \in I$ and let $P = \cup_{i\in I} F_i$ .

Let us suppose that the Lie algebra P is generated by set $\{x_1, \dots, x_n, \dots\}$. If I is an ideal of P generated by $\{x_3, \dots, x_n, \dots\}$, then $P\big/_I = \cup_{i=1}^{\infty}\left(F_i + I\big/_I\right)$. Since the quotient algebra $P\big/_I$ is freely generated, for some $i \in I$, we have $P\big/_I = F_i + I\big/_I$.

Now consider the homomorphism $\varphi : P\big/_I \to F_i + I\big/_I$ defined by $x_1 \to a_i + I$, $x_2 \to b_i + I$.

The homomorphism $\varphi$ is an isomorphism, Therefore we have ${}^{P}/_{I} = {}^{F_i + I}/_{I}$ . Since the free Lie algebra $F_i$ is freely generated by the set $\{a_i, b_i\}$ then the algebra ${}^{P}/_{I}$ is generated by $\{a_i + I, b_i + I\}$ . Therefore ${}^{P}/_{I}$ is generated by two elements.

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**INTERNATIONAL CONFERENCE ON MATHEMATICS**
**"An Istanbul Meeting for World Mathematicians"**
**3-5 July 2019, Istanbul, Turkey**

# On DNA Codes based on additive self-dual $F_4$-codes

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**Abstract**

In this paper we translate in terms of coding theory constraints that are used in designing DNA codes for use in DNA computing. We focus in particular on additive self-dual $F_4$-codes and we present some results for constructed DNA codes satisfying the Hamming distance constraint, the reverse-complement constraint and the GC-content constraint.

**Keywords: DNA code, additive self-dual code**

## 1. Introduction

Coding theory has several applications in Genetics and Bioengineering. Every DNA molecule consists of two complementary strands which are sequences of four different nucleotide bases: adenine (A), cytosine (C), guanine (G) and thymine (T). The problem of designing DNA codes (sets of words of fixed length $n$ over the alphabet {A, C, G, T} that satisfy certain combinatorial constraints has applications for reliably storing and retrieving information in synthetic DNA strands.

In current work the constraints used in designing DNA codes are translated in terms of coding theory. Using a method developed in our previous work we construct DNA codes satisfying a Hamming distance constraint, reverse-complement constraint and a GC-content constraint. This method is based on usage of the representation form of a generator matrices of special class additive self-dual $F_4$-codes.

The paper is organized as follows: in Section 2 some basic notions for DNA codes and additive $F_4$-codes are presented. Also, the constraints on DNA codes are translated into the terms of additive $F_4$-codes. In Section 3 the constructive method is presented and the obtained results are given. In the conclusion, the results of the work are summarized.

* This research was partially supported by FSD31-340-12/2019 Project, University of Veliko Tarnovo

## 2. Preliminaries

The terms in this section are mostly taken from [1, 4].

A *DNA code* of length $n$ is a set of codewords $(x_1, \ldots, x_n)$ with $x_i \in$ {A, C, G, T} (representing the four nucleotides in DNA). We use a hat to denote the Watson-Crick complement of a nucleotide, so $\hat{A}$ = T, and $\hat{C}$ = G (and vice versa).

The *Hamming distance* $H(x, y)$ between two codewords $x$ and $y$ is the number of coordinates in which $x$ and $y$ differ. The reverse of a codeword $y = (y_1, \ldots, y_n)$ is denoted by $y_R = (y_n, \ldots, y_1)$, and the reverse-complement of $y = (y_1, \ldots, y_n)$ is denoted by $y_{RC} = (\hat{y}_n, \ldots, \hat{y}_1)$.

In this paper we shall identify codes over {A, C, G, T} with codes over other four-letter alphabet. In our case this is the Galois field $F_4 = \{0, 1, w, w^2\}$, with $w^2 + w + 1 = 0$. The four symbols in {A, C, G, T} are identified with the four elements in $F_4$ in the orders given above, so that $\hat{y} = y + 1$, for $y \in F_4$.

Let $F_4^n$ be the $n$-dimensional vector space over $F_4$. The *Hamming weight* of a vector $x \in F_4^n$, written $wt(x)$, is the number of nonzero entries of $x$, and Hamming distance $d(x,0) = wt(x)$. A *linear* $[n, k]$ $F_4$-code $C$ is a $k$-dimensional linear subspace of $F_4^n$. Any $k \times n$ matrix $G$ (with entries in $F_4$) whose rows are a basis of the code $C$ is a *generator matrix* of $C$. A *minimum weight* (or *minimum distance*) of a linear code is the smallest weight among all nonzero codewords. A linear $[n, k, d]$ $F_4$-code $C$ is an $[n, k]$ $F_4$-code with minimum distance $d$. A *weight enumerator* of a code $C$ is the polynomial $C(z) = \sum_{i=0}^{n} A_i z^i$, where $A_i$ is the number of codewords of weight $i$.

An additive $(n, 2^k)$ $F_4$-code of length $n$ is an additive subgroup of $F_4^n$ with $2^k$ codewords. The definitions for Hamming weight, generator matrix, minimum weight, and weight enumerator are the same as the definitions about linear codes. By $(n, 2^k, d)$ we denote an additive $F_4$-code of length $n$ with $2^k$ codewords that has minimum weight $d$. About additive codes over $F_4$, there is an inner product arising from the trace map. The trace map $Tr : F_4 \to F_2$ is given by $Tr(x) = x + x^2$. We can define the *trace inner product* of two vectors $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $F_4^n$ as:

$$x * y = \sum_{i=1}^{n} Tr(x_i . y_i^2) \tag{1}$$

If $C$ is an additive code, its *dual code* with respect to (1) is the code $C^{\perp} = \{x \in F_4^n \mid x*c = 0$ for all $c \in C\}$. If $C$ is an $(n, 2^k)$ code, then $C$ is an $(n, 2^{2n-k})$ code. The code $C$ is *self-orthogonal* if $C$ is a subset of $C^{\perp}$, and *self-dual* if $C = C^{\perp}$. In particular, if $C$ is self-dual, then $C$ is an $(n, 2^n)$ code.

In our work we will the following map: $0 \to$ A, $1 \to$ T, $2 \to$ C, and $3 \to$ G. In this case the Watson-Crick complement (the transpositions A $\leftrightarrow$ T and C $\leftrightarrow$ G) is presented as $\hat{y} = y + 1$, for $y \in F_4$.

About DNA codes we consider the following constraints:

- *Hamming distance constraint*: the Hamming distance constraint for a DNA code $C$ is that $H(x,y) \geq d$ for all $x, y \in C$ with $x \neq y$, for some prescribed minimum distance $d$. This constraint will be enforced in all of the codes we consider, in addition to some combination of the constraints described below.
- *Reverse constraint:* the reverse constraint is that $H(x^R, y) \geq d$ for all $x, y \in C$, including $x = y$. It is useful as an intermediate step in constructing codes with the reverse-complement constraint. A natural idea is to start with a code that is fixed by the reverse permutation $R$, which exchanges column $i$ and column $n - i + 1$, for $1 \leq i \leq n$.
- *Reverse-complement constraint*: this constraint is that $H(x^{RC}, y) \geq d$ for all $x, y \in C$, including $x = y$. To construct codes satisfying the reverse-complement constraint, it can be useful to begin with codes over $F_4$ that contain the all-one codeword $j$. Note that $x^{RC} = x^R + j$, so an additive code containing $j$ that is fixed by the permutation $x \to x^R$ is also fixed by the map $x \to x^{RC}$.
- *GC-content constraint*: this constraint is that each codeword $x \in C$ has the same $GC$-weight (the number of entries that are G or C). Starting from a linear code, the question is how to compute the $GC$-weight enumerator (finding the complete weight enumerator may in itself take a long time). Below a simple way to compute the number of codewords with fixed $GC$-weight of a special class of additive self-dual codes over $F_4$ is described.

## 3. Main Results

In the current work we consider DNA codes with fixed GC-content that satisfy given Hamming distance constraint and reverse-complement constraint. By $A_4^{GC,RC}(n,d,u)$ we denote the maximum size of a DNA code of length $n$ with constant $GC$-content $u$ that satisfies the Hamming distance constraint and the reverse-complement constraint for a given $d$.

A *graph code* is an additive self-dual $F_4$-code with generator matrix $G = \Gamma + wI$ where $I$ is the identity matrix and $\Gamma$ is the adjacency matrix of a simple undirected graph, which must be symmetric with 0's along the diagonal.

**Example:**

$$\Gamma = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \qquad G = \begin{pmatrix} w & 1 & 1 \\ 1 & w & 1 \\ 1 & 1 & w \end{pmatrix}$$

It is proved [5] that for any self-dual quantum code, there is an equivalent graph code. This means that there is a one-to-one correspondence between the set of simple undirected graphs and the set of additive self-dual codes over $F_4$.

A matrix $B$ of the form

$$B = \begin{pmatrix} b_0 & b_1 & \ldots\ldots & b_{n-2} & b_{n-1} \\ b_{n-1} & b_0 & b_1 & \ldots\ldots & b_{n-2} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ b_2 & \ldots\ldots & b_{n-1} & b_0 & b_1 \\ b_1 & b_2 & \ldots\ldots & b_{n-1} & b_0 \end{pmatrix}$$

is called a *circulant* matrix. The vector $(b_0, b_1, \ldots, b_{n-1})$ is called generator vector for the matrix B. An additive code with circulant generator matrix is called *circulant code*.

An *additive circulant graph* (ACG) code is a code corresponding to graph with circulant adjacency matrix. Circulant graphs must be regular, i.e., all vertices must have the same number of neighbours. It is easy to see that such matrix has the following property: $b_i = b_{n-i}$ , for all $i = 1, \ldots, n-1$, and $b_0 = w$. Then, the entries in the generator matrix of ACG code depend on the coordinates $(b_1, b_2, \ldots, b_{\lfloor n/2 \rfloor})$ only.

The graph codes are proper to construct DNA codes with fixed GC-content $u$ that satisfy Hamming distance constraint for given $d$. If we know already that the minimum distance of the code is at least $d$, then $H(x,y) \geq d$ (for any two codewords $x$ and $y$), and the Hamming distance constraint is satisfied. Other good property is that the generator matrix $G$ of the code has just one position in any row (and column) that is neither 0 nor 1. It is easy to see that any codeword that is a sum of $u$ rows has GC-weight $u$. Then, the corresponding DNA code with $H(x,y) \geq d$ and fixed GC-content $u$ consists of all codewords that are linear combinations of $u$ rows of the generator matrix $G$.

The generator matrix $G$ of an ACG code has another special property (useful about the RC-constraint). The $i^{\text{th}}$ row is a reverse of the $(n - i + 1)^{\text{th}}$ row, for any $1 \leq i \leq n/2$. Let $R$ be a reverse permutation (a permutation that exchanges column $i$ and column $n - i + 1$ of the code, for $1 \leq i \leq n$). Then any codeword that is a linear combination of even number of pairwise reversed rows of $G$ is fixed by $R$. Also, any ACG code contains all-$w$ or all-$w^2$ codeword (that is the linear combination of all rows of $G$). Therefore, by a multiplication of all columns of $G$ by $w^2$ or $w$ (respectively) we can get a code that contains all-one codeword.

According to these properties, in our work we use the following:

**Theorem 1 [6]:** *Any ACG code of length n with minimum distance d consists of DNA codes of length n with $H(x,y) \geq d$, fixed GC-content $u = n / 2$, and* $A_4^{GC,RC}(n,d,u) = \left(\binom{n}{u} - \binom{u}{k}\right)/2$ (where $k = u / 2$), except for n ≡ 2 (mod 4), where $A_4^{GC,RC}(n,d,u) = \binom{n}{u}/2$

Based on the results about additive self-dual $F_4$-codes obtained in [3] we construct many DNA codes. Also, by shortening and lengthening (see [2] for these methods) we construct additive self-dual $F_4$-codes with other lengths. All bounds for DNA codes of length n ≥ 36 are new [4, 6]. In Table 1 we summarize the obtained results. The main problem was to check experimentally these results. In some cases the number of codewords was too large for the primitive data types in the programming languages and we used JAVA language and the class BigInteger (used for mathematical operation which involves very big integer calculations that are outside the limit of all available primitive data types) in order to check them.

## 4. Conclusion

In this work we have presented some connections between DNA codes and additive $F_4$-codes. We use a special class of additive self-dual $F_4$-codes and we use a constructive method on DNA codes based on the form of generator matrices of the codes in this special class. By this construction we improve the lower bounds on DNA codes that satisfy some necessary constraints.

Table 1. New bounds for $A_4^{GC,RC}(n,d,u)$ for $55 \leq n \leq 100$

| $n$ | $D$ | $A_4^{GC,RC}(n,d,u)$ | $n$ | $d$ | $A_4^{GC,RC}(n,d,u)$ |
|---|---|---|---|---|---|
| 55 | 14 | $1{,}912 \times 10^{15}$ | 71 | 18 | $1{,}106 \times 10^{20}$ |
| 56 | 15 | $3{,}824 \times 10^{15}$ | 79 | 19 | $2{,}687 \times 10^{22}$ |
| 57 | 15 | $7{,}517 \times 10^{15}$ | 82 | 19 | $2{,}124 \times 10^{23}$ |
| 58 | 16 | $3{,}007 \times 10^{16}$ | 83 | 20 | $4{,}197 \times 10^{23}$ |
| 62 | 16 | $2{,}327 \times 10^{17}$ | 87 | 20 | $6{,}562 \times 10^{24}$ |
| 63 | 16 | $4{,}581 \times 10^{17}$ | 88 | 20 | $1{,}312 \times 10^{25}$ |
| 67 | 17 | $7{,}113 \times 10^{18}$ | 89 | 21 | $2{,}595 \times 10^{25}$ |
| 70 | 18 | $1{,}122 \times 10^{20}$ | 95 | 20 | $1{,}608 \times 10^{27}$ |

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# A STRONG LOWER BOUND ON BLOWING-UP SOLUTIONS OF THE 3D NAVIER–STOKES EQUATIONS IN H^{5/2}

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We prove a lower bound for the blow-up rate of smooth solution of the 3D Navier-Stokes equations in the H^{5/2}-norm, both on the whole space and in the periodic case. This result gives a positive answer to a question left open by James et al (2012, J. Math. Phys.). and Mccormick et al (2016, J. Math. Analysis).

**Keyword(s): Blow-up rate; Strong lower bound; Navier-Stokes equations**

A Strong lower bound on the blow-up of solutions to the 3D Navier-Stokes equations in $\dot{H}^{5/2}$
Blow-up solutions for the 3D Navier-Stokes equations
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Blow-up rate; Strong lower bound; Navier-Stokes equations

Under the assumption that $T^*$ is the first time of blow up of smooth solutions of the 3D Navier-Stokes equations in the Sobolev space $\dot{H}^{5/2}$, we prove a strong lower bound for the blow-up rate of the type $c(T^*-t)^{-1}$, in $\dot{H}^{5/2}$. Moreover, we give an explicit estimate of the value of the constant $c$. This result completes the works of James et al (2012, J. Math. Phys.) and Mccormick et al (2016, J. Math. Analysis). We extend this result to general nonlinear ordinary differential equations.

## 1.Introduction

We consider, in this paper, the 3D incompressible Navier-Stokes equations

$$\frac{\partial u}{\partial t} + u.\nabla u = -\nabla p + \triangle u, \text{ in } \Omega \times (0,\infty)$$
$$\operatorname{div} u = 0, \text{ in } \Omega \times (0,T) \text{ and } u(x,0) = u_0, \text{ in } \Omega,$$

where $u = u(x,t)$ is the velocity vector field, $p$ is the pressure. The domain $\Omega$ may have periodic boundary conditions or $\Omega = \mathbb{R}^3$.

For small data $\|\nabla u_0\|_{L^2} \leq c$ the global existence of strong solutions for the 3D Navier-Stokes equations it is well known, see Constantin and Foias [2]. But for the 3D Navier-Stokes equations with large data, we don't have a result of global existence. Under the assumption that the solution of the three-dimensional Navier-Stokes equations becomes irregular at finite time $T^*$ Leray 1934 [5] proved that there exists a constant $c > 0$ such that

$$\|\nabla u(.,t)\|^4_{\dot{H}^1(\mathbb{R}^3)} \geq \frac{c}{(T^* - t)}.$$

In 2010, Benameur [1] showed in the whole space

$$\|u(.,t)\|_{\dot{H}^s(\mathbb{R}^3)} \geq c(s)\frac{\|u(.,t)\|^{\frac{3-2s}{3}}_{L^2(\mathbb{R}^3)}}{(T^* - t)^{\frac{s}{3}}} \text{ with } s > \frac{5}{2}.$$

The result above was improved by Robinson, Sadowski, and Silva in [7] to

$$\|u(.,t)\|_{\dot{H}^s(\Omega)} \geq c(s)\frac{\|u_0\|^{\frac{5-2s}{5}}_{L^2(\Omega)}}{(T^* - t)^{\frac{2s}{3}}}$$

with $\Omega = [0,1]^3$ or $\mathbb{R}^3$. In the homogeneous Sobolev space $\dot{H}^{5/2}(\mathbb{T}^3)$ of real valued periodic functions, Cortissoz, Montero, & Pinilla 2014 [3] proved lower bounds on the blow up with logarithmic corrections,

$$\|u(.,t)\|_{\dot{H}^{5/2}(\mathbb{T}^3)} \geq \frac{c}{(T^* - t)|\lg(T^* - t)|}.$$

Recently, McCormick et al (2016) [6] proved the blow up estimate

$$\lim_{t\to T^*}(T^* - t)\|u(.,t)\|_{\dot{H}^{5/2}} \geq c.$$

The estimates of the blow up rate in different spaces see [1, 3, 4, 6, 7]. The interesting question left open is the strong blow up estimate

$$(T^* - t)\|u(.,t)\|_{\dot{H}^{5/2}} \geq c,$$

in both the whole space and the periodic case. Motivated by the previous works, the goal of this paper is to establish a strong lower bound (ref: 7) for the blow-up rate in $\dot{H}^{5/2}(\mathbb{T}^3)$ of a possible blow-up solution to the 3D Navier-Stokes equations. Based on a contradictory argument,we prove that is possible to obtain a rate of blow up of the type (ref: 7) in $\dot{H}^{5/2}$ space for $t \leq T^*$.

## 2. Preliminaries

Let $Q = [0,2\pi]^3$, we write $\mathbb{Z}^3 = \mathbb{Z}^3/\{0,0,0\}$, let $\dot{H}^s(Q)$ be the subspace of the Sobolev space $H^s$ consisting of divergence-free, zero-average, periodic real functions,

$$\dot{H}^s(Q) = \left\{ u = \sum_{\xi \in \dot{\mathbb{Z}}^3} \hat{u}_\xi e^{-i\xi.x} \ : \ \overline{\hat{u}_\xi} = \hat{u}_{-\xi} \, , \sum_{\xi} |\xi|^{2s} |\hat{u}_\xi|^2 < \infty \text{ and } \xi . \hat{u}_\xi = 0 \right\}$$

and equip $\dot{H}^s(Q)$ with the norm

$$\|u\|^2_{\dot{H}^s} = \|u\|^2_{\dot{H}^s(Q)} = \sum_{\xi} |\xi|^{2s} |\hat{u}_\xi|^2 .$$

On the whole space the corresponding definition of the $\dot{H}^s(\mathbb{R}^3)$ norm is

$$\|u\|^2_{\dot{H}^s(\mathbb{R}^3)} := \int_{\mathbb{R}^3} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi < \infty,$$

where

$$F[u](\xi) = \hat{u}(\xi) = \int_{\mathbb{R}^3} e^{-2i\pi x.\xi} u(x) dx$$

is the Fourier transform of $u$, for more details see [7]. We prove our estimate in the periodic case, but it also holds in the full space. Throughout the paper, $c_i$, $i \in \mathbb{N}$, denotes a positive constant.

# 3.Main result

In the proof we shall use the following lemma

**Lemma** *Let* $u$ *be a smooth solution of the 3D Navier-Stokes equations* $(\text{ref: } 1)$ *with a maximal time interval of existence* $(0, T^*)$ *,* $T^* < \infty$ *. Then* $u$ *satisfies*

$$(T^* - t)^{\frac{1}{2}} \leq c_1 \|u(.,t)\|_{\dot{H}^{5/2}} (T^* - t)$$

*for all* $t \leq T^*$.

. Our main result is the following

**Theorem** *Suppose that* $u$ *is a classical solution of the 3D Navier-Stokes equations* $(\text{ref: } 1)$ *with a maximal time interval of existence* $(0, T^*)$ *,* $T^* < \infty$ *. For all* $\tau$ *in* $[0, T^*]$ *, if* $t \in [\tau, T^*]$ *then*

$$\|u(.,t)\|_{\dot{H}^{5/2}} (T^* - t) \geq C_\tau$$

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# Reproducing Kernel Functions for Solution of Linear and Nonlinear Third order Fractional Differential Equations

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**Abstract**

We construct some special spaces to obtain numerical solution of the third order fractional differential equation. We obtain very useful reproducing kernel functions in these spaces. These functions will be very helpful for researchers to investigate the nonlinear third order fractional differential equations.

**Keywords:** Reproducing kernel method, third order fractional differential equation, Hilbert spaces.

## 1. Introduction

Fractional differential equations have taken considerable importance recently in the literature. These equations have many implementations in the areas of finance, applied sciences, seismology engineering, physics and biology [1-3]. Fractional differential equations can be solved separately depending on the time and space variables. There are some techniques for approximate solutions of fractional differential equations due to space and time variables [4, 5]. These techniques are the radial basis function, Chebyshew Tau method, thin plate splines method, variational iteration method, finite difference schemes method and Daftardar-Gejii-Jafaris method [6-10]. In recent years, many works have been constructed on modeling of fractional differential equations [11-15]. In this paper, we will study the initial-boundary value problems of the third-order fractional differential equations defined by Caputo derivative.

$$\frac{\partial^3\eta(t,x)}{\partial t^2}+\frac{\partial^\alpha\eta(t,\alpha)}{\partial t^\alpha}+\eta(t,x)=\lambda\frac{\partial^2\eta(t,x)}{\partial\alpha^2}+f(t,x)$$

$$0<x<L,\ 0<t<T,\ 0<\alpha\le 1, \tag{1}$$

$$\eta(0,x)=g_1(x),\ \eta_t(0,x)=g_2(x),\ \eta(0,x)=g_3(x),\ 0\le t\le T$$

$$\eta(t,X_L)=r_1(t),\ \eta(t,X_R)=r_2(t),\quad X_L<x<X_R$$

Where $\lambda$ is known constant coefficient, $g_1$, $g_2$, $g_3$, $r_1$ and $r_2$ are known functions and $\eta$ is the unknown function.

**Definition 1**. The definition of Gamma function is presented as:

$$\Gamma(z)=\int_0^t e^{-t}\,t^{z-1}dt,$$

for all $z\in C$.

**Definition 2**. The Caputo fractional derivative $D_t^\alpha\eta(t,x)$ of order $\alpha$ with respect to time is given as:

$$\frac{\partial^\alpha\eta(t,x)}{\partial t^\alpha}=D_t^\alpha\eta(t,x)$$

$$=\frac{1}{\Gamma(n-\alpha)}\int_0^t\frac{1}{(t-p)^{\alpha-n+1}}\frac{\partial^\alpha\eta(p,x)}{\partial p^\alpha}dp,(n-1<\alpha<n)$$

and for $\alpha=n\in N$ defined as:

$$D_t^\alpha\eta(t,x)=\frac{\partial^\alpha\eta(t,x)}{\partial t^\alpha}=\frac{\partial^n\eta(t,x)}{\partial t^n}$$

**Definition 3.** By using Gamma function, we have

$$D_t^\alpha\eta(t,x)=\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}t^{\beta-\alpha}.$$

## 2.Reproducing Kernel Functions

We define the reproducing kernel Hilbert space $H_2^4[0,1]$ as:

$$H_2^4[0,1]=\left\{\begin{matrix} h(z),\ h'(z),\ h''(z),\ h'''(z)\ are\ absolutely\ continious\ functions \\ h^{(4)}(z)\in L^2[0,1], z\in[0,1], h(z)=h'(z)=h''(z)=0 \end{matrix}\right\}$$

For any functions $h(z), s(z) \in H_2^4[0,1]$, we have

$$\langle h,s\rangle_{H_2^4}=\ \sum_{i=0}^{3} h^{(i)}(0)s^{(i)}\,(0)+\int_0^1\ h^{(m)}\,s^{(m)}(z)dz,$$

and

$$\|h\|_{H_2^4}=\sqrt{\langle h,h\rangle_{H_2^4}}$$

Lets find the reproducing kernel function $A_m(z)$

$$\langle s,A_m\rangle_{H_2^4}=\sum_{i=0}^{3} A_m^{(i)}\,(0)s^{(i)}(0)+\int_0^1 A_m^{(4)}\,(z)s^{(4)}(z)dz$$

$$=A_m(0)s(0)+A'_m(0)s'(0)+A''_m(0)s''(0)+A'''_m(0)s'''(0)$$

$$+\int_0^1 A_m^{(4)}\,(z)s^{(4)}(z)dz$$

We will apply integrations by parts and obtain

$$\langle s,A_m\rangle_{H_2^4}=A_m(0)s(0)+A'_m(0)s'(0)+A''_m(0)s''(0)+A'''_m(0)s'''(0)$$

$$+A_m^{(4)}(1)s^{(3)}(1)-A_m^{(4)}(0)s^{(3)}(0)-A_m^{(5)}(1)s''(1)+A_m^{(5)}(0)s''(0)$$

$$+A_m^{(6)}(1)s'(1)-A_m^{(6)}(0)s'(0)-A_m^{(7)}(1)s(1)+A_m^{(7)}(0)s(0)$$

$$+\int_0^1 A_m^{(8)}\,(z)s(z)dz=s(m)$$

Since $s(0)=s'(0)=s''(0)=0$, we get

$$\langle s,A_m\rangle_{H_2^4}=A'''_m(0)s'''(0)+A_m^{(4)}(1)s^{(3)}(1)-A_m^{(4)}(0)s^{(3)}(0)-A_m^{(5)}(1)s''(1)$$

$$+A_m^{(6)}(1)s'(1)-A_m^{(7)}(1)s(1)+\int_0^1 A_m^{(8)}\,(z)s(z)dz$$

If we have

1. $A_m'''(0) - A_m^{(4)}(0)$ =0,

2. $A_m^{(4)}(1)$=1,

3. $A_m^{(5)}(1) = 0$,

4. $A_m^{(6)}(1) = 0$,

5. $A_m^{(7)}(1) = 0$,

then we will find

$$\langle s, A_m \rangle_{H_2^4} = \int_0^1 A_m^{(8)}(z) s(z) dz$$

By reproducing feature we know

$$\langle s, A_m \rangle_{H_2^4} = s(m).$$

Therefore, we get

$$\int_0^1 A_m^{(8)}(z) s(z) dz = s(m).$$

Thus, we reach

$$A_m^{(8)}(z) = \delta(m - z).$$

When $m \neq z,$ we get

$$A_m^{(8)}(z) = 0.$$

Then, we obtain the reproducing kernel function $A_m(z)$ as

$$A_m(z) = \begin{cases} \sum_{k=1}^{8} a_k(m) z^{k-1}, & z \leq m, \\ \sum_{k=1}^{8} b_k(m) z^{k-1}, & z > m. \end{cases}$$

The reproducing kernel function should satisfy the conditions. Therefore, we have

6. $A_m(0) = 0$,

7. $A'_m(0) = 0$,
8. $A''_m(0) = 0$.

We have 16 unknown coefficients and 8 equations .We need 8 more equations to obtain these coefficients. If we use the properties of Dirac-Delta function we will find

9. $A_{m^+}(m) = A_{m^-}(m)$ ,

10. $A'_{m^+}(m) = A'_{m^-}(m)$ ,

11. $A''_{m^+}(m) = A''_{m^-}(m)$ ,

12. $A'''_{m^+}(m) = A'''_{m^-}(m)$ ,

13. $A^{(4)}_{m^+}(m) = A^{(4)}_{m^-}(m)$ ,

14. $A^{(5)}_{m^+}(m) = A^{(5)}_{m^-}(m)$ ,

15. $A^{(6)}_{m^+}(m) = A^{(6)}_{m^-}(m)$ ,

We have

$$A^{(8)}_m(z) = \delta(m - z).$$

If we take integral from both sides, we will find

16. $A^{(7)}_{m^+}(m) - A^{(7)}_{m^-}(m) = 1$ .

Lets find these unknown coefficients. We have

$$A_m(z) = \begin{cases} a_1 + a_2 z + a_3 z^2 + a_4 z^3 + a_5 z^4 + a_6 z^5 + a_7 z^6 + a_8 z^7 & z \le m \\ b_1 + b_2 z + b_3 z^2 + b_4 z^3 + b_5 z^4 + b_6 z^5 + b_7 z^6 + b_8 z^7 & z > m \end{cases}$$

$$A'_m(z) = \begin{cases} a_2 + 2a_3 z + 3a_4 z^2 + 4a_5 z^3 + 5a_6 z^4 + 6a_7 z^5 + 7a_8 z^6 & z < m \\ b_2 + 2b_3 z + 3b_4 z^2 + 4b_5 z^3 + 5b_6 z^4 + 6b_7 z^5 + 7b_8 z^6 & z > m \end{cases}$$

$$A''_m(z) = \begin{cases} 2a_3 + 6a_4 z + 12a_5 z^2 + 20a_6 z^3 + 30a_7 z^4 + 42a_8 z^5 & z < m \\ 2\, b_3 + 6b_4 z + 12b_5 z^2 + 20b_6 z^3 + 30b_7 z^4 + 42b_8 z^5 & z > m \end{cases}$$

$$A'''_m(z) = \begin{cases} 6a_4 + 24a_5 z + 60a_6 z^2 + 120a_7 z^3 + 210a_8 z^4 & z < m \\ 6\, b_4 + 24b_5 z + 60b_6 z^2 + 120b_7 z^3 + 210b_8 z^4 & z > m \end{cases}$$

$$A_m^{(4)}(z) = \begin{cases} 24a_5 + 120a_6 z + 360a_7 z^2 + 840a_8 z^3 & z < m \\ 24b_5 + 120b_6 z + 360b_7 z^2 + 840b_8 z^3 & z > m \end{cases}$$

$$A_m^{(5)}(z) = \begin{cases} 120a_6 + 720a_7 z + 2520a_8 z^2 & z < m \\ 120b_6 + 720b_7 z + 2520b_8 z^2 & z > m \end{cases}$$

$$A_m^{(6)}(z) = \begin{cases} 720a_7 + 5040a_8 z & z < m \\ 720b_7 + 5040b_8 z & z > m \end{cases}$$

$$A_m^{(7)}(z) = \begin{cases} 5040a_8 & z < m \\ 5040b_8 & z > m \end{cases}$$

Then, we get

1. $a_1 + 5040a_8 = 0$
2. $a_2 - 720a_7 = 0$
3. $2a_3 + 120a_6 = 0$
4. $6a_4 - 24a_5 = 0$
5. $24b_5 + 120b_6 + 360b_7 + 840b_8 = 0$
6. $120b_6 + 720b_7 + 2520b_8 = 0$
7. $720b_7 + 5040b_8 = 0$
8. $5040b_8 = 0$
9. $b_1 + b_2m + b_3m^2 + b_4m^3 + b_5m^4 + b_6m^5 + b_7m^6 + b_8m^7 = a_1 + a_2m + a_3m^2 + a_4m^3 + a_5m^4 + a_6m^5 + a_7m^6 + a_8m^7$
10. $b_2 + 2b_3m + 3b_4m^2 + 4b_5m^3 + 5b_6m^4 + 6b_7m^5 + 7b_8m^6 = a + 2a_3m + 3a_4m^2 + 4a_5m^3 + 5a_6m^4 + 6a_7m^5 + 7a_8m^6$
11. $2b_3 + 6b_4m + 12b_5m^2 + 20b_6m^3 + 30b_7m^4 + 42b_8m^5 = 2b_a + 6a_4m + 12a_5m^2 + 20a_6m^3 + 30a_7m^4 + 42a_8m^5$
12. $6b_4 + 24b_5m + 60b_6m^2 + 120b_7m^3 + 210b_8m^4 = 6a_4 + 24a_5m + 60a_6m^2 + 120a_7m^3 + 210a_8m^4$
13. $24b_5 + 120b_6m + 360b_7m^2 + 840b_8m^3 = 24a_5 + 120a_6m + 360a_7m^2 + 840a_8m^3$
14. $120b_6 + 720b_7m + 2520b_8m^2 = 120a_6 + 720a_7m + 2520a_8m^2$
15. $720b_7 + 5040b_8m = 720a_7 + 5040a_8m$
16. $5040b_8 - 5040a_8 = 1$

Thus, we find

$$a_1 = 1 \qquad b_1 = 1 - \frac{m^7}{5040}$$

$$a_2 = m \qquad b_2 = m + \frac{m^6}{720}$$

$$a_3 = \frac{m^2}{4} \qquad b_3 = \frac{m^2}{4} - \frac{m^5}{240}$$

$$a_4 = \frac{m^3}{36} \qquad b_4 = \frac{m^3}{36} + \frac{m^4}{144}$$

$$a_5 = \frac{m^3}{144} \qquad b_5 = 0$$

$$a_6 = \frac{-m^2}{240} \qquad b_6 = 0$$

$$a_7 = \frac{m}{720} \qquad b_7 = 0$$

$$a_8 = \frac{-1}{5040} \qquad b_8 = 0$$

Therefore, our reproducing kernel function is obtained as:

$$A_m(z) = \begin{cases} 1 + mz + \dfrac{m^2z^2}{4} + \dfrac{m^3z^3}{36} + \dfrac{m^3z^4}{144} - \dfrac{m^2z^5}{240} + \dfrac{mz^6}{720} - \dfrac{z^7}{5040}, & z \leq m, \\ 1 + mz + \dfrac{m^2z^2}{4} + \dfrac{m^3z^3}{36} + \dfrac{m^4z^3}{144} - \dfrac{m^5z^2}{240} + \dfrac{m^6z}{720} - \dfrac{m^7}{5040}, & z > m. \end{cases}$$

Now, we are ready to give the second reproducing kernel Hilbert space $H_2^3[0,1]$ as:

$$H_2^3[0,1] = \left\{ \begin{matrix} h(z), h'(z), h''(z), \quad \mathit{are\ absolutely\ continious\ functions} \\ h^{(3)}(z) \in L^2[0,1], z \in [0,1], h(z) = h'(z) = 0 \end{matrix} \right\}$$

For any functions $h(z), s(z) \in H_2^3[0,1]$

$$\langle h, s \rangle_{H_2^3} = \sum_{i=0}^{2} h^{(i)} s^{(i)} + \int_0^1 h^{(m)}\, s^{(m)}(z) dz$$

and

$\|h\|_{H_2^3}=\sqrt{\langle h,h\rangle_{H_2^3}}$

Lets find the reproducing kernel function $A_m(z)$

$$\langle s,A_m\rangle_{H_2^3}=\sum_{i=0}^{2} A_m^{(i)}(0)s^{(i)}(0)+\int_0^1 A_m^{(3)}(z)s^{(3)}(z)dz$$

$$=A_m(0)s(0)+A_m'(0)s'(0)+A_m''(0)s''(0)+\int_0^1 A_m^{(3)}(z)s^{(3)}(z)dz$$

We will apply integrations by parts

$$\langle s,A_m\rangle_{H_2^3}=A_m(0)s(0)+A_m'(0)s'(0)+A_m''(0)s''(0)+A_m'''(1)s^{('')}(1)$$

$$-A_m'''(0)s''(0)-A_m^{(4)}(1)s'(1)+A_m^{(4)}(0)s'(0)+A_m^{(5)}(1)s(1)$$

$$-A_m^{(5)}(0)s(0)-\int_0^1 A_m^{(6)}(z)s(z)dz=s(m)$$

Since $s(0)=s'(0)=0$, we obtain

$$\langle s,A_m\rangle_{H_2^3}=A_m'''(0)s'''(0)+A_m^{(4)}(1)s'''(1)-A_m^{(4)}(0)s'''(0)-A_m^{(5)}(1)s''(1)$$

$$+A_m^{(6)}(1)s'(1)-A_m^{(7)}(1)s(1)+\int_0^1 A_m^{(8)}(z)s(z)dz$$

If we have

1. $A_m''(0)-A_m'''(0)=0$
2. $A_m'''(1)=0$
3. $A_m^{(4)}(1)=0$
4. $A_m^{(5)}(1)=0$,

then we will find

$$\langle s,A_m\rangle_{H_2^3}=\int_0^1 A_m^{(6)}(z)s(z)dz$$

By reproducing feature we know

$$\langle s,A_m\rangle_{H_2^3}=s(m)$$

Therefore, we get

$$\int_0^1 A_m^{(6)}(z)s(z)dz = s(m)$$

Thus, we reach

$$A_m^{(6)}(z) = \delta(m - z)$$

When $m \neq z$ we get

$$A_m^{(6)}(z) = 0$$

Then we obtain the reproducing kernel function $A_m(z)$ as

$$A_m(z) = \begin{cases} \sum_{k=1}^{6} a_k(m)z^{k-1} & z \leq m, \\ \sum_{k=1}^{6} b_k(m)z^{k-1} & z > m. \end{cases}$$

The reproducing kernel function should satisfy the conditions. Therefore we have

5. $A_m(0) = 0$

6. $A'_m(0) = 0$

We have 12 unknown coefficients and 6 equations .We need 6 more equations to obtain these coefficients. If we use the properties of Dirac-Delta function we will find

7. $A_{m^+}(m) = A_{m^-}(m)$

8. $A'_{m^+}(m) = A'_{m^-}(m)$

9. $A''_{m^+}(m) = A''_{m^-}(m)$

10. $A'''_{m^+}(m) = A'''_{m^-}(m)$

11. $A^{(4)}_{m^+}(m) = A^{(4)}_{m^-}(m)$.

We have

$$A_m^{(6)}(z) = -\delta(m - z)$$

If we take integral from both sides, we will find

12. $A_{m^+}^{(5)}(m) - A_{m^-}^{(5)}(m) = -1$

Lets find these unknown coefficients. We have

$$A_m(z) = \begin{cases} a_1 + a_2 z + a_3 z^2 + a_4 z^3 + a_5 z^4 + a_6 z^5 & z \le m \\ b_1 + b_2 z + b_3 z^2 + b_4 z^3 + b_5 z^4 + b_6 z^5 & z > m \end{cases}$$

$$A'_m(z) = \begin{cases} a_2 + 2a_3 z + 3a_4 z^2 + 4a_5 z^3 + 5a_6 z^4 & z \le m \\ b_2 + 2b_3 z + 3b_4 z^2 + 4b_5 z^3 + 5b_6 z^4 & z > m \end{cases}$$

$$A''_m(z) = \begin{cases} 2a_3 + 6a_4 z + 12a_5 z^2 + 20a_6 z^3 & z \le m \\ 2\,b_3 + 6b_4 z + 12b_5 z^2 + 20b_6 z^3 & z > m \end{cases}$$

$$A'''_m(z) = \begin{cases} 6a_4 + 24a_5 z + 60a_6 z^2 & z \le m \\ 6\,b_4 + 24b_5 z + 60b_6 z^2 & z > m \end{cases}$$

$$A_m^{(4)}(z) = \begin{cases} 24a_5 + 120a_6 z & z \le m \\ 24b_5 + 120b_6 z & z > m \end{cases}$$

$$A_m^{(5)}(z) = \begin{cases} 120a_6 & z \le m \\ 120b_6 & z > m \end{cases}$$

Then, we can write

1. $a_1 - 120a_6 = 0$
2. $a_2 + 24a_5 = 0$
3. $2a_3 - 6a_4 = 0$
4. $6b_4 + 24b_5 + 60b_6 = 0$
5. $24b_5 + 120b_6 = 0$
6. $120b_6 = 0$
7. $b_1 + b_2 m + b_3 m^2 + b_4 m^3 + b_5 m^4 + b_6 m^5 = a_1 + a_2 m + a_3 m^2 + a_4 m^3 + a_5 m^4 + a_6 m^5$
8. $b_2 + 2b_3 m + 3b_4 m^2 + 4b_5 m^3 + 5b_6 m^4 = a + 2a_3 m + 3a_4 m^2 + 4a_5 m^3 + 5a_6 m^4$
9. $2b_3 + 6b_4 m + 12b_5 m^2 + 20b_6 m^3 = 2b_a + 6a_4 m + 12a_5 m^2 + 20a_6 m^3$
10. $6b_4 + 24b_5 m + 60b_6 m^2 = 6a_4 + 24a_5 m + 60a_6 m^2$
11. $24b_5 + 120b_6 m = 24a_5 + 120a_6 m$

$12.\ 120b_6 - 120a_6 = -1$

Thus, we get coefficients as:

$a_1 = 1$ $\qquad$ $b_1 = 1 + \frac{m^5}{120}$

$a_2 = m$ $\qquad$ $b_2 = m - \frac{m^4}{24}$

$a_3 = \frac{m^2}{4}$ $\qquad$ $b_3 = \frac{m^2}{4} + \frac{m^3}{12}$

$a_4 = \frac{m^2}{12}$ $\qquad$ $b_4 = 0$

$a_5 = \frac{-m}{24}$ $\qquad$ $b_5 = 0$

$a_6 = \frac{1}{120}$ $\qquad$ $b_6 = 0$

Therefore our second reproducing kernel function is obtained as:

$$A_m(z) = \begin{cases} 1 + mz + \dfrac{m^2z^2}{4} + \dfrac{m^3z^2}{12} + \dfrac{m^4z}{24} + \dfrac{z^5}{120}, & z \le m, \\ 1 + mz + \dfrac{m^2z^2}{4} + \dfrac{m^2z^3}{12} + \dfrac{mz^4}{24} - \dfrac{z^5}{120}, & z > m. \end{cases}$$

We can solve the following problems in the $H_2^4[0,1]$.

$$\frac{\partial^3\eta(t,x)}{\partial t^3} + \frac{\partial^{\frac{1}{2}}\eta(t,\alpha)}{\partial t^{\frac{1}{2}}} + \eta(t,x) = \frac{\partial^2\eta(t,x)}{\partial x^2} + e^x(6 + 6\frac{t^{\frac{5}{2}}}{\Gamma\left(\frac{7}{2}\right)})$$

$$0 < x,\ \ 0 < t < 1, \qquad 0 < \alpha \le 1,$$

$$\eta(0,x) = \eta_t(0,x) = \eta_{tt}(0,x) = 0,\ \ 0 \le t.$$

and

$$\frac{\partial^3\eta(t,x)}{\partial t^3}+\frac{\partial^{\frac{1}{2}}\eta(t,\alpha)}{\partial t^{\frac{1}{2}}}=3\frac{\partial^2\eta(t,x)}{\partial x^2}\eta(t,x)+6(x-x^2)\left(t^6+\frac{t^{\frac{5}{2}}}{\Gamma(\frac{7}{2}}+1\right),$$

$$0<x,\ 0<t<1,\ 0<\alpha\leq 1,$$

$$\eta(0,x)=\eta_t(0,x)=\eta_{tt}(0,x)=0,\ 0\leq t.$$

$$\eta(t,0)=\eta(t,1)=0,\ 0\leq x\leq 1.$$

## 4. Conclusion

In this paper, we constructed some important special Hilbert Spaces. We obtained very useful reproducing kernel functions to investigate nonlinear third order fractional differential equation.

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**INTERNATIONAL CONFERENCE ON MATHEMATICS**
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**3-5 July 2019, Istanbul, Turkey**

## The Second Order q-Haar Distribution

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**Abstract**

We define a $p$-adic distribution called the second order $q$-Haar distribution which is closely related to the well-known $q$-Haar distribution. We will give some basic functional equations involving the integrals against the second order $q$-Haar distribution in analogy with the functional equations obtained from the classical $q$-Haar distribution.

**Keywords: p-adic distribution, p-adic integration, q-Haar distribution**

## 1. Introduction

In recent years, $q$-Volkenborn integral has been extensively used especially to give $p$-adic integral representations and recurrence relations of some important families of polynomials and numbers. These include Bernoulli, Apostol-Bernoulli, Euler, Frobenius-Euler and Stirling polynomials/numbers and their q-analogues. The reader is referred to [3,4,5,6,7] and references therein for different applications of $p$-integration regarding the families of these numbers. For the special case $q$=-1, the corresponding $q$-integral is also called as the fermionic $p$-adic integral and is used to construct generating functions for Euler and Genochhi type numbers [8]. Also taking the limit as $q \rightarrow 1$, we recover the the Volkenborne integral. The reader is referred to [9] and also to [1,2,10]).

$q$-Volkenborne integral can easily be defined by replacing the Haar distribution on the ring of $p$-adic integers in the classical Volkenborne integral by the $q$-Haar distribution introduced by T. Kim in [1,2] (See **2. Preliminaries** for details). As the classical Volkenborne integral gives expressions for Bernoulli numbers/polynomials, the $q$-Volkenborne integral provides a generalization of such expressions by introducing a parameter $q$. It is also common to use the $q$-Volkenborne integral of some appropriate functions to define new classes of numbers/polynomials similar to Bernoulli numbers/polynomials.

Here we will define another $p$-adic distribution, namely the second order $q$-Haar distribution. We derive some important functional equations related to integrals with respect to this new distribution. Through these functional equations, we will see that this new $p$-adic distribution has close relations to Kim's $q$-Volkenborn integral.

## 2. Preliminaries

Let $\mathbb{Z}_p$ denote the ring of $p$-adic integers and $\mathbb{Q}_p$ denote its quotient field. We denote the completion of a fixed algebraic closure of $\mathbb{Q}_p$ by $\mathbb{C}_p$ normalized as $|p| = 1/p$.

The Haar distribution on $\mathbb{Z}_p$ is defined as

$$\mu_{Haar}(a+(p^N)) = 1/p^N \in \mathbb{C}_p$$

on the compact open subsets of $\mathbb{Z}_p$ of the form $a+(p^N)$ and extended to all compact open subsets additively. Then the Volkenborn integral of a strictly differentiable function $f : \mathbb{Z}_p \to \mathbb{Z}_p$ is defined as

$$\int_{\mathbb{Z}_p} f(t)\,dt := \lim_{N\to\infty} \sum_{a=0}^{p^N-1} f(a)\mu_{Haar}(a+(p^N)) = \lim_{N\to\infty} \sum_{a=0}^{p^N-1} \frac{f(a)}{p^N}.$$

A similar integral, namely the $q$-Volkenborn integral has been defined by T. Kim by replacing the Haar distribution $a+(p^N) \mapsto 1/p^N$ by the $q$-Haar distribution

$$\mu_q(a+(p^N)) = \frac{(1-q)q^a}{1-q^{p^N}} \tag{1}$$

where $q \in \mathbb{C}_p$ with $|q-1|_p < p^{-1/(p-1)}$. Note that the condition on $|q-1|_p$ is required for the equality $e^{\log q} = \log e^q = q$. Explicitly the $q$-Volkenborn integral is defined as

$$\int_{\mathbb{Z}_p} f(t)\,d\mu_q := \lim_{N\to\infty} \frac{q-1}{q^{p^N}-1} \sum_{a=0}^{p^N-1} f(a)q^a.$$

The term $(q^x-1)/(q-1)$ is commonly called as a $q$-integer and denoted by $[x]_q$. Then the $q$-Volkenborn integral can be written as

$$\int_{\mathbb{Z}_p} f(t)\,d\mu_q = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{a=0}^{p^N-1} f(a)q^a.$$

Note that in various applications $q$ may be also taken as an indeterminate or a complex number. But here we will always consider $q \in \mathbb{C}_p$. The reader is referred to [1, 3, 4] for further results on $q$-Haar distribution and $q$-Volkenborn integral.

## 3. Main Results

Here we will provide another $p$-adic distribution, namely the second order $q$-Haar distribution denoted by $\mu_{q,2}$ and also give a basic equation similar to Kim's equation (See Equation (3) below).

**Definition.** *Let $q \in \mathbb{C}_p$ such that $q \neq 1$. We define $\mu_{q,2}$ on the compact open subsets of $\mathbb{Z}_p$ of the form $a+(p^N)$, $0 \leq a \leq p^N - 1$ as*

$$\mu_{q,2}(a+(p^N)) = \frac{q^a}{[p^N]_q}\left(\frac{(a-p^N)q^{p^N}-a}{q^{p^N}-1}\right) \tag{2}$$

*Proof.* Equality trivially holds for $k=1$. So let $k \geq 2$. Then we have

$$\sum_{b=0}^{k-1} bx^b = x\sum_{b=1}^{k-1} bx^{b-1} = x\frac{d}{dx}\left(\sum_{b=0}^{k-1} x^b\right) = x\frac{d}{dx}\left(\frac{x^k-1}{x-1}\right) = \frac{kx^k}{x-1} - \frac{x(x^k-1)}{(x-1)^2}$$

as desired. □

**Theorem 1.** *Let $\mu_{q,2}$ be given as in (2) with $|q-1|_p < p^{-1/(p-1)}$. Then $\mu_{q,2}$ extends to a p-adic distribution on $\mathbb{Z}_p$.*

*Proof.* It is enough to show that $\mu_{q,2}$ satisfies the following condition (Section II.3 of [11]);

$$\sum_{b=0}^{p-1} \mu_{q,2}(a+bp^N+(p^{N+1})) = \mu_{q,2}(a+(p^N)).$$

Now the sum on the left hand side is equal to

$$\begin{aligned}
&\sum_{b=0}^{p-1} \frac{q^{a+bp^N}}{[p^{N+1}]_q}\left(\frac{(a+bp^N-p^{N+1})q^{p^{N+1}}-a}{q^{p^{N+1}}-1}\right) \\
&= \frac{aq^a}{[p^{N+1}]_q}\sum_{b=0}^{p-1}\left(q^{p^N}\right)^b + \frac{p^N q^a}{[p^{N+1}]_q}\sum_{b=0}^{p-1} b\left(q^{p^N}\right)^b - \frac{q^a}{[p^{N+1}]_q}\frac{p^{N+1}q^{p^{N+1}}}{q^{p^{N+1}}-1}\sum_{b=0}^{p-1}\left(q^{p^N}\right)^b
\end{aligned}$$

Then we have

$$\sum_{b=0}^{p-1} b\left(q^{p^N}\right)^b = \frac{p\,q^{p^{N+1}}}{q^{p^N}-1} - \frac{q^{p^N}(q^{p^{N+1}}-1)}{(q^{p^N}-1)^2}.$$

So summing up the finite geometric series above we obtain

$$\begin{aligned}
&\sum_{b=0}^{p-1} \mu_{q,2}(a+bp^N+(p^{N+1})) = \\
&\frac{aq^a}{[p^N]_q} + \frac{p^{N+1}q^{a+p^{N+1}}}{[p^{N+1}]_q(q^{p^N}-1)} - \frac{p^N q^{a+p^N}(q^{p^{N+1}}-1)}{[p^{N+1}]_q(q^{p^N}-1)^2} + \frac{p^{N+1}q^{a+p^{N+1}}}{(1-q^{p^{N+1}})[p^N]_q}.
\end{aligned}$$

The second and fourth terms cancel out each other since

$$[p^{N+1}]_q(q^{p^N}-1)=\frac{(q^{p^{N+1}}-1)(q^{p^N}-1)}{q-1}=-[p^N]_q(1-q^{p^{N+1}}).$$

Also the third term equals to

$$\frac{p^N\,q^{a+p^N}(q-1)}{(q^{p^N}-1)^2}=\frac{p^N\,q^{a+p^N}}{[p^N]_q(q^{p^N}-1)}.$$

Hence we obtain

$$\sum_{b=0}^{p-1}\mu_{q,2}(a+bp^N+(p^{N+1}))=\frac{q^a}{[p^N]_q}\left(\frac{(a-p^N)q^{p^N}-a}{q^{p^N}-1}\right)=\mu_{q,2}(a+(p^N)).$$

□

Since $\mu_{q,2}$ is a $p$-adic distribution the following definition makes sense.

**Definition.** *Let $f$ be a uniformly differentiable function on $\mathbb{Z}_p$. We define the $q$-integral of $f$ with respect to $\mu_{q,2}$ denoted by $I_{q,2}$ as*

$$I_{q,2}(f)=\int_{\mathbb{Z}_p}f(x)\,d\mu_{q,2}(x)=\lim_{N\to\infty}\sum_{a=0}^{p^N-1}f(a)\mu_{q,2}(a+(p^N)).$$

Now we we will drive an important functional equation for $I_{q,2}$. This equation will be analogue of the one provided by Kim (See Theorem 1 in [2]). But first we recall and set the following notation. Let $f$ be a uniformly differentiable function on $\mathbb{Z}_p$. We put $f_n(x)=f(x+n)$ for any integer $n$. Below $\mu_q$ denotes the $q$-Haar distribution, and let

$$I_q(f)=\int_{\mathbb{Z}_p}f(x)d\mu_q(x).$$

Recall that for $n\geq 1$, Kim's basic functional equation related to $\mu_q$ is

$$q^nI_q(f_n)=I_q(f)+\frac{q-1}{\log q}\sum_{j=0}^{n-1}q^jf'(j)+(q-1)\sum_{j=0}^{n-1}q^jf(j). \tag{3}$$

**Theorem 2.** *For any positive integer n, we have*

$$q^n I_{q,2}(f_n) = I_{q,2}(f) - \sum_{j=1}^{n} q^j I_q(f_j) - \frac{q-1}{(\log q)^2} \sum_{j=0}^{n-1} q^j f'(j) \tag{4}$$

*Proof.* First we show the equality for $n = 1$. So we have

$$\begin{aligned}
qI_{q,2}(f_1) &= q \int_{\mathbb{Z}_p} f(x+1)d\mu_{q,2}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{a=0}^{p^N-1} f(a+1)q^{a+1} \left( \frac{(a-p^N)q^{p^N} - a}{q^{p^N}-1} \right) \\
&= \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{a=1}^{p^N} f(a)q^a \left( \frac{(a-1-p^N)q^{p^N} - a + 1}{q^{p^N}-1} \right) \\
&= \lim_{N\to\infty} \frac{1}{[p^N]_q} S_N + \lim_{N\to\infty} \frac{1}{[p^N]_q} L_N
\end{aligned} \tag{5}$$

where we set

$$\begin{aligned}
S_N &:= \sum_{a=0}^{p^N-1} f(a)q^a \left( a - 1 - \frac{p^N q^{p^N}}{q^{p^N}-1} \right) \\
L_N &:= \left[ f(p^N)q^{p^N} \left( p^N - 1 - \frac{p^N q^{p^N}}{q^{p^N}-1} \right) - f(0) \left( -1 - \frac{p^N q^{p^N}}{q^{p^N}-1} \right) \right]
\end{aligned}$$

to simplify the notation. Now by definition, we have that

$$\lim_{N\to\infty} \frac{1}{[p^N]_q} S_N = I_{q,2}(f) - I_q(f). \tag{6}$$

For the term $L_N$ we have that

$$
\begin{aligned}
L_N &= f(p^N)q^{p^N}\left(p^N - 1 - \frac{p^N q^{p^N}}{q^{p^N}-1}\right) - f(0)\left(-1 - \frac{p^N q^{p^N}}{q^{p^N}-1}\right)\\
&= f(p^N)q^{p^N}\left(\frac{p^N}{1-q^{p^N}} - 1\right) - f(0)\left(\frac{p^N q^{p^N}}{1-q^{p^N}} - 1\right)\\
&= \left(\frac{f(p^N)-f(0)}{\frac{(1/q)^{p^N}-1}{p^N}}\right) - \left(f(p^N)q^{p^N} - f(0)\right)
\end{aligned}
$$

Multiplying by $1/[p^N]_q$ we obtain

$$
\frac{1}{[p^N]_q}L_N = (q-1)\left[\frac{f(p^N)-f(0)}{q^{p^N}-1}\frac{1}{\frac{(1/q)^{p^N}-1}{p^N}} - \frac{f(p^N)q^{p^N}-f(0)}{q^{p^N}-1}\right].
$$

Then by passing to limit and utilizing the definition of derivative we obtain

$$
\begin{aligned}
\lim_{N\to\infty}\frac{1}{[p^N]_q}L_N &= (q-1)\left[\frac{f'(0)}{\log q}\frac{(-1)}{\log q} - \frac{f'(0)+f(0)\log q}{\log q}\right]\\
&= -(q-1)f(0) - \frac{q-1}{\log q}f'(0) - \frac{q-1}{(\log q)^2}f'(0).
\end{aligned} \tag{7}
$$

We plug the expressions (6) and (7) in (5), and then use (3) for $n=1$ to obtain

$$
\begin{aligned}
qI_{q,2}(f_1) &= I_{q,2}(f) - I_q(f) - (q-1)f(0) - \frac{q-1}{\log q}f'(0) - \frac{q-1}{(\log q)^2}f'(0)\\
&= I_{q,2}(f) - qI_q(f_1) - \frac{q-1}{(\log q)^2}f'(0)
\end{aligned}
$$

as desired. This proves the identity (4) for $n=1$. Applying it successively, we easily obtain the functional equation (4) for any $n \geq 1$. □

## 4. Conclusion

A new $p$-adic distribution $\mu_{q,2}$ has been introduced and a related equation involving the $p$-adic integrals with respect to $\mu_{q,2}$ is proven. Equation 4 given in Theorem 2 is similar to T. Kim's equation relying on $q$-Haar dsitribution. Equation 4 may be used to prove new idenditites involving special numbers (e.g. Bernoulli numbers) in number theory.

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**3-5 July 2019, Istanbul, Turkey**

# Analytical Solution of Falkner-Skan Equation with Heat and Mass Transfer

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***Keywords**:* Shooting method, Decomposition Method, MHD flow, Boundary layer, Temperature profile, concentration profile.

## Abstract

In this paper, numerical and analytical methods were proposed to study the steady two-dimensional and viscous flow of the laminar boundary layer over a flat plate under the effect of a transverse magnetic field in the presence of an adverse pressure gradient. The main purpose of this study is to show the influence of the magnetic field on the boundary layer flow. The investigated problem was handled analytically using the decomposition method. Furthermore, a numerical study was performed using the fourth order Runge Kutta method featuring shooting technique. Analytical and numerical results obtained for the temperature profile are in excellent agreement.

## Introduction

The invention of the electromagnetic pump in 1918 by Hartmann [1], fuelled a lot of interest in the field of magnetohydrodynamic (MHD) flow. This field of study has become essential for many industries; it was originally applied to astrophysical and geophysical problems, where it is still very important. But more recently, it has been applied to the problem of fusion power, where the application is the creation and containment of hot plasmas by electromagnetic forces, since material walls would be otherwise destroyed. Astrophysical problems include solar structure, especially in the outer layers, the solar wind bathing the earth and other planets, and interstellar magnetic fields. The primary geophysical problem is planetary magnetism, produced by currents deep in the planet, a problem that has not been solved to any degree of satisfaction. It seems that the first work investigating the flow over a flat plate using the Rung Kutta numerical methods was done by Howarth [2]. Later, numerical methods to investigate the two- dimensional motion of a viscous incompressible fluid impulsively started past a flat plate were employed by D.B. Ingham [3]. Kumari. M, Nath [4] considered the effect of the magnetic field on the stagnation point flow and heat transfer on a linearly stretching sheet. The effect of a uniform transverse magnetic field over a stretching surface with heat transfer was studied by Chakrabati A, Gupta [5]. Another study of the momentum and the heat transfer of a hydromagnetic fluid past a stretching sheet was carried out by Liu IC [6]. On his part, Sakiadis [7] introduced the concept of boundary layer flow over a stretching surface. Recently, Mahsud, Y [8], Xenos [9] and Shahmohamad [10] investigated different problems of boundary layer flow.

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## Results

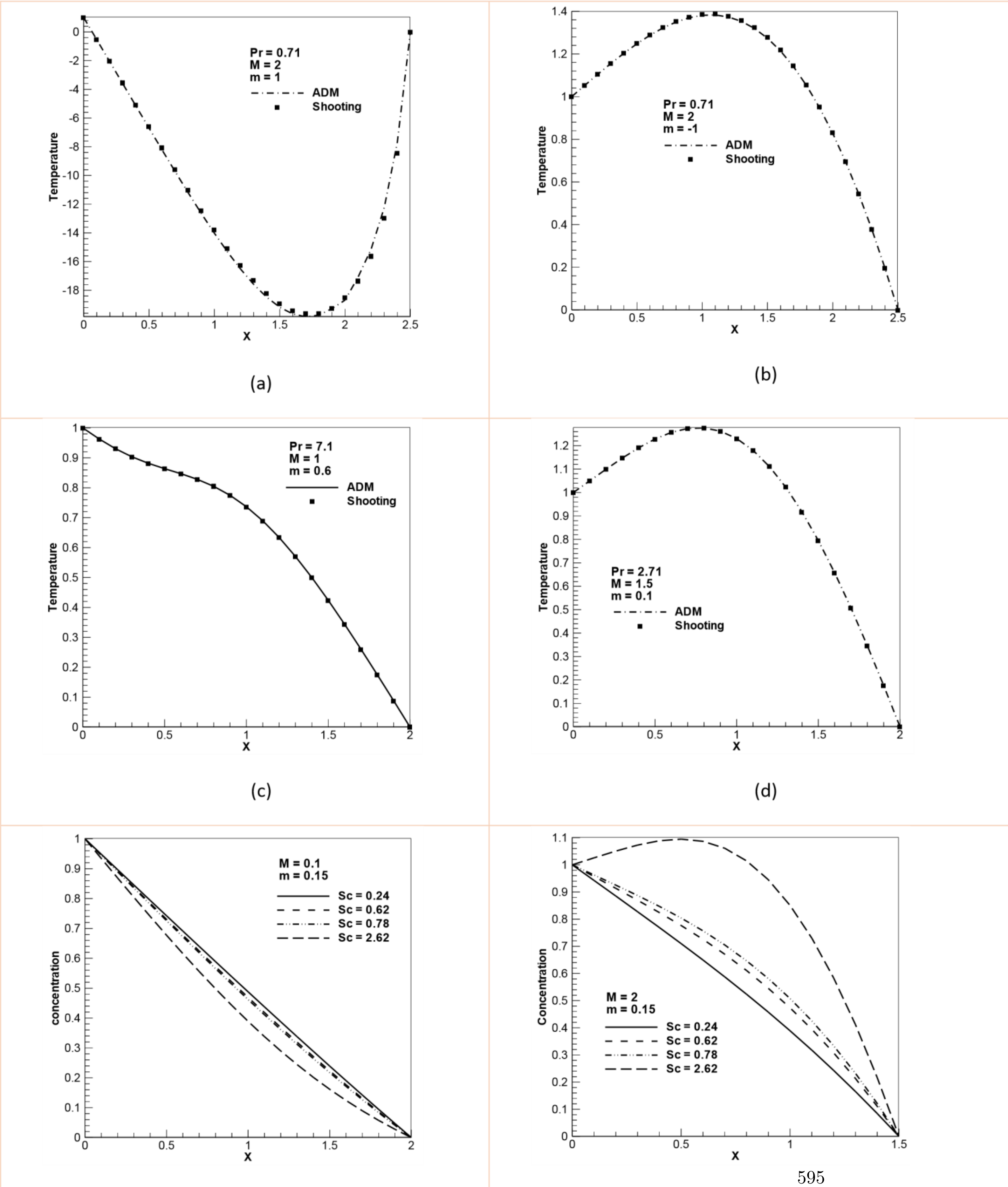




*Figure 1 Concentration and temperature profile obtained analytically and numerically from various parameter*

**Governing equation**:

Once the system of partial differential equations, governing the dynamic, thermal and concentration field of the MHD boundary layer flow is translated into a system of ordinary equations and after some simplifications, we end up with a given equation of magnetohydrodynamic Falkner-Skan (equation 5). The followed approach is as continuation:

Introducing the similarity variable $X$ and the dimensionless stream line function $f$, gives:

$$X = y\sqrt{\frac{u_\infty}{\nu x}} = \frac{y}{x}\sqrt{Re_x} \ , \quad u = u_\infty . f', u_\infty = a_0 . x^m \ , \quad \Psi = \sqrt{\nu . u_\infty . x} . f(X)$$

$$\frac{d^3 f}{dX^3} + \frac{(m+1)}{2} f \frac{d^2 f}{dX^2} + m\left(1 - \left(\frac{df}{dX}\right)^2\right) - \left(\frac{df}{dX} - 1\right) M = 0 \qquad (1)$$

The parameter $M = \frac{\sigma B_0^2 x}{\rho u_\infty}$ characterizes the number of Hartmann.

$$\theta'' + \frac{1}{2} . Pr . (m+1) . f . \theta' - Pr . m . f' . \theta + Pr . M . Ec . f'^2 = 0 \qquad (2)$$

$$\phi'' + \frac{1}{2} . Sc . (m+1) . f . \phi' - Sc . m . f' . \phi + Sc . M . Ec . f'^2 = 0 \qquad (3)$$

Where $Sc = \frac{\nu}{D}$, is the Schmidt number.

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**3-5 July 2019, Istanbul, Turkey**

# A simple mathematical approach for the determination of the optimal insulation thickness of cryogenic tanks

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**Abstract**

Thermal insulation of cryogenic gas storage tanks is of great importance. It is governed by the control of heat transfer mechanisms and requires a specific analysis, because to isolate this category of tanks we have a large number of insulating materials. However, technical constraints of required performance prevent for the application of some materials. In addition to the thermophysical characteristics, other criteria must also be considered to choose the suitable insulation, namely the implementation, the lifetime, the reliability and the cost of achieved insulation. This article is based on a mathematical method for optimizing the thickness of cylindrical tanks with elliptical bottom intended for cryogenic fluid storage. The considered insulation is a uniformly applied outer layer whose thickness varies according to the boundary conditions of the external and internal tank surfaces. Radial heat transfer, based on heat conduction equation, is taken into consideration. An expression of the optimal insulation thickness derived from the total cost function and depending on the geometrical parameters of the container is presented.

**Keyword(s):** Cryogenics, Insulation, Mathematical method.

## Nomenclature

| | | |
|---|---|---|
| $D$ | Tank diameter | [m] |
| $e_i$, $X_i$ | Insulation thickness | [m] |
| $L$ | Length of the cylinder | [m] |
| $n'$ | Mass unit cost of stored fluid | [€/kg] |
| $\dot{Q}$ | Thermal flux | [W] |
| $\dot{Q}_1$ | Cylinder side thermal flux | [W] |

| | | |
|---|---|---|
| $\dot{Q}_2$ | Lateral side thermal flux | [W] |
| $R_i$ | Internal radius | [m] |
| $R_e$ | External radius | [m] |
| $S_F$ | Tank bottom surface | [m$^2$] |
| $S_L$ | Tank side surface | [m$^2$] |
| $\overline{S}$ | Thermal surface of reference | [m$^2$] |
| $T_{\text{ext}}$ | External temperature | [K] |
| $T_{\text{liq}}$ | Liquefaction temperature | [K] |
| $V_{\text{cyl}}$ | Cylinder volume | [m$^3$] |
| $x$ | Reduced thickness | |
| $x_{\text{opt}}$ | Dimensionless thickness | |
| $X_{\text{opt}}$ | Optimal thickness | [m] |
| $\lambda_i$ | Insulator thermal conductivity | [W/(m·K)] |
| $\Delta T_{\text{éq}}$ | Equivalent thermal gradient | [K] |
| $\Delta H_v$ | Enthalpy of vaporization | [kJ/kg] |

## 1. Introduction

In designing and manufacturing cryogenic tanks for transport and storage of liquid nitrogen, oxygen, and argon, special attention is given to improving their technical characteristics and, in particular, to reducing the specific loss of the liquid caused by evaporation. However, this often contradicts the technical and economic considerations. A technical and economic model of optimization was constructed [1] which takes into account both the manufacturing and service conditions of the tanks. Two compulsory requirements must be fulfilled in this case: Comparison of the tanks with insulation of various types should be carried out for the same holding capacity of the tanks to avoid the effect of the scale factors; optimum thickness must be determined for each type of insulation. The optimality criterion was represented by the corrected expenses consisting of the sum of net cost of the product and the proportion of capital investment taken into account by the norm factor. The results and execution of many full scale

fire engulfment tests on LPG tanks, protected with thin sublimation and The results and execution of many full scale fire engulfment tests on LPG tanks, protected with thin sublimation and intumescent coatings, mineral wool with steel jackets, vermiculite coatings and water spray systems, are presented [2]. Additional furnace tests with coated steel plates have been carried out to optimize the necessary coating thickness. Similarly, an overview of the test conditions for the recognition based on technical rules and BAM requirements for fire protection systems of LPG tanks are shown and explained.
In order to evaluate the thermal performance of the MLI fabricated in the horizontal cryostats of superconducting magnets, it is important to investigate the contact pressure in the MLI [7]. At first, a single thin film wound around the horizontal cylinder was analyzed to evaluate the contact pressure acting on the cylinder. The analysis has been extended to the multiply wound film around horizontal cylinder, in order to investigate the distribution of contact pressure between adjacent layers. By using experimental data obtained with a flat panel calorimeter, the results of this analysis have been applied to evaluate the thermal performance of MLI around a horizontal cylinder.

In this article we first propose to formulate heat losses for a cylindrical tank. Then, using a simple mathematical procedure, an analytical expression giving the optimal thickness of insulation will be achieved. It will be assumed that insulation is of one layer only. The insulation cost and thermophysical properties are known, so is the nature of the stored fluid. The thickness and the thermal resistance of the studied tank are neglected.

## 2. Preliminaries

### 2.1 Method of calculation of tank heat losses

The sphere and hollow cylinder are the two geometric shapes used in the chemical industry for their storage capacities. To determine the radial heat flux through a cylindrical tank of elliptical bottoms, it is assumed that the container is homogeneous and the temperatures of the inner ($T_{\text{liq}}$) and outer ($T_{\text{ext}}$) surfaces are kept constant.

Conduction heat transfer is supposed unidirectional and stationary. The heat flux, evaluated at mid-thickness of the insulation, can be written:

$$\dot{Q} = \dot{Q}_1 + \dot{Q}_2 \tag{1}$$

The heat flux through the cylindrical surface of length $L$:

$$\dot{Q}_1 = \frac{2\pi\lambda_i L}{ln\,\frac{R_e}{R_i}}\Delta T_{eq} \tag{2}$$

$\Delta T_{eq}$ is the equivalent thermal gradient given by Buhler [8], as:

$$\Delta T_{eq} = 0{,}9\left(T_{ext} - T_{liq}\right) \quad (3)$$

Tank thickness is negligible and the layer of insulation adheres perfectly to the wall. Eq. (2) can easily be written in the form:

$$\dot{Q}_1 = \frac{2\pi\lambda_i L}{\ln\left(1+\frac{2e_i}{D}\right)}\Delta T_{eq} \quad (4)$$

Heat conduction flux through the side faces of the container:

$$\dot{Q}_2 = \frac{\lambda_i}{e_i} S_L \Delta T_{eq} \quad (5)$$

with,

$$S_L = 2S_F = 0{,}69\pi D^2 \quad (6)$$

Whence:

$$\dot{Q}_2 = \frac{\lambda_i}{e_i} 0{,}69\pi D^2 \Delta T_{eq} \quad (7)$$

Finally:

$$\dot{Q} = 2\pi\lambda_i \Delta T_{eq}\left[\frac{L}{ln\left(1+\frac{2e_i}{D}\right)} + \frac{0{,}69D^2}{2e_i}\right] \quad (8)$$

## 3. Main Results

### 3.1 Determination of optimum thickness

We seek to determine the optimal insulation thickness of a sized tank. We will establish an analytical expression of the optimal thickness, based on mathematical approximations, according to geometric, thermal and economic parameters. Only insulation thermal resistance is taken into account, $X_i$ and $\lambda_i$ are respectively thickness and conductivity of insulator. The non-wetted walls are assumed to be dry and the previous thermal gradient ($\Delta T_{eq}$) is adopted.

The thermal flux is expressed through an average surface $\bar{S}$ evaluated at insulator mid-thickness, Conte [9]:

$$\dot{Q} = \frac{\lambda_i}{X_i}\bar{S}0,9\Delta T_{eq} \tag{9}$$

Geometrical parameters of the cylindrical reservoir considered (Fig. 1) are expressed by the mathematical equations and approximations below [9].

Dimensionless configuration factor:

$$\theta = \frac{L}{D} \tag{10}$$

Dimensionless reduced thickness:

$$x = \frac{X_i}{D} \tag{11}$$

Tank volume:

$$V_{tk} = (\theta - 0,1667)\frac{\pi D^3}{4} \tag{12}$$

Tank surface:

$$S_{tk} = (\theta + 0,19)\pi D^2 \tag{13}$$

Insulation volume:

$$V_{in} = \left[(\theta + 0,25)x + (\theta - 0,5)x^2 - \frac{x^3}{3}\right]\pi D^3 \tag{14}$$

External surface of insulation:

$$S_{in} = [(\theta + 0,19) + (2\theta + 2,76)x + 0,76x^2]\pi D^2 \tag{15}$$

Thermal surface of reference:

$$\bar{S} = [(\theta + 0,19) + (2\theta + 1,38)x + 0,19x^2]\pi D^2 \tag{16}$$

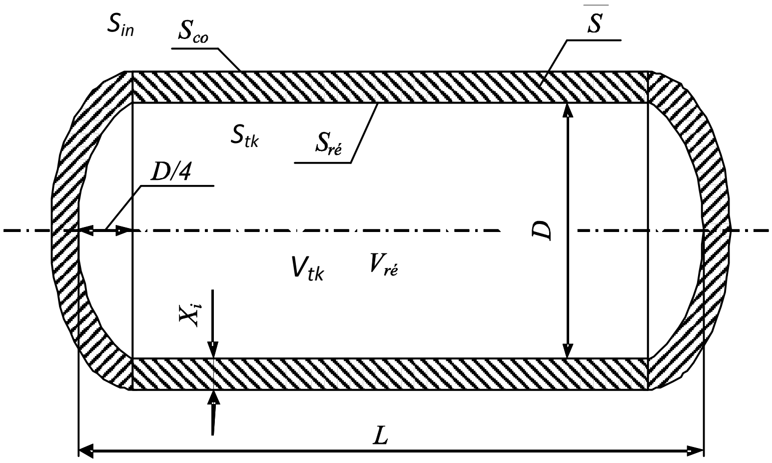


Fig. 1. Main geometrical characteristics of the tank.

The total cost of the installation ($C_T$) is based on geometrical approximations of the container and depends on three necessary costs.

Insulation cost:

$$C_{in} = V_{in} \cdot C_{in/v} \tag{17}$$

Cost of shell:

$$C_{sh} = S_{in} \cdot C_{sh/s} \tag{18}$$

Operating energy cost:

$$C_{en} = \frac{\bar{S}}{xD} C_{en/L} \tag{19}$$

And, the energy cost per unit length of insulation:

$$C_{en/L} = \lambda_i \frac{3{,}156 \cdot 10^7}{\Delta H_v} n' N \Delta T_{eq} \tag{20}$$

Equation in which $N$ is the coating lifetime and $n'$ the unit cost of 1 kg of the stored fluid.

The total cost is expressed as:

$$C_T = C_{in} + C_{sh} + C_{en} \tag{21}$$

Hence, equations (17), (18) and (19) become:

$$C_{in} = \left[(\theta + 0{,}25)x + (\theta - 0{,}5)x^2 - \frac{x^3}{3}\right]\pi D^3 C_{in/v} \tag{22}$$

$$C_{sh} = [(\theta + 0{,}19) + (2\theta + 2{,}76)x + 0{,}76x^2]\pi D^2 C_{sh/s} \tag{23}$$

$$C_{en} = [(\theta + 0{,}19) + (2\theta + 1{,}38)x + 0{,}19x^2]\frac{\pi D C_{en/L}}{x} \tag{24}$$

Finally, the equation of the total cost:

$$\begin{aligned} C_T = {} & -\pi D^3 C_{in/v}\frac{x^3}{3} + [(\theta - 0{,}5)\pi D^3 C_{in/v} + 0{,}76\pi D^2 C_{sh/s}]x^2 \\ & + [(\theta + 0{,}25)\pi D^3 C_{in/v} + (2\theta + 2{,}76)\pi D^2 C_{sh/s} + 0{,}19\pi D C_{en/L}]x \\ & + (\theta + 0{,}19)\frac{\pi D C_{en/L}}{x} \\ & + [(\theta + 0{,}19)\pi D^2 C_{sh/s} + (2\theta + 1{,}38)\pi D C_{en/L}] \end{aligned} \tag{25}$$

In order to obtain an analytical expression giving the optimal insulation thickness, the interpolation of the type below is defined with an approximation precision of $10^{-3}$.

$$x_{opt} = [x_0^{-n} + x_\infty^{-n}]^{\frac{-1}{n}}, \quad x_{opt} \in ]0,1[ \tag{26}$$

$n$ being a parameter that depends on the configuration factor ($\theta$), $x_o$ is estimated for $C_{en/L} \to 0$ and $x_\infty$ estimated for $C_{en/L} \to \infty$. The derivative of the total cost must be zero:

$$\begin{aligned} \frac{dC_T}{dx} = {} & -x^4 + [(2\theta - 1) + 1{,}52b]x^3 + [0{,}19a + (2\theta + 2{,}76)b + \theta + 0{,}25]x^2 - (\theta + 0{,}19)a \\ & = 0 \end{aligned} \tag{27}$$

Where,

$$a = \frac{C_{en/L}}{C_{in/v}D^2} \; and \; b = \frac{C_{sh/s}}{C_{in/v}D}$$

The coefficient $n$ is calculated as:

$$n = 0{,}54\left[1 + \frac{1}{\sqrt{\theta}}\right] \tag{28}$$

If $X_i << D$, terms in $x^3$ and $x^4$ are negligible.

$$\lim_{a\to 0}\frac{dC_T}{dx} \Rightarrow x_0 = \sqrt{\frac{(\theta+0{,}19)a}{0{,}19a+(2\theta+2{,}76)b+\theta+0{,}25}} \tag{29}$$

and,

$$\lim_{a\to\infty}\frac{dC_T}{dx} \Rightarrow x_\infty = \sqrt{1+\frac{\theta}{0{,}19}} \tag{30}$$

Replacing $x_o$ and $x_\infty$ by their respective values in equation (26), one obtains for $\theta = 3$:

$$x_{opt} = \left\{\left[\frac{3{,}19\left(\frac{C_{en/L}}{C_{in/v}D^2}\right)}{0{,}19\left(\frac{C_{en/L}}{C_{in/v}D^2}\right)+8{,}76\left(\frac{C_{sh/s}}{C_{in/v}D}\right)+3{,}25}\right]^{-0{,}4259}+0{,}3008\right\}^{-1{,}1737} \tag{31}$$

## 4. Conclusion

The problem of optimal design has been formulated taking into account the total cost (Shell + insulation + exploitation) presented as a function of insulation thickness (Eq. 31). Obviously, this is a limited simplification especially for vacuum multilayer insulation systems (MLI) using super-insulators. In such systems the cost of isolation is a function of many parameters and requires a more complex optimization [10]. Nevertheless, the obtained formulation with the adopted technical-economic criterion remains applicable for the insulation of tanks containing cryogenic fluids whose liquefaction temperature does not exceed 100 K.

To minimize heat leaks into storage tanks and transfer lines, high-performance materials are needed to provide high levels of thermal isolation. Complete knowledge of thermal insulation is a key part of enabling the development of efficient, low-maintenance cryogenic systems. The choice of insulation often depends on the tank size, in fact, for big containers it is possible to admit less efficient insulators, on the other hand, smaller is the tank, good must be the thermal characteristics. The complexity due to heat transfer processes and physico-mechanical constraints imposes a large number of laboratory and workshop tests before launching a product on the market.

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oINTERNATIONAL CONFERENCE ON MATHEMATICS
"An Istanbul Meeting for World Mathematicians"
3-5 July 2019, Istanbul, Turkey
numerical approaches by finite volumes of a model and its renewal

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**Abstract**. In this work, we study solutions of an evolution problem with a fractional power of the laplacien in the principal part and algebraic degree in nonlinear part,such equation butt in naturally in continuum mechanics area.Our results includ existence, uniqueness of solutions problem with fractional term. These problems arise in a variety of engineering analysis and design situations .

**Introduction**:

The nonlinear diffusion equation represent the most important phenomena occurring in the world. Manipulation of nonlinear phenomena is of great importance in applied mathematics, physics, and issues related to engineering.The nonlinear diffusion equation was known to Forsy [1] and had been discussed by Bateman in connection with various viscous flows [2]. J.M.Burgers considered this equation as a model of turbulence [3 - 4], for example of nonlinear equation with frational term is fractional burgers equation . Burgers equations involving in their linear parts fractional powers $\Delta_\alpha = -(\Delta)^{\alpha/2}$ of the laplacien, $0 \prec \alpha \leq 2$, have been investigated in connection with certain models of hydrodynamical phenomena; see shlesinger and al [5] , Funaki [6] and Biler [7] . Funaki and Woyczynski studied existence[8], uniqueness, regularity and asymptotic behavior of solutions to the multidimensional fractal Burgers-type equation

$$\frac{\partial u\left(t,x\right)}{\partial t} = v\Delta_\alpha u\left(t,x\right) - a\nabla u^r\left(t,x\right)$$

where $x \in \mathbb{R}^d,\ d \geq 1,\ \alpha \in [0,2],\ r \geq 1,$ and $a \in \mathbb{R}^d$.For $\alpha \succ 3/2$ and $d = 1$ they prove existence of a unique regular weak solution for initial conditions in $H^1(\mathbb{R})$.

A large variety of physically motivated (linear) fractal differential equation can be found in Shlesinger et al [9], including applications to hydrodynamics, statistical mechanics, physiology and molecular biology. Fractal relaxation models are described in Saichev and Woyczynski [10] . Models of several other hydrodynamical phenomena (including hereditary and viscoelastic behavior and propagation of nonlinear acoustic waves in a tunnel with an array of Helmholtz resonators) employing the Burgers equation involving the fractional Laplacian have also been developed (Sugimoto and Kakutani [11], Sugimoto ([12][13]).

In this work, we study solutions of an evolution problem governed by equations of fractional term.

Here we consider the nonlinear equation with fractional term:

$$\frac{\partial u\left(x,t\right)}{\partial t}+u\left(x,t\right)\frac{\partial u\left(x,t\right)}{\partial x}+D_x^{\alpha}u\left(x,t\right)+d\left(x\right)u\left(x,t\right)=f\left(x,t\right) \qquad (1)$$

where $u(x,t)$ is the unknown function and f, d are functions satisfying given condition of regularity .

where $x\in I\subset\mathbb{R};t\succ 0;u:I\times\mathbb{R}^{+}\rightarrow\mathbb{R}$.

we will study the existence of the solution of the problem (1).

Using a priori elementary estimates, we prove results for problem (1).This will show the role of dissipative operator -$D_x^{\alpha}$ and, in particular, its strength compared to the nonlinearity $uu_x$.

We look for weak solutions of (1) supplemented with the initial condition $u(x,0)=u_0(x)$ in $V_2$ such that

$$V_2=L^{\infty}(]0,T[\,;L^2(I))\cap L^2(]0,T[\,;H^1(I))$$

satisfying the identity

$$\int u(x;t)\phi(x,t)dx-\int_0^t\int u(x,t)\phi_t(x,t)dxdt+\int_0^t\int D_x^{\frac{\alpha}{2}}u(x,t)D_x^{\frac{\alpha}{2}}\phi(x,t)dxdt$$

$$-\int_0^t\int\frac{1}{2}u^2(x,t)\phi_x(x,t)dxdt+\int_0^t\int d(x)u(x,t)\phi(x,t)dxdt$$

$$=\int u_0(x)\phi(x,0)dx+\int_0^t\int f(x,t)dxdt \quad (1.1)$$

$$for\ t\in\,]0,T[\ \ and\ \phi(x,t)\in H^1(I\times\,]0,T[)$$

Let $\frac{3}{2}\prec\alpha\prec 2,T\succ 0, and\ u_0(x)\in H^1(I)$.Then problem (1) has an unique weak solution $u\in V_2$.Moreover , u satisfies the following regularity proptrties:

$$u\in L^{\infty}(]0,T[\,;H^1(I))\cap L^2(]0,T[\,;H^{1+\frac{\alpha}{2}}(I))$$

and

$$u_t\in L^{\infty}(]0,T[\,;L^2(I))\cap L^2(]0,T[\,;H^{\frac{\alpha}{2}}(I))$$

Proof: suppose u is a weak solution of (1) ;Multiplying (1) by $u_n(x;t)$, after applying the definition of the diffusion operator $D^\alpha$ we arrive at

$$\frac{d}{dt}\int u_n(x;t)u_n(x;t)dx+\int(D_x^{\frac{\alpha}{2}}u_n)(x;t)(D_x^{\frac{\alpha}{2}}u_n)(x;t)dx+\int u_n(x;t)(u_n)_x(x;t)u_n(x;t)dx$$

$$+\int d(x)u_n(x;t)u_n(x;t)dx=\int f(x;t)u_n(x;t)dx$$

which implies

$$\frac{d}{dt}\int u_n^2(x;t)dx+\int(D_x^{\frac{\alpha}{2}}u_n)^2(x;t)dx+\int(u_n)_x(x;t)u_n^2(x;t)dx+\int d(x)u_n^2(x;t)dx$$

$$=\int f(x,t)u_n(x;t)dx \quad (1.2)$$

one has

$$\frac{d}{dt}\int u_n^2(x;t)dx=\frac{d}{dt}\left|u_n(t)\right|_2^2 \quad (1.3)$$

$$\int(D_x^{\frac{\alpha}{2}}u_n)^2(x;t)dx=\left|D_x^{\frac{\alpha}{2}}u_n\right|_2^2 \quad (1.4)$$

$$\int(u_n)^2(x,t)(u_n)_x(x,t)dx=(\frac{1}{3}u_n^3(x,t)\mid_I=cst\leq C_1\left|u_n\right|_2 \quad (1.5)$$

$$\int d(x)u_n^2(x;t)dx\geq\min d(x)\int u_n^2(x;t)dx\geq\min d(x)\left|u_n(t)\right|_2^2 \quad (1.6)$$

$$\int f(x,t)u_n(x;t)dx\leq\left|f\right|_2\left|u_n\right|_2 \quad (1.7)$$

then is holds that

$$\frac{d}{dt}\left|u_n\right|_2^2+\left|D_x^{\frac{\alpha}{2}}u_n\right|_2^2+\min d(x)\left|u_n(t)\right|_2^2\leq(\left|f\right|_2+C_1)\left|u_n\right|_2 \quad (1.8)$$

Similarly, differentiating (1) with respect to $x$ and multiplying by $(u_n)_x$ we obtain

$$\frac{d}{dt}\int u_n(x,t)(u_n)_x(x,t)dx+\int(D_x^{\frac{\alpha}{2}}u_n)(x,t)(u_n)_x(x,t)dx+\int\frac{1}{2}(u_n^2)_x(x,t)(u_n)_x(x,t)dx$$

$$+\int d(x)(u_n)(x,t)(u_n)_x(x,t)dx=\int f(x,t)(u_n)_x(x,t)dx$$

which implies

$$\frac{1}{2}\frac{d}{dt}\int\left(u_n^2)_x(x,t)\right)dx \quad +\int(D_x^{\frac{\alpha}{2}}u_n(x,t))(u_n)_x(x,t)+\int\frac{1}{2}(u_n^2)_x(x,t)(u_n)_x(x,t)dx$$

$$+\int d(x)\frac{1}{2}\left(u_n^2)_x(x,t)\right)dx=\int f(x,t)(u_n)_x(x,t)dx \quad (1.9)$$

$$\tfrac{d}{dt}\int u_n(x,t)(u_n)_x(x,t)dx=\tfrac{1}{2}\tfrac{d}{dt}\int\left(u_n^2)_x(x,t)\right)dx$$

$$|(u_n)_x|_3^3\le\|u_n\|_{1,3}^3\le C\|u_n\|_{1+\frac{\alpha}{2}}^{\frac{7}{2+\alpha}}|u_n|_2^{3-\frac{7}{2+\alpha}}\le\|u_n\|_{1+\frac{\alpha}{2}}^2+C|u_n|^m$$

because

$$-\int\frac{1}{2}(u_n^2)_x(x,t)(u_n)_x(x,t)dx=\int u_n(x,t)(u_n)_x(x,t)(u_n)_{xx}(x,t)dx$$

$$=\frac{1}{2}\int u_n(x,t)((u_n)_x^2)_x(x,t)dx=-\frac{1}{2}\int(u_x)_x^3(x,t)dx$$

$$\int d(x)(u_n)(x,t)(u_n)_x(x,t)dx=\frac{1}{2}\int d(x)(u_n^2)_x(x,t)dx\le\sup(d(x))\left[\frac{1}{2}u_n^2\right]_I=C$$

$$\int f(x,t)(u_n)_x(x,t)dx\le|f|_2\,|(u_n)_x|_2$$

it holds that

$$\frac{d}{dt}|(u_n)_x|_2^2+2\left|D_x^{1+\frac{\alpha}{2}}u_n\right|_2^2\le|(u_n)_x|_3^3+(2|f|_2+C)|u_n|_2 \quad (1.10)$$

Note that the assumption $\alpha>3/2$ has been used in the interpolation of the $W^{1.3}$ norm of $u$ by the norms of its fractional derivatives to have $7/(2+\alpha)<2$. Indeed, this follows from Henry rivatives like in,e.g., Triebel (1983, 1992). Combining this with (1),(1.2) and(1.3) we get

$$\frac{d}{dt}\left\|u_n\right\|_1^2 + \left\|u_n\right\|_{1+\frac{\alpha}{2}}^2 \leq C(|f|_2\,|u_n|_2 + |u_n|_2^2 + |u_n|^m + C_1) \quad (1.11)$$

and by (1.8) implies

$$\frac{d}{dt}\left|u_n\right|_2^2 \leq (|f|_2 + C_1 + C_2)\,|u_n|_2 \Longrightarrow |u_n(t)|_2 \leq M + |(u_n)_0|_2\,,\ \forall t \in [0,T]$$

hence we obtain

$$\left\|u_n\right\|_1^2 + \int_0^t \left\|u_n(t)\right\|_{1+\frac{\alpha}{2}}^2 ds \leq C = C(T, f, \left\|(u_n)_0\right\|_1).(1.12)$$

To get the estimate for the time derivative of the solution, let us differentiate (1) with respect to t and multiply by $u_t$.

$$\frac{d}{dt}\int u_n(x,t)(u_n)_t(x,t)dx + \int (D_x^{\frac{\alpha}{2}} u_n)(x,t)(u_n)_t(x,t)dx + \int \frac{1}{2}(u_n^2)_x(x,t)(u_n)_t(x,t)dx$$

$$+\int d(x)(u_n)(x,t)(u_n)_t(x,t)dx = \int f(x,t)(u_n)_t(x,t)dx$$

$$\frac{1}{2}\frac{d}{dt}\int \left((u_n^2)(x,t)\right)_t dx \ + \frac{1}{2}\int (D_x^{\frac{\alpha}{2}}(u_n^2)_t(x,t)dx + \int \frac{1}{2}(u_n^2)_x(x,t)(u_n)_t(x,t)dx$$

$$+\frac{1}{2}\int d(x)(u_n^2)_t(x,t)dx = \int f(x,t)(u_n)_t(x,t)dx$$

such as

$$\int d(x)(u_n^2)_t(x,t)dx \leq C_1\,|(u_n)_t|_2^2$$

and

$$\int f(x,t)(u_n)_t(x,t)dx \leq |f|_2\,|(u_n)_t|_2$$

$$\frac{1}{2}\int |(u_n)_x|\,(u_n)_t^2 \leq C\left\|(u_n)_t\right\|_{\frac{\alpha}{2}}^{\frac{1}{\alpha}}\ |(u_n)_t|_2^{2-\frac{1}{\alpha}}\ |(u_n)_x|_2 \leq \frac{1}{2}\left\|(u_n)_t\right\|_{1+\frac{\alpha}{2}}^2 + C\,|(u_n)_t|_2^2$$

since

$$-\int (u_n(u_n)_x)_t(u_n)_t = -\int (u_n)_x(u_n)_t^2 - \frac{1}{2}\int u_n((u_n)_t^2)_x = -\frac{1}{2}\int (u_n)_x(u_n)_t^2$$

it holds that

$$\frac{1}{2}\frac{d}{dt}\left|(u_n)_t\right|_2^2+\frac{1}{2}\left|D_x^{\frac{\alpha}{2}}(u_n)_t\right|_2^2 \leq \frac{1}{2}\left\|(u_n)_t\right\|_{1+\frac{\alpha}{2}}^2+C\left|(u_n)_t\right|_2^2+|f|_2\left|(u_n)_t\right|_2+C_1\left|(u_n)_t\right|_2^2 \quad (1.13)$$

A classical Gronwall inequality gives

$$\left|(u_n)_t\right|_2^2+\int_0^T\left\|(u_n)_t(s)\right\|_{\frac{\alpha}{2}}^2 ds \leq C(T) \quad (1.14)$$

It holds,from (1.12) and (1.14),that a solution $u_n$ is bounded.Then it is sufficient in order to apply approximation Galerkin's procedure. Hence we can extract a subsequence which converge to a limit $u$ in $L^\infty(]0,T[\,;H^1(I)\cap L^2(]0,T[\,;H^{1+\frac{\alpha}{2}}(I))$. To finish, it remains to know if $u$ is a solution of problem ?

Since injection of $H^1(I)$ into $L^2(I)$ is compact, we can apply Ascoli theorem and conclude a strongly convergence of $(u_n)_{n\in\mathbb{N}}$ to $u$ in $L^2(]0,T[\,;L^2(I)$.

In order to conclude, it is enough to prove that $(u_n)^2$ converges strongly to $u^2$ in $L^1(]0,T[\,;L^2(I))$.remark that

$$\left\|(u_n)^2-u^2\right\|_{L^1(]0,T[;L^2(I))} \leq \|u_n-u\|_{L^1(]0,T[;L^4(I))}\left(\|u_n\|_{L^1(]0,T[;L^4(I))}+\|u\|_{L^1(]0,T[;L^4(I))}\right)$$

It is enough to prove that $u_n-u$ converges strongly in $L^1(]0,T[\,;L^4(I))$.this last result holds by Gagliardo-Nirenberg's inequality$(7-8)$

$$\begin{aligned}
\|u_n-u\|_{L^1(]0,T[;L^4(I))} &\leq C\|u_n-u\|_{L^2(]0,T[;L^4(I))}^{1-\frac{1}{4}}\|\nabla(u_n-u)\|_{L^2(]0,T[;L^4(I))}^{\frac{1}{4}} \\
&\leq C\|u_n-u\|_{L^2(]0,T[;L^4(I))}^{1-\frac{1}{4}}
\end{aligned}$$

and to prove that $D^\alpha u_n$ converges strongly to $D^\alpha u$ in $L^1(]0,T[\,;L^2(I))$.

In the same way, we remark that

$$\begin{aligned}
&\|D^\alpha u_n-D^\alpha u\|_{L^1(]0,T[;L^2(I))} \\
\leq\ &\left\|\frac{\partial^2 u_n}{\partial x^2}-\frac{\partial^2 u}{\partial x^2}\right\|_{L^1(]0,T[;H^{1+\frac{\alpha}{2}}(I))}\left(\left\|\frac{\partial^2 u_n}{\partial x^2}\right\|_{L^1(]0,T[;H^{1+\frac{\alpha}{2}}(I))}+\left\|\frac{\partial^2 u}{\partial x^2}\right\|_{L^1(]0,T[;H^{1+\frac{\alpha}{2}}(I))}\right)
\end{aligned}$$

and since the term $\partial^2/\partial x^2$ is linear, approch problem converges weakly to a limit point, then the existence holds.

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INTERNATIONAL CONFERENCE ON MATHEMATICS
"An Istanbul Meeting for World Mathematicians"
3-5 July 2019, Istanbul, Turkey

# PMSM optimal design parameters by using inverse problem

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**Abstract**

In electrical machine design, the direct problem is to find the design specifications by using given input. In the inverse problems we try to find the shape of the machine that produce the itemize performance from a given output by taking into account constrained variables. From the mathematical standpoint, the inverse problems are further artful than direct problems for several reasons. By using inverse problem, we have to carry out a preset optimal design. An outline of the machine design as well as PMSM simulation results is presented. An approach for using inverse problem in electrical machine design is presented and overview of optimal machine design via inverse problems is given.

**Keywords:** PMSM, inverse problem, Monte Carlo approach, design parameters.

## 1. Introduction

Brushless DC motors (BLDC) in use in the low and medium power range show increased efficiency with smaller size and higher speed compared to conventional motor. Thus, permanent magnet electric machines (PMSM) are used more and more in several industrial activities. The development of this type of machine is linked to the growth of permanent magnets, nano-composite materials for energy-efficient electric motors and advances in the field of modern electronics [1-3]. However, the problem of the optimal use of electrical energy would require an optimal design of the machine and is suitable in cases of high speed and high accuracy. So the problem of optimization plays a very important role in the production and conversion of electrical energy. Currently, engineers and researchers are continuously developing methods using mathematical approaches [4] to improve engine efficiency while reducing power losses, energy costs and increased efficiency [2].

Numerous conception algorithms are proposed in the literature including direct and stochastic search algorithms for both single and multi-objective design optimization cases on a permanent-magnet-synchronous-motor (PMSM). The obtained results show the effectiveness of the recent artificial algorithm especially when more design candidates are considered.

This paper is based on the choice and determination of the design parameters of the machine using the inverse problem approach. This methodology is effective, but the solutions provided can not perfectly meet the specifications due to some simplifying assumptions of the models used. The reverse-problem approach makes it possible to solve design problems in a more reasonable manner and to propose appropriate solutions to imposed conditions.

## 2. Preliminaries

### 2.1 Overview on inverse problems

In science, an inverse problem is a situation in which we try to determine the causes of a phenomenon based on the experimental observations of its effects. The resolution of the inverse problem generally involves an initial stage of modeling the phenomenon, called the direct problem, which describes how the parameters of the model translate into effects observable experimentally.

The most general way of describing an information state is to define a probability density on the space parameters. Thus, the results of measurements of observable parameters (data), prior information on the model, parameters and information on physical correlations between observable parameters and model parameters can all be described using probability. The general inverse problem can then be defined as a problem of "combinations" of all this information. In all usual cases, the results obtained with this method are reduced to those obtained by more conventional approaches [5].

The scientific procedure for the study of a physical system S can be divided into three stages:

i) **System configuration:** choice of the model parameters to be used to describe a system, whose values characterize the system completely. So defining a parameterization means defining a set of experimental procedures that allow us to measure a set of physical quantities that characterize the system.

ii) **Direct modeling:** determine the physical state u (physical laws) generated by an environment from knowledge of its parameters and constrains. The physical laws allowing us, for given values of the model parameters, to predict the results of measurements on certain observable parameters

iii) **Inverse modeling:** Determine some physical parameters $x \in U$ from measurements $y \in V$ related to the physical state u. Using the actual results of some observable parameters measurements to derive the actual values from the model parameters.

Strong feedback exists between the three stages. Whereas the first two stages are mainly inductive (rules of thought that are difficult to explain), the third stage is deductive (application of the mathematical theory of logic).

In most cases, the description of a system is given in the form of a set of mathematical equations (differential equations, integral equations ...), containing some parameters. The analysis of the physical process given by these equations can be separated into three types of distinct problems (Direct; Reconstruction and Identification).

**I.** The basic principle for determining the inverse problem begins with the study and observation of the behavior of the physical system where identification is considered as the key of the inverse theory with a clear definition of the basic variables on the system and a purpose of using data that are easily observable to infer some geometric parameters that are not directly observable. So, we look for mathematical equations that accurately describe the values of the parameters of the physical system. A general approach to identification seeks, therefore, to define an objective function that would reach these limits (lower and upper) on the components of the vector X given by an assumed configuration [6-7].

### 2.2. Motor electric circuit:

The motor geometry and rotor configuration having high energy magnets of the 16 poles and 24 slots studied machine is shown in Figure 1.

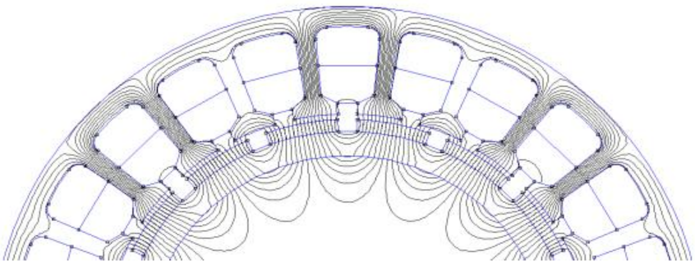

Figure 1: Interior-rotor Brushless dc motors.

The determination of parameters equivalent electric circuit such as the magnetic flux density in the air gap is obtained by analysis and simulation of a magnetic circuit. Using Norton equivalent circuit [8] of the motor, the value of reluctance can be calculated by using law as shown beneath :

$$R_g = \frac{g^{'}}{\mu_0 A_g}$$

Where $\mathbf{A_g}$ is the cross-sectional area per pole of the air gap, $L_g$, is the air gap length and $\mathbf{g}^{'}$ is represented by relation as:

$$g^{'} = K_c L_g \; .$$

Here $K_c$ design the Carter's coefficient whit analytical expressions are given by:

$$K_c = \left[1 - \frac{2W_{slot}}{\pi\tau_s}\left(tan^{-1}\left(\frac{W_{slot}}{2g'}\right) - \frac{g'}{W_{slot}} ln\left(1 + \frac{1}{4}\left(\frac{W_{slot}}{g'}\right)^2\right)\right)\right]^{-1}$$

The air gap $A_g$ is the area, through which the flux passes is given by [9]:

$$A_g = \left[\theta_p\left(R_{si} - \frac{L_g}{2}\right)\right].L$$

The equation of magneto-motive force across the air gap is done by:

$$F_m = \frac{(\phi_r - \phi_g)}{P_m} = \phi_g R_g \phi_g = \frac{\phi_r}{(1 + P_m R_g)}$$

With $P_m$ is the magnet permeance and is given by:

$$P_m = P_{m0} + P_{r1} = P_{m0}(1 + P_{r1})$$

where $\phi_r$ is given by :

$$\phi_r = B_r A_m$$

with $B_r$ is the remanant magnetization .

The ratio of magnet pole area ( $A_m$) to air gap area ($A_g$ ) or flux concentration factor is done by [10]:

$$C_\emptyset = \frac{A_m}{A_g}$$

Thus, the air gap flux density is done by :

$$B_g = \frac{C_\emptyset}{(1 + P_m R_g)} = \mu_0 H_g$$

and the corresponding magnet flux density is determined by :

$$B_m = \mu_0\mu_{rm}.H_m + B_r = -\mu_0\mu_{rm}.\frac{H_g.L_g}{L_m} + B_r = -\frac{\mu_{rm}.L_g}{L_m}.B_g + B_r$$

Where $\mu_0$ and $\mu_{rec}$ represent,respectivelly, the permeability of the free space and the relative recoil permeability of the magnet.

Rearranging the above equations, it gives that the length of magnet thickness as,

$$L_m \approx \frac{B_g . L_g . \mu_r}{B_r - B_g}$$

In the programming phase, we must first define the objective function and the constraints in the form of equations and variables which represent the parameters of the motor to be optimized.

The geometrical design parameters of the machine are optimized by changing the variables until the desired parameters are obtained.
The vectors of variables $X= [L_m , W_{slot} , L_g , A_m , A_g , C_\Phi , R_g , P_{mo} , B_g]$ varies between minimum and maximum values. The obtained optimum design parameters must satisfies the objective function that match to desired magnetic flux density in the air-gap. However, the inverse problem can be presented as follows:

$$Min\ (F(X)) = \left(\frac{B_{gmax}}{B_g^c} - 1\right)^2$$

Where $B_g^c$ is the computed flux density and $B_{gmax}$ is the desired flux density at the $i^{th}$ point on a path of the air gap BLDC motor respectively.

### 2.3. Optimization by using the Sequential quadratic programming (SQP) :

Sequential quadratic programming (SQP) is an iterative approach method for the optimization of nonlinear constraints problems. The conceptual method is used on mathematical problems for which the objective function and the constraints are differentiable twice in a continuous way [11]. The SQP method solves the optimization of a series of sub-problems and allows assuming directly the Newton's method for the optimization under stress. In various steps, this estimate is constructed from the hessian function of the Lagrangian function by means of a quasi-Newton updating method. Nevertheless, the common SQP method consists of minimizing or maximizing the following objective function encountered in several areas of engineering, science and management defined by:

Minimize f(x)

Subject to

$h_i (x) = 0 \quad i =1,2,....m_e$

$h_i (x) \leq 0 \quad i = m_e + 1 ; ......$

where $f$ is $R^n \rightarrow R$, and $h: R^n \rightarrow R^m$.

*With f* is linear of quadratic objective function, *h* is the constraint function and *x* is the vector with length *n.* Here the objective function *f is substituted by* the Lagrangian quadratic approximation function, such as:

$$L(x,\lambda) = f(x) + \sum_{i=1}^{m} \lambda_i \,.\, h_i(x)$$

By expressing the constraints bound in the form of inequality constraints, the precedent equation will be simplified which leads to a linearization of the nonlinear constraints in quadratic programming sub-problem.

$$Min \frac{1}{2} d^T H_k d + \nabla f(x_k)^T d \qquad d \in R^n$$

$$\nabla h_i(x_k)^T d + h_i(x_k) = 0, \quad i = 1, \dots, m_e$$

$$\nabla h_i(x_k)^T d + h_i(x_k) \leq 0, \quad i = m_e + 1, \dots, m$$

### 3. Main Results

For a given torque, magnetic and electric loading,and the machine length the overall machine rotor diameter can be determined as follows [8].

$$T = \frac{\pi}{2} . B_g . \Delta I . L . (2R_{si})^2$$

Where B is called magnetic loading ; ΔI is called electrical loading ; L is motor length and R is rotor radius.
Therefore, the radius of the machine rotor is then,

$$R = \sqrt{\frac{T}{2\pi \; B \; \Delta I \; L}}$$

We note that a small increase in the thickness of the magnet with a small decrease in air gap had a good and clear effect in increasing the flux density inside the motor and thus an increase in torque and also an increase in motor efficiency.
From the last equation, the value of the primary full load torque was T = 39.32 N · m and after optimization is became 46.47 N · m with $R_{si}$ = 0.043 m and ΔI= 50 kAmps /m .

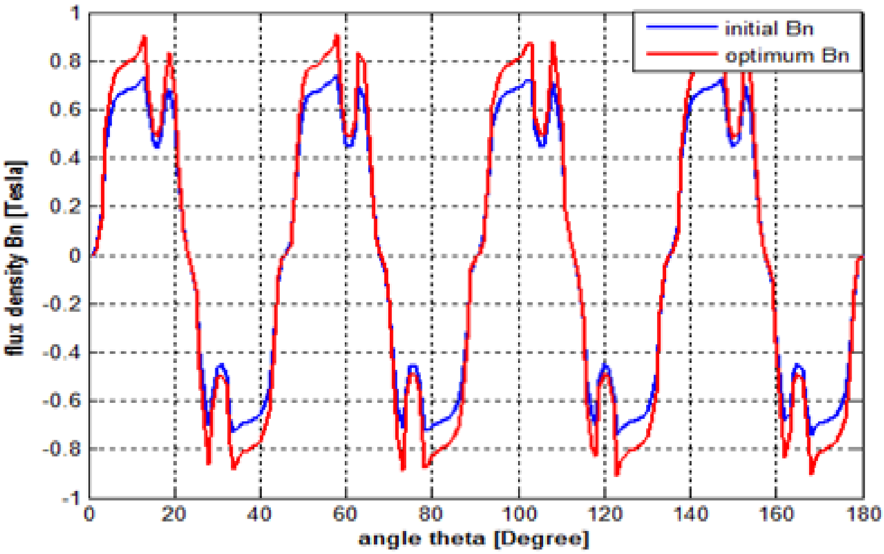


Figure. 2: Initial and optimized air gap magnetic flux

## 4. Conclusion

We have presented in this paper a suitable method of an optimal and suitable design for motor type BLDC in order to increase the air gap magnetic flux density and therefore the motor efficiency by using an inverse problem. The optimized approach used in this paper is based on the minimum constrained for nonlinear programming by using 'fmincon' function of Matlab. The obtained results based on finite element shows that the objective of this work is achieved, the density of the flux in the air gap is improved and therefore the torque and the motor efficiency have been enhanced. Finally, the formulation of the problem must be well defined by using efficiently the local algorithm proposed by Matlab's 'Fmincon' function.

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**3-5 July 2019, Istanbul, Turkey**

# Triangles in the Suborbital Graphs of the Normalizer of $\Gamma_0(N)$

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**Abstract**

In this paper, we investigate a suborbital graph for the normalizer of $\Gamma_0(N)$ in $PSL(2,\mathbb{R})$, where $N$ will be of the form $2^2 3^2 p^2$, $p$ is a prime and $p > 3$. Then we give edge and circuit conditions on graphs arising from the non-transitive action of the normalizer.

**Keywords:** Fuchsian groups, imprimitive action, suborbital graphs

## 1. Introduction

The modular group acts transitively on $\widehat{\mathbb{Q}}$ and in a paper of Jones, Singerman, Wicks, the suborbital graphs were studied and the most basic one turn out to be the well-known Farey graph [5]. Suborbital graphs of the normalizer $Nor(N)$ were studied by same idea. All circuits in the suborbital graph were found when $N$ is a square-free positive integer [6] and when $N$ satisfies the condition of transitive action [7]. Then, non-transitive cases have been examined to reach the general statement [4]. In this study, we continue to examine the combinatorial properties of the normalizer.

## 2. Preliminaries

$\Gamma_0(N)$= {g ∈ Γ : c ≡ 0 (mod $N$)} is a well known congruence subgroup of the classical modular group Γ. The normalizer turns to be a very important group in the study of moonshine and for this reason has been studied by many authors [3]. It consists exactly of the matrices $\begin{pmatrix} ae & b/h \\ cN/h & de \end{pmatrix}$; $ade^2 - bcN/h^2 = e$,

where $e \parallel (N/h^2)$ and $h$ is the largest divisor of 24 for which $h^2|N$ with understandings that the determinant $e$ of the matrix is positive, and that $r \parallel s$ means that $r|s$ and $(r, s/r) = 1$ ($r$ is called an exact divisor of $s$).

## 3. Main Results

In this study, we take $N = 2^2 3^2 p^2$ , where $p$ is a prime and $p > 3$. Since $h = 2^{\min\{3,[\![\alpha/2]\!]\}} 3^{\min\{1,[\![\beta/2]\!]\}}$, then $h = 6$ for $N = 2^\alpha 3^\beta p_3^{\alpha_3} \cdots p_r^{\alpha_r}$. Hence $e$ must be 1 or $p^2$. As a corollary, we get two types of the element of $Nor(2^2 3^2 p^2)$ as follows

$$T_1=\begin{pmatrix} a & b/6 \\ 6p^2c & d \end{pmatrix},\ ad-bcp^2=1 \quad \text{ve} \quad T_2=\begin{pmatrix} ap^2 & b/6 \\ 6p^2c & dp^2 \end{pmatrix},\ adp^4-bcp^2=p^2$$

**Lemma 3.1.** Let $N$ have the prime power decomposition as $2^{\alpha}3^{\beta}p_3^{\alpha_3}\cdots p_r^{\alpha_r}$. Then $Nor(N)$ acts transitively

on $\widehat{\mathbb{Q}}$ if and only if $\alpha\le 7, \beta\le 3, \alpha_i\le 1$ for $i=3,\cdots,r$.

**Corollary 3.2.** $Nor(2^2 3^2 p^2)$ is not transitive on $\widehat{\mathbb{Q}}$.

Therefore, we will find a maximal subset of $\widehat{\mathbb{Q}}$ on which $Nor(2^2 3^2 p^2)$ acts transitively.

**Lemma 3.3.** Let $d|N$. Then the orbit $\binom{a}{d}$ of $a/d$ with $(a,d)=1$ under $\Gamma_0(N)$ is the set $\{x/y\in\widehat{\mathbb{Q}}:\ (N,y)=d, a\equiv xy/d\ mod(d,N/d)\}$. Furthermore, the number of orbits $\binom{a}{d}$ with $d|N$ under $\Gamma_0(N)$ is just $\varphi(d,N/d)$ where $\varphi(N)$ is Euler's totient function which is the number of positive integers less than or equal to $N$ that are coprime to $N$.

Hence, we can give following tables by above lemma:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | $p$ | $p^2$ | 3 | $3p$ | $3p^2$ | 9 | $9p$ | $9p^2$ |
| 2 | $2p$ | $2p^2$ | 6 | $6p$ | $6p^2$ | 18 | $18p$ | $18p^2$ |
| 4 | $4p$ | $4p^2$ | 12 | $12p$ | $12p^2$ | 36 | $36p$ | $36p^2$ |

*Table 1. Divisors of N/h^2*

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | $p$-1 | 1 | 2 | $2p$-2 | 2 | 1 | $p$-1 | 1 |
| 1 | $p$-1 | 1 | 2 | $2p$-2 | 2 | 1 | $p$-1 | 1 |
| 1 | $p$-1 | 1 | 2 | $2p$-2 | 2 | 1 | $p$-1 | 1 |

*Table 2. Number of Orbits*

**Theorem 3.4.** The set $\widehat{\mathbb{Q}}(2^2 3^2 p^2)=\binom{1}{1}\cup\binom{1}{2}\cup\binom{1}{6}\cup\binom{1}{9}\cup\binom{1}{18}\cup\binom{1}{36}\cup\binom{1}{p^2}\cup\binom{1}{2p^2}\cup\binom{1}{6p^2}\cup\binom{1}{9p^2}\cup\binom{1}{18p^2}\cup\binom{1}{36p^2}$ is a maximal orbit of $Nor(2^2 3^2 p^2)$ on $\widehat{\mathbb{Q}}$.

*Proof*. Let us consider the orbit $\binom{1}{1}$ under the action of the elements of $Nor(2^2 3^2 p^2)$:

i. If $d$-odd, $c$-even and $3|b$ , then $T_1\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}2a+b_0\\2(6p^2c+d)\end{pmatrix}=\begin{pmatrix}1\\2\end{pmatrix}$

ii. If $(a,b)=1$ , then $T_1\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}6a+b\\6(6p^2c+d)\end{pmatrix}=\begin{pmatrix}1\\6\end{pmatrix}$

iii. If $b$-even and $3|d$ , then $T_1\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}3a+b_0\\3^2(2p^2c+d_0)\end{pmatrix}=\begin{pmatrix}1\\3^2\end{pmatrix}$

iv. If $b$-odd, $d$-even and $3|d$ , then $T_1\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}2.3a+b_0\\2.\,3^2(6p^2c+d)\end{pmatrix}=\begin{pmatrix}1\\2.\,3^2\end{pmatrix}$

v. If $b$-odd, $c$-odd and $6|d$ , then $T_1\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}2.3a+b_0\\2^2.\,3^2(6p^2c+d)\end{pmatrix}=\begin{pmatrix}1\\2^2.\,3^2\end{pmatrix}$

vi. If $d$-odd, $c$-odd and $6|d$ , then $T_2\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}ap^2+b_0\\p^2(6c+d)\end{pmatrix}=\begin{pmatrix}1\\p^2\end{pmatrix}$

vii. If $d$-odd, $c$-even and $3|b$ , $b$-odd, then $T_2\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}2ap^2+b_0\\2p^2(6c+d)\end{pmatrix}=\begin{pmatrix}1\\2p^2\end{pmatrix}$

viii. If $(a,b)=1$, then $T_2\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}6ap^2+b\\6p^2(6c+d)\end{pmatrix}=\begin{pmatrix}1\\6p^2\end{pmatrix}$

ix. If $b$-even, $c$-odd and $3|d$ , then $T_2\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}3ap^2+b_0\\3^2p^2(2c+d)\end{pmatrix}=\begin{pmatrix}1\\3^2p^2\end{pmatrix}$

x. If $b$-odd, $c$-even and $3|d$ , then $T_2\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}2.3ap^2+b_0\\2.3^2p^2(2c+d_0)\end{pmatrix}=\begin{pmatrix}1\\2.3^2p^2\end{pmatrix}$

xi. If $b$-odd, $c$-odd and $6|d$ , then $T_2\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}2.3ap^2+b_0\\2^2.\,3^2p^2(2c+d_0)\end{pmatrix}=\begin{pmatrix}1\\2^2.\,3^2p^2\end{pmatrix}$

**Lemma 3.5.** [2], Let $(G,\Delta)$ be a transitive permutation group. $(G,\Delta)$ is primitive if and only if $G_\alpha$, the stabilizer of $\alpha\in\Delta$, is a maximal subgroup of $G$, for each $\alpha\in\Delta$.

From the above lemma we see that whenever, for some $\alpha$, $G_\alpha<H<G$, then $\Omega$ admits some $G$ -invariant equivalence relation other than the trivial cases. Because of the transitivity, every element of $\Omega$ has the form $g(\alpha)$ for some $g\in G$. Thus one of the non-trivial $G$ -invariant equivalence relation on $\Omega$ is given as follows:

$$g(\alpha)\approx g'(\alpha) \text{ if and only if } g'\in gH$$

The number of blocks (equivalence classes) is the index $|G\colon H|$ and the block containing $\alpha$ is just the orbit $H(\alpha)$. We can apply these ideas to the case where $G$ is the $Nor(2^2 3^2 p^2)$, $\Delta$ is $\widehat{\mathbb{Q}}\left(2^2 3^2 p^2\right)$, $G_\alpha$ is the stabilizer of $\infty$ in $\widehat{\mathbb{Q}}\left(2^2 3^2 p^2\right)$, that is $Nor(2^2 3^2 p^2)_\infty=\langle\begin{pmatrix}1&1/6\\0&1\end{pmatrix}\rangle$, and $H=\Gamma_C(N)$. Clearly, the relation $Nor(2^2 3^2 p^2)_\infty<\Gamma_C(N)<Nor(2^2 3^2 p^2)$ produce an imprimitive action as desired.

**Theorem 3.6.** There are only two blocks which are $[\infty]$ and $[0]$. These are as following:

$$[0]=\binom{1}{1}\cup\binom{1}{2}\cup\binom{1}{6}\cup\binom{1}{9}\cup\binom{1}{18}\cup\binom{1}{36}$$

$$[\infty]=\binom{1}{p^2}\cup\binom{1}{2p^2}\cup\binom{1}{6p^2}\cup\binom{1}{9p^2}\cup\binom{1}{18p^2}\cup\binom{1}{36p^2}.$$

*Proof*. Since $|Nor(2^2 3^2 p^2) : \Gamma_C(N)| = 2$, we have that $Nor(2^2 3^2 p^2)= \Gamma_C(N) \cup T_2\Gamma_C(N)$. The result is obvious.

In [8], Sims introduced the idea of the suborbital graphs of a permutation group $G$ acting on a set $\Delta$, these are graphs with vertex-set $\Delta$, on which $G$ induces automorphisms. We summarize Sims'theory as follows: Let $(G, \Delta)$ be transitive permutation group. Then $G$ acts on $\Delta\times\Delta$ by $g(\alpha,\beta) = (g(\alpha), g(\beta))$ for $g\in G$ and $\alpha,\beta\in\Delta$. The orbits of this action are called *suborbitals* of $G$. The orbit containing $(\alpha,\beta)$ is denoted by $\mathrm{O}(\alpha,\beta)$. From $\mathrm{O}(\alpha,\beta)$ we can form a *suborbital graph* of $G(\alpha,\beta)$ : its vertices are the elements of $\Delta$, and there is a directed edge from $\gamma$ to $\delta$ if $(\gamma,\delta)\in \mathrm{O}(\alpha,\beta)$. A directed edge from $\gamma$ to $\delta$ is denoted by $(\gamma\to\delta)$. If $(\gamma,\delta)\in\mathrm{O}(\alpha,\beta)$, then we will say that there exists an edge $(\gamma\to\delta)$ in $G(\alpha,\beta)$ and represent them as hyperbolic geodesics in the upper half plane $\mathrm{IH} := \{\mathrm{z}\in\mathbb{C} : \mathrm{Im(z)} > 0\}$. If $\alpha=\beta$, the corresponding suborbital graph $G(\alpha,\alpha)$, called the trivial suborbital graph, is *self-paired*: it consists of a loop based at each vertex $\alpha\in\Delta$. By a circuit of length $m$ (or a closed edge path), we mean a sequence $v_1\to v_2\to\cdots\to v_m\to v_1$ such that $v_i\neq v_j$ for $i\neq j$, where $m\geq 3$. If $m=3$ or 4 then the circuit is called a triangle or rectangle.

In this study, $G$ and $\Delta$ will be the normalizer of $\Gamma_0(N)$ in $PSL(2.\mathbb{R})$ and the extended rational $\widehat{\mathbb{Q}}=\mathbb{Q}\cup\{\infty\}$, respectively. Since rational numbers are well ordered, we also used the notations $\gamma\overset{>}{\to}\delta$ or $\gamma\overset{<}{\to}\delta$ according to the order of vertices.

$Nor(2^2 3^2 p^2)$ acts transitively on $\widehat{\mathbb{Q}}(2^2 3^2 p^2)$, every suborbital $\mathrm{O}(\alpha,\beta)$contains a pair $(\infty, u/p^2)$ for $u/p^2\in\widehat{\mathbb{Q}}\left(2^2 3^2 p^2\right)$. As $Nor(2^2 3^2 p^2)$ permutes the blocks transitively, all subgraphs corresponding to blocks are isomorphic. Therefore we will only consider the subgraph $F(\infty, u/p^2)$ of $G(\infty, u/p^2)$ whose vertices form the block $[\infty]$.

**Theorem 3.7.** (Edge condition) Let $r/s$ and $x/y$ be in the block $[\infty]$. Then there is an edge $r/s\to x/y$ in $F(\infty, u/p^2)$ if and only if

(i) $If\;\; 36p^2 \parallel s$, then $x\equiv\pm ur\ (mod\ p^2)$, $y\equiv\pm us\ (mod\ p^2)$, $ry-sx=\pm p^2$

(ii) $If\;\; 6p^2 \parallel s$ , then $x\equiv\pm 6ur\ (mod\ p^2)$, $y\equiv\pm 6us\ (mod\ p^2)$, $ry-sx=\pm 6p^2$

(iii) $If\;\; p^2 \parallel s$, then $x\equiv\pm 36ur\ (mod\ p^2)$, $y\equiv\pm 36us\ (mod\ p^2)$, $ry-sx=\pm p^2$,

(Plus and minus sign correspond to $r/s > x/y$ and $r/s < x/y$, respectively).

$Proof$. Assume first that $r/s \overset{>}{\rightarrow} x/y$ is an edge in $F(\infty, u/p^2)$. It means that there exists some $T$ in the normalizer $Nor(2^2 3^2 p^2)$ such that $T$ sends the pair $(\infty, u/p^2)$ to the pair $(r/s, x/y)$, that is $T(\infty) = r/s$ and $T(u/p^2) = x/y$.

$Case$ 1. If $36p^2 \parallel s$, taking into account that $T = \begin{pmatrix} a & b \\ 36p^2c & d \end{pmatrix}$ and $ad - 36bcp^2 = 1$. $T(\infty) = \frac{a}{36p^2c} = \frac{r}{s}$ gives that $r = a$ and $s = 36p^2c$.

$T(u/p^2) = \frac{au+bp^2}{36p^2cu+dp^2} = \frac{x}{y}$ gives that $x \equiv ur \ (mod\ p^2)$, $y \equiv us \ (mod\ p^2)$, Furthermore, we get $ry - sx = p^2$, from the equation

$$\begin{pmatrix} a & b \\ 36p^2c & d \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & p^2 \end{pmatrix} = \begin{pmatrix} a & au + bp^2 \\ 3p^2c & 36p^2cu + dp^2 \end{pmatrix} = \begin{pmatrix} r & s \\ x & y \end{pmatrix}.$$

$Case$ 2. If $6p^2 \parallel s$, taking into account that $T = \begin{pmatrix} a & b/6 \\ 6p^2c & d \end{pmatrix}$, suppose that $a$-odd by the equation $ad - bcp^2 = 1$. $T(\infty) = \frac{a}{6p^2c} = \frac{r}{s}$ gives that $r = a$ and $s = 6p^2c$.

$T(u/p^2) = \frac{au+bp^2/6}{6p^2cu+dp^2} = \frac{6au+bp^2}{36p^2cu+6dp^2} = \frac{x}{y}$ gives that $x \equiv 6ur \ (mod\ p^2)$, $y \equiv 6us \ (mod\ p^2)$. Furthermore, we get $ry - sx = 6p^2$ from the equation

$$\begin{pmatrix} 6a & b \\ 6p^2c & d \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & p^2 \end{pmatrix} = \begin{pmatrix} 6a & 6au + bp^2 \\ 6p^2c & 6p^2cu + dp^2 \end{pmatrix} = \begin{pmatrix} 6r & x \\ s & y/6 \end{pmatrix}.$$

$Case$ 3. If $p^2 \parallel s$, taking into account that $T = \begin{pmatrix} a & b/6 \\ 6p^2c & d \end{pmatrix}$, suppose that $6 \parallel a$ by the equation $ad - bcp^2 = 1$. $T(\infty) = \frac{a}{6p^2c} = \frac{a_0}{p^2c} = \frac{r}{s}$ gives that $r = a_0$ and $s = p^2c$.

$T(u/p^2) = \frac{au+bp^2/6}{6p^2cu+dp^2} = \frac{36a_0u+bp^2}{36p^2cu+6dp^2} = \frac{x}{y}$ gives that $x \equiv 36ur \ (mod\ p^2)$, $y \equiv 36us \ (mod\ p^2)$. Furthermore, we get $ry - sx = p^2$ from the equation

$$\begin{pmatrix} a & b/6 \\ 6p^2c & d \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & p^2 \end{pmatrix} = \begin{pmatrix} 36a_0 & b \\ 36p^2c & 6d \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & p^2 \end{pmatrix} = \begin{pmatrix} 36a_0 & 36a_0u + bp^2 \\ 36p^2c & 36p^2cu + dp^2 \end{pmatrix} = \begin{pmatrix} r & x \\ s & y \end{pmatrix}.$$

For the opposite direction, we assume that $36p^2 \parallel s$ and $x \equiv ur \ (mod\ p^2)$, $y \equiv us \ (mod\ p^2)$, $ry - sx = p^2$ . In this case, there exist $b, d \in \mathbb{Z}$ such that $x = ur + bp^2$and $y = us + dp^2$. If we put these equivalences in $ry - sx = p^2$, we obtain $rd - bs = 1$. So the element $T = \begin{pmatrix} r & b \\ s & d \end{pmatrix}$ is clearly in $H$. For minus sign and another conditions, similar calculations are done. ■

It is known that a graph which contains no circuit is called a forest. In introduction part, we also mentioned that the trivial suborbital graphs are self-paired ones. In this section, we will be mainly interested in the remaining non-trivial suborbital graphs.

**Theorem 3.8.** Let $F(\infty, u/p^2)$ contains a triangle if and only if $36u^2 \pm 6u + 1 \equiv 0\ (mod\ \ p^2)$.

*Proof.* We suppose that there is a triangle such as $\frac{k}{l} \to \frac{m}{n} \to \frac{x}{y} \to \frac{k}{l}$ in $F(\infty, u/p^2)$.Since $H$ permutes the vertices transitively, we may suppose that the triangle has the form $\frac{1}{0} \to \frac{r_0}{s_0 p^2} \to \frac{x_0}{y_0 p^2} \to \frac{1}{0}$. Furthermore, without loss of generality, suppose $\frac{r_0}{s_0 p^2} < \frac{x_0}{y_0 p^2}$. From Theorem 3.7.(i), we have that $r_0 \equiv u\ \ (mod\ \ p^2)$ and $s_0 = 1$ from the first edge. Hence, we get the second vertex as $\frac{u}{p^2}$. Applying to Theorem 3.7 to third edge, we have two possibilities of the configuration as follows: $\frac{1}{0} \to \frac{u}{p^2} \to \frac{x_0}{p^2} \to \frac{1}{0}$ or $\frac{1}{0} \to \frac{u}{p^2} \to \frac{x_0}{6p^2} \to \frac{1}{0}$.

$Case$ 1. From the second edge $up^2 - x_0 p^2 = -p^2$, that is $x_0 = u + 1$. By Theorem 3.7, we have that $u + 1 \equiv -36u^2\ \ (mod\ \ p^2)$, then $36u^2 + u + 1 \equiv 0\ \ (mod\ \ p^2)$. From the third edge, $1 \equiv -36u(u + 1)\ (mod\ \ p^2)$, then $36u^2 + 36u + 1 \equiv 0\ \ (mod\ \ p^2)$. These equivalences gives a contradiction taking into account that $5.7. u \equiv 0\ \ (mod\ \ p^2)$ and that $(u, p^2) = 1$.

$Case$ 2. From the second edge $6up^2 - x_0 p^2 = -p^2$, that is $x_0 = 6u + 1$. Applying to Theorem 3.7 to $\frac{6u+1}{6p^2} \to \frac{1}{0}$, we obtain that $36u^2 + 6u + 1 \equiv 0\ (mod\ \ p^2)$. If the inequalities $\frac{r_0}{s_0 p^2} > \frac{x_0}{y_0 p^2}$ hold then we conclude that $36u^2 - 6u + 1 \equiv 0\ \ (mod\ \ p^2)$.

For the opposite direction, we assume that $36u^2 \pm 6u + 1 \equiv 0\ \ (mod\ \ p^2)$. Using Theorem 3.7, it is clear that $\frac{1}{0} \to \frac{u}{p^2} \to \frac{6u\pm1}{6p^2} \to \frac{1}{0}$ is a triangle in $F(\infty, u/p^2)$.■

## 4. Conclusion

**Theorem 3.9.** The prime divisors $p$ of $36u^2 \pm 6u + 1$, for any $u \in \mathbb{Z}$, are of the form $p \equiv 1(mod 3)$.

*Proof.* Let $u$ be any integer and $p$ a prime divisor of $36u^2 \pm 6u + 1$. Then, without any difficulty, it can be easily seen that the normalizer $Nor(2^2 3^2 p)$, like $Nor(2^2 3^2 p^2)$, has the elliptic element $\varphi = \begin{pmatrix} -48u & 36u^2 \pm 6u + 1/p \\ -48p & -48u + 6 \end{pmatrix}$ of order 3. By [1], we obtain $p \equiv 1(mod 3)$.

**Acknowledgment.** This work is supported by the Scientific and Technical Research Council of Turkey (TUBITAK) under Grant No. 118F018.

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INTERNATIONAL CONFERENCE ON MATHEMATICS
"An Istanbul Meeting for World Mathematicians"
3-5 July 2019, Istanbul, Turkey

# p-Martingales on Lattice Normed Vector Lattices

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**Abstract**

A measure free approach to stochastic processes has been developed for a long time. Various order theoretic settings for stochastic processes is given. In this talk, we introduced the concepts of conditional expectations and p-martingales on lattice-normed vector lattices. Here we formulate and prove p-theoretic analogue of the Hopf ergodic theorem in a measure free context.

**Keywords:** Lattice-normed vector lattices, conditional expectation, p-martingales

**2010 MSC:** 46A40, 47B60, 60G48, 28A20, 46E30

## 1. Introduction and Preliminaries

Let $X$ be a vector space, $E$ be a vector lattice, and $p : X \to E_+$ be a vector norm (i.e. $p(x) = 0 \Leftrightarrow x = 0$, $p(\lambda x) = |\lambda| p(x)$ for all $\lambda \in \mathbb{R}$, $x \in X$, and $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$), then the triple $(X, p, E)$ is called a *lattice-normed space*, as LNS. We say that elements $x$ and $y$ of an LNS $X$ are $p$-disjoint if their lattice norms are disjoint, and shown by $x \perp_p y$. The lattice norm $p$ in an LNS $(X, p, E)$ is said to be *decomposable* if, for all $x \in X$ and $e_1, e_2 \in E_+$, from $p(x) = e_1 + e_2$ it follows that there exist $x_1, x_2 \in X$ such that $x = x_1 + x_2$ and $p(x_k) = e_k$ for $k = 1, 2$.

We abbreviate the convergence $p(x_\alpha - x) \xrightarrow{o} 0$ as $x_\alpha \xrightarrow{p} x$ and say in this case that $x_\alpha$ $p$-converges to $x$.

If, in addition, $X$ is a vector lattice and the vector norm $p$ is monotone (i.e. $|x| \leq |y| \Rightarrow p(x) \leq p(y)$), then the triple $(X, p, E)$ is called a *lattice-normed vector lattice*, as LNVL. In an LNVL $(X, p, E)$, $p$-disjointness implies disjointness. Indeed, let $x \perp_p y$. Then $p(|x| \wedge |y|) \leq p(x) \wedge p(y) = 0$ and hence $x \perp y$. We shall make difference between two notions of bands in an LNVL $X = (X, p, E)$. More precisely, a subset $B$ of $X$ is called a *band* if it is a band in the vector lattice $X$ in the usual sense. A subset $B$ of $X$ is a *p-band* if

$$B = M^{\perp_p} = \{x \in X : (\forall m \in M)\ x \perp_p m\}$$

for some $M \subseteq X$. In general, there are many bands which are not $p$-bands. To see this, consider the normed lattice $(\mathbb{R}^2, \|\cdot\|, \mathbb{R})$. It has four bands, but only two of them are $p$-bands. It is easy to see that any $p$-band is an order ideal.

Note that let $X$ be a normed lattice with a norm $\|\cdot\|$. Then $X$ is the LNL $(X, \|\cdot\|, \mathbb{R})$. Moreover, let $X$ be a vector lattice. Then $X$ is the LNVL $(X, |\cdot|, X)$. If $X = (X, \|\cdot\|)$ be a normed lattice. Consider the closed unit ball $B_{X'}$ of the dual Banach lattice $X'$. Let $E = \ell_\infty(B_{X'})$ be the vector lattice of all bounded real-valued functions on $B_{X'}$. Define an $E$-valued norm $p$ on $X$ by

$$p(x)[f] := |f|(|x|) \quad (f \in B_{X'})$$

for any $x \in X$. The Hahn-Banach theorem ensures that $p(x) = 0$ iff $x = 0$. All other properties of lattice norm are obvious for $p$. Thus $(X, p, E)$ is an LNVL. For the more terminology, the authors refer to reader to [1].

## 2. Conditional Expectation on Riesz Spaces

In the classical setting, let $(\Omega, \Sigma, \mu)$ is a probability space.

**Definition 1.** *An increasing sequence of $\sigma$-algebras is called a filtration.*

**Definition 2.** *A $\Sigma$-measurable function $E[f|\Sigma]$, defined for a random variable $f \in L_1(\Omega, \Sigma, \mu)$ and a sub-$\sigma$-algebra $\Sigma$ is called a conditional expectation if*

$$\int_A E[f|\Sigma] d\mu = \int_A f d\mu, \quad \forall A \in \Sigma$$

Since $E$ is a positive order continuous linear functional on $L_1(\Omega, \Sigma, \mu)$, it follows that $T$, defined by $T(f) = E(f)1$ for each $f \in L_1(\Omega, \Sigma, \mu)$ (where 1 denotes the constant 1 function), defines an order continuous operator on $L_1(\Omega, \Sigma, \mu)$. This prompts the definition for expectation operators on a Riesz spaces with weak order unit.

A pair $(X_n, B_n)$, where $(X_n) \subseteq L_1(\Omega, \Sigma, \mu)$ and $(B_n)$ is a filtration with $X_n$ $B_n$-measurable for each $n \in \mathbb{N}$ is called a martingale if

$$E(X_n|B_n) = X_n, \forall n \in \mathbb{N}.$$

Researchers generalized the above definitions and develop further martingale theory in the abstract Riesz space setting. The conditional expectations in the classical setting are the only positive contractive projections on $L_1(\Omega, \Sigma, \mu)$ and thus it is natural in the new setting to replace conditional expectations with positive contractive projections.

**Definition 3.** *Let $T$ be a positive order continuous projection on a Riesz space $E$ with weak order unit such that the range, $R(T)$, is a Dedekind complete Riesz subspace of $E$. If $T$ maps weak order units to weak order units in $E$, then we call $T$ an RS-conditional expectation*

## 3. Conditional Expectation on LNVL

In this section, we give the generalization of a conditional expectation on LNVL. Before the conditional expectation definition, we need to give some definitions also.

**Definition 4.** *Let $(X, p, E)$ be an LNVL. A vector $e \in X$ is called a $p$-*order unit *if, for any $x \in X_+$, we have $p(x - x \wedge ne) \xrightarrow{o} 0$.*

**Definition 5.** *Let $(X, p, E)$ and $(Y, q, F)$ be two LNVLs. A linear operator $T : X \to Y$ is called p-continuous if $x_\alpha \xrightarrow{\mathrm{p}} 0$ in $(X, E)$ implies $Tx_\alpha \xrightarrow{\mathrm{p}} 0$ in $(Y, F)$.*

The following result, Theorem 1, motivate Definition 6 for conditional expectation operators on LNVL.

**Theorem 1.** *Let (X,p,E) be a LNVL with a p-unit and $T$ be a positive p-continuous and projection on X. $T$ maps p-units to p-order units is equivalent to the existence of a p-unit e in E for which $Te = e$.*

**Definition 6.** *Let $(X, p, E)$ be a LNVL and $T$ be a p-continuous dominated by positive projection on LNVL with p-order unit such that the range, $R(T)$, is a Dedekind complete Riesz subspace of $E$. Then $T$ is called conditional expectation on LNVL.*

**Theorem 2.** *Let $T$ be a conditional expectation on a p-complete LNVL $(X, p, E)$, with p-unit and let $P$ be the band projection of $X$ onto the band $B$, in $X$ generated by $0 \leq g \in R(T)$. Then $TP = PT$ .*

## 4. Conclusion

The classical paper of Garsia [4] gives a version of the Hopf Ergodic Theorem which was generalized to Riesz spaces in [5]. Our aim is to generalize Hopf Ergodic Theorem to LNVL. Therefore it is way to generalize Maximal Ergodic Theorem of Wiener, Kakutani and Yoshida. It might help us to prove Birkhoff Ergodic Theorem for LNVLs.

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INTERNATIONAL CONFERENCE ON MATHEMATICS
"An Istanbul Meeting for World Mathematicians"
3-5 July 2019, Istanbul, Turkey

# Unbounded Pseudonorm Convergence on Vector Lattices

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## Abstract

The notion of unbounded order convergence in vector lattices is a generalization of almost everywhere convergence. Last years, the unbounded order convergence in Banach lattices was deeply investigated by many mathematicians. In this proceeding, we define pseudonorm convergence on an Archimedean vector lattice $X$. Moreover also we define unbounded pseudonorm convergence and give boundedness condition on them. Moreover connection between order convergence and pseudonorm convergence is studied.

**Keywords:** Vector lattices, pseudonorm convergence, boundedness, order convergence

**2010 MSC:** 46B42, 46A03

## 1. Introduction and Preliminaries

Let $X$ be a vector lattice. We say that a net $(x_\alpha)$ in $X$ order converges to $x \in X$ if there exists a net $(y_\alpha)$ such that $y_\beta \downarrow 0$ and there exists $\alpha_0$ such that $|x_\alpha - x| \leq y_\alpha$ for all $\alpha \geq \alpha_0$. In this case, we write $x_\alpha \xrightarrow{o} x$. A net $(x_\alpha)$ in $X$ unbounded order convergent to $x \in X$ if $|x_\alpha - x| \wedge u \xrightarrow{o} 0$ for all $x \in X^+$. In this case, we say that $(x_\alpha)$ *uo*-converges to $x$, and, we write $x_\alpha \xrightarrow{uo} x$, see [?, ?, ?, ?, ?, ?, ?] and the references therein. For unexplained terminology, we refer to arbitrary vector lattice book as [?, ?]. In this note, we assume that all vector lattices are Archimedean.

The notion of unbounded order convergence in vector lattices is a generalization of almost everywhere convergence on $L_p(\Omega)$ where $(\Omega, \Sigma, \mu)$ is a $\sigma$-measure space and $1 \leq p \leq \infty$. Also the spaces $c_0$ and $\ell_p$, $1 \leq p \leq \infty$ are well-known examples in functional analysis. In these spaces, uo-convergence is coordinatewise convergence, see [?].

A linear topology on a vector lattice $X$ is said to be *locally solid* if zero has a neighborhood basis consisting of solid sets. A non-negative function $\rho\colon X \to \mathbb{R}$ is said to be a pseudonorm on $X$ if $\rho(x+y) \leq \rho(x) + \rho(y)$ and $\lim_{\theta \to 0} \rho(\theta x) = 0$ for all $x, y \in X$. A pseudonorm $\rho$ is said to be a Riesz pseudonorm if $|x| \leq |y|$ implies $\rho(x) \leq \rho(y)$ for $x, y \in X$.

Let $X$ be a vector lattice. It follows from classical definitions that every seminorm on $X$ is a pseudonorm. If $\rho$ is a Riesz seminorm on $X$ then $\rho$ is a Riesz pseudonorm. If $f$ is a functional on $X$ then $\rho_f(x) = |f(x)|$ is a seminorm on $X$. If the functional $f\colon X \to \mathbb{R}$ is order bounded then $\rho_f(x) = |f(|x|)|$ is a Riesz seminorm on $X$.

## 2. Pseudonorm Convergence on Vector Lattices

We assume that reader is familiar with some elementary topological notions.

**Definition 1.** *A net $x_\alpha$ in $X$ pseudonorm converges to some $x$, denoted by $x_\alpha \xrightarrow{\rho} x$, if $\rho(x_\alpha - x) \to 0$.*

The set $U_\rho = \{x \in X\colon \rho(x) \leq 1\}$ is the closed unit ball with respect to $\rho$. It is pseudonorm closed.

**Lemma 1.** *Suppose that $X$ is a vector lattice and $\rho$ is a Riesz pseudonorm on $X$. Let $x_\alpha$ and $y_\beta$ be two nets in $X$ such that $x_\alpha \xrightarrow{\rho} x$ and $y_\beta \xrightarrow{\rho} y$. Then $x_\alpha \wedge y_\beta \xrightarrow{\rho} x \wedge y$. In particular, if $x_\alpha \xrightarrow{\rho} x$ then $x_\alpha^+ \xrightarrow{\rho} x^+$.*

*Proof.* The topology on $X$ generated by the Riesz pseudonorm $\rho$ is a locally solid topology. It follows from [**?**, Theorem 2.17] that the lattice operations $(x, y) \mapsto x \wedge y$ and $x \mapsto x^+$ are uniformly continuous. $\square$

**Corollary 1.** *Suppose that $X$ is a vector lattice and $\rho$ is a Riesz norm on $X$. If $B$ is a band in $X$ then $B$ is closed with respect to the locally solid topology induced by $\rho$.*

*Proof.* By Lemma **??**, lattice operations on $X$ are continuous with respect to the locally solid topology induced by $\rho$. Let $x_\alpha$ be a net in $B$ such that $x_\alpha \xrightarrow{\rho} x$ for some $x \in X$. For every $y \in B^\perp$ we have $|x_\alpha| \wedge |y| \xrightarrow{\rho} |x| \wedge |y|$. Since $|x_\alpha| \wedge |y| = 0$ for all $\alpha$, it follows that $|x| \wedge |y| = 0$. Hence, $x \in B$. $\square$

Remark that if $\rho$ is a Riesz pseudonorm then we have $x \in U_\rho$ if and only if $|x| \in U_\rho$. Further, because lattice operations are continuous with respect to the locally solid topology induced by the Riesz pseudonorm $\rho$, the generating cone $X^+$ of the vector lattice $X$ is not closed in general.

## 3. Boundedness on Pseudonorm

For the following definition we need first of all the notion of pseudonorm bounded set definition is the following:

**Definition 2.** *A nonempty subset $B \subseteq X$ is said to be pseudonorm bounded if the set $\rho(B)$ is bounded.*

If $\rho$ is a Riesz pseudonorm on $X$ and $B \subseteq X$ is order bounded then $B$ is pseudonorm bounded. In this case, i.e., when $\rho$ is a Riesz pseudonorm, a set $B$ is pseudonorm bounded if and only if the set $|B| = \{|b| \colon b \in B\} \subseteq X^+$ is pseudonorm bounded.

**Definition 3.** *A pseudonorm $\rho$ on a vector lattice $X$ is said to be a bounded pseudonorm, if every pseudonorm bounded set is order bounded in $X$.*

Suppose that $X$ is an $AM$-space with a strong norm unit. Let $f \in X'$, the topological dual of $X$, and $\rho_f(x) = |f(x)|$. If $B \subseteq X$ is pseudonorm bounded with respect to $\rho_f$ then it is norm bounded in $X$. Hence, $B$ is order bounded. It follows that $\rho_f$ is a bounded pseudonorm.

The following result is motivated from [**?**, Theorem 2.1]. It provides an analytical technique to derive a Riesz pseudonorm from a Riesz seminorm. The resulting Riesz pseudonorm is known to have topological relationships with the initial Riesz seminorm.

**Lemma 2.** *Let $u \in X^+$ be arbitrary. If $\rho$ is a Riesz seminorm on the vector lattice $X$ then $\rho_u(x)\rho(|x| \wedge u)$ is a Riesz pseudonorm on $X$.*

*Proof.* It is clear that $\rho_u(x) \geq 0$ for all $x \in X$ and that $\rho_u(0) = 0$. Because $\rho$ is a Riesz seminorm, it follows from $|x+y| \leq |x|+|y|$ for $x, y \in X$ that $\rho_u(x+y) \leq \rho_u(x)+\rho_u(y)$. Let $\lambda_n$ be a sequence of real numbers such that $\lambda_n \to 0$. It follows from

$$\rho_u(\lambda_n x) = \rho(|\lambda_n x| \wedge u) = |\lambda_n|\rho(x)$$

that $\rho_u(\lambda_n x) \to 0$ for all $x \in X$. Hence, $\rho_u$ is a Riesz pseudonorm on $X$. □

**Proposition 1.** *Suppose that $X$ is a vector lattice and $\rho$ is a Riesz seminorm on $X$. Let $u \in X^+$ be arbitrary, and put $\rho_u(x) = \rho(|x| \wedge u)$ for $x \in X$.*

*(i) If $\rho_u$ is such that $U_{\rho_u}$ is order bounded in $X$ then $U_\rho$ is order bounded in $X$.*
*(ii) If $\rho_u$ is a bounded pseudonorm then $\rho$ is bounded seminorm.*

*Proof.* $(i)$ Let $x \in U_\rho$ so that $\rho(x) \leq 1$ holds. Because $\rho$ is a Riesz seminorm, $\rho(|x|) = \rho(x) \leq 1$ holds. It follows that $\rho(|x| \wedge u) \leq \rho(|x|) \leq 1$. Hence, $U_\rho \subseteq U_{\rho_u}$ for all $u \in X^+$. Because the set $U_{\rho_u} \subseteq X$ is order bounded, the set $U_\rho$ is order bounded.

$(ii)$. Let $A$ be a nonempty subset of $X$ which is pseudonorm bounded with respect to the Riesz seminorm $\rho$. Because the set $\rho(A)$ is bounded, the set $\rho_u(A)$ is also bounded. As $\rho_u$ is a bounded pseudonorm, the set $A$ is order bounded. Hence, $\rho$ is a bounded seminorm.

□

## 4. Connection between order convergence and pseudonorm convergence

The following lemma shows us that under monotonicity pseudonorm convergence implies order convergence.

**Lemma 3.** *Suppose that $X$ is a vector lattice, and, $\rho$ is a Riesz pseudonorm on $X$ such that the generating cone $X^+$ is closed with respect to locally solid topology induced by $\rho$. Any monotone and pseudonorm convergent net in $X$ order converges to its pseudonorm limit in $X$.*

*Proof.* Let $x_\alpha$ be a net such that $x_\alpha \uparrow$ and $x_\alpha \xrightarrow{\rho} x$. The generating cone $X^+$ of $X$ is closed with respect to the locally solid topology on $X$ induced by $\rho$. Fix an arbitrary index $\alpha$. Then $x_\beta - x_\alpha \in X^+$ whenever $\beta \geq \alpha$. By taking limit of $x_\beta - x_\alpha$ over $\beta$ we conclude that $x - x_\alpha \in X^+$, and hence, $x \geq x_\alpha$ for any $\alpha$. Since $\alpha$ is arbitrary, $x$ is an upper bound of $x_\alpha$. If $y \geq x_\alpha$ for all $\alpha$ then $y - x_\alpha \xrightarrow{\rho} y - x \in X^+$. Hence, $y \geq x$. Hence, $x_\alpha \uparrow x$. $\square$

## 5. Conclusion

Our aim is to continue to study on pseudonorm continuous and compact operators defined on pseudonorm vector lattices.

**Acknowledgments** This work is supported by TUBITAK Project 118F204.

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