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Chairman's Welcome Speech

Dear Guests,

On behalf of the organizing committee, welcome to International Conference on Mathematics: *An Istanbul Meeting for World Mathematicians*, 3-6 July 2018, Istanbul, Turkey. First of all, we present our deepest thanks to Fatih Sultan Mehmet Vakif University Management due to their great hospitality and understanding.

The conference aims to bring together leading academic scientists, researchers and research scholars to exchange and share their experiences and research results about mathematical sciences.

Besides these academic aims, we also have some social programs for introducing our culture and Istanbul to you. We hope that you will have nice memories in Istanbul for conference days.

We wish to all participants efficient conference and nice memories in Istanbul.

Thank you very much for your interest in International Conference on Mathematics: *An Istanbul Meeting for World Mathematicians*.

Kenan YILDIRIM, Ph. D.

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On the Behavior of Solutions of Second Order Linear Autonomous Delay Differential Equations

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Abstract

In this paper, our aim is to establish a new result for the solutions to second order linear delay differential equations with constant coefficients and constant delay. We used two different real roots of the corresponding characteristic equation. So we obtained a new result on the behavior of the solutions.

Keywords: Delay differential equation, Characteristic equation, Roots, Asymptotic behavior.

1. Introduction

In many fields of the contemporary science and technology systems with delaying links are often met and the dynamical processes in these are described by systems of delay differential equations [1,3,4]. The delay appears in complicated systems with logical and computing devices, where certain time for information processing is needed. The theory of linear delay differential equations has been developed in the fundamental monographs [1], [3-7]. Analogous results for the solutions to second order linear delay differential equations has recently been obtained by the authors [2], [15-17] and [20]. Our work in the present paper is essentially motivated by the results in the papers by Ch. G. Philos and I. K. Purnaras [9-14]. Let us consider initial value problem for second order delay differential equation

$$y''(t) = py'(t) + \sum_{i \in I} p_i y'(t - \tau_i) + qy(t) + \sum_{i \in I} q_i y(t - \tau_i) , \quad t \ge 0,$$
(1.1)

$$y(t) = \phi(t) , \quad -\tau \le t \le 0,$$
 (1.2)

where I is an initial segment of natural numbers, p,q, p_i, q_i for $i \in I$ are real constants, and τ_i for $i \in I$ positive real numbers such that $\tau_{i_1} \neq \tau_{i_2}$ for $i_1, i_2 \in I$ with $i_1 \neq i_2$. Let's define $\tau = \max_{i \in I} \tau_i$. (τ is a positive real number.) The equation of form of (1.1) is of interest in biology in explaining self-balancing of the human body and in robotics in consructing biped robots (see [8], [18]). These are illustrations of inverted pendulum problems. A typical example is the balancing of a stick (see [19]). As usual, a twice continuously differentiable real-valued function y defined on the interval $[-\tau, \infty)$ is said to be a *solution* of the initial value problem (1.1) and (1.2) if y satisfies (1.1) for all $t \ge 0$ and (1.2) for all $-\tau \le t \le 0$. It is known that (see, for example, [4]), for any given initial function ϕ , there exists a unique solution of the initial problem (1.1)-(1.2) or, more briefly, the solution of (1.1)-(1.2). Along with the second order delay differential equation (1.1), we associate the following equation

$$\lambda^{2} = p\lambda + \lambda \sum_{i \in I} p_{i} e^{-\lambda \tau_{i}} + q + \sum_{i \in I} q_{i} e^{-\lambda \tau_{i}}, \qquad (1.3)$$

which will be called the *characteristic equation* of (1.1). Equation (1.3) is obtained from (1.1) by looking for solutions of the form $y(t) = e^{\lambda t}$ for $t \in IR$, where λ is a root of the equation (1.3). For a given solution λ of the characteristic equation (1.3), we consider the (first order) delay differential equation

$$z'(t) = (p - 2\lambda_0)z(t) + \sum_{i \in I} p_i e^{-\lambda_0 \tau_i} z(t - \tau_i) - \lambda_0 \sum_{i \in I} p_i e^{-\lambda_0 \tau_i} \int_{t - \tau_i}^{t} z(s) ds - \sum_{i \in I} q_i e^{-\lambda_0 \tau_i} \int_{t - \tau_i}^{t} z(s) ds . \quad (1.4)$$

With the first order delay differential equation (1.4), we associate the equation

$$\delta = p - 2\lambda_0 + \sum_{i \in I} p_i e^{-\lambda_0 \tau_i} e^{-\delta \tau_i} - \delta^{-1} \sum_{i \in I} \left\{ (1 - e^{-\delta \tau_i}) (p_i \lambda_0 + q_i) e^{-\lambda_0 \tau_i} \right\},$$
(1.5)

which is said to be the characteristic equation of (1.4). The last equation is obtained from (1.4) by seeking solutions of the form $z(t) = e^{\delta t}$ for $t \in IR$, where δ is a root of the equation (1.4). The proofs in the following lemmas and theorems can be made in the same way as in the article [20].

2. An Asymptotic Result

In this section, we will present an asymptotic result for the solutions of the second order delay differential equation (1.1), which is closely related to the main result of this paper.

Theorem 2.1. Let λ_0 be real root of the characteristic equation (1.3) and let δ_0 be real root of the characteristic equation (1.5), and set

$$\begin{split} \beta_{\lambda_0} &\equiv \sum_{i \in I} \left(\lambda_0 p_i + q_i\right) \tau_i \, e^{-\lambda_0 \tau_i} + 2\lambda_0 - p - \sum_{i \in I} p_i e^{-\lambda_0 \tau_i} \neq 0 \ , \\ \eta_{\lambda_0, \delta_0} &\equiv 1 + \sum_{i \in I} p_i e^{-(\lambda_0 + \delta_0) \tau_i} \tau_i - \delta_0^{-2} \sum_{i \in I} \left(\lambda_0 p_i + q_i\right) \left(1 - e^{-\delta_0 \tau} - \delta_0 \tau_i e^{-\delta_0 \tau_i}\right) e^{-\lambda_0 \tau_i} \ . \end{split}$$

Also, define

$$\begin{split} L(\lambda_{0};\phi) &= \phi'(0) + (2\lambda_{0} - p)\phi(0) - \sum_{i \in I} p_{i}\phi(-\tau_{i}) + \sum_{i \in I} \left(\lambda_{0}p_{i} + q_{i}\right)e^{-\lambda_{0}\tau_{i}} \int_{-\tau_{i}}^{0} e^{-\lambda_{0}s}\phi(s)ds \ , \\ R(\lambda_{0},\delta_{0};\phi) &= \phi(0) - \frac{L(\lambda_{0};\phi)}{\beta_{\lambda_{0}}} + \sum_{i \in I} p_{i}e^{-(\lambda_{0}+\delta_{0})\tau_{i}} \int_{-\tau_{i}}^{0} e^{-\delta_{0}s} \left(e^{-\lambda_{0}s}\phi(s) - \frac{L(\lambda_{0};\phi)}{\beta_{\lambda_{0}}}\right)ds \\ &- \sum_{i \in I} \left(\lambda_{0}p_{i} + q_{i}\right)e^{-\lambda_{0}\tau_{i}} \int_{0}^{\tau_{i}} e^{-\delta_{0}s} \left\{\int_{-s}^{0} e^{-\delta_{0}u} \left(e^{-\lambda_{0}u}\phi(u) - \frac{L(\lambda_{0};\phi)}{\beta_{\lambda_{0}}}\right)du\right\}ds. \end{split}$$

(Note that, because of $\beta_{\lambda_0} \neq 0$, we always have $\delta_0 \neq 0$.) Assume that

$$\mu_{\lambda_{0},\delta_{0}} \equiv \sum_{i \in I} \left| p_{i} \right| \left|^{-(\lambda_{0}+\delta_{0})\tau_{i}} \tau_{i} + \delta_{0}^{-2} \sum_{i \in I} \left| \lambda_{0} p_{i} + q_{i} \right| \left| 1 - e^{-\delta_{0}\tau_{i}} - \delta_{0}\tau_{i} e^{-\delta_{0}\tau_{i}} \right| e^{-\lambda_{0}\tau_{i}} < 1$$

$$(2.1)$$

(This assumption guarantees that $\eta_{\lambda_0,\delta_0} > 0$.) Then, for any $\phi \in C([-\tau,0], IR)$, the solution y of (1.1)-(1.2) satisfies

$$\lim_{t \to \infty} \left\{ e^{-(\lambda_0 + \delta_0)t} y(t) - \frac{L(\lambda_0; \phi)}{\beta_{\lambda_0}} e^{-\delta_0 t} \right\} = \frac{R(\lambda_0, \delta_0; \phi)}{\eta_{\lambda_0, \delta_0}} \,. \tag{2.2}$$

2

3. Three Lemma

Lemma 3.1. Let λ_0 and δ_0 be real roots of the characteristic equations (1.3) and (1.5), respectively, and let $\eta_{\lambda_0,\delta_0}$ be defined as in Theorem 2.1. Suppose that

$$\sum_{i\in I} p_i < 0 \quad and \quad \sum_{i\in I} (\lambda_0 p_i + q_i) > 0. \tag{3.1}$$

Then $\eta_{\lambda_0,\delta_0} > 0$ if (1.5) has another real root less than δ_0 , and $\eta_{\lambda_0,\delta_0} < 0$ if (1.5) has another real root greater than δ_0 .

Lemma 3.2. Let λ_0 be real root of the characteristic equations (1.3). Assume that

$$-\sum_{i\in I} p_i \tau_i e^{-\left(p-\lambda_0-\frac{1}{\tau_i}\right)\tau_i} + \sum_{i\in I} (\lambda_0 p_i + q_i)\tau_i e^{-\lambda_0\tau_i} \int_{\tau_i}^{\tau_i} e^{-\left(p-2\lambda_0-\frac{1}{\tau_i}\right)s} ds < 1 , \quad (3.2)$$

$$\sum_{i \in I} |p_i| \ \tau_i \ e^{-\left(p - \lambda_0 - \frac{1}{\tau_i}\right)\tau_i} + \sum_{i \in I} |\lambda_0 p_i + q_i| \tau_i \ e^{-\lambda_0 \tau_i} \int_0^{\tau_i} s e^{-\left(p - 2\lambda_0 - \frac{1}{\tau_i}\right)s} ds \le 1.$$
(3.3)

Then, in the interval $\left(p - 2\lambda_0 - \frac{1}{\tau}, \infty\right)$, the characteristic equation (1.5) has a unique root δ_0 ; this root satisfies (2.1), and the root δ_0 is less than $p - 2\lambda_0 - \frac{1}{\tau}$, provided that

$$-\sum_{i\in I} p_i \tau_i e^{-\left(p-\lambda_0+\frac{1}{\tau_i}\right)\tau_i} + \sum_{i\in I} (\lambda_0 p_i + q_i)\tau_i e^{-\lambda_0\tau_i} \int_0^{\tau_i} e^{-\left(p-2\lambda_0+\frac{1}{\tau_i}\right)s} ds > -1.$$
(3.4)

Lemma 3.3. Let λ_0 be real root of the characteristic equations (1.3). Suppose that statement (3.1) is true. Then we have:

- a) In the interval $[p 2\lambda_0, \infty)$, the characteristic equation (1.5) has no roots.
- b) Assume that (3.2) holds. Then: (i) $\delta = p 2\lambda_0 \frac{1}{\tau}$ is not a root of the characteristic equation (1.5). (ii) In the interval $\left(p 2\lambda_0 \frac{1}{\tau}, p 2\lambda_0\right)$, (1.5) has a unique root. (iii) In the interval $\left(-\infty, p 2\lambda_0 \frac{1}{\tau}\right)$, (1.5) has a unique root.

4. The Main Result

Theorem 4.1. Let λ_0 and δ_0 be real roots of the characteristic equation (1.3) and (1.5), respectively, and let β_{λ_0} , $\eta_{\lambda_0,\delta_0}$, $L(\lambda_0;\phi)$ and $R(\lambda_0,\delta_0;\phi)$ be defined as in Theorem 2.1. Suppose that statement (3.1) is true. Also, let δ_1 be real root of (1.5) with $\delta_1 \neq \delta_0$. (Note that, because of $\beta_{\lambda_0} \neq 0$, we have $\delta_0 \neq 0$ and $\delta_1 \neq 0$. Moreover, Lemma 3.1. guarantees that $\eta_{\lambda_0,\delta_0} \neq 0$.) Then the solution y of the IVP (1.1) and (1.2) satisfies

$$C_1(\lambda_0, \delta_0, \delta_1; \phi) \le e^{-\delta_1 t} \left[e^{-\lambda_0 t} y(t) - \frac{L(\lambda_0; \phi)}{\beta_{\lambda_0}} - e^{-\delta_0 t} \frac{R(\lambda_0, \delta_0; \phi)}{\eta_{\lambda_0, \delta_0}} \right]$$

$$\leq C_2(\lambda_0, \delta_0, \delta_1; \phi) \tag{4.1}$$

for all $t \ge 0$, where

$$C_{1}(\lambda_{0},\delta_{0},\delta_{1};\phi) = \min_{-\tau \le t \le 0} \left\{ e^{-\delta_{1}t} \left[e^{-\lambda_{0}t}\phi(t) - \frac{L(\lambda_{0};\phi)}{\beta_{\lambda_{0}}} - e^{-\delta_{0}t} \frac{R(\lambda_{0},\delta_{0};\phi)}{\eta_{\lambda_{0},\delta_{0}}} \right] \right\}, (4.2)$$

$$C_2(\lambda_0, \delta_0, \delta_1; \phi) = \max_{-\tau \le t \le 0} \left\{ e^{-\delta_1 t} \left[e^{-\lambda_0 t} \phi(t) - \frac{L(\lambda_0; \phi)}{\beta_{\lambda_0}} - e^{-\delta_0 t} \frac{R(\lambda_0, \delta_0; \phi)}{\eta_{\lambda_0, \delta_0}} \right] \right\}.$$
(4.3)

We see immediately that inequalities (4.1) can equivalently be written as follows $\begin{bmatrix} I & I_{\lambda}(\lambda_{0}; \phi) \end{bmatrix} = R(\lambda_{0}, \delta_{0}; \phi)$

$$C_{1}(\lambda_{0}, \delta_{0}, \delta_{1}; \phi) e^{(\delta_{1} - \delta_{0})t} \leq e^{-\delta_{0}t} \left[e^{-\lambda_{0}t} y(t) - \frac{L(\lambda_{0}; \phi)}{\beta_{\lambda_{0}}} \right] - \frac{R(\lambda_{0}, \delta_{0}; \phi)}{\eta_{\lambda_{0}, \delta_{0}}}$$
$$\leq C_{2}(\lambda_{0}, \delta_{0}, \delta_{1}; \phi) e^{(\delta_{1} - \delta_{0})t} , \quad t \geq 0.$$

Hence, if $\delta_1 < \delta_0$, then the solution **y** of the IVP (1.1) and (1.2) satisfies (2.2).

Also, we observe that (4.1) is equivalent to

$$e^{\lambda_0 t} \left[C_1(\lambda_0, \delta_0, \delta_1; \phi) e^{\delta_1 t} + \frac{L(\lambda_0; \phi)}{\beta_{\lambda_0}} + \frac{R(\lambda_0, \delta_0; \phi)}{\eta_{\lambda_0, \delta_0}} e^{\delta_0 t} \right] \le y(t)$$

$$\le e^{\lambda_0 t} \left[C_2(\lambda_0, \delta_0, \delta_1; \phi) e^{\delta_1 t} + \frac{L(\lambda_0; \phi)}{\beta_{\lambda_0}} + \frac{R(\lambda_0, \delta_0; \phi)}{\eta_{\lambda_0, \delta_0}} e^{\delta_0 t} \right] \quad \text{for all } t \ge 0.$$

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Analysis of Engineering Elasticity Problems by Finite Elements Based on the Strain Approach

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Abstract

The finite element method is the most practical tool for the analysis of structures whatever the geometrical shape, applied loads and behavior. In addition, practice shows that engineers prefer to model their structures with the simplest finite elements. Also, in the numerical analysis, it is well known that according to the choice of the interpolation field, several models of finite elements can be used and with a good displacement pattern, convergence towards the correct value will be much faster than with a poor pattern, thus resulting in saving of the computing time. In this paper, the procedure of the development of finite elements based on the strain based approach (S.B.A) is described. Through some applications and validation tests; using some membrane elements, an excellent convergence can be obtained when the results are compared with those given by corresponding displacement-based elements.

Keywords: Elasticity Problems, Finite Element Method, Membrane Elements, Strain Based Approach.,

1. Introduction

In the field of structural analysis, the most common approach, to finite element modelling of structure, is to consider that the displacements at the nodal points are the main unknown parameters of the problem [1], [2] and [3]. Earliest, numerical tests with strain based approach were carried out by Ashwell, Sabir and Roberts [4], on simple circular arches with different aspect ratios, the results obtained show that a better convergence can be obtained when assumed strain based elements are used instead of assumed displacement models. The development of elements based on the strain approach has continued and many elements were developed for general plane elasticity problems by Sabir and al [5], [6] and [7]. The extension of the work to the development of finite elements in polar coordinates has continued [8], [9]. After that, a new rectangular element was elaborated for the general plane elasticity by Belarbi & Maalem [10]. An improved Sabir triangular element with drilling rotation was developed; this triangular element, with three nodes and three degrees of freedom, presents very good performance and may be used in various practical problems [11]. In this paper, the procedure of the development of finite elements based on the strain based approach (S.B.A) is described. Some applications and validation tests; using two membrane elements, are presented. The results obtained are compared with those given by corresponding displacement-based elements and the closed form solution.

2. Procedure of the development of finite element based on the strain approach

In the Strain based approach, we first formulate the strain field; the displacement field is than obtained by integration. Compared to the classic formulation where deformations are obtained by derivation of the chosen displacement fields.

The main advantages of this approach are cited in reference [12]. To illustrate the procedure for the development of displacement field when the strain approach is used; the simple example of a rectangular element of plan elasticity is examined. In general, for the problems of plan elasticity, the relations between the strains ε_x , ε_y , γ_{xy} and the displacements of translation U and V in the plane are given as follows:

$$\varepsilon_{x} = U_{,x} = (\partial U/\partial x) ; \ \varepsilon_{y} = V_{,y} ; \ \gamma_{xy} = U_{,y} + V_{,x}$$
 (1)

The three above strains are equated to zero (Rigid Body Mode) and the resulting differential equations are integrated, the following expressions are obtained for the displacements U and V:

$$U_{R} = a_{1} - a_{3} y \qquad (2a)$$

$$V_{\rm R} = a_2 + a_3 x \tag{2b}$$

The above equations represent the displacement fields corresponding to the rigid body mode (RBM) relative to the element. We note that equations (2) contain three constants: a_1 and a_2 represent the translational movements in the X and Y directions; a_3 represents the rotation in the plan. If a rectangular element is required with four nodes and each node has two DOF, the final displacement fields must contain eight independent constants. We have used three constants for the representation of the RBM; we have five constants that can be distributed over the strains as follows:

$$\begin{aligned} \varepsilon_{\mathbf{x}} &= \mathbf{a}_{4} + \mathbf{a}_{5} \mathbf{y} \\ \varepsilon_{\mathbf{y}} &= \mathbf{a}_{6} + \mathbf{a}_{7} \mathbf{x} \\ \gamma_{\mathbf{xy}} &= \mathbf{a}_{8} \end{aligned}$$
 (3 a,b,c)

Equations (3) can now be integrated:

$$U_{S} = a_{4}x + a_{5}xy - a_{7}\frac{y^{2}}{2} + a_{8}\frac{y}{2}$$

$$V_{S} = a_{6}y - a_{5}\frac{x^{2}}{2} + a_{7}xy + a_{8}\frac{x}{2}$$
(4a,b)

The final displacement functions can be obtained by summing equations (2) and (4), hence:

U =
$$a_1 - a_3 y + a_4 x + a_5 xy - a_7 \frac{y^2}{2} + a_8 \frac{y}{2}$$
 (5a,b)
V = $a_2 + a_3 x + a_6 y - a_5 \frac{x^2}{2} + a_7 xy + a_8 \frac{x}{2}$

Firstly, the strains are independent of each other, so it will be useless to couple between the bending and shearing actions, from which a pure bending state can be obtained. In addition to this, the displacement functions contain quadratic terms, which allow correct bending. It is interesting to compare certain peculiarities of this element with its equivalent based on the displacement model, that is, the most commonly used bilinear rectangular element deriving its name from its ability to represent linear displacements on both sides of the rectangle and the displacement functions are given by:

$$U = a_1 + a_2 x + a_3 y + a_4 x y$$
(6a)

$$V = a_5 + a_6 x + a_7 y + a_8 x y$$
(6b)

Since there are only two nodes on each side of the element, only linear displacements will be possible for the interpolation of the continuity between the elements that will be guaranteed. Thus, under loading, the bilinear element deforms while ensuring inter-element continuity without overlapping, and the strains are given by deriving the displacement functions. Thus:

$$\varepsilon_{x} = a_{2} + a_{4}y$$

$$\varepsilon_{y} = a_{7} + a_{8}x$$

$$\gamma_{xy} = a_{3} + a_{4}x + a_{6} + a_{8}y$$

(7a,b,c)

These strains are not independent, as long as they are linked by the constants a_4 and a_8 . It is clear, however, that the bilinear element cannot represent an independent shear state unless $a_4 = a_8 = 0$, and which will give ε_x and ε_y as constants. Hence a state of pure bending associated with direct linear deformations and without shear strains cannot be obtained with the bilinear element.

3. Presentation of two finite elements based on the S.B.A. for plane Elasticity Problems **3.1** Sector Element SBMS-BH [13]

This element has four nodes in addition to the central node, and two degrees of freedom per node U and V and uses the static condensation. The displacement functions for the sector element in polar coordinates will be (Fig.1) :

$$U_{r} = a_{1} - a_{3} \theta + a_{4} r + a_{5} r \theta - 0.5 a_{7} \theta^{2} + 0.5 a_{8} \theta + 0.5 a_{9} r^{2}$$

$$V_{\theta} = a_{2} + a_{3} r - 0.5 a_{5} r^{2} + a_{6} \theta + a_{7} r \theta + 0.5 a_{8} r + 0.5 a_{10} \theta^{2}$$
(8a)
(8b)



Fig.1: Coordinate system and displacements. (a) The sector element "SBMS-BH". (b) The quadrilateral element "Q4SBE5"

3.2. Quadrilateral membrane element Q4SBE5 [14]

Figure 1 (b) shows the geometry of the element "Q4SBE5" (Strain Based Quadrilateral Element) and the corresponding nodal displacements. The quadrilateral element has five nodes, four corner nodes in addition to an internal node, verifies the local equilibrium and uses the static condensation. Each node (i) is attributed to two degrees of freedom (d.o.f) Ui, and Vi. Therefore, the displacement field should include ten independent constants. The strain field can be defined as follows:

$$\begin{cases} \varepsilon_{x} = a_{4} + a_{5}y + a_{9}x \\ \varepsilon_{y} = a_{6} + a_{7}y + a_{10}x \\ \gamma_{xy} = -a_{5}xR - a_{7}yR + a_{8} - a_{9}Hy - a_{10}Hx \end{cases}$$
(9)
Where: $H = \frac{2}{(1-\nu)}$; $R = \frac{2\nu}{(1-\nu)}$

By integrating equations (9) and adding the rigid body mode, we obtain the final displacement functions:

$$U = a_1 - a_3 y + a_4 x + a_5 x y - a_7 y^2 (R+1) / 2 + a_8 y / 2 + a_9 (x^2 - Hy^2) / 2$$
(10a)

$$V = a_2 + a_3 x - a_5 x^2 (R+1)/2 + a_6 y + a_7 x y + a_8 x/2 + a_{10} (y^2 - Hx^2)/2$$
(10.b)

4 Validation Tests **4.1 SBMS-BH element**

The performance of the developed sector element SBMS-BH is tested by applying it to a thick cylinder under internal pressure. The results obtained for the radial deflections Ur are compared to the analytical solution given by Rekatch [15]. Figure 2 gives the convergence curve for the results obtained from elements SBMS-BH and SBS4 [8] for the radial deflection, it convergence to the analytical results when the cylinder is divided into a small number of elements (2x2), which illustrates the high degree of accuracy obtained from element SBMS-BH, the error accounts is equal to 0.063 % of the exact solution. Good performance is also given for the radial and tangential stresses σ_r and σ_{θ}



Fig.2: Convergence curve for the radial deflection U_r

4.2. Q4SBE5 element

In this test problem, the behaviour of finite elements with a significant geometrical irregular shape is examined. This problem was critically analysed in [16] to test the behaviour and accuracy of elements 07β and $07\beta^*$, and consists of a cantilever beam having a rectangular section (1 x t x h = 10 x 1 x 2), and subjected to two nodal forces (P =1000) forming a couple.



Fig.3: Convergence curve for vertical displacement at the end of the Cantilever Beam

Figure 3 shows the stability, the reliability and the good performance of "Q4SBE5" element no matter what the geometrical distortion might be, this is in part probably explained by the nature of analytical integration carried out. These results confirm that the formulated element Q4SBE5 satisfies the High Order Patch Test. 9

5. Conclusions

From the results obtained above, the following conclusions can be drawn:

- The results obtained from SBMS-BH and Q4SBE5 elements are shown to converge to the theoretical solution for the problems considered.

- It should be mentioned here that the convergence is monotone for both deflections and stresses. The inclusion of the internal node and the verification of the equilibrium equations ameliorate the results obtained.

The efficiency and good performance of the Strain Based Approach is confirmed. For recommendation, it is of importance to extend the existing formulated elements to non linear analysis, dynamics behaviour and thermal effect.

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More Advanced Finite Elements for the Analysis of Rectangular and Circular Plates

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Abstract: Modeling of structures composed of plates with different geometrical shapes becomes a very practical tool in engineering problems, whatever the type of the material used, applied loadings, thickness and boundary conditions. The mostly used method is the finite element; which is well known and with a good displacement field, the convergence towards the correct solution will be much faster than with a poor one. In this paper, two finite elements based on two different theories for the analysis of rectangular and circular plates are formulated and examined. Good convergence to the analytical solutions has been observed when compared to those given by the corresponding elements. According to the results obtained, the efficiency and performance of first order shear deformation theory and the strain based approach is demonstrated.

Keywords: Rectangular and Circular Plates, Strain Based Approach, Finite Element Method, First Order Theory, Laminated Plates, Sector Element.

1. Introduction

Modelling engineering structures should be chosen to represent the real structure as closely as possible with regard to materiel properties, the geometrical shape and applied loadings. Another factor in the idealisation process is the type of the finite element used in the numerical analysis. This, however, depends on many other parameters, such as the efficiency of the elements and the importance of local features in the structure particularly stress concentrations [1], [2] and [3]. Composite laminated materials are those meet the above requirements due to their low density, high strength and excellent durability. There are several theories for the study of the multilayer structures behaviour; among them we can cite the classical theory of laminates plates which is the extension of Kirchhoff's theory to composite materials. The first order theory is the extension of that of Reissner-Midlin and recently the higher order theories. In this paper, a four-node rectangular finite element with five degrees of freedom per node, based on the first order theory taking into account the transverse shear effect has been formulated. This element can be used for modelling symmetrical thick rectangular plates made of laminated composite materials. Also, the strain based approach is one of the efficient approaches used for developing finite elements for the numerical analysis of plates and shells in engineering structures [4], [5]. The numerical analysis of thin circular plates with openings, a few works are carried out with different numerical approaches, among them, modelling circular plates with quadrilaterals elements. This technique is not good enough to fit the curve surface properly; especially for circular plates with openings, only if the mesh size of elements are too small and increased number mashing are used near the opening; while the sector elements will model quite good the structures properly. In this paper, a sector finite element based on the strain approach developed recently by Abderrahamani et al, [6] is presented. The development₁element is

based on the Kirchhoff theory and is used for the numerical of thin circular plates with opening. The results obtained are very significant for both deflections and stresses.

2. Formulation of rectangular plate element based on the first order shear deformation theory (R4FDST)

The element (R4FDST) is a rectangular in shape with four nodes based on Lagrangian type i.e. the variables are independent. Each node has five degrees of freedom, two degrees in the plane (x, y) that are $u_0(x, y), v_0(x, y)$ and three out of plane $w_0(x, y), \varphi_x(x, y), \varphi_y(x, y)$ (see Figure 1)



Fig.1: Geometry of the element R4FDST and corresponding nodal variables

The element used is based on first order theory, so the displacements in the plane and out of plane are given by Berthelot [7].

$$\begin{cases} u(x, y, z) \\ v(x, y, z) \\ w(x, y, z) \end{cases} = \begin{cases} u_0(x, y) + z\varphi_x(x, y) \\ v_0(x, y) + z\varphi_y(x, y) \\ w_0(x, y) \end{cases}$$
(1)

The variables of the displacements are: $u_0(x, y), v_0(x, y), w_0(x, y), \varphi_x(x, y), \varphi_y(x, y)$ and φ_x , φ_y are the rotations of the normal around the (x, y) axes respectively.

The displacement vector for all coordinates points (x, y) of the plate are expressed by:

$$\delta(x, y) = \sum_{i=1}^{4} N_i(x, y) \cdot \delta_i$$
⁽²⁾

Where:

 $\delta(x, y)$: Is the displacement vector.

<

 $N_i(x, y)$: Is the element shape functions.

 $\{\delta_i\}$: Is the nodal vector displacement

The potential energy of plate deformation is given by:

$$U = \frac{1}{2} \int_{v} \sigma \ \varepsilon^{T} \ dv \tag{3}$$

The total potential energy of plate deformation subjected to transverse loading distributed across its surface is given by:

$$\Pi = U + W$$

The equilibrium configuration is defined by the minimization of the total potential energy which means the cancellation of its first variation, namely:

$$\partial \Pi = \delta U - \delta W = 0$$
¹²

This allows obtaining the following equilibrium equation:

$$[k^e] \{q\} = \{F^e\}$$

$$\tag{4}$$

Where the element stiffness matrix:

$$[k^{e}] = \int_{A} \left(\{B_{m}\}^{T} [A] \{B_{m}\} + \{B_{m}\}^{T} [B] \{B_{f}\} + \{B_{f}\}^{T} [B] \{B_{m}\} \right)$$
(5)

$$+ \left\{ B_{f} \right\}^{T} \left[D \right] \left\{ B_{f} \right\} + \left\{ B_{c} \right\}^{T} \left[H \right] \left\{ B_{c} \right\} \quad \left) dA \\ \left[K \right] \left\{ q \right\} = \left\{ F \right\} \tag{6}$$

With [K] is the global stiffness matrix, $\{F\}$ is the global force vector and $\{q\}$ is the vector of global displacements of the plate nodes.

3. Formulation of sector plate bending element based on the strain approach and Kirchhoff theory (SBSPK)

The developed element presented here is called SBSPK and has four nodes and three degrees of freedom per node (3 d.o.f./node).

$$w = a_1 - a_2 r - a_3 \theta - a_4 \frac{r^2}{2} - a_5 \frac{r^3}{6} - a_6 \frac{r^2 \theta}{2} - a_7 \frac{r^3 \theta}{6} - a_8 \frac{\theta^2}{2} - a_9 \frac{r \theta^2}{2} - a_{10} \frac{\theta^3}{6} - a_{11} \frac{r \theta^3}{6} - a_{12} \frac{r \theta}{6} - a_{12} \frac{r \theta}{2}$$
(7a)

$$\beta_r = a_2 + a_4 r + a_5 \frac{r^2}{2} + a_6 r\theta + a_7 \frac{r^2\theta}{2} + a_9 \frac{\theta^2}{2} + a_{11} \frac{\theta^3}{6} + a_{12} \frac{\theta}{2}$$
(7b)

$$\beta_{\theta} = a_3 + a_6 \frac{r^2}{2} + a_7 \frac{r^3}{6} + a_8 \theta + a_9 r \theta + a_{10} \frac{\theta^2}{2} + a_{11} \frac{r \theta^2}{2} + a_{12} \frac{r}{2}$$
(7c)



Fig 2. Geometry of the sector element SBSPK

Where:

 r_1 : Internal radius; r_2 : External radius; R: The radius of curvature of the central circumferential line of the element.

The element stiffness matrix $[k^e]$ of SBSPK element can be obtained by the well known expressions of the finite element method.

$$[k^{e}] = [A]^{-T} \left(\int_{-\beta}^{\beta} \int_{r_{1}}^{r_{2}} [B]^{T} [D] [B] r \, dr \, d\theta \right) [A]^{-1} = [A]^{-T} [k_{0}] [A]^{-1}$$
(8)

For an isotropic material, the constitutive relationships between stress and strain for the Kirchhoff theory are given by:

$$\begin{bmatrix} M_r \\ M_{\theta} \\ M_{r\theta} \end{bmatrix} = \begin{bmatrix} d_{11} & d_{12} & 0 \\ d_{12} & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix} \begin{bmatrix} k_r \\ k_{\theta} \\ k_{r\theta} \end{bmatrix}$$
(9)

With: $M_r, M_{\theta}, M_{r\theta}$: The bending moments

[D]: Matrix contains the values of d_{ij} which are defined by:

$$d_{11} = d_{22} = \frac{Eh^3}{12(1-v^2)}, d_{12} = \frac{vEh^3}{12(1-v^2)}, d_{33} = d_{11}\frac{(1-v)}{2}$$
With:

With:

E: Young modulus, h: Plate thickness and v: Poisson's ration.

4. Validations

4.1 Validation of R4FDST element

This application consists in studying a square laminated plate simply supported on its four edges and subjected to uniformly distributed load ($q = 1 \text{ N} / \text{m}^2$. The plate is composed with three layers with the following geometrical and mechanical properties:

 $e_1 = e_2 = e_3$ and a / h = 20, the orientation of fibers are (0/90/0)

Young's module: $E_{11} = 25 Mpa$, $E_{22} = 1Mpa$.

Shear modulus: $G_{12}=G_{13}=0.2\,Mpa$, $G_{23}=0.5Mpa$

Table 1 : The maximum deflection of laminated	plate	(0/90/0) simply	supported
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Mesh	Maximum deflection	Error (%)
2 x 2	0.2480	67.2520
4 x 4	0.5270	30.4106
6 x 6	0.6516	13.9574
8 x 8	0.7088	6.4043
10 x 10	0.7381	2.5353
12 x 12	0.7548	0.3301
Analytical solution [8]	0.7573	

4.2 Validation of SBSPK element:

The ring plate shown in Fig.3 is subjected to a uniformly distributed load P = 1 N.mm along the inner diameter. The geometrical and mechanical properties are taken as:

 $h = 0.2 \text{ mm}, R = 40 \text{ mm}, r = 20 \text{ mm}, E = 2x10^5, v = 0.3.$



Fig.3: Clamped ring plate under uniformly distributed load along the inner circle

The analytical solution of the lateral displacement W is given by Geminard and Giet [9].

$$W = \frac{PR^2}{8D} \beta [(1+2K_2)(1-\alpha^2) + 4K_2 \log \alpha + 2\alpha^2 \log \alpha]$$

$$K_2 = \beta^2 \frac{1+(1-\nu)\log\beta}{1-\nu+(1+\nu)\beta^2}$$

Where:

anserns acca roua arong	
Mesh	Wmax
	(SBSPK)
1x1	0.03183
2x2	0.09322
3x3	0.2028
4x4	0.3791
4x5	0.4492
Analytical solution[9]	0.4391

 Table 2 The lateral displacement W_{max} for the clamped ring plate under uniformly distributed load along the inner circle

5. Conclusion

We examined the performance of the developed element **R4FDST** based on the first order shear deformation through a comparative study on the maximum deflection. The comparative study shows the good behavior of the element, even only one test is presented above. The results of the numerical analysis with the **R4FDST** element are very acceptable compared to the analytical solution. The new strain-based element "SBSPK" is proposed for the analysis of circular thin plate bending problems. Several numerical examples were studied to evaluate the performance of the present element and only one test is presented here. It has been shown that with only small numbers of elements, good results can be obtained compared to the analytical solution. Both elements are very suitable for modeling several civil engineering applications.

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Generalized Boolean Sum Operators of Bivariate (p,q)-Balazs-Szabados Operators

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We intoduce a bivariate operators of (p,q)-Balazs-Szabados operators and obtain the degree of approximation for these operators in terms of the partial moduli of continuity, the complete modulus of continuity and the Lipschitz class functions. Also, we construct the generalized Boolean sum (GBS) operators of bivariate (p,q)-Balazs-Szabados operators in terms of the mixed modulus of smoothness and theLipschitz class of Bögel continuous functions

Keyword(s): Balazs-Szabadosoperators, (p,q)-calculus, rate of convergence

1. Introduction

Mursaleen et al. applied (p,q)-calculus in approximation theory and introduced Bernstein operators based on (p,q)-integers. Hence the studies in approximation theory have been extended from q-calculus to (p,q)-calculus. We recall some notation of (p,q)- calculus.

Let $0 \le q \le p \le 1$. For each nonnegative integer *n*, (p,q)-integer of *n* and (p,q)- factorial of *n* are defined by

$$[n]_{p,q} := \frac{p^n - q^n}{p - q},$$

$$[n]_{p,q} !:= \begin{cases} [n]_{p,q} [n - 1]_{p,q} \dots [2]_{p,q} [1]_{p,q}, & if \quad n = 1, 2, 3 \dots \\ 1, & if \quad n = 0 \end{cases},$$

and for integers *n*, *k* satisfying $n \ge k \ge 0$, (p,q)-binomial coefficients are defined by

$${n \brack k}_{p,q} \coloneqq \frac{[n]_{p,q}!}{[n-k]_{p,q}! [k]_{p,q}!}$$

If we take p=1, they reduce to q-analogues. Further, we have

$$(ax + by)_{p,q}^{n} \coloneqq \sum_{k=0}^{n} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} {n \brack k}_{p,q}^{n} a^{n-k} b^{k} x^{n-k} y^{k},$$
$$(ax + by)_{p,q}^{n} = (ax + by)(pax + qby)(p^{2}ax + q^{2}by) \dots (p^{n-1}ax + q^{n-1}by).$$

(p,q)-analogue of Balázs-Szabados operators is defined by Yıldız Özkan and İspir as follows

$$T_n^{(p,q)}(f;x) = \frac{1}{(1+a_n x)_{p,q}^n} \sum_{k=0}^n f\left(\frac{[k]_{p,q}}{q^{k-1}b_n}\right) p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} {n \brack k}_{p,q}^n (a_n x)^k.$$

The rate of convergence of (p,q)-Balázs-Szabados operators was obtained by using the Lipschitz class functions and the Peetre's K-functional functions, and the degree of asymptotic approximation was given by means of Voronovskaja type theorem. Also, Comparisons associated the convergence of Balázs-Szabados, q-Balázs-Szabados and (p,q)-Balázs-Szabados operators to certain functions we given by illustration.

2. Construction of Operators

We introduce a bivariate operators of (p,q)-Balázs-Szabados operators

$$\begin{split} T_{n_{1},n_{2}}^{(p_{1},q_{1},p_{2},q_{2})}(f;x,y) &= \sum_{k=0}^{n_{1}} \sum_{j=0}^{n_{2}} v_{n_{1},k}(x;p_{1},q_{1}) s_{n_{2},m}(y;p_{2},q_{2}) f\left(\frac{[k]_{p_{1},q_{1}}}{q_{1}^{k-1}b_{n}},\frac{[j]_{p_{2},q_{2}}}{q_{2}^{j-1}d_{n}}\right). \\ \text{Here f is a real valued function on } [0,\infty) \times [0,\infty), \text{ for } 0 < q_{1} < p_{1} \leq 1, 0 < q_{2} < p_{2} \leq 1, \\ n_{1},n_{2} \in \mathbb{N}, \ x,y \in [0,\infty) \times [0,\infty), \ a_{n_{1}} = [n_{1}]_{p_{1},q_{1}}^{\beta_{1}-1}, \ b_{n_{1}} = [n_{1}]_{p_{1},q_{1}}^{\beta_{1}}, \ c_{n_{2}} = [n_{2}]_{p_{2},q_{2}}^{\beta_{2}-1} \text{ and} \\ d_{n_{2}} = [n_{2}]_{p_{2},q_{2}}^{\beta_{2}} \text{ such that } 0 < \beta_{1} \leq \frac{2}{3} \text{ and } 0 < \beta_{2} \leq \frac{2}{3}. \text{ And also} \\ v_{n_{1},k}(x;p_{1},q_{1}): = \frac{p_{1}^{\frac{(n_{1}-k)(n_{1}-k-1)}{2}}q_{1}^{\frac{k(k-1)}{2}} {n_{1} \choose k}_{p_{1},q_{1}}}{(1+a_{n_{1}}x)_{n_{1}}^{n_{1}}} \end{split}$$

$$s_{n_2,k}(y;p_2,q_2):=\frac{p_2^{\frac{(n_2-j)(n_2-j-1)}{2}}q_2^{\frac{j(j-1)}{2}}{n_2 \choose j}}{(1+c_{n_2}y)_{p_2,q_2}^{n_2}}(c_{n_2}y)^j}{(1+c_{n_2}y)_{p_2,q_2}^{n_2}}.$$

These operators are tensorial product of $T_{n_1}^{(p_1,q_1)}$ and $T_{n_2}^{(p_2,q_2)}$.

Let I = [0, r] for r > 0, $I^2 = I \times I$ and $C(I^2)$ be the space of all real valued continuous functions on I^2 with the norm $||f|| = sup\{|f(x, y)|: (x, y) \in I^2\}$.

In order to obtain the uniform convergence of bivariate operators $T_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(f;x,y)$, we choose the sequences $(p_{1,n_1}), (q_{1,n_1}), (p_{2,n_2})$ and (q_{2,n_2}) satisfying $q_{1,n_1}, q_{2,n_2} \in (0,1)$ and $p_{1,n_1} \in (q_{1,n_1}, 1], p_{2,n_2} \in (q_{2,n_2}, 1]$ such that $\lim_{n_1 \to \infty} p_{1,n_1} = \lim_{n_1 \to \infty} (p_{1,n_1})^{n_1} = \lim_{n_1 \to \infty} q_{1,n_1} = 0,$ 17 (1)

$$\lim_{n_1 \to \infty} (q_{1,n_1})^{n_1} = l_1, 0 < l_1 < 1,$$
(2)

$$\lim_{n_2 \to \infty} p_{2,n_2} = \lim_{n_2 \to \infty} (p_{2,n_2})^{n_2} = \lim_{n_2 \to \infty} q_{2,n_2} = 0,$$
(3)

$$\lim_{n_2 \to \infty} (q_{2,n_2})^{n_2} = l_2, 0 < l_2 < 1,$$
(4)

Theorem1. Let be the sequences (p_{1,n_1}) , (q_{1,n_1}) , (p_{2,n_2}) and (q_{2,n_2}) satisfying the conditions (1-4). Then the bivariate operators $T_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(f; x, y)$ converges uniformly to f on I^2 , for all $f \in C(I^2)$.

For $f \in C(I^2)$, the complete modulus of continuity for the bivariate case is defined as

$$\omega(f; \delta_1, \delta_2) = \sup\{|f(t, s) - f(x, y)| : |t - x| \le \delta_1, |s - y| \le \delta_2\}$$

where $\delta_1 > 0$, $\delta_2 > 0$ and (t, s), $(x, y) \in I^2$. It is clear that,

$$\lim_{\delta_1,\delta_2\to 0^+}\omega(f;\delta_1,\delta_2)=0.$$

Theorem2. Let be the sequences (p_{1,n_1}) , (q_{1,n_1}) , (p_{2,n_2}) and (q_{2,n_2}) satisfying the conditions (1-4) and $f \in C(I^2)$, then for all $(x, y) \in I^2$, it holds the following inequality

$$\left|T_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(f;x,y) - f(x,y)\right| \le 4 \,\omega \left(f;\delta_{n_1}(x),\delta_{n_2}(y)\right),$$

where $\delta_{n_1}(x)$ and $\delta_{n_2}(y)$ are as in Theorem 1.

The partial moduli of continuity with respect to x and y are given by

$$\omega_1(f;\delta) = \sup\{|f(x_1,y) - f(x_2,y)| : y \in I, |x_1 - x_2| \le \delta\}$$

and

$$\omega_2(f;\delta) = \sup\{|f(x,y_1) - f(x,y_2)| : x \in I, |y_1 - y_2| \le \delta\}.$$

It is clear that they satisfy the properties of the usual modulus of continuity.

Theorem3. Let be the sequences (p_{1,n_1}) , (q_{1,n_1}) , (p_{2,n_2}) and (q_{2,n_2}) satisfying the conditions (1-4) and $f \in C(I^2)$, then for all $(x, y) \in I^2$, it holds the following inequality

$$\left| T_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(f;x,y) - f(x,y) \right| \le 2 \left\{ \omega_1 \left(f; \delta_{n_1}(x) \right) + \omega_2 \left(f; \delta_{n_2}(y) \right) \right\},$$

where $\delta_{n_1}(x)$ and $\delta_{n_2}(y)$ are as in Theorem 1.

The Lipschitz class $Lip_M(f; \alpha_1, \alpha_2)$ for bivariate case is defined by $f \in Lip_M(f; \alpha_1, \alpha_2)$ if and only if $|f(t,s) - f(x,y)| \le M|t - x|^{\alpha_1}|s - y|^{\alpha_2}$, for all $f \in C(I^2)$, where $0 < \alpha_1, \alpha_2 \le 1$, $(t,s), (x,y) \in I^2$ are arbitrary.

Theorem4. Let be the sequences (p_{1,n_1}) , (q_{1,n_1}) , (p_{2,n_2}) and (q_{2,n_2}) satisfying the conditions (1-4) and $Lip_M(f; \alpha_1, \alpha_2)$ then for all $(x, y) \in I^2$, it holds the following inequality

$$\left|T_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(f;x,y) - f(x,y)\right| \le M\left(\delta_{n_1}(x)\right)^{\alpha_1} \left(\delta_{n_2}(x)\right)^{\alpha_2}$$

where $\delta_{n_1}(x)$ and $\delta_{n_2}(y)$ are as in Theorem 1.

Bögel defined Bögel-continuous and Bögel-bounded functions.

Let X and Y be compact subset of \mathbb{R} . A function $f: X \times Y \to \mathbb{R}$ is called Bögel-continuous function at $(x_0, y_0) \in X \times Y$ if

$$\lim_{(x,y)\to(x_0,y_0)}\Delta_{(x,y)}f[x_0,y_0;x,y]=0,$$

where $\Delta_{(x,y)} f[x_0, y_0; x, y]$ denotes the mixed difference defined by

$$\Delta_{(x,y)}f[x_0, y_0; x, y] = f(x, y) - f(x, y_0) - f(x_0, y) + f(x_0, y_0)$$

Let *A* be a subset of \mathbb{R}^2 . The function $f: A \to \mathbb{R}$ is Bögel-bounded function on *A* if there exists *M*>0 such that $|\Delta_{(x,y)}f[x_0, y_0; x, y]| \le M$, for every $(t, s), (x, y) \in A$. If *A* is a compact subset of \mathbb{R}^2 , then each Bögel-continuous function is a Bögel-bounded function.

Let denote by $C_b(A)$, the space of all real valued Bögel-continuous functions defined on A with the norm $||f||_B = sup\{|\Delta_{(x,y)}f[t,s;x,y]|:(x,y),(t,s) \in A\}$. And also, we denote with C(A) and B(A) the space of all real valued continuous and bounded functions on A, respectively. C(A) and B(A) are Banach spaces with the norm $||f|| = sup\{|f(x,y)|:(x,y) \in A\}$. It is known that $C(A) \subset C_b(A)$.

3. GBS Operators

We define generalized Boolean sum (GBS) operators of bivariate (p,q)-Balázs-Szabados operators as follows:

$$B_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(f(t,s);x,y) = T_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(f(t,y) + f(x,s) - f(t,s);x,y),$$

for all $(t, s), (x, y) \in I^2$ and $f \in C(I^2)$.

The mixed modulus of smoothness of $f \in C(I^2)$ is defined as

$$\omega_{mixed}(f;\delta_1,\delta_2) = \sup\{\left|\Delta_{(x,y)}f[t,s;x,y]\right|: |t-x| \le \delta_1, |s-y| \le \delta_2\},\$$

for all $(t, s), (x, y) \in I^2$ and $\delta_1 > 0, \delta_2 > 0$.

Theorem5. Let be the sequences (p_{1,n_1}) , (q_{1,n_1}) , (p_{2,n_2}) and (q_{2,n_2}) satisfying the conditions (1-4) and $C_b(I^2,)$ then for all $(x, y) \in I^2$, it holds the following inequality 19

$$\left|B_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(f;x,y) - f(x,y)\right| \le 4 \,\omega_{mixed} \left(f;\delta_{n_1}(x),\delta_{n_2}(y)\right),$$

where $\delta_{n_1}(x)$ and $\delta_{n_2}(y)$ are as in Theorem 1.

The Lipschitz class for B-continuous functions is denoted by $B - Lip_M(f; \alpha_1, \alpha_2)$, and it is defined by $f \in Lip_M(f; \alpha_1, \alpha_2)$ if and only if $|f(t, s) - f(x, y)| \le M|t - x|^{\alpha_1}|s - y|^{\alpha_2}$, for all $f \in C(I^2)$, where $0 < \alpha_1, \alpha_2 \le 1$, $(t, s), (x, y) \in I^2$ are arbitrary.

Theorem6. Let be the sequences (p_{1,n_1}) , (q_{1,n_1}) , (p_{2,n_2}) and (q_{2,n_2}) satisfying the conditions (1-4) and $f \in B - Lip_M(f; \alpha_1, \alpha_2)$ then for all $(x, y) \in I^2$, it holds the following inequality

$$\left| B_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(f;x,y) - f(x,y) \right| \le M \left(\delta_{n_1}(x) \right)^{\alpha_1} \left(\delta_{n_2}(x) \right)^{\alpha_2},$$

where $\delta_{n_1}(x)$ and $\delta_{n_2}(y)$ are as in Theorem 1.

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Approximation by Kantorovich Type q-Balazs-Szabados Operators

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Abstract

We introduce Kantorovich type q-Balazs-Szabados operators called q-BSK operators. We give weighted statistical approximation theorem and the rate of convergence of the q-BSK operators with the help of the weighted modulus of smoothness. Moreover, we investigate the local approximation results. Further, we give some comparisons associated the convergence of q-BSK operators.

Keywords: Balazs-Szabadosoperators, q-calculus, rate of convergence, Peetre's K-functional

1. Introduction

Bernstein type rational functions was defined by Balázs. He gave an estimate for the order of its convergence and proved an asymptotic approximation theorem and a convergence theorem concerning the derivative of these operators. Balázs and Szabados modified these operators. They obtained best possible estimate under more restrictive conditions, in which both the weight and the order of convergence would be better than Balázs operators.

For any non-negative integer r, the q-integer of the number r is defined by

$$[r]_q = \begin{cases} \frac{1-q^r}{1-q} & if \quad q \neq 1\\ r & if \quad q = 1, \end{cases}$$

where q is a fixed positive real number. The q-factorial is defined by

$$[r]_q! = \begin{cases} [1]_q [2]_q \dots [r]_q & if \quad r = 1, 2, \dots \\ 1 & if \quad r = 0, \end{cases}$$

For integers *n*, *r* with $0 \le r \le n$, the *q*-binomial coefficients are defined by

$${n \brack r}_q = \frac{[n]_q!}{[r]_q! [n-r]_q!}.$$

The definite *q*-integral is defined by

$$\int_{0}^{b} f(t)d_{q}t = (1-q)b\sum_{j=0}^{\infty} f(q^{j}b)q^{j}$$
21

and

$$\int_{a}^{b} f(t)d_{q}t = \int_{0}^{b} f(t)d_{q}t - \int_{0}^{a} f(t)d_{q}t,$$

for $0 < q < 1$.

The *q*-analogue of Balázs -Szabados operators is defined by O. Dogru. He investigated Korovkin type statistical approximation properties of these operators for the functions of one and two variables.

We consider q-Balázs-Szabados-Kantorovich operators

$$R_n(f;q,x) = \frac{b_n}{\prod_{s=0}^{n-1}(1+q^s a_n x)} \sum_{j=0}^n q^{-j} q^{\frac{j(j-1)}{2}} {n \brack j}_q (a_n x)^j \int_{\frac{[j]_q}{b_n}}^{\frac{[j+1]_q}{b_n}} f(t) d_q t$$

where $f:[0,\infty) \to \mathbb{R}$ is a nondecreasing and continuous function, $x \in \mathbb{R}_+ = [0,\infty)$, $a_n = [n]_q^{\beta-1}$, $b_n = [n]_q^{\beta}$ for $q \in (0,1]$, $0 < \beta \le \frac{2}{3}$ and $n \in \mathbb{N}$. The operators R_n are lineer and

positive.

2. Main Results

The concept of the statistical convergence was introduced by Fast . We recall some definitions about the statistical convergence. The density of a set $K \subseteq \mathbb{N}$ is defined by

$$\delta\{k \le n : k \in K\}.$$

The natural density, δ , of a set $K \subseteq \mathbb{N}$ is defined by

$$\lim_{n \to \infty} \frac{1}{n} |\{k \le n : k \in K\}|,$$

provided the limits exist.

A sequence $x = x_k$ is called statistically convergent to a number *L* if

 $\delta\{k: |x - x_k| \ge \varepsilon\}$ for every $\varepsilon > 0$ and it is denoted as $st - limx_k = L$. The weighted space is defined by

 $B_{\rho_0}(\mathbb{R}_+) \coloneqq \{f: f \text{ is real valued on } \mathbb{R}_+ \text{ such that } |f(x)| \le M_f \rho_0(x) \text{ for all } x \in \mathbb{R}_+\},\$ where $\rho_0(x) = 1 + x^2$ is weight function and M_f is a constant depend on the function f. We also denote by

$$C_{\rho_0}(\mathbb{R}_+) \coloneqq \{ f \in B_{\rho_0}(\mathbb{R}_+) \colon f \text{ is continuous on } \mathbb{R}_+ \}$$

The weighted subspace of $B_{\rho_0}(\mathbb{R}_+)$. $B_{\rho_0}(\mathbb{R}_+)$ and $C_{\rho_0}(\mathbb{R}_+)$ are Banach spaces with the norm

$$||f||_{\rho_0} = \sup_{x \in \mathbb{R}_+} \frac{|f(x)|}{\rho_0(x)}.$$

Theorem1. Let $q = (q_n)$ be a sequence satisfying the following conditions

$$st - limq_n = 1$$
 and $st - lim(q_n)^n = c$, $(0 \le c < 1)$. 22

If f is nondecreasing function in $C_{\rho_0}(\mathbb{R}_+)$, then it holds

$$st - lim ||R_n(f; q, .) - f||_{\rho_0} = 0$$

The weighted modulus of smoothness for the functions $f \in B_{\rho_0}(\mathbb{R}_+)$ is defined as

$$\Omega_{\rho_0}(f;\delta) \coloneqq \sup_{x \in \mathbb{R}_+, 0 < h \le \delta} \frac{|f(x+h) - f(x)|}{1 + (x+h)^2}$$

for all $\delta > 0$. It is clear that

$$\lim_{\delta \to 0^+} \Omega_{\rho_0}(f;\delta) = 0.$$

Theorem2. Let $q = (q_n)$ be a sequence satisfying the following conditions

 $st - limq_n = 1$ and $st - lim(q_n)^n = c$, $(0 \le c < 1)$. For all nondecreasing functions f in $B_{\rho_0}(\mathbb{R}_+)$, we have

$$|R_n(f;q_n,x) - f(x)| \le 2\sqrt{R_n(\kappa_x^2(t);q_n,x)}\Omega_{\rho_0}(f;\mu_n(x)),$$

where $x \in \mathbb{R}_+, \delta > 0, n \in \mathbb{N}, \kappa_x(t) \coloneqq 1 + (x + |t - x|)^2$ for $t \in \mathbb{R}_+$ and $\mu_n(x) = (R_n((t - x))^2)$ $(x)^2; q_n, x)$ ^{1/2}.

Let $C_B(\mathbb{R}_+)$ be the space of all real valued continuous bounded functions define on \mathbb{R}_+ . The norm on the space $C_B(\mathbb{R}_+)$ is the supremum norm $||f|| = \sup\{|f(x)|: x \in \mathbb{R}_+\}$. The usual modulus of continuity is defined by

$$\omega(f; \delta^{1/2}) \coloneqq \sup\{|f(x+h) - f(x)| : x \in \mathbb{R}_+, 0 < |h| < \delta^{1/2}\}.$$

The second order modulus of continuity is defined by

$$\omega_2(f;\delta^{1/2}) \coloneqq \sup\{|f(x+2h) - 2f(x+h)f(x)| : x \in \mathbb{R}_+, 0 < |h| < \delta^{1/2}\}.$$

Peetre's K-funtional is defined by

$$K_2(f;\delta) \coloneqq \inf_{g \in W^2} \{ \|f - g\| + \delta \|g''\| \},\$$

where

 $W^{2} = \{ g \in C_{B}(\mathbb{R}_{+}) : g', g'' \in C_{B}(\mathbb{R}_{+}) \}.$ There exist a positive constant C > 0 such that

$$K_2(f;\delta) \leq C\omega(f;\delta^{1/2}), \delta > 0$$

Theorem3. Let $q = (q_n)$ be a sequence satisfying the following conditions st – li

$$mq_n = 1 \text{ and } st - lim(q_n)^n = c, (0 \le c < 1)$$

and the function f in $C_{\mathcal{B}}(\mathbb{R}_+)$. Then for all $n \in \mathbb{N}$, there exist a positive constant C > 0 such that

$$|R_n(f;q_n,x) - f(x)| \le C\omega_2 \left(f;\sqrt{\delta_n(x)}\right) \omega(f;\alpha_n(x)),$$

Where $\delta_n(x) = R_n((t-x)^2;q_n,x) + \left(R_n((t-x);q_n,x)\right)^2$ and $\alpha_n(x) = |R_n((t-x);q_n,x)|.$

Let E any subset of \mathbb{R}_+ and $\alpha \in (0, 1]$. The Lipschitz function class $\operatorname{Lip}_{M_f}(f; E, \alpha)$ denotes the space of functions f in $C_B(\mathbb{R}_+)$.satisfying the condition

$$|f(t) - f(x)| \le M_f |t - x|^{\alpha}, t \in \overline{E} \text{ and } x \in \mathbb{R}_+,$$
²³

where M_f is a constant depend on f and \overline{E} denotes the closure of E in \mathbb{R}_+ . **Theorem4.** Let $q = (q_n)$ be a sequence satisfying the following conditions

 $st - limq_n = 1 \text{ and } st - lim(q_n)^n = c, (0 \le c < 1)$

and the function f in $C_B(\mathbb{R}_+) \cap \operatorname{Lip}_{M_f}(f; E, \alpha)$ for $\alpha \in (0, 1]$ and E be any bounded subset of \mathbb{R}_+ . Then for each $x \in \mathbb{R}_+$, we have

$$|R_n(f;q_n,x) - f(x)| \le M_f \{ (\mu_n(x))^{\alpha} + 2(d(x,E))^{\alpha} \}$$

where $\mu_n(x) = (R_n((t-x)^2; q_n, x))^{1/2}$, M_f is a constant depend on f and d(x, E) is the distance between the point x and the set E, that is $d(x, E) = inf\{|t-x|: t \in E\}$. **3. An Illustrative Example**

In case of $\beta = 0.5$, for $f(x) = x \left(\frac{x+1}{2}\right) \left(\frac{x+1}{3}\right)$, the convergence of the operators $R_n(f; q, x)$ to f(x) can be illustrated for increasing values of q and n. It is clear that, for increasing values of q and n, the degree of approximation become better (see in maple program as graphical.).



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Complete Rewriting Systems of Some Types of Amalgamated Free Product of Groups

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In this work, by considering rewriting system procedure, we obtained complete rewriting systems of some types of amalgamated free product of groups. This complete rewriting system is important in respect to find normal forms of elements of given algebraic structure and thus have a solvable word problem. With the help of these complete rewriting systems, we attained normal form of elements of each group structures.

Keywords: Rewriting System, Normal Form, Amalgamated Free Product.

1. Introduction and Preliminaries

Presentations arise in various areas of mathematics such as knot theory, topology and geometry. Another motivation for studying presentations is the advent of softwares for symbolic computations like GAP. Providing algorithms to compute presentations of given monoids is a great help for the developers of these softwares. So, in this work, we consider monoid presentations of amalgamated free products W_3, W_4 and W_{n+1} and find complete rewriting systems for these monoid presentations. Thus, by these complete rewriting systems we characterize the structure of elements of these groups. Therefore, we obtain solvability of the word problem. To catch up our aim let us give briefly some preliminaries that will be needed in this paper.

Let X be a set and let X^* be the free monoid consisting of all words obtained by the elements of X. A string rewriting system, or simply a rewriting system, on X^* is a subset $R \subseteq X^* \times X^*$ and an element $(u,v) \in R$ also can be written as $u \to v$ is called a rule of R. In general, for a given rewriting system R we write $x \to y$ for $x, y \in X^*$ if $x = uv_1w, y = uv_2w$ and $(v_1, v_2) \in R$. Also we write $x \to^* y$ if x = y or $x \to x_1 \to x_2 \dots \to y$ for some finite chain of reductions and \to^* is the reflexive, symmetric, and transitive closure of \to . Furthermore an element $x \in X^*$ is called irreducible with respect to R if there is no possible rewriting (or reduction) $x \to y$ otherwise x is called reducible. The rewriting system R is called

- Confluent if whenever $x \to y_1$ and $x \to y_2$ there is a $z \in X^*$ such that $y_1 \to z$ and $y_2 \to z$.
- Complete if *R* is both Noetherian and confluent.

Critical pair of a rewriting system R is a pair of overlapping rules such that one of the forms

i)
$$(r_1r_2, s), (r_2r_3, t) \in R$$
 with $r_2 \neq 1$ or ii) $(r_1r_2r_3, s), (r_2, t) \in R$

is satisfied. Also a critical pair is resolved in *R* if there is a word *z* such that $sr_3 \rightarrow^* z$ and $r_1t \rightarrow^* z$ in the first case or $s \rightarrow^* z$ and $r_1tr_3 \rightarrow^* z$ in the second. A Noetherian rewriting system is complete if and only if every critical pair is resolved. We also note that if a rewriting system is complete then it has a solvable word problem (Adian and Durnev,2000). We finally note that the reader is referred to (Book and Otto,1993) and (Sims,1994) for a detailed survey on (complete) rewriting systems and (Çetinalp et al.,2019) and (Karpuz, 2010) for compute of complete rewriting system of some structures .

The free amalgamated products on which the aim of this paper will be presented is given as follows:

$$W_{3} = \langle w_{0}, w_{1}, w_{2}; w_{0}^{2} = w_{1}^{2} = w_{2}^{2} = (w_{0}w_{1})^{2} = (w_{0}w_{2})^{2} = (w_{1}w_{2})^{2} = 1 >$$

$$W_{4} = \langle w_{0}, w_{1}, w_{2}, w_{3}; w_{0}^{2} = w_{1}^{2} = w_{2}^{2} = w_{3}^{2} = (w_{0}w_{1})^{2} = (w_{0}w_{2})^{2} = (w_{0}w_{3})^{2}$$
(1)

$$w_{2}, w_{3}, w_{0} - w_{1} - w_{2} - w_{3} - (w_{0}w_{1}) - (w_{0}w_{2}) - (w_{0}w_{3})$$
$$= (w_{0}w_{1})^{2} = (w_{0}w_{2})^{2} = (w_{0}w_{3})^{2} = 1 >$$
(2)

$$(w_1 w_2)^2 = (w_1 w_3)^2 = (w_2 w_3)^2 = 1 >$$
(2)

$$W_{n+1} = \langle w_i (0 \le i \le n); w_i^2 = 1 (0 \le i \le n), (w_i w_j)^2 = 1 (0 \le i \ne j \le n) \rangle$$
(3)

2. Complete Rewriting System for Amalgamated Free Products W_3, W_4 and W_{n+1}

In this section, we obtain complete rewriting systems for monoid presentations of amalgamated free products W_3, W_4 and W_{n+1} given in (1), (2) and (3). Firstly, let us consider the monoid presentation of amalgamated free product W_3 given in (1) and use the ordering $w_0^{-1} > w_0 > w_1^{-1} > w_1 > w_2^{-1} > w_2$ among generators.

Theorem 2.1: A complete rewriting system for the monoid presentation of amalgamated free product W_3 consists of the following rules:

$$1) w_0^2 \to 1, \qquad 2) w_1^2 \to 1, \qquad 3) w_2^2 \to 1, \qquad 4) w_0 w_1 \to w_1 w_0, \qquad 5) w_0 w_2 \to w_2 w_0, \qquad 27$$

6) $w_1 w_2 \to w_2 w_1$, 7) $w_0^{-1} \to w_0$, 8) $w_1^{-1} \to w_1$, 9) $w_2^{-1} \to w_2$.

Proof: This rewriting system is Noetherian since there is no infinite chain of rewritings of overlapping words for the lexicographic order induced by the order on $w_0^{-1} > w_0 > w_1^{-1} > w_1 > w_2^{-1} > w_2$. It remains to show that the confluent property holds. To do that we have the following overlapping words and corresponding critical pairs, respectively.

$$1 \cap 1: w_0^3 (w_0, w_0), \qquad 1 \cap 4: w_0^2 w_1 (w_1, w_0 w_1 w_0), \qquad 1 \cap 5: w_0^2 w_2 (w_2, w_0 w_2 w_0), \\ 2 \cap 2: w_1^3 (w_1, w_1), \qquad 2 \cap 6: w_1^2 w_2 (w_2, w_1 w_2 w_1), \qquad 3 \cap 3: w_2^3 (w_2, w_2), \\ 4 \cap 2: w_0 w_1^2 (w_1 w_0 w_1, w_0), \qquad 4 \cap 6: w_0 w_1 w_2 (w_1 w_0 w_2, w_0 w_2 w_1), \qquad 5 \cap 3: w_0 w_2^2 (w_2 w_0 w_2, w_0), \\ 6 \cap 3: w_1 w_2^2 (w_2 w_1 w_2, w_1), \qquad 7 \cap 7: w_0^{-2} (w_0 w_0^{-1}, w_0^{-1} w_0), \qquad 8 \cap 8: w_1^{-2} (w_1 w_1^{-1}, w_1^{-1} w_1), \\ 9 \cap 9: w_2^{-2} (w_2 w_2^{-1}, w_2^{-1} w_2). \end{cases}$$

All these above critical pairs are resolved by reduction steps. We show one of them as

 $4 \cap 6: w_0 w_1 w_2 \ (w_1 w_0 w_2, w_0 w_2 w_1)$ follows: $w_0 w_1 w_2 \rightarrow \begin{cases} w_1 w_0 w_2 \rightarrow w_1 w_2 w_0 \rightarrow w_2 w_1 w_0 \\ w_0 w_2 w_1 \rightarrow w_2 w_0 w_1 \rightarrow w_2 w_1 w_0 \end{cases}$

After all these above processes, since rewriting system is Noetherian and confluent it is complete. Hence the result.

Now let us consider the monoid presentation of amalgamated free product W_4 given in (2) and use the ordering $w_0^{-1} > w_0 > w_1^{-1} > w_1 > w_2^{-1} > w_2 > w_3^{-1} > w_3$ among generators.

Theorem 2.2: A complete rewriting system for the monoid presentation of amalgamated free product W_4 consists of the following rules:

$$1) w_{0}^{2} \rightarrow 1, \qquad 2 \ \psi_{1}^{2} \rightarrow \qquad 3 \ \psi_{2}^{2} \rightarrow \qquad 4 \ \psi_{3}^{2} \rightarrow \qquad 5 \ \psi_{0} \ w_{1} \rightarrow w_{1}$$

$$6) w_{0} w_{2} \rightarrow w_{2} w_{0}, \ 7) w_{0} w_{3} \rightarrow w_{3} w_{0}, \qquad 8) w_{1} w_{2} \rightarrow w_{2} w_{1}, \qquad 9) w_{1} w_{3} \rightarrow w_{3} w_{1}, \qquad 10) w_{2} w_{3} \rightarrow w_{3} w_{2},$$

$$11) w_{0}^{-1} \rightarrow w_{0}, \qquad 12) w_{1}^{-1} \rightarrow w_{1}, \qquad 13) w_{2}^{-1} \rightarrow w_{2}, \qquad 14) w_{3}^{-1} \rightarrow w_{3}.$$

Proof: This rewriting system is Noetherian since there is no infinite chain of rewritings of overlapping words for the lexicographic order induced by the order on $w_0^{-1} > w_0 > w_1^{-1} > w_1 > w_2^{-1} > w_2 > w_3^{-1} > w_3$. It remains to show that the confluent property 28

holds. To do that we have the following overlapping words and corresponding critical pairs, respectively.

$$\begin{split} &1 \cap 1: \ w_0^3 \ (w_0, w_0), &1 \cap 5: \ w_0^2 \ w_1 \ (w_1, w_0 w_1 w_0), &1 \cap 6: \ w_0^2 \ w_2 \ (w_2, w_0 w_2 w_0), \\ &1 \cap 7: \ w_0^2 \ w_3 \ (w_3, w_0 w_3 w_0), &2 \cap 2: \ w_1^3 \ (w_1, w_1), &2 \cap 8: \ w_1^2 \ w_2 \ (w_2, w_1 w_2 w_1), \\ &2 \cap 9: \ w_1^2 \ w_3 \ (w_3, w_1 w_3 w_1), &3 \cap 3: \ w_2^3 \ (w_2, w_2), &3 \cap 10: \ w_2^2 \ w_3 \ (w_3, w_2 w_3 w_2), \\ &4 \cap 4: \ w_3^3 \ (w_3, w_3), &5 \cap 2: \ w_0 w_1^2 \ (w_1 w_0 w_1, w_0), &5 \cap 8: \ w_0 w_1 w_2 \ (w_1 w_0 w_2, w_0 w_2 w_1), \\ &5 \cap 9: \ w_0 w_1 w_3 \ (w_1 w_0 w_3, w_0 w_3 w_1), &6 \cap 3: \ w_0 w_2^2 \ (w_2 w_0 w_2, w_0), &7 \cap 4: \ w_0 w_3^2 \ (w_3 w_0 w_3, w_0), \\ &6 \cap 10: \ w_0 w_2 w_3 \ (w_2 w_0 w_3, w_0 w_3 w_2), &8 \cap 2: \ w_1 w_2^2 \ (w_2 w_1 w_2, w_1), &9 \cap 4 \ w_1 \ w_3^2 \ (w_3 \ w_1 \ w_3, w_3, w_3, w_3), \\ &10 \cap 4: \ w_2 w_3^2 \ (w_3 w_2 w_3, w_2), &11 \cap 11: \ w_0^{-2} \ (w_0 w_0^{-1}, w_0^{-1} w_0), &12 \cap 12: \ w_1^{-2} \ (w_1 w_1^{-1}, w_1^{-1} w_1), \\ &13 \cap 13: \ w_2^{-2} \ (w_2 w_2^{-1}, w_2^{-1} w_2), &14 \cap 14: \ w_3^{-2} \ (w_3 w_3^{-1}, w_3^{-1} w_3). \end{split}$$

All these above critical pairs are resolved by reduction steps. Hence rewriting system is

$$7 \cap 4: w_0 w_3^2 (w_3 w_0 w_3, w_0),$$

confluent. We show one of them as follows:
$$w_0 w_3^2 \rightarrow \begin{cases} w_3 w_0 w_3 \rightarrow w_3^2 w_0 \rightarrow w_0 \\ w_0 \end{cases}$$

Since rewriting system is Noetherian and confluent it is complete. Hence the result.

Finally, let us consider the monoid presentation of amalgamated free product W_{n+1} given in (3) and use the ordering $w_0^{-1} > w_0 > w_1^{-1} > ... > w_n^{-1} > w_n$ among generators.

Theorem 2.3: A complete rewriting system for the monoid presentation of amalgamated free products W_{n+1} consists of the following rules:

1)
$$w_i^2 \to 1 \ (0 \le i \le n), \quad 2) w_i w_j \to w_j w_i \ (i < j, \ 0 \le i, j \le n), \quad 3) w_i^{-1} \to w_i \ (0 \le i \le n),$$

Proof: This rewriting system is Noetherian since there is no infinite chain of rewritings of overlapping words for the lexicographic order induced by the order on $w_0^{-1} > w_0 > w_1^{-1} > ... > w_n^{-1} > w_n$. It remains to show that the confluent property holds. To do that we have the following overlapping words and corresponding critical pairs, respectively.

 $1 \cap 1: w_i^3(w_i, w_i) \quad (0 \le i \le n), \qquad 1 \cap 2: w_i^2 w_j(w_j, w_i w_j w_i) \quad (i < j, \ 0 \le i, j \le n), \\ 2 \cap 1: w_i w_j^2(w_j w_i w_j, w_i) \quad (i < j, \ 0 \le i, j \le n), \qquad 2 \cap 2: w_i w_j w_k \quad (w_j w_i w_k, w_i w_k w_j)_2 (i < j < k)$

 $3 \cap 3: w_i^{-2} (w_i w_i^{-1}, w_i^{-1} w_i) (0 \le i \le n).$

All these above critical pairs are resolved by reduction steps. We show one of them as

$$2 \cap 2: w_i w_j w_k \ (w_j w_i w_k, w_i w_k w_j) \ (i < j < k)$$

follows:
$$w_i w_j w_k \rightarrow \begin{cases} w_j w_i w_k \rightarrow w_j w_k w_i \rightarrow w_k w_j w_i \\ w_i w_k w_j \rightarrow w_k w_i w_j \rightarrow w_k w_j w_i \end{cases}$$

Thus rewriting system is confluent. Since rewriting system is Noetherian and confluent it is complete. Hence the result.

By considering Theorems 2.1, 2.2 and 2.3, we can give following result.

Corollary 2.4: Let C(u), C(u') and C(u'') be normal forms of words $u \in W_3, u' \in W_4$

and $u^{"} \in W_{n+1}$, respectively. Then, $C(u) = w_2^{j_2} w_1^{j_1} w_0^{j_0}$, $C(u') = w_3^{j_3} w_2^{j_2} w_1^{j_1} w_0^{j_0}$ and

$$C(u^{"}) = w_n^{i_n} w_{n-1}^{i_{n-1}} \dots w_2^{i_2} w_1^{i_1} w_0^{i_0}, \text{ where } i_p \in \{0,1\}, 0 \le p \le n.$$

By considering Corollary 2.4, we can give another result.

Theorem 2.5: The word problem for amalgamated free products W_3, W_4 and W_{n+1} is solvable.

We note that if we apply some operations on relators given the presentations in (1), (2) and (3), we the obtain some known important group types, namely elliptic Weyl groups of types $A_1^{(1,1)*}, A_1^{(1,1)*}$ and n-extended affine Weyl group of type A_1 , respectively.

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Existence theorems and weighted pseudo almost periodic solutions of a

Generalized Volterra equation

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Abstract

In this paper various types of fixed point theorems such as Banach, Krasnoselskii, Leray-Schauder and Krasnoselskii-Schaefer will be used in order to study the existence of the weighted pseudo almost periodic solutions of a class of Volterra equation

$$x(t) = f\left(t, x(t), x\left(t - \tau(t)\right)\right) + \int_{t}^{\infty} k\left(t, s\right) g\left(s, x(s)\right) ds$$

Keywords: Weighted pseudo almost periodic, fixed point theorems, Volterra- equation.

1. Introduction

The existence of periodic solutions is one of the most interesting and important topics in the qualitative theory of differential equations. Many authors have made important contributions to this theory. In [5], the authors studied the existence of periodic solution to the following equation

$$x(t) = f(t, x(t), x(t-h)) + \int_t^\infty c(t, s)g(s, x(s), x(s-h))ds$$

Where $f, g: R \times R^2 \to R$ and $c: R \times R \to R$ are continuous functions, and the delay h is a positive constant.

As we all know, if we consider the effects of the environmental factors, almost periodicity is sometimes more realistic and more general than periodicity. The almost periodic functions are closely connected with differential equations, dynamical systems etc. They are the natural generalization of continuous periodic functions. Besides, several newconcepts were introduced as generalizations of almost periodicity, such as pseudo almost periodicity (see

[4]), weighted pseudo almost periodicity (see [2, 3]). In particular, the properties of weighted pseudo almost periodic functions are more complicated and changeable than the almost periodic functions and the pseudo almost periodic functions because of the influence of the weight. Many problems in applied science are treated using differential and integral equations. Motivated by the discussion above, in this paper, we shall study the existence, uniqueness of the weighted pseudo almost periodic solutions to the following integral equation

$$x(t) \hspace{0.1in} = \hspace{0.1in} f(t,x(t),x(t-\tau(t))) + \int_t^\infty K(t,s)g(s,x(s))ds$$

Where $f: R_+ \times R^2 \to R$, $g: R_+ \times R \to R$ are weighted pseudo almost periodic functions and $\tau: R_+ \to R_+$ is weighted pseudo almost periodic function. More precisely by Banach, Krasnoselskii, Krasnoselskii-Schaefer and Leray-Schauder we study the existence of weighted pseudo almost periodic solutions to the above integral equation. It should be mentioned that the main results of this paper include Theorems 2,4,7.

2.Preliminaries

Throughout this paper, we make the following assumptions

(H1) There exist positive constant $0 < L_f^1, L_f^2, L_f^3, L_g^1, L_g^2 < 1$ such that for all $x_1, x_2, y_1, y_2 \in R$ and $t, s \in R$,

$$|g(t, x_1) - g(t, y_1)| < L_g^1 |t - s| + L_g^2 |x_1 - y_1|,$$

$$|f(t, x_1, x_2) - f(t, y_1, y_2)| < L_f^1 |t - s| + L_f^2 |x_1 - y_1| + |x_2 - y_2|.$$

(H2) There exist $\alpha > 1$ such that $K(t,s) \le e^{\alpha(t-s)}$, and for every $\varepsilon > 0$ very small and $\gamma \in R$

$$\int_t^\infty |K(t+\gamma,s+\gamma)-K(t,s)|ds<\varepsilon.$$

 $(H3)L_f^2 + L_f^3 < 1 - \frac{L_g^2}{\alpha}.$

(H4) $\tau_{ij}(.)$ is nonnegative continuously differentiable functions, such that $inf(l - \tau'_{ij}(t)) > 0, \tau^- = sup(\tau(t)) \in BC(R, R_+)$ and $\rho: R \to (0, \infty)$, $\rho \in U_\infty$ is continuous and assume $sup(\frac{\rho(s+\delta)}{\rho(s)}) < \infty$, $sup(\frac{\mu(T+\delta,\rho)}{\mu(T,\rho)}) < \infty$ for each $\delta \in \mathbb{R}$ and $sup(\rho(t)) < \infty$.

4. Existence of solutions

Let $S = \{x \in PAP(R, R, \rho), \|x\|_{\infty} \le M\}.$

4.1 .Banach'sFixed point

Theorem.4.1.Suppose that assumptions (H1)-(H5) hold. Then the equation

(4.1)
$$x(t) = f(t, x(t), x(t - \tau(t))) + \int_{t}^{\infty} K(t, s)g(s, x(s))ds$$

has a unique weighted pseudo almost periodic solution $x^*(.)$ in S.

4.2. Krasnoselskii'sFixed point theorem

Theorem.4.2. LetD be a closed convex nonempty subset of a Banach space (S,||. ||). Suppose that A and B map into S such that

- $Ax + By \in D(\forall x, y \in D).$
- Ais compact and continuous.
- *B* is a contraction mapping.

Then there exists $y \in D$ such that Ay + By = y.

Theorem.4.3. Assume that (H1)-(H4) holds. Then there exist a weighted pseudo almost periodic solution of equation (4.1) in S.

4.3.Leray-Schauder Alternative Theorem

Theorem.4.4.Let D be a closed convex of a Banach space X such that $0 \in D$. Let $\mathbf{F}: D \to D$ be a completely continuous map. Then the set $\{x \in D: x = \alpha F(x), 0 < \alpha < 1\}$ is unbounded or the map F has a fixed point in D.

Theorem.4.5. Suppose that (H1)-(H4) holds and if the operator $T: PAP(R, R, \varrho) \rightarrow$

 $PAP(R, R, \varrho)$ is completely continuous. Then the equation (4.1) has a fixed weighted pseudo almost periodic solution.

4.4.Krasnoselskii-Schaefer Fixed point Theorem

Theorem.4.6. Let $(S, \|.\|)$ be a Banach space. Suppose $B: S \to S$ is a contraction map, and A $: S \to S$ is continuous and maps bounded sets into compact sets. Then either

- $x = \delta B\left(\frac{x}{\delta}\right) + \delta Ax$ has a solution in S for $\delta = 1$, or
- The set of all such solution $0 < \delta < 1$, is unbounded.

Theorem.4.7.Assume that (H1)-(H4) holds. Then (4.1) has a weighted pseudo almost periodic solution.

6. Application

Let

$$x'(t) = -ax(t) - q(t, x(t)) + ax'(t) + r(t).$$

Then

$$x(t) = \alpha x(t) + p(t) + \int_t^\infty e^{-a(t-s)} [q(s, x(s)) - a\alpha x(s)] ds$$

Here $f(t,x(t),x(t-\tau(t))) = \alpha x(t) + p(t)$, $g(t,x(t))=q(s,x(s))-a\alpha x(s)$. The function q and p are weighted pseudo almost periodic and satisfy the following conditions: there exist $0 < L, L^1, L^2 < 1$ such that $|p(t) - p(s)| \le L|t - s|$, $|q(s,x(s))-q(t,x(t))| \le L^1|t-s|+L^2|x(s)-x(t)|$ and

a>1, in addition $0 < \alpha < 1$, such that $L^2 + \alpha \alpha < 1$ and $\frac{\alpha \alpha + L^2}{\alpha} < 1$.

Clearly all the conditions of Theorem (4.1) are satisfied, then the equation (5.1) has a unique weighted pseudo almost periodic solution.

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Compact Operators on the Sets of Fractional Difference Sequences Faruk Özger¹

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Abstract

The sets of fractional difference sequences have been studied in the literature recently. In this work, some identities or estimates for the operator norms and the measures of noncompactness of some operators on difference sets of sequences of fractional orders are established. Some classes of compact operators on those spaces are characterized. This study gives general and comprehensive results.

Keywords: measure of noncompactness, fractional difference sequence spaces, compact operators

1. Introduction

Fractional difference sets of sequences have been shown up in literature like fractional derivatives and fractional integrals. The gamma function which can be written by the improper integral is used to construct the fractional difference operators. One of the main goal of this study is to consider fractional operators and fractional sets of sequences $c_0(\Delta^{(\tilde{\alpha})})$, $c(\Delta^{(\tilde{\alpha})})$ and $\ell_{\infty}(\Delta^{(\tilde{\alpha})})$ in addition to determine the operator norms, find the β duals and characterize corresponding matrix transformations.

The gamma function of a real number x (except zero and the negative integers) is defined by an improper integral:

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt.$$

It is known that for any natural number n, $\Gamma(n + 1) = n!$ and $\Gamma(n + 1) = n\Gamma(n)$ holds for any real number $n \notin \{0, -1, -2, ...\}$. The fractional difference operator for a fraction $\tilde{\alpha}$ have been defined in [23] as

$$\Delta^{(\widetilde{\alpha})}(\rho_k) = \sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(\widetilde{\alpha}+1)}{i!\Gamma(\widetilde{\alpha}-i+1)} \rho_{k-i}.$$
(1.1)

It is assumed that the series defined in (1.1) is convergent for $\rho \in \omega$.

The inverse of fractional difference matrix

$$\Delta_{nk}^{(\widetilde{\alpha})} = \begin{cases} (-1)^{n-k} \frac{\Gamma(\widetilde{\alpha}+1)}{(n-k)!\Gamma(\widetilde{\alpha}-n+k+1)} & (0 \le k \le n) \\ 0 & (k > n) \end{cases}$$

is given by

$$\Delta_{nk}^{(-\tilde{\alpha})} = \begin{cases} (-1)^{n-k} \frac{\Gamma(-\tilde{\alpha}+1)}{(n-k)!\Gamma(-\tilde{\alpha}-n+k+1)} & (0 \le k \le n) \\ 0 & (k > n). \end{cases}$$

For some values of $\tilde{\alpha}$, we have

$$\Delta^{1/2}\rho_{k} = \rho_{k} - \frac{1}{2}\rho_{k-1} - \frac{1}{8}\rho_{k-2} - \frac{1}{16}\rho_{k-3} - \frac{5}{128}\rho_{k-4} - \dots$$

$$\Delta^{-1/2}\rho_{k} = \rho_{k} + \frac{1}{2}\rho_{k-1} + \frac{3}{8}\rho_{k-2} + \frac{5}{16}\rho_{k-3} + \frac{35}{128}\rho_{k-4} + \dots$$

$$\Delta^{2/3}\rho_{k} = \rho_{k} - \frac{2}{3}\rho_{k-1} - \frac{1}{9}\rho_{k-2} - \frac{4}{81}\rho_{k-3} - \frac{7}{243}\rho_{k-4} - \dots$$

Remark 1.1 The following results hold:

a) $\Delta^{(\widetilde{\alpha})} \Delta^{(\widetilde{\beta})} = \Delta^{(\widetilde{\alpha} + \widetilde{\beta})}.$ b) $\Delta^{(\widetilde{\alpha})} (\Delta^{(-\widetilde{\alpha})} \rho_k) = \rho_k.$

Proof. Since the proofs of Part (a) can similarly be obtained, we only prove Part (b).

$$\begin{split} \Delta^{(\widetilde{\alpha})}(\Delta^{(-\widetilde{\alpha})}\rho_{k}) &= \Delta^{(\widetilde{\alpha})}\left\{\rho_{k} + \rho_{k-1}\alpha + \rho_{k-2}\frac{\alpha(\alpha+1)}{2!} + \rho_{k-3}\frac{\alpha(\alpha+1)(\alpha+2)}{3!} + \rho_{k-4}\frac{\alpha(\alpha+1)(\alpha+2)(\alpha+3)}{4!} + \cdots\right\} \\ &= \rho_{k} + \rho_{k-1}\{-\alpha + \alpha\} + \rho_{k-2}\left\{\frac{\alpha(\alpha-1)}{2!} + \alpha^{2} + \frac{\alpha(\alpha+1)}{2!}\right\} + \\ &\qquad \rho_{k-3}\left\{-\frac{\alpha(\alpha-1)(\alpha-2)}{3!} + \frac{\alpha^{2}(\alpha-1)}{2!} - \frac{\alpha^{2}(\alpha+1)}{2!} + \frac{\alpha(\alpha+1)(\alpha+2)}{3!}\right\} + \\ &\qquad \rho_{k-4}\left\{\frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{4!} - \frac{\alpha^{2}(\alpha-1)(\alpha-2)}{3!} + \frac{\alpha^{2}(\alpha+1)(\alpha-1)}{2!2!} - \frac{\alpha^{2}(\alpha+1)(\alpha+2)}{3!}\right\} + \cdots \\ &= \rho_{k}. \end{split}$$

We refer to [23] for more properties of the fractional difference operators.

2. Sets of Sequences

Consider now the fractional difference sequence spaces $c_0(\Delta^{(\widetilde{\alpha})}) = \{\rho \in \omega : \lim_k \Delta^{(\widetilde{\alpha})}(\rho_k) = 0\},\ c(\Delta^{(\widetilde{\alpha})}) = \{\rho \in \omega : \lim_k \Delta^{(\widetilde{\alpha})}(\rho_k) | \text{exists}\} \text{ and } \ell_{\infty}(\Delta^{(\widetilde{\alpha})}) = \{\rho \in \omega : \sup_k |\Delta^{(\widetilde{\alpha})}(\rho_k)| < \infty\}.$

Those spaces can be considered as the matrix domains of the triangle $\Delta^{(\tilde{\alpha})}$ in the classical sequence spaces c_0, c, ℓ_{∞} . The set λ_T is a *BK* space with $\|.\|_T = \|T(.)\|$ whenever $(\lambda, \|.\|)$ is a *BK* space. By this fact, defined fractional difference sequence spaces are complete, linear *BK*-spaces with the norm $\|\rho\| = \sup_n \left|\sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(\tilde{\alpha}+1)}{i!\Gamma(\tilde{\alpha}-i+1)} \rho_{n-i}\right|$.

3. Operator Norms

Let us now establish identities and inequalities of operator norms for fractional sequence spaces. We refer to [1-7, 14-20] for the needed notion, notations and definitions for this study. **Theorem 3.1** Let $\lambda = c_0(\Delta^{(\widetilde{\alpha})})$ or $\lambda = \ell_{\infty}(\Delta^{(\widetilde{\alpha})})$.

• Let $\mu = c_0, c, \ell_{\infty}$. If $A \in (\lambda_T, \mu)$ then, putting

$$\|A\|_{(\lambda_{T},\infty)} = \sup_{n} \|\hat{A}_{n}\|_{1} = \sup_{n} \sum_{k} \left| \sum_{j=k}^{\infty} (-1)^{j-k} \frac{\Gamma(-\tilde{\alpha}+1)}{(j-k)!\Gamma(-\tilde{\alpha}-j+k+1)} a_{nj} \right|,$$

we have $||L_A|| = ||A||_{(\lambda_T,\infty)}$.

• If $A \in (\lambda_T, \ell_1)$. Then we have

$$\|A\|_{(\lambda_T,1)} = \sup_{\substack{N \subset \mathbb{N} \\ N \text{finite}}} \left(\sum_k \left| \sum_{n \in \mathbb{N}} \sum_{j=k}^{\infty} (-1)^{j-k} \frac{\Gamma(-\widetilde{\alpha}+1)}{(j-k)!\Gamma(-\widetilde{\alpha}-j+k+1)} a_{nj} \right| \right) \le \|L_A\| \le 4 \|A\|_{(\lambda_T,1)}.$$

Theorem 3.2 The operator norm of the set $c(\Delta^{(\tilde{\alpha})})$ is given.

• Let $A \in (c(\Delta^{(\widetilde{\alpha})}), \mu)$, where μ is any of the spaces c_0, c or ℓ_{∞} . Then we have

$$\|L_A\| = \|A\|_{\left(c(\Delta^{(\widetilde{\alpha})}),\infty\right)} = \sup_{n} \left(\sum_{k} \left|\sum_{j=k}^{\infty} (-1)^{j-k} \frac{\Gamma(-\widetilde{\alpha}+1)}{(j-k)!\Gamma(-\widetilde{\alpha}-j+k+1)} a_{nj}\right| + |\gamma_n|\right),$$

where $\gamma_n = \lim_m \sum_{k=0}^m w_{mk}^{(A_n)}$ for n = 0, 1, ...

• Let $A \in (c(\Delta^{(\widetilde{\alpha})}), \ell_1)$, then, putting,

$$\|A\|_{\left(c(\Delta^{(\widetilde{\alpha})}),1\right)} = \sup_{\substack{N \subset \mathbb{N} \\ N \text{ finite}}} \left(\sum_{k} \left| \sum_{n \in N} \sum_{j=k}^{\infty} (-1)^{j-k} \frac{\Gamma(-\widetilde{\alpha}+1)}{(j-k)!\Gamma(-\widetilde{\alpha}-j+k+1)} a_{nj} \right| + \left| \sum_{n \in N} \gamma_n \right| \right)$$

we have $\|A\|_{(c(\Delta^{(\widetilde{\alpha})}),1)} \le \|L_A\| \le 4 \|A\|_{(c(\Delta^{(\widetilde{\alpha})}),1)}$.

4. Conclusions

We establish necessary and sufficient conditions for a matrix operator to be a compact operator from fractional difference sequence spaces into μ , where $\mu \in \{c_0, c, \ell_{\infty}, \ell_1\}$. This is achieved applying the results about Hausdorff measure of noncompactness.

Theorem 4.1 The identities or estimates for L_A when $A \in (\lambda(\Delta^{(\tilde{\alpha})}, \mu), where \mu \in \{\ell_{\infty}, c_0, c, \ell_1\}$ and $\lambda \in \{\ell_{\infty}, c_0, c\}$ can be read from the following table:

The following Theorem gives necessary and sufficient conditions for an operator from our fractional sets to classical sets of sequences to be compact.

Theorem 4.2 The identities or estimates for L_A when $A \in (\lambda(\Delta^{(\tilde{\alpha})}, \mu), where \mu \in \{\ell_{\infty}, c_0, c, \ell_1\}$ and $\lambda \in \{\ell_{\infty}, c_0, c\}$ can be read from the following table: 38

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(**p**, **q**) **bivariate Cholodowsky variant of Bernstein Schurer operators** Faruk Özger¹ and Khursheed J. Ansari

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Abstract

In this study, we introduce a bivariate Cholodowsky variant of Bernstein Schurer operators based on (p,q) —integers. We examine certain approximation properties of defined bivariate operator. We give some of our results without proofs.

Keywords: (p, q) -integers, Voronovskaja type theorem, rate of convergence

1. Introduction

Approximation theory is fast becoming a key instrument not only in classical approximation theory but also in other fields of mathematics such as differential equations, orthogonal polynomials and geometric design. Since Korovkin's famous theorem was first published in 1950, the issue of approximation by linear positive operators has become increasingly important area as part of approximation theory.

Let us recall some definitions and notations regarding the concept of (p,q) –calculus. The (p,q) –integer of the number n is defined by

$$[n]_{p,q} := \frac{p^n - q^n}{p - q}, \ n = 1, 2, 3 ..., \ 0 < q < p < 1.$$

Further, the (p,q) –binomial expansions are given as

$$(ax + by)_{p,q}^{n} = \sum_{k=0}^{n} p^{\binom{n-k}{2}} q^{\binom{k}{2}} a^{n-k} b^{k} x^{n-k} y^{k}.$$

and

$$(x - y)_{p,q}^{n} = (x - y)(px - qy)(p^{2}x - q^{2}y) \cdots (p^{n-1}x - q^{n-1}y).$$

Further information related to (p, q) calculus can be found in [12, 13].

2. Construction of the Operators

Recently, Ansari and Karaisa [4] have defined and studied (p,q) analogue of Chlodowsky operators

$$C_{n,p,q}(f;x) = \frac{1}{p^{n(n-1)/2}} \sum_{k=0}^{n} {n \brack k}_{p,q} p^{k(k-1)/2} \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)_{p,q}^{n-k-1} f\left(\frac{[k]_{p,q}}{[n]_{p,q}p^{k-n}}b_n\right),$$
(2.1)

where $\left(1 - \frac{x}{b_n}\right)_{p,q}^{n-k-1} = \prod_{s=0}^{n-k-1} \left(p^s - q^s \frac{x}{b_n}\right)$.

Cholodowsky variant of Bernstein Schurer operators based on (p,q) -integers defined as

$$\overline{C}_{n,s}(f;p,q;x) = \sum_{k=0}^{n+s} {n+s \brack k} p_{p,q} p^{\frac{k(k-1)}{2} - \frac{(n+s-1)(n+s)}{2}} \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)_{p,q}^{n+s-k-1} f\left(\frac{p^{n-k}[k]_{p,q}}{[n]_{p,q}}b_n\right).$$
(2.2)

The bivariate form of Bernstein Schurer operators and modified Bernstein Schurer operators studied by Muraru and Acu, and Cai. Let us assume that $n + d_1 = \hat{n}$, $m + d_2 = \overline{m}$ for $d_1, d_2 \in \mathbb{N}$ and $0 < q_1, q_2 < p_1, p_2 1$. We now define two dimensional Cholodowsky variant of Bernstein Schurer operators based on (p, q) —integers as follows:

$$\tilde{C}_{n,m}^{(p_{1,2};q_{1,2})}(f;x,y) = \sum_{k_{2}=0}^{\hat{m}} \Phi_{k_{1},n}^{(p_{1},q_{1})}(x) \Phi_{k_{2},n}^{(p_{2},q_{2})}(y) f\left(\frac{[k_{1}]_{p_{1},q_{1}}}{[n]_{p_{1},q_{1}}p_{1}^{k_{1}-\hat{n}}}b_{n},\frac{[k_{2}]_{p_{2},q_{2}}}{[m]_{p_{2},q_{2}}p_{2}^{k_{2}-\overline{m}}}b_{m}\right)$$

$$(2.3)$$

 $\begin{array}{ll} \text{for all } n,m\in\mathbb{N} \ , \ f\in C(I_{b_nb_m}) \ \text{ with } \ I_{b_nb_m}=\{(x,y)\colon 0xb_n \ , 0yb_m\} \ \text{ and } \ C(I_{b_nb_m})=\{f\colon I_{b_nb_m}\to R \ \text{is continuous}\}. \end{array}$

Here (b_n) and (b_m) are increasing unbounded sequences of positive real numbers such that

$$\lim_{n \to \infty} \frac{b_n}{[n]_{p_1, q_1}} = 0, \tag{2.4}$$

$$\lim_{m \to \infty} \frac{b_m}{[m]_{p_2, q_2}} = 0$$
(2.5)

and $\Phi_{k_1,n}^{(p_1,q_1)}(x)$ and $\Phi_{k_2,m}^{(p_2,q_2)}(y)$ are

$$\Phi_{k_{1},n}^{(p_{1},q_{1})}(x) = p_{1}^{\frac{k_{1}(k_{1}-1)-\tilde{n}(\tilde{n}-1)}{2}} \begin{bmatrix} \hat{n} \\ k_{1} \end{bmatrix}_{p_{1},q_{1}} \left(\frac{x}{b_{n}}\right)^{k_{1}} \prod_{s_{1}=0}^{\hat{n}-k_{1}-1} \left(p_{1}^{s_{1}}-q_{1}^{s_{1}}\frac{x}{b_{n}}\right),$$

$$\Phi_{k_{2},m}^{(p_{2},q_{2})}(y) = p_{2}^{\frac{k_{2}(k_{2}-1)-\tilde{m}(\tilde{m}-1)}{2}} \begin{bmatrix} \tilde{m} \\ k_{2} \end{bmatrix}_{p_{2},q_{2}} \left(\frac{y}{b_{m}}\right)^{k_{2}} \prod_{s_{2}=0}^{\tilde{m}-k_{2}-1} \left(p_{2}^{s_{2}}-q_{2}^{s_{2}}\frac{y}{b_{m}}\right).$$

3. Main Results

Theorem 3.1 Let $0 < q_n < p_n 1$ and $0 < q_m < p_m 1$ be sequences, then the sequence $\tilde{C}_{n,m}^{(p_{1,2};q_{1,2})}(f;x,y)$ convergences uniformly to f(x,y) on $[0,a] \times [0,b] = I_{ab}$ for each $f \in \frac{43}{43}$

 $C(I_{ab})$, where a, b be reel numbers such that ab_n , bb_m and $C(I_{ab})$ be the space of all real valued continuous function on I_{ab} with the norm $|| f ||_{C(I_{ab})} = \sup_{(x,y) \in I_{ab}} |f(x,y)|$.

Theorem 3.2 Let $f \in C(I_{ab})$. For all $x \in I_{ab}$, we have

$$\left|\tilde{C}_{n,m}^{(p_{1,2};q_{1,2})}(f;x,y) - f(x,y)\right| 4\omega(f;\delta_{\widehat{n}}(x),\delta_{\overline{m}}(y)),$$

where

$$\delta_{\hat{n}}^{2}(\mathbf{x}) := \frac{x p_{1}^{\hat{n}+3} b_{n}[\hat{n}]_{p_{1},q_{1}}}{[n]_{p_{1},q_{1}}^{2}} + \frac{x^{2} \{q_{1}[\hat{n}]_{p_{1},q_{1}}[\hat{n}-1]_{p_{1},q_{1}}-2[\hat{n}]_{p_{1},q_{1}}[n]_{p_{1},q_{1}}+[n]_{p_{1},q_{1}}^{2}\}}{[n]_{p_{1},q_{1}}^{2}};$$
(3.1)

$$\delta_{\overline{m}}^{2}(y) := \frac{y p_{2}^{\overline{m}+3} b_{m}[\overline{m}]_{p_{2},q_{2}}}{[m]_{p_{2},q_{2}}^{2}} + \frac{y^{2} \{q_{2}[\overline{m}]_{p_{2},q_{2}}[\overline{m}-1]_{p_{2},q_{2}}-2[\overline{m}]_{p_{2},q_{2}}[m]_{p_{2},q_{2}}+[m]_{p_{2},q_{2}}^{2}\}}{[m]_{p_{2},q_{2}}^{2}}.$$
(3.2)

Theorem 3.3 Let $f \in C(I_{ab})$, then the following inequalities satisfy

$$\left|\tilde{C}_{n,m}^{(p_{1,2};q_{1,2})} - f(x,y)\right| 2\left[\omega_1(f;\delta_{\widehat{n}}^{1/2}(x)) + \omega_2(f;\delta_{\overline{m}}^{1/2}(y))\right],$$

where $\delta_{\hat{n}}(x)$ and $\delta_{\bar{m}}(y)$ are defined in 3.1 and 3.2.

Theorem 3.4 Let $f \in \text{Lip}_{M}(\hat{\beta}_{1}, \hat{\beta}_{2})$. Then, for all $(x, y) \in I_{ab}$, we have

$$|\tilde{C}_{n,m}^{(p_{1,2};q_{1,2})}(f;x,y) - f(x,y)|M\delta_{\widehat{n}}^{\widehat{\beta}_{1}/2}(x)\delta_{\overline{m}}^{\widehat{\beta}_{2}/2}(y),$$

where $\delta_{\hat{n}}(x)$ and $\delta_{\overline{m}}(y)$ are defined in 3.1 and 3.2, respectively.

Theorem 3.5 Let $f \in C^1(I_{ab})$ and $0 < q_n, q_m < p_n, p_m 1$. Then, we have

$$|\tilde{C}_{n,m}^{(p_{1,2};q_{1,2})}(f;x,y) - f(x,y)| \parallel f_x \parallel_{C(I_{ab})} \delta_{\widehat{n}} + \parallel f_y \parallel_{C(I_{ab})} \delta_{\overline{m}}$$

We now study the convergence of the sequence of defined linear positive operator to a functions of two variables which defined on weighted space and compute rate of convergence via weighted modulus continuity.

Theorem 3.6 Let $\tilde{C}_{n,m}^{(p_{1,2};q_{1,2})}$ be sequence of linear positive operators defined (2.3), then for each $f \in C_{\rho}^{0}$ and for all $(x; y) \in I_{b_n b_m}$, we have

$$\lim_{n\to\infty} \| \widetilde{C}_{n,m}^{(p_n,q_n)}(f;x,y) - f(x,y) \|_{\rho} = 0.$$

Theorem 3.7 Let $f \in C_{\rho}^{0}$ and C_{2} is a constant independent of n, m, then we have

$$\sup_{(x,y)\in\mathbb{R}^2_+} \frac{\left|\frac{\tilde{C}_{n,m}^{(p_n,q_n)}(f;x,y)-f(x,y)}{\rho^3(x,y)}\right|}{\rho^3(x,y)} C_2\omega_{\rho}(f;\delta_{\widehat{n}},\delta_{\overline{m}}),$$

where $\delta_{\widehat{n}}^2 = O\left(\frac{p_n^{\widehat{n}+3}b_n[\widehat{n}]_{p_n,q_n}}{[n]_{p_n,q_n}^2}\right)$ and $\delta_{\overline{m}}^2 = O\left(\frac{p_m^{\overline{m}+3}b_m[\overline{m}]_{p_m,q_m}}{[m]_{p_m,q_m}^2}\right)$.

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Öğretmen Adaylarının Kavram Yanılgıları İle İlgili Farkındalıklarının Alan Bilgileri Bağlamında Değerlendirilmesi

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Özet

Bu çalışmanın amacı ilköğretim matematik öğretmeni adaylarının verilen bir problem karşısında öğrencilerde oluşabilecek kavram yanılgılarını tahmin edebilme ve etkin öğretim yöntemlerini kullanabilme bilgilerinin incelenmesidir. Araştırma verilerini, 44 matematik öğretmen adayının olası öğrenci hatalarını tahmin etme ve bu hatalara yönelik çözüm önerilerini ortaya koyma amacı güden kar-zarar konusuyla ilgili bir probleme verdikleri yanıtlar oluşturmaktadır. Verilerin betimsel analizi sonucunda öğretmen adaylarının çoğu verilen problem karşısında öğrencilerin düşebilecekleri kavram yanılgılarını tahmin edebilmesine karşın çok azı bu kavram yanılgılarını önlemeye yönelik etkin öğretim yöntemlerini kullanabilmiştir. Bu durum öğretmen adaylarının pedagojik alan bilgilerinin yeterince gelişmemesi ve onlardaki deneyim eksikliği ile ilişkilendirilebilir.

Keywords: Kavram yanılgısı, Pedagoji, Alan bilgisi, Öğretmen adayı

1. Giriş

Kavram yanılgısı yanlış bir kavramın doğru imiş gibi görülmesi olup öğrenciler bu yanılgılara sık sık düşebilmektedir. Öğrencinin kavram hatasına düşmesi öğretimde sık rastlanan bir durum olmakla birlikte öğrencinin hatasının farkına varmaması ve öğretmenin bunu fark etmemesi tercih edilmeyen bir durum olarak karşımıza çıkmaktadır. Öğretmenler öğrencilerin hataya düşebilecekleri durumları tahmin etme ve gerekli önlemleri alma ile matematik öğretiminde anlamlı öğrenmeyi gerçekleştirebilirler. Özellikle problem çözümlerinde öğrencilerin yanlış çözüm yapmalarına imkân vermeden problemi anlamlaştırma ve önemli noktaları vurgulama yoluna gidebilirler. Bu ise çoğu kez öğretmenin pedagojik alan bilgisi ile ilişkilendirilebilir.

Öğrenci hatalarının nereden kaynaklandığını anlamak ve nasıl önleneceğini planlamak, öğrencinin cevabını analiz etmek ve bu cevabın doğru olup olmadığına karar vermekle başlar (Boz, 2004). Çünkü öğrenciler çoğu kez yanlış bir kavramı doğru gibi zannedip yanılgıya

düşebilirler. Özellikle problem çözme esnasında bazı kavramları yanlış ilişkilendirerek kavramsal hatalar yapma yoluna gidebilirler. Bu yanlış ilişkilendirme çoğu kez öğrencinin problemi ezbere çözmesinden veya kavramsal anlamayı yeterince gerçekleştirememiş olmasından kaynaklanabilir (Yenilmez ve Yaşa, 2008). Kavramsal anlamanın daha iyi gerçekleştirilmesi akıl yürütme, eleştirel düşünme ve ilişkilendirme gibi zihinsel becerilerin öğrencilere kazandırılması ile yakından ilişkilidir (Karaağaç ve Köse, 2008). Dolayısı ile öğretmen öğrencilerin kavramsal hatalarını önlemek ve öğrencilerde daha anlamlı bir öğrenme sağlamak için bu hataların farkına varabilmeli ve etkin öğretim yöntemlerini kullanabilmelidir. Bu çalışmanın amacı ilköğretim matematik öğretmeni adaylarının verilen bir problem karşısında öğrencilerde oluşabilecek kavram yanılgılarını tahmin edebilme ve etkin öğretim metotlarını kullanabilme bilgilerinin incelenmesidir.

2. Yöntem

Bu çalışmada, öğretmen adaylarının öğrencilerde oluşabilecek kavram yanılgılarını tahmin edebilme ve etkin öğretim yöntemlerini kullanabilme bilgilerinin incelenmesi amacı ile nitel araştırma yöntemlerinden durum çalışması deseni kullanılmıştır. Araştırmanın çalışma grubunu 44 ilköğretim matematik öğretmeni adayı oluşturmaktadır. Verilerin toplanması, öğretmen adaylarının olası öğrenci hatalarını tahmin etme ve bu hatalara yönelik çözüm önerilerini ortaya koyma amacı güden kar-zarar konusuyla ilgili bir problemin öğretmen adaylarına yöneltilmesi ile gerçekleştirilmiştir. Öğretmen adaylarının bu probleme verdikleri cevaplar nitel analiz yöntemlerinden betimsel analiz tekniğiyle analiz edilmiştir. Önce araştırmacılar tarafından bireysel analizler yapılmış, sonra bir araya gelinerek yapılan analizler tartışılmış, farklılıklar üzerinde durulmuş ve uzlaşılarak analize son şekli verilmiştir.

3. Bulgular ve Yorum

Öğretmen adaylarının verilen problem karşısında öğrencilerin düşebilecekleri hatalara karşı çözüm önerilerine yer vermeden önce kaç tane öğretmenin problemi doğru anladığı ve doğru çözüm yolu ürettiği belirlenmiştir. Öğretmen adaylarının bir kısmı sorulan soruyu yanlış veya yetersiz cevaplandırıken, bir kısmı doğru cevaplandırmıştır. Cevaplara ilişkin istatistikler Tablo 1'deki gibidir:

Tablo 1. Ö	ğretmen	adaylarının	soruya	ilişkin	vermiş	oldukları
cevaplara y	önelik fre	ekanslar				

Öğretmen adaylarının cevapları	f
Yanlış çözüm	9
Yetersiz çözüm	13
Doğru çözüm	22

Tablo 1'de görüldüğü gibi 44 öğretmen adayının yarısı soruya doğru cevap verirken diğerleri yanlış ya da yetersiz çözümlerde bulunmuştur. Yetersiz çözümde bulunan öğretmen adayları genellikle doğru çözüm yolunu takip etmelerine rağmen işlem hatası yaptıkları için yanlış sonuca ulaşmışlardır. Yanlış çözüm yapanlar ise yüzdelik veya indirim hesaplarken çözüm yolunda problem yaşadıkları için doğru çözüme ulaşamamışlardır. Aşağıdaki tabloda öğretmen adaylarının olabilecek öğrenci hatalarına yönelik tahminlerine ve bunların giderilmesine yönelik yaptıkları çözüm önerilerinin sayısal istatistiklerine yer verilmiştir.

Tablo 2. Öğretmen adaylarının hatayı tahmin edebilme ve çözüm önerilerinde bulunabilmelerine yönelik frekanslar

Kodlar	f
Hatayı yanlış tespit etme ve yanlış çözüm önerisi getirme	7
Hatayı kısmen doğru tespit etme ve yetersiz çözüm önerisi getirme	2
Hatayı kısmen doğru tespit etme ve doğru çözüm önerisi getirme	1
Hatayı doğru tespit etme ve kısmen doğru çözüm önerisi getirme	21
Hatayı doğru tespit etme ve doğru çözüm önerisi getirme	13

Tablo 2'de tüm öğretmen adaylarının çözüm önerisinde bulundukları ancak bunlardan bazılarının önerilerinin yetersiz ya da yanlış olduğu görülmüştür. Öğretmen adaylarının çözümlerine ilişkin istatistikler incelendiğinde 7 öğretmen adayının hatayı yanlış tespit ettiği ve dolayısıyla yanlış çözüm önerisi getirdiği, 2 öğretmen adayının ise hatayı kısmen doğru tespit ettiği ancak yetersiz çözüm önerisi getirdiği görülmüştür. Bazı öğretmen adaylarının doğru olan çözümleri yanlış algılaması ve öğrencinin hata yaptığı algısına kapılması onların hatayı yanlış tespit etmelerinin bir nedeni olabilir. Hatayı kısmen doğru tespit edenlerin yetersiz çözüm önerisi getirdik bir durum olarak karşımıza çıkmaktadır. Adaylarının yetersiz ya da yanlış çözüm önerisi getirmelerinin sebebinin onların soruyu yeteri kadar incelememesi veya anlamlaştıramamasından kaynaklandığı düşünülmektedir.

Tablo incelendiğinde 21 adayın hatayı doğru tespit etmelerine rağmen kısmen doğru çözüm önerisi getirdikleri görülmektedir. Bu öğretmen adaylarının soruyu doğru çözmekerine rağmen

yetersiz öneri getirmeleri aslında onların net ifadeler kullanmamış ve yeterli örneklendirmeler yapmamış olmalarından kaynaklanmıştır. Bu durum öğretmenlerin düşüncelerini ifade ederken ayrıntıya girmek istemediklerinden kaynaklanabilir. Başka bir deyişle çoğu adayın bazı şeylerin bilincinde oldukları ancak düşüncelerini kısa ve kestirme yollardan ifade etmeyi tercih ettikleri söylenebilir.

Tabloya göre adayların 14'ünün doğru çözüm önerisi getirdikleri ancak bunların 1'inin hatayı kısmen doğru tespit ettiği 13'ünün ise hatayı doğru bir şekilde tespit ettiği görülmüştür. Yeterli örneklendirme yapmadan bir açıklamada bulunulması hatanın kısmen doğru tespit edildiğinin bir göstergesidir. Hatayı doğru bir şekilde tahmin edip yeterli bir şekilde çözüm önerisi getirenler ise genellikle öğrencilerin hatasını fark ettirecek şekilde yeterli örneklendirmelere ve açıklamalara başvurmuşlardır. Bu durum öğretmen adaylarının pedagojik bilgilerinin yeterli düzeyde olmasının onların ifade güçlerini iyi temsil etmesi ile ilişkilendirilebilir.

4. Sonuç ve Öneriler

Araştırma sonuçlarına göre hatayı doğru belirleyen ve yeterli çözüm önerisinde bulunan adayların sayısı tüm öğretmen adaylarına nazaran yeterli düzeye ulaşmamıştır. Öğretmen adaylarının büyük çoğunluğu hatayı doğru tespit edebilmelerine rağmen kısmen doğru çözüm önerisi getirmişlerdir. Dolayısıyla öğretmen adaylarının çoğu verilen problem karşısında öğrencilerin düşebilecekleri kavram yanılgılarını tahmin edebilmesine karşın çok azı bu kavram yanılgılarını önlemeye yönelik etkin öğretim yöntemlerini kullanabilmişlerdir.

Adayların yetersiz çözüm önerilerinde bulunmaları ve yeterli örneklendirme yapamamaları, onların soruyu anlamalarına rağmen üstünkörü çözümler ürettiklerini göstermektedir. Bu durum öğretmen adaylarının pedagojik alan bilgilerinin yeterince gelişmemesi ve onlardaki deneyim eksikliği ile ilişkilendirilebilir. Dahası bu öğretmen adaylarının öğrenciye hatasını buldurmak yerine daha çok anlatma yolunu tercih ettikleri düşünülmektedir. Bu ise öğretimde çok fazla tercih edilmemektedir. Öğretmen adaylarına öğrencilerde oluşabilecek kavram yanılgılarını araştırma ve tahmin etmenin yanı sıra, bu hataları giderebilecekleri öğrenme ortamlarının sağlanması ile bu eksikliğin giderilebileceği düşünülmektedir.

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Compact Soft Elements and Algebraic Soft Domains

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Abstract

Soft Set Theory was introduced by Molodtsov to deal with uncertainties. Also, there are increasingly many studies about the soft set theory. Scott topology is well known in theoretical computer science and topological lattice theory. Soft Scott topology was introduced by using soft set relation. To define soft Scott topology, directed and directed complete soft sets were introduced by Tanay and Yaylali. We know that, way-below soft set relation has a very important role in the soft Scott topology. Since Auxiliary soft set relation is a general form for the way-below soft set relation, we study relation between auxiliary soft set relation and soft Scott topology. We obtain some results.

Keywords: Soft Set Theory: Soft Set Relation: Way-Below Soft Set Relation: Soft Set Relation: Soft Scoot Topology

1 Preliminaries and basic definitions

Definition 1. [4] Let U be an initial universe and E be a set of parameters. Let P(U) be the set of all subsets of U and A be a subset of E. A pair (F,A) is called a soft set over U where F: A \rightarrow P(U) is a set-valued function.

In some studies a soft set (F,A) was shown as $(F,A) = \{(a, F(a)) \mid a \in A\}$, but in some studies F(a) was written instead of (a, F(a)) just as a notation for make it shorter. In this paper, we will use F(a) as a notation instead of (a, F(a)).

Definition 2. [3] A soft set (F,A) over U is said to be a Null soft set denoted by Φ , if for every $\varepsilon \in A$, $F(\varepsilon) = \emptyset$

Definition 3. [3] For two soft sets (F,A) and (G,B) over a common universe U, we say that is a soft subset of (G, B) and is denoted by (F,A) \subseteq (G, B) if

- (i) $A \subset B$ and,
- (ii) $\forall \epsilon \in A, F(\epsilon) \text{ and } G(\epsilon) \text{ are identical approximations, which means } F(\epsilon) = G(\epsilon)$

(G,B) is said to be a soft super set of (F,A), if (F,A) is a soft subset of (G,B).

Definition 4. [3] Union of two soft sets (F,A) and (G,B) over the common universe U is the soft set (H,C), where $C = A \cup B$, and for each $e \in C$,

$$H(e) = \begin{cases} F(e), \text{ if } e \in A - B\\ G(e), \text{ if } e \in B - A\\ F(e) \cup G(e), \text{ if } e \in A \cap B \end{cases}$$

We write $(F, A) \widetilde{U} (G, B) = (H, C)$.

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Definition 5. [3] Intersection of two soft sets (F,A) and (G,B) over a common universe U is the soft set (H,C), where $C=A\cap B$ and for each $e \in C$, $H(e) = F(e) \cap G(e)$. We write (F,A) \cap (G,B) = (H,C).

Definition 6. [1] Let (F,A) and (G,B) be two soft sets over U, then the cartesian product of (F, A) and (G,B) is defined as, $(F,A) \times (G,B) = (H,A \times B)$ where $H : A \times B \to P(U \times U)$ and $H(a,b) = F(a) \times G(b)$, where $(a, b) \in A \times B$ i.e. $H(a,b) = \{(h_i, h_j) \mid h_i \in F(a), h_j \in G(b)\}$

Definition 7. [1] Let (F, A) and (G, B) be two soft sets over U, then a soft set relation R from (F, A) to (G, B) is a soft subset of (F, A) × (G, B). In other words, a soft set relation R from (F,A) to (G,B) is of the form $R = (H_1, S)$ where $S \subset A \times B$ and $H_1(a, b) = H(a, b)$ for all $(a, b) \in S$ where $(H, A \times B) = (F, A) \times (G, B)$.

Definition 8. [1] Let R be a soft set relation on (F, A), then

- 1. R is reflexive if $H_1(a, a) \in R, \forall a \in A$.
- 2. R is symmetric if $H_1(a, b) \in R \Rightarrow H_1(b, a) \in R$.
- 3. R is transitive if $H_1(a, b) \in R$, $H_1(b, c) \in R \Rightarrow H_1(a, c) \in R$ for every a, b, $c \in R$

A.

Definition 9. [2] A soft set relation R on (F,A) is antisymmetric if $F(a) \times F(b) \in R$ and $F(b) \times F(a) \in R$ for every $F(a), F(b) \in (F, A)$ imply F(a) = F(b).

Definition 10. [2] A soft set relation \leq on (F,A) which is reflexive, antisymmetric and transitive is called a partial ordering of (F,A). The triple (F,A, \leq) is called a partially ordered soft set.

Definition 11. [6] Consider a soft set (F,A) equipped with reflexsive, transitive soft set relation \leq . This soft set relation is called preorder and (F,A) is called a preordered soft set.

Definition 12. [2] Let (G, B, \leq) be a partially ordered soft set. Then, for $b \in B$, G(b) is the least element of (G,B) in the ordering ' \leq ' if G(b) \leq G(x) for all $x \in B$ and for $b \in B$,G(b) is the greatest element of (G,B) in the ordering ' \leq ' if $G(x) \leq G(b)$, $\forall x \in B$.

Definition 13. [6] Let \leq be an ordering of (F,A), let (*G*, *B*) \cong (*F*, *A*). For $a \in A$, F(a) is an upper bound of (G,B) in the ordered soft set (F,A, \leq) if G(x) \leq F(a) separate for all $x \in B$. For $a \in A$, F(a) is called supremum of (G,B) in (F,A, \leq) (or the least upper bound) if it is the separate element of the set of all upper bounds of (G,B) in (F,A, \leq).

Definition 14. [6] Let (F,A) be a soft set. (F,A) is called a finite soft set, if it is a soft set with a soft set set.

Definition 15. [6] Let (F,A) be a preordered soft set. A soft subset (G,B) of (F,A) is directed provided it is nonnull and every finite soft subset of (G,B) has an upperbound in (G,B).

Definition 16. Let (F,A) be a soft set with a preorder soft set relation \leq . For $(G, B) \cong (F, A)$ i) [6] \downarrow (G,B)=(H,C) where $C = \{a \in A : F(a) \leq G(b) \text{ for some } b \in B\}$ and H =F|_C.

ii)[6] \uparrow (G,B)=(K,D)where $D = \{a \in A : G(b) \le F(a) \text{ for some } b \in B\}$ and K=FID.

iii) [6] (G,B) is a lower soft set iff (G,B) = \downarrow (G,B).

iv) [6] (G,B) is an upper soft set iff $(G,B) = \uparrow (G,B)$.

v)[6] (G,B) is an soft ideal iff it is a directed lower soft set.

vi) [7] A soft ideal is principal iff it has a maximum element.

Definition 17. [7] A soft inf-semilattice is a partially ordered soft set (F, A, \leq) in which F (a),

F (b) have infimum for any two elements a, $b \in A$ soft sup-semilattice is a partially

ordered soft set (F, A) in which any F (a), F (b) in (F, A) have a supremum. A partially ordered soft set (F, A) which is both soft inf-semilattice and soft sup-semilattice is called a soft lattice.

Definition 18.

1. [6] A partially ordered soft set is said to be directed complete soft sets if every directed soft subset has a supremum.

1. [7] A partially ordered soft set which is an soft inf-semilattice and directed complete will be called a directed complete soft inf-semilattice.

2. [7] A soft lattice is called complete soft lattice in which every soft subset has a supremum and infimum. A totally ordered complete soft lattice is called a complete soft chain.

3. [7]Apartiallyorderedsoftsetiscalledacompletesoftinfsemilatticeiffeverynonnullsoftsubset has an infimum and every directed soft subset has a supremum.

4. [7] A posset is called bounded complete, if every soft subset that is bounded above has a least upper bound.

Definition 19. [8] Let (F, A, \leq) be a partially ordered soft set. We say that F (a) way-below F (b) iff for all directed soft subsets (G, B) \subseteq (F, A) for which sup(G, B) exists, the soft set relation F (b) \leq sup(G,B) always implies the existence of a G(d) in (G,B) with F(a) \leq G(d). F(a) way-below F(b) is denoted by F(a) \ll F(b).

This definition was expressed simultaneously by Sayed [5] as "Let (F,A) be a posset. For any two elements $F(x),F(y) \in (F,A)$. F(x) is approximate to F(y), and write $F(x) \ll F(y)$, if for any directed soft subset (G,B) \cong (F,A) with V(G,B) existing and $F(y) \leq V(G,B)$, there exists $G(z) \in (G,B)$ such that $F(x) \leq G(z)$."

Definition 20.

i [5] A partially ordered soft set (F, A, \leq) is called soft continuous if it satisfies the axiom of approximation:

 $(\forall F(a)in(F,A)) F(a) = \bigvee - \downarrow F(a)$ i.e. for all F(a) in (F,A), the soft set $\downarrow F(a)$ which is

(H,C) such that $C = \{b \in A | F(b) \ll F(a)\}$ and $H = F|_C$, is directed and $F(a) = \sup(H,C)$.

Sefii [5] A directed complete partially ordered soft set is soft continuous as a partially ordered soft set will be called soft set domain.

iii [8] A soft set domain which is complete soft set lattice is called continuous soft lattice.

iv [8] A complete soft semilattice which is a soft set domain as a partially ordered soft set is called a complete continuous soft semilattice or alternatively bounded complete soft domain.

v [8] A soft domain in which every principal soft ideal $\frac{1}{2}$ F (x) is complete soft lattice is called an L-soft domain.

3 Compact soft elements and Algebraic soft domains

Definition 21. In any partially ordered soft set (F, A), an element F (k) is called compact soft element (or isolated soft element) iff F (k) \ll F (k), i.e. whenever (D, C) is directed soft subset of (F, A) such that sup(D, C) exists and F (k) \leq sup(D, C), then F (k) \leq D(c) for some c \in C. The soft subset of all compact soft elements is denoted by K(F,A).

Definition 22.

a. A partially ordered soft set (F, A) is called algebraic iff it satisfies the Axiom of Compact Approximation

 $(\forall F(a) \text{ in } (F,A)) F(a) = \bigvee (\downarrow F(a) \cap K(F,A)) \xrightarrow{[i]}{\text{ssp}} \text{ i.e. for all } a \in A \text{ the soft set}$ $\downarrow F(a) \cap K(F,A) \text{ is directed and } F(a) = \sup(\downarrow F(a) \cap K(F,A)).$

b. A directed complete algebraic partially ordered soft set (F, A) is called algebraic soft domain.

c. An algebraic soft domain which is a soft lattice is called an algebraic soft lattice.

d. An algebraic soft domain which is a soft semilattice is called an algebraic soft semilattice.

e. A complete soft semilattice which is an algebraic soft domain as a partially ordered soft set is called a bounded complete algebraic soft domain.

f. An algebraic soft domain in which every principal soft ideal $\downarrow F(a)$ is a complete soft lattice is called an algebraic soft L-domain

Theorem 3. In a partially ordered soft set (F, A), the following statements are equivalent:

a. (F,A)isalgebraic;

b. (F,A) is soft continuous and F(a) \ll F(b) iff there is a F(k) in K(F,A) with F(a) \leq F(b). 55

In particular every algebraic partially ordered soft set is continuous partially ordered soft set and every algebraic soft (semi)lattice is continuous soft (semi)lattice.

Theorem 4. Let (F, A) be a directed complete partially ordered soft set. If (F,A) has a least element F(0), then F(0) is in K(F,A). If two elements F(a), F(b) in K(F, A) have supremum in (F,A), then sup{F(a), F(b)} is in K(F,A).

Definition 23. A soft semilattice (F, A) is called an arithmetic soft semilattice iff it is algebraic and if K(F, A) is a soft semilattice of (F, A), i.e., if sup{F(a), F(b)} is in K(F,A) for all F(a), F(b) in K(F,A). An arithmetic soft lattice is an algebraic soft lattice in which the soft set of compact soft elements is a soft subsemilattice.

Theorem 5. Let (F,A) be an algebraic soft semilattice. Then the following statements are equivalent:

- a. (F,A) is arithmetic;
- b. K(F,A) is a soft semilattice.

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Relation Between Meet Continuous Soft Sets and Soft Scott Topology

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Abstract

Soft Set theory was introduced by Molodtsov in 1999 to deal with uncertainties in the economics, engineering, environmental and related scientific areas which need of certain mathematical solutions but classical mathematical tools are inadequate to satisfy their needs related to the uncertainties derived from complicated problems. Moreover there are increasingly many studies about the soft set theory. Babitha and Sunil gave soft set relations. By using soft set relations, directed soft set, soft Scott topology and meet continuous soft set were defined. In this study, alternative definition for meet continuous soft set is given by using soft Scott topology. Also we showed that these two definitions are equivalent. Furthermore some results about meet continuous soft set and soft Scott topology are proved.

Keywords: Soft set, Soft set relation, Soft topology, Soft Scott topology

1 Preliminaries and basic definitions

Definition 1. [5] Let U be an initial universe and E be a set of parameters. Let P(U) be the set of all subsets of U and A be a subset of E. A pair (F,A) is called a soft set over U where F: A \rightarrow P(U) is a set-valued function.

Definition 2. [4] A soft set (F,A) over U is said to be a Null soft set denoted by Φ , if for every $\varepsilon \in A$, $F(\varepsilon) = \emptyset$

Definition 3. [4] For two soft sets (F,A) and (G,B) over a common universe U, we say that (F,A) is a soft subset of (G, B) and is denoted by (F,A) \cong (G, B) if A \subset B and $\forall \varepsilon \in A$, F(ε) and G(ε) are identical approximations, which means F(ε) = G(ε). (G,B) is said to be a soft super set of (F,A), if (F,A) is a soft subset of (G,B).

Definition 4. [4] Union of two soft sets (F,A) and (G,B) over the common universe U is the soft set (H,C), where $C = A \cup B$, and for each $e \in C$,

$$H(e) = \begin{cases} F(e), \text{ if } e \in A - B\\ G(e), \text{ if } e \in B - A\\ F(e) \cup G(e), \text{ if } e \in A \cap B \end{cases}$$
. We write (F, A) $\widetilde{U}(G, B) = (H, C)$.

Intersection of two soft sets (F,A) and (G,B) over a common universe U is the soft set (H,C),

where C=A \cap B and for each $e \in C$, $H(e) = F(e) \cap G(e)$. We write (F, A) $\widetilde{\cap}$ (G, B) = (H, C).

Definition 5. [1] Let (F,A) and (G,B) be two soft sets over U, then the cartesian product of (F,A) and (G,B) is defined as, $(F,A) \times (G,B) = (H,A \times B)$ where $H : A \times B \to P(U \times U)$ and $H(a,b) = F(a) \times G(b)$, where $(a, b) \in A \times B$ i.e. $H(a,b) = \{(h_i, h_j) \mid h_i \in F(a), h_j \in C(D)\}$

G(b). A soft set relation R from (F, A) to (G, B) is a soft subset of (F, A) × (G, B).

Definition 6. Let R be a soft set relation on (F, A), then

1. [1] R is reflexive if $H_1(a, a) \in R, \forall a \in A$.

2. [1] R is symmetric if $H_1(a, b) \in R \Rightarrow H_1(b, a) \in R$.

3. [1] R is transitive if $H_1(a, b) \in R$, $H_1(b, c) \in R \Rightarrow H_1(a, c) \in R$ for every a, b, $c \in A$.

4. [2] R is antisymmetric if $F(a) \times F(b) \in R$ and $F(b) \times F(a) \in R$ for every $F(a), F(b) \in (F, A)$ imply F(a) = F(b).

Definition 7. [2] A soft set relation \leq on (F,A) which is reflexive, antisymmetric and transitive is called a partial ordering of (F,A). The triple (F,A, \leq) is called a partially ordered soft set.

Definition 8. [7] Consider a soft set (F,A) equipped with reflexsive, transitive soft set relation \leq . This soft set relation is called preorder and (F,A) is called a preordered soft set.

Definition 9. [2] Let (G,B,\leq) be a partially ordered soft set. Then, for $b \in B$, G(b) is the least element of (G,B) in the ordering ' \leq ' if $G(b) \leq G(x)$, $\forall x \in B$ and for $b \in B, G(b)$ is the greatest element of (G,B) in the ordering ' \leq ' if $G(x) \leq G(b)$, $\forall x \in B$.

Definition 10. [7] Let \leq be an ordering of (F,A), let $(G,B) \cong (F,A)$. For $a \in A$, F(a) is an upper bound of (G,B) in the ordered soft set (F,A, \leq) if G(x) \leq F(a) $\underset{\text{EP}}{\overset{\text{def}}{\text{for all }}}$ for all $x \in B$. For $a \in A$, F(a) is called supremum of (G,B) in (F,A, \leq) (or the least upper bound) if it is the $\underset{\text{EP}}{\overset{\text{def}}{\text{feast}}}$ least element of the set of all upper bounds of (G,B) in (F,A, \leq).

Definition 11. [7] Let (F,A) be a preordered soft set. A soft subset (G,B) of (F,A) is directed [see] provided it is nonnull and every finite soft subset of (G,B) has an upperbound in (G,B).

Definition 12. Let (F,A) be a soft set with a preorder soft set relation \leq . For $(G, B) \cong (F, A)$ 1. [7] \downarrow (G,B)=(H,C) where $C = \{a \in A : F(a) \leq G(b) \text{ for some } b \in B\}$ and H =F|C. 2. [7] \uparrow (G,B)=(K,D)where $D = \{a \in A : G(b) \leq F(a) \text{ for some } b \in B\}$ and K=F|D.

3. [7] (G,B) is a lower soft set iff (G,B) = \downarrow (G,B).

4. [7] (G,B) is an upper soft set iff (G,B) = \uparrow (G,B).

5. [7] (G, B) is a soft filter iff it is a filtered upper soft set.

6. [8] A soft filter is principal iff it has a minimum element.

Definition 13. [8] A soft inf-semilattice is a partially ordered soft set (F,A,\leq) in which F(a),

F(b) have infimum for any two elements a, $b \in A$.

soft set (F,A) in which any F(a), F(b) in (F,A) have a supremum. A partially ordered soft set (F,A) which is both soft inf-semilattice and soft sup-semilattice is called a soft lattice.

Definition 14. [7] A partially ordered soft set is said to be directed complete soft sets if every directed soft subset has a supremum.

Definition 15. [9] A soft inf-semilattice (F, A) is called soft meet continuous if it is directed complete soft set and satisfying F(x)sup(G, B) = sup(F(x)(G,B)) for all $x \in A$ and all directed soft subsets $(G, B) \cong (F, A)$.

Definition 16. [6] A soft topology $\tilde{\tau}$ on a soft set (F,A) is a family of soft subsets of (F,A) satisfying the following properties

i)
$$\Phi, (F, A) \in \tilde{\tau};$$

ii) If $(G, B), (H, C) \in \tilde{\tau}$, then $(G, B) \cap (H, C) \in \tilde{\tau}$;
iii) If $(F_{\alpha}, A_{\alpha}) \in \tilde{\tau}$ for all $\alpha \in \Lambda$, an index set, then $\widetilde{U}_{\alpha \in \Lambda} (F_{\alpha}, A_{\alpha}) \in \tilde{\tau}.$

If $\tilde{\tau}$ is a soft topology on a soft set (F, A), then (F,A, $\tilde{\tau}$) is called the soft topological space.

Definition 17. [3] Let $(F, A, \tilde{\tau})$ be a soft topological space and $(F, B) \cong (F, A)$. Then, the soft interior of (F,B) is defined as the soft union of all soft open subsets of (F,B) and the soft closure of (F, B) is defined as the soft intersection of all soft closed supersets of (F, B).

Definition 18. [6] A collection $\tilde{\beta}$ of some soft subsets of (F, A) is called a soft open base or simply a base for some soft topology on (F, A) if $\Phi \in \tilde{\beta}$, $\tilde{U} \in \tilde{\beta} = (F, A)$ and if (G,B),(H,C) $\in \tilde{\beta}$

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then for each $e \in B \cap C$ and $x \in G(e) \cap H(e)$ there exists $(I,D) \cong \tilde{\beta}$ such $\underset{sep}{\boxtimes}$ that $(I,D) \cong$

 $(G,B)\widetilde{\cap}(H,C)$ and $x \in I(e)$, where $D \subseteq B \cap C$.

Definition 19. [11] Let $(F,A,\tilde{\tau})$ be a soft topological space and \tilde{S} be a collection of nonnull soft open subsets of (F,A). If finite intersection of the elements of \tilde{S} is a base for $\tilde{\tau}$ then \tilde{S} is called soft subbase.

2 Soft Scott topology and Soft Lawson Topology

Definition 20. [7] Let (F,A) be a directed complete partially ordered soft set and $(G,B) \cong (F,A)$. Then (G,B) is called a Scott soft open set iff $(G,B)=\uparrow(G,B)$ and $sup(D,C) \in (G,B)$ implies $(D,C) \cap (G,B) \neq \Phi$ for all directed complete soft sets $(D,C) \cong (F,A)$. The collection of all Scott soft open sets of (F,A) is called soft Scott topology on (F,A) and this topology will be denoted by $\sigma(F,A)$.

Definition 21. [10] Let (F,A) be a partially ordered soft set. The soft topology generated by the soft complements of principal filters (F,A) $-\uparrow$ {F(a)} (as subbasic open soft sets) is called the soft Lower topology and denoted by ω (F,A).

Definition 22. [10] Let (F,A) be directed complete partially ordered soft set. Then common refinement of $\sigma((F,A))V\omega((F,A))$ of the soft Scott topology and the Lower soft topology is called the soft Lawson topology and denoted by $\lambda((F,A))$.

3 Relation Between Meet Continuous Soft Sets and Soft Scott Topology

Definition 23. A directed complete partially ordered soft set (F, A) is soft meet continuous if for any $a \in A$, F(a) and any directed soft set (D,C) with F(a) $\leq \sup(D,C)$, then F(a) is in the Scott soft closure of $\downarrow(D,C)\widetilde{\cap}\downarrow F(a)$.

Theorem 1. For directed complete soft semilattice the preceding definition of meet continuity is equivalent to the standard one.

Theorem 2. A directed complete partially ordered soft set (F,A) is soft meet continuous iff for any Scott open soft set (G,B) and any $a \in A$, $\uparrow((G,B) \cap \downarrow F(a))$ is Scott open soft set.

Theorem 3. For a meet continuous directed complete partially ordered soft set (F, A) we have:

i)
$$(G,B) \in \lambda(F,A)$$
, then $\uparrow (G,B) \in \sigma(F;A)$;

- ii) if (H,C) is an upper soft set, then $int_{\sigma}(H,C)=int_{\lambda}(H,C)$;
- iii) if (H,C) is a lower soft set, $cl_{\sigma}(H,C)=cl_{\lambda}(H,C)$.

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Some Characterizations of Translation Surfaces in Lorentzian 3- Dimensional Heisenberg Group

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Abstract

In this paper, some characterizations of translation surfaces in 3-dimensional Heisenberg group generated by two spacelike curves have been studied. **Keywords:** Heisenberg group, Lorentz metric, translation surface.

1. Introduction

The theory of minimal surfaces in three-dimensional Euclidean space has its roots in the calculus of variations developed by Euler and Lagrange in the 18-th century and in later investigations by Enneper, Scherk, Schwarz, Riemann and Weierstrass in the 19-th century. Then, C. B. Morrey studied minimal surface in Riemannian manifold. Using the direct methods of calculus of variations, he was able to give an existence proof for a large class of Riemannian manifolds of differentiability C^1 which, especially, contains all compact C^1 manifolds. During the years, many great mathematicians have contributed to this theory.

Translation surfaces in E^3 , firstly studied by H. F. Scherk. He proved that, besides the planes, the only minimal translation surfaces are the surfaces given by

$$z = \frac{1}{a}\log\left|\frac{\cos(ax)}{\cos(ay)}\right| = \frac{1}{a}\log|\cos(ax)| - \frac{1}{a}\log|\cos(ay)|,$$

where *a* is a non-zero constant, [8]. Then, the study of translation surfaces in the Euclidean space was extended when the second fundamental form was considered as a metric on a non-developable surface. M. I. Munteanu and A. I. Nistor have studied the second fundamental form of translation surfaces in E^3 , [1]. The translation surfaces in 3-dimensional Euclidean space generated by two space curves have been investigated by Çetin. Also they showed that Scherk surface is not only minimal translation surface. [5] Some classification of

the translation surfaces with constant mean curvature or constant Gauss curvature in 3dimensional Euclidean space E^3 or 3-dimensional Minkowski space E_1^3 have given by Liu [3]. D. W. Yoon has studied translation surfaces in the 3-dimensional Minkowski space whose Gauss map G satisfies the condition $\Delta G = \Delta A$, where Δ denotes the Laplacian of the surface with respect to the induced metric and A is a 3 x 3 real matrix, [8]. Translation surfaces in the 3-dimensional hyperbolic space H³ have been studied by Lopez in [4] and he classified minimal translation surfaces. Translation surfaces can be defined in any 3dimensional Lie groups equipped with left invariant Riemannian metric. Translation surfaces in the 3-dimensional Heisenberg group *Nil*³ in terms of a pair of two planar curves lying in orthogonal planes defined by J. Inoguchi, R. López and M. I. Munteanu, [2]. They classified minimal translation surfaces in *Nil*³. Translation surfaces in Sol3 constructed by R. López and M. I. Munteanu and they investigated properties of minimal one, [5].

The purpose of this paper is to study and classify modifieded translation surfaces in $Heis_3$ and investigate conditions of being minimal surface. Also, obtain characterizations of points on this surface.

2. Materials and Methods

The Heisenberg group $Heis_3$ is a Lie group which is diffeomorphic to R^3 and the group operation is defined as

$$(x, y, z) * (x_1, y_1, z_1) = \left(x + x_1, y + y_1, z + z_1 + \frac{1}{2}(xy_1 - x_1y)\right).$$
 (2.1)

The identity of the group is (0,0,0) and the inverse of (x, y, z) is given by (-x, -y, -z). The left-invariant Lorentz metric on *Heis*₃ is

$$g = ds^{2} = -dx^{2} + dy^{2} + (xdy + dz)^{2}.$$
(2.2)

The following set of left-invariant vector fields forms an orthonormal basis for the corresponding Lie algebra:

$$e_1 = \frac{\partial}{\partial z}, e_2 = \frac{\partial}{\partial y} - x \frac{\partial}{\partial z}, e_3 = \frac{\partial}{\partial x}.$$
 (2.3)

The characterising properties of this algebra are the following commutation relations:

$$[e_2, e_3] = e_1, [e_1, e_2] = [e_1, e_3] = 0$$
(2.4)

with

$$g(e_1, e_1) = g(e_2, e_2) = 1, g(e_3, e_3) = -1.$$
 (2.5)

Proposition 2.1. For the covariant derivatives of the Levi-Civita connection of the left -invariant metric g, defined above the following is true: Levi Civita connections are

$$2\nabla_{e_i} e_j = \begin{bmatrix} 0 & e_3 & e_2 \\ e_3 & 0 & e_1 \\ e_2 & -e_1 & 0 \end{bmatrix},$$
 (2.6)

where the (i,j)-element in the table above equals for $\nabla_{e_i} e_j$ for our basis

$$\{e_k, k=1,2,3\} = \{e_1, e_2, e_3\}.$$

Let $\gamma: I \to Heis_3$ be a unit speed spacelike curve with timelike binormal and {**T**, **N**, **B**} are Frenet vector fields, then Frenet formulas are as follows

$$\nabla_{\mathbf{T}} \mathbf{T} = \boldsymbol{\kappa} \mathbf{N},$$

$$\nabla_{\mathbf{T}} \mathbf{N} = -\boldsymbol{\kappa} \mathbf{T} + \boldsymbol{\tau} \mathbf{B} \qquad (2.7)$$

$$\nabla_{\mathbf{T}} \mathbf{B} = \boldsymbol{\tau} \mathbf{N},$$

where κ , τ are curvature function and torsion function, respectively and

$$g(\mathbf{T}, \mathbf{T}) = g(\mathbf{N}, \mathbf{N}) = 1, g(\mathbf{B}, \mathbf{B}) = -1$$

$$g(\mathbf{T}, \mathbf{N}) = g(\mathbf{T}, \mathbf{B}) = g(\mathbf{N}, \mathbf{B}) = 0.$$
(2.8)

With respect to the orthonormal basis $\{e_1, e_2, e_3\}$, we can write

$$\mathbf{T} = t_1 e_1 + t_2 e_2 + t_3 e_3,$$

$$\mathbf{N} = n_1 e_1 + n_2 e_2 + n_3 e_3,$$

$$\mathbf{B} = b_1 e_1 + b_2 e_2 + b_3 e_3.$$
(2.9)

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3. Results and Discussions

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Some Characterizations of Translation Surfaces in Heis,

Let $\psi(x, y)$ be a spacelike translation surface in 3-dimensional Heisenberg Group which is endowed with the Lorentzian metric g. Then $\psi(x, y)$ parametrized as

$$\psi(x, y) = \alpha(x) + \beta(y) \tag{3.1}$$

where α and β are unit-speed spacelike curves in $Heis_3$, x and y, arclength parameters, respectively. Let $\{\mathbf{T}_{\alpha}, \mathbf{N}_{\alpha}, \mathbf{B}_{\alpha}\}$ and $\{\mathbf{T}_{\beta}, \mathbf{N}_{\beta}, \mathbf{B}_{\beta}\}$ be the Frenet frame field of α and β , respectively, where $g_1(\mathbf{B}_{\alpha}, \mathbf{B}_{\alpha}) = g_1(\mathbf{B}_{\beta}, \mathbf{B}_{\beta}) = -1$.

Let $\psi(x, y)$ be a translation surface in *Heis*₃. Then, from (3.1) the translation surface

$$\psi(x, y) = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \alpha_3 + \beta_3), \tag{3.2}$$

where, $\alpha(x) = (\alpha_1(x), \alpha_2(x), \alpha_3(x))$ and $\beta(y) = (\beta_1(y), \beta_2(y), \beta_3(y)).$

The coefficients of the first fundamental form are

$$E = g(\psi_x, \psi_x) = 1,$$

$$F = g(\psi_x, \psi_y) = \|\mathbf{T}_{\alpha}\| \|\mathbf{T}_{\beta}\| \cos\varphi = \cos\varphi,$$

$$G = g(\psi_y, \psi_y) = 1.$$

So, the first fundamental form is

$$I = dx^2 + 2\cos\varphi dxdy + dy^2.$$

Let unit tangent vector fields of $\alpha(x)$ and $\beta(y)$ be

$$\mathbf{T}_{\alpha} = t_1 e_1 + t_2 e_2 + t_3 e_3,$$
$$\mathbf{T}_{\beta} = \widetilde{t_1} e_1 + \widetilde{t_2} e_2 + \widetilde{t_3} e_3.$$

From above equations, the Gaussian curvature K is given by

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$$K = \frac{1}{1 - \cos^2 \varphi} \{ \sinh \theta_{\alpha} \sinh \theta_{\beta} (t_1^{'2} + (t_3 t_1 + t_2^{'})^2 - (t_1 t_2 + t_3^{'})^2) (\tilde{t_1}^{'2} + (\tilde{t_1} \tilde{t_3} + \tilde{t_2}^{'})^2 - (\tilde{t_2} \tilde{t_1} + \tilde{t_3}^{'})^2) - (\sinh \theta_2 \sinh \theta_1 ([\frac{1}{2} (t_2 \tilde{t_3} - t_3 \tilde{t_2}) + \tilde{t_1}^{'}]^2 + [\frac{1}{2} (t_1 \tilde{t_3} + t_3 \tilde{t_1}) + \tilde{t_2}^{'}]^2 - [\frac{1}{2} (t_1 \tilde{t_2} + t_2 \tilde{t_1}) + \tilde{t_3}^{'}]^2)) \\ ([\frac{1}{2} (t_3 \tilde{t_2} - t_2 \tilde{t_3}) + t_1^{'}]^2 + [\frac{1}{2} (t_3 \tilde{t_1} + t_1 \tilde{t_3}) + t_2^{'}]^2 - [\frac{1}{2} (t_2 \tilde{t_1} + t_1 \tilde{t_2}) + t_3^{'}]^2] \}$$

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Some New Characterizations of A-Net Parallel Surfaces in Riemannian Heisenberg Group

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Abstract

In this paper, some new properties of A-net surfaces and parallel surfaces of A-net surfaces in Riemannian three dimensional Heisenberg group are obtained **Keywords:** Heisenberg group, parallel surface, A-net surface.

1. Introduction

Creation of parallel surfaces is useful in design and manufacture. Making of dies for forging and castings require modeling of parallel surfaces. Enhancing or reducing the size of free-from surfaces requires calculation of curvature and other properties of the new surface, which is parallel to the original surface. A surface M^r whose points are at a constant distance along the normal from another surface M is called parallel to M. Since choosing of the constant distance along the normal is arbitrarily, there are infinite numbers of surfaces. From the definition, the parallel surface can be declared the locus of point which are on the normals to M at a nonzero constant distance r from M, [1].

If a isometric representation between two surfaces preserves the principal curvatures of these surface. The name of these surfaces are Bonnet surfaces. In [12], he deal with the Bonnet problem of finding the surfaces in E^3 which can acknowledge at least one nontrivial isometry that preserves principal curvatures. This problem considered locally and for the general case. Then, to find a Bonnet surface a method is deduced. In according to this, A-net on a surface such that, when this net is parametrized, the conditions E = G, $F = 0, M = c = const. \neq 0$ are satisfied, is called an A-net, where E, F, G are the coefficients of the first fundamental form of the surface and h_{11} , h_{12} , h_{22} are the coefficients of the second fundamental form. And necessary and sufficient condition for a surface to be a Bonnet surface is that the surface can have an A-net. Then, in [7], it is considered the Bonnet ruled surfaces which accept only one non-trivial isometry that preserves the principal curvatures, then, she gave the definition of the A-net surface and determined the Bonnet ruled surfaces whose generators and orthogonal trajectories form a special net called an A-net.

2. Materials and Methods

In this paper we deal with the Riemannian metric

$$g = ds^2 = dx^2 + dy^2 + (xdy + dz)^2.$$

which has the covariant derivatives of the Levi-Civita connections;

$$\nabla_{\mathbf{e}_{i}}\mathbf{e}_{j} = \frac{1}{2} \begin{bmatrix} 0 & \mathbf{e}_{3} & -\mathbf{e}_{2} \\ -\mathbf{e}_{3} & 0 & \mathbf{e}_{1} \\ -\mathbf{e}_{2} & \mathbf{e}_{1} & 0 \end{bmatrix},$$

where the (i, j)-element in the table above equals $\nabla_{\mathbf{e}_i} \mathbf{e}_j$. Then, we study A- net parallel surface in this group. Then, we give some characterizations of this surface.

3. Results and Discussions

A- net and A- net Parallel Surfaces in Heisenberg Group

In this section, we characterize A-net surfaces in Euclidean 3- space E^3 . Then, we obtain constant mean curvature and Gaussian curvature of this surface. A-net on a surface is defined following conditions

$$E = G, F = 0, h_{12} = constant \neq 0,$$

where E, F, G are the coefficients of the first fundamental form of the surface and h_{11} , h_{12} , h_{22} are the coefficients of the second fundamental form.

Then, parallel surfaces of the surface $\theta(u, v)$ is

$$\varphi(u,v) = \theta(u,v) + a\mathbf{U}(u,v)$$

where a is a constant and U(u, v) is unit normal vector field of the surface $\theta(u, v)$.

In this section, we will study following surface and its parallel surfaces;

$$\theta(u,v) = (f(u),g(v),h(u,v), \tag{3.1})$$

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where f(u), g(v) and h(u, v) be smooth functions.

Theorem 3.1. Let $\theta(u, v)$ be a smooth surface in Heisenberg group which is parametrized (3.1). If the surface $\theta(u, v)$ is a A-net surface, then following conditions are provided;

i) h(u,v) = g(v)f(u) + c(u),ii) $f'^2 + (g(v)f'(u) + c'(u))^2 = g'^2,$ iii) $(g(v)f'(u) + c'(u))^2(\lambda + 1) = f'^2(u)(1 - \lambda).$

where c(u) is a smooth function, λ is a constant.

Proof. If we take derivatives of the surface, which is given with the parametrization (3.1), we have

$$\begin{array}{rcl}
\theta_{u} &=& f'(u)e_{1} + h_{u}(u,v)e_{3}, \\
\theta_{v} &=& g'(v)e_{2} + (h_{v}(u,v) - g'(v)f(u))e_{3}.
\end{array}$$
(3.2)

Then, components of the first fundamental form of the surface are

$$E = f'^{2}(u) + h_{u}^{2}(u,v),$$

$$F = h_{u}(u,v)(h_{v}(u,v) - g'(v)f(u)),$$

$$G = g'^{2}(v) + (h_{v}(u,v) - g'(v)f(u))^{2}.$$
(3.2)

From equations (3.3), if

F=0,

we have

$$h_u(u,v) = 0 \text{ or } h_v(u,v) - g'(v) f(u) = 0.$$

Let assume that

$$h_{v}(u,v)-g'(v)f(u)=0.$$

So,

$$h(u,v) = g(v)f(u) + c(u).$$
(3.4)

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Then, because of E = G, the following differential equation obtains;

 $f'^{2}(u) + (g(v)f'(u) + c'(u))^{2} = g'^{2}(v).$

The unit normal vector field of the $\theta(u, v)$ is

$$\mathbf{U} = \frac{1}{g'} \left(-(g(v)f'(u) + c'(u))e_1 + f'e_3 \right).$$
(3.5)

Then, components of the second fundamental form of the surface are

$$h_{12} = \frac{1}{2} (f'^{2}(u) - g(v)f'(u) + c'(u)),$$

$$h_{21} = \frac{1}{2} (f'^{2}(u) + g(v)f'(u) + c'(u)),$$
(3.6)

$$h_{12} = const. = \lambda_1$$

$$\Rightarrow \frac{1}{2} (f'^2(u) - g(v)f'(u) + c'(u)) = \lambda_1$$
(3.7)

and

$$h_{21} = const. = \lambda_2$$

$$\Rightarrow \frac{1}{2} (f^{'2}(u) + g(v)f'(u) + c'(u)) = \lambda_2.$$
(3.8)

From (3.7), (3.8), we have

$$(g(v)f'(u)+c'(u))^{2}(\lambda+1)=f'^{2}(u)(1-\lambda),$$
(3.9)

where $\lambda = \frac{\lambda_1}{\lambda_2}$.

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A Survey on Being A Sober Space that Fulfills the Conditions of the Metric Space

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A common supercategory of topological spaces and metric spaces is defined as approach space by Lowen [1] in 1989. Sober approach space is introduced in [2] as the equivalent of sober topological space in metric texture. After that, sober metric approach space is defined in [3]. In the works of Smyth [4,5], smyth completeness introduced in 1994. The aim of this work is research to provide a relation between smyth complete metric space and sober metric approach space by findings in [3].

Keywords: Metric space, Sober, Approach space, Metric approach space, Sober approach space.

1. Introduction

In this article we will study a notion of sobriety for approach spaces that fulfills the conditions of the metric Space. I give an account of the basic facts concerning the results about sobriety. These results bring to the foreground a completeness-aspect of the notion of sobriety which is somewhat hidden in the topological setting.

2. Definitions, Results and Discussions

Approach spaces, introduced by Lowen [1], are a common extension of topological spaces and metric spaces. Sober approach spaces, a counterpart of sober topological spaces in the metric setting. An approach space X can be characterized by means of various defining structures. One of these is the so-called regular function frame RX. An approach frame L is a frame equipped with two families of unary operations, addition and subtraction of $\alpha \in [0,\infty]$. It is proved there that a topological space is sober as an approach space, if and only if it is sober as a topological space. So, it is natural to ask what kind of metric approach spaces are sober? The answer is obtained in [3] and a bit surprising: a metric space is sober, as an approach space, if and only if it is Smyth complete. In this work in this work I will present the results handeled in [3].

Definition : ([3]) An approach space (X, δ) consists of a set X and a map $\delta : X \times 2^X \to [0, \infty]$, subject to the following conditions:

(A1) $\delta(x, \{x\}) = 0$,

(A2) $\delta(x, \emptyset) = \infty$,

(A3) $\delta(x, A \cup B) = \min{\{\delta(x, A), \delta(x, B)\}},$

(A4) $\delta(x, A) \ge \delta(x, B) + \sup_{b \in B} \delta(b, A),$

for all $x \in X$ and $A, B \in 2^X$. The map δ is called an approach distance on X.

Given an approach space (X,δ) , define $\Omega(\delta)$: $X \times X \to [0, \infty]$, by $\Omega(\delta)(x, y) = \delta(x, \{y\})$, then $\Omega(\delta)$ is a metric on X, called the specialization metric of (X, δ) . The correspondence $(X, \delta) \to (X, \Omega(\delta))$ defines a functor

 Ω : App \rightarrow Met.

This functor is a counterpart of Ω : Top \rightarrow Ord in the metric setting.

Approach spaces can be equivalently described in many ways [3], one of them we need is the description by regular functions. A regular function of an approach space (X, δ) is a contraction $\phi : (X, \delta) \to P$, where P is the opposite of the Lawvere distance on $[0, \infty]$, i.e., $\Omega(\delta_P) = d_R$.

The following proposition says that an approach space is uniquely determined by its regular functions:

Proposition : ([3]) Let (X, δ) be an approach space. Then the set *RX* of regular functions of (X, δ) satisfies the following conditions:

(R1) For each subset $\{\varphi_i\}_{i \in I}$ of *RX*, $\sup_{i \in I} \varphi^{\varphi_i} \in RX$.

(R2) For all $\varphi, \psi \in \mathbf{RX}$, min $\{\varphi, \psi\} \in \mathbf{RX}$.

(R3) For all $\varphi \in \mathbf{RX}$ and $\alpha \in [0, \infty]$, both $\varphi + \alpha$ and $\varphi_g \alpha$ are in \mathbf{RX} . Conversely, suppose that $S \subseteq [0, \infty]^X$ satisfies the conditions(R1) - (R3). Define a function $\delta : X \times 2^X \to [0, \infty]$ by

 $\delta(x, A) = \sup \{ \varphi(x) | \varphi \in S, \forall a \in A, \varphi(a) = 0 \}.$

Then (X, δ) is an approach space with S being its set of regular functions.

Contractions between approach spaces can be characterized in terms of regular functions.

Definition : A metric space is Smyth complete if every forward Cauchy net in it converges in its symmetrization.

Smyth completeness originated in the works of Smyth that aimed to provide a common framework for the domain approach and the metric space approach to semantics in computer science. As advocated in [4–5], in this paper we emphasize that the relationship between approach spaces and metric spaces is analogous to that between topological spaces and ordered sets. This point of view has proved to be fruitful, and is well in accordance with the thesis of Smyth.

Theorem : ([3]) Let (X, δ) be an approach space.

(1) (X, δ) is a sober approach space.

(2) For each contraction f from (X, δ) to a sober approach space (Y, ρ) , there is a unique contraction

 $f: (X, \delta) \rightarrow (Y, \rho)$ such that $f = f \circ \eta X$.

Lemma : For each metric space (X,d), the approach primes of $\Gamma(X, d)$ are exactly the flat weights of (X, d).

Proof: Given an approach prime φ of $\Gamma(X, d)$, we show that φ is a flat weight of (X, d). It suffices to check that φ satisfies the flat weight conditions: Suppose $\varphi(xi) < \varepsilon_i(i=1, 2)$. Consider the functions $\psi(x) = \max\{0, \varepsilon_1 - d(x_1, x)\}$ and $\xi(x) = \max\{0, \varepsilon_2 - d(x_2, x)\}$. It is easy to check that ψ and ξ are regular functions satisfying $\psi \le \varphi$ and $\xi \le \varphi(\psi(x_1) = \varepsilon_1, \xi(x_2) = \varepsilon_2)$.

Proposition : ([3]) If (X, δ) and (Y, ρ) are approach spaces and $f: X \to Y$ is a map, then f is a contraction if and only if for each $\varphi \in \mathbf{R}Y$, $\varphi \circ f \in \mathbf{R}X$.

A map $\varphi : X \to [0, \infty]$ is a weight of (X, d) if and only if it is a regular function of $\Gamma(X, d)$, as stated in the following conclusion:

Proposition : ([3]) For a metric space (X, d), a function $\varphi : X \to [0, \infty]$ is a weight of (X, d) if and only if it is a regular function of the approach space $\Gamma(X, d)$.

3. Conclusions

Corollary : ([3]) Let (X, d) be a symmetric metric space. Then the sobrification of (X, $\Gamma(d)$) is a metric approach space and is generated by the Cauchy completion of (X, d). **Proof.** This follows from that every symmetric metric space is Smyth completable.

Theorem : ([3]) Let (X, d) be a metric space. The following are equivalent:

- (1) The approach space $(X, \Gamma(d))$ is sober.
- (2) (X, d) is Smyth complete.

(3) (X, d) is a fixed point of the Yoneda completion, i.e., y_x : (X, d) \rightarrow (FX, d) is an isomorphism.

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Characterization of Fuzzy Topology by Fuzzy Relations

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In this paper I have studied the concept of fuzzy topological space generated by a fuzzy relation as an extension of the corresponding concepts in [1], [2] and [3] respectively, for the crisp case. Then several related results have been shown. I have given some information about separation axioms in this fuzzy topology that some specific problems related to compactness can be found out in the future works.

Keywords: Fuzzy set, Fuzzy relation, Fuzzy topological space.

1. Introduction

In this presentation, fuzzy topologies generated by fuzzy relations are studied. Several related results are proved. In particular, characterizations of a fuzzy topology generated by a fuzzy relation, a fuzzy topology generated by a fuzzy interval order, a preorderable fuzzy topology and an orderable fuzzy topology are obtained. Fuzzy relations have been studied by several authors, e.g., Chakraborty and Sarkar (1987), Chakraborty and Das (1983, 1985), Jayaram and Mesiar (2009), Kundu (2000) and Figueira et al. (2005). Apart form definitions and theorems are numbered, known concepts are mentioned in the text along with the reference [3].

2. Definitions, Results and Discussions

Definition (Zadeh 1965) : A fuzzy set in X is a function $A : X \to I$, where I is the closed unit interval [0,1].

• Now we define some basic fuzzy set operations as follows:

Let *A* and *B* be two fuzzy sets in *X*. Then :

- 1. A = B if A(x) = B(x), $\forall x \in X$.
- 2. $A \subseteq B$ if $A(x) \leq B(x)$, $\forall x \in X$.
- 3. $(A \cup B)(x) = \max \{A(x), B(x)\}, \forall x \in X.$
- 4. $(A \cap B)(x) = \min\{A(x), B(x)\}, \forall x \in X.$
- 5. $A^{c}(x) = 1 A(x)$, $\forall x \in X$ (here A^{c} denotes the complement of A).
- The constant fuzzy set in X taking value $\alpha \in [0, 1]$ will be denoted by α_x .

Definition (Chang 1968) A fuzzy topological space is a pair (X, τ) consisting of a nonempty set X and a family τ of fuzzy sets in X satisfying the following conditions:

- 1. 0_x , $1_x \in \tau$;
- 2. If $\{A_i : i \in \}$ is an arbitrary family of fuzzy sets in τ , then $i \in A_i \in \tau$. 76
- 3. If $A, B \in \tau$, then $A \cap B \in \tau$.

• Members of τ are called fuzzy open sets (or τ – fuzzy open sets) and a fuzzy set A in X is called closed if $A^c \in \tau$.

Definition (Srivastava et al. 1981) A fuzzy point x_{λ} ($0 < \lambda < 1$) in X is a fuzzy set in X such that

- $x_{\lambda}(x') = \lambda$, if x' = x
- 0, otherwise.

Here x and λ are, respectively, called the support and value of x_{λ} .

A fuzzy point x_{λ} is said to belong to a fuzzy set A if $\lambda < A(x)$ and two fuzzy points x_r and y_s in X are said to be distinct if x = y.

Definition (Zadeh 1965) Let *X* be a non-empty set.

Then a <u>fuzzy relation</u> R on X is a mapping $R: X \times X \rightarrow I$.

Definition (Klir and Yuan 1997) The transpose of a fuzzy relation *R* on a set *X* is the mapping $R^t : X \times X \rightarrow I$

given by $R^t(x, y) = R(y, x)$, for each $(x, y) \in X \times X$.

Definition (Zadeh 1971) The complement of a fuzzy relation *R* on a set *X* is the mapping $R^c: X \times X \rightarrow I$ given by

 $R^{c}(x, y) = 1 - R(x, y)$, for each $(x, y) \in X \times X$.

Knoblauch (2009) had introduced a topology generated by a binary relation. Here I study this concept mentioned in [3] in the case of fuzzy topology.

Definition [3]: Let *R* be a fuzzy relation on a set *X*. Then for $x \in X$, the fuzzy sets L_x and R_x , which are defined as

- Lx(y) = R(y, x), for all $y \in X$,
- Rx(y) = R(x, y), for all $y \in X$,

are called lower and upper contour, respectively of the element $x \in X$.

The fuzzy topology generated by the collection S_1 of all lower contours (i.e., $S_1 = \{L_x : x \in X\}$) will be denoted by τ_1 , and the fuzzy topology generated by the collection S_2 of all upper contours (i.e., $S2 = \{Rx : x \in X\}$) will be denoted by τ_2 .

<u>Definition</u> [3] : The fuzzy topology which is generated by the subbase $S = \{Lx\}_{x \in X} \cup \{Rx\}_{x \in X}$

is called the fuzzy topology generated by *R* and is denoted by τ_R .

3. Examples

1) Let R be a fuzzy relation on $X = \{x, y, z\}$, which is given as follows:

R	x	У	Z
x	1	0.5	0
У	0	1	0.8
Z	0.7	0	1

_ _ _ _ _ _ _

Then the fuzzy topology τ_R is generated by the following subbase S:

$$S = \{L_x, L_y, L_z, R_x, R_y, R_z\},\$$

where L_x , L_y , L_z , R_x , R_y , R_z are given by:

$L_x = \frac{1}{x} + \frac{0}{y} + \frac{0.7}{z}, L_y =$	$\frac{0.5}{x}$	$+\frac{1}{y}$	$-\frac{0}{z}$
$L_z = \frac{0}{x} + \frac{0.8}{y} + \frac{1}{z}R_x =$	$\frac{1}{x} + 0$	$\frac{0.5}{y}$ -	$+\frac{0}{z}$,

2) Let R be a fuzzy relation on $X = \{x, y, z\}$, which is given as follows:

R	Х	У	Ζ
Х	1	0.3	0.5
У	0	1	0
Ζ	0	0.9	1

Then the fuzzy topology δ_R is generated by the following subbase S:

 $S = \{L_{x}, L_{y}, L_{z}, R_{x}, R_{y}, R_{z}\},\$

where L_x , L_y , L_z , R_x , R_y , R_z are given by:

$$L_x = \frac{1}{x} + \frac{0}{y} + \frac{0}{z}L_y = \frac{0.3}{x} + \frac{1}{y} + \frac{0.9}{z},$$
$$L_z = \frac{0.5}{x} + \frac{0}{y} + \frac{1}{z}, R_x = \frac{1}{x} + \frac{0.3}{y} + \frac{0.5}{z},$$

4. Conclusions

In this work, I have studied the concept of a fuzzy topological space generated by a fuzzy relation as an extension of the corresponding concepts in Knoblauch (2009), Induráin and Knoblauch (2013) and Mishra and Srivastava (2018) respectively, for the crisp case. As a further study, the topological versions of representation theorems by using the fuzzy topologies induced by fuzzy relations and their interrelationship may be studied. Allam et al. (2008),have introduced some new methods, which are used to generate topologies by relations and studied the interrelationship between these methods and other methods.

Such type of problems may be studied in the context of the topological structures induced by fuzzy relations. 79

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A New Graph for Crossed Product of Groups

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In this study, we consider crossed product construction from view of Combinatorial Group Theory and define a new graph based on the crossed product of finite cyclic groups. Then we give some graph properties on this new graph, namely diameter, girth, maximum and minimum degrees, domination number, chromatic number and clique number. **Keywords:** Group Presentation, Graph Theory, Crossed Product

1. Introduction and Preliminaries

Crossed product construction appears in different areas of algebra such as Lie algebras, C*algebras and group theory. This product has also many applications in other fields of mathematics like group representation theory and topology. This product is important than the known group constructions since it contains direct, semi-direct, twisted and knit products of groups (Agore and Fratila, 2010). Here, by considering crossed product construction from view of Combinatorial Group Theory, we investigate the interplay between the crossed product over finite cyclic groups and the graph-theoretic properties of this extension in terms of its relations. By graph-theoretic properties, we are interested in diameter, girth, maximum, minumum degrees, domination number, chromatic number and clique number of the corresponding graph of this crossed product. In the literature, there are some important graph varieties and works related with them, for instance Cayley graphs, zero-divisor graphs. But the graph constructed here is different and interesting in terms of using the relations and normal form structure of elements of the crossed product of finite cyclic groups. We refer the reader to (Karpuz et al., 2013(a); Karpuz et al., 2013(b)) for some other new graphs obtained by presentations of given algebraic structures and to (Karpuz and Cetinalp, 2018) for studies on crossed product.

Let C_n and C_m be finite cyclic groups presented by $\langle a; a^n = 1 \rangle$ and $\langle b; b^m = 1 \rangle$, respectively. Then the crossed product of C_n by C_m , $C_n \#_a^f C_m$, has the following presentation

$$C_n \#^f_\alpha C_m = \left\langle a, b; a^n = 1, b^m = a^i, b^{-1}ab = a^j \right\rangle, \tag{1}$$

where $0 \le i \le n-1, 0 \le j \le m-1, i.(j-1) \equiv 0 \pmod{n}$ and $j^m \equiv 1 \pmod{n}$ (Agore and Fratila, 2010).

For any simple graph Γ , the distance (length of the shortest path) between two vertices u, v of Γ denoted by $d_{\Gamma}(u,v)$. The *diameter* of Γ is is defined by $diam(\Gamma) = \sup \{ d_{\Gamma}(u, v) : u \text{ and } v \text{ are vertices of } \Gamma \}$. The girth of a graph Γ is the length of a shortest cycle contained in Γ . However, if the graph does not contain any cycle, then the girth of it is assumed to be infinity. The degree $\deg_{\Gamma}(v)$ of a vertex v of Γ is the number of vertices adjacent to v. Among all vertices, the maximum degree $\Delta(\Gamma)$ (or the minumum degree $\delta(\Gamma)$) of Γ is the number of the largest (or the smallest) degrees in Γ . A subset D of the vertex set $V(\Gamma)$ is called a dominating set if every vertex $V(\Gamma) - D$ is joined to at least one vertex of D by an edge. Additionally, the *domination number* $\gamma(\Gamma)$ is the number of vertices in a smallest dominating set for the graph. The minumum number n for which Γ is *n*-colorable is called *chromatic number* of Γ , and is denoted by $\chi(\Gamma)$. The largest number of vertices in any clique (each of the maximal complete subgraphs) of Γ is called the *clique number* and denoted by $\omega(\Gamma)$. In general, it is well known that $\chi(\Gamma) \ge \omega(\Gamma)$ for any graph Γ (Gross, 2004).

2. A New Graph Based on Crossed Product of Finite Cyclic Groups

In the following, we define an undirected graph $\Gamma_{C_n \#_{\alpha}^f C_m} = (V, E)$ associated with presentation given in (1) which all results will be constructed on it. The vertex set V of the graph $\Gamma_{C_n \#_{\alpha}^f C_m}$ consists of the followings:

- generators of $C_n \#^f_{\alpha} C_m$ (*a* and *b*) and the identity element $1_{C_n \#^f_{\alpha} C_m}$,
- words of the form a^i $(1 \le i \le n-1)$ and b^j $(1 \le j \le m-1)$,
- words of the form $a^i b^j$ $(1 \le i \le n-1, 1 \le j \le m-1)$.

In fact this vertex set consists of the normal form structure of elements of $C_n \#^f_{\alpha} C_m$. The edge *E* of the graph $\Gamma_{C \#^f C}$ consists of the followings:

• connect each vertex a^i with a^{i+1} for all $1 \le i \le n-2$ $(a^i \square a^{i+1})$, 82

- connect each vertex b^{j} with b^{j+1} for all $1 \le j \le m-2$ $(b^{j} \square b^{j+1})$,
- connect each vertices a^i and b^j with the vertex $a^i b^j$ from both sides for all $1 \le i \le n-1$ and $1 \le j \le m-1$ $(a^i \square a^j b^j \square b^j)$,

• connect the unique vertex
$$1_{C_n \#_{lpha}^f C_m}$$
 with all vertices

$$(a^{i} \Box 1_{C_{n} \#_{\alpha}^{f} C_{m}} (1 \le j \le n-1), b^{j} \Box 1_{C_{n} \#_{\alpha}^{f} C_{m}} (1 \le j \le m-1) \text{ and } a^{i} b^{j} \Box 1_{C_{n} \#_{\alpha}^{f} C_{m}} (1 \le i \le n-1, 1 \le j \le m-1))$$



Figure 1. Crossed product graph $\Gamma_{C \# C}$

As seen in Figure 1, the numbers of vertex and edge sets depend on the orders of generators of $C_n \#^f_\alpha C_m$. Therefore, we have

$$V = \{1, a, a^{2}, ..., a^{n-1}, b, b^{2}, ..., b^{m-1}, ab, ab^{2}, ..., ab^{m-1}, a^{2}b, a^{2}b^{2}, ..., a^{2}b^{m-1}, a^{3}b, a^{3}b^{2}, ..., a^{3}b^{m-1}, ..., a^{n-1}b, a^{n-1}b^{2}, ..., a^{n-1}b^{m-1}\}$$

and thus $\left|V\left(\Gamma_{C_{n}\#_{\alpha}^{f}C_{m}}\right)\right| = \left|C_{n}\#_{\alpha}^{f}C_{m}\right| = mn$ and $\left|E\left(\Gamma_{C_{n}\#_{\alpha}^{f}C_{m}}\right)\right| = 3mn - m - n - 3$.

3. Graph Theoretical Results over $\Gamma_{C_n \#_{c_n}^{f}}$

By considering the graph $\Gamma_{C_n\#_{\alpha}^f C_m}$ drawn in Figure 1, we will mainly deal with some graph properties, namely diameter, girth, maximum, minumum degrees, domination number, chromatic number and clique number of $\Gamma_{C_n\#_{\alpha}^f C_m}$.

Theorem 3.1 The diameter of the graph $\Gamma_{C,\#/C_{-}}$ is 2.

Proof. By Figure 1, it is seen that the vertex $1_{C_n \#_{\alpha}^f C_m}$ is connected with all other vertices of the forms $a^i (1 \le j \le n-1)$, $b^j (1 \le j \le m-1)$ and $a^i b^j (1 \le i \le n-1, 1 \le j \le m-1)$. Thus we can reach to all vertices in $\Gamma_{C_n \#_{\alpha}^f C_m}$ by using the vertex $1_{C_n \#_{\alpha}^f C_m}$. So we get $diam(\Gamma_{C_n \#_{\alpha}^f C_m}) = 2$.

Theorem 3.2 The girth of the graph $\Gamma_{C_{a}\#_{a}^{L}C_{a}}$ is 3.

Proof. By Figure 1, we have four types cyclec of the forms $1_{C_n \#_{\alpha}^f C_m} \square a^i \square a^{i+1} \square 1_{C_n \#_{\alpha}^f C_m}$ $(1 \le i \le n-2)$, $1_{C_n \#_{\alpha}^f C_m} \square b^j \square b^{j+1} \square 1_{C_n \#_{\alpha}^f C_m}$ $(1 \le j \le m-2)$, $1_{C_n \#_{\alpha}^f C_m} \square a^i b^j \square a^i \square 1_{C_n \#_{\alpha}^f C_m}$ and $1_{C_n \#_{\alpha}^f C_m} \square a^i b^j \square b^j \square 1_{C_n \#_{\alpha}^f C_m}$ $(1 \le i \le n-1, 1 \le j \le m-1)$. So the length of the shortest cycle contained in the graph $\Gamma_{C, \#_{\alpha}^f C_m}$ is 3.

Theorem 3.3 The maximum and minumum degrees of the graph $\Gamma_{C_n \#_{\alpha}^f C_m}$ are

$$\Delta(\Gamma_{C_n \#_{\alpha}^f C_m}) = mn - 1 \quad \text{and} \quad \delta(\Gamma_{C_n \#_{\alpha}^f C_m}) = 3, \text{ respectively.}$$

Proof. By the vertex definition of the graph $\Gamma_{C_n\#_a^f C_m}$, we have $\left| V(\Gamma_{C_n\#_a^f C_m}) \right| = \left| C_n \#_a^f C_m \right| = mn$. By Figure 1, we have $v \Box 1_{C_n\#_a^f C_m}$ for all $v \in V(\Gamma_{C_n\#_a^f C_m})$. Because of this, we get mn-1 vertex adjacent to the vertex $1_{C_n\#_a^f C_m}$. Hence $\deg_{\Gamma_{C_n\#_a^f C_m}}(1_{C_n\#_a^f C_m}) = mn-1$. Besides, the number of the largest degrees in a simple graph which has mn vertex is mn-1, we have $\Delta(\Gamma_{C_n\#_a^f C_m}) = mn-1$. For the minumum degree of $\Gamma_{C_n\#_a^f C_m}$, we consider the vertex of the form $a^i b^j$ $(1 \le i \le n-1, 1 \le j \le m-1)$. By Figure 1, we have $a^i \Box a^i b^j \Box b^j$ for this type of vertex. Every vertex is also connected with the vertex $1_{C_n\#_a^f C_m}$. For the graph $\Gamma_{C_n\#_a^f C_m}$, since the minumum degree depends on the vertex of the form $a^i b^j$, we obtain $\delta(\Gamma_{C_n\#_a^f C_m}) = 3$.

Theorem 3.4 The domination number of the graph $\Gamma_{C,\#^{\ell}C_{m}}$ is 1.

Proof. By Figure 1, we have $v \Box 1_{C_n \#_{\alpha}^f C_m}$ for all $v \in V(\Gamma_{C_n \#_{\alpha}^f C_m})$. So $\left\{1_{C_n \#_{\alpha}^f C_m}\right\}$ is a dominating set. By the definition of the graph $\Gamma_{C_n \#_{\alpha}^f C_m}$, the smallest dominating set is $\left\{1_{C_n \#_{\alpha}^f C_m}\right\}$. Hence, we get $\gamma(\Gamma_{C_n \#_{\alpha}^f C_m})=1$.

Theorem 3.5 The chromatic number $\chi(\Gamma_{C,\#/C_{-}})$ is equal to 4.

Proof. By Figure 1, we have $v \Box 1_{C_n \#_a^f C_m}$ for all $v \in V(\Gamma_{C_n \#_a^f C_m})$. That means if we label the vertex $1_{C_n \#_a^f C_m}$ by color A, then all other vertices have different colors. Let us suppose that the colors for the vertices a^i and b^j are labeled by B and C, respectively. Since these vertices are connected to $a^i b^j$ we have a different color labeled by D for the vertex $a^i b^j$. So the graph $\Gamma_{C,\#_a^f C_m}$ has 4 minumum number of colors for its vertices. Thus, $\chi(\Gamma_{C,\#_a^f C_m})=4$.

Theorem 3.6 The clique number $\omega(\Gamma_{C_n \#_d C_m})$ is equal to 4.

Proof. According to Figure 1, there exist one type of complete subgraph which has the largest number of vertices. This subgraph has edges of the form $1 \square a^i \square a^i b^j \square b^j \square 1$. Thus, we get $\omega(\Gamma_{C_n \#_a^j C_m}) = 4$.

4. Conclusions

In this study, we defined a new graph based on the crossed product of finite cyclic groups. Then we obtained some graph properties on this new graph. The importance of this graph is that it has been defined by using the normal form structure of elements of crossed product construction.

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The use of textbook in teaching practices : Case of mathematical proof teaching

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Abstract

Our study falls within the general theoretical framework of the dual didactic and ergonomic approach developed by Aline Robert and Janine Rogalski (2002). It aims to identify the representations of Tunisian teachers about the official mathematics textbook and their didactic choices relative to its use in the development of a lesson or during its concretization in class, particularly in the case of teaching mathematical proof.

In order to be able to bring replies to our questions, we adopted a methodology of research held in two times:

- In a first time, we have developped a global study focused on "speech on the practices" whose objective is to inform us on teacher representations and some of their choices relatively to the use of the textbook.

- In a second time, we have conducted a local study: it concerns the effective teaching practices based on the observation of a sequence presenting a mathematical proof.

Keywords: Teaching practices, Textbook, Mathematical proof.

1. Introduction

Before 1990, we find few French and Tunisian researches explicitly interested in the study of the teaching practices where the favored pole of study is the teacher. This kind of researches are developed gradually from year 1989 and much more clearly since 1993 (Comiti and Grenier, 1995; Bosch and al, 2003). It's also the case of the general theoretical framework of the dual didactic and ergonomic approach developed by Aline Robert and Janine Rogalski (Robert, 2001 ; Robert and Rogalski, 2002 ; Rogalski, 2003). These authors chose to close to a didactic analysis of the work of the pupil and the organization of this work by the teacher, an analysis of the teacher as exercising a job, by using concepts of cognitive ergonomics to interpret the practices of the teachers as the expression of a work in an open dynamic environment. The complexity of the practices brings the authors to be a study of the study of the study of the study of the practices brings the authors to be a study of the study of the study of the study of the practices brings the authors to be a study of the study of the study of the study of the practices brings the authors to be a study of the study of the study of the study of the practices brings the authors to be a study of the st

distinguish five components of practices: the study of the two first components (cognitive and mediative components) can provide information on the operating logic of the teacher and the study of the three other components (institutional, personal and social components) can help to enlighten on regularities in the practices.

Our study falls within this general theoretical framework. We suggest to center, more particularly, on the teacher representation of the textbook and on his didactic choices relative to its use in the elaboration of a lesson or during its realization in class. Note that, in Tunisia, the studies in didactics of mathematics approaching the use of the textbook are rare and the theme, in spite of its importance remains untidy. So, let us consider, by this work, to put the light on the relationship of maths teachers to the textbook, what could contribute to a better use of this support.

2. Materials and Methods

Our interest concerns the textbook of the level third « Maths » (17-year-old pupils specialized in mathematics). When appeared in 2009, this latter was highly contested and has generated debate between teachers and authors due to the new presentation of the course contents, based exclusively on activities. In addition, this textbook is full of various mathematical notions introduced at this crucial level. Furthermore, it's the unique official book intented for the pupils and represents, in this way, the only curriculum reading defining teaching contents and circumscribing pedagogic orientations. Therefore, it has an important scientific, institutional and social power. As already said, we aim to identify the representations of teachers about this official textbook and their didactic choices relative to its use in the development of a lesson or during its concretization in class. To this end, we aspire to bring replies to the following question:

"What are the transpositive choices of the teachers *(margin of discretion)* to reconcile institutional injunctions and those imposed by the reality of the class *(constraints)*?"

The research methodology adopted is held in two times:

- In a first time, we develop a global study focused on "speech on the practices" whose objective is to inform us on teacher representations and some of their choices relatively to the use of the textbook. The tool of data collection is a questionnaire proposed to the teachers

- In a second time, we conduct a local study : it concerns the effective teaching practices based on the observation of a sequence related to the lesson « Derivative number ». The objective of this study is to refine the results stemming from the global study.

In this article, we present only results concerning the analytic study based on a questionnaire. This latter is composed of 21 questions which can be divided into three great groups QT1, QT2 and QT3. The questions QT1 relate mainly to the position of textbook, as an institutional standard. The questions QT2 concern the kind of teacher management of different parts composing the textbook and the questions QT3 are specific to the lesson « Derivative number ».

3. Results and Discussions

For conciseness, we limit to give, in what follows, only the results relative to the question 13 of the questionnaire, concerning the working modality of proof activities in the classroom. Note that the textbook of the level third Maths presents the proof of theorems by means of activities in order to give to the pupils the opportunity to take part in building mathematical knowledge, to reason, to learn new methods and tricks and to get used to the mathematical rigor. It's important for us to identify the importance of the proof activities in the practices of the quizzed teachers.

The question 13, proposed to 93 teachers quizzed, runs as follows :

"In most cases, in the classroom, the modality followed when performing a proof activity is :

- The teacher gives the solution of the activity without prior individual work of pupils ;

- The teacher gives the solution of the activity after prior individual work of pupils ;

- The solution of the activity is the result of collaborative work without prior individual work of pupils;

- The solution of the activity is the result of collaborative work after prior individual work of pupils.

- Otherwise ;

specify:	
Justify this choice	"

The teachers' sample composed by the 93 teachers answering to the questionnaire can be described by the following pie charts: 88



The answers of the teachers to the question 13 can be so summarized:

Code of modality	Modality of work of proof activities	Percentage
MDı	The teacher gives the solution of the activity without prior individual work of pupils	3%
MDa	The teacher gives the solution of the activity after prior individual work of pupils	7%
MD3	The solution of the activity is the result of collaborative work without prior individual work of pupils	14%
MD4	The solution of the activity is the result of collaborative work after prior individual work of pupils.	63%
MD23	Mixture of modalities 2 : 3	2%
MD24	Mixture of modalities 2;4	8%
MD34	Mixture of modalities 3:4	2%
MD234	Mixture of modalities 2; 3; 4	1%

As indicated in the table above, the greatest choice of the teachers is the modality MD4 (63%). These latter prefer to do the proof activities in the classroom after prior individual work of pupils and the solution of the activity is proposed after collaborative discussion and work. These results can be interpreted as follows:

- In terms of mediative component : the teachers attach importance to the engagement of pupils.
- In terms of cognitive component : the knowledge building is seriously considered.
- The justifications given by the teachers show that they award a particular importance to the pupils work and they want to provide them opportunities for reasoning, using acquired knowledge, exchanging and correcting ideas.

Seventeen percent of teachers declare that they don't let pupils work on proof activities individually, before proposing a solution to these activities (MD1, MD3). They explain their choice by constraints, which are mainly institutional: Lack of time, heaviness of the

curriculum, great staff of pupils in the class, difficulty and lenght of proof activities, willingness to propose rigorous proofs well formulated.

4. Conclusions

Based on a little part of our research, this article aim to show how, via a questionnaire proposed to tunisian maths teachers of level third Maths, concerning how do they use the textbook in the classroom, we were able to identify some components of their practices, in the particular case of proof teaching and learning. The questionnaire informed us mainly and partly on the three first components of practices concerning the use of textbook :

The cognitive component,

The mediative component,

The institutional component.

Note that it's difficult to highlight the personal and the social components of practices just from the results of the analytic study. These components need to be identified from the invariants of cognitive and mediative components detected in the observation of teacher practices on a long period, what we aspire to make in our later researches.

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An Induced Isometry on a Total Space of a Vector Bundle

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Abstract

Let (E, π, M) be a vector bundle. There exists a specific Riemannian metric on E which is induced from a given Riemannian metric g on M. In this study, we use this special metric and define an isometry on the total space of the vector bundle E. We find these structures on the tangent bundle of S^1 .

Keyword(s): Fiber bundles, Isometry Group, Vector Bundles, Principal Bundles.

1. Introduction

Let (E, π, M, V) be a vector bundle, where M is an n- dimensional Riemannian manifold. It is shown in [5] that one can construct a Riemannian metric on an abstract bundle by using the notion of partitions of unity. Since vector bundles are special kind of bundles, such metric can be constructed on E as follows:

$$\overline{g}_E(\overline{V}, \overline{W}) = g(\pi_*(V), \pi_*(W)) + g_V(\gamma_*(V), \gamma_*(W))$$
(1)

where g is the Riemannian metric on M, and $\gamma = pr_2 \circ \Phi$ where Φ represents the local trivialization of the bundle. Here \bar{g}_E is called the "induced metric" on E. It is shown that if E is a vector bundle, then g_V is the usual metric (or inner product) on the vector space $TV = V \times V$. Therefore equation (1) transfers into following equation:

$$\overline{g}_E(\overline{V}, \overline{W}) = g(\pi_*(V), \pi_*(W)) + \langle \gamma_*(V), \gamma_*(W) \rangle$$
(2)

where \langle , \rangle represents the usual inner product on $V \times V$.

The main purpose of this paper is to use this special metric to define an isometry on a trivial bundle E which is induced from the base manifold M. As an example, we also construct such structures on tangent bundle of S^1 . We assume all manifolds are Hausdorff, second countable, connected.

2. Theory

In this section, we apply the definitions and theorems in [6] to trivial bundles. Suppose that, Φ is a trivialization on a trivial bundle E. Let f be an isometry on M, and $h_1, h_2 \epsilon E$. If $h_1 = h_2$, then they have to be in the same fiber. Suppose that $h_1, h_2 \epsilon \pi^{-1} \{x\}$, then we have⁹¹

$$(f \circ \pi, pr_2 \circ \Phi)(h_1) = (f \circ \pi, pr_2 \circ \Phi)(h_2).$$
(3)

Since Φ is a diffeomorphism, then the following equation holds:

$$\Phi^{-1}((f \circ \pi, pr_2 \circ \Phi))(h_1) = \Phi^{-1}((f \circ \pi, pr_2 \circ \Phi))(h_2).$$
(4)

Equation (4) shows that $\Phi^{-1}((f \circ \pi, pr_2 \circ \Phi))$ is well defined, and can be expressed as a function from E to itself. Now we give the formal definition of above function:

Definition 1. Let f be an isometry on M. We define a function $F: E \rightarrow E$ as follows:

$$F(h) = \Phi^{-1}((f \circ \pi, pr_2 \circ \Phi))(h).$$
(5)

Theorem 2. F is an isometry with respect to the induced metric g_E on the total space E.

Proof. The more general proof of this theorem can be found in reference [6].

3. Application on a Cylinder

Now we apply the concepts on cylinder. It is well known that a cylinder is the tangent bundle of S^1 , and tangent bundle is a vector bundle.

We view S^1 as the unit circle in IR^2 . Then is a submanifold of IR^4 , where

$$T(S^{1}) = \{(x, v) \in S^{1} \times IR^{2} : v \in T_{x}(S^{1})\} = \{(x, v) \in S^{1} \times IR^{2} : \langle x, v \rangle = 0\}.$$

The right hand side of the formulation suggests that $v \perp x$ in hence in IR^2 . Therefore $\langle x, v \rangle = 0$ and $x_1^2 + x_2^2 = 1$ suggests that $v = (v_1, v_2) = \lambda(x_2, -x_1)$. The last equation shows that x and λ completely determines the tangent vector v. This defines the trivialization Φ of $T(S^1)$.

The trivialization $\Phi: T(S^1) \longrightarrow S^1 \times IR$, is defined as follows:

$$(x, v) \rightarrow \Phi(x, v) = (x, \lambda)$$

Suppose that $f: S^1 \to S^1$ be an isometry. The corresponding function

 $F:TS^1 \to TS^1$ is as follows:

$$F(x, v) = \Phi^{-1}((f \circ \pi)(x, v), (pr_2 \circ \Phi)(x, v))$$

= $\Phi^{-1}(f(x), \lambda)$
= $(f(x), \lambda(y_2, -y_1))$

where $x = (x_1, x_2)$, $v = (v_1, v_2) = \lambda(x_2, -x_1)$, and $(y_1, y_2) = f(x)$. So, we define

 $F:TS^1 \to TS^1$ as $F(x,v) = (f(x), \overline{v})$, where $\overline{v} = (\overline{v}_1, \overline{v}_2) = \lambda(y_2, -y_1)$.

Now we give an example in finding the induced isometry of a given function f on S^1 .

Example: Let $f: S^1 \to S^1$ defined as

 $f(x_1, x_2) = (x_1 \cos\theta - x_2 \sin\theta, x_1 \sin\theta + x_2 \cos\theta)$

It is indeed an isometry on S^1 . Then, by definition 1, we have

 $F(x, v) = (y_1, y_2, \bar{v}_1, \bar{v}_2)$

where $(\bar{v}_1, \bar{v}_2) = \lambda(y_2, -y_1)$, and $(y_1, y_2) = (x_1 \cos\theta - x_2 \sin\theta, x_1 \sin\theta + x_2 \cos\theta)$. It is

indeed a function on S^1 , because $y_1^2 + y_2^2 = 1$ and

$$\begin{aligned} \langle y, \bar{v} \rangle &= (x_1 cos\theta - x_2 sin\theta)(\lambda y_2) + (x_1 sin\theta + x_2 cos\theta)(-\lambda y_1) \\ &= \lambda [(x_1 cos\theta - x_2 sin\theta)(x_1 sin\theta + x_2 cos\theta) - (x_1 sin\theta + x_2 cos\theta)(x_1 cos\theta - x_2 sin\theta)] \\ &= 0 \end{aligned}$$

On the other hand, if we let f(x) = y, f(z) = t, and $(\overline{v}_1, \overline{v}_2) = \lambda(y_2, -y_1)$, $(\overline{w}_1, \overline{w}_2) = \sigma(t_2, -t_1)$. we have

On the other hand,

Since right side of the equations are equal, then $\langle F(x, v), F(z, w) \rangle = \langle (x, v), (z, w) \rangle$ which concludes that F is an isometry.

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An Algorithm for Classifying Nilsoliton Metrics with Singular Gram Matrix

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In this work, we introduce an algorithmic approach in classifying nilpotent Lie algebras endowed with soliton metrics. This paper is a continuation of our paper in Journal of Symbolic Computation 50 (2013), 350 - 373. In our previous paper, we classified nilsoliton metric Lie algebras with nonsingular Gram matrix. In this paper, we consider same type of metric with singular Gram matrix. We give some detailed algorithms regarding to compute some desired properties of a Lie algebra.

Keyword(s): Nilsoliton Metrics, Nilpotent Lie algebras, Simple Derivation

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1. Introduction

In this paper, we focus on metric nilpotent Lie algebras. Considering all metrics on a nilpotent Lie algebra, nilsoliton metrics are the most preferable since non abelian nilpotent Lie groups do not admit Einstein metrics. A metric on a nilpotent Lie algebra is call "nilsoliton metric" if its Ricci endomorphism differs from a derivation D by a scalar multiple of the identity map, that is $D = Ric - \beta Id$ where D and β is called "nilsoliton derivation" and "nilsoliton constant" respectively. Also if the nilsoliton derivation has n different real eigenvalues, we call the derivation as a "simple derivation". Here, our main purpose is to develop an algorithm for the classification of finite dimensional nilpotent Lie algebras endowed with nilsoliton metric with simple nilsoliton derivation. For more information regarding to the nilmanifolds, please refer to [5,6,7,8, and 9]. Classification results in various dimensions can be found in [1,2,3 and 4].

Throughout this paper, we give some detailed algorithms named **Badpairs**, **Goodpairs**, **Invertible**, and **SimpleD**. We haven't finished some of the algorithms for the complete classification yet. Thus the algorithms given in this paper is not the entire list of algorithms that are needed in classifying nilsoliton metrics with simple derivations. We plan to take care of this issue in our continuation paper(s).

2. Basic Notations

Now let's define some combinatorial objects associated to a set of integer triples:

$$\Lambda = \{ (i, j, k) \colon 1 \le i, j, k \le n \}$$
(1)

- i. For $1 \le i, j, k \le n$, we define $1 \times n$ matrix (a row vector) $y_{ij}^k = e_i + e_j e_k$ where $\{e_i : 1 \le i \le n\}$ is the standard orthonormal basis for IR^n . We call y_{ij}^k as the 'root vectors for Λ '.
- ii. Let y_1, y_2, \dots, y_m be be an enumeration of the root vectors in dictionary order. We define root matrix Y_{Λ} to be the $m \times n$ matrix whose rows are the root vectors.
- iii. Gram matrix U_{Λ} is $m \times m$ symmetric matrix, whose (i,j) th entry is the inner product of the i th and j th root vectors.

3. The Algorithm

In order to use the computational procedure, we need to represent the Lie algebras, subalgebras, or any other element regarding to the Lie algebra itself such a way that they can be dealt with by computer. For this purpose, we use structure constants. It is well known that any n-dimensional Lie algebra can be represented by its structure constants c_{ij}^k for $1 \le i, j, k \le n$, that satisfies the Jacobi identity condition. The Lie product of two elements of this Lie algebra is completely determined by these structure constants. In this section, we also mention about theorems/ lemmas on which the algorithm based.

Algorithm for Listing $\Lambda = \{(i, j, k): 1 \le i, j, k \le n\}$ and Its Subsets:

Before mentioning these algorithms, we need to write all possible triples as the rows of a matrix (which will be of type $\binom{n}{3} \times 3$). We call it as Z matrix. By using this matrix, one can easily compute greatest Gram matrix U_{Λ} . Additionally, we compute all possible sub matrices of this Z matrix by coding each row of Z as 1 in a logical matrix W. That means, for example: If sub matrix of Z is the first 3 rows of Z when $|\Lambda| = 6$, then the corresponding row of W matrix will be $[1 \ 1 \ 1 \ 0 \ 0 \ 0]$.

Now we give the algorithms:

a. **Badpairs:** This algorithm is designed for computing a matrix with two columns such that the rows of this matrix correspond to the entry 2 in the Gram matrix U. This algorithm is based on Lemma 2.7 in [3].

Input: Z matrix

<u>Output:</u> A $r \times 2$ matrix such that for any row [i j], the (i,j) th entry of corresponding Gram matrix is 2.

Step1: Compute the Gram matrix U for Z. Step2: Find #(i,j) such that (i,j) th entry of U is 2.

b. Goodpairs: This algorithm is for computing the matrix with two columns whose rows correspond to -1 entry in the Gram matrix. This algorithm is based on Lemma 2.8 in [3].

Input: Z matrix

<u>Output:</u> A $r \times 2$ matrix such that for any row [i j], the (i,j) th entry of corresponding Gram matrix is 2.

Step1: Compute the Gram matrix U for Z.

Step2: Find #(i,j) *such that* (i,j) *th entry of* U *is -1.*

c. Invertible: This algorithm is for pruning the rows of W such that the row corresponds to an invertible Gram matrix.

Input: A logical matrix W.

Output: Eliminated W matrix.

Step1: Use the # *of rows of* W*, name it as* q*. Use for loop for* $1 \le i \le q$ *, and compute corresponding Gram matrix* U*.*

Step2: If U is invertible, then eliminate that row from W matrix.

d. SimpleD

Input: Logical matrix W.

<u>Output:</u> A logical matrix whose rows corresponds to the Lie algebras with simple nilsoliton derivation.

Step1: Use the # of rows of W, name it as q. Use for loop for $1 \le i \le q$, and compute corresponding Gram matrix U, and the vector v; which is one of the solutions of the system $Uv = [1]_m$. Here $[1]_m$ represents $m \times 1$ matrix such that all the entries are 1s.

Step2: Compute the eigenvalues of nilsoliton derivation D.

Step3: Eliminate rows of W if the eigenvalues are not distinct.

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The Solution of Water Wave Equation Under the Galerkin Method

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Abstract

We consider the existence of solutions of water wave equation using the standard Faradeo–Galerkin method. Using the Galerkin method, we establish the existence of solutions of the problem. In particular we deal with the water wave equations with a logarithmic nonlinearity. We see the main theorem for the existence of solutions for this type of equations. Using an orthogonal base of the space, we search for an approximate solution and we prove the theorem.

Keywords: existence of solution; approximate solution; numerical-type of models; standard Galerkin method; water wave equation

1. Introduction

We consider the water wave equation. We consider the finite volume method, in particular the Galerkin method. We consider the integral form of the conservation law

$$\frac{d}{dt} \int_{x_1}^{x_2} q(x,t) dx + f(q(x_2,t)) - f(q(x_1,t)) = 0$$
(1.1)

We want to obtain a model for non-linear equations. In the first part using the fully non-linear model for irrotational water waves in the form (see [1], [2]) given as

$$0 = \delta L = \delta \iint L \, dx \, dt \tag{1.2}$$

Dingemans (1997) describes several methods with positive-definite Hamiltonian, but these methods are quite tedious and have certain ambiguities regarding the order of certain operators, (see [3], [4]). The present method leads to a positive-definite Hamiltonian and can be fully non-linear if desired. The present model is an additional elliptic equation in the horizontal plane has to be solved (see [6]). High-order non-linear models solve free-surface evolution equations derived from a Hamiltonian under the constraint that the Laplace equation is satisfied exactly in the interior of the fluid domain (see [7]).

In the second part we deal with the existence and decay of solutions of the following problem

$$u_{tt} + Au + u + h(u_t) = kuln|u|$$

$$(1.3)$$

with boundary conditions

$$u(x,t) = \frac{\partial u}{\partial v}(x,t) = 0, \quad x \in \partial \Omega, \quad t > 0$$

$$u(x,0) = u_0(x); \qquad u_t(x,0) = u_1(x) \tag{1.4}$$

M. Al-Gharabli And S. A. Messaoudi J. Evol. Equ. and established the existence and uniqueness of the solution for the Cauchy problem. Hiramatsu et al. [9] introduced the following equation

 $utt - u + u + ut + |u|2u = u \ln |u|$ (1.5)

to study the dynamics, Q-ball in theoretical physics.

2. Materials and Methods

Definition 2.1. (weak solution of eq. (2.1))

$$u_{tt} + Au + u + h(u_t) = kuln|u|$$
 (2.1)

A continuous function u = u(t, x) is a global weak solution to the Cauchy problem (1.2) if: $u = u(t, x) \in C((0, \infty) \times \Omega) \cap L^{\infty}(R, H^{m}(\Omega))$ and $||u||_{H^{m}(\Omega)} \leq ||u_{0}||_{H^{m}(\Omega)} \quad \forall t > 0$ u(t, x) satisfies equation (1-2) in the sense of distributions.

Lemma 2.2..Logarithmic Sobolev inequality

(see [13,14]). Let **u** be any function in $H_0^m(\Omega)$ and a > 0 be any number. Then

$$2\int_{\Omega} |\mathbf{u}|^2 \ln|\mathbf{u}| \, dx \le \frac{1}{2} \|\mathbf{u}\|^2 \ln\|\mathbf{u}\|^2 + \frac{ca^2}{2\pi} \|A\mathbf{u}\|^2 - (1+\ln a)\|\mathbf{u}\|^2$$
(2.2)

Lemma 2.3. Logarithmic Gronwall inequality

(see [8]). Let c > 0 and $\gamma \in (0, T, \Omega)$. Let ω be any function $\omega: [0, T[\rightarrow [1, \infty[$ satisfies

$$\omega \le c \left(1 + \int_0^t \gamma(s) \,\omega(s) \ln \omega(s) \,ds\right), \ 0 \le t \le T \text{, then}$$
$$\omega \le c \exp\left(c \int_0^t \gamma(s) \,ds\right), \ 0 \le t \le T \tag{2.3}$$

Lemma 2.4. The Cautchy – Schwartz inequlity

Recall: For the Hilbert space with a norm (u, v) and its resulted norm $||(u, v)|| = \sqrt{(u, v)}$, than the Cauchy-Schwartz inequation is $|u(x), v(x)| \le ||u|| ||v||$.

3. Results and Discussions Galerkin method for existence of solutions

Theorem 3.1

Let $(u_0, u_1) \in H_0^m(\Omega) \times L^2(\Omega)$. Then, problem of equations (2.1) has a global week solution as $u = u(t, x) \in C((0, T), H_0^m(\Omega) \cap C^1(0, T), L^2(\Omega) \cap C^2(0, T), H^m(\Omega))$

Proof: To proof the theorem we consider the standard Faedo-Galerkin method. We take an orthogonal basis of the space $H_0^m(\Omega)$ in the form $\{\omega_j\}_{j=1}^{\infty}$. This is othonormal in $L^2(\Omega)$. Let $V_m = span\{\omega_1, \omega_2, \dots, \omega_m\}$ and let the projections of the initial data on the subspace V_m be given by

$$u_0^m(x) = \sum_{j=1}^m a_j \omega_j(x) , \ u_1^m(x) = \sum_{j=1}^m b_j \omega_j(x)$$

where $u_0^m \to u_0$ in $H_0^m(\Omega)$ and $u_1^m \to u$ in $L^2(\Omega)$, as $m \to \infty$.

We search for an approximate solution $u^m(x,t) = \sum_{j=1}^m g_j^m(t)\omega_j(x)$ of the approximate problem in V_m

$$\begin{cases} \int_{\Omega} (u_{II}^{m} w + \Delta u^{m} \Delta w + u^{m} w + h(u_{t}^{m}) w) dx = \int_{\Omega} w u^{m} ln |u^{m}|^{k} dx, \ w \in V_{m} \\ u^{m}(0) = u_{0}^{m} = \sum_{j=1}^{m} (u_{0,} w_{j}) w_{j} \\ u_{1}^{m}(0) = u_{1}^{m} \sum_{j=1}^{m} (u_{1,} w_{j}) w_{j} \end{cases}$$
(3,1)

This leads to a system of ODE_s for unknown functions $g_j^m(t)$. Based on standard excistence theory for ODE, one can obtain functions:

$$g_j: [0, t_m) \to R, \qquad j = 1, 2, ..., m,$$

which satisfy (3,4) in a maximal interval $[0, t_m), t_m \in (0, T]$. Then, using Cauchy-Schwarz' inequality, we get

$$\|u^{m}(t)\|_{2}^{2} \leq 2\|u^{m}(0)\|_{2}^{2} + 2\left\|\int_{0}^{t} \frac{\partial u^{m}}{\partial_{s}}(s)ds\right\|_{2}^{2}$$
$$\leq 2\|u^{m}(0)\|_{2}^{2} + 2T\int_{0}^{t}\|u_{t}^{m}(s)\|_{2}^{2}ds \qquad (3.2)$$

$$\|u^{m}(t)\|_{2}^{2} \leq 2\|u^{m}(0)\|_{2}^{2} + 2TC\left(1 + \int_{0}^{t} \|u^{m}\|_{2}^{2}\ln\|u^{m}\|_{2}^{2}\,ds\right)^{101}$$
(3.3)

Applying the Logarithmic Gronwall inequality to the last inequality, we obtain the following estimate

 $\|u^m\|_2^2 \le 2C_1 e^{2C_1 T} \le 2C_2$

Hence, from the inequality (3.8) it follows that:

$$\|u_t^m\|_{L^2(\Omega)}^2 + \|\Delta u^m\|_{L^2(\Omega)}^2 + \|u^m\|_{L^2(\Omega)}^2 \le C_3$$

where C_3 is a positive constant independent of m and t. This implies

$$\sup_{t \in (0,t_m)} \|u_t^m\|_{L^2(\Omega)}^2 + \sup_{t \in (0,t_m)} \|\Delta u^m\|_{L^2(\Omega)}^2 + \sup_{t \in (0,t_m)} \|u^m\|_{L^2(\Omega)}^2 \le C_4$$
(3.4)

Therefore, $u^m(x, 0)$ makes sense and $u^m(x, 0) \rightarrow u(x, 0)$ in $L^2(\Omega)$ Also, we have $u^m(x, 0) = u_0^m(x) \rightarrow u_0(x)$ in $H_0^m(\Omega)$ Hence, $u(x, 0) = u_0(x)$ So, $u_t^m(x, 0)$ makes sense and $u_t^m(x, 0) \rightarrow u_t(x, 0)$ in $H^{-m}(\Omega)$ But $u_t^m(x, 0) = u_1^m(x) \rightarrow u_1(x)$ in $L^2(\Omega)$ Hence, $u_t(x, 0) = u_1(x)$.

4. Conclusions

The finite element and Galerkin methods are currently the standard numerical technique in use to solve various nonlinear problems.

We show that $t_m = T$ and that the local solution is uniformly bounded independent of m and t. So, the approximate solution is uniformly bounded independent of m and t. Therefore, we can extend t_m to T.

The methods retain the advantages of weak formulations, which lower the continuity requirements of matching elements and permits to use simple basis functions.

However, these methods demand a great amount of numerical integration.
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Numerical Methods on Approximation of Solutions Using a Linear Operator

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Abstract

We consider water wave equation. We use a linear combination and a linear operator to the solution of this equation. Using the Galerkin method we give the idea to approximate the solution with a polynomial on water wave equation. The polynomial satisfies the water wave equation using a set of parameters. Giving the exact solution and the approximation of solution we can compare the exact error.

Keywords: Linear operator; approximate, solutions; water wave equation; linear parameters; Galerkin method

1. Introduction

We consider the idea to approximate the solution with a polynomial involving a set of parameters. The polynomial is made to satisfy both the differential equation and the associated boundary conditions. Using orthogonality of polynomials, we can approximate the solution to the differential equation on shallow water wave equation. The method has been used to solve problems in mechanical engineering such as structural mechanics, dynamics, fluid flow, heat and mass transfer, acoustics and other related fields.

Milder, Miles and Broer deal with the water waves on the Hamiltonian theory of surface waves (see [1], [2], [3]). The Galerkin method permitted finite-element techniques to be extended into areas such as fluid mechanics and series solution of some problems in elastic equilibrium of rods and plates (see [10]).

Waves in a surf zone were studied by Svendsen, Madsen and Hansen (see [7]). The idea of approximation for long wave equations is given by Broer (see [4]). To get exact calculation on approximation technique we use algebraic software as Mathematica ([11]).

Several other researchers have tested the validity of the KdV equation and variants in laboratory experiments (Remoissenet [9], Helfrich and Melville [8]).

2. Materials and Methods

Prior to the development of the finite element method, there existed an approximation technique for solving differential equations. The basic idea is to use a function with a number of unknown parameters to approximate the solution. Then a weighted average over the interior and boundary is set to zero.

Let we have a linear equation L u = v (2.1)

Where u is the unknown function and v is the given function. L is a linear operator (differential operators, matrices etc.). An approximate solution to eq. (1.1) is given by a linear combination of *N* base vectors in the

linear space as $U_i = \sum_{i=1}^N u_i e_i$ (2.2)

where u_i is the unknown coefficient and e_i is the base vector in the linear space.

Define E, the error between the approximate solution and the exact solution as

$$E = LU_i - c = L\sum_{i=1}^N u_i e_i - c = \sum_{i=1}^N u_i Le_i(x) - c(x)$$
(2.3)

3. Results and Discussions of Approximation

The finite-element method is a special case of the Galerkin method in which the base functions are chosen such that each base function becomes 1 at the corresponding nodes but otherwise 0 at other nodes. The link with the Galerkin method permitted finite-element techniques to be extended into areas such as fluid mechanics.

Let's we have the forth differential equation for a plate space as follow

$$\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p_z(x,y)}{D} + \frac{\partial^2 w}{\partial x^2}$$
(3.1)

where, p_z is the lateral pressure that is being applied, D is the flexural rigidity of the wave.

Since Eq (2.1) is a fourth-order differential equation, two boundary conditions, either for the displacements or for the internal forces, are required at each boundary. This equation can be rewritten using the two dimensional Laplacian operator

$$\left(\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \text{ as:} \qquad 105$$

$$\nabla^2 \nabla^2 w = \frac{p_z}{D} \frac{\partial^2 w}{\partial x^2} \tag{3.2}$$

Consider a differential operator L, defined as

$$L = \left[\frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}\right]$$
(3.3)

We can express the solution to the above equation in terms of the eigenfunction and eigenvalues, which are defined as:

$$Le_{nm}(x, y) = \lambda_{nm} Le_{nm}(x, y)$$
(3.4)

where $e_{nm}(x, y)$ are the eigenfunctions and λ_{nm} are the corresponding eigenvalues. Once the eigenfunctions and eigenvalues are known, it is possible to express w(x, y) as,

$$w(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} w_{nm} e^{\lambda_{nm}} e_{nm}(x, y)$$
(3.5)

We can write the above equation in the following form,

$$A\bar{c} = \lambda_{nm} B\bar{c}, \tag{3.6}$$

where

$$a_{ij} = \int_{0}^{a} \int_{0}^{b} LLf_{i}(x, y)f_{j}(x, y)dydx$$

$$b_{ij} = \int_{0}^{a} \int_{0}^{b} Lf_{i}(x, y)f_{j}(x, y)dydx$$
(3.7)

The quantities A and B are N x N square matrices as shown below:

$$A = \begin{bmatrix} a_{11} \dots \dots \dots a_{1N} \\ \dots \dots \dots \\ a_{NN} \dots \dots \dots a_{NN} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} \dots \dots \dots b_{1N} \\ \dots \dots \dots \\ b_{NN} \dots \dots \dots \\ b_{NN} \end{bmatrix}$$

where λ_{nm} , is the eigenvalue and \bar{c} will be the corresponding eigenvectors. 106

Simply Supported Boundary Condition on Approximation

At first, considering the simply supported boundary condition for a square plate dimensions a and b subjected to lateral loads.

The governing differential equation of the free zone subjected to lateral loads is:

$$\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p_z(x,y)}{D} + \frac{\partial^2 w}{\partial x^2}$$
(3.8)

where, p_z is the lateral pressure that is being applied, *D* is the flexural rigidity of the wave. A boundary that is prevented from deflecting but free to rotate about a line along the boundary edge, is defined as a simply supported edge. The conditions on a simply supported edge parallel to the y-axis at x = a, are.

$$W|_{x=a} = 0 \tag{3.9}$$

$$w|_{x=a} = -D\left(\frac{\partial^2 w}{\partial x^2} + \vartheta \frac{\partial^2 w}{\partial y^2}\right)_{x=a} = 0$$
(3.10)

(3.11)

The first step in solving this problem is to systematically choose a trial function e that satisfies the plate's boundary conditions. Polynomial approximating functions will be used to represent the lateral displacement of the plate. In this discussion, the trial function $e_i(x, y)$ will be represented as:

$$\phi_i(x,y) = \sum_{i=1}^N \sum_{j=1}^N a[i,j] u_j(x,y)$$
(3.12)

where

$$u_j(x,y) = x^{L_j} \cdot y^{M_j} \tag{3.13}$$

and, L_i and M_i are positive integers and $\phi_i(x, y)$ are coefficients to be determined

In the simply supported boundary condition example, it is found that an eight order polynomial is the lowest order possible to satisfy the boundary conditions

$$\phi_i(x,y) = a[1,1] + a[2,1]x + a[3,1]y + a[4,1]xy + a[5,1]x^2 + a[6,1]y^2 + a[7,1]xy^2 + \dots + a[42,1]x^3y^5 + a[43,1]x^2y^6 + a[44,1]xy^7 + a[45,1]y^8$$
(3.14)

4. Conclusions

The numerical method can be used to approximate the solution to ordinary differential equations, partial differential equations and integral equations.

Several other researchers have tested the validity of the KdV equation and variants in laboratory experiments (Remoissenet [9], Helfrich and Melville [8]). Their study includes a numerical scheme with error estimates, a convergence test of the computer code, a

comparison between the predictions of the theoretical model and the results of laboratory experiments

The Galerkin method is used to determine he coefficients of approximated polynomials. The finite element and Galerkin methods are currently the standard numerical technique in use to solve various nonlinear problems on water wave equations and shallow water wave equations.

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Commutativity Conditions of Some Time-varying Systems

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ABSTRACT

In an unpublished work, explicit differential equations representing the commutative pairs of some well-known second-order linear time-varying systems have been derived. In this contribution, the commutativity of these systems are investigated by considering 27 linear second-order differential equations. It is shown that the system modeled by each one of these equations has commutative pair or not with (or without) any condition.

Keywords: Commutativity, differential equations

1. INTRODUCTION

It is well known that cascade connection is the connection of two subsystems one after the other so that the output of the first system behaves as the input of the second one. The order of connection becomes important sometimes and one must decide which subsystem should be the first one. For this aim, secondary characteristics of the combined system concerning such as sensitivity, disturbance, robustness should be considered; then engineering skill, experience, and most possibly some mathematical analysis are needed equipment. Although, the input-output relation of two interconnections are the same in ideal conditions, one of them comes out to be preferable when the mentioned secondary performance characteristics are of concern. Hence, commutativity, that is the invariance of the main inputoutput characteristics with the order of connection in a cascade structure, comes out as an important subject for scientists and engineers.

Two different cascade connections AB and BA of the subsystems A and B where A and B are assumed to be continuous time systems with time-varying parameters are considered. If both of AB and BA have the same input-output relation we say that the pair (A, B) constitutes a commutative pair.

E. Marshall was the first scientist who defined and studied the above commutativity concept [1]. He mainly released the commutativity conditions for first-order linear time-varying systems. His work also includes the reality that first and higher order systems can be commutative only if they are of the same kind, time varying or invariant. Although elementary and simple, Marsall's work carries importance of introducing a future research subject in the literature.

After that, commutativity has been investigated by a few scientists only and several important developments have been realized theoretically. M. Koksal [2] and S. V. Saleh [3] are the only scientists until 2011 since Marshall's paper in 1977. They derived commutativity conditions of second-order linear time-varying systems. These conditions are studied in [4] for systems of any order. A summary of the previous results including the case of non-zero initial conditions, and explicit commutativity conditions for linear systems described by a fifth-order differential equation with time-varying coefficients were presented in [5] by M. Koksal and M. E. Koksal.

The importance of cascade-connection becomes prominent in electrical circuits [6, 7]. Further, one of the basic tools of modulation in communication theory is the use of linear time-varying electronic circuits. Therefore, the subject of commutativity when linear timevarying differential systems are of concern becomes crucial in applications as well.

In this study, many of the second-order linear differential equations in the literature are reviewed in Chapter 2. In Chapter 3, the theoretical results of [4] are applied to these second-order differential equations for finding their commutative conjugates. Finally, the paper ends up with Conclusions which constitute Chapter 4.

2. SECOND-ORDER DIFFERENTIAL EQUATIONS

Let the system *A* be described by

$$a_2(t)\ddot{y}_A(t) + a_1(t)\dot{y}_A(t) + a_0(t)y_A(t) = x_A(t); t \ge 0$$
(1)

which is a second-order linear differential equation with time-varying coefficients. Assume the initial conditions are $y_A(0)$ and $\dot{y}_A(0)$. The coefficients $a_2(t)$, $a_1(t)$ and $a_0(t)$ are such that $a_2(t) \neq 0$. The input and output of System A are represented by $x_A(t)$ and $y_A(t)$,

respectively. We notate that $\ddot{y}_A(t) = y_A''(t) = \frac{d^2}{dt^2} y_A(t)$, $\dot{y}_A(t) = y_A'(t) = \frac{d}{dt} y_A(t)$, and these notations will be also used for some other variables be in the rest of the paper.

Some famous differential equations are listed below and treated in this paper. In general, they are famed by the name of the person who introduced them in the literature [8-9].

Among these 27 equations, some of them are special or general forms of the others. Further, although they have different names, some can be transformed to others. Which one is special or general form being not important in this study. We study them for the existences of their commutative pairs, and search for the commutative pairs if they exist.

Line #	Name of Equation	Formula
1	Baer wave DE	$(x - a_1)(x - a_2)y'' + \frac{1}{2}[2x - (d_1 + d_2)]y' - (k^2x^2 - p^2x + q^2)y = 0$
2	Confluent Hyp. DE	xy'' + (c-x)y' - ay = 0
3	Coulomb wave DE	$y'' + \left[1 - \frac{2\eta}{x} - \frac{L(L+1)}{x^2}\right]y = 0$
4	Halm's DE	$(1+x^2)^2 y'' + \lambda y = 0$
5	Hermite DE	$y^{\prime\prime} - 2xy^{\prime} + \lambda y = 0$
6	Ince's DE-first	$y'' + \delta \sin(2x) y' + [\mu - p\delta \cos(2x)]y = 0$
7	Ince's DE- second	$y'' + \frac{\alpha + \beta \cos 2t + \gamma \cos 4t}{(1 + \alpha \cos 2t)^2} y = 0$
8	Kelvin DE	$x^2y'' + xy' - (ix^2 + v^2)y = 0$
9	Lame's DE- first	$(x^{2} - b^{2})(x^{2} - c^{2})y'' + x(x^{2} - b^{2} + x^{2} - c^{2})y' -[m(m - 1)x^{2} - (b^{2} + c^{2})p]y = 0$
10	Lame's DE- second	$y'' + \frac{1}{2} \left(\frac{1}{x} + \frac{1}{x-a} + \frac{1}{x-b} \right) y' + \frac{1}{4} \left[\frac{(a^2 + b^2)q - p(p+1)x + kx^2}{x(x-a)(x-b)} \right] y = 0$
11	Legendre DE- first	$(1 - x^2)y'' - 2xy' + \mu(\mu + 1)y = 0$
12	Legendre DE- second	$(1-x^2)y'' - 2xy' + \left[\mu(\mu+1) - \frac{m^2}{1-x^2}\right]y = 0$
13	Legendre DE- third	$(1-x^2)y'' - 2xy' + \left[k^2a^2(x^2-1) - p(p+1) - \frac{q^2}{x^2-1}\right]y = 0$
14	Lommel DE	$x^{2}y'' + xy' + (x^{2} - v^{2})y = kx^{\mu+1}$

 Table 1: Well-known second-order linear differential equations

15	Malmsten's DE	$y'' + \frac{\Gamma}{z}y' = \left(Az^m + \frac{s}{z^2}\right)y$
16	Mathieu DE- first	$y'' + (a - 2q\cos 2x)y = 0$
17	Mathieu DE- second	$y'' + [(1 - 2r)cotx]y' + (a + k^2cos^2x)y = 0$
18	Mathieu DE- third	y'' - (a - 2qcosh2x)y = 0
19	Pöschl–Teller DE-first	$y'' + \left[a^{2}\left(\frac{k(k-1)}{\sin^{2}(ax)} + \frac{\lambda(\lambda-1)}{\cos^{2}(ax)}\right) - b^{2}\right]y = 0$
20	Pöschl–Teller DE-second	$y'' + \left[a^2 \left(\frac{k(k-1)}{\sin h^2(ax)} + \frac{\lambda(\lambda-1)}{\cos h^2(ax)}\right) - b^2\right] y = 0$
21	Sharpe's DE	xy'' + y' + (x + A)y = 0
22	Spheroidal wave DE-first	$(1-x^2)y'' - 2xy' + \left(\lambda + c^2x^2 - \frac{m^2}{1-x^2}\right)y = 0$
23	Spheroidal wave DE- second	$(1+x^2)y'' + 2xy' + \left(\lambda - c^2x^2 - \frac{m^2}{x^2 + 1}\right)y = 0$
24	Sturm-Liouville DE	$p(x)y'' + p'(x)y' + [\lambda w(x) - q(x)]y = 0$
25	Ultraspherical DE	(1 - x2)y'' - (2a + 1)xy' + n(n + 2a)y = 0
26	Whittaker DE	$y'' + \left(-\frac{1}{4} + \frac{k}{x} + \frac{\frac{1}{4} - m^2}{x^2}\right)y = 0$
27	Whittaker-Hill DE	$y'' + [A + B\cos 2x + C\cos 4x]y = 0$

3. COMMUTATIVE PAIRS OF SECOND-ORDER DIFFERENTIAL EQUATIONS

by

We now consider another second-order time-varying system B of type A represented

$$b_2(t)\ddot{y}_B(t) + b_1(t)\dot{y}_B(t) + b_0(t)y_B(t); (t); t \ge 0,$$
(2)

with the inputs and outputs $x_B(t)$ and $y_B(t)$, respectively:; with the initial conditions $y_B(0)$ and $\dot{y}_B(0)$; and with the time-varying coefficients $b_2(t) \neq 0, b_1(t), b_0(t)$.

The set of necessary and sufficient conditions that systems *A* and *B* are commutative are

$$\begin{bmatrix} b_2 \\ b_1 \\ b_0 \end{bmatrix} = \begin{bmatrix} a_2 & 0 & 0 \\ a_1 & a_2^{0.5} & 0 \\ a_0 & a_2^{-0.5}(2a_1 - \dot{a}_2)/4 & 1 \end{bmatrix} \begin{bmatrix} c_2 \\ c_1 \\ c_0 \end{bmatrix},$$
(3a)

$$-a_2^{0.5} \frac{d}{dt} \left[a_0 - \frac{1}{16a_2} \left(4a_1^2 + 3\dot{a}_2^2 - 8a_1\dot{a}_2 + 8\dot{a}_1a_2 - 4a_2\ddot{a}_2 \right) \right] c_1 = 0.$$
(3b)

Conditions (3a) requires three relations between time-varying coefficients of systems A and B, where c_2, c_1, c_0 are arbitrary constants. Condition (3b) implies that not all second order systems have commutative pairs; and to have a commutative pair like B, the coefficients of A must satisfy Eq. (3b) with the constant c_1 in Eq. (3a). Note that if c_2 in Eq. (3a) is chosen as zero, order of system B reduces to one. When both c_2 and c_1 are zero, B becomes a scalar (algebraic) system with gain $\frac{1}{c_2}, c_0 \neq 0$.

The commutativity property of the linear second-order time-varying differential equations listed in Section 2 are investigated next by using (3b) and the results are listed Table 2.

Table 2: Commutativity property of differential systems described by some famous DE

Not commutative Conditionally commutative		Commutative
2, 5, 8, 11, 13-14, 21, 24	1, 3, 6-7, 9-10, 12, 15-20, 22-23, 25-27	4

Example: The example is for the case of "Conditionally commutative". We assume a system **A** which is modelled by the second equation (Lame's differential equation-first type). The coefficients of Eq. (1) are

 $a_2 = (x^2 - b^2)(x^2 - c^2), a_1 = x(x^2 - b^2 + x^2 - c^2), a_0 = m(m+1)x^2 + (b^2 + c^2)p$ for Lame's differential equation. The expression in the parenthesis in Eq. (3b) should be a constant for the existence of the commutative pair of a system. For m = 0 or m = -1, this expression is constant then the equation has commutative pair. Using Eq. (3a), the coefficients of its commutative pair are found as follows: 113

$$b_{2} = c_{2}a_{2} = c_{2}(x^{2} - b^{2})(x^{2} - c^{2}),$$

$$b_{1} = c_{2}a_{1} + c_{1}a_{2}^{0.5} = c_{2}x(x^{2} - b^{2} + x^{2} - c^{2}) + c_{1}\sqrt{(x^{2} - b^{2})(x^{2} - c^{2})},$$

$$b_{0} = c_{2}a_{0} + c_{1}a_{2}^{-0.5}(2a_{1} - \dot{a}_{2})\frac{1}{4} + c_{0}$$

$$= -(b^{2} + c^{2})p + \frac{c_{1}}{c_{2}(x^{2} - b^{2})(x^{2} - c^{2})}[2x(x^{2} - b^{2} + x^{2} - c^{2}) - 4x^{3} + 2x(c^{2} + b^{2}) + c_{0}].$$

Then, commutative pair is written as follows:

$$c_{2}(x^{2}-b^{2})(x^{2}-c^{2})y'' + \left[c_{2}x(x^{2}-b^{2}+x^{2}-c^{2})+c_{1}\sqrt{(x^{2}-b^{2})(x^{2}-c^{2})}\right]y' + \left\{-(b^{2}+c^{2})p + \frac{c_{1}}{c_{2}(x^{2}-b^{2})(x^{2}-c^{2})}\left[2x(x^{2}-b^{2}+x^{2}-c^{2})-4x^{3}+2x(c^{2}+b^{2})+c_{0}\right]\right\}y' = x_{2}$$

In the following Table 3, commutativity conditions are given and the final forms of the equations are presented in the case of the fact that condition is used in the stated equation. In Table 4, commutativity conjugates of all equations are presented after finding them by using (3a).

Line #	Name of Equation	Condition for Commutativity	Final Forms of Equations
1	Baer wave DE	k = p = 0	$(x-d_1)(x-d_2)y'' + [x-0.5(d_1+d_2)]y' - q^2y = 0$
2	Coulomb wave DE	<i>i</i>) $\eta = 0, L = 0$ <i>ii</i>) $\eta = 0, L = 1$	y'' + y = 0
3	Halm's DE	no condition	$(1+x^2)^2 y^{\prime\prime} + \lambda y = 0$
4	Ince's DE-first	$\delta = 0$	$y'' + \mu y = 0$
5	Ince's DE- second	$\alpha = \beta = \gamma = 0$	<i>y</i> '' = 0
6	Lame's DE-first	i) m = 0 ii)m = -1	$(x^2 - b^2)(x^2 - c^2)y'' + x(x^2 - b^2 + x^2 - c^2)y' + (b^2 + c^2)y'$
7	Lame's DE- second	a, b, k = 0 i) p = 0.5 ii) p = -1.5	$y'' + \frac{3}{2x}y' - \frac{3}{16x^2}y = 0$
8	Legendre DE- second	<i>i</i>) $m = -0.5$ <i>ii</i>) $m = 0.5$	$(1-x^2)y''-2xy'+\left[\mu(\mu+1)-\frac{1}{4(1-x^2)}\right]y=0$

 Table 3: Commutativity conditions

9	Malmsten's DE	i) $a = 0, r^2 - 2r = 4s$ ii) $m = 0, r^2 - 2r = 4s$ iii) $m = -2,$ $r^2 - 2r = 4s - 4a$	$y'' + \frac{r}{z}y' - \frac{r^2 - 2r}{4z^2}y = 0$ $y'' + \frac{r}{z}y' - \left(a + \frac{r^2 - 2r}{4z^2}\right)y = 0$
10	Mathieu DE- first	<i>q</i> = 0	
11	Mathieu DE- second	r = 0.5, k = 0	y' + ay = 0
12	Mathieu DE- third	q = 0	y'' - ay = 0
13	Pöschl–Teller DE-first	<i>i</i>) $a = 0$ <i>ii</i>) $k = \lambda = 0$	
14	Pöschl–Teller DE-second	$iii)k = 0, \lambda = 1$ $iv) k = 1, \lambda = 0$ $v) k = \lambda = 1$	$y'' - a^2 b^2 y = 0$
15	Spheroidal wave DE-first	<i>i</i>) $c = 0, m = -0.5$	$(1-x^2)y'' - 2xy' + \left[\lambda - \frac{1}{4(1-x^2)}\right]y = 0$
16	Spheroidal wave DE-second	$ii)\ c=0,m=0.5$	$(1+x^2)y'' + 2xy' + \left[\lambda - \frac{1}{4(1+x^2)}\right]y = 0$
17	Ultraspherical DE	i) a = -1 $ii)a = -2$	(1 - x2)y'' - 3xy' + n(n - 2)y = 0 (1 - x ²)y'' + 3xy' + n(n - 4)y = 0
18	Whittaker DE	$k=0,m=\pm0.5$	$y'' - \frac{1}{4}y = 0$
19	Whittaker-Hill DE	B = C = 0	$y^{\prime\prime} + Ay = 0$

 Table 4: Commutative conjugates of differential equations in Table 3

Line #	Name of Equation	Commutativity Conjugates
1	Baer wave DE	$c_{2}(x-d_{1})(x-d_{2})y'' + \left\{c_{2}[x-0.5(d_{1}+d_{2})] + c_{1}\sqrt{(x-d_{1})(x-d_{2})}\right\}y' + (-c_{2}q^{2} + c_{0})y = x_{2}$
2	Coulomb wave DE	$c_2y'' + c_1y' + (c_2 + c_0)y = x_2$
3	Halm's DE	$c_2(1-x^2)^2y'' + c_1(1-x^2)y' + (c_2\lambda - c_1x + c_0)y = x_2$
4	Ince's DE-first	$c_2y'' + c_1y' + (c_2\mu + c_0)y = x_2$
5	Ince's DE-second	$c_2 y'' + c_1 y' + c_0 y = x_2$
6	Lame's DE-first	$c_{2}(x^{2}-b^{2})(x^{2}-c^{2})y'' + [c_{2}x(x^{2}-b^{2}+x^{2}-c^{2})+c_{1}\sqrt{(x^{2}-b^{2})(x^{2}-c^{2})}]y' + [-(b^{2}+c^{2})p + \frac{c_{1}}{c_{2}(x^{2}-b^{2})(x^{2}-c^{2})}[2x(x^{2}-b^{2}+x^{2}-c^{2})-4x^{3}+2x(c^{2}+b^{2}+b^{2})(x^{2}-c^{2})]y' + [-(b^{2}+c^{2})p + \frac{c_{1}}{c_{2}(x^{2}-b^{2})(x^{2}-c^{2})}[2x(x^{2}-b^{2}+x^{2}-c^{2})-4x^{3}+2x(c^{2}+b^{2}+b^{2})(x^{2}-c^{2})]y' + [-(b^{2}+c^{2})p + \frac{c_{1}}{c_{2}(x^{2}-b^{2})(x^{2}-c^{2})}[2x(x^{2}-b^{2}+x^{2}-c^{2})-4x^{3}+2x(c^{2}+b^{2}+b^{2})(x^{2}-c^{2})]y' + [-(b^{2}+c^{2})p + \frac{c_{1}}{c_{2}(x^{2}-b^{2})(x^{2}-c^{2})}[2x(x^{2}-b^{2}+x^{2}-c^{2})-4x^{3}+2x(c^{2}+b^{2}+b^{2})(x^{2}-c^{2})]y' + [-(b^{2}+c^{2})p + \frac{c_{1}}{c_{2}(x^{2}-b^{2})(x^{2}-c^{2})}[2x(x^{2}-b^{2}+x^{2}-c^{2})-4x^{3}+2x(c^{2}+b^{2}+b^{2})(x^{2}-c^{2})]y' + [-(b^{2}+c^{2})p + \frac{c_{1}}{c_{2}(x^{2}-b^{2})(x^{2}-c^{2})}[2x(x^{2}-b^{2}+x^{2}-c^{2})-4x^{3}+2x(c^{2}+b^{2}+b^{2})(x^{2}-c^{2})]y' + [-(b^{2}+c^{2})p + \frac{c_{1}}{c_{2}(x^{2}-b^{2})(x^{2}-c^{2})}]y' + [-(b^{2}+c^{2})p + \frac{c_{1}}{c_{2}(x^{2}-b^{2})(x^{2}-c^{2})}]y' + [-(b^{2}+c^{2})p + \frac{c_{1}}{c_{2}(x^{2}-b^{2})(x^{2}-c^{2})}]y' + [-(b^{2}+c^{2})p + \frac{c_{1}}{c_{2}(x^{2}-b^{2})(x^{2}-c^{2})}]y' + [-(b^{2}+c^{2})p + \frac{c_{1}}{c_{2}(x^{2}-b^{2})(x^{2}-c^{2})}]y' + [-(b^{2}+c^{2})p + \frac{c_{1}}{c_{2}(x^{2}-b^{2})(x^{2}-c^{2})}]y' + [-(b^{2}+c^{2})(x^{2}-c^{2})]y' + [-(b^{2}+c^{2})(x^{2}-c^{2})(x^{2}-c^{2})]y' + [-(b^{2}+c^{2})(x^{2}-c^{2})(x^{2}-c^{2})]y' + [-(b^{2}+c^{2})(x^{2}-c^{2})(x^{2}-c^{2})]y' + [-(b^{2}+c^{2})(x^{2}-c^{2})(x^{2}-c^{2})]y' + [-(b^{2}+c^{2})(x^{2}-c^{2})(x^{2}-c^{2})(x^{2}-c^{2})(x^{2}-c^{2})]y' + [-(b^{2}+c^{2})(x^{2}-c^{2})(x^{2}-c^{2})(x^{2}-c^{2})(x^{2}-c^{2})(x^{2}-c^{2})(x^{2}-c^{2})]y' + [-(b^{2}+c^{2})(x^{2}-c^{2})($
7	Lame's DE-third	$c_2 y'' + \left(\frac{3c_2}{2x} + c_1\right) y' - \left(\frac{3c_2}{16x^2} + \frac{3c_1}{4x} + c_0\right) y = x_2$
8	Legendre DE- second	$c_2(1-x^2)y'' - (2c_2x - c_1\sqrt{1-x^2})y'$ ¹¹⁵

		$+ \left[c_2 \left(\mu(\mu+1) - \frac{0.25}{1-x^2} \right) + c_1 \frac{-1-2x}{4\sqrt{1-x^2}} + c_0 \right] y = x_2$
9	Malmsten's DE	$c_{2}y'' + \left(c_{2}\frac{r}{z} + c_{1}\right)y' + \left[-c_{2}\left(Az^{m} + \frac{s}{z^{2}}\right) + c_{1}\frac{r}{2z} + c_{0}\right]y = x_{2}$
10	Mathiue DE-first	
11	Mathieu DE-second	$c_2 y + c_1 y + (c_2 a + c_0) y = x_2$
12	Mathieu DE-third	$c_2y'' + c_1y' + (-c_2a + c_0)y = x_2$
13	Pöschl–Teller DE- first	
14	Pöschl–Teller DE- second	$c_2y + c_1y + (-c_2a \ b + c_0)y = x_2$
15	Spheroidal wave DE-first	$c_{2}(1-x^{2})y'' - \left(2c_{2}x - c_{1}\sqrt{1-x^{2}}\right)y' + \left[c_{2}\left(\lambda - \frac{0.25}{1-x^{2}}\right) + c_{1}\frac{-1-2x}{4\sqrt{1-x^{2}}} + c_{0}\right]y = x_{2}$
16	Spheroidal wave DE-second	$c_{2}(1+x^{2})y'' + (2c_{2}x + c_{1}\sqrt{1+x^{2}})y' + \left[c_{2}\left(\lambda - \frac{0.25}{1+x^{2}}\right) + c_{1}\frac{-1+2x}{4\sqrt{1+x^{2}}} + c_{0}\right]y = x_{2}$
17	Ultraspherical DE	$ \begin{bmatrix} c_2(1-x^2)y'' - \left[3c_2x - c_1\sqrt{1-x^2}\right]y' + \left[c_2(n^2-n) - c_1\frac{x}{\sqrt{1-x^2}} + c_0\right]y = x_2 \\ c_2(1-x^2)y'' + \left[3c_2x + c_1\sqrt{1-x^2}\right]y' + \left[c_2(n^2-4n) + c_1\frac{2x}{\sqrt{1-x^2}} + c_0\right]y = x_2 \end{bmatrix} $
18	Whittaker DE	$c_2 y'' + c_1 y' + \left(-\frac{c_2}{4} + c_0\right) y = x_2$
19	Whittaker-Hill DE	$c_2y'' + c_1y' + (c_2A + c_0)y = x_2$

4. CONCLUSION

In this study, second-order linear differential equations are searched in the literature and 27 second-order linear differential equations are presented. The existences of their commutative pairs are investigated with (or without) condition and which equation' commutativity pair exits is presented with (or without) any condition. Commutativity conjugates of differential equations whose commutativity pairs exit are constructed.

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A version of the Saks-Henstock Lemma for ordered integrals in the Riesz space

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Abstract

In this paper we prove a version of the Saks-Henstock Lemma for (oM)-integral ((oH)integral, strongly (oM)-integral,) integrals with values Banach lattice. We have used it to prove that one interesting difference between these kind of integration is the fact that the (oH)-integral possess the properties represented by Hake theorem.

Keyword(s) Banach lattice, (oH) and (oM)-integral, Hake theorem.

1. Introduction and preliminaries

It is known that the McShane integral and the Henstock-Kurzweil integral are two kinds of the Riemann-type integral. Relations of different generalizations of Riemann-type integral was done in the last decades and afterwards the notions of order-type integrals were introduced and studied for functions taking their values in ordered vector spaces, and in Banach lattices. In particular we can see [5], [4], [2], [10], [9], [6], [7], [8], [11] [12]. We are inspired from the works of Candeloro and Sambucini [5] as well as Boccuto et al. [1], [3] about order –type integrals. In this paper a definition of strongly (oM)-integral, ((oH)-integral,) is given and a version of the Saks-Henstock Lemma for (oM)-integral ((oH)-integral, strongly (oM)-integral,) integrals with values Banach lattice are proved. We have used it to prove that one interesting difference between these kind of integration is the fact that the (oH)-integral possess the properties represented by Hake theorem.

A sequence $(r_n)_n$ is said to be order-convergent (or (o)-convergent) to r, if there exists a sequence $(p_n)_n \in R$, such that $p_n \downarrow 0$ and $|r_n - r| \le p_n$, $\forall n \in \mathbb{N}$.

(see also [9], [11]), and we will write(o) $\lim_{n \to \infty} r_n = r$.

A gage is any map $\gamma: T \to \mathbb{R}^+$. A partition Π of T is a finite family $\Pi = \{(E_i, t_i): i = 1, ..., k\}$ of pairs such that the sets E_i are pairwise disjoint sets whose union is T and the points t_i are called *tags*. If all tags satisfy the condition $t_i \in E_i$ then the partition is said to be of *Henstock* type, or a *Henstock* partition. Otherwise, if t_i is not necessary to be in E_i , we say that it is a *free* or *McShane* partition.

Given a gage γ , we say that Π is γ -fine if $d(w, t_i) < \gamma(t_i)$ for every $w \in E_i$ and $i = 1, \ldots, k$. Clearly,

a gage γ can also be defined as a mapping associating with each point $t_i \in T$ an open ball centered at t_i and cover E_i .

Let us assume now that X is any Banach lattice with an order-continuous norm. For the sake of completeness we recall the main notions of integral we are interested in.

Definition 1.1.

A function $f: T \to X$ is called (o)- McShane integrable ((oH)-integrable) and $J \in X$ is its (o)-McShane integral ((oH)-integral) if for every (o)- sequence $(b_n)_n$ in X, there is a corresponding sequence $(\gamma_n)_n$ of gauges $(\gamma_n(t): T \to]0, +\infty[$ such that for every n and (γ_n) fine M-partition (H-partition) { $(E_i, t_i), i = 1, ..., p$ } of T holds the inequality

$$|\sigma(f,\Pi) - J| \le b_n$$

Where $\sigma(f, \Pi) = \sum_{i=1}^{p} f(t_i) \mu(E_i)$. We denote

$$J = (oM) \int_{T} f$$
, respectively $J = (oH) \int_{T} f$.

Theorem 1.2 [5].

Let $f: T \to X$ be any mapping. Then f is (o)– Henstock integrable ((o)- McShane integrable) if and only if there exist an (o) –sequence $(b_n)_n$ and a corresponding sequence $(\gamma_n)_n$ of gages, such that for every n, as soon as Π'', Π' are two $-\gamma_n$ fine Henstock (McShane) partitions, the following holds true:

$$\mid \sigma(f, \Pi^{''}) - \sigma(f, \Pi^{'}) \mid \leq b_n$$

Definition 1.3.

A function $f: T \to X$ is said strongly oM(oH)-integral on T if there is an additive function $F: \mathfrak{B} \to X$, such that for every (o)- sequence $(b_n)_n$ in X there is a corresponding sequence $(\gamma_n)_n$ of gauges $(\gamma_n(t): T \to]0, +\infty[$ on T such that for every γ_n -fine M- partition(H-partition) $\Pi = \{(E_i, t_i): i = 1, ..., s\}$ of T holds the inequality

$$\sum_{i=1}^{s} |f(t_i)\mu(E_i) - F(E_i)| \le b_n$$

where $F(E_i) = (oM) \int_{E_i} f \quad (F(E_i) = (oH) \int_{E_i} f)$

Denote SoM (SoH) the set of functions which are strogly oM(oH)-integrable on T.

2. The Saks-Henstock Lemma

Lemma 2.1 (Saks-Henstock).

Assume that $f: T \to X$ is (o)-McShane integrable. Given (o)- sequence $(b_n)_{n_1}$ assume that a corresponding sequence $(\gamma_n)_n$ of gauges $(\gamma_n(t): T \to]0, +\infty[$ on T such that for every n

and for every γ_n -fine M- partition $\Pi = \{(E_i, t_i): i = 1, ..., k\}$, of T holds the inequality

$$\left|\sum_{i=1}^{k} f(t_i) \,\mu(E_i) - (oH) \int_T f\right| \le b_n$$

Then if $\{(F_j, \tau_j): j = 1, ..., m\}$ is an arbitrary γ_n -fine M-system we have $\left|\sum_{j=1}^m (f(\tau_j) \mu(F_j) - (oH) \int_{F_j} f)\right| \le b_n$

Proof. Since $\{(F_j, \tau_j): j = 1, ..., m\}$ is a γ_n -fine McShane system the set $I \setminus \bigcup_{j=1}^m (F_j^0)$ contains of a finite system $K_l \ l = 1, ..., p$ of non-overlapping intervals in T.The function f is (o) -McShane integrable and in virtue of Bolzano –Cauchy theorem (1.2), the integral $(oM) \int_{K_l} f$ exists. By the definition of the integrals, for any (o) – sequence $(a_n)_n$ there is a sequence $(\gamma'_n)_n$ of gauges on M_n , such that $(\gamma'_n(t)) < (\gamma_n(t))$ for $t \in K_l$ such that for every l = 1, ..., p, we have

$$\left|\sum_{i=1}^{k_l} f\left(s_i^l\right) \mu\left(E_i^l\right) - (oM) \int_{K_l} f\right| \le \frac{a_n}{p+1}$$

Provided $\{(E_i^l, s_i^l): i = 1, ..., k_l\}$ is γ_m -fine M-partition of the interval K_l . The sum

$$\sum_{j=1}^{m} f(\tau_j) \mu(F_j) + \sum_{l=1}^{p} \sum_{i=1}^{k_l} f(s_i^l) \mu(E_i^l)$$

represents an integral sum corresponds one M-partition γ_n – fine of the interval T and consequently by the assumption we have

$$\left|\sum_{j=1}^{m} f(\tau_i) \mu(F_i) + \sum_{l=1}^{p} \sum_{i=1}^{k_l} f(s_i^l) \mu(E_i^l) - (oM) \int_T f \right| < b_n.$$

Hence

$$\begin{split} \left| \sum_{j=1}^{m} f(\tau_{i}) \, \mu(F_{i}) - (oM) \int_{F_{j}} f \right| \leq \\ \leq \left| \sum_{j=1}^{m} f(\tau_{j}) \, \mu(F_{i}) + \sum_{l=1}^{p} \sum_{i=1}^{k_{l}} f(s_{i}^{l}) \mu(E_{i}^{l}) - (oM) \int_{T} f \right| + \\ + \sum_{l=1}^{p} \left| \sum_{i=1}^{k_{l}} f(s_{i}^{l}) \mu(E_{i}^{l}) - (oM) \int_{M_{l}} f \right| \leq b_{n} + p \frac{a_{n}}{p+1} < b_{n} + a_{n} \end{split}$$

We obtain the proof of the theorem.

If we replace M- partition in the proof of Lemma 2.1 by H- partition we obtain the following result for the (o)- Henstock integral.

Lemma2.2 (Saks-Henstock).

Assume that $f: T \to X$ is (o)– Henstock integrable. Given (o)- sequence $(b_n)_n$ assume that a corresponding sequence $(\gamma_n)_n$ of gauges $(\gamma_n(t): T \to]0, +\infty[$ on T such that for every n

and for every γ_n -fine H- partition $\Pi = \{(E_i, t_i): i = 1, ..., k\}$, of T holds the inequality

$$\left|\sum_{i=1}^{k} f(t_i) \,\mu(E_i) - (oH) \int_T f\right| \le b_n$$

Then if $\{(F_j, \tau_j): j = 1, ..., m\}$ is an arbitrary γ_n -fine H-system we have

$$\left|\sum_{j=1}^{m} (f(\tau_j) \mu(F_j) - (oH) \int_{F_j} f)\right| \le b_n$$

The following variant of the Saks-Henstock lemma is for the strong version oM(oH)-integrable.

Lemma 2.3 (Saks-Henstock)

Assume that $f: T \to X$ is SoM (SoH)- integrable then to every (o)- sequence $(b_n)_n$ in X there is a corresponding sequence $(\gamma_n)_n$ of gauges $(\gamma_n(t): T \to]0, +\infty[$ on T such that for every n and for every arbitrary γ_n -fine M- system (H-system) $\Pi = \{(E_i, t_i): i = 1, ..., s\}$ of T holds the inequality

$$\sum_{i=1}^{s} |f(t_i)\mu(E_i) - F(E_i)| \le b_n$$

Proposition 2.4

If $f: T \to X$ is SoM (SoH)– integrable with the additive interval function $F: \mathfrak{B} \to X$. Then for every $E \in \mathfrak{B}$

$$F(E) = (oM) \int_{F} f \qquad (F(E) = (oH) \int_{F} f)$$

Assume that $f: T \to X$ is SoM (SoH)- integrable and $E \in \mathfrak{B}$ then to every (o)- sequence $(b_n)_n$ in X there is a corresponding sequence $(\gamma_n)_n$ of gauges $(\gamma_n(t): T \to]0, +\infty[$ on T such that for every n and for every arbitrary γ_n -fine M- system (H-system) $\Pi = \{(E_i, t_i): i = 1, \dots, s\}$ of T holds the inequality

$$\sum_{i=1}^{s} |f(t_i)\mu(E_i) - F(E_i)| \le b_n$$

If $\{(E_i, t_i)\}$ is an arbitrary γ_n -fine M- partition (H-partition) of the interval E then by Lemma 2.4 we have

$$\begin{aligned} \left| \sum_{i} f(t_i) \mu(E_i) - F(E) \right| &= \left| \sum_{i} \left[f(t_i) \mu(E_i) - F(E_i) \right] \right| \\ &\leq \sum_{i} \left| f(t_i) \mu(E_i) - F(E_i) \right| \leq b_n \end{aligned}$$

And this shows that $F(E) = (oM) \int_E f$ ($F(E) = (oH) \int_E f$).

Theorem 2.5 (Hake)

Let $[a, b] \subset \mathbb{R}$, $f: [a, b] \to X$. If the integral $(oH) \int_c^b f d\mu$ and $(o)-\lim_{c \to a^+} (oH) \int_c^b f = L \in X$ exists for every $a < c \le b$ then the integral $(oH) \int_a^b f$ exists and holds the equality:

$$(oH)\int_a^b f = L.$$

for any γ_n -fine H- partition { $(u_j, [a_{j-1}, \alpha_j]): j = 1, ..., m$ } of the interval T.

By the definition 1.1 the integral (*oH*) $\int_{a}^{b} f$ exists and holds the equality:

$$(oH)\int_{a}^{b}f = L \in X$$

3.Conclusions

We prove a version of the Saks-Henstock Lemma for ordered integrals integrals with values Banach lattice. We have used it to prove that one interesting difference between these kind of integration is the fact that the (o)- Henstock integral possess the properties represented by Hake theorem.

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The (o)-convergence properties of ordered integrals in Riesz space

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Abstract

Recently, there are many paper paying attention to the integration in Riesz space. There is introduced and studied the notions of order-type integrals, for functions taking their values in ordered vector spaces, and in Banach lattices. In this paper we prove some convergence theorems of order- Mcshane (Henstock- Kurzweil) equi –integrals on Banach lattice and arrive same result in L-space as on Mcshane (Henstock Kurzweil) norm-integrals.

Keyword(s) Riesz space, Henstock integral, Mcshane integral, (o)-convergence.

1. Introduction and preliminaries

Recently, there are many papers paying attention to the integration in Riesz space. There are introduced and studied the notions of order-type integrals, for functions taking their values in ordered vector spaces, and in Banach lattices. In particular we can see [6], [7], [3], [10], [9], [5], [4], [8]. We are affected from the works of Candeloro and Sambucini [6] as well as Boccuto et al.[1-2] about order –type integrals. In this paper a definition of strongly (oM)-equi-integral (strongly (oH)- equi-integral) is given, some convergece theorems for the order - equi-integrals with values Banach lattice are proved in particular, we give here some convergence results for the strong version of order –equi-integrals.

A sequence $(r_n)_n$ is said to be order-convergent (or (o)-convergent) to r, if there exists a sequence $(p_n)_n \in R$, such that $p_n \downarrow 0$ and $|r_n - r| \le p_n, \forall n \in \mathbb{N}$.

(see also [9], [11]), and we will write(o) $\lim_{n \to \infty} r_n = r$.

A gage is any map $\gamma: T \to \mathbb{R}^+$. A partition Π of T is a finite family $\Pi = \{(E_i, t_i): i = 1, ..., k\}$ of pairs such that the sets E_i are pairwise disjoint sets whose union is T and the points t_i are called *tags*. If all tags satisfy the condition $t_i \in E_i$ then the partition is said to be of *Henstock* type, or a *Henstock partition*. Otherwise, if t_i is not necessary to be in E_i , we say that it is a *free* or *McShane* partition. Given a gage γ , we say that Π is γ -fine if $d(w, t_i) < \gamma(t_i)$ for every $w \in E_i$ and i = 1, ..., k. Clearly, a gage γ can also be defined as a mapping associating with each point $t_i \in T$ an open ball centered at t_i and cover E_i .

Let us assume now that X is any Banach lattice with an order-continuous norm₂₃For the sake of completeness we recall the main notions of integral we are interested in.

Definition 1.1.

A function $f: T \to X$ is called (o)- McShane integrable ((oH)-integrable) and $J \in X$ is its (o)-McShane integral ((oH)-integral) if for every (o)- sequence $(b_n)_n$ in X, there is a corresponding sequence $(\gamma_n)_n$ of gauges $(\gamma_n(t): T \to]0, +\infty[$ such that for every n and (γ_n) fine M-partition (H-partition) { $(E_i, t_i), i = 1, ..., p$ } of T holds the inequality

 $|\sigma(f,\Pi) - J| \le b_n.$ Where $\sigma(f,\Pi) = \sum_{i=1}^p f(t_i) \mu(E_i)$. We denote

 $J = (oM) \int_T f$, respectively $J = (oH) \int_T f$.

Theorem 1.2. [6].

Let $f: T \to X$ be any mapping. Then f is (*o*)– Henstock integrable ((o)- McShane integrable) if and only if there exist an (*o*) –sequence $(b_n)_n$ and a corresponding sequence $(\gamma_n)_n$ of gages, such that for every n, as soon as Π'', Π' are two $-\gamma_n$ fine Henstock (McShane) partitions, the following holds true:

$$|\sigma(f,\Pi'') - \sigma(f,\Pi')| \le b_n$$

Definition 1.3.

A collection \mathcal{F} of functions $f: T \to X$ is called (oM)-equi-integrable ((oH)-equi-integrable) if every $f \in \mathcal{F}$ is (o) -McShane integrable ((o) -Henstock-Kurzweil integrable) and for any (o)- sequence $(b_n)_n$ there is a corresponding sequence $(\gamma_n)_n$ of gauges such that for any $f \in \mathcal{F}$ the inequality holds provided $\{(E_i, t_i), i = 1, ..., p\}$ is (γ_n) -fine M-partition (Hpartition) of T.

$$\left| \sum_{i=1}^{s} f(t_i) \, \mu(E_i) - (oM) \int_T f \right| \le b_n$$
$$\left(\left| \sum_{i=1}^{s} f(t_i) \, \mu(E_i) - (oH) \int_T f \right| \le b_n \right)$$

Lemma 1.4. (Saks-Henstock)

Assume that an (oM) -equi- integrable ((oH)-equi-integrable) collection \mathcal{F} of functions of $f: T \to X$ is given. For every (o)- sequence $(b_n)_n$ assume that the sequence $(\gamma_n)_n$ of gauges on T is such that for every n and for every γ_n -fine M- partition (H- partition) $\Pi = \{(E_i, t_i): i = 1, ..., s\}$, of T holds the inequality

$$\left|\sum_{i=1}^{s} f(t_i) \,\mu(E_i) - (oM) \int_T f\right| \le b_n$$
¹²⁴

$$\begin{aligned} \left| \sum_{i=1}^{s} f(t_i) \, \mu(E_i) - (oH) \int_T f \right| &\leq b_n \\ \text{Then if } \left\{ \left(F_j, \tau_j \right) : j = 1, \dots, p \right\} \text{ is an arbitrary } \gamma_n \text{-fine M- system (H- system) we have} \\ \left| \sum_{j=1}^{p} (f(\tau_j) \, \mu(F_j) - (oM) \int_{F_j} f) \right| &\leq b_n \\ \left| \sum_{j=1}^{p} (f(\tau_j) \, \mu(F_j) - (oM) \int_{F_j} f) \right| &\leq b_n \end{aligned}$$

For any $f \in \mathcal{F}$.

2. The (o)-convergence properties of ordered equi –integrals Theorem 2.1.

If $\mathcal{F} = \{f_u: T \to X; u \in \mathbb{N}\}$ is (oM)-equi-integrable sequence such that.

(o)
$$-\lim_{u\to\infty} f_u(t) = f(t), \qquad t\in T$$

Then the function $f: T \to X$ is (o)-McShane integrable and holds the equation

$$((o) - \lim_{u \to \infty} (oM) \int_T f_u = (oM) \int_T f).$$

Definition 2.2. A collection \mathcal{F} of functions $f: T \to X$ is called strongly (oM)-equi-integrable (strongly (oH)-equi-integrable) if every $f \in \mathcal{F}$ is strongly (o) –McShane integrable (strongly (o) –Henstock-Kurzweil integrable) and for any (o)- sequence $(b_n)_n$ there is a corresponding sequence $(\gamma_n)_n$ of gauges such that for every n and for every γ_n -fine M- partition(H-partition) $\Pi = \{(E_i, t_i): i = 1, ..., s\}$, of T and any $f \in \mathcal{F}$ holds the inequality

$$\left|\sum_{i=1}^{s} f(t_i) \, \mu(E_i) - F(E_i)\right| \le b_n$$

F is the additive *X*-valued interval function corresponding to $f \in \mathcal{F}$.

Theorem 2.3.

If $\mathcal{F} = \{f_u: T \to X; u \in \mathbb{N}\}$ is a strongly (oM)-equi-integrable sequence such that.

(o)
$$-\lim_{u\to\infty} f_u(t) = f(t), \qquad t\in T$$

Then the function $f: T \to X$ is strongly (o)-McShane

$$(o) - \lim_{u \to \infty} F_u(T) = F(T).$$

 F_u , F are the additive X-valued interval function corresponding to f_u , f respectively.

Proof. From the Definition 2.2. implies the (oM)-equi-integrability in dhe sense of Definition 1.3, Theorem 2.1 implies the Mcshane integrability of f as well as the relation

$$(o) - \lim_{u \to \infty} F_u(T) = F(T).$$

for every interval $E \subset T$.

Let (o)- sequence $(b_n)_n$ be given and let $(\gamma_n)_n$ be the corresponding sequence of gauges from the definition of strong (oM)-equi-integrability of the sequence f_u . Suppose that $\Pi = \{(E_i, t_i): i = 1, ..., s\}$ is an arbitrary γ_n -fine M- partition, of T and consider the sum

$$\sum_{i=1}^{s} |f(t_i)\mu(E_i) - F(E_i)| \leq \sum_{i=1}^{s} |f(t_i)\mu(E_i) - f_u(t_i)\mu(E_i)| + \sum_{i=1}^{s} |f_u(t_i)\mu(E_i) - F_u(E_i)| + \sum_{i=1}^{s} |F_u(E_i) - F(E_i)| \leq b_n + b_n + b_n$$

We obtain

$$\sum_{i=1}^{s} |f(t_i)\mu(E_i) - F(E_i)| \le 3b_n$$

and the strong (o)- McShane integrability of f is proved.

Analogously a similar convergence result for the strong (o)- Henstock-Kurzweil integrable of f can be proved.

3.Conclusions

Some convergece theorems for the order - equi-integrals with values Banach lattice are proved in particular, we give here some convergence results for the strong version of order – equi-integrals on Banach lattice and arrive same result in L-space as on Mcshane (Henstock Kurzweil) norm-integrals.

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Multistep Hermit Collocation Method for Solving Stochastic Fractional Integro-Differential Equationss

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Abstract

The main concern of this study is to propose high order multistep collocation method for evaluating the numerical solution of stochastic fractional integro-differential equations. For this purpose the unknown function is approximated by Hermit interpolation and its Caputo fractional derivatives are calculated and substituted in main equation. Also using some concepts of financial mathematics, It'o integral in main problem transform to classic Stieltjes integral. Then utilizing multistep collocation method, obtained equation reduces to some algebraic system. Illustrative examples are given for showing the efficiency and accuracy of the method.

Keywords: Hermit interpolation, Stochastic integro-differential equation, Fractional calculus, Newton Cotes quadrature

1. Introduction

The aim of this research is to present a high order multistep collocation method for numerically solving the stochastic fractional integro-differential equations of the form

$$D^{\alpha}\xi(t) = \nu(t) + \int_{0}^{t} \kappa_{1}(s,t)\xi(t)dt + \int_{0}^{t} \kappa_{2}(s,t)\xi(t)dB(t), \qquad 0 \le t \le 1, \qquad (1)$$

with initial conditions $\xi^{(j)}(0) = \xi_j$, $j = 0, 1, ..., [\alpha]$, where ξ, ν and $\kappa_i, i = 1, 2$ are the stochastic processes defined on same probability space, B(t) is a Brownian motion and $\alpha \in R$ and all Lebesgue and It'o integrals in the integral form of (1) are well defined. First the unknown function $\xi(t)$ is interpolated by Hermit interpolation and its Caputo fractional derivatives are calculated and substituted in equation (1). Also using some concepts in financial mathematics, It'o integral in main problem transforms to classic Stieltjes integral. Then multistep collocation method is applied to reduce the obtained equation to some algebraic system and for evaluating some integrals Gaussian quadrature is utilized. Illustrative examples are given for showing the efficiency and accuracy of the method.

2. Some Preliminaries

In this section we briefly mention some necessary definition and concepts for following discussion.

2.1. Brownian motion process

Definition 1: [1] A real-valued stochastic process $B(t), t \in [0, T]$ is called Brownian motion, if it satisfies the following properties:

- (Independence of increments) B(t) B(s), for t > s, is independent of the past.
- (Normal increments) B(t) B(s), has normal distribution with mean 0 and variance t - s.
- (Continuity of paths) B(t), $t \ge 0$ is a continuous function of t.

Note 1: In this paper we consider B(0) = 0 (with probability 1).

Lemma 1: (Integration by parts [1]) Suppose $f(s, \omega) = f(s)$ only depends on s and f is continuous and of bounded variation in [0, t]. Then

$$\int_{0}^{t} f(s) dB_{s} = f(t)B_{t} - \int_{0}^{t} B_{s} df_{s}.$$
 (3)

2.2. Fractional calculus

Definition 5: The Caputo definition of the fractional-order derivative is defined as:

$$D^{\alpha}f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{f^n(t)}{(x-t)^{\alpha+1-n}} dt, \quad n-1 < \alpha \le n, \ n \in N,$$

where $\alpha > 0$ is the order of the derivative and n is the smallest integer greater than α . For the Caputo derivative we have [2]:

- $D^{\alpha}C = 0$, (C is a constant),
- $D^{\alpha}x^{\beta} = 0, \ \beta \in N_0, \ \beta < [\alpha],$ $D^{\alpha}x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}x^{\beta-\alpha}, \ \beta \in N_0, \ \beta \ge [\alpha] \ or \ \beta \in R N_0, \ \beta > [\alpha],$

where $[\alpha]$ denotes the smallest integer greater than or equal to α and $|\alpha|$ denotes the largest integer less than or equal to α and $N_0 = \{0, 1, 2, ...\}$.

3. Multistep Hermit collocation method

In this section, we apply multistep Hermit collocation method to approximate the solution of stochastic fractional integro-differential equations (1). Let Δ be a uniform partition of the interval I with grid points $t_n = nh$, n = 0, 1, ..., N and let h be the step size, so we get N subinterval $\Delta_i = [t_i, t_{i+1}], i = 0, 1, \dots, N - 1$. For given real numbers ρ_i with

$$0 \le \rho_1 < ... < \rho_M \le 1$$
, we choose the following collocation points in the subinterval Δ_i :
 $t_{i,j} = t_i + \rho_j h, \ j = 1, 2, ..., M, \ i = 1, 2, ..., N - 1.$

Now suppose ψ be the interpolant polynomial of ξ evaluated by Hermit interpolation and defined in the interval Δ_i as

$$\psi_i(s) = \psi(t_i + sh) = \sum_{k=0}^{1} \sum_{j=1}^{M} h^k \,\beta(i, j, k) \gamma_{k,j}(s), \quad 0 \le s \le 1, \tag{4}$$

where $\gamma_{k,j}$ are polynomials of degree 2M - 1 to be determined by imposing the interpolation conditions

$$\psi^{(k)}(t_{i,j}) = \beta(i,j,k), \ k = 0,1, \quad j = 1, 2, \dots, M.$$
 ¹²⁹ (5)

The method is constructed by imposing the collocation conditions, which will be described in the following. The approximation of ξ' in the interval Δ_i is given by

$$h\psi'_{i}(s) = h\psi'(t_{i} + sh) = \sum_{k=0}^{1} \sum_{j=1}^{M} h^{k} \beta(i, j, k)\gamma'_{k,j}(s).$$
(6)

Applying the interpolation conditions (5), we get

 $\gamma_{0,j}(\rho_r) = \delta_{jr}, \quad \gamma'_{0,j}(\rho_r) = 0, \quad \gamma_{1,j}(\rho_r) = 0, \quad \gamma'_{1,j}(\rho_r) = \delta_{jr},$ where δ_{jr} is the Kronecker delta. In Hermit interpolation for j = 1, 2, ..., M we have

$$\gamma_{0,j}(s) = L_{0,j}(s) - L'_{0,j}(\rho_j)L_{1,j}(s), \qquad \gamma_{1,j}(s) = L_{1,j}(s),$$

where

$$L_{0,j}(s) = \prod_{i=1, i \neq j}^{M} (\frac{s - \rho_i}{\rho_j - \rho_i})^2, \qquad L_{1,j}(s) = (s - \rho_j) L_{0,j}(s).$$

Now we put

$$\beta_{i} = (\beta(i, 1, 0), \dots, \beta(i, M, 0), \beta(i, 1, 1), \dots, \beta(i, M, 1))^{T}, \Phi(s) = (\gamma_{0,1}(s), \dots, \gamma_{0,M}(s), \gamma_{1,1}(s), \dots, \gamma_{1,M}(s))^{T},$$

thus we can write the vector form of equations (4), as $\psi_i(s) = \beta_i^T \Phi(s)$. Utilizing property I (integration by parts) we can rewrite equation (1) as

$$D^{\alpha}\xi(t) = \nu(t) + k_2(t,t)\xi(t) + \int_0^t k_1(\tau,t)\xi(\tau)d\tau - \int_0^t K_2(\tau,t)\psi_i(\tau)B(\tau)d\tau,$$
(7)
$$K_i(\tau,t) = \int_0^{\theta} (k_i(\tau,t)\xi(\tau)) \text{ New by substituting the instead of }\xi \text{ in equation (7) and}$$

where $K_2(\tau, t) = \frac{\partial}{\partial \tau} (k_2(\tau, t)\xi(\tau))$. Now by substituting ψ_i instead of ξ in equation (7) and discritizing the obtained equation in mesh points $t_{i,j}$, we get

$$D^{\alpha}\psi_{i}(s)|_{\rho_{j}} = \nu(t_{i,j}) + k_{2}(t_{i,j}, t_{i,j})\psi_{i}(s)|_{\rho_{j}} + \int_{0}^{t_{i,j}} k_{1}(\tau, t_{i,j})\psi_{i}(\tau)d\tau - \int_{0}^{t_{i,j}} K_{2}(\tau, t_{i,j})\psi_{i}(\tau)B(\tau)d\tau, \qquad (8)$$

also we can write

$$\int_{0}^{t_{i,j}} k_{1}(\tau, t_{i,j}) \psi_{i}(\tau) d\tau = \sum_{r=0}^{i-1} \int_{t_{r}}^{t_{r}+1} k_{1}(\tau, t_{i,j}) \psi_{i}(\tau) d\tau + \int_{t_{i}}^{t_{i}+\rho_{j}h} k_{1}(\tau, t_{i,j}) \psi_{i}(\tau) d\tau$$
$$= h \sum_{r=0}^{i-1} \int_{0}^{1} k_{1}(\tau, t_{i,j}) \psi_{i}(\tau) d\tau + h \int_{0}^{\rho_{j}} k_{1}(\tau, t_{i,j}) \psi_{i}(\tau) d\tau, \quad (9)$$

similarly

$$\int_{0}^{t_{i,j}} K_{2}(\tau, t_{i,j}) \psi_{i}(\tau) B(\tau) d\tau = \sum_{r=0}^{i-1} \int_{t_{r}}^{t_{r}+1} K_{2}(\tau, t_{i,j}) \psi_{i}(\tau) B(\tau) d\tau + \int_{t_{i}}^{t_{i}+\rho_{j}h} K_{2}(\tau, t_{i,j}) \psi_{i}(\tau) B(\tau) d\tau = h \sum_{r=0}^{i-1} \int_{0}^{1} K_{2}(\tau, t_{i,j}) \psi_{i}(\tau) B(\tau) d\tau + h \int_{0}^{\rho_{j}} K_{2}(\tau, t_{i,j}) \psi_{i}(\tau) B(\tau) d\tau,$$
(10)
Therefore using (11)-(12), we have

Therefore using (11)-(12), we have

$$D^{\alpha}\psi_i(s)|_{\rho_j} = \nu(t_{i,j}) + k_2(t_{i,j}, t_{i,j})\psi_i(\rho_j)$$

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$$+h\sum_{r=0}^{i-1}\int_{0}^{1} \left(k_{1}(\tau,t_{i,j})-K_{2}(\tau,t_{i,j})B(\tau)\right)\psi_{i}(\tau)d\tau$$
$$+h\int_{0}^{\rho_{j}}\left(k_{1}(\tau,t_{i,j})-K_{2}(\tau,t_{i,j})B(\tau)\right)\psi_{i}(\tau)d\tau,$$

Applying vector form of
$$\psi_i(s)$$
 we get
 $\beta_i^T D^{\alpha} \Phi_i(s)|_{\rho_j} = v(t_{i,j}) + \beta_i^T k_2(t_{i,j}, t_{i,j}) \Phi_i(\rho_j)$
 $+ h \beta_i^T \sum_{\substack{r=0 \ \rho_j}}^{i-1} \int_{0}^{1} \left(k_1(\tau, t_{i,j}) - K_2(\tau, t_{i,j}) B(\tau) \right) \Phi_i(\tau) d\tau$
 $+ h \beta_i^T \int_{0}^{\tau} \left(k_1(\tau, t_{i,j}) - K_2(\tau, t_{i,j}) B(\tau) \right) \Phi_i(\tau) d\tau.$ (11)

Now for calculating the unknown coefficients β_i in equation (11) we should approximate the integral terms by some appropriate quadrature rule. For this purpose we apply Newton cotes rule, using suitable variable change in the second integral of (11), we get

$$\beta_{i}^{T} D^{\alpha} \Phi_{i}(s)|_{\rho_{j}} = \nu(t_{i,j}) + \beta_{i}^{T} k_{2}(t_{i,j}, t_{i,j}) \Phi_{i}(\rho_{j}) + h \beta_{i}^{T} \sum_{r=0}^{i-1} \sum_{j=1}^{Q} \left(k_{1}(\tau_{j}, t_{i,j}) - K_{2}(\tau_{j}, t_{i,j}) B(\tau) \right) \Phi_{i}(\tau_{j}) \omega_{j} + \frac{h \beta_{i}^{T}}{\rho_{j}} \sum_{j=1}^{Q} \left(k_{1}(\tau_{j}, t_{i,j}) - K_{2}(\tau_{j}, t_{i,j}) B(\tau_{j}) \right) \Phi_{i}(\tau_{j}).$$
(12)

where τ_j and ω_j , j = 1, 2, ..., Q are Newton cotes points and weights, respectively. It should be considered that for evaluating the values of Brownian motion B(.) in Newton cotes points, we utilize the definition of Brownian motion. We know that B(t) has normal distribution $B(t) - B(s) \sim \sqrt{t - sN(0, 1)}, t > s$. So we set step length $\Delta \gamma = \frac{1}{T}$ for some positive integer T and let $B_j = B(\gamma_j)$ and $\gamma_j = j\Delta\gamma$. Applying Note 1, we have $B_j = B_{j-1} + dB_j, j = 1, 2, ..., T$. Also each dB_j is an independent random variable of the form $\sqrt{\Delta\gamma}N(0, 1)$. Now, using linear spline interpolation at point (γ_j, B_j) approximate function for $B(\gamma)$ is obtained [3].

4. Illustrative Example

In this section for showing the validity and accuracy of proposed method we calculate the absolute error $|\xi(t, B(t)) - \psi_N(t, B(t))|$, where $\xi(t, B(t))$ and $\psi_N(t, B(t))$ are the solutions of (1) obtained for T = 1000 and approximate solution of (12) by using N-points Newton cotes rule. Consider the stochastic fractional integro-differential (1) with

$$\nu(t) = \frac{7}{12}t^4 - \frac{5}{6}t^3 + \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}, \quad \kappa_1(s,t) = s+t, \quad \kappa_2(s,t) = s, \quad \alpha = 0.5,$$

with initial condition $\xi(0) = 0$. Table 1 shows the absolute error of presented method for some different values of *t* and *N*.

t	Q = 5	Q = 10	N = 5 [4]	N = 10 [4]
0.0	0.00000	0.00000	0.0000	0.0000
0.2	0.00014	0.00006	0.0016	0.0004
0.4	0.00075	0.00007	0.0013	0.0011
0.6	0.00082	0.00005	0.0081	0.0041
0.8	0.00033	0.00009	0.0055	0.0011
1.0	0.00058	0.00012	0.0168	0.0099

Table 1. Absolute errors for example 1

5. Conclusion

In this study a new approach based on multistep Hermit collocation method and Newton cotes quadrature is introduced for solving stochastic fractional integro-differential equation. For evaluating the Brownian motion in the points of quadrature spline interpolation is used. Absolute errors in table 1 show the high accuracy of the method. This method can be extended for numerical solution nonlinear stochastic fractional integro-differential equations with additional work.

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Utilizing B-Spline Operational Matrices for Solving a Class of Nonlinear Boundary Value Problems Arising in Chemical Reactor Modeling

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Abstract

In this work cubic B-spline wavelets and their operational matrices of derivative and integration are applied for numerical solution of some nonlinear boundary value problems which arise in modeling a tabular adiabatic chemical reactor. Properties of these wavelets via some projection methods leads to the nonlinear problem transforms to some algebraic system. For showing the accuracy and efficiency of the introduced methods one case study of main problem is given and findings are compared with the results of alternative methods for numerical solving of this class of equations.

Keywords: Boundary value problems, Cubic B-spline wavelets, Operational matrix of derivative and integration, Spectral methods

1. Introduction

The mathematical model for an adiabatic tubular chemical reactor which processes an irreversible exothermic chemical reaction can be reduced to

 $u''(x) - \lambda u'(x) + \omega(x, u(x)) = 0, \quad 0 \le x \le 1,$ subject to the following boundary conditions
(1)

 $u'(0) = \lambda u(0),$ u'(1) = 0, (2) where $\omega(x, u(x)) = \lambda \mu (\beta - u)e^u$ and λ, μ and β are the Peclet, Damkohler number and the dimensionless adiabatic temperature rise, respectively, which are impressive in determination of the steady state temperature of the reaction [1]-[2]. In fact the steady state temperature of the reaction is equivalent to a positive solution u of equation (1). Authors of [3] have studied on the existence of the positive solutions of this class of equations. Numerous researchers presented various numerical methods for solving this problem [4]-[6]. In this paper we present two new approaches for numerically solving of this kind of problems. In the first approach of this study using cubic B-spline wavelets and their operational matrix of derivative and spectral methods BVP (1)-(2) is converted to an algebraic system. Second method is divided to applying operational matrix of integration of B-spline wavelets and similarly to second method, the main problem is transformed to some algebraic system.

2. Cubic B-Spline Wavelets

Cubic B-spline scaling function $\varphi_4(x)$ is given by

$$\varphi_{4}(x) = \frac{1}{6} \begin{cases} x^{3}, & x \in [0,1) \\ -3x^{3} + 12x^{2} - 12x + 4, & x \in [1,2) \\ 3x^{3} - 24x^{2} + 60x - 44, & x \in [2,3) \\ (4-x)^{3}. & x \in [3,4) \end{cases}$$
(3)

2.1. Boundary scaling adaptation

• Left boundary cubic B-spline scaling functions: We define the boundary near functions at the left boundary by

 $\phi_{3,k}(x) = \varphi_4(8x - k)\chi_{[0,1]}(x), \qquad k = -3, -2, -1,$ (4) and for other levels of J we have

$$\phi_{J,k}(x) = \varphi_4(2^J x - k)\chi_{[0,1]}(x), \qquad k = -3, -2, -1, \qquad J = 3, 4, \dots$$
(5)

• Right boundary cubic B-spline scaling functions: For the right end of the interval, note that, by symmetry we have the following relations

$$\phi_{3,5}(x) = \phi_{3,-1}(1-x)\chi_{[0,1]}(x), \tag{6}$$

$$\phi_{3,6}(x) = \phi_{3,-2} (1-x)\chi_{[0,1]}(x), \tag{7}$$

$$\phi_{3,7}(x) = \phi_{3,-3}(1-x)\chi_{[0,1]}(x), \tag{8}$$

and for other levels of J, we have

$$\phi_{J,2^J-k-3}(x) = \phi_{3,k}(2^J x - k)\chi_{[0,1]}(x), \qquad k = -3, -2, -1, J = 3, 4, \dots$$
(9)

2.2. Interior scalings

Five interior cubic B-spline scaling functions are chosen as

$$\phi_{3,k}(x) = \varphi_4(8x - k)\chi_{[0,1]}(x), \qquad k = 0,1,2,3,4,$$
(10)
and for other levels of *J*, we get

$$\varphi_{J,k}(x) = \varphi_4(2^J x - k)\chi_{[0,1]}(x), \qquad k = 0,1,\dots,2^J - 4, \quad J = 3,4,\dots$$
(11)
Two scale dilation relation for cubic B-Spline wavelet is given by

$$\Psi_4(x) = \sum_{k=0}^{10} \frac{(-1)^k}{8} {4 \choose l} \varphi_8(k-l+1) \varphi_4(2x-k), \tag{12}$$

Other inner and boundary wavelets are constructed similarly as in [7].

2.3. Function Approximation

A function $f(x) \in L^2(R)$ may be approximated by cubic B-spline wavelets in arbitrary scale J_u as

$$f(x) \simeq \sum_{i=-3}^{2^{J_0-1}} c_{J_{0,i}} \phi_{J_{0,i}}(x) + \sum_{j=J_0}^{J_u} \sum_{k=-3}^{2^{J}-4} d_{j,k} \psi_{j,k}(x) = = C^{\mathrm{T}} \Upsilon(x),$$
(13)

where *C* and Υ are $2^{(J_{u}+1)}+3$ column vectors given by

$$C = (c_{J_0,-3}, \dots, c_{J_0,2^{J_0}-1}, d_{J_0,-3}, \dots, d_{J_u,2^{J_u}-4})^T,$$

$$Y = (\phi_{J_0,-3}, \dots, \phi_{J_0,2^{J_0}-1}, \psi_{J_0,-3}, \dots, \psi_{J_u,2^{J_u}-4})^T,$$

with

$$c_{J_0,i} = \int_0^1 f(x) \ \tilde{\phi}_{J_0,i}(x) dx, \qquad d_{j,k} = \int_0^1 f(x) \ \tilde{\psi}_{j,k}(x) dx,$$

where $\tilde{\phi}_{J_0,i}$ and $\tilde{\psi}_{j,k}$ are dual of cubic B-spline scaling functions and wavelets, respectively and can be obtained by linear combination of cubic B-spline scaling and wavelet functions [7].

3. Description of the numerical methods

In this section we purpose two new and computational wavelets based methods for solving BVP (1)-(2). In the first method cubic B-spline wavelets operational matrix of derivative (BS-OMD) and in the second method cubic B-spline wavelets operational matrix of integration (BS-OMI) are applied via some projection procedures for solving the main problem.

3.1. Approach I: : BS-OMD

In this method the unknown functions in equation (1) are expanded by using cubic B-spline wavelets as equation (13),

$$u(x) = C_u^T \Upsilon(x), \qquad \omega(x, u(x)) = \omega(x) = C_\omega^T \Upsilon(x), \qquad (14)$$

also by using operational matrix of derivative, we can write $u'(x) = C_u^T D\Upsilon(x), \qquad u''(x) = C_u^T D^2 \Upsilon(x),$

substituting equations (14)-(15) in equation (1), we get $C_{u}^{T}(D^{2} - \lambda D)Y(x) + C_{u}^{T}Y(x) = 0,$

applying Galerkin method with
$$\Upsilon(x)$$
 as weighting functions in the interval [0,1] we get

$$\int_{0}^{1} (C_{u}^{T}(D^{2} - \lambda D) + C_{\omega}^{T})\Upsilon(x)\Upsilon^{T}(x) = (C_{u}^{T}(D^{2} - \lambda D) + C_{\omega}^{T})\Pi = 0,$$

where Π is invertible product matrix, so we have

$$C_{\omega} = -(D^2 - \lambda D)^{\mathrm{T}} C_{\mathrm{u}}, \qquad (16)$$

equation (16) is a linear system of algebraic equations with $2 \times (2^{J_u+1}+3)$ unknowns and $2^{J_u+1}+3$ equations. On the other hand we have

$$\omega\left(\mathbf{x}, \mathbf{C}_{\mathbf{u}}^{\mathrm{T}} \mathbf{\Upsilon}(\mathbf{x})\right) = \mathbf{C}_{\omega}^{\mathrm{T}} \mathbf{\Upsilon}(\mathbf{x}), \tag{17}$$

for having another $2^{j_u+1} + 3$ equations, which will complete system of equations (16), we collocate the equation (17) in the support points $\zeta_j = \frac{j}{2^{J_u+1}+3}$, $j = 1, 2, ..., 2^{J_u+1} + 1$. Considering the following boundary conditions

$$C_{u}^{T}D\Upsilon(0) - \lambda C_{u}^{T}\Upsilon(0) = 0, \qquad C_{u}^{T}D\Upsilon(1) = 0,$$

we obtain two equations, too. Therefore we have a system includes $2 \times (2^{J_u+1}+3)$ equations with the same number of unknowns which could be solved by some iteration methods.

3.2. Approach II: BS-OMI

In this scheme we put $\omega(x, u(x)) = \Omega(x)$ and approximate the functions describing the equation (1) by B-spline wavelets as (13)

$$\Omega(\mathbf{x}) = \mathbf{C}_{\Omega}^{\mathrm{T}} \Upsilon(\mathbf{x}), \qquad \qquad \mathbf{u}''(\mathbf{x}) = \mathbf{C}_{\mathbf{u}}^{\mathrm{T}} \Upsilon(\mathbf{x}), \qquad (18)$$

$$u'(x) = C_u^T \Re \Upsilon(x) + \eta_1, \qquad u(x) = C_u^T \Re^2 \Upsilon(x) + \eta_1 x + \eta_2$$
 (19)

where \Re is operational matrix of integration. Substituting equations (18)-(19) in equation (1), we have

$$C_u^T \Upsilon(x) - \lambda (C_u^T \Re \Upsilon(x) + \eta_1) + C_\Omega^T \Upsilon(x) = 0,$$

now utilizing Galerkin method via $\widetilde{\Upsilon}^T(x)$, we get

$$C_u^T(I - \lambda \mathfrak{R}) \int_0^1 \Upsilon(x) \tilde{Y}^T(x) dx + C_\Omega^T \int_0^1 \Upsilon(x) \tilde{Y}^T(x) dx = \lambda \eta_1 \int_0^1 \tilde{Y}^T(x) dx,$$

so

(15)

$$C_u^T(I - \lambda \mathfrak{R}) + C_\Omega^T = \lambda \eta_1 \Pi^{-1} \int_0^1 \tilde{Y}^T(x) dx, \qquad (20)$$

on the other hand we can write

$$\omega(x,u(x)) = \omega(x,C_u^T \mathfrak{R}^2 \Upsilon(x) + \eta_1 x + \eta_2) = C_\Omega^T \Upsilon(x),$$
⁽²⁵⁾

collocating the equation (25) and boundary conditions (2) in the support points $\zeta_j = \frac{J}{2^{J_u+1}+3}$ $j = 1, ..., 2^{J_u+1} + 3$, we get

- $\omega\left(\zeta_{j}, C_{u}^{T} \Re^{2} \Upsilon(\zeta_{j}) + \eta_{1}\zeta_{j} + \eta_{2}\right) = C_{\Omega}^{T} \Upsilon(\zeta_{j}),$
- $\eta_1 = -C_u^T \Re \Upsilon(1),$ • $\eta_2 = \frac{C_u^T \Re}{\lambda} (I - \lambda \Re - \Upsilon(1)).$

considering system of equations (20) and current equations, the nonlinear boundary value problem (1)-(2) reduces to an algebraic system which could be solved easily by some iteration method.

4. Case Study

Consider equations (1)-(2) with $\lambda = 10$, $\beta = 3$ and $\mu = 0.02$. Existence of the unique solution of this equation with these values was proved in [4] by the contraction mapping principle. This equation was solved by mentioned methods in pervious section. For having a geometric understanding of the solution and comparison of the effect of operational parameters in main problem the results of first method are given by plots and the results of other method are shown in tables 1 in some arbitrary mesh points, also in order to comparison results of some other methods are given in relevant tables. Also For this case by mentioned theorem in this section, our results satisfy the following inequality.

$$2.72399 \times 10^{\{-6\}} < u(x) \le 3, \ 0 \le x \le 1.$$



Figure 1.Numerical solution of case 1 obtained by approach I for $\beta = 3$ and $\mbox{}mu = 0.02$ and some values of λ .

5. Conclusions

In the present study, two interesting techniques have been developed for solving nonlinear boundary value problems arising in chemical reactor modeling. In these methods some projection methods such as Galerkin and collocation methods via compactly supported cubic B-spline wavelets, as testing and weighting functions, and their operational matrices of derivative and integration are applied. The purposed methods are applied for reducing the nonlinear boundary value problem to some algebraic system.

x _j	$J_u = 3$	$J_u = 4$	[5] N = 10	[6] N = 9
0.0	0.006045	0.006048	0.006049	0.006046
0.2	0.018184	0.018192	0.018197	0.018190
0.4	0.030419	0.030424	0.030437	0.030423
0.6	0.042660	0.042669	0.042649	0.042666
0.8	0.054365	0.054371	0.054383	0.054369
1.0	0.061451	0.061458	0.061459	0.061459

Table 1. Comparison of the results of case 1 obtained by approach II

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Analyzing Textbooks to Teach Proof Related Activities at Middle School Level

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Abstract

Proof has received significant attention in mathematics at all grade levels, and is expected to be an important part of every student's education. Despite the importance given to proof, the corpus of existing literature has demonstrated that many secondary school teachers and pre-service mathematics teachers have difficulties constructing and understanding proofs. Textbooks are a crucial link between intended and implemented curriculum because they provide teachers to identify content to be taught, to choose appropriate instructional strategies, and to assess students' learning. The purpose of this study is to examine how pre-service mathematics teachers identify and modify tasks/problems/activities in middle school textbook in Turkey. Twenty four pre-service teachers participated in this study and all attended a course called mathematical reasoning, justification, and proof for 14 weeks. The primary sources of data were students reports on textbook analysis and their classroom presentation on their reports. The analysis showed that pre-service teachers were somehow successful while identifying tasks/problems/activities related to proof in the textbooks, but had a hard time while modifying them to teach proof. The students found that there were almost none tasks that specifically focus on proof related activities, but found some tasks that can be modified into proof-related activities.

Keywords: Reasoning and Proof, Textbook Analysis, Teacher Education

1. Introduction

The importance of the teaching and learning of justification and justification is undisputed within the mathematics education community. That is mainly because proof is an essential part of mathematics given its roles in establishing the truth of mathematical statements (Tall & Mejia-Ramos, 2006), explaining why such statements are true and convincing (e.g. Hersh, 1993; Hanna, 2000; Harel & Sowder, 1998), and promoting mathematical communication and development (Schoenfeld, 1994). Proof can also be seen as

a way of problem solving that removes doubt about the validity of mathematical statements (e.g. Selden & Selden, 2003; Weber, 2005; Harel & Sowder, 1998) and as a tool for learning mathematics (Knuth, 2002a). Therefore, stakeholders (e.g., National Council of Teachers of Mathematics [NCTM], 2000; Council of Chief State School Officers [CCSSO], 2010; Mathematical Association of America [MAA], 2004) as well as mathematics education researchers (e.g. Harel & Sowder, 1998; Hanna, 1995, 2000; Knuth, 2002a, 2002b) advocate for the increased prominence of proof and reasoning in the mathematics education of students at all levels. Despite the importance given to proof, the corpus of existing literature has demonstrated that many secondary school teachers and pre-service mathematics teachers have difficulties constructing and understanding proofs (Dogan, 2015). The mean reasons that many teachers find the teaching of proof difficult, often due to their beliefs about teaching proof and their perceptions that proof is not a mathematical practice that can be integrated into the curriculum at all grade levels (Knuth, 2002a). Textbooks are a crucial link between intended and implemented curriculum because they provide teachers to identify content to be taught, to choose appropriate instructional strategies, and to assess students' learning. Considering several researchers that link between curricula and students' learning, in order to have an effective students learning of mathematics, especially proof, teachers need to understand and analyze their textbooks with a specific goal. For this reason, the purpose of this study is to examine how pre-service mathematics teachers identify and modify tasks/problems/activities in middle school textbook in Turkey.

2. Materials and Methods

Twenty four pre-service teachers participated in this study and all attended a course called mathematical reasoning, justification, and proof for 14 weeks. At the end of the semester, the students were asked to analyze a unit of the textbook with the focus of numbers, algebra, geometry, and probability and statistics. Their main assignment was to identify tasks/problems/activities in middle school textbooks and modify tasks/problems/activities to teach mathematics reasoning, justification, and proof. The primary sources of data were students reports on textbook analysis and their classroom presentation on their reports. Open coding (Glaser and Strauss, 1967) used to analyze the data.

3. Results and Discussions

The preliminary analysis showed that pre-service teachers were somehow successful while identifying tasks/problems/activities related to proof in the textbooks, but had a hard time while modifying them to teach proof. The students found that there were almost none tasks that specifically focus on proof related activities, but found some tasks that can be modified into proof-related activities. The students identified 309 tasks in the middle school textbook. They claimed that 105 of 309 tasks can be modified to a proving task. However, while modifying the tasks, they correctly modified only 46 of 105 tasks as a proving task. For example, the following task asks to find $m(\widehat{DAE})$.



In order to modify the task, the participant first solved it and found the angle as $m(DAE) = 15^{\circ}$, and then edited the question as "explain that the (DAE) is equal to 15° ?". He claimed that just having the word "explain" would make the task a proving task. Another participant modified the following task: "What is the area of a rectangle with sides 7 cm and 4 cm?". He changed the question as: "The area of a rectangle with sides 7 cm and 4 cm is 28 cm^2 . Prove?" As seen from these examples, the main criteria for modifying the tasks for the students was to change the wording of the problem and add the word "explain", "proof" or "justification" on the problem root as the question.

4. Conclusions

The research literature clearly shows that pre-service teachers' conceptions of proof are not well aligned with both researchers' and policy documents' expectations that proof ought to be central to mathematics education and a learning tool at all grade levels, and that teachers must possess a sound understanding of justifications and proof. If teachers have a robust understanding of justification and proof, they might be able to help their students develop a better understanding of proof. Afflowever, one

of the biggest challenges in learning and teaching of proof is how to help teachers develop their knowledge of proof so that their instruction supports their students to develop a better understanding of proof. Thus, having opportunities for pre-service teachers to engage in reasoning and proving activities might allow us to have teachers who are well-equipped for teaching proof in future. It is important to note that integrating their course with middle school textbook was a new idea for them. Thus, even though they presented a limited understanding of proof in textbooks, providing the opportunity helped them enhance their understanding of learning and teaching proof at the middle school level.

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Pre-service Teachers' Notion of Generic Example Proof

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Abstract

Proof is an essential aspect of mathematical activity and central to mathematics. Generic example proofs have received special attention in the literature and are seen as an important aspect of producing justifications and proofs since generic example serves not only to confirm an instance of a statement but to convey an argument to explain why such a statement is always true. Therefore, generic example proofs can be a crucial part of learning and teaching proof, especially at the K-12 level. This study explores how preservice secondary mathematics teachers evaluate arguments that consist of generic examples. The data was collected at an undergraduate mathematics course designed to engage pre-service middle school teachers in proof-related activities as a means for developing the mathematical knowledge and skills needed for effectively teaching proof in middle school mathematics classrooms. The primary source of data was transcribed video data of class session, with additional sources of data consisting of classroom artifacts and tasks. The results show that while evaluating arguments, students have a hard time to identify whether an argument that includes examples is an example-based argument or is a generic example proof. They did not see the difference between example-based reasoning and generic example proof. The result suggests that most pre-service teacher notion of generic example does not fit with the desired outcome.

Keywords: Proof, Generic Example, Teacher Education

1. Introduction

Proof is an essential aspect of mathematical activity and central to mathematics. Yet despite proof being viewed as a crucial mathematical activity, neither its various roles in mathematics nor its nature has permeated K-12 education or have been well understood by students. As discussed by Hersh (1993), the primary purpose of proof in the classroom is to explain and to

show why something is the case rather than just provide formal logic/proof. However, even though proof is crucial in terms of how mathematics is learned and done, neither the roles of proof in mathematics nor the nature of proof itself has been well understood by students at all levels. Thus, the teaching and learning of justification and proof is of major concern to K-16, especially to secondary and tertiary level mathematics education. Generic example proofs have received special attention in the literature and are seen as an important aspect of producing justifications and proofs (e.g., Balacheff, 1988; Dogan, 2015; Stylanides, 2007) since generic example serves not only to confirm an instance of a statement but to convey an argument to explain why such a statement is always true. As defined by Pimm and Mason (1984), generic example that given as a particular does not rely on any specific properties of that number. "A generic example is an actual example, but one presented in such a way as to bring out its intended role as the carrier of general" (Pimm and Mason, 1984, p.284). Thus, if the presenters' purpose of example is to provide a general argument without relying on that actual example, the argument may count as a generic proof since it serves not only to confirm instance of a statement but to convey an argument to explain why such a statement is always true. Similarly, Balacheff (1988) describes generic examples as:

"The generic example serves not only to present a confirming instance of a proposition-which it certainly is-but to provide insight as to *why* the proposition holds true for that single instance. The transparent presentation of the example is such that analogy with other instances is readily achieved, and their truth is thereby made manifest. Ultimately the audience can conceive of no possible instance in which the analogy could not be achieved" (p.219)

Therefore, generic example proofs can be a crucial part of learning and teaching proof, especially at the K-12 level. However, there is no consensus on whether an argument using a generic example is a mathematical proof or not. In this paper, I see generic example as a viable argument and acceptable justification at K-12 grade levels. This study explores how preservice secondary mathematics teachers evaluate arguments that consist of generic examples.

2. Materials and Methods

The data was collected at an undergraduate mathematics course designed to engage preservice middle school teachers in proof-related activities as a means for developing the mathematical knowledge and skills needed for effectively teaching proof in middle school

mathematics classrooms. The primary source of data was transcribed video data of class session, with additional sources of data consisting of classroom artifacts and tasks. The students were given the following task and three hypothetical student arguments that justified the claim in the task (Adapted from Isler, 2015).

The sum of any three consecutive numbers is equal to three times the middle number. For example, 4, 5 and 6 are consecutive numbers and 4 + 5 + 6 equals 15, which equals three times the middle number, 5. Show that the sum of any three consecutive numbers is always equal to three times the middle number.

Three students gave the responses below. Please read all of the student responses, and then respond to the questions below.

Student 1: Emir

I found a way using marbles. I can make three columns of marbles representing any three consecutive numbers. The first column represents the first number; the second column represents the middle number, and the third <u>column represent</u> the last number. I can take the top marble from the last column and move it to the first column. This makes the number of marbles in each column the same as the number of marbles in the middle column. Since the total number of marbles is always three times the number in the middle column, I know the conjecture is always true.



Student 2: Damla

5, 6, and 7 are three consecutive numbers and 5 + 6 + 7 = 18, and $3 \ge 6 = 18$. 7, 8, and 9 are three consecutive numbers and 7 + 8 + 9 = 24, and $3 \ge 8 = 24$. 569, 570, and 571 are three consecutive numbers and 569 + 570 + 571 = 1710, and $3 \ge 570 = 1710$. Since it works in these three examples, I know the conjecture is always true.

Student 3: Kenan

I'll show you using 4, 5 and 6. I can write 4 as (5-1) and 6 as (5+1). So, it will be (5-1) + 5 + (5+1). Since adding 1 and taking away 1 cancels each other, there will be three 5's. So, you see that it equals adding three times the middle number that is 5.

Student 1 (Emir) and Student 3 (Kenan)' responses can be seen as valid arguments, but Student 2 (Damla)'s response can be seen as example based reasoning, but not a valid proof. I adapted Glaser and Strauss's (1967) constant comparison method to analyze the data.

3. Results and Discussions

The results show that while evaluating arguments, students have a hard time to identify whether an argument that includes examples is an example-based argument or is a generic example proof. 3 of the 23 participants found Student 1's argument as a valid proof. One of the participants claimed that "this is a valid proof because the number of marbles is not important here; from his model, we can see the structure of the argument..." Another one claimed that "it is not a valid argument because students only visualize the problem by just using one example, thus you cannot generalize it". As seen from the excerpts, most of the students claimed that the generic example does not count as proof since it uses a specific case and does not use a general/formal language. This was the case for the Student 3's argument as well, only 4 of the participants found that as a valid proof. Also, 3 of them found Student 2's argument as a valid proof. Students mostly said 'having a few examples is not enough to have a valid proof'. Considering all participants evaluations, it can be claimed that they did not see the difference between example-based reasoning and generic example proof. However, one student stated that if a specific example is used to convey a general argument, then that can be count as viable proof. She went further and argued that a generic example can be expressed in general terms. Her justification helped other students develop a more robust understanding of generic example. This is important because pre-service teachers need to identify what counts as a valid argument in order to teach proof.

4. Conclusions

One important goal of generic example proofs is to reduce the abstraction of the argument by not using variables and to make the argument accessible to all students. However, it is not clear how this goal can be achieved if teachers have some important misconceptions about generic example type proofs. The result suggests that most pre-service teacher notion of generic example does not fit with the desired outcome, yet if they have enough opportunities to engage in this kind of activities, they may well be prepared for teaching proof at the middle school level.

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On the Bertrand Dual Bezier Curve Pairs

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Bezier curves are the curves that we have seen the examples of their applications in the (CAD) (CAM) systems at very large field. Bezier curves have the property of staying in the convex hull, which is formed by a group of points called control points. Furthermore, when any of the control points of these curves is changed, the curve changes locally in a neighborhood of that point.

Keyword(s): Bertand curve, Bezier curve, dual vectors.

0.1 Dual Vectors and Dual Bezier Curves

A dual number A is defined as $A = a + \epsilon a^*$, where a and $a^* \in R$ and $\epsilon^2 = 0$ ($\epsilon \neq 0$). The set of all dual numbers is denoted by D. Similarly a dual vector X is defined as $X = x + \epsilon x^*$, where x and $x^* \in R^3$ and $\epsilon^2 = 0$. The set of all dual vectors is denoted by D^3 .

Let $A=a+\epsilon a^*~$ and $B=b+\epsilon b^*$ be two dual vectors. Then the inner product od these vectors is defined by

$$\langle A, B \rangle = \langle a, b \rangle + \epsilon \left(\langle a, b^* \rangle + \langle a^*, b \rangle \right) \tag{1}$$

If $||a|| \neq 0$ then the norm of a dual vector $A = a + \epsilon a^*$ is defined by

$$\|A\| = \|a\| + \epsilon \left(\frac{\langle a, a^* \rangle}{\|a\|}\right) \tag{2}$$

A dual Bezier curves is a Bezier curves which control points are dual vectors.

So A dual Bezier curve of degree n is defined by

$$B(t) = \sum_{i=0}^{n} \overline{b_i} B_i^n(t) \tag{3}$$

where the parameter $t \in [0, 1]$, the dual control points $\overline{b_i} = b_i + \epsilon b_i^*$ and the functions $B_i^n(t)$ are called Bersntein polynomials or Bernstein basis functions and defined by if $0 \le i \le n$ then $B_i^n(t) = {n \choose i} (1-t)^{n-i} t^i$; orherwise $B_i^n(t) = 0$.

Theorem 3.3. The First derivative of a dual Bezier curve B(t) of degree n with control points $\overline{b}_0, \overline{b}_1, ..., \overline{b}_n$ is

$$\frac{d}{dt}B(t) = \sum_{i=0}^{n-1} \bar{b}_i^{(1)} B_i^{n-1}(t) \tag{4}$$

where $b_i^{(1)} = n(b_{i+1} - b_i)$.

Theorem 3.4. A dual Bezier curve B(t) of degree n with control points $\bar{b}_0, \bar{b}_1, ..., \bar{b}_n$ is satisfied these properties as follows 1

$$B(0) = \bar{b}_0, \quad B(1) = \bar{b}_n \text{ (end point interpolation property)},$$

$$B'(0) = \frac{d}{dt}B(0) = n(\bar{b}_1 - \bar{b}_0) \text{ (end point tangent property)}, \quad (5)$$

$$B'(1) = \frac{d}{dt}B(1) = n(\bar{b}_n - \bar{b}_{n-1}).$$

Theorem 4.1. Let a dual Bezier curve B(t) of degree n with control points

 $\bar{b}_0, \bar{b}_1, ..., \bar{b}_n$ be given. Then the dual Frenet vector fields $\{\mathbf{T} = \mathbf{t} + \epsilon t^*, \mathbf{N} = \mathbf{n} + \epsilon n^*, \mathbf{B} = \mathbf{b} + \epsilon b^*\}$ of the dual Bezier curve B(t) at the point for t = 0 can be written instead of control points as

$$\begin{aligned} \mathbf{T} &| \quad t=0 = \frac{\tilde{b}_{1} - \tilde{b}_{0}}{\|\tilde{b}_{1} - \tilde{b}_{0}\|} = \frac{b_{1} - b_{0}}{\|b_{1} - b_{0}\|} + \epsilon \left(\frac{b_{1}^{*} - b_{0}^{*}}{\|b_{1} - b_{0}\|^{3}} + \frac{(b_{1} - b_{0}, b_{1}^{*} - b_{0}^{*})(b_{1} - b_{0})}{\|b_{1} - b_{0}\|^{3}}\right), \\ \mathbf{N} &| \quad t=0 = \frac{\tilde{b}_{2} - \tilde{b}_{1}}{\|\tilde{b}_{2} - \tilde{b}_{1}\|} \csc \Psi + \frac{\tilde{b}_{1} - \tilde{b}_{0}}{\|\tilde{b}_{1} - \tilde{b}_{0}\|} \cot \Psi \\ &= \left(\frac{b_{2} - b_{1}}{\|b_{2} - b_{1}\|} \csc \varphi + \frac{b_{1} - b_{0}}{\|b_{1} - b_{0}\|} \cot \varphi\right) + \\ + \epsilon \left[\csc \varphi \left(\frac{b_{2}^{*} - b_{1}^{*}}{\|b_{2} - b_{1}\|} + \frac{(b_{2} - b_{1}, b_{2}^{*} - b_{1}^{*})(b_{2} - b_{1})}{\|b_{2} - b_{1}\|^{3}} \right) - \varphi^{*} \frac{b_{2} - b_{1}}{\|b_{2} - b_{1}\|} \cot \varphi \right] + \\ + \epsilon \left[\left(\frac{b_{1}^{*} - b_{0}^{*}}{\|b_{1} - b_{0}\|} + \frac{(b_{1} - b_{0}, b_{1}^{*} - b_{0}^{*})(b_{1} - b_{0})}{\|b_{1} - b_{0}\|^{3}} \right) \cot \varphi - \frac{b_{1} - b_{0}}{\|b_{1} - b_{0}\|}(1 + \cot^{2}\varphi) \right], \\ \mathbf{B} &| \quad t=0 = \frac{(\tilde{b}_{1} - \tilde{b}_{0}) \Lambda (\tilde{b}_{2} - \tilde{b}_{1})}{\|(\tilde{b}_{1} - \tilde{b}_{0}) \Lambda (\tilde{b}_{2} - \tilde{b}_{1})\|} = \frac{(b_{1} - b_{0}) \times (b_{2} - b_{1})}{\|(b_{1} - b_{0}) \times (b_{2} - b_{1})\|} + \\ + \epsilon \left(\frac{(b_{1} - b_{0}) \times (b_{2}^{*} - b_{1}^{*}) + (b_{1}^{*} - b_{0}^{*}) \times (b_{2} - b_{1})}{\|(b_{1} - b_{0}) \times (b_{2} - b_{1})\|^{2}} \right) \\ + \epsilon \left(\frac{\langle (b_{1} - b_{0}) \times (b_{2} - b_{1}) + (b_{1}^{*} - b_{0}^{*}) \times (b_{2} - b_{1})}{\|(b_{1} - b_{0}) \times (b_{2} - b_{1})\|^{2}} \right) \\ + \epsilon \left(\frac{\langle (b_{1} - b_{0}) \times (b_{2} - b_{1}) + (b_{1}^{*} - b_{0}^{*}) \times (b_{2} - b_{1})}{\|(b_{1} - b_{0}) \times (b_{2} - b_{1})\|^{2}} \right) \\ + \epsilon \left(\frac{\langle (b_{1} - b_{0}) \times (b_{2} - b_{1}) + (b_{1}^{*} - b_{0}^{*}) \times (b_{2} - b_{1})}{\|(b_{1} - b_{0}) \times (b_{2} - b_{1})\|^{2}}} \right) \\ + \epsilon \left(\frac{\langle (b_{1} - b_{0}) \times (b_{2} - b_{1}) + (b_{1}^{*} - b_{0}^{*}) \times (b_{2} - b_{1})}{\|(b_{1} - b_{0}) \times (b_{2} - b_{1})\|^{2}}} \right) \\ + \epsilon \left(\frac{\langle (b_{1} - b_{0}) \times (b_{2} - b_{1}) + (b_{1}^{*} - b_{0}^{*}) \times (b_{2} - b_{1})}{\|(b_{1} - b_{0}) \times (b_{2} - b_{1})\|^{2}}} \right) \\ \left(\frac{b_{1} - b_{0} \otimes (b_{2} - b_{1}) + (b_{1}^{*} - b_{0}^{*}) \times (b_{2} - b_{1})}{\|(b_{1} - b_{0}) \times (b_{2} - b_{1})}} \right) \\ + \epsilon \left(\frac{b_{1} - b_{0} \otimes (b_{2} - b_{1}) + (b_{1}^{*} - b_{0}^{*}) \times (b_{2} - b_$$

where $\Psi = \varphi + \epsilon \varphi^*$ is dual angle between the the vectors $\bar{b}_1 - \bar{b}_0$ and $\bar{b}_2 - \bar{b}_1$.

Theorem 4.2. Let a dual Bezier curve B(t) of degree n with control points

 $\bar{b}_0, \bar{b}_1, ..., \bar{b}_n$ be given. Then the dual curvature and torsion of given dual Bezier curve B(t) at the point for t = 0 are

$$\bar{\kappa} = \frac{n-1}{n} \frac{\|b_2 - b_1\|}{\|\bar{b}_1 - \bar{b}_0\|^2} \sin \Psi,
\bar{\tau} = \frac{n-2}{n} \frac{\langle (\bar{b}_1 - \bar{b}_0) \times (\bar{b}_2 - \bar{b}_1), (\bar{b}_3 - \bar{b}_2) \rangle}{\|(\bar{b}_1 - \bar{b}_0) \times (\bar{b}_2 - \bar{b}_1)\|^2},$$
(7)

where Ψ is the angle between the vectors $ar{b}_1 - ar{b}_0$ and $ar{b}_2 - ar{b}_1$.

0.2 The De Casteljau Algorithm

The De Casteljau algorithm for dual Bezier curves is used to calculate the value $B(t_0)$ of Bezier curves B(t) at any parameter $t_0 \in [0, 1]$ and also used to divide

the dual Bezier curve into two curve segments called B_{left} and B_{right} .

Theorem 1 Let a dual Bezier curve B(t) of degree n with control points $\bar{b}_0, \bar{b}_1, ..., \bar{b}_n$ and any parameter $t_0 \in [0, 1]$ be given Then

$$B(t_0) = \bar{b}_0^n \tag{8}$$

where $\bar{b}_i^0 = \bar{b}_i$ for i = 0, 1, ..., n and

$$\bar{b}_i^j = (1 - t_0)\bar{b}_i^{j-1} + t_0\bar{b}_{i+1}^{j-1} \tag{9}$$

for j = 0, 1, ..., n and for i = 0, 1, ..., n - j [?]

As a result of this algorithm control points of devided dual Bezier curves B_{left} and B_{right} are $\{\bar{b}_0^0, \bar{b}_0^1, ..., \bar{b}_n^n\}$ and $\{\bar{b}_0^n, \bar{b}_1^{n-1}, ..., \bar{b}_{n-1}^1, \bar{b}_n^0\}$ respectively.

0.3 The dual Frenet Frame Fields of dual Bezier Curves

at Arbitrary Point t_0

Theorem 2 Let a dual Bezier curve B(t) of degree n with control points $\overline{b}_0, \overline{b}_1, ..., \overline{b}_n$ be given. Then the dual curvature and torsion of given dual Bezier curve B(t)at the point $B(t_0)$ for $t = t_0$ as

$$\kappa = \frac{n-1}{n} \frac{\left\| \bar{b}_2^{n-2} - \bar{b}_1^{n-1} \right\|}{\left\| \bar{b}_1^{n-1} - \bar{b}_0^n \right\|^2} \sin \bar{\Psi}$$
(10)

$$\tau = \frac{n-2}{n} \frac{\left\langle \left(\bar{b}_1^{n-1} - \bar{b}_0^n\right) \times \left(\bar{b}_2^{n-2} - \bar{b}_1\right), \left(\bar{b}_3^{n-3} - \bar{b}_2^{n-2}\right) \right\rangle}{\left\| \left(\bar{b}_1^{n-1} - \bar{b}_0^n\right) \times \left(\bar{b}_2^{n-2} - \bar{b}_1^{n-1}\right) \right\|^2} \tag{11}$$

where \bar{b}_i^j are control points of the dual Bezier segment B_{right} obtained by subdivision algorithm formulated by (??), $\bar{\Psi}$ is the angle between the vectors $\bar{b}_2^{n-2} - \bar{b}_1^{n-1}$ and $\bar{b}_1^{n-1} - \bar{b}_0^n$.

Theorem 3 Let a dual Bezier curve B(t) of degree n with control points $\bar{b}_0, \bar{b}_1, ..., \bar{b}_n$ be given. Then the dual Frenet vector fields $\{T, N, B\}$ of the dual Bezier curve B(t) at the point $B(t_0)$ for $t = t_0$ as

$$T \mid_{t=t_0} = \frac{\bar{b}_1^{n-1} - \bar{b}_0^n}{\left\| \bar{b}_1^{n-1} - \bar{b}_0^n \right\|}$$
(12)

$$B|_{t=t_0} = \frac{\left(\bar{b}_1^{n-1} - \bar{b}_0^n\right) \times \left(\bar{b}_2^{n-2} - \bar{b}_1^{n-1}\right)}{\left\| \left(\bar{b}_1^{n-1} - \bar{b}_0^n\right) \times \left(\bar{b}_2^{n-2} - \bar{b}_1^{n-1}\right) \right\|}$$
(13)

$$N|_{t=t_0} = \frac{\bar{b}_2^{n-2} - \bar{b}_1^{n-1}}{\|\bar{b}_2^{n-2} - \bar{b}_1^{n-1}\|} \csc{\bar{\Psi}} - \frac{\bar{b}_1^{n-1} - \bar{b}_0^n}{\|\bar{b}_1^{n-1} - \bar{b}_0^n\|} \cot{\bar{\Psi}}$$
(14)

where \bar{b}_i^j are control points of the dual Bezier segment B_{right} obtained by subdivision algorithm formulated by (??) and $\bar{\Psi}$ is angle between the the vectors $\bar{b}_1^{n-1} - \bar{b}_0^n$ and $\bar{b}_2^{n-2} - \bar{b}_1^{n-1}$.

1 Bertand curve Pairs

In literature it is well known that a Bertrand curve α is a curve whose principal normal is the principal normal of another curve β [??]. These curve pairs $\{\alpha, \beta\}$ are called Bertrand curve pairs. Now let these curve pairs $\{\alpha, \beta\}$ be considered as each of them of these curve is Bezier curve in \mathbb{R}^3 .

Let α be a dual Bezier curve of degree n with control points $\overline{b}_0, \overline{b}_1, \dots, \overline{b}_n$ and β be a dual Bezier curve of degree n with control points $\overline{c_0}, \dots, \overline{c_n}$ and $\{\alpha, \beta\}$ be Bertrand curve pairs. The dual Frenet-Serret frame on these curves can be stated as $\{T_\alpha, N_\alpha, B_\alpha\}$ and $\{T_\beta, N_\beta, B_\beta\}$ respectively.

It is clear that

$$N_{\alpha} = N_{\beta}$$

since $\{\alpha, \beta\}$ are Bertrand curve pairs

Theorem: Let α be a dual Bezier curve of degree n with control points $\overline{b_0}, \dots, \overline{b_n}$ and β be a dual Bezier curve of degree n with control points $\overline{c_0}, \dots, \overline{c_n}$. If the control points of these curves

$$\overline{c_i} = \overline{b_i} + a$$

for i = 0, 1, 2 are satisfied for $a \in D^3$ then these curves α and β are Bertrand pairs. This means If the control points of these curves are Tr(3, D) – equivalent then these curves α and β are Bertrand pairs where Tr(3, D) is the *translation* group in dual Euclidean space D^3

Theorem: Let α be a dual Bezier curve of degree n with control points $\overline{b_0}, \ldots, \overline{b_n}$ and β be a dual Bezier curve of degree n with control points $\overline{c_0}, \ldots, \overline{c_n}$. If the control points of these curves

$$\overline{c_i} = k\overline{b_i}$$

for i = 0, 1, 2 are satisfied for $k \in D$ then these curves α and β are Bertrand pairs. This means if the control points of these curves are LH(3, D) – equivalent then these curves α and β are Bertrand pairs where LH(3, D) is the *linear* homotethies' group in dual Euclidean space D^3

Theorem: Let α be a dual Bezier curve of degree n with control points $\overline{b_0}, ..., \overline{b_n}$ and β be a dual Bezier curve of degree n with control points $\overline{c_0}, ..., \overline{c_n}$. If these curves are Bertrand curve pairs then

$$\frac{\overline{b_2^{n-2}} - \overline{b_1^{n-1}}}{\left\|\overline{b_2^{n-2}} - \overline{b_1^{n-1}}\right\|} \csc \bar{\Psi} - \frac{\overline{b_1^{n-1}} - \overline{b_0^n}}{\left\|\overline{b_1^{n-1}} - \overline{b_0^n}\right\|} \cot \bar{\Psi} = \frac{\overline{c_2^{n-2}} - \overline{c_1^{n-1}}}{\left\|\overline{c_2^{n-2}} - \overline{c_1^{n-1}}\right\|} \csc \Phi - \frac{\overline{c_1^{n-1}} - \overline{c_0^n}}{\left\|\overline{c_1^{n-1}} - \overline{c_0^n}\right\|} \cot \Phi$$

is satisfied where $\overline{b_i^j}$ are control points of the dual Bezier segment B_{right} obtained

by subdivision algorithm formulated by $(\ref{eq:subdivision})$ and $\bar{\Psi}$ is angle between the the vectors $\overline{b_1^{n-1}} - \frac{\overline{b_0^n}}{c_1^{n-2}} - \overline{b_1^{n-1}}$ and Φ is angle between the the vectors $\overline{c_1^{n-1}} - \overline{c_0^n}$ and $\overline{c_2^{n-2}} - \overline{c_1^{n-1}}$.

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Neighborhood system of soft identity element of soft topological group

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Abstract

The aim of this study is to introduce the neighborhood system of soft identity element of soft topological groups by using the soft element which is defined by Wardowski in 2013. **Keywords:** Soft set, Soft element, Soft topology.

1. Introduction

Soft set theory is defined by Molodtsov in 1999. After this definition many autors has been contribute to this research area by defining soft groups [4,10], soft topologies [6,7] and soft element [3]. After having the definition of soft topological space and soft group, the axiomatization of the concept is soft topological group is a natural procedure. The aim of this study is to introduce the neighborhood system of soft identity element of soft topological groups by using the soft element which is defined by Wardowski in 2013.

2. Preliminaries

In this section, we recall some basic notions in soft set theory.

Definition 2.1: [1] Let U be an initial universal set and E be a set of parameters. Let P(U) denote the power set of U and $A \subseteq E$. A soft set F_A is called a soft set over U, where f_A is a mapping given by $f_A: E \to P(U)$ such that $f_A(x) = \emptyset$ if $x \notin A$.

Note that the set of all soft sets over U will be denoted by S(U) and all nonempty soft sets over U will be denoted by $S_f(U)$.

Definition 2.2: [8] A soft set F_A over U is said to be an empty soft set denoted by F_{\emptyset} , if for all $e \in E$, $f_A(x) = \emptyset$.

Definition 2.3: [8] A soft set F_A over U is said to be an A-universal soft set denoted by $\widetilde{F_A}$, if for all $e \in E$, $f_A(x) = A$. If A = E; then the A-universal soft set is called a universal soft set, denoted by $\widetilde{F_E}$.

Definition 2.4: [8] Let $F_A, F_B \in S(U)$. Then, F_A is a soft subset of F_B , denoted by $F_A \cong F_B$, if $f_A(e) \subseteq f_B(e)$ for all $e \in E$.

Definition 2.5: [8] Let $F_A, F_B \in S(U)$. Then, the soft union $F_A \widetilde{\cup} F_B$, the soft intersection $F_A \widetilde{\cap} F_B$ and the soft difference $F_A \widetilde{\setminus} F_B$ of F_A and F_B are defined by the approximate functions as;

 $f_{A \widetilde{\cup} B}(x) = f_A(x) \cup f_B(x), \qquad f_{A \widetilde{\cap} B}(x) = f_A(x) \cap f_B(x), \qquad f_{A \widetilde{\setminus} B}(x) = f_A(x) \setminus f_B(x)$

respectively. The soft complement $F_A^{\tilde{C}}$ of F_A is defined by the approximate function $f_{A\tilde{C}} = (f_A(e))^C$, where $(f_A(e))^C$ is the complement of the set $f_A(e)$; that is $f_{A\tilde{C}} = (f_A(e))^C = U/f_A(e)$ for all $e \in E$.

Definition 2.6 : Let $F_A \cong F_B \in S(U)$. The soft complement $(F_B^{\ C})_A$ in F_A is defined by the approximate function $f_{B^{\widetilde{C}}} = (f_B(e))^C$, where $(f_B(e))^C$ is the complement of the set $f_B(e)$ in the soft set F_A ; that is $f_{B^{\widetilde{C}}} = (f_B(e))^C = f_A(e)/f_B(e)$ for all $e \in E$.

Definition 2.7: [7] Let $F_A \in S(U)$. A soft topology on F_A , denoted by $\tilde{\tau}$, is a collection of soft subsets of F_A having the following properties, the pair $(F_A, \tilde{\tau})$ is called a soft topological space.

- i) $F_{\emptyset} \widetilde{F_A} \in \tilde{\tau}$,
- ii) $F_B, F_C \in \tilde{\tau}$, then $F_B \cap F_C \in \tilde{\tau}$,
- iii) Λ indexed set and for all $\alpha \in \Lambda$, $F_{B_{\alpha}} \in \tilde{\tau}$ then $\bigcup_{\alpha \in \Lambda}^{\sim} F_{B_{\alpha}} \in \tilde{\tau}$.

Definition 2.8: [7] Let $(F_A, \tilde{\tau})$ be a soft topological space. Then every element of $\tilde{\tau}$ is called soft open set. Clearly F_{\emptyset} and $\widetilde{F_A}$ are soft open sets.

Definition 2.9: [3] Let $F_A \in S(U)$. We say that $\alpha = (e, \{u\})$ is nonempty soft element of F_A , if $e \in E$ and $u \in F(e)$. The pair (e, \emptyset) , where $e \in E$, will be called an empty soft element of F_A . The fact that $(e, \{u\})$ is a soft element of F_A will be denoted by $(e, \{u\}) \in F_A$. We denote the set of all nonempty soft elements of F_A by F_A^{\bullet} .

Example 1: Let $U = \{h_1, h_2, h_3\}$, $E = \{e_1, e_2, \}$. Take a soft set $F_A \in S(U)$ of the form $F_A = \{(e_1, \{h_1\})\}$. Then all the soft elements of F_A are (e_1, \emptyset) , $(e_1, \{h_1\})$, (e_2, \emptyset) .

Definition 2.10: Let $(F_A, \tilde{\tau})$ be a soft topological space, $F_B \cong F_A$ and $\alpha \in F_B$. If there exist a soft open set F_C such that $\alpha \in F_C \cong F_B$ then F_B is called soft neighborhood of soft element α .

If F_B is soft open set then F_B is called soft open neighborhood. We denote the set of all soft open neighborhoods of α by $\widetilde{\mathcal{N}}_{\alpha}$.

Example 2: $U = \{u_1, u_2, u_3\}, A = \{x_1, x_2, x_3, \}$ and

 $F_A = \{(x_1, \{u_1, u_3\}), (x_2, \{u_1\}), (x_3, \{u_1, u_2\})\}$ is a soft set over U. Some of the soft subsets of F_A are listed below.

$$F_{A_1} = \{(x_1, \{u_1\})\}, F_{A_2} = \{(x_1, \{u_1, u_3\})\}, F_{A_3} = \{(x_1, \{u_1\}), (x_2, \{u_1\})\},$$

$$F_{A_4} = \{(x_1, \{u_1, u_3\}), (x_2, \{u_1\})\},$$

The collections of $F_A \tilde{\tau}_1 = \{ \tilde{\emptyset}, F_A \}, \tilde{\tau}_2 = \tilde{P}(F_A)$ ve $\tilde{\tau}_3 = \{ \tilde{\emptyset}, F_A, F_{A_1}, F_{A_2}, F_{A_3}, F_{A_4} \}$ are soft topology. Nonempty soft elements of F_A are

{((x_1 , { u_1 }), (x_1 , { u_3 }), (x_2 , { u_1 }), (x_3 , { u_1 }), (x_3 , { u_2 })}. So set of all Soft neighborhoods for each soft element is given in the below fort he soft topology $\tilde{\tau}_3$.

$$\begin{split} \widetilde{\mathcal{N}}_{(x_1,\{u_1\})} &= \{F_A, F_{A_1}, F_{A_2}, F_{A_3}, F_{A_4}\}. \ \ \widetilde{\mathcal{N}}_{(x_1,\{u_3\})} &= \{F_{A_2}, F_{A_4}\}. \ \widetilde{\mathcal{N}}_{(x_2,\{u_1\})} &= \{F_A, F_{A_4}\}. \\ \widetilde{\mathcal{N}}_{(x_3,\{u_1\})} &= \{F_A\}. \ \widetilde{\mathcal{N}}_{(x_3,\{u_2\})} &= \{F_A\} \ . \end{split}$$

Definition 2.11: [3] Let $(F_A, \tilde{\tau})$ be a soft topological space and $\tilde{\beta} \subseteq \tilde{\tau}$. If every element of $\tilde{\tau}$ can be written as the union of elements of $\tilde{\beta}$, then $\tilde{\beta}$ is called a soft basis for the soft topology $\tilde{\tau}$. Each element of $\tilde{\beta}$ is called a soft basis element.

Definition 2.12: Let $(F_A, \tilde{\tau})$ be a soft topological space $\alpha \in F_A$ and $\tilde{\beta}_{\alpha}$ be the family of soft open neighborhoods of α . If for every soft open neighborhoods F_B of α there exist a soft set $F_C \in \tilde{\beta}_{\alpha}$ such that $F_C \subseteq F_B$ then $\tilde{\beta}_{\alpha}$ is called soft local base at the soft element α .

Definition 2.13: [10] Let (E, \circ) and (U, *) be two groupoids, $A \subseteq E$ and $F_A \in S_f(U)$. The binary operation $\tilde{*}$ on F_A^{\bullet} is defined by; $(e_i, \{u_k\}) \tilde{*} (e_j, \{u_l\}) = (e_i \circ e_j, \{u_k * u_l\})$ for all $(e_i, \{u_k\}), (e_j, \{u_l\}) \in F_A^{\bullet}$.

 F_A^{\bullet} is said to be closed under the binary composition $\tilde{*}$ if and only if $e_i \circ e_j \in A$ and $u_k * u_l \in F(e_i \circ e_j)$ for all $(e_i, \{u_k\}), (e_j, \{u_l\}) \in F_A^{\bullet}$.

Definition 2.14: [10] If F_A^{\bullet} is closed under the binary composition $\tilde{*}$, then the algebraic system $(F_A^{\bullet}, \tilde{*})$ is said to be a soft groupoid over (E,U).

Definition 2.15: [10] Let $(F_A^{\bullet}, \tilde{*})$ be a soft groupoid over (E,U), the binary composition $\tilde{*}$ is said to be i)commutative if $(e_i, \{u_k\}) \tilde{*} (e_j, \{u_l\}) = (e_j, \{u_l\}) \tilde{*} (e_i, \{u_k\})$, for all $(e_i, \{u_k\}), (e_j, \{u_l\}), (e_m, \{u_n\}) \in F_A^{\bullet}$, ii)associative if $((e_i, \{u_k\}) \tilde{*} (e_j, \{u_l\})) \tilde{*} (e_m, \{u_n\}) = (e_i, \{u_k\}) \tilde{*} ((e_j, \{u_l\})) \tilde{*} (e_m, \{u_n\})$.

Definition 2.16: [10] A soft element $(e, \{u\}) \in F_A^{\bullet}$ is said to be a soft identity element in a soft groupoid $(F_A^{\bullet}, \tilde{*})$ if for all $(e_i, \{u_k\}) \in F_A^{\bullet}$, 157

 $(e_i, \{u_k\}) \approx (e_i, \{u_k\})^{-1} = (e_i, \{u_k\})^{-1} \approx (e_i, \{u_k\}) = (e, \{u\})$

Definition 2.17:[10] Let $(F_A^{\bullet}, \tilde{*})$ be a soft groupoid with soft identity element $(e, \{u\})$. A soft element $(e_i, \{u_k\}) \in F_A^{\bullet}$ is said to be invertible if there exists a soft element $(e_i, \{u_k\})^{-1} \in F_A^{\bullet}$ such that

 $(e_i, \{u_k\}) \approx (e_i, \{u_k\})^{-1} = (e_i, \{u_k\})^{-1} \approx (e_i, \{u_k\}) = (e, \{u\}).$ Then $(e_i, \{u_k\})^{-1}$ is called the soft inverse of $(e_i, \{u_k\})$.

Definition 2.18:[10] Let (E, \circ) and (U, *) be two groups, $A, B \subseteq E$ and $F_A \in S_f(U)$. A soft groupoid $(F_A^{\bullet}, \tilde{*})$ is said to be a soft group over (E, U) if i) $\tilde{*}$ is associative,

ii) there exist a soft element $(e, \{u\}) \in F_A^{\bullet}$ such that

 $(e_i, \{u_k\}) \approx (e, \{u\}) = (e, \{u\}) \approx (e_i, \{u_k\}) = (e_i, \{u_k\}) \text{ for all } (e_i, \{u_k\}) \in F_A^{\bullet}$

iii) for each soft element $(e_i, \{u_k\}) \in F_A^{\bullet}$, there exists a soft element $(e_i, \{u_k\})^{-1} \in F_A^{\bullet}$ such that $(e_i, \{u_k\}) \approx (e_i, \{u_k\})^{-1} = (e_i, \{u_k\})^{-1} \approx (e_i, \{u_k\}) = (e, \{u\})$. We often refer to a soft group F_A^{\bullet} , rather than use the binary structure notion (F_A^{\bullet}, \approx) , with the understanding that there is of course a binary operation on the set F_A^{\bullet} .

3. Soft Topological Group Based on Soft Element

Throughout this section, let (E, \circ) and (U, *) be two groups, $A \subseteq E$ and $F_A \in S_f(U)$.

Definition 3.1: Let $(F_A^{\bullet}, \tilde{*})$ be a soft group and $(F_A, \tilde{\tau})$ be a soft topological space. Then $(F_A, \tilde{*}, \tilde{\tau})$ is called a soft topological group if;

1) For each soft neighborhood F_B of $(e_i, \{u_k\}) \approx (e_j, \{u_l\})$, there exists a soft neighborhood F_C of $(e_i, \{u_k\})$ and a soft neighborhood F_D of $(e_i, \{u_l\})$ such that $F_C \approx F_D \cong F_B$.

2) For each soft neighborhood F_B of $(e_i, \{u_k\})^{-1}$, there exists a soft neighborhood F_C f $(e_i, \{u_k\})$ such that $F_c^{-1} \cong F_B$.

Theorem 3.2: Let $\alpha = (e_j, \{u_l\})$ be a fixed element of a soft topological group $(F_A, \tilde{*}, \tilde{\tau})$. Then the mappings

 $r_{(e_j,\{u_l\})}: (e_i,\{u_k\}) \to (e_j,\{u_l\}) \stackrel{\sim}{*} (e_i,\{u_k\}), \quad l_{(e_j,\{u_l\})}: (e_i,\{u_k\}) \to (e_i,\{u_k\}) \stackrel{\sim}{*} (e_j,\{u_l\}).$ of F_A onto F_A are soft homeomorphisms of F_A .

Proposition 3.3: Let $(F_A, \tilde{*}, \tilde{\tau})$ be a soft topological group and $e = (e_1, \{e_2\})$, be the soft identity soft element of F_A . If $\tilde{\beta}_e$ is a soft local base at the soft identity then $\tilde{\beta}_{\alpha} = \{F_B \tilde{*} \alpha : F_B \in \tilde{\beta}_e\}$ is a soft local base at the soft element α .

Proposition 3.4: Let $(F_A, \tilde{*}, \tilde{\tau})$ be a soft topological group and $e = (e_1, \{e_2\})$, be the soft identity soft element of F_A . If $\tilde{\beta}_e$ is a soft local base at the soft identity then $\tilde{\beta} = \{F_B \tilde{*} \alpha : F_B \in \tilde{\beta}_e, \alpha \in F_A\}$ is a soft base for the soft topology $\tilde{\tau}$.

Theorem 3.5: Let $(F_A, \tilde{*}, \tilde{\tau})$ be a soft topological group and β_e is a soft local base at the soft identity $e = (e_1, \{e_2\})$ of F_A , then β_α satisfies the following conditions: a) If $F_B, F_C \in \beta_e$ then there exist a soft set $F_D \in \beta_e$ such that $F_D \cong F_B \cap F_C \in \beta_e$. b) If $\alpha \in F_B \in \beta_e$ then there exist a soft set $F_C \in \beta_e$ such that $F_C \tilde{*} \alpha \cong F_B$. c) If $F_B \in \beta_e$ then there exist a soft set $F_C \in \beta_e$ such that $F_C \tilde{*} F_C^{-1} \cong F_B$. d) If $F_B \in \beta_e$ and $\alpha \in F_C$ then there exist a soft set $F_D \in \beta_e$ such that $\alpha^{-1} \tilde{*} F_D \tilde{*} \alpha \cong F_B$ e) For all $F_B \in \beta_e$ there exist a soft set $F_C \in \beta_e$ such that $F_C \tilde{*} F_C \subseteq F_B$. **Acknowledgement**: This work is supported by the Scientific Research Project of Muğla Sıtkı Koçman University, SRPO (no:16/001) and SRPO (no:18/062)

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Some Results on Soft element and Soft Topological Space

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Abstract

Soft set theory is a topic of interest for many authors working in diverse areas due to its rich potential for applications in several directions since the day it was defined by Molodtsov in 1999. To contribute this research area, in this paper we examine some properties and results on soft element and soft topological space such as soft cluster and soft isoleted points of a soft set, boundary of soft sets and so on. Moreover we give some examples to clarify our definitions.

Keywords: Soft set, Soft element, Soft topology.

1. Introduction

Some operations on soft sets were defined by Maji et al. [9]. After that, Çağman and Enginoğlu [8] studied several soft operations to make them more functional for improving several new results. Çağman et. al. [7] defined the soft topology. Also, Shabir et al. Studied on soft topology too [6]. Wardowski [3] approached soft sets as classical mathematics by giving definition of soft element. Following to these studies our purpose for this study to examine some properties and results on soft element and soft topological space such as as soft cluster and soft isoleted points of a soft set, boundary of soft sets and so on. Moreover we give some examples to clarify our definitions.

2. Preliminaries

In this section, we recall some basic notions in soft set theory.

Definition 2.1: [1] Let U be an initial universal set and E be a set of parameters. Let P(U) denote the power set of U and $A \subseteq E$. A soft set F_A is called a soft set over U, where f_A is a mapping given by $f_A: E \to P(U)$ such that $f_A(x) = \emptyset$ if $x \notin A$.

Note that the set of all soft sets over U will be denoted by S(U).

Definition 2.2: [8] A soft set F_A over U is said to be an empty soft set denoted by F_{\emptyset} , if for all $e \in E$, $f_A(x) = \emptyset$.

Definition 2.3: [8] A soft set F_A over U is said to be an A-universal soft set denoted by $\widetilde{F_A}$, if for all $e \in E$, $f_A(x) = A$. If A = E; then the A-universal soft set is called a universal soft set, denoted by $\widetilde{F_E}$.

Definition 2.4: [8] Let $F_A, F_B \in S(U)$. Then, F_A is a soft subset of F_B , denoted by $F_A \cong F_B$, if $f_A(e) \subseteq f_B(e)$ for all $e \in E$.

Definition 2.5: [8] Let $F_A, F_B \in S(U)$. Then, the soft union $F_A \widetilde{\cup} F_B$, the soft intersection $F_A \widetilde{\cap} F_B$ and the soft difference $F_A \widetilde{\setminus} F_B$ of F_A and F_B are defined by the approximate functions as;

$$f_{A\widetilde{\cup}B}(x) = f_A(x) \cup f_B(x), \qquad f_{A\widetilde{\cap}B}(x) = f_A(x) \cap f_B(x), \qquad f_{A\widetilde{\setminus}B}(x) = f_A(x) \setminus f_B(x)$$

respectively.

The soft complement $F_A^{\tilde{C}}$ of F_A is defined by the approximate function $f_{A\tilde{C}} = (f_A(e))^C$, where $(f_A(e))^C$ is the complement of the set $f_A(e)$; that is $f_{A\tilde{C}} = (f_A(e))^C = U/f_A(e)$ for all $e \in E$.

It is easy to see that $(F_A^{\tilde{C}})^{\tilde{C}} = F_A$ and $F_{\phi}^{\tilde{C}} = \widetilde{F_E}$.

Definition 2.6 : Let $F_A \cong F_B \in S(U)$. The soft complement $(F_B^{\tilde{C}})_A$ in F_A is defined by the approximate function $f_{B\tilde{C}} = (f_B(e))^C$, where $(f_B(e))^C$ is the complement of the set $f_B(e)$ in the soft set F_A ; that is $f_{B\tilde{C}} = (f_B(e))^C = f_A(e)/f_B(e)$ for all $e \in E$.

Definition 2.7: [7] Let $F_A \in S(U)$. A soft topology on F_A , denoted by $\tilde{\tau}$, is a collection of soft subsets of F_A having the following properties:

- i) $F_{\emptyset} \widetilde{F_A} \in \tilde{\tau}$,
- ii) $F_B, F_C \in \tilde{\tau}$, then $F_B \cap F_C \in \tilde{\tau}$,
- iii) Λ indexed set and for all $\alpha \in \Lambda$, $F_{B_{\alpha}} \in \tilde{\tau}$ then $\bigcup_{\alpha \in \Lambda}^{\sim} F_{B_{\alpha}} \in \tilde{\tau}$.

The pair $(F_A, \tilde{\tau})$ is called a soft topological space.

Definition 2.8: [7] Let $(F_A, \tilde{\tau})$ be a soft topological space. Then every element of $\tilde{\tau}$ is called soft open set. Clearly F_{\emptyset} and $\widetilde{F_A}$ are soft open sets.

Definition 2.9: [7] Let $(F_A, \tilde{\tau})$ be a soft topological space and $F_B \cong F_A$. Then F_B is said to be soft closed if the soft complement of F_B in the soft set F_A is soft open. F_{\emptyset} and $\widetilde{F_A}$ are soft closed sets.

Definition 2.10: [3] Let $F_A \in S(U)$. We say that $\alpha = (e, \{u\})$ is nonempty soft element of F_A , if $e \in E$ and $u \in F(e)$. The pair (e, \emptyset) , where $e \in E$, will be called an empty soft element of F_A . The fact that $(e, \{u\})$ is a soft element of F_A will be denoted by $(e, \{u\}) \in F_A$. We denote the set of all nonempty soft elements of F_A by F_A^* .

Example 1: Let $U = \{h_1, h_2, h_3\}$, $E = \{e_1, e_2, \}$. Take a soft set $F_A \in S(U)$ of the form $F_A = \{(e_1, \{h_1\})\}$. Then all the soft elements of F_A are the following:

 $(e_1, \emptyset), (e_1, \{h_1\}), (e_2, \emptyset).$

Proposition 1: [3] For each $F_A \in S(U)$, the followings holds:

 $F_A = \bigcup_{(e_i, \{u_j\}) \in F_A}^{\sim} \{ (e_i, \{u_j\}) \} \text{ and } F_A = \bigcup_{(e_i, \{u_j\}) \in F_A^{\bullet}}^{\sim} \{ (e_i, \{u_j\}) \}.$

Definition 2.11: Let $(F_A, \tilde{\tau})$ be a soft topological space, $F_B \cong F_A$ and $\alpha \in F_B$. If there exist a soft open set F_C such that $\alpha \in F_C \cong F_B$ then F_B is called soft neighborhood of soft element α .

If F_B is soft open set then F_B is called soft open neighborhood. We denote the set of all soft open neighborhoods of α by $\widetilde{\mathcal{N}}_{\alpha}$.

Example 2: $U = \{u_1, u_2, u_3\}, A = \{x_1, x_2, x_3, \}$ and

 $F_A = \{(x_1, \{u_1, u_3\}), (x_2, \{u_1\}), (x_3, \{u_1, u_2\})\}$ is a soft set over U. Some of the soft subsets of F_A are listed below.

$$F_{A_1} = \{(x_1, \{u_1\})\},\$$

$$F_{A_2} = \{(x_1, \{u_1, u_3\})\},\$$

$$F_{A_3} = \{(x_1, \{u_1\}), (x_2, \{u_1\})\},\$$

$$F_{A_4} = \{(x_1, \{u_1, u_3\}), (x_2, \{u_1\})\},\$$

The collections of F_A $\tilde{\tau}_1 = \{ \tilde{\emptyset}, F_A \}$, $\tilde{\tau}_2 = \tilde{P}(F_A)$ ve $\tilde{\tau}_3 = \{ \tilde{\emptyset}, F_A, F_{A_1}, F_{A_2}, F_{A_3}, F_{A_4} \}$ are soft topology. Nonempty soft elements of F_A are $\{((x_1, \{u_1\}), (x_1, \{u_3\}), (x_2, \{u_1\}), (x_3, \{u_1\}), (x_3, \{u_2\})\}$. So set of all Soft neighborhoods for each soft element is given in the below fort he soft topology $\tilde{\tau}_3$.

$$\begin{split} \widetilde{\mathcal{N}}_{(x_1,\{u_1\})} &= \{F_A, F_{A_1}, F_{A_2}, F_{A_3}, F_{A_4}\}, \\ \widetilde{\mathcal{N}}_{(x_1,\{u_3\})} &= \{F_{A_2}, F_{A_4}\}, \\ \widetilde{\mathcal{N}}_{(x_2,\{u_1\})} &= \{F_A, F_{A_4}\}, \\ \widetilde{\mathcal{N}}_{(x_3,\{u_1\})} &= \{F_A\} \\ \widetilde{\mathcal{N}}_{(x_3,\{u_2\})} &= \{F_A\} \\ . \end{split}$$

Definition 2.12: Let $(F_A, \tilde{\tau})$ be a soft topological space, $F_B \cong F_A$ and $\alpha \in F_B$. If there exist a soft open set F_C such that $\alpha \in F_C \cong F_B$ i.e., F_B is soft neighborhood of soft element α , then α is called s soft interior element of F_B .

The set of all soft interior elements of F_B is called soft interior of F_B and it is denoted by F_B° .

Definition 2.13: Let $(F_A, \tilde{\tau})$ be a soft topological space, $F_B \cong F_A$ and $\alpha \in F_B$. If for every soft open neighborhood F_C of α , $F_B \cap F_C \neq F_{\emptyset}$, then α is a soft closure element of F_B . The set of all soft closure elements of F_B is called soft closure of F_B and denoted by $\overline{F_B}$.

Example 4: If we consider the soft topology $(F_A, \tilde{\tau}_3)$ given in Example 2 and the soft subset o

 $F_B = \{(x_1, \{u_3\}), (x_3, \{u_1\})\} \text{ of } F_A,$

Then $\overline{F_B} = \{(x_1, \{u_3\}), (x_3, \{u_1\}), (x_3, \{u_2\})\}$.

Definition 2.14. Let $(F_A, \tilde{\tau})$ be a soft topological space, $F_B \cong F_A$ and $\alpha \in F_B$. If for all open soft neighborhood F_C of α , $F_B \cap F_C \tilde{/}\alpha \neq F_{\emptyset}$ then α is called a soft cluster element of F_B . The set of all soft cluster elements of F_B is denoted by \tilde{F}_B .

Definition 2.15. Let $(F_A, \tilde{\tau})$ be a soft topological space, $F_B \cong F_A$ and $\alpha \in F_B$. α is soft called soft isolated element of F_B if and only if $\{\alpha\}$ is soft open set. A soft topological space is called discrete if and only if every soft element in the soft set F_A is soft isolated.

Definition 2.16 Let $(F_A, \tilde{\tau})$ be a soft topological space, and $F_B \cong F_A$. The soft boundary of F_B is $Bd(F_B) = \overline{F_B} \cap \overline{F_A \setminus F_B}$.

3. Conclusions

In this paper we studied on soft topological structure based on the definition soft element. One can expand this work by searching more topological structures from the same point of view. **Acknowledgement**: This work is supported by the Scientific Research Project of Muğla Sıtkı Koçman University, SRPO (no:16/001) and (no:18/062)

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On Close-To-Convexity of Normalized Analytic Functions

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Abstract

The object of this article is to derive sufficient conditions for close-to-convexity of certain (normalized) analytic functions. Furthermore, it is shown that a convex function is close-to-convex of order 2^{-r} in unit disk \mathcal{U} where r is a positive integer.

Keywords: Analytic functions, Univalent functions, Starlike functions, Close-to-convex functions, Convex functions and Subordination principle.

1. Introduction

Let the class \mathcal{A} of functions f analytic and univalent in the unit disk $\mathcal{U} = \{z : |z| < 1\}$, normalized by the conditions f(0) = 0 and f'(0) = 1. Thus each $f \in \mathcal{U}$ has a Taylor series expansion of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

Also let by $\mathcal{C}(2^{-r})$ denote the subclasses of \mathcal{A} consisting of functions which are close-toconvex of order 2^{-r} in \mathcal{U} . We know that ([1, 2] and [3, 7])

$$\mathcal{C}(2^{-r}) = \left\{ f : f \in \mathcal{A} \text{ and } \mathcal{Re}\left(\frac{f'(z)}{g'(z)}\right) > 2^{-r}, (z \in \mathcal{U}; g \text{ is convex function in } \mathcal{U}) \right\}$$

Now, we recalled the principle of subordination between analytic functions, for two functions f and g analytic in \mathcal{U} , we say that f is subordinate to g in \mathcal{U} , written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function h(z) analytic in \mathcal{U} , with h(0) = 0 and |h(z)| < 1 such that $f(z) = g(h(z)), z \in \mathcal{U}$.

In particular, if the function g is univalent in \mathcal{U} then $f \prec g$ if and only if f(0) = g(0) and

$$f(\mathcal{U}) \subset g(\mathcal{U})$$

The following lemma which is known as Jack's Lemma will be required for our present paper.

Lemma 1. Let the (nonconstant) function w(z) be analytic in \mathcal{U} with w(0) = 0. If |w(z)| attains its maximum value on the circle |z| = r < 1 at the point $z_0 \in \mathcal{U}$, then

$$Z_0 w'(Z_0) = C w(Z_0)$$

where c is a real number and $c \ge 1([5,6])$.

2. Materials and Methods

We consider the following theorem providing a sufficient condition for close-to-convexity of functions $f \in A$.

Theorem 1. Let the function $f \in A$ satisfy the inequality

$$\mathcal{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > \frac{3+2^r}{2+2^{1+r}}, (z \in \mathcal{U})$$

then

$$\mathcal{Re}(f'(z)) > \frac{1+2^r}{2^{1+r}}, (z \in \mathcal{U})$$

 $f \in \mathcal{C}\left(\frac{1+2^r}{2^{1+r}}\right)$

or equivalently,

Proof. The proof requires to define a function w(z) as follow

$$f'(z) = \frac{1 + 2^{-r} w(z)}{1 + w(z)}, (w(z) \neq -1; z \in \mathcal{U}).$$

Then, clearly, w(z) is analytic in $z \in \mathcal{U}$ with w(0) = 0. We also find from that

$$1 + \frac{zf''(z)}{f'(z)} = 1 + \frac{2^{-r} zw'(z)}{1 + 2^{-r} w(z)} - \frac{zw'(z)}{1 + w(z)}, \ (z \in \mathcal{U}).$$
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Suppose now that there exists a point $z_{\scriptscriptstyle 0} \in \mathcal{U}$, such that

$$|w(z_0)| = 1$$
 and $|w(z)| < 1$, when $|z| < |z_0|$.

Then, by applying Lemma 1, we have

$$z_0w'(z_0) = cw(z_0), \quad (c \ge 1; w(z_0) = e^{i\theta}; \theta \in R).$$

Thus we can obtain that

$$\mathcal{Re}\left(1 + \frac{z_0 f''(z_0)}{f'(z_0)}\right) = 1 + \frac{2^{-r} c \left(2^{-r} + \cos\theta\right)}{1 + 2 \cdot 2^{-r} \cos\theta + \left(2^{-r}\right)^2} - \frac{c}{2}.$$
$$\mathcal{Re}\left(1 + \frac{z_0 f''(z_0)}{f'(z_0)}\right) \le 1 + \frac{2^{-r} \left(1 + 2^{-r}\right)}{1 + 2 \cdot 2^{-r} + \left(2^{-r}\right)^2} - \frac{1}{2} = \frac{1}{2} + \frac{2^{-r}}{\left(1 + 2^{-r}\right)}.$$

then,

or equivalently, $\mathcal{Re}\left(1+\frac{z_0f''(z_0)}{f'(z_0)}\right) \leq \frac{3+2^r}{2+2^{1+r}}, \ (z_0 \in \mathcal{U}),$

which obviously contradicts the hypothesis. It follows that

$$|w(z)| < 1, (z \in \mathcal{U})$$

That is,
$$f'(z) = \frac{1 + 2^{-r} w(z)}{1 + w(z)} = \left| \frac{1 - f'(z)}{f'(z) - 2^{-r}} \right| < 1, \ (z \in \mathcal{U}).$$

This evidently completes the proof of the theorem.

Theorem 2. Let the function $f \in A$ satisfy the inequality

$$\mathcal{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > \frac{2+3.2^r}{1+2^{1+r}}, (z \in \mathcal{U}),$$

then

Proof. The proof of theorem is also based upon Lemma 1. We let the function
$$w(z)$$
 be
by

given

 $\left|f'(z)-1\right| < \frac{1+2^r}{2^r}, \ \left(z \in \mathcal{U}\right).$

$$f'(z) = (1+2^{-r})w(z)+1, (z \in \mathcal{U})$$

Then, clearly, w(z) is analytic in $z \in \mathcal{U}$ with w(0) = 0.

$$1 + \frac{zf''(z)}{f'(z)} = 1 + \frac{z\left\{\left(1 + 2^{-r}\right)w'(z)\right\}}{\left(1 + 2^{-r}\right)w(z) + 1}, \ \left(z \in \mathcal{U}\right)$$

Suppose that there exists a point $z_0 \in \mathcal{U}$, such that $|w(z_0)| = 1$ and |w(z)| < 1, when $|z| < |z_0|$.

Then, by applying Lemma 1, we have

$$z_{0}w'(z_{0}) = cw(z_{0}), \quad (c \ge 1; w(z_{0}) = e^{i\theta}; \theta \in \mathbb{R}).$$
$$\mathcal{Re}\left(1 + \frac{z_{0}f''(z_{0})}{f'(z_{0})}\right) \ge \frac{2 + 3.2^{r}}{1 + 2^{1+r}}, \quad (z \in \mathcal{U}),$$

As a result,

which obviously contradicts the hypothesis. It follows that

$$w(z) | < 1, (z \in \mathcal{U}),$$

that is,

$$f'(z) = (1 + 2^{-r})w(z) + 1 \Longrightarrow w(z) = \frac{f'(z) - 1}{(1 + 2^{r})2^{-r}}$$

$$\left|f'(z)-1\right| < \frac{1+2^r}{2^r}, \ \left(z \in \mathcal{U}\right).$$

This evidently completes the proof.

Theorem 3. If the function $f \in A$ satisfies the inequality

$$\left|f'(z)-1\right|^{\beta}\left|zf''(z)\right|^{\gamma} < \frac{\left(1-2^{-r}\right)^{\beta+\gamma}}{2^{\beta+2\gamma}}, \ \left(z \in \mathcal{U}; \beta, \gamma \ge 0\right)$$

then,

$$\mathcal{Re}(f'(z)) > \frac{1+2^r}{2^{1+r}}, \ (z \in \mathcal{U}).$$

Results and Discussions

Remark 1. Since the inequality used Theorem 2 implies that

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$$\mathcal{Re}(f'(z)) > -\frac{1}{2}, \ (z \in \mathcal{U}),$$

by setting r = 1 in Theorem 2.

Corollary 1. Let the function $f \in A$ satisfy the inequality

$$\mathcal{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) < \frac{8}{5}, \ (z \in \mathcal{U})$$

then

 $\left|f'(z)-1\right| < \frac{3}{2}, \ (z \in \mathcal{U})$

hence $f \in \mathcal{C}$.

Corollary 2. If the function $f \in A$ satisfies the inequality

$$\left|zf''(z)\right| < \frac{2^r - 1}{2^{2+r}}, \ \left(z \in \mathcal{U}\right),$$

then,

$$\mathcal{Re}(f'(z)) > \frac{1+2^r}{2^{2+r}}, \ (z \in \mathcal{U}).$$

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Matematik Öğretmenlerinin Somut İşlemler Döneminde Problem Çözme Yetkinlikleri

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Özet

Daha nitelikli, yaratıcı ve çağa uyum sağlayabilen öğrencilerin yetişmesini amaçlayan bir eğitim sisteminin en temel bileşeni şüphesiz öğretmendir. Öğretmenler bu amaç doğrultusunda öğrenci seviyesini dikkate alarak, nitelikli ve etkili eğitim süreçleri gerçekleştirmelidir.

2012 yılında yapılan "4+4+4 Eğitim Sistemi" değişikliği ile, 5. sınıf düzeyi ortaokula dahil edilmiş ve okula başlama yaşı 72 aydan 66 aya, veli izniyle de 60 aya kadar indirilmiştir. Sistem değişikliklerinin sonucunda, ortaokul matematik öğretmenleri, ilk defa somut işlemler dönemindeki yaş seviyesinden öğrencilerle karşılaşmıştır. Dolayısıyla, ortaokul matematik öğretmenlerinin, 10-11 yaş öğrencilerinin bilişsel özelliklerine uygun olarak problem çözme stratejileri ve matematiksel temsiller kullanmaları önemli hale gelmiştir.

Bu çalışmada, ortaokul matematik öğretmenlerinin somut işlemler döneminde problem çözme becerilerini ve yetkinliklerini tespit etmek amaçlanmaktadır. Nitel araştırma yöntemlerinden durum çalışması olarak yürütülen araştırma, farklı mesleki deneyime sahip beş öğretmen ile gerçekleştirilmiştir. Verileri toplamak üzere, Problem Çözme Testi ve Görüşme Formları olarak iki tür veri toplama aracı geliştirilmiştir. İlk aşamada öğretmenlerin 5. sınıf düzeyinde derslerde ve problem çözmede karşılaştığı zorlukları tespit etmek için birinci görüşme formu uygulanmıştır. İkinci aşamada, öğretmenlerin ortaokul seviyesinde problem çözüm yaklaşımlarını belirlemek amacıyla, somut ve soyut işlemler dönemine uygun olarak çözülebilen dört problemden oluşan Problem Çözme Testi uygulanmıştır. Son olarak, öğretmenlerin problem çözmede kullandıkları strateji ve temsiller dikkate alınarak, bu yetkinlikleri nasıl kazandıklarını tespit etmek amacıyla ikinci görüşme gerçekleştirilmiştir. Öğretmenlerin bireysel problem çözme süreçleri ve yarı yapılandırılmış görüşmeler kaydedilmiş, elde edilen veriler betimsel analize tabi tutulmuştur.

Bulgular, öğretmenlerin genel olarak her iki bilişsel düzeyde çözüm geliştirebildiklerini gösterirken, somut dönem uygulamalarında deneyim kazanmalarına farklı faktörlerin etki ettiğini ortaya çıkarmıştır.

Anahtar Kelimeler: Problem Çözme Yetkinlikleri, Ortaokul Matematik Öğretmeni, Somut İşlemler Dönemi

1. Giriş

Eğitim, ülkelerin her alanda ilerleyebilmesinde önemli role sahiptir. Eğitimin en önemli bileşenleri şüphesiz öğrenci ve öğretmendir. Daha nitelikli, yaratıcı ve çağa uyum sağlayabilen öğrencilerin yetişmesini amaçlayan eğitim sistemleri, öğretmenlerin niteliklerini artıracak ortamlarla desteklenmelidir. Diğer bir deyişle, değişen eğitim sistemlerine uygun ve beklentiler doğrultusunda eğitim verecek öğretmenlerin yetkin olması gerekmektedir. Bu yetkinlik "öğrenci seviyesini dikkate alarak, nitelikli ve etkili eğitim süreci gerçekleştirmek" olarak tanımlanabilir.

Türkiye'de 2012-2013 eğitim-öğretim döneminde uygulamaya geçilen 2012 Eğitim Sistemi Değişikliği [2012 ESD] ile, 5. sınıf düzeyi ortaokula dahil edilmiş ve okula başlama yaşı 72 aydan 66 aya, veli izniyle de 60 aya kadar indirilmiştir (MEB, 2012). Bu değişiklik ile ortaokul kademesinin (10-13 yaş) somut ve soyut işlem dönemlerini bir arada bulundurmasından dolayı, öğretmenlerin farklı bilişsel seviyelerde öğretim gerçekleştirme becerilerine sahip olması önemlidir. 2012 ESD öncesinde eğitim fakültelerinde verilen matematik alan eğitimi derslerinde 6., 7. ve 8. sınıf (soyut işlemler dönemi) öğrencileri için düzenlenmiş öğretim faaliyetlerine odaklanılmıştır (Altun, 2008). Ortaokul matematik öğretmenlerinin, gerek aldıkları eğitim, gerekse sistem değişikliği öncesinde kazandıkları mesleki deneyim göz önünde bulundurulduğunda, genel olarak somut işlemler dönemi öğretim yaklaşımlarına yeterince sahip olmadıkları varsayılabilir. Öğretmenler ortaokul seviyesinde ilk defa somut işlemler dönemindeki 10-11 yaş seviyesinden öğrencilerle karşılaşmıştır. Dolayısıyla, öğretmenlerin, bu yaştaki öğrencilerin bilişsel özelliklerine uygun olarak problem çözme stratejileri ve matematiksel temsiller kullanmaları önemli hale gelmiştir.

Öğretmenlerin hizmet öncesi eğitimlerindeki eksiklikler, mesleki görev sürecinde hizmet içi eğitimlerle giderilir. 2012 yılından itibaren düzenlenen hizmet içi eğitim faaliyetleri incelendiğinde, 5.sınıf öğrencileri veya somut işlemler dönemi özelliklerini içeren herhangi bir eğitime rastlanmamıştır (ÖYEGM, 2018). Bununla birlikte öğretmenlerin mesleki gelişimleriyle ilgili Özmantar ve Önala'nın (2017) yaptığı araştırmada katılımcı matematik öğretmenlerinin % 66'sı kendi alanlarıyla ilgili hiçbir eğitim almamıştır. Aynı çalışmada, matematik öğretmenlerinin % 44'ünün matematik öğretimi ile ilgili hizmet içi eğitime

katılmalarına rağmen, yarıdan fazlası etkili matematik öğretimi konusunda hala eğitime ihtiyacı olduğunu belirtmiştir.

Öğretmenlerin nitelikli bir eğitim almaları öz yeterliğini de etkileyecektir. Öz yeterlik, belirli bir performansa ulaşmayı sağlayacak eylemleri organize etme ve sergileme becerisine ilişkin düşünceler olarak tanımlanabilir (Bandura, 1994). Matematik başarısı ile öz yeterlik arasında ortaya koyulan doğru orantılı ilişki daha özele inildiğinde problem çözme ve öz yeterlik arasında da görülmektedir. Matematik dersindeki öz yeterlik, problem çözme sürecindeki kabiliyeti de etkilemektedir (Pajares & Kranzler, 1995). Bu çalışmada, ortaokul matematik öğretmenlerinin somut işlemler dönemi öğrencilerinin seviyesine uygun nasıl problem çözdükleri ve bu konuda kullandıkları temsillerin yeterliliği incelenmektedir.

2. Yöntem

Bu çalışmada, ortaokul matematik öğretmenlerinin somut işlemler döneminde problem çözme becerilerini ve yetkinliklerini tespit etmek amaçlanmaktadır. Nitel araştırma yöntemlerinden durum çalışması olarak yürütülen araştırmanın katılımcı grubu MEB'e bağlı ortaokullarda görev yapan matematik öğretmenlerinden oluşmaktadır. Öğretmen seçiminde, mesleki eğitim ve deneyim durumlarına göre çeşitlilik göz önünde bulundurulmuştur. Öğretmenlerin Eğitim Fakültesi mezunu olma, mezuniyet yılı (4+4+4 eğitim sisteminin uygulanmaya başlandığı 2012 ESD öncesi ve sonrası mezunları içermesi) ve 5. sınıf düzeyinde görev almaları ölçüt alınmıştır.

Verileri toplamak üzere, Görüşme Formları ve Problem Çözme Testi olarak iki tür veri toplama aracı geliştirilmiştir. İlk aşamada öğretmenlerin 5. sınıf öğrencileriyle ilk deneyimleri, öğrencilerin beceri ve davranış özelliklerini tanıma, lisans eğitimlerinin yeterliği, problem çözme ve problem çözümlerini bilişsel dönemlere göre sınıflama durumlarını tespit etmek için yarı yapılandırılmış ön görüşme formu uygulanmıştır. İkinci aşamada, öğretmenlerin ortaokul seviyesinde problem çözüm yaklaşımlarını belirlemek amacıyla, somut ve soyut işlemler dönemine uygun olarak çözülebilen dört problemden oluşan Problem Çözme Testi uygulanmıştır. Son olarak, öğretmenlerin problem çözmede kullandıkları strateji ve temsiller, ilgili alan yazın (Janvier, 1987; Hiebert & Carpenter, 1992; Posamentier & Krulik, 2016; Van De Walle, 2004; Wadsworth, 2015) dikkate alınarak, bu yetkinlikleri nasıl

kazandıklarını tespit etmek amacıyla ikinci görüşme gerçekleştirilmiştir. Öğretmenlerin bireysel problem çözme süreçleri ve yarı yapılandırılmış görüşmeler kaydedilmiş, elde edilen veriler betimsel analize tabi tutulmuştur.

3. Bulgular

Bulgular, 2012 ESD öncesi mezun öğretmenlerin 5. sınıf düzeyindeki ilk deneyimlerini yetersiz olarak tanımladığını ortaya çıkarmıştır. Öğretmenler, bu eksikliklerini 5. sınıf düzeyinde daha uzun süre derse girerek giderilebileceği inancı taşımaktadır. Öğretmenler problem çözme stratejilerini, zamanla ve öğrencilerin çözümü anlama durumuna göre şekillendirdiklerini ifade etmişlerdir. Diğer bir deyişle öğretmenler, problem çözümlerinde farklı stratejileri deneyerek uygun olanları kullanma yoluna gitmişlerdir.

2012 ESD sonrası mezun ile çift ana dal programı (ilköğretim matematik ve sınıf öğretmenliği) ve mezunu öğretmenler ise, lisans eğitimlerinin yeterli olduğunu, öğretimde herhangi bir sorun yaşamadıklarını, fakat 5. sınıf öğrencilerinin diğer sınıf düzeylerinde karşılaşmadıkları davranışlar sergiledikleri için zorluk yaşadıklarını ifade etmişlerdir.

Öğretmenlerin, somut\soyut işlemler dönemine uygun olarak problem çözümü sergileyebildikleri, çözümleri sınıf düzeylerine göre kategorize edebildikleri, çözümlerde farklı temsiller kullandıkları ve bu süreçte Polya (1957) problem çözme adımlarını takip edebildikleri belirlenmiştir. Bu yetkinliği kazanmada 2012 ESD öncesi mezun öğretmenlerin, sınıf öğretmeni meslektaşlarından destek almadığı, sosyal ağları ve zaman içinde deneme yanılma yolunu benimsedikleri belirlenmiştir. 2012 ESD sonrası mezun öğretmenlerin ise öğretim ve problem çözmede yetkinlikleri lisans eğitimlerinden kaynaklanmaktadır. Bu öğretmenler, somut işlemler dönemi öğrencilerinin bilişsel seviyesine uygun olarak psikomotor becerileri öne çıkaran kesme, boyama ve geometride araç gereç kullanımına önem verdikleri belirlenmiştir.

2012 ESD öncesi mezun öğretmenler kendilerini 7. ve 8. sınıf düzeylerinde yetkin görmekte ve problem çözümlerinde soyut işlemler dönemine uygun cebirsel işlemlere öncelik vermektedir. 2012 ESD sonrası mezun öğretmenler ise kendilerini en çok 5. sınıf düzeyinde yetkin bulduklarını, bu becerileri büyük ölçüde lisans eğitiminde ve sosyal ağlardan edindiklerini ifade etmiştir. 5.sınıf öğrencileriyle ilk deneyimlerini genel olarak¹⁷⁴ bilgisizlik ve
belirsizlik olarak tanımlayan 2012 ESD öncesi mezun öğretmenler, ilk yıllardaki 5. sınıf öğrencilerinin sonraki yıllara göre daha başarılı olduklarını belirtmiş, bu durumu yaş değişkeni ile açıklamışlardır. 5. sınıf öğrencilerinin diğer sınıflara göre daha yavaş yazma, öğrenme ve problem çözme hızına sahip olmasının gelişimsel bir farklılık olduğunu, bu farklılıktan doğan zorluğun aşılabilmesi ve sınıf içi bütünlük sağlanabilmesi için derste defter tutma zorunlu tutulmuştur. 2012 ESD sonrası mezun öğretmenler ise defter tutmayı derste disiplini sağlayabilmek ve sözel çözümlerin kalıcılığı için zorunlu tuttuklarını belirtmiştir.

4. Sonuç ve Tartışma

Ortaokul matematik öğretmenlerinin somut işlemler dönemi yetkinliklerine sahip olduğu tespit edilmiştir. Öğretmenlerden 2012 ESD öncesi mezun olanların bu yetkinliği 5. sınıfın dersine girerek ve deneme yanılma yoluyla kazandıklarını ifade etmeleri, Gökkurt Özdemir, Erdem, Örnek ve Soylu'nun (2017) çalışmasındaki, öğretmenlerin somut işlemler dönemi problem çözümlerinde deneme yanılma yolunu kullanma eğiliminde olmaları ile desteklenebilir. 2012 ESD sonrası mezun öğretmenler ise bu problem çözme yetkinliğini lisans eğitimleriyle kazanmıştır. Ayrıca öğretmenlerin problem çözümlerine cebirsel yollarla başlama eğilimde olması; problem çözme alan yazınında soyut işlemlerin tercih edilmesi ile ilgili bulguları desteklemektedir (Tatar, İşleyen ve Okur, 2005).

Ortaokul matematik öğretmenlerinin somut işlemler dönemi problem çözme becerilerinin gelişmesinde sırasıyla, lisans eğitimi, tecrübe, sosyal ağlar, daha önceki öğretmenlerin problem çözme stratejileri, öğrencilerin problem çözme stratejileri ve kaynak kitapların etkili olduğu belirlenmiştir. Sosyal ağlar ve internetin mesleki gelişimdeki önemi, Baş'ın (2016) çalışmasındaki bulgularla benzerlik göstermektedir. Öğretmenlerin, ortaokul düzeyinde yetkin olmalarına rağmen, etkili öğretim ve diğer öğretmenlerin tecrübelerini kazanabilmesi için, hizmet içi eğitim ve ders izleme faaliyetlerine istekli olmaları, EARGED (2008) raporu ve alan yazındaki (Özmantar ve Önala, 2017) bulgularla desteklenebilir.

Bu sonuçlardan yola çıkarak, araştırma kapsamının genişletilerek daha fazla öğretmene 5. sınıf öğrencilerinin bilişsel gelişim özelliklerini, döneme uygun problem ve etkinlikleri, uzaktan hizmet içi ya da atölye eğitimleriyle aktarılması faydalı olacaktır. Eğitimlerin yanında

öğretmenlerin birbirine tecrübelerini aktarabilecekleri ortamların ve ders gözlemleme faaliyetlerinin yürütülmesi önerilebilir.

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Maximal Formally Normal Differential Operators for First Order

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Abstract: In this work, the general form of all maximal formally normal extensions of the minimal operator generated by differential-operator expression for first order in the weighted Hilbert spaces of vector-functions on right semi-axis in terms of boundary conditions has been found. Furthermore, structure of spectrum of these extensions was investigated.

Keywords: Formally normal and normal operator, Differential operator, Spectrum. **2010 AMS Classification:** 47A10, 47B25

1. Introduction

It is known that a densely defined closed operator N in any Hilbert space is called formally normal if $D(N) \subset D(N^*)$ and $||Nf|| = ||N^*f||$ for all $f \in D(N)$, where N^* is the adjoint to the operator N. If a formally normal operator has no formally normal extension, then it is called maximal formally normal operator. If a formally normal operator N satisfied the condition $D(N) = D(N^*)$, then it is called a normal operator [1].

Generalization of J. von Neumann's theory to the theory of normal extensions of formally normal operators in Hilbert space has been done by E. A. Coddington in work [1]. And also the first results in the area of normal extension of unbounded formally normal operators in a Hilbert space are due to Y. Kilpi [2]-[4] and R. H. Davis [5]. Some applications of this theory to two-point regular type first order differential operators in Hilbert space of vector functions can be found in [6] (also see references therein).

In this work, in the third section all maximal formally normal extensions of the minimal formally normal operator generated by a linear differential expression in weighted Hilbert space of vector-functions defined in right half-infinite interval are described. Furthermore, the spectrum of such extensions is investigated.

2. Statement of the Problem

Let *H* be a separable Hilbert space and $\alpha \in \mathbb{R}$. And also assumed that $\alpha : (a, \infty) \to (0, \infty), \ \alpha \in C(a, \infty)$ and $\alpha^{-1} \in L^1(a, \infty)$. In the weighted Hilbert space $L^2_{\alpha}(H, (a, \infty))$ of *H* – valued vector-functions defined on the right semi-axis consider the following linear quasi-differential expression with operator coefficient for first order in a form

$$l(u) = (\alpha u) + Au(t)$$

where $A: H \rightarrow H$ is a selfadjoint bounded operator with condition $A \ge E: H \rightarrow H$ is an identity operator.

By a standard way the minimal L_0 and maximal L operators corresponding to quasidifferential expression $l(\cdot)$ in $L^2_{\alpha}(H,(a,\infty))$ can be defined (see [6],[7]). In this case the minimal operator L_0 is formally normal, but it is not maximal in $L^2_{\alpha}(H,(a,\infty))$.

The main purpose of this work is to describe of all maximal formally normal extensions of the minimal operator L_0 in terms of boundary conditions in $L^2_{\alpha}(H,(a,\infty))$. Moreover, structure of the spectrum of these extensions will be surveyed.

3. The General Form of the Maximal Formally Normal Extensions

In this section the general form of all maximal formally normal extensions of the minimal operator L_0 in $L^2_{\alpha}(H,(a,\infty))$ will be investigated.

In a similar way the minimal operator L_0^+ generated by quasi-differential-operator expression

$$l^+(v) = -(\alpha v)' + Av(t)$$

can be defined in $L^2_{\alpha}(H,(a,\infty))$ (see [6],[7]).

In this case the operator $L^+ = (L_0)^*$ in $L^2_{\alpha}(H, (a, \infty))$ is called the maximal operator generated by $l^+(\cdot)$. It is clear that $L_0 \subset L$, $L^+_0 \subset L^+$.

In this case the following assertion is true.

Lemma 3.1. If \tilde{L} is any maximal formally normal extension of the minimal operator L_0 in $L^2_{\alpha}(H,(a,\infty))$, then

$$\alpha D(\tilde{L}) \subset W^1_{2,a}(H,(a,\infty)),$$

where $W_{2,\alpha}^{1}(H,(a,\infty))$ is a weighted Sobolev space.

The minimal operator M_0 generated by following differential expression

$$m(u) = -i(\alpha u)$$

in $L^2_{\alpha}(H,(a,\infty))$ is a symmetric. And also a operator $M = M^*_0$ in $L^2_{\alpha}(H,(a,\infty))$ it will be indicated a maximal operator corresponding to differential expression $m(\cdot)$.

Lemma 3.2. The deficiency indices of the minimal operator M_0 in $L^2_{\alpha}(H,(a,\infty))$ are in form

$$(n_{+}(M_{0}), n_{-}(M_{0})) = (dimH, dimH).$$

For the description of all maximally symmetric extensions of the minimal operator M_0 in $L^2_{\alpha}(H,(a,\infty))$ we must be construct space of boundary values of M_0 .

Definition 3.3. [8] Let \mathcal{H} be any Hilbert space and $S: D(S) \subset \mathcal{H} \to \mathcal{H}_{179}$ be a closed densely defined symmetric operator in the Hilbert space \mathcal{H} having equal finite or infinite

deficiency indices. A triplet (H, γ_1, γ_2) , where H is a Hilbert space, γ_1 and γ_2 are linear mappings from $D(S^*)$ into H, is called a space of boundary values for the operator S if for any $f, g \in D(S^*)$

$$\left(S^*f,g\right)_{\mathcal{H}} - \left(f,S^*g\right)_{\mathcal{H}} = \left(\gamma_1(f),\gamma_2(g)\right)_{\mathsf{H}} - \left(\gamma_2(f),\gamma_1(g)\right)_{\mathsf{H}}$$

while for any $F_1, F_2 \in H$, there exists an element $f \in D(S^*)$ such that $\gamma_1(f) = F_1$ and $\gamma_2(f) = F_2$.

Lemma 3.4. The triplet (H, γ_1, γ_2) ,

$$\gamma_1: D(M) \to H, \ \gamma_1(u) = \frac{1}{\sqrt{2}} ((\alpha u)(\infty) - (\alpha u)(a)) \text{ and}$$

 $\gamma_2: D(M) \to H, \ \gamma_2(u) = -\frac{1}{i\sqrt{2}} ((\alpha u)(\infty) + (\alpha u)(a)), \ u \in D(M)$

is a space of boundary values of the minimal operator M_0 in $L^2_{\alpha}(H,(a,\infty))$.

Theorem 3.5. If \widetilde{M} is a maximally symmetric extension of the minimal operator M_0 in $L^2_{\alpha}(H,(a,\infty))$, then it generates by the differential-operator expression $m(\cdot)$ and boundary condition

$$(\alpha u)(\infty) = V(\alpha u)(a),$$

where $V: H \to H$ is an isometric operator. Moreover, the isometric operator V in H is determined uniquely by the extension \widetilde{M} i.e. $\widetilde{M} = M_V$ and vice versa.

Now we describe the general form of all maximal formally normal extensions of minimal operator L_0 in $L^2_{\alpha}(H,(a,\infty))$.

Theorem 3.6. Let $A^{1/2}W_{2,\alpha}^1(H,(a,\infty)) \subset W_2^1(H,(a,\infty))$. Each maximal formally normal extension $\tilde{L}, L_0 \subset \tilde{L} \subset L$ of the minimal operator L_0 in $L^2_{\alpha}(H,(a,\infty))$ generates by the differential-operator expression $l(\cdot)$ with boundary condition

$$(\alpha u)(\infty) = V(\alpha u)(a),$$

where V and $A^{1/2}VA^{-1/2}$ are isometric operators in H. The isometric operator V is determined uniquely by the extension \tilde{L} , i.e. $\tilde{L} = L_V$.

On the contrary, the restriction of the maximal operator L to the linear manifold of vector-functions $(\alpha u) \in W_{2,\alpha}^1(H,(a,\infty))$ that satisfy mentioned above condition for some

isometric operator V, where $A^{1/2}VA^{-1/2}$ also isometric operator in H, is a maximal formally normal extension of the minimal operator L_0 in $L^2_{\alpha}(H,(a,\infty))$.

Proof. If \tilde{L} is any maximal formally normal extension of the minimal operator L_0 in $L^2_{\alpha}(H,(a,\infty))$, then

$$Re(\tilde{L}) = A \otimes E, Re(\tilde{L}): D(\tilde{L}) \to L^{2}_{\alpha}(H, (a, \infty)),$$
$$Im(\tilde{L}) = \overline{E \otimes \iota \frac{d}{dt}(\alpha)}, Im(\tilde{L}): D(\tilde{L}) \to L^{2}_{\alpha}(H, (a, \infty)),$$

where the symbol \otimes denotes a tensor product, are selfadjoint extensions of $Re(L_0)$ and $Im(L_0)$ in $L^2_{\alpha}(H,(a,\infty))$, respectively. Then the extension $Im(\tilde{L})$ is generated by differential expression $m(\cdot)$ and boundary condition

$$(\alpha u)(\infty) = V(\alpha u)(a),$$

where V is an isometric operator in H such that it determined uniquely by the extension \tilde{L} , i.e. $\tilde{L} = L_V$ [8].

On the other hand since the extension \tilde{L} is a maximal formally normal operator, then for every $u \in D(\tilde{L})$ the following equality holds

$$(Re(\tilde{L})u, Im(\tilde{L})u)_{L^{2}_{\alpha}(H,(a,\infty)} = (Im(\tilde{L})u, Re(\tilde{L})u)_{L^{2}_{\alpha}(H,(a,\infty)}.$$

In other words, for every $u \in D(\tilde{L})$ we have $\left(\left(\alpha u\right)^{\prime}, Au\right)_{L^{2}_{\alpha}(H,(a,\infty))} + \left(Au,\left(\alpha u\right)^{\prime}\right)_{L^{2}_{\alpha}(H,(a,\infty))} = 0.$

From last relation and condition of theorem $A^{1/2}W_{2,\alpha}^1(H,(a,\infty)) \subset W_2^1(H,(a,\infty))$ we have

 $\left(\left(\alpha A^{1/2}u\right)^{\prime}, \alpha A^{1/2}u\right)_{L^{2}(H,(a,\infty))} + \left(\alpha A^{1/2}u, \left(\alpha A^{1/2}u\right)^{\prime}\right)_{L^{2}(H,(a,\infty))} = 0, \text{ that is, for every } u \in D(\tilde{L})$ $\int_{a}^{\infty} \left(\alpha A^{1/2}u, \alpha A^{1/2}u\right)^{\prime}_{H} dt = \left\|\left(\alpha A^{1/2}u\right)(\infty)\right\|_{H}^{2} - \left\|\left(\alpha A^{1/2}u\right)(a)\right\|_{H}^{2} = 0. \text{ Hence there exists a isometry operator } K \text{ in } H \text{ , such that } A^{1/2}(\alpha u)(\infty) = KA^{1/2}(\alpha u)(a), \text{ that is,}$

$$(\alpha u)(\infty) = A^{-1/2} V A^{1/2}(\alpha u)(a), \ u \in D(\tilde{L}).$$

Since the isometric operator K in H uniquely is determined by the extension \tilde{L} , then from last equation it is obtained that $A^{-1/2}KA^{1/2} = V$, that is, $K = A^{1/2}VA^{-1/2}$ is isometric in H.

On the other hand, a sufficient part of this theorem can be easily to check. 18

Hence the proof of theorem is completed.

4. Spectrum of the maximal formally normal extensions

Here the spectrum of the maximal formally normal extension of the minimal operator L_0 generated by linear differential expression $l(\cdot)$ with corresponding boundary condition in Theorem 3.6 in $L^2_{\alpha}(H,(a,\infty))$ will be investigated.

Firstly let us prove the following results.

Theorem 4.1. The spectrum of any maximal formally normal extension L_v in $L^2_{\alpha}(H,(a,\infty))$ of the minimal operator L_0 has a form

$$\sigma(L_V) = \left\{ \lambda \in \mathbb{C} : \lambda = \left(\int_a^\infty \frac{ds}{\alpha(s)} \right)^{-1} (\ln|\mu|^{-1} + 2n\pi i - iarg\mu), n \in \mathbb{Z}, \mu \right\}$$
$$\in \sigma\left(V^* exp\left(-A \int_a^\infty \frac{ds}{\alpha(s)} \right) \right) \left\}.$$

Proof. Consider a problem for the spectrum for the any maximal formally normal extension L_V , that is,

$$(\alpha u)'(t) + Au(t) = \lambda u(t) + f(t), \ \lambda \in \mathbb{C}, \ Re\lambda = \lambda_r \ge 1, \ u, \ f \in L^2_{\alpha}(H, (a, \infty))$$

with boundary condition $(\alpha u)(\infty) = V(\alpha u)(a)$, where V and $A^{1/2}VA^{-1/2}$ are the isometric operators in H.

Then it is clear that a general solution of above differential equation is in form

$$u(t;\lambda) = \frac{1}{\alpha(t)} exp\left(\left(\lambda E - A\right)\int_{a}^{t} \frac{ds}{\alpha(s)}\right) f_{\lambda} + \frac{1}{\alpha(t)}\int_{a}^{t} exp\left(\left(\lambda E - A\right)\int_{s}^{t} \frac{d\tau}{\alpha(\tau)}\right) f(s) ds, f_{\lambda} \in H.$$

In this case we can easily see that $u(\cdot, \lambda) \in L^2_{\alpha}(H, (a, \infty))$ for $\lambda \in \mathbb{C}$, $\lambda_r \ge 1$. Hence the boundary condition we get the following relation

$$\left(exp\left(-\lambda\int_{a}^{\infty}\frac{ds}{\alpha(s)}\right)-V^{*}exp\left(-A\int_{a}^{\infty}\frac{ds}{\alpha(s)}\right)\right)f_{\lambda}=exp\left(-\lambda\int_{a}^{\infty}\frac{ds}{\alpha(s)}\right)V^{*}\int_{a}^{\infty}exp\left((\lambda E-A)\int_{s}^{\infty}\frac{d\tau}{\alpha(\tau)}\right)f(s)ds.$$

From this it is seen that in order to $\lambda \in \sigma(L_{\nu})$ the necessary and sufficient condition is

$$exp\left(-\lambda\int_{a}^{\infty}\frac{ds}{\alpha(s)}\right) = \mu \in \sigma\left(V^{*}exp\left(-\Lambda\int_{a}^{\infty}\frac{ds}{\alpha(s)}\right)\right).$$

Therefore

$$\lambda = \left(\int_{a}^{\infty} \frac{ds}{\alpha(s)}\right)^{-1} (\ln|\mu|^{-1} + 2n\pi i - i \arg \mu), n \in \mathbb{Z}, \mu \in \sigma\left(V^* exp\left(-A \int_{a}^{\infty} \frac{ds}{\alpha(s)}\right)\right).$$

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Singular Numbers of Lower Triangular One-Band Block Operator Matrices

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In this work the boundedness and compactness properties of lower triangular oneband block operator matrices in the infinite direct sum of Hilbert spaces have been studied. And also belonging to Schatten-von Neumann classes this type operators it will be investigated.

 Keywords: Direct sum of Hilbert spaces, Lower triangular one-band block operator matrix, Compact operator, Schatten-von Neumann classes.
 2010 AMS Classification: 47A05, 47A10.

1. Introduction

The general theory of singular or characteristic numbers for linear compact operators in Hilbert spaces has been explained in I. Z. Gohberg and M. G. Krein [1]. Actually Schmidt E. [2] and J. von Neumann, R. Schatten [3] have used there important results in the theory of non-selfadjoint integral operators.

It is known that the traditional infinite direct sum of Hilbert spaces H_n , $n \ge 1$ is defined as

$$H = \bigoplus_{n=1}^{\infty} H_n = \left\{ u = (u_n) : u_n \in H_n, \ n \ge 1, \ \sum_{n=1}^{\infty} \|u_n\|_{H_n}^2 < \infty \right\}$$

Note that H is a Hilbert space with norm induced by the inner product

$$(u,v)_{H} = \sum_{n=1}^{\infty} (u_{n},v_{n})_{H_{n}}, u,v \in H \text{ (see [4])}.$$

Throughout the paper we will use the notations as follows

$$(\cdot,\cdot)_{H} \coloneqq (\cdot,\cdot), \|\cdot\|_{H} \coloneqq \|\cdot\| \text{ and } (\cdot,\cdot)_{H_{n}} \coloneqq (\cdot,\cdot)_{n}, \|\cdot\|_{H_{n}} \coloneqq \|\cdot\|_{n}, n \ge 1.$$

It is known that a lot of physical problems of today arising in the modelling of processes of multiparticle quantum mechanics, quantum field theory and in the physics of rigid bodies support to study a theory of linear operators in the direct sum of Hilbert spaces (see [5], [6] and references in it).

Investigation of these problems in direction of spectral analysis of finite or infinite dimensional real and complex entires special matrices (upper and lower triangular double-band or third-band or Toeplitz types) in sequences spaces ω , c, c_0 , bs, $b\omega_p$, l_p have been widely studied in current literature (for example, see [7] - [15]).

On the other hand some spectral analysis of 2×2 and 3×3 types block operator matrices have been provided in [16], [17]. Note that the structure of spectrum of diagonal operator matrices has been surveyed in [18]. The compactness properties and belonging to Schattenvon Neumann classes of diagonal operator matrices in the direct sum of Hilbert spaces has been researched in [19].

In this work, firstly the compactness properties of lower triangular one-band operator matrices in the infinite direct sum of Hilbert spaces have been investigated. Lastly, belonging to Schatten-von Neumann classes of this type operators will be studied.

Throughout the paper algebra of linear bounded operators, linear compact operators from any Hilbert space \mathcal{H}_1 to another Hilbert space \mathcal{H}_2 , Schatten-von Neumann classes and operator singular numbers of any compact will be denoted by $L(\mathcal{H}), L(\mathcal{H}_1, \mathcal{H}_2), C_{\infty}(\mathcal{H}_1, \mathcal{H}_2), C_p(\mathcal{H}_1, \mathcal{H}_2), 1 \le p < \infty \text{ and } s_n(\cdot), n \ge 1, \text{ respectively. And}$ also $L(\mathcal{H}) = L(\mathcal{H}, \mathcal{H}), \ C_{\infty}(\mathcal{H}) = C_{\infty}(\mathcal{H}, \mathcal{H}) \text{ and } C_{p}(\mathcal{H}) = C_{p}(\mathcal{H}, \mathcal{H}), \ 1 \le p < \infty.$

2. Boundedness and compactness of lower triangular one-band block operator

matrices

Here it will be investigated the continuity and compactness properties of lower one-band diagonal block operator matrices in form

$$A = \begin{pmatrix} 0 & & & \\ A_1 & 0 & & & \\ & A_2 & 0 & 0 & \\ & & \ddots & \ddots & \\ & 0 & & A_n & 0 & \\ & & & & \ddots & \ddots \end{pmatrix}$$

in direct sum $H = \bigoplus_{n=1}^{\infty} H_n$ of Hilbert Spaces H_n , $n \ge 1$ in case when $A_n \in L(H_n, H_{n+1})$, $n \ge 1$. Note that $A = A_d V$ and $A_d = AK$, where

Firstly, prove the following propositions.

Theorem 2.1. In order to $A \in L(H)$ the necessary and sufficient condition is

$$\sup_{n\geq 1} \|A_n\| < \infty$$

Proof: Now consider the boundedness of the operator A_d under the condition $\sup_{\substack{n \\ n \notin 5}} ||A_n|| < \infty$. Indeed for any $x = (x_n) \in H$ we have

$$\|A_{d}x\|_{H}^{2} = \sum_{n=1}^{\infty} (A_{n}x_{n}, A_{n}x_{n})_{H_{n+1}} \leq \sum_{n=1}^{\infty} \|A_{n}\|^{2} \|x_{n}\|_{H_{n}}^{2} \leq \left(\sup_{n\geq 1} \|A_{n}\|\right)^{2} \sum_{n=1}^{\infty} \|x_{n}\|_{H_{n}}^{2} = \left(\sup_{n\geq 1} \|A_{n}\|\right)^{2} \|x\|_{H_{n}}^{2}$$

Hence $A_{d} \in L(H_{0}, H)$. From this and relation $A = A_{d}V$ it is implies that $A \in L(H)$.

On the contrary, if $A \in L(H)$, then via an equality $A_d = AK$ it is obtained that $A_d \in L(H_0, H)$.

Now we will shown that $\sup_{n\geq 1} ||A_n|| < \infty$. Let $A \in L(H)$ but $\sup_{n\geq 1} ||A_n|| = \infty$. In this case, there is a $(k_n) \subset \mathbb{N}$ sequence such that

$$\|A_{k_n}\| = \sup\left\{\frac{\|A_{k_n}u_{k_n}\|_{k_{n+1}}}{\|u_{k_n}\|_{k_n}} : u_{k_n} \in H_{k_n}, \ u_{k_n} \neq 0, \ n \ge 1\right\} \to \infty, \ as \ n \to \infty.$$

In that case there is a $(u_{k_n}^*) \subset H_{k_n}$ sequence such that $\frac{\|A_{k_n}u_{k_n}^*\|_{k_{n+1}}}{\|u_{k_n}^*\|_{k_n}} \to \infty$, as $n \to \infty$. Then,

the $(m_n) \subset (k_n)$ subsequence can be found such that for any $n \ge 1$, $\frac{\|A_{m_n} u_{m_n}^*\|_{m_{n+1}}}{\|u_{m_n}^*\|_{m_n}} > n$. That is,

for any $n \ge 1$, the following inequality is obtained $\frac{\left\|u_{m_n}^*\right\|_{m_n}}{\left\|A_{m_n}u_{m_n}^*\right\|_{m_{n+1}}} \le \frac{1}{n}$. In this case for the

element in form $v := (0, 0, ..., 0, v_{m_1}, 0, ..., 0, v_{m_2}, 0, ..., 0, v_{m_n}, 0, ...), v_{m_n} = \frac{u_{m_n}^*}{\|A_{m_n} u_{m_n}^*\|_{m_{n+1}}}, n \ge 1$ is true $\|v\|_H^2 = \sum_{n=1}^{\infty} \|v_{m_n}\|_{m_n}^2 \le \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$ Unfortunately, $\|Av\|_H^2 = \sum_{n=1}^{\infty} \frac{\|A_{m_n} u_{m_n}^*\|_{m_{n+1}}}{\|A_{m_n} u_{m_n}^*\|_{m_n}} = \sum_{n=1}^{\infty} 1 < \infty.$

Consequently, $v \notin D(A)$ and so $D(A) \neq H$. This contradiction indicates $\sup_{n \neq 1} ||A_n|| < \infty$.

Theorem 2.2. If $A \in C_{\infty}(H)$, then for any $n \ge 1$, $A_n \in C_{\infty}(H_n, H_{n+1})$. Moreover, in case $A_n \in C_{\infty}(H_n, H_{n+1})$, $n \ge 1$ in order for $A \in C_{\infty}(H)$ the necessary and sufficient condition is $\lim_{n \to \infty} ||A_n|| = 0.$

Proof. The first part of theorem it is clear. The validity second part of theorem it is implies from

$$A = A_d V, A_d = A K$$

and important results on compactness of operators in direct sum of Hilbert spaces [19].

Theorem 2.3. The spectrum of the operator $A \in C_{\infty}(H)$ with $dimH = +\infty$ is in form

$$\sigma(A) = \sigma_r(A) = \{0\}.$$
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Proof. Consider the following problem to point spectrum $Ax = \lambda x$, $x \neq 0$, $\lambda \neq 0$, $x \in H$. Then

$$0 = \lambda x_1, \ A_1 x_1 = \lambda x_2, \ A_2 x_2 = \lambda x_3, \dots, A_n x_n = \lambda x_{n+1}, \dots$$

Hence, for any $n \ge 1$, $x_n = 0$. That is, x = 0. Consequently, if $\lambda \ne 0$, then $\lambda \notin \sigma_p(A)$.

On the other hand $\lambda = 0 \in \sigma_r(A)$. Indeed, since $\dim H_1 \ge 1$ and $(AH)^{\perp} \supset H_1$ or $\overline{(AH)} \neq H$, then $0 \in \sigma_r(A)$.

Theorem 2.4. Let $A \in C_{\infty}(H)$. Then for the singular numbers of A is true

$$\left\{s_m(A): m \ge 1\right\} = \bigcup_{n=1}^{\infty} \left\{s_k(A_n): k \ge 1\right\}.$$

Proof. In this case $A^*A = \bigoplus_{n=1}^{\infty} A_n^*A_n$. Therefore $\sqrt{A^*A} = \bigoplus_{n=1}^{\infty} \sqrt{A_n^*A_n}$. From this and Theorem 2.3.

[18] it is obtained that
$$\sigma_p(\sqrt{A^*A}) = \bigcup_{n=1}^{\infty} \sigma_p(\sqrt{A_n^*A_n})$$
. Hence

 $\left\{s_m(A):m\geq 1\right\}=\bigcup_{n=1}^{\infty}\left\{s_k(A_n):k\geq 1\right\}.$

From this it is implies the validity the following results.

Corollary 2.5. If $A \in C_p(H)$, $1 \le p \le \infty$, then for every $n \ge 1$, $A_n \in C_p(H_n, H_{n+1})$. *Proof.* For $p = \infty$ this proposition has been proved in Theorem 2.2. Let $1 \le p < \infty$. Since $A \in C_p(H)$, then the series $\sum_{m=1}^{\infty} s_m^p(A)$ is convergent. Consequently, from inequality $\sum_{k=1}^{\infty} s_k^p(A_n) \le \sum_{m=1}^{\infty} s_m^p(A), n \ge 1$

it is implies that the series $\sum_{k=1}^{\infty} s_k^p(A_n)$ is convergent. This means that for every $n \ge 1, A_n \in C_p(H_n, H_{n+1})$.

Theorem 2.6. In order to $A \in C_p(H)$, $1 \le p < \infty$ if and only if the series $\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} s_k^p(A_n)$ is convergent.

Proof. If $A \in C_p(H)$, then the series $\sum_{m=1}^{\infty} s_m^p(A)$ is convergent. In this case by the Theorem 2.4. and important theorem on the convergence of rearrangement series it is obtained that the series $\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} s_k^p(A_n)$ is convergent.

On the contrary, if the series $\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} s_k^{p}(A_n)$ is convergent, the series $\sum_{m=1}^{\infty} s_m^{p}(A)$ being a rearrangement of the above series is also convergent. So $A \in C_p(H)$.

Corallary 2.7. For $n \ge 1$, $A_n \in C_{p_n}(H_n, H_{n+1})$ and $p = \sup_{n\ge 1} p_n < \infty$. Then $A \in C_p(H)$ if and only if the series $\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} s_k^p(A_n)$ converges.

Indeed, in this case for each $n \ge 1$, $A_n \in C_{p_n}(H_n, H_{n+1})$. Then from Theorem 2.6. it is obtained that the validity of proposition.

Corallary 2.8. For each $n \ge 1$, $A_n \in C_{p_n}(H_n, H_{n+1})$ and $p = \sup_{n\ge 1} p_n < \infty$. If the series $\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} s_k^{p_n}(A_n)$ is convergent, then $A \in C_p(H)$.

Remark 2.9. The similar problems can be considered for the block operator-matrices in form

$$A = \begin{pmatrix} 0 & & & & & \\ . & 0 & & & & \\ . & 0 & & 0 & & \\ . & & \ddots & & & \\ A_1 & & & \ddots & & \\ A_2 & & & \ddots & & \\ & A_2 & & & \ddots & & \\ & & \ddots & & & \ddots & \\ & 0 & A_n & & 0 & \\ & & & \ddots & & \ddots \end{pmatrix}, A : H \to H.$$

Note that in this case $A^*A = diag(A_1^*A_1, A_2^*A_2, \dots, A_n^*A_n, \dots)$. Then $s_n(A) = s_k(A_m), k \ge 1, m \ge 1$.

Example 2.10. Let us $H_n = (\mathbb{C}^2, |.|), A_n = \begin{pmatrix} 0 & \alpha^n \\ \alpha^n & 0 \end{pmatrix}, n \ge 1, \alpha \in \mathbb{C}, |\alpha| < 1.$ Then

 $||A_n|| = |\alpha|^n$ and $\lim_{n \to \infty} ||A_n|| = 0$. In this case the operator

in direct sum $H = \bigoplus_{n=1}^{\infty} H_n$ is compact and singular numbers of A are in form

$$s_n(A) = |\alpha|^n, n \ge 1.$$

Moreover, $A \in C_p(H)$ for any $p, 1 \le p < \infty$.

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Local Existence of Solutions to Initial Boundary Value Problem for the Higher Order Boussinesq Equation

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Abstract

We study the initial and boundary value problem for the higher order Boussinesq equation. The existence of a local solution is proved.

Keywords: The higher order Boussinesq equation; Initial and boundary value problem; Existence of local solution

1 Introduction

In this study, we consider the initial and boundary value problem of the higher order Boussinesq (HBq) equation

$$u_{tt} = u_{xx} + \eta_1 u_{xxtt} - \eta_2 u_{xxxxtt} + (f(u))_{xx}, \qquad 0 < x < 1, \quad t > 0, \tag{1}$$

 $u(0,t) = 0, \ u(1,t) = 0, \ u_{xx}(0,t) = 0, \ u_{xx}(1,t) = 0, \qquad t > 0,$ (2)

$$u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \qquad 0 \le x \le 1,$$
(3)

where u(x,t) denotes the unknown function, η_1 and η_2 are real positive constants, f(u) is the given nonlinear function and $u_0(x)$ and $u_1(x)$ are known initial conditions. The higher order Boussinesq equation was first derived by Rosenau in [1] for continuum limit of a dense chain of particles with elastic couplings. The same equation was also used to model water waves with surface tension in [2] and the bi-directional propagation of longitudinal waves in an infinite, non-locally elastic medium in [3]. Eliminating the higher order dispersive effects $(\eta_2 = 0)$, the HBq equation reduces to the well-known improved Boussinesq (IBq) equation.

Zhijian and Guowang et al. have proved the global well posedness of solutions to the initial and boundary value problem of the IBq equation in [4, 5]. The local and global well-posedness of the Cauchy problem for the HBq equation has been proved in [3] and [6] in the Sobolev space H^s with any s > 1/2. The numerical investigation of the HBq equation is studied in [7] by using Fourier pseudo-spectral method. It is therefore natural to ask how the higher-order dispersive term affects the local solution to the initial and boundary value problem. For this aim, we will study the local existence of solutions to the initial and boundary value problem for the HBq equation. In this study, we denote the Sobolev space $W^{k,2}(0,1)$ by $H^k(0,1)$. We use notations as follows: $\|.\|, \|.\|_{\infty}, \|.\|_{k,2}$ and $\|.\|_{k,\infty}$ denote the norms of the spaces $L^2(0,1), L^{\infty}(0,1), H^k(0,1)$ and $W^{k,\infty}(0,1)$, respectively.¹⁹⁰

2 Local existence of solution

In this section, we will prove the existence and uniqueness of the local solution for the initial and boundary value problem (1)-(3) by using the contraction mapping principle. If we define $v(x,t) = \int_0^x u(\xi,t) d\xi$, then v satisfies

$$v_{xtt} = v_{xxx} + \eta_1 v_{xxxtt} - \eta_2 v_{xxxxxtt} + (f(v_x))_{xx}, \qquad 0 < x < 1, \quad t > 0, \tag{4}$$

and initial and boundary conditions

$$v_x(0,t) = 0, \ v_x(1,t) = 0, \ v_{xxx}(0,t) = 0, \ v_{xxx}(1,t) = 0, \qquad t > 0,$$

$$v(x,0) = v_0(x), \ v_t(x,0) = v_1(x), \qquad 0 \le x \le 1,$$
(5)

where $v_0(x) = \int_0^x u_0(\xi) d\xi$ and $v_1(x) = \int_0^x u_1(\xi) d\xi$. Now we consider the initial and boundary value problem (5) of the following equation

$$v_{tt} = v_{xx} + \eta_1 v_{xxtt} - \eta_2 v_{xxxxtt} + (f(v_x))_x, \qquad 0 < x < 1, \quad t > 0.$$
(6)

Assume that $f \in C^2(\mathbb{R})$ and $v \in C^2([0,T]; H^5(0,1))$ is a smooth solution of problem (5)-(6), then v is a generalized solution of problem (4)-(5). We define the function space $\tilde{H}^4(0,1)$ by

$$\tilde{H}^4(0,1) = \{ u \in H^4(0,1) | \ u(0) = u(1) = 0, u_{xx}(0) = u_{xx}(1) = 0 \}$$

Therefore $u \in C^2([0,T]; \tilde{H}^4(0,1))$ is a solution of problem (1)-(3). We rewrite the equation (6) as

$$(I - \eta_1 \frac{\partial^2}{\partial x^2} + \eta_2 \frac{\partial^4}{\partial x^4})v_{tt} = f_1(v_x)_x, \quad 0 < x < 1, \quad t > 0, \tag{7}$$

where $f_1(v_x) = f(v_x) + v_x$.

The Green's function $G(x,\xi)$ of the boundary value problem for the ordinary differential equation

$$y - \eta_1 \frac{d^2 y}{dx^2} + \eta_2 \frac{d^4 y}{dx^4} = 0, \qquad y'(0) = y'(1) = y'''(0) = y'''(1) = 0$$

is given by

$$G(x,\xi) = \begin{cases} \frac{1}{2\eta_2(\alpha_1^2 - \alpha_2^2)} [\frac{e^{\alpha_2(2-\xi)} + e^{\alpha_2\xi}}{\alpha_2(e^{2\alpha_2} - 1)} (e^{\alpha_2 x} + e^{-\alpha_2 x}) - \frac{e^{\alpha_1(2-\xi)} + e^{\alpha_1\xi}}{\alpha_1(e^{2\alpha_1} - 1)} (e^{\alpha_1 x} + e^{-\alpha_1 x})], 0 \le x < \xi \\ \frac{1}{2\eta_2(\alpha_1^2 - \alpha_2^2)} [\frac{e^{\alpha_2(2-x)} + e^{\alpha_2 x}}{\alpha_2(e^{2\alpha_2} - 1)} (e^{\alpha_2 \xi} + e^{-\alpha_2 \xi}) - \frac{e^{\alpha_1(2-x)} + e^{\alpha_1 x}}{\alpha_1(e^{2\alpha_1} - 1)} (e^{\alpha_1 \xi} + e^{-\alpha_1 \xi})], \xi < x \le 1, \end{cases}$$

for $\eta_1^2 > 4\eta_2$ where $\alpha_1 = \sqrt{\frac{\eta_1 + \sqrt{\eta_1^2 - 4\eta_2}}{2\eta_2}}$ and $\alpha_2 = \sqrt{\frac{\eta_1 - \sqrt{\eta_1^2 - 4\eta_2}}{2\eta_2}}$. The problem (5)-(6) is equivalent to the problem

$$v(x,t) = v_0(x) + v_1(x)t - \int_0^t \int_0^1 (t-\tau)G_{\xi}(x,\xi)f_1(v_{\xi}(\xi,\tau)) \, d\xi d\tau \quad 0 < x < 1,$$

$$v_x(0,t) = 0, \ v_x(1,t) = 0, \ v_{xxx}(0,t) = 0, \ v_{xxx}(1,t) = 0, \qquad t > 0,$$
 (8)

i.e. $v \in C^2([0,T]; H^4(0,1))$ is a solution of (5)-(6) if and only if $v \in C^2([0,T]; H^4(0,1))$ is a solution of problem (8).

Theorem 2.1 Assume that $v_0, v_1 \in W^{3,\infty}(0,1)$, $v'_0(0) = v'_0(1) = v'_1(0) = v'_1(1) = v''_0(0) = v'''_0(1) = v'''_1(0) = v'''_1(1) = 0$ and $f(s) \in C^1(\mathbb{R})$ with f(0) = 0. Then problem (5)-(6) has unique weak solution $v \in C^2([0,T^0); W^{3,\infty}(0,1))$ where $[0,T^0)$ is the maximal interval of existence of v(t), and if

$$\sup_{0 \le t < T^0} (\|v(t)\|_{\infty} + \|v_x(t)\|_{\infty} + \|v_{xx}(t)\|_{\infty} + \|v_{xxx}(t)\|_{\infty}) < +\infty,$$
(9)

then $T^0 = +\infty$.

Proof 1 : We consider the Banach space

$$X(T) = \{ v \in C([0,T]; W^{3,\infty}(0,1)) | v_x(0,t) = v_x(1,t) = v_{xxx}(0,t) = v_{xxx}(1,t) = 0 \}$$

which endowed with the norm

$$\|v\|_{X(T)} = \sup_{0 \le t \le T} \|v(t)\|_{3,\infty} = \sup_{0 \le t \le T} (\|v(t)\|_{\infty} + \|v_x(t)\|_{\infty} + \|v_{xx}(t)\|_{\infty} + \|v_{xxx}(t)\|_{\infty}).$$
(10)

We define the set

$$B(M,T) = \{ v | v \in X(T), \|v\|_{X(T)} \le M \}.$$
(11)

If the equation (7) is integrated twice with respect to t, the solution of the initial and boundary value problem satisfies the integral equation v = Sw where

$$(\mathcal{S}w)(x,t) = v_0(x) + v_1(x)t - \int_0^t \int_0^1 (t-\tau)G_{\xi}(x,\xi)f_1(w_{\xi}(\xi,\tau)) \, d\xi d\tau.$$
(12)

We first prove S maps B(M,T) into B(M,T) for T small enough. Using the triangle inequality, we have

$$|Sw(x,t)| \le |v_0(x)| + |v_1(x)|t + \int_0^t (t-\tau)| \int_0^1 G_{\xi}(x,\xi) f_1(w_{\xi}(\xi,\tau)) \ d\xi | d\tau.$$

Using $|G_{\xi}(x,\xi)| \leq \frac{2}{\sqrt{\eta_1^2 - 4\eta_2}}$ and locally Lipschitz continuity of f, we obtain

$$||Sw(t)||_{\infty} \leq ||v_0||_{\infty} + ||v_1||_{\infty}t + \frac{(L+1)M}{\sqrt{\eta_1^2 - 4\eta_2}} t^2.$$
(13)

Similarly
$$|G_{\xi x}(x,\xi)| \le \sqrt{\frac{2(\eta_1 + \sqrt{\eta_1^2 - 4\eta_2})}{\eta_2(\eta_1^2 - 4\eta_2)}}$$
 and $|G_{\xi xx}(x,\xi)| \le \frac{\eta_1 + \sqrt{\eta_1^2 - 4\eta_2}}{\eta_2\sqrt{\eta_1^2 - 4\eta_2}}$, we have

$$\|(Sw)_{x}(t)\|_{\infty} \leq \|v_{0}'\|_{\infty} + \|v_{1}'\|_{\infty}t + \sqrt{\frac{2(\eta_{1} + \sqrt{\eta_{1}^{2} - 4\eta_{2}})}{\eta_{2}(\eta_{1}^{2} - 4\eta_{2})}}(L+1)M\frac{t^{2}}{2} \\ \|(Sw)_{xx}(t)\|_{\infty} \leq \|v_{0}''\|_{\infty} + \|v_{1}''\|_{\infty}t + \frac{\eta_{1} + \sqrt{\eta_{1}^{2} - 4\eta_{2}}}{\eta_{2}\sqrt{\eta_{1}^{2} - 4\eta_{2}}}(L+1)M\frac{t^{2}}{2}.$$

Since $G_{\xi xxx}(x,\xi)$ has jump discontinuity at $\xi = x$, we get

$$(\mathcal{S}w)_{xxx}(x,t) = v_0'''(x) + v_1'''(x)t - \int_0^t (t-\tau) \int_0^1 G_{\xi xxx}(x,\xi) f_1(w_{\xi}(\xi,\tau)) \, d\xi d\tau - \int_0^t (t-\tau) (f_1(w_x(x,\tau))) d\tau.$$
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Using $|G_{\xi xxx}(x,\xi)| \le \sqrt{\frac{(\eta_1 + \sqrt{\eta_1^2 - 4\eta_2})^3}{2\eta_2^3(\eta_1^2 - 4\eta_2)}}$, we obtain

$$\|\mathcal{S}w_{xxx}(t)\|_{\infty} \le \|v_0'''\|_{\infty} + \|v_1'''\|_{\infty}t + (\sqrt{\frac{(\eta_1 + \sqrt{\eta_1^2 - 4\eta_2})^3}{2\eta_2^3(\eta_1^2 - 4\eta_2)}} + 1)(L+1)M\frac{t^2}{2}$$

By choosing M as

 $M \ge 2(\|v_0\|_{\infty} + \|v_1\|_{\infty} + \|v_0'\|_{\infty} + \|v_1'\|_{\infty} + \|v_0''\|_{\infty} + \|v_1''\|_{\infty} + \|v_0'''\|_{\infty} + \|v_1'''\|_{\infty})$ and sufficiently small T satisfying

$$T \le \min\{1, [(\frac{2(1+\alpha_1+\alpha_1^2+\alpha_1^3)}{\sqrt{\eta_1^2-4\eta_2}}+1)(L+1)]^{-1/2}\}$$

we have $|| Sw ||_{X(T)} \leq M$. This proves that S maps B(M,T) into B(M,T). The next step is to prove that S is contractive, namely,

$$\|\mathcal{S}w_1 - \mathcal{S}w_2\|_{X(T)} \le \frac{1}{2} \|w_1 - w_2\|_{X(T)}, \qquad \forall w_1, w_2 \in B(M, T).$$

We have

$$\mathcal{S}w_1 - \mathcal{S}w_2 = \int_0^t (t-\tau) \int_0^1 G_{\xi}(x,\xi) (f_1(w_{1\xi}(\xi,\tau)) - f_1(w_{2\xi}(\xi,\tau))) \, d\xi d\tau.$$

We define $\overline{f} : [0, \infty) \to [0, \infty)$ by $\overline{f}(t) = \max_{|s| \le t} \{|f_1(s)|, |f'_1(s)|\}$. Using the Mean Value Theorem for f_1 and integration by parts, we obtain

$$\|\mathcal{S}w_1(t) - \mathcal{S}w_2(t)\|_{\infty} \le \frac{1}{\sqrt{\eta_1^2 - 4\eta_2}}\overline{f}(M) \ t^2 \|w_1(t) - w_2(t)\|_{1,\infty}.$$
 (14)

Similarly, we have

$$\|(\mathcal{S}w_1)_x(t) - (\mathcal{S}w_2)_x(t)\|_{\infty} \le \frac{\alpha_1}{\sqrt{\eta_1^2 - 4\eta_2}} \overline{f}(M) \ t^2 \|w_1(t) - w_2(t)\|_{1,\infty}$$
(15)

$$\|(\mathcal{S}w_1)_{xx}(t) - (\mathcal{S}w_2)_{xx}(t)\|_{\infty} \le \frac{\alpha_1^2}{\sqrt{\eta_1^2 - 4\eta_2}} \overline{f}(M) \ t^2 \|w_1(t) - w_2(t)\|_{1,\infty}$$
(16)

$$\|(\mathcal{S}w_1)_{xxx}(t) - (\mathcal{S}w_2)_{xxx}(t)\|_{\infty} \le (\frac{\alpha_1^3}{\sqrt{\eta_1^2 - 4\eta_2}} + 1)\overline{f}(M) \ t^2 \|w_1(t) - w_2(t)\|_{1,\infty}.$$
(17)

Adding the inequalities (14)-(17), we have

$$\|\mathcal{S}w_1(t) - \mathcal{S}w_2(t)\|_{3,\infty} \le \overline{f}(M) \ t^2 (\frac{1 + \alpha_1 + \alpha_1^2 + \alpha_1^3}{\sqrt{\eta_1^2 - 4\eta_2}} + 1) \|w_1(t) - w_2(t)\|_{3,\infty}.$$
 (18)

By taking the supremum with respect to t, we get

$$\|\mathcal{S}w_1 - \mathcal{S}w_2\|_{X(T)} \le \frac{1}{2} \|w_1 - w_2\|_{X(T)},$$

where $T \le \sqrt{\frac{\sqrt{\eta_1^2 - 4\eta_2}}{2\overline{f}(M)(1 + \alpha_1 + \alpha_1^2 + \alpha_1^3 + \sqrt{\eta_1^2 - 4\eta_2})}}.$

We conclude that S is a contraction mapping from B(M,T) to itself when M is sufficiently large and T is sufficiently small relative to M. The problem (8) has at most one solution $v \in C([0,T']; W^{3,\infty}(0,1))$ for each $T': 0 < T' < T^0$, where [0,T'] is the maximal interval of existence of v(t). **Theorem 2.2** Assume that $v_0, v_1 \in H^{m+3}(0,1), m \ge 1, v'_0(0) = v'_0(1) = v'_1(0) = v'_1(1) = v''_0(0) = v'''_0(1) = v'''_1(0) = v'''_1(1) = 0, f(s) \in C^m(\mathbb{R}) and f(0) = 0.$ Then problem (8) has a unique solution $v \in \bigcap_{k=1}^m C^{m+2-k}([0,T^0);$

 $H^{k+3}(0,1)$, where $[0,T^0)$ is the maximal interval of existence of v(t). Moreover if (9) holds, then $T^0 = +\infty$.

Corollary 2.3 Asume that $u_0, u_1 \in H^{m+2}(0,1) \cap \tilde{H}^4(0,1), f \in C^m(\mathbb{R}), (m \ge 2), f(0) = 0$. Then problem (1)-(3) has a solution $u \in \bigcap_{k=2}^m C^{m+2-k}([0,T^0); H^{k+2}(0,1) \cap \tilde{H}^4(0,1))$ where $[0,T^0)$ is the maximal interval of existence of u(t). Moreover, if

$$\sup_{0 \le t < T^0} (\operatorname*{ess\,sup}_{x \in [0,1]} | \int_0^x u(\xi, t) d\xi | + ||u(t)||_{\infty} + ||u_x(t)||_{\infty} + ||u_{xx}(t)||_{\infty}) < +\infty,$$
(19)

then $T^0 = +\infty$.

Now, we will obtain the stability of problem (1)-(3) as in the following theorem.

Theorem 2.4 Assume that $u_0, u_1 \in \tilde{H}^4(0, 1)$, f(0) = 0. Then the solution u of problem (1)-(3) belongs to $C^2([0, T^0); \tilde{H}^4(0, 1))$, and continuously depends upon the initial data, i.e. if $\bar{u} \in C^2([0, T^0); \tilde{H}^4(0, 1))$ is another solution of problem (1)-(3) corresponding to the initial data $\bar{u}_0, \bar{u}_1 \in \tilde{H}^4(0, 1)$, then for any $\epsilon > 0$, then there is a $\delta > 0$ such that $||u(t) - \bar{u}(t)||_{3,2}^2 + ||u_t(t) - \bar{u}_t(t)||_{3,2}^2 < \epsilon$ for all $t \in [0, T](T < T^0)$ as $||u_0 - \bar{u}_0||_{3,2}^2 + ||u_1 - \bar{u}_1||_{3,2}^2 < \delta$, where $[0, T^0)$ is the maximal interval of existence of u(t).

According to Corollary (2.3) and Theorem (2.4), we have the following corollary.

Corollary 2.5 Under the assumptions of Theorem (2.4), problem (1)-(3) has a unique solution $u \in C^2([0, T^0); \tilde{H}^4(0, 1))$. And if (19) holds, then $T^0 = +\infty$.

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Semi-classical Analysis for the Long Wave-Short Wave Interaction Equations

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Abstract

In this study, we applied WKB analysis of the three coupled long wave-short wave interaction (LSI) equations. We consider the semi-classical limit of the LSI for initial data with Sobolev regularity, before the shocks appear in the limit system.

Keywords: The long wave-short wave interaction equation; Dispersive perturbation: Quasilinear hyperbolic system

1 Introduction

In this study, we investigate the semi-classical limit, i.e., $\epsilon \to 0$, of the three coupled long wave-short wave interaction (LSI) equations are given by

$$i\epsilon\partial_t\psi_1^\epsilon + \alpha\epsilon^2\partial_x^2\psi_1^\epsilon = \beta(u^\epsilon + |\psi_1^\epsilon|^2)\psi_1^\epsilon \tag{1}$$

$$i\epsilon\partial_t\psi_2^\epsilon + \alpha\epsilon^2\partial_x^2\psi_2^\epsilon = \beta(u^\epsilon + |\psi_2^\epsilon|^2)\psi_2^\epsilon \tag{2}$$

$$\partial_t u^{\epsilon} = \beta \partial_x (|\psi_1^{\epsilon}|^2 + |\psi_2^{\epsilon}|^2) \,. \tag{3}$$

Here, the real valued function $u^{\epsilon}(x,t)$ is the amplitude of long longitudinal wave and the complex valued functions $\psi_1(x,t)$ and $\psi_2(x,t)$ are the slowly varying envelopes of short transverse waves. The coupling parameters α and β are real constants and ϵ is a small parameter ($0 < \epsilon \ll 1$). The initial data are the WKB states (or Lagrangian states)

$$\psi_k^{\epsilon}(x,0) = \psi_{k,0}^{\epsilon}(x) = A_{k,0}^{\epsilon}(x) \exp\left(\frac{i}{\epsilon}S_{k,0}^{\epsilon}(x)\right), \qquad k = 1,2$$
(4)

for the short waves and

$$u^{\epsilon}(x,0) = u_0^{\epsilon}(x) \tag{5}$$

for the long wave. Note that $S_{k,0}^{\epsilon}$, k = 1, 2 are functions of $H^{s}(\mathbb{R})$ (Sobolev space) for s large enough, $A_{k,0}^{\epsilon}$, k = 1, 2 are functions, polynomials in ϵ , with coefficients of Sobolev regularity in x. More precisely, we are concerned with the behaviour of the solutions to¹² SI equations (1)–(5) as $\epsilon \to 0$ with rapid oscillating initial data for the short waves.

2 Modified Madelung transformation

We will use the modified Madelung transformation to convert LSI equations (1)-(3) into a linear dispersive perturbation of a quasi-linear hyperbolic system. Therefore we can apply for the Lax-Friedrich-Kato theory. We look for the solutions of the form

$$\psi_k^{\epsilon} = A_k^{\epsilon}(x, t) \exp\left[\frac{i}{\epsilon} S_k^{\epsilon}(x, t)\right], \qquad k = 1, 2.$$
(6)

Here the amplitudes $A_1^{\epsilon}(x,t)$ and $A_2^{\epsilon}(x,t)$ are allowed to be complex-valued. Substituting the ansatz (6) into (1), we obtain

$$\begin{split} i\epsilon\partial_t A_1^\epsilon &- A_1^\epsilon\partial_t S_1 + \alpha\epsilon^2\partial_{xx}A_1^\epsilon + i2\alpha\epsilon\partial_x A_1^\epsilon\partial_x S_1^\epsilon + i\alpha\epsilon A_1^\epsilon\partial_{xx}S_1^\epsilon \\ &- \alpha A_1^\epsilon(\partial_x S_1^\epsilon)^2 = \beta(u+|A_1^\epsilon|^2)A_1^\epsilon \end{split}$$

We can decompose the above equation as

$$\partial_t A_1^\epsilon + 2\alpha \partial_x A_1^\epsilon \partial_x S_1^\epsilon + \alpha A_1^\epsilon \partial_{xx} S_1^\epsilon = i\alpha\epsilon \partial_{xx} A_1^\epsilon \tag{7}$$

$$\partial_t S_1^{\epsilon} + \alpha (\partial_x S_1^{\epsilon})^2 + \beta (u^{\epsilon} + |A_1^{\epsilon}|^2) = 0.$$
(8)

Similarly, substituting the ansatz (6) into (2), we have

$$i\epsilon\partial_t A_2^\epsilon - A_2^\epsilon \partial_t S_2^\epsilon + \alpha \epsilon^2 \partial_{xx} A_2^\epsilon + i2\alpha\epsilon\partial_x A_2^\epsilon \partial_x S_2^\epsilon + i\alpha\epsilon A_2^\epsilon \partial_{xx} S_2^\epsilon -\alpha A_2^\epsilon (\partial_x S_2^\epsilon)^2 = \beta (u^\epsilon + |A_2^\epsilon|^2) A_2^\epsilon$$

and it can be decomposed the above equation as

$$\partial_t A_2^\epsilon + 2\alpha \partial_x A_2^\epsilon \partial_x S_2^\epsilon + \alpha A_2^\epsilon \partial_{xx} S_2^\epsilon = i\alpha \epsilon \partial_{xx} A_2^\epsilon \tag{9}$$

$$\partial_t S_2^{\epsilon} + \alpha (\partial_x S_2^{\epsilon})^2 + \beta (u^{\epsilon} + |A_2^{\epsilon}|^2) = 0.$$
⁽¹⁰⁾

Since $A_1^{\epsilon}(x,t)$ and $A_2^{\epsilon}(x,t)$ are complex valued functions, we take $A_k^{\epsilon} = a_k^{\epsilon} + ib_k^{\epsilon}$, k = 1, 2. We use the substitutions $v_k^{\epsilon} = 2\alpha \partial_x S_k^{\epsilon}$, k = 1, 2 which yield the following equations

$$\partial_t a_1^\epsilon + v_1^\epsilon \partial_x a_1^\epsilon + \frac{a_1^\epsilon}{2} \partial_x v_1^\epsilon = -\alpha \epsilon \partial_{xx} b_1^\epsilon \tag{11}$$

$$\partial_t b_1^\epsilon + v_1^\epsilon \partial_x b_1^\epsilon + \frac{b_1^\epsilon}{2} \partial_x v_1^\epsilon = \alpha \epsilon \partial_{xx} a_1^\epsilon \tag{12}$$

$$\partial_t v_1^\epsilon + v_1^\epsilon \partial_x v_1^\epsilon + 2\alpha\beta \partial_x u^\epsilon + 4\alpha\beta (a_1^\epsilon \partial_x a_1^\epsilon + b_1^\epsilon \partial_x b_1^\epsilon) = 0$$
(13)

$$\partial_t a_2^\epsilon + v_2^\epsilon \partial_x a_2^\epsilon + \frac{a_2^\epsilon}{2} \partial_x v_2^\epsilon = -\alpha \epsilon \partial_{xx} b_2^\epsilon \tag{14}$$

$$\partial_t b_2^\epsilon + v_2^\epsilon \partial_x b_2^\epsilon + \frac{b_2^\epsilon}{2} \partial_x v_2^\epsilon = \alpha \epsilon \partial_{xx} a_2^\epsilon \tag{15}$$

$$\partial_t v_2^\epsilon + v_2^\epsilon \partial_x v_2^\epsilon + 2\alpha\beta \partial_x u^\epsilon + 4\alpha\beta (a_2^\epsilon \partial_x a_2^\epsilon + b_2^\epsilon \partial_x b_2^\epsilon) = 0$$
(16)

and the long wave satisfies the conservation law

$$\partial_t u^{\epsilon} - \beta \partial_x \left(|a_1^{\epsilon}|^2 + |b_1^{\epsilon}|^2 + |a_2^{\epsilon}|^2 + |b_2^{\epsilon}|^2 \right) = 0 \tag{17}$$

This system can be written in the vector form

$$\partial_t U^\epsilon + \mathcal{A}(U^\epsilon) \partial_x U^\epsilon + V^\epsilon = \alpha \epsilon \mathcal{L}(U^\epsilon) \tag{18}$$

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where

$$U^{\epsilon}(x,t) = \left(a_1^{\epsilon}(x,t), b_1^{\epsilon}(x,t), v_1^{\epsilon}(x,t), a_2^{\epsilon}(x,t), b_2^{\epsilon}(x,t), v_2^{\epsilon}(x,t)\right)^T$$
$$V^{\epsilon}(x,t) = 2\alpha\beta\left(0, 0, u_x^{\epsilon}(x,t), 0, 0, u_x^{\epsilon}(x,t)\right)^T$$

and the initial datum are given by

$$u_0^{\epsilon}(x) = u^{\epsilon}(x,0), \tag{19}$$

and

$$U_{0}^{\epsilon}(x) = \left(a_{1}^{\epsilon}(x,0), b_{1}^{\epsilon}(x,0), v_{1}^{\epsilon}(x,0), a_{2}^{\epsilon}(x,0), b_{2}^{\epsilon}(x,0), v_{2}^{\epsilon}(x,0)\right)^{T} \\ \equiv \left(a_{1,0}^{\epsilon}(x), b_{1,0}^{\epsilon}(x), v_{1,0}^{\epsilon}(x), a_{2,0}^{\epsilon}(x), b_{2,0}^{\epsilon}(x), v_{2,0}^{\epsilon}(x)\right)^{T}.$$
(20)

The matrices \mathcal{A} and \mathcal{L} are given respectively by

$$\mathcal{A}(U^{\epsilon}) = \begin{pmatrix} v_1^{\epsilon} & 0 & a_1^{\epsilon}/2 & 0 & 0 & 0\\ 0 & v_1^{\epsilon} & b_1^{\epsilon}/2 & 0 & 0 & 0\\ 4\alpha\beta a_1^{\epsilon} & 4\alpha\beta b_1^{\epsilon} & v_1^{\epsilon} & 0 & 0 & 0\\ 0 & 0 & 0 & v_2^{\epsilon} & 0 & a_2^{\epsilon}/2\\ 0 & 0 & 0 & 0 & v_2^{\epsilon} & b_2^{\epsilon}/2\\ 0 & 0 & 0 & 4\alpha\beta a_2^{\epsilon} & 4\alpha\beta b_2^{\epsilon} & v_2^{\epsilon} \end{pmatrix}$$
(21)

and

Obviously the matrix $\mathcal{A}(U^{\epsilon})$ can be symmetrized by

$$S = \begin{pmatrix} 8\alpha\beta & 0 & 0 & 0 & 0 & 0 \\ 0 & 8\alpha\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8\alpha\beta & 0 & 0 \\ 0 & 0 & 0 & 0 & 8\alpha\beta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$
(23)

which is symmetric and positive definite for $\alpha\beta > 0$. Therefore, we have the existence and uniqueness of the classical solution of the dispersive perturbation of the quasilinear symmetric hyperbolic system (18)–(20).

Theorem 2.1 Let $s > \frac{5}{2}$ and $\alpha\beta > 0$. Suppose $M_0 \ge 1$, M and T are given such that

$$[M_0 + (cM_0^2 + M)T]e^{cM_0T} \le 2M_0.$$
(24)

- (1) If $V^{\varepsilon} \in L^{\infty}([0,T]; H^{s}(\mathbb{R})) \cap C([0,T]; H^{s-2}(\mathbb{R}))$ such that $\|V^{\varepsilon}\|_{H^{s}} \leq M$ and the initial data $U_{0}^{\varepsilon} \in H^{s}(\mathbb{R})$ satisfying $\|U_{0}^{\varepsilon}\|_{H^{s}(\mathbb{R})} \leq M_{0}$ are given then, the initial value problem for the (18)–(20) has a unique solution $U^{\varepsilon} \in C([0,T]; H^{s}(\mathbb{R})) \cap C^{1}([0,T]; H^{s-2}(\mathbb{R}))$ such that $\|U^{\varepsilon}\|_{H^{s}} \leq 2M_{0}$.
- (2) If $U^{\varepsilon} = U^{\varepsilon}_{(0)}, U^{\varepsilon}_{(1)}$ are the solutions corresponding with $V^{\varepsilon} = V^{\varepsilon}_{(0)}, V^{\varepsilon}_{(1)}$ for the same initial conditions U^{ε}_{0} satisfying the condition (1), then 197

$$|U_{(0)}^{\varepsilon} - U_{(1)}^{\varepsilon}||_{H^{s-2}} \le c e^{cM_0 T} T ||V_{(0)}^{\varepsilon} - V_{(1)}^{\varepsilon}||_{H^{s-3}}, \qquad (25)$$

(3) Let k = 1, 2. If $\rho_k^{\epsilon}(x, 0) = (a_k^{\epsilon}(x, 0))^2 + (b_k^{\epsilon}(x, 0))^2 > 0$ then $\rho_k^{\epsilon}(x, t) > 0$ for all $t \ge 0$; if $\rho_k^{\epsilon}(x, 0)$ has a compact support, then $\rho_k^{\epsilon}(\cdot, t)$ does too for any $t \in [0, T]$ and

$$R\{\rho_k^{\epsilon}(\cdot,t)\} \le R\{\rho_k^{\epsilon}(\cdot,0)\} + (1+\epsilon)CT.$$
(26)

where $R{f} \equiv \sup\{|x| : f(x) \neq 0\}.$

The proof is similar to that given in [1] (Theorem 3.1), the key step is to show

$$V^{\epsilon} \in L^{\infty}([0,T];H^{s}(\mathbb{R})) \bigcap C([0,T];H^{s-2}(\mathbb{R}))$$

such that $\|V^{\epsilon}\|_{H^s} \leq M$, which is equivalent to show

$$u^{\epsilon} \in L^{\infty}([0,T];H^{s+1}(\mathbb{R})) \bigcap C([0,T];H^{s-1}(\mathbb{R})) = C([0,T]$$

To this end, we represent the long wave equation (17) as an integral form

$$u^{\epsilon}(x,t) = u_0^{\epsilon}(x) + \beta \int_0^t \partial_x \left(|a_1^{\epsilon}|^2 + |b_1^{\epsilon}|^2 + |a_2^{\epsilon}|^2 + |b_2^{\epsilon}|^2 \right) d\tau \,,$$

then by Cauchy-Schwarz, Minkowski integral inequalities and the imbedding $H^1 \hookrightarrow L^2$ we have

$$\begin{split} \|u^{\epsilon}(t)\|_{L^{2}} &\leq \|u_{0}^{\epsilon}\|_{L^{2}} + \beta \bigg(\int_{-\infty}^{\infty} \bigg| \int_{0}^{t} \partial_{x} \big(|a_{1}^{\epsilon}|^{2} + |b_{1}^{\epsilon}|^{2} + |a_{2}^{\epsilon}|^{2} + |b_{2}^{\epsilon}|^{2} \big) d\tau \bigg|^{2} dx \bigg)^{\frac{1}{2}} \\ &\leq \|u_{0}^{\epsilon}\|_{L^{2}} + \beta \int_{0}^{t} \bigg(\int_{-\infty}^{\infty} \bigg| \partial_{x} \big(|a_{1}^{\epsilon}|^{2} + |b_{1}^{\epsilon}|^{2} + |a_{2}^{\epsilon}|^{2} + |b_{2}^{\epsilon}|^{2} \big) \bigg|^{2} dx \bigg)^{\frac{1}{2}} d\tau \\ &\leq \|u_{0}^{\epsilon}\|_{L^{2}} + 2\beta T \big(\|a_{1}^{\epsilon}\|_{H^{1}}^{2} + \|b_{1}^{\epsilon}\|_{H^{1}}^{2} + \|a_{2}^{\epsilon}\|_{H^{1}}^{2} + \|b_{2}^{\epsilon}\|_{H^{1}}^{2} \big) \,. \end{split}$$

Similarly, for higher derivatives $(s \ge 1)$

$$\|u^{\epsilon}(t)\|_{H^{s-1}} \leq \|u_0^{\epsilon}\|_{H^{s-1}} + 2\beta T \left(\|a_1^{\epsilon}\|_{H^s}^2 + \|b_1^{\epsilon}\|_{H^s}^2 + \|a_2^{\epsilon}\|_{H^s}^2 + \|b_2^{\epsilon}\|_{H^s}^2\right).$$

Moreover, since $a_k^{\epsilon}, b_k^{\epsilon} \in C([0,T]; H^s(\mathbb{R})) \cap C^1([0,T]; H^{s-2}(\mathbb{R})), \ k = 1, 2$, the integral formulation also shows that

$$u^{\epsilon} \in \operatorname{Lip}([0,T]; H^{s-1}(\mathbb{R}))$$

Theorem 2.2 Assume the hypothesis of Theorem 2.1. Given initial datum $U_0^{\epsilon}, U_0 \in H^s(\mathbb{R})$ and $U_0^{\epsilon}(x)$ converges to $U_0(x)$ in $H^s(\mathbb{R})$ as $\epsilon \to 0$. Let [0,T] be the fixed interval determined in Theorem 2.1. Then as $\epsilon \to 0$, there exists $U \in L^{\infty}([0,T]; H^s(\mathbb{R}))$ and $u \in C^1([0,T]; H^{s-1}(\mathbb{R}))$ such that for all $\sigma > 0$

$$U^{\epsilon} \rightarrow U \quad in \quad C([0,T]; H^{s-\sigma}(\mathbb{R})),$$

$$(27)$$

$$u^{\epsilon} \rightarrow u \quad in \quad C^1([0,T]; H^{s-\sigma-1}(\mathbb{R})).$$
 (28)

The function U(x,t) belongs to $C([0,T]; H^s(\mathbb{R})) \cap C^1([0,T]; H^{s-1}(\mathbb{R}))$ and is a classical solution of limit system.

The proof is the standard compactness argument, Arzela-Ascoli theorem (applied in the time variable) and the Rellich lemma (applied in the space variable).

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An investigation of Teachers' Qualifications of Designing Interdisciplinary Mathematical Modeling Activities¹

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Abstract

This study examines the criteria that science and mathematics teachers use when developing interdisciplinary mathematical modeling activities. In the study, the interdisciplinary dimension is limited to mathematics and science and teachers choice of subjects while integrating two disciplines were also investigated. This study was carried out by 9 science teachers and 9 mathematics teachers who participated in a three-month workshop training in which the theoretical knowledge about mathematical modeling and interdisciplinary mathematical modeling was presented. The teachers were asked to develop interdisciplinary mathematical modeling activities in groups of two, one being science and one being mathematics. As a result of the preliminary analysis, it was seen that teachers related the two disciplines before the training mainly in the context of science that requires mathematical calculations such as physics. However, after the workshop, they related two disciplines more in biology oriented subjects while developing interdisciplinary mathematical modeling activities. The teachers included real-life scenario that may be meaningful for the students in the developed-activities that enable them to provide different models effectively. The results are thought to be important in terms of seeing the contributions of the workshop training to the teachers.

Keywords: Interdisciplinary Mathematical Modeling, Mathematical Modeling Teacher Competencies, Activity Dimension

1. Introduction

With the rapid changes in technology and knowledge, the competencies that individuals possess vary as well. One of the important aims of the education system is to enable individuals to solve the problems they have in real life with the knowledge and skills they have (MEB, 2018). Mathematical modeling may have an important effect in acquiring these kinds of skills (Lesh and Doerr, 2003). Mathematical modeling is a complex process involving solving a problem situation in real life with the aid of a mathematical model and

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interpreting the obtained solution in real life and evaluating the final outcome (Berry and Houston, 1995). By engaging in mathematical modeling, students discover the mathematics in real life and have the opportunity to see that mathematics is not a separate discipline from life, and that it is intertwined with life. Students realize how mathematics is needed in real life (Borromeo Ferri, 2018). It helps students better understand the relationship between mathematical concepts and allows them to develop different perspectives on a problem situation (Chamberlin and Moon, 2005). Because of it is nature, mathematical modeling may bring together many disciplines (Lingefjard 2007). Thus, mathematical modeling can be seen as a bridge to STEM education (English, 2015). From this perspective, interdisciplinary mathematical modeling (IMM), which can be expressed similar ideas as the understanding of mathematical modeling that deals with different disciplines together. It is very important to integrate IMM activities that enhance problem-solving skills by providing the opportunity to combine different disciplines in the instructional program and use it in practice in order to support the development of learning at the conceptual level. For this purpose, it is necessary to increase teachers' awareness and skills of IMM activities. Ferri (2018) stated that teachers should have some competencies for effective modeling teaching, and they have dealt with the qualifications in four different dimensions as theoretical, activity, practice and diagnosis. In this study, the competencies of the teachers in the aspect of activity were discussed with regard to task design.

2. Materials and Methods

This study, which addresses IMM in the context of science and mathematics disciplines, was conducted by 9 mathematics and 9 science teachers. All teachers participated in a three-month workshop training in which the theoretical knowledge on IMM and mathematical modeling was presented. At the end of the training, the teachers were asked to develop IMM activities in groups of two, one being science and the other being mathematics.

3. Results and Discussions

The results showed that, at the beginning of the workshop, teachers associate science concepts such as force-motion, pressure, simple machines, which are required mathematical computation with mathematics. Yet, when they were asked to design an IMM activity at the end of the workshop, they mainly associate mathematics and science disciplines in biologyrelated concepts such as environmental problems, renewable energy sources, conscious

consumption, recycling. The main topics that teachers associate with are "force-motion", "pressure", "simple machines", "electrical circuits". When the activities that teachers have developed after training are examined, it has been determined that the science standards included in the activities are biology-oriented. The standards associated with the activities are presented in Table 1.

Problem	Concepts in Science
Water Saving Problem	Saving use of resources - Conscious consumption - Environmental education
Heat Insulation Problem	Head conductivity and insulation of materials
Apricot Problem	Nutrition Balance - digestion
Green Classes Problems	Photosynthesis- Oxygenated Respiration
Water Storage Problem	Fluid Pressure - Energy Conversion
Pet Bottle House Problem	Recycling- Sustainable Development
Forest Problem	Photosynthesis - Oxygen respiration
Farm Problem	Diet
Carbon Footprint Problem	Fossil fuels - air pollution - greenhouse effect - environmental education

The science curriculum focuses on teaching the science concepts without entering the mathematical competition, which may be the reason why teachers prefer more biology-related concepts while designing the activities. Other reason might be the nature of IMM which focuses on real-world activities in association with mathematics and science. The aspects that teachers paid attention to when designing their activities included a real-life scenario that might be meaningful for the students with a strong story to encourage them to solve the problem (§ahin, Doğan, Gürbüz ve Çavuş Erdem, 2017). The emergence of different models or solutions in IMM activities was another important criterion that teachers consider when designing their activities. One of the most important features that distinguishes mathematical modeling problems from other problems is that they includes different solutions (English, 2006, Antonius, Haines, Jensen, Niss and Burkhardt, 2007). While

considering teachers' limited conceptions of IMM at the beginning of the study, their progress of understanding and designing IMM activities were impressive and valuable.

4. Conclusions

Mathematical modeling and IMM are fairly new concepts for our country and gradually begin to take place in teachers training programs. Of course, it is very important for teachers to gain proficiency and to develop their awareness of IMM in order to take such practices in their classroom. While considering teachers' limited conceptions of IMM at the beginning of the study, their progress of understanding and designing IMM activities were impressive and valuable. Thus, proving teachers this kind of training programs are important for teachers development.

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STEM from Social Sciences Perspective: Mathematics-Turkish Interdisciplinary Model Eliciting Activity

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Abstract

STEM education is an educational approach designed to meet the need for educating creative individuals who systematically think in science, technology, engineering and mathematics, giving them a critical perspective, and transferring their learnings to new and different problems. STEM education is a constantly evolving field, and many different views have been raised in this area. According to some researchers, "Engineering" in the STEM section means "engineering" as well as "design and production". The letter "S" describing the word "science" includes not only natural sciences but also humanities and social sciences. The letter "E" is considered as design and production and "S" as social sciences.

The aim of the study is to see to what level students learn some concepts related to mathematics and social sciences together. For this purpose, the researchers worked together with the teachers of Mathematics and Turkish to develop the "Reading Problem" which is an activity to create interdisciplinary model. Before this problem was solved, the students worked on modeling problems for 4 weeks with collaborative learning approach and produced explanations, representations, mathematical forms, diagrams and mathematical models from the solutions of these problems. The Reading Problem includes learning areas of both mathematics and Turkish disciplines. This problem has been applied to 7th grade students in groups of 3-4 studying in a city center in the eastern region of Turkey. In the interdisciplinary problem-solving process, students have learned some concepts related to Turkish and developed a mathematical model.

Keywords: Model Development Process, Interdisciplinary Model Building Activity, Interdisciplinary Problem Solving

1. Introduction

In today's dynamic and digital societies, mathematics, science, medicine, social sciences, finance, engineering, economics and many other areas consist of complex systems. Complexity, formed of interconnected and hard to understand parts, has led to significant scientific methodological developments (Sabelli, 2006). With the spread of complex systems, new technologies have emerged for communication, collaboration and conceptualization, and these technologies have led to significant changes in the mathematical and scientific thinking styles required outside the classroom environment, such as producing, analyzing, working on and transforming complex data (English & Sriraman, 2010). These changes brought along new educational approaches. STEM is also one of these educational approaches. STEM education is an educational approach designed to meet the need of educating creative individuals in science, technology, engineering and mathematics fields, who think systematically, provide a critical perspective, transfer their learning to new and different problems, and are increasingly needed.

One of the tools that make the transition to STEM education is Model Eliciting Activities (MEA) (Maiorca and Stohlmann, 2016). Model Eliciting Activities (MEA) is an open-ended interdisciplinary problem solving activity that encourages students to build models to solve complex real-life problems and encourage them to test their models.

2. Method

In this study, multi-tiered teaching experiment (Lesh & Kelly, 2000) was used to see the development of the students in the process of solution of "Reading Problem". In addition, a modified version of the design research method (Dolk, Widjaja, Zonneveld, & Fauzan, 2010) was used to support the interpretation of research findings and the analysis of data.

3. Results

Findings in the study were handled within the framework of the semi-structured preliminary interview with teachers and the semi-structured final interviews with teachers and students. In a preliminary meeting with the teachers, they were asked "Do you teach your lessons by associating them with everyday life?", "Would you tell the lesson by associating

the topics with other disciplines? "Are you able to explain the topics with examples?" in order to see whether they relate their lessons to daily life. Below are some of the answers given by teachers to these questions:

 $\ddot{O}_{M:}$... Some students say they do not need to learn some subjects because they do not use them in everyday life. Therefore, we had better choose daily life examples after the topic is explained. Let me give an example: Let's say on a 100 km road, a car spends 5 lt of fuel. According to this, how much fuel do you spend on the 400 km road? So, I think it would be better to give examples of daily life like this.

 \ddot{O}_T : In the explanation of some subjects, we use mathematics. It helps make this lesson more permanent and effective. For example, I use the concept of clusters when teaching similarities and differences.

In the last meeting, teachers were asked "Have you ever seen such kind of problems (Reading Problem) before?", "Do you think these problems have improved the ability to associate disciplines?", "Do you think such problems should be included in the curriculum?", "If these problems are included in the curriculum, what kind of benefits can be provided to the students? ". Below are some of the answers given by teachers to these questions:

 \ddot{O}_{M} : I have never encountered such comprehensive problems before. One of the questions that students constantly ask is, "How does this matter benefits us in daily life?" Once students have encountered such problems, they no longer feel the need to ask such questions. Interdisciplinary learning certainly develops with such problems. The Mathematics Practice course needs to be composed of such problems.

 \ddot{O}_{T} : I think such problems improve the ability to associate Turkish-Mathematics. I also think such problems should be included in the curriculum. I always believed that there must be transitions between courses. I believe that such problems increase permanence, motivation and students' confidence in themselves.

Students were asked such questions as "Have you ever encountered such problems before?", "What benefits did these problems provide you?", "Would you like these types of

problems to appear in school books?" Below are some of the answers given by students to these questions:

Ümit: No. I have never encountered such problems before. After solving these problems, I learned to make connections between lessons.

Zehra: I generally did not like every lesson, but thanks to these problems, I started to like Mathematics and Turkish especially. I did not use to love mathematics before but now I do. I like Turkish more than before. Fortunately, I have seen these good problems.

4. Conclusion

The two teachers who participated in the study process stated that they already made everyday life and interdisciplinary associations of the subjects they taught. Mathematics teacher said that such problems should be included in the course of Mathematics Practices. Turkish Teacher thought that such problems should be included in the curriculum and stated that these problems provide interdisciplinary passage. Turkish Teacher stated that such problems affect permanent learning, students' self-confidence and motivation in a positive way. As for students' opinions, the students reported that they had never encountered such problems before, and that such problems had positively improved the ability of interdisciplinary association and attitude towards other disciplines. In addition, there are students who pointed out that DAMOEs increased self-confidence and influence mathematical attitude positively..

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Entropy of Weighted Tree Structures

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Abstract

Among the large number of existing indices, an important class of such measures relies on Shannon's entropy to characterize graphs by determining their structural information content. The Sackin index of a rooted tree is defined as the sum of the depths of its leaves. In this paper, we study the entropy of weighted tree structures with the Sackin index as weights. Exact formulas for the entropy of paths, stars, comets and dendrimers are given.

Keywords: Weighted trees, Entropy, Sackin index.

1. Introduction

Studies of the information content of complex networks and graphs have been initiated in the late 1950s based on the seminal work due to Shannon. Numerous measures for analyzing complex networks quantitatively have been contributed. A variety of problems in, e.g., discrete mathematics, computer science, information theory, statistics, chemistry, biology, etc., deal with investigating entropies for relational structures. For example, graph entropy measures have been used extensively to characterize the structure of graph-based systems in mathematical chemistry, biology and in computer science-related areas [1]. Rashevsky is the first who introduced the so-called structural information content based on partitions of vertex orbits [9]. Mowshowitz used the the same measure and proved some properties for graph operations (sum, join, etc.) [8]. Moreover, Rashevsky used the concept of graph entropy to measure the structural complexity of graphs. Mowshowitz introduced the entropy of a graph as an information-theoretic quantity, and he interpreted it as the structural information content of a graph. Mowshowitz later studied mathematical properties of graph entropies measures thoroughly and also discussed special applications thereof. Dehmer and Kraus [3] have studied extremal properties of graph entropies based on so-called information functionals. They obtained some extremality results for the resulting graph entropies which rely on the Shannon entropy. Also by applying these results, they infered some entropy bounds for certain graph classes. Kazemi [6] has studied the entropy of weighted graphs with the degree-

based topological indices as weights (see [4] for the degree–based topological indices). The goal of this paper is to study of entropy with a distance–based index as weights.

2. Sackin Index

The distance D(v) between the rooted root and node v (the depth of node v) in a tree of order n has been studied by many authors. Sackin index is one of the oldest measure that summarizes the shape of a tree [10]. It adds the number of internal nodes between each leaf of the tree and the root to form the following index $S(n) = \sum_{i=1}^{n} N_i$, where the sum runs over the n leaves of the tree and N_i is the number of internal nodes crossed in the path from i to the root (including the root). An equivalent formulation of S(n) is by counting the number of leaves under each internal nodes $S(n) = \sum_{j=1}^{n-1} \overline{N_j}$, where $\overline{N_j}$ is the number of leaves that descend from the ancestor j. In fact, the Sackin index S(n) of a tree of order n is defined as the sum of the depths of its leaves.

3. Entropy

For a given graph G and vertex v_i , let d_i be the degree of v_i . For an edge $v_i v_j$, one defines:

$$p_{ij} = \frac{w(v_i v_j)}{\sum_{j=1}^{d_j} w(v_i v_j)},$$
(1)

where $w(v_i v_j)$ is the weight of the edge $v_i v_j$ and $w(v_i v_j) > 0$. The node entropy has been defined by $H(v_i) = -\sum_{j=1}^{d_i} p_{ij} \log p_{ij}$. Motivated by this method, Chen *et al.* [2] introduced the definition of the entropy of edge-weighted graphs, which also can be interpreted as multiple graphs. For an edge-weighted graph, G = (V, E, w), where V, E and w denote the vertex set, the edge set and the edge weight of G, respectively.

Definition 1 For an edge weighted graph G = (V, E, w), the entropy of G is defined by:

$$I(G,w) = -\sum_{uv \in E} p_{u,v} \log p_{u,v},$$
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where
$$p_{u,v} = \frac{w(uv)}{\sum_{uv \in E} w(uv)}$$
.

The above definition of the entropy for edge-weighted graphs is based on the probability function (1).

4. Main Results

For a tree T_n of order n, assume that L_n is the set of its leaves.

Theorem 1 Let

$$w(uv) = \begin{cases} 0, & v \notin L_n \\ D(v), & v \in L_n. \end{cases}$$
(2)

Then for $n \ge 3$,

$$I(T_n, w) = \log S(n) - \frac{1}{S(n)} \sum_{v \in L_n} D(v) \log D(v),$$

where S(n) is the Sackin index.

Corollary 1 Assume $n \ge 3$. Let $D_{\min}(v)$ and $D_{\max}(v)$ be the minimum and maxumum of depth of node v in a rooted tree. By Theorem 1,

$$\log(\frac{S(n)}{D_{\max}(v)}) \le I(T_n, w) \le \log(\frac{S(n)}{D_{\min}(v)}).$$

In the rest of the paper, we use the weight defined in equation (2).

Theorem 2 Let P_n and S_n be the path and star of order n, respectively. Then

$$I(P_n, w) = \begin{cases} 0, & pendentnode = root \\ \log(n-1) - \frac{i\log(i) + (n-i-1)\log(n-i-1)}{n-1}, & pendentnode \neq root \end{cases}$$

$$I(S_n, w) = \begin{cases} \log(n-1), & central node = root \\ \log(n-2), & pendent node = root \end{cases}$$
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where i = 1, ..., n - i - 1.

Theorem 3 Let D(t;r) be a monocentric dendrimer with the progressive degree t and the radius r.

a) If the center is the root node, then $I(D(t;r), w) = \log(t^r(t+1))$.

b) If a leaf or branching node in *i* -th orbit is the root node, then

$$I(D(t;r),w) = \log(t^{r}(r(t+1)+i(t-1))) - \frac{t^{r+1}(r+i)\log(r+i)+t^{r}(r-i)\log(r-i)}{t^{r}(r(t+1)+i(t-1))}, i = 1,...,r.$$

Theorem 4 Let D(t;r) be a dicentric dendrimer with the progressive degree t and the radius r.

a) If one of the centers is the root node, then

$$I(D(t;r),w) = \log(t^{r+1}(2r+1)) - \frac{t^{r+1}r\log r + t^{r+1}(r+1)\log(r+1)}{t^{r+1}(2r+1)}$$

b) If a laef or branching node in *i* -th orbit is the root node, then

$$I(D(t;r),w) = \log((t-1)t^{r}(r+i) + t^{r}(r-i) + t^{r+1}(r+i))$$

$$-\frac{t^{r}(t-1)(r+i)\log(r+i)+t^{r}(r-i)\log(r-i)+t^{r+1}(r+i)\log(r+i)}{(t-1)t^{r}(r+i)+t^{r}(r-i)+t^{r+1}(r+i)},$$

where i = 1, ..., 2r.

Theorem 5 Let C(n;t) be a comet of order n with t pendent nodes.

a) If a pendent node in star $\,S_{\iota+1}\,$ is the root, then

$$I(C(n;t),w) = \log(n+t-3) - \frac{(t-2)\log 2 + (n-t+1)\log(n-t+1)}{n+t-3}$$

b) If the central node in star S_{i+1} is the root, then $I(C(n;t),w) = \log(n-1) - \frac{(n-t)\log(n-t)}{212} - \frac{(n-t)\log(n-t)}{n-1}$.

c) If the pendent node in path P_{n-i} is the root, then

$$I(C(n;t),w) = \log(t-1)(n-t+1) - \frac{\log(n-t+1)}{t-1}.$$

d) If a non-pendent node in path P_{n-t} is the root, then

$$I(C(n;t),w) = \log((t-1)(n-t-(i-2)) + (i-1)) - \frac{\log(i-1) + (n-t-(i-2))\log(n-t-(i-2))}{(t-1)(n-t-(i-2)) + (i-1)},$$

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Energy of Dynamical Force Fields in Minkowski Space via Parallel Vectors

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Abstract

In this study, energy of the moving timelike particle in different force fields are computed by considering fundamental definitions of the differential geometry and kinematics of the moving particle.

Keywords: Dynamics System, Force, Energy, Parallel Vector Field.

1. Introduction

Many researches have been done on the energy of a vector field under certain circumstances. Unit vector field's energy on a Riemannian manifold M is described to be equal to the energy of the mapping $M \rightarrow T_1M$, where T_1M is defined as unit tangent bundle equipped with Sasaki metric, [3]. By similar argument volume of a unit vector field Ξ is described as the volume of the submanifold in the unit tangent bundle defined by X(M) [4].

Then studies on the energy of unit vector fields are divesified by investigating energy on the special vector fields in the last couple years. For instance, Altin [1], computed energy of a Frenet vector fields for a given non-lightlike curves in semi Euclidean space. Körp nar [7], discussed timelike biharmonic particle's energy in Heisenberg spacetime.

Motion of a particle in space has always drawn attention of scientists due to wide range application of the subject. Motion of the particle in absolute space and time was defined firstly by Newtonian dynamics. Then geometric generalization of this action which includes terms belonging to curvature of the moving particle's trajectory in different spaces are given in [5,8].

The equations of particle motion's in the certain vector field are thought that utilizes generalized accelarations, velocities, and coordinates and is appropriate for obtaining motion's integral. For instance, [10] investigated Lagrangian equations of particle's and photon's motion in Schwarzschild field to show that attraction exerted on photons and particles through gravitational field is proportional to kinetic energies of photons and particles. Trajectories of charged particle moving on M under the influence of a magnetic field B corresponds to

magnetic curves such that Lorentz force can be defined as skew-symmetric (1,1)-type tensor field for B thanks to Riemannian metric.

2. Materials and Methods

Let Γ be a particle moving in a space such that the precise location of the particle is specified by $\Gamma = \Gamma(t)$, where t is a time parameter. Changing time parameter describes the motion and hereby the trajectory corresponds to a curve ζ in the space for a moving particle. It is defined by $\frac{ds}{dt} = ||\mathbf{v}||$, where $\mathbf{v} = \mathbf{v}(t) = \frac{d\zeta}{dt}$ is the velocity vector and $\frac{d\zeta}{dt} \neq 0$. In particle dynamics, the arc-length parameter s is considered as a function of t. Thanks to the arclenght, it is also determined Serret-Frenet frame, which allow us determining characterization of the intrinsic geometrical features of the regular curve. This coordinate system is constructed by four orthonormal vectors $\mathbf{e}_{(\alpha)}^{\mu}$ assuming the curve is sufficiently smooth at each point. The index within the parameters is the tetrad index that describes particular member of the tetrad. In particular, $\mathbf{e}_{(0)}^{\mu}$ is the unit tangent vector, $\mathbf{e}_{(1)}^{\mu}$, $\mathbf{e}_{(2)}^{\mu}$, $\mathbf{e}_{(2)}^{\mu}$ are the first, second, third unit normals to the curve ζ , respectively. Orthonormality conditions are summarized by $\mathbf{e}_{(\alpha)}^{\mu} \mathbf{e}_{(\beta)}^{\mu} = \eta_{\alpha\beta}$, where $\eta_{\alpha\beta}$ is Lorentzian metric such that: diag(-1,1,1,1). For a nonnegative coefficients κ , τ , δ , which are known as first, second and third curvature of the curve ζ , and vectors $\mathbf{e}_{(\mu)}^{\mu}(i = 0,1,2,3)$ following equations and properties satisfy:

$$\frac{D\mathbf{e}_{(0)}^{\mu}}{ds} = \kappa \mathbf{e}_{(1)}^{\mu}, \ \frac{D\mathbf{e}_{(1)}^{\mu}}{ds} = \kappa \mathbf{e}_{(0)}^{\mu} + \tau \mathbf{e}_{(2)}^{\mu},$$
$$\frac{D\mathbf{e}_{(2)}^{\mu}}{ds} = \delta \mathbf{e}_{(3)}^{\mu} - \tau \mathbf{e}_{(1)}^{\mu}, \ \frac{D\mathbf{e}_{(3)}^{\mu}}{ds} = -\delta \mathbf{e}_{(2)}^{\mu}$$

Since we identify $\mathbf{e}_{(0)}^{\mu}$ as a unit vector tangent to the the curve's trajectory at each point on the curve, we have $\mathbf{e}_{(0)}^{\mu} = d\Gamma^{\mu}/ds$, where Γ^{μ} is the point on the trajectory of curve ζ . Thus $\mathbf{e}_{(0)}^{\mu}, \mathbf{e}_{(1)}^{\mu}, \mathbf{e}_{(2)}^{\mu}$ and $\mathbf{e}_{(3)}^{\mu}$ generate the Frenet frame [2].

As is known, Frenet frame is not defined when the first curvature vanishes. Parallel frame of the curve is determined as an alternative way to handle with this problem. This method is based on the choice of first, second and third normal vectors of the curve which are choosen as a perpendicular to unchanged tangent vectors of the curve such that derivatives of the first, second and third normal of the curve depend only fixed tangent vector. In other words, vectors of Frenet frame $\{\mathbf{e}_{(0)}^{\mu}, \mathbf{e}_{(1)}^{\mu}, \mathbf{e}_{(2)}^{\mu}, \mathbf{e}_{(3)}^{\mu}\}$ is replaced by $\{\mathbf{e}_{(0)}^{\mu^*}, \mathbf{e}_{(1)}^{\mu^*}, \mathbf{e}_{(2)}^{\mu^*}, \mathbf{e}_{(3)}^{\mu^*}\}$ due to

Lorentzian rotation. We should remind that $\mathbf{e}_{(0)}^{\mu^*} = \mathbf{e}_{(0)}^{\mu}$. By the fact that rotation of Lorentzian preserve the feautres and characters of the vectors then we have similar construction for parallel frame vectors as Frenet frame vectors. That is, $\mathbf{e}_{(\alpha)}^{\mu^*} \mathbf{e}_{(\beta)}^{\mu^*} = \eta_{\alpha\beta}$, where $\eta_{\alpha\beta}$ is Lorentzian metric such that: diag (-1,1,1,1). For a nonnegative coefficients k_1, k_2, k_3 , which are known as first, second and third principal curvature of the curve ζ with respect to parallel frame, and vectors $\mathbf{e}_{(i)}^{\mu^*}$ (i = 0,1,2,3) following equations and properties satisfy:

$$\frac{D\mathbf{e}_{(0)}^{\mu}}{ds} = k_1 \mathbf{e}_{(1)}^{\mu^*} + k_2 \mathbf{e}_{(2)}^{\mu^*} + k_3 \mathbf{e}_{(3)}^{\mu^*},$$

$$\frac{D\mathbf{e}_{(1)}^{\mu^*}}{ds} = k_1 \mathbf{e}_{(0)}^{\mu}, \ \frac{D\mathbf{e}_{(2)}^{\mu^*}}{ds} = k_2 \mathbf{e}_{(0)}^{\mu}, \ \frac{D\mathbf{e}_{(3)}^{\mu^*}}{ds} = k_3 \mathbf{e}_{(0)}^{\mu^*}$$

where $\kappa = \sqrt{k_1^2 + k_2^2 + k_3^2}$ and $k_1 = \kappa \cos\psi \cos\gamma$, $k_2 = \kappa (\sin\gamma \cos\zeta - \sin\psi \cos\gamma \sin\zeta)$

 $k_3 = \kappa(\sin\gamma\sin\zeta + \sin\psi\cos\gamma\cos\zeta) \, [6].$

Energy on the Unit Vector Field in Space

Definition 1 Let (M,ρ) and (N,\tilde{h}) be Riemannian manifolds. Then the energy of a differentiable map $f:(M,\rho) \rightarrow (N,\tilde{h})$ can be defined as

$$\varepsilon nergy(f) = \frac{1}{2} \int_{M} \sum_{a=1}^{n} \widetilde{h}(df(e_{a}), df(e_{a})) v,$$

where $\{e_a\}$ is a local basis of the tangent space and v is the canonical volume form in M [3].

Proposition 2 Let $Q: T(T^1M) \rightarrow T^1M$ be the connection map. Then following two conditions hold:

i) $\omega \circ Q = \omega \circ d\omega$ and $\omega \circ Q = \omega \circ \widetilde{\omega}$, where $\widetilde{\omega} : T(T^1M) \to T^1M$ is the tangent bundle projection;

ii) for $\rho \in T_x M$ and a section $\xi: M \to T^1 M$; we have $Q(d\xi(\rho)) = D_{\rho}\xi$, where D is the

Levi-Civita covariant derivative [1,3].

Definition 3 Let $\varsigma_1, \varsigma_2 \in T_{\varepsilon}(T^1M)$, then we define 216

 $\rho_{S}(\varsigma_{1},\varsigma_{2}) = \rho(d\omega(\varsigma_{1}),d\omega(\varsigma_{2})) + \rho(Q(\varsigma_{1}),Q(\varsigma_{2})).$

Theorem 4 Let the moving particle on the space be a timelike, then we can derive following relations on the energy of tangent, first, second and third normal of the Frenet vectors respectively;

$$\varepsilon nergy \mathbf{e}_{(0)}^{\mu} = \frac{1}{2} \int_{0}^{s} (-1+\kappa^{2}) ds, \quad \varepsilon nergy \mathbf{e}_{(1)}^{\mu^{*}} = \frac{1}{2} \int_{0}^{s} (-1+k_{1}^{2}) ds,$$

$$\varepsilon nergy \mathbf{e}_{(2)}^{\mu^{*}} = \frac{1}{2} \int_{0}^{s} (-1+k_{2}^{2}) ds, \quad \varepsilon nergy \mathbf{e}_{(3)}^{\mu^{*}} = \frac{1}{2} \int_{0}^{s} (-1+k_{3}^{2}) ds,$$

where k_1, k_2, k_3 are known as first, second and third principal curvature of the curve ζ , [9].

3. Results and Discussions

If the moving particle ζ has a unit timelike tangent vector $\mathbf{e}_{(0)}^{\mu}$, then $\mathbf{v}(t(s)) = \frac{d\zeta}{dt} = \frac{ds}{dt} \mathbf{e}_{(0)}^{\mu}$

and
$$\mathbf{a}(t(s)) = \frac{d\mathbf{v}}{dt} = \frac{d^2s}{dt^2} \mathbf{e}_{(0)}^{\mu} + k_1 \left(\frac{ds}{dt}\right)^2 \mathbf{e}_{(1)}^{\mu^*} + k_2 \left(\frac{ds}{dt}\right)^2 \mathbf{e}_{(2)}^{\mu^*} + k_3 \left(\frac{ds}{dt}\right)^2 \mathbf{e}_{(3)}^{\mu^*}$$
. According to Newton's

second law, the resultant force acting on the particle, which has the mass m, is defined by

$$\mathbf{F} = m\mathbf{a} = m\frac{d^2s}{dt^2}\mathbf{e}_{(0)}^{\mu} + mk_1\left(\frac{ds}{dt}\right)^2\mathbf{e}_{(1)}^{\mu^*} + mk_2\left(\frac{ds}{dt}\right)^2\mathbf{e}_{(2)}^{\mu^*} + mk_3\left(\frac{ds}{dt}\right)^2\mathbf{e}_{(3)}^{\mu^*}.$$

For the set of parallel vectors $\{\mathbf{e}_{(0)}^{\mu}, \mathbf{e}_{(1)}^{\mu^{*}}, \mathbf{e}_{(2)}^{\mu^{*}}, \mathbf{e}_{(3)}^{\mu^{*}}\}$, we may write $\mathbf{F} = \mathbf{F}_{0}\mathbf{e}_{(0)}^{\mu} + \mathbf{F}_{1}\mathbf{e}_{(1)}^{\mu} + \mathbf{F}_{2}\mathbf{e}_{(2)}^{\mu} + \mathbf{F}_{3}\mathbf{e}_{(3)}^{\mu}$, where

$$\mathbf{F}_{0} = \mathbf{F}_{0} \cdot \mathbf{e}_{(0)}^{\mu}, \ \mathbf{F}_{1} = \mathbf{F}_{1} \cdot \mathbf{e}_{(1)}^{\mu}, \ \mathbf{F}_{2} = \mathbf{F}_{2} \cdot \mathbf{e}_{(2)}^{\mu}, \ \mathbf{F}_{3} = \mathbf{F}_{3} \cdot \mathbf{e}_{(3)}^{\mu}.$$

Using this result we have following notations we have

$$\mathbf{F}_0 = m \frac{d^2 s}{dt^2}, \quad \mathbf{F}_1 = m k_1 \left(\frac{ds}{dt}\right)^2, \quad \mathbf{F}_2 = m k_2 \left(\frac{ds}{dt}\right)^2, \quad \mathbf{F}_3 = m k_3 \left(\frac{ds}{dt}\right)^2.$$

Energy on a Particle in Dynamics

Theorem 5. Energy on the unit timelike particle in the vector field of the resultant force \mathbf{F} by using Sasaki metric is given by

$$\varepsilon nergy \mathbf{F} = \frac{1}{2} \int_{0}^{s} (-1 + m^{2} (-(s + \kappa^{2} s^{3})^{2} + (3k_{1} s s + k_{1} s^{3})^{2} + (3k_{2} s s + k_{2} s^{3})^{2} + (3k_{3} s s + k_{3} s^{3})^{2}))ds,$$

where superposed dot denotes the time derivative of the function.

4. Conclusions

We believe that this study also will lead up to further research on the relativistic dynamics of the particle in Minkowski space in terms of computing the energy on a paricle in different force fields.

Finally, application of the energy on the moving particle of the dynamics and electrodynamics can be done in different spacetimes.

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Timelike Dynamical Magnetic Curves on 3D Semi-Riemannian Manifolds

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Abstract

In this study, we investigate a special type of timelike magnetic trajectories associated with a magnetic field B defined on a 3D semi-Riemannian manifold. Firstly, we consider a moving charged timelike particle, which is assumed to be under the action of a particular external force in the magnetic field B on the 3D semi-Riemannian manifold. Then, we assume that timelike trajectories of the particle associated with the magnetic field B correspond to a particular timelike dynamical magnetic curve of the magnetic vector field B. Furthermore, we compute energy of each dynamical magnetic curve by considering the least action principle. Then, the radius of gyration and the gyrofrequency of each timelike magnetic trajectory are investigated to comprehend the exact movement of the charged particle in the given uniform magnetic field B. Finally, we give the physical interpretations of the study.rite your abstract here. Please do not exceed 5 pages with your references.

Keywords: Magnetic field, force field, timelike magnetic curve, energy, magnetic force.

1. Introduction

A magnetic field B defined on a n-dimensional Riemannian manifold is a closed 2-form such that its Lorentz force is a one-to-one tensor field.

The magnetic trajectories associated with the magnetic field B are magnetic curves ζ in *n*-dimensional Riemannian manifold such that they satisfy $\nabla_{z'}\zeta' = \phi(\zeta')$.

In three dimension, we know that vector fields and 2-forms are the same thing, magnetic fields correspond to divergence free vector fields, and uniform magnetic fields mean parallel vector fields. These facts help us define Lorentz force equation by the cross product. In other words, equations of Lorentz force (ϕ) associated with a magnetic field B can be computed by $\phi(\Psi) = B \times \Psi$.

As a result, Lorentz force equations for magnetic curves ζ can be stated by

$$\nabla_{\boldsymbol{\zeta}'}\boldsymbol{\zeta}' = \boldsymbol{\phi}(\boldsymbol{\zeta}') = \mathbf{B} \times \boldsymbol{\zeta}'.$$
²¹⁹

In the literature, one of the major goals is to obtain magnetic curves associated with the magnetic field B on a n-dimensional Riemannian manifold. Thus, intrinsic geometrical features of the n-dimensional Riemannian manifold can be used to determine the curvature of the magnetic curves. Consequently, magnetic curves can be figured out completely depending on the particular structure of the manifold. For example, [1,3,8,9,10,11,12] demonstrated that trajectories of magnetic fields defined on the Riemannian surface having a constant Gaussian curvature K could easily be determined. These research efforts were expanded to distinct ambient spaces. For instance, [8,9] obtained explicit trajectories associated with Kahler magnetic fields by assuming the ambient space is a complex form of space. Furthermore, [10] gave detailed prescription for normal flowlines of the contact magnetic field by assuming ambient space is contact magnetic field by assuming the field by assuming the field by assuming the field by assuming the field by assuming

Studies on the theory have been extended by defining Killing magnetic fields with the help of Killing vector fields. A variational approach method on magnetic flows of the Killing magnetic field in 3D was developed. Hence, [11] investigated that solution of the equation of Lorentz force can be considered as Kirchoff elastic rods and vice versa by studying magnetic flow on a Killing magnetic field in 3D. Then, [12] dealt with the magnetic flowlines associated with Killing magnetic fields on the unit sphere in 3D. Finally, [13] described N-magnetic and B-magnetic curves as the trajectories of the certain magnetic field and they revealed their magnetic flows associated with Killing magnetic field in 3D.

By considering the dynamical evolution of a charged particle in the existence of an electromagnetic field and a succession of the Lorentz equation together with the Lorentz force law it is possible to obtain original magnetic trajectories belonging to the particle, which is under the action of external forces i.e. frictional force, normal force, and gravitational force, in an associated magnetic field B. In this study, we use geometry of a semi-Riemannian spacetime to obtain the timelike magnetic curves.

2. Materials and Methods

We consider a moving particle, whose timelike worldline is described by the embedding $\alpha^{\mu} = \alpha^{\mu}(t)$, where α^{μ} are local coordinates in a 3-dimensional Minkowski space with the given metric $\pi_{\mu\nu} = diag(-1,1,1)$ for an arbitrary parameter *t*. Tangent vector of the worldline of the particle is defined by $\alpha^{\mu} = \frac{d\alpha^{\mu}}{dt}$, where α^{μ} are the embedding functions. One-dimensional metric is defined as $\phi = \pi_{\mu\nu} \alpha^{\mu} \alpha_{22\overline{11}}^{\nu} \alpha \cdot \alpha$ along

the curve. The arc-lenght parameter of the trajectory is also computed by $ds = (-\phi)^{\frac{1}{2}} dt$. We take $\phi = 1$ to take the advantege of using intrinsic geometry of the unit speed curve. Now, we introduce normal and binormal vectors of the trajectory, which are denoted by $\mathbf{e}_{(1)}$ and $\mathbf{e}_{(2)}$. Thus, we obtain an orthonormal basis such that it satisfies $D\alpha^{\mu} \cdot D\alpha^{\mu} = -1, D\alpha^{\mu} \cdot \mathbf{e}_{(i=1,2)} = 0$, and $\mathbf{e}_{(i)} \cdot \mathbf{e}_{(j)} = \Omega_{ij}, i, j = 1, 2$,

Here *D* denotes differentiation with respect to *s* and $\Omega_{ij} = \begin{cases} 1, i = j \\ 0, i \neq j. \end{cases}$

It also obeys following 3-dimensional Frenet-Serret equations

$$D_{\mathbf{e}_{(0)}} \mathbf{e}_{(0)} = \kappa \mathbf{e}_{(1)},$$

$$D_{\mathbf{e}_{(0)}} \mathbf{e}_{(1)} = \kappa \mathbf{e}_{(0)} + \pi \mathbf{e}_{(2)},$$

$$D_{\mathbf{e}_{(0)}} \mathbf{e}_{(2)} = -\pi \mathbf{e}_{(1)}.$$

where κ is the curvature and τ is the torsion of the curve. Here $\mathbf{e}_{(0)}$ is selected to represent to the tangent vector of the timelike worldline in place of $D\alpha^{\mu}$ [14].

Now, we assume that the moving particle is an under the action of an external force in a given space. Let Γ be a moving particle with a positive mass m such that it slides downward on a surface. Then the forces acting over a particle sliding on a surface are the normal force, the weight force, and the frictional force. These forces are expressed in terms of Frenet-Serret elements as the following way. The normal force is $\mathbf{C} = aC\mathbf{e}_{(1)}$, where $C = \|\mathbf{C}\|$ and $a = \pm 1$; the gravitational force is $\mathbf{D} = m(l_0\mathbf{e}_{(0)} + l_1\mathbf{e}_{(1)})$, where $l_{i=0,1}$ is gravitational coefficient; the frictional force is $\mathbf{A} = -bC\mathbf{e}_{(0)}$, where b is frictional coefficient [15].

3. Results and Discussions

Timelike frictional magnetic curves in E_1^3

Now, let Γ be a moving charged particle with a positive mass *m* such that there exists a frictional force acting on the particle in a given magnetic field B in E_1^3 .

Theorem 1. The curve ζ is called as a timelike frictional magnetic curve if the frictional force field of the curve satisfies the following equation.

$$D_{\zeta} \mathbf{A} = \phi(\mathbf{A}) = \mathbf{B} \times \mathbf{A}.$$

Timelike normal force magnetic curves in E_1^3

Now, let Γ be a moving charged particle with a positive mass *m* such that there exists a normal force acting on the particle in a given magnetic field B in E_1^3 .

Theorem 2. The curve ζ is called as a timelike normal force magnetic curve if the normal force field of the curve satisfies the following equation.

$$D_{\zeta'}\mathbf{C} = \phi(\mathbf{C}) = \mathbf{B} \times \mathbf{C}.$$

Timelike gravitational magnetic curves in E_1^3

Now, let Γ be a moving charged particle with a positive mass *m* such that there exists a gravitational force acting on the particle in a given magnetic field B in E_1^3 .

Theorem 3. The curve ζ is called as a timelike gravitational magnetic curve if the gravitational force field of the curve satisfies the following equation;

$$D_{\zeta'} \mathbf{D} = \phi(\mathbf{D}) = \mathbf{B} \times \mathbf{D}.$$

4. Conclusions

In the future studies, we will concentrate on other special magnetic curves associated to magnetic field B by considering some other important spaceitme structures. By doing this, we are hoping that we have a better understanding on dynamics of the moving charged particle in any magnetic.

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On the Solution to Nonlinear Wave Type Equations via Lie's Approach

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Abstract

In this study we consider a general family of nonlinear wave type equations and investigate their equivalence transformations. The method depends on generating appropriate transformation groups between various types of different equations each can be expressed as a member of the same family of equations. We show the necessary conditions to be able obtain transformations between linear and nonlinear wave type equations, so that the exact solutions can be written for nonlinear equations via the linear ones. An example is also given for a particular nonlinear wave equation which is equivalent to the simple constant coefficient wave equation. An an exact solution to nonlinear wave equations is also given via the transformation group.

Keywords: Lie Group Application, Admissible Transformations, Equivalence Transformations, Exact Solution, Nonlinear Wave Equation

1. Introduction

The behavior of nonlinear partial differential equations has significant concern in Applied Mathematics and Mathematical Physics as they express many physical problems. Lie group analysis to nonlinear differential equations has many applications. Recently, group classification of differential equations, invariant solutions of some group of equations and equivalence groups of differential equations has been examined by many researchers to understand the behavior of the equations and many different approaches have been developed for this purpose.

Systems of partial differential equations containing some arbitrary functions or parameters which we may call family of differential equations have a great interest in both mathematical and physical sciences. Equivalence transformations preserve the family of equations while changing the functional dependencies of their arbitrary functions. So that the transformations can generate maps between different types of equations belonging to the same family. Those different types of equations can be constant coefficient or variable coefficient, homogeneous or nonhomogeneous or more interestingly linear and nonlinear equations. Lie group of equivalence transformations plays a special role, since the computation technique is straight forward. Some interesting studies can be found in the references Özer(2018a,b), Long(2017).

The aim of the present work is to investigate the admissible point transformations for a family of 2 dimensional wave equation which represents a great variety 22 wave type equations. In the second section we generate the determining equations for the Equivalence group of transformations of the family, then obtain the explicit solutions of the determining equations with the certain relations between each other. In the last section we consider a particular application to nonlinear equation which can be transformed into the very well known constant coefficient wave equation.

2. Equivalence Transformations

In the present paper we shall investigate the equivalence group a general family of two dimensional wave equation

$$u_{tt} = f(x, t, u, u_x, u_y, u_t)_x + g(x, t, u, u_x, u_y, u_t)_y.$$
(1)

which will be transformed into

$$\bar{u}_{\bar{t}\bar{t}} = \bar{f}(\bar{x}, \bar{t}, \bar{u}, \bar{u}_{\bar{x}}, \bar{u}_{\bar{y}}, \bar{u}_{\bar{t}})_{\bar{x}} + \bar{g}((\bar{x}, \bar{t}, \bar{u}, \bar{u}_{\bar{x}}, \bar{u}_{\bar{y}}, \bar{u}_{\bar{t}})_{\bar{y}}.$$

where u is the dependent variable of the independent variables x, y, t and f, g are smooth nonconstant functions of their variables, subscripts denote the partial derivatives with respect to the corresponding variables and all over bars represent the transformed variables and functions via an appropriate point transformation.

Let M be a 3 dimensional manifold with a local coordinate system $\mathbf{x} = (x_i) = (x, y, t)$ which we call the space of independent variables. A vector field on the tangent space of the manifold M can be written as:

$$V = X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + T\frac{\partial}{\partial t} + U\frac{\partial}{\partial u}$$
(2)

where X, Y, T and U are coordinate functions covered by the manifold M.

$$X = X(x, y, t, u), \quad Y = Y(x, y, t, u), \quad T = T(x, y, t, u), \quad U = U(x, y, t, u).$$

To construct the equivalence groups we first write the equation as a set of first order equations by defining

$$v_1 = u_x, \quad v_2 = u_y, \quad v_3 = u_t$$

then consider the functional dependecies of the free functions as auxiliary variables:

$$s_1^1 = f_x, \ s_2^1 = f_y, \ s_3^1 = f_t, \ s_1^2 = g_x, \ s_2^2 = g_y, \ s_3^2 = g_t, \ \sigma^1 = f_u, \ \sigma^2 = g_u,$$

 $s^{11} = f_{v_1}, \ s^{12} = f_{v_2}, \ s^{21} = g_{v_1}, \ s^{22} = g_{v_2}$

can now be The extended manifold then has the coordinate cover as:

$$\tilde{\mathcal{K}} = \{x, y, t, u, f, g, v_1, v_2, v_3, s_1^1, s_2^1, s_3^1, s_1^2, s_2^2, s_3^2, \sigma^1, \sigma^2, s^{11}, s^{12}, s^{21}, s^{22}\}$$
(3)

The prolongation vector \tilde{V} over the extended manifold covered by $\tilde{\mathcal{K}}$ is in the form:

$$\tilde{V} = V + S^1 \frac{\partial}{\partial f} + S^2 \frac{\partial}{\partial g} + \sum_{j=1}^3 (V_j \frac{\partial}{\partial v_j} + \sum_{i=1}^2 S^i_j \frac{\partial}{\partial s^i_j}) + \sum_{i,j=1}^2 (S^{ij} \frac{\partial}{\partial s^{ij}} + S^i \frac{\partial}{\partial \sigma^i})$$
(4)

where the coefficients are assumed to be in the form:

$$V_j = V_j(x, y, t, u, v_1, v_2, v_3), \quad S^i = S^i(x, y, t, u, v_1, v_2, v_3, f, g)$$
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and the coefficients related to the new variables representing the functional dependencies of the free functions f and g are functions of all the coordinates of (3). Transformations

between the members of the family of given family of 2 dimensional wave type equations are determined by solving the system of ordinary differential equations.

$$\frac{d\bar{x}}{d\epsilon} = X(\bar{x}, \bar{y}, \bar{t}, \bar{u}), \quad \frac{d\bar{y}}{d\epsilon} = Y(\bar{x}, \bar{y}, \bar{t}, \bar{u}), \quad \frac{d\bar{t}}{d\epsilon} = T(\bar{x}, \bar{y}, \bar{t}, \bar{u}), \quad \frac{d\bar{u}}{d\epsilon} = U(\bar{x}, \bar{y}, \bar{t}, \bar{u}), \\
\frac{d\bar{f}}{d\epsilon} = S^1(\bar{x}, \bar{y}, \bar{t}, \bar{u}, \bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{f}, \bar{g}), \quad \frac{d\bar{g}}{d\epsilon} = S^2(\bar{x}, \bar{y}, \bar{t}, \bar{u}, \bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{f}, \bar{g}),$$
(5)

under the initial conditions

$$\bar{x}(0) = x, \ \bar{y}(0) = y, \ \bar{t}(0) = t, \ \bar{u}(0) = u, \ \bar{f}(0) = f, \ \bar{g}(0) = g.$$
 (6)

2.1. Determining Equations

The family of two dimensional wave type equations (1) considered in the present work can be written as:

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} - \frac{\partial v_3}{\partial t} = 0 \tag{7}$$

In general a first order pde can be expressed

$$\frac{\partial \Sigma^1}{\partial x} + \frac{\partial \Sigma^2}{\partial y} + \frac{\partial \Sigma^3}{\partial t} + \Sigma = 0$$
(8)

(7) and (8) can be matched by defining

$$\Sigma^1 = f, \quad \Sigma^2 = g, \quad \Sigma^3 = -v_3, \quad \Sigma = 0$$

Thus the general formula determined for (8) by Suhubi (2000) now can be applied to (7) by the following determining equations:

$$S^3 + V_3 = 0, \qquad \mathcal{S} = 0 \tag{9}$$

the first is written from

$$\frac{\partial}{\partial v_3} \to \frac{\partial}{\partial v_3} + \frac{\partial}{\partial \Sigma^3} \frac{\partial \Sigma^3}{\partial v_3} = \frac{\partial}{\partial v_3} - \frac{\partial}{\partial \Sigma^3}.$$

Solution of the determining equations (9) give the infinitesimal generators for the family of wave type equation as:

$$\begin{split} X &= X(x, y, t, u), \quad Y = Y(x, y, t, u), \quad T = T(t), \quad U = U(x, y, t, u), \\ V_1 &= U_x + (U_u + X_x + X_u v_1 + Y_u v_2) v_1 + Y_x v_2, \\ V_2 &= U_y + X_y v_1 + (U_u + Y_y + X_u v_1 + Y_u v_2) v_2, \\ V_3 &= U_t + X_t v_1 + Y_t v_2 + (U_u + \dot{T} + X_u v_1 + Y_u v_2) v_3 \\ S^1 &= (\alpha^{33} + 2\dot{T} + U_u + Y_u v_2 - X_x) f - (X_y + X_u v_2) g + \alpha^{11} v_1 + \alpha^{12} v_2 + \\ &(2X_t + X_u v_3) v_3 + \beta^1, \\ S^2 &= (\alpha^{33} + 2\dot{T} + U_u + X_u v_1 - Y_y) g - (Y_x + Y_u v_1) f - \alpha^{12} v_1 + \alpha^{22} v_2 + \\ &(2Y_t + Y_u v_3) v_3 + \beta^2 \end{split}$$
(10)

where $\alpha^{ij} = \alpha^{ij}(x, y, t), \ \beta^i = \beta^i(x, y, t, u)$ and the followings must hold:

$$\alpha_x^{11} - \alpha_y^{12} = X_{tt} - \beta_u^1 = Y_{tt} - \beta_u^2, \quad \beta_x^1 + \beta_y^2 = U_{tt}, \qquad 226$$
$$U_u - X_x - Y_y = -\frac{1}{2}\dot{T} + \gamma(x, y), \quad \alpha^{33} = \frac{1}{2}\dot{T} + \alpha(t) + \gamma(x, y). \qquad (11)$$

Theorem 1. Any map between the linear and nonlinear members for the family of wave type equation that can be expressed as

$$u_{tt} = f(x, t, u, u_x, u_y, u_t)_x + g(x, t, u, u_x, u_y, u_t)_y$$

is only possible if at least one the transformation of the local coordinates involves the dependent variable.

Proof. One can simply see that from the first equation of (11).

3. Application

Let us consider a particular example for the transformation between linear and nonlinear members of the wave type equation given by (1) by choosing

$$X = u, \quad Y = T = U = 0 \tag{12}$$

The infinitesimal generators then can be written from (10) as:

$$V_1 = v_1^2, \quad V_2 = v_1 v_2, \quad V_3 = v_1 v_3, \quad S^1 = -v_2 g + v_3^2, \quad S^2 = v_1 g.$$
 (13)

By integrating these equations under their initial condition (6) we have the equivalence transformations

$$\bar{x} = x - \epsilon u, \ \bar{y} = y, \ \bar{t} = t,
\bar{v}_1 = \frac{v_1}{1 - \epsilon v_1}, \ \bar{v}_2 = \frac{v_2}{1 - \epsilon v_1}, \ \bar{v}_3 = \frac{v_3}{1 - \epsilon v_1},
\bar{f} = f - \frac{\epsilon (g \ v_2 - v_3^2)}{1 - \epsilon v_1}, \ \bar{g} = \frac{g}{1 - \epsilon v_1}.$$
(14)

One can simply see that the transformation written above is an admissible equivalence transformation. The partial derivatives can be written as:

$$\frac{\partial}{\partial \overline{x}} = \frac{\partial}{\partial x} \frac{\partial x}{\partial \overline{x}} + \frac{\partial}{\partial x} \frac{\partial x}{\partial \overline{u}} \frac{\partial \overline{u}}{\partial \overline{x}} = (1 + \epsilon \overline{v}_1) \frac{\partial}{\partial x},$$

$$\frac{\partial}{\partial \overline{y}} = \frac{\partial}{\partial y} \frac{\partial y}{\partial \overline{y}} + \frac{\partial}{\partial x} \frac{\partial x}{\partial \overline{u}} \frac{\partial \overline{u}}{\partial \overline{y}} = \frac{\partial}{\partial y} + \epsilon \overline{v}_2 \frac{\partial}{\partial \overline{x}},$$

$$\frac{\partial}{\partial \overline{t}} = \frac{\partial}{\partial t} \frac{\partial t}{\partial \overline{t}} + \frac{\partial}{\partial x} \frac{\partial x}{\partial \overline{u}} \frac{\partial \overline{u}}{\partial \overline{t}} = \frac{\partial}{\partial t} + \epsilon \overline{v}_3 \frac{\partial}{\partial \overline{x}}.$$

Using these derivatives with the transformed functions obtained (14) one can simply satisfy

$$f_{\bar{x}} + \bar{g}_{\bar{y}} = \bar{v}_{3_{\bar{t}}} \Rightarrow f_x + g_y = v_{3_{\bar{t}}}$$

That proves the transformations (14) to be an appropriate equivalence transformations. Thus the transformed free functions \overline{f} , \overline{g} , \overline{h} and $\overline{v}3$ become in terms of the transformed variables:

$$\bar{f} = \tilde{f} + \epsilon \left(\frac{\bar{u}_{\bar{t}}^2}{1 + \epsilon \bar{u}_{\bar{x}}} - \bar{u}_{\bar{y}} \tilde{g} \right), \quad \bar{g} = (1 + \epsilon \bar{u}_{\bar{x}}) \tilde{g}, \quad \bar{v}_3 = \bar{u}_{\bar{t}}$$
⁽¹⁵⁾

where the functions \tilde{f} , \tilde{g} are the functions in terms of the transformed form of particular choice for f, g.

Example 1: Let us consider the classical constant coefficient wave equation in the plane

$$u_{xx} = u_{tt} \tag{16}$$

by choosing $f = u_x$, g = 0 whose general solution can be written as $u = \psi(y)(t - x) + \phi(y)(t + x)$. Equation (16) is transformed into via (15)

$$\frac{1}{(1+\epsilon\bar{u}_{\bar{x}})^2} \left[2\epsilon\bar{u}_{\bar{t}}(1+\epsilon\bar{u}_{\bar{x}})\bar{u}_{\bar{x}\bar{t}} + (1-\epsilon^2\bar{u}_{\bar{t}}^2)\bar{u}_{\bar{x}\bar{x}} \right] = \bar{u}_{\bar{t}\bar{t}}.$$

A solution of for this nonlinear equation can be determined by applying the equivalence transformations (14) to the general solution of the constant coefficient equation (16) as:

$$\bar{u} - \psi(\bar{y})(\bar{t} - \bar{x} - \epsilon\bar{u}) - \phi(\bar{y})(\bar{t} + \bar{x} + \epsilon\bar{u}) = 0.$$

We should warn the reader that this implicit solution is not a general solution to the nonlinear problem.

Many other applications can be done by choosing different infinitesimal generators X, Y, Tand U as long as they satisfy the equalities (11) between each other by simply running the same procedure give above.

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An Algorithm of Application of Lie Groups to Family of Differential Equations

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Abstract

Differential equations, involving some free functions of their variables represent actually, a set of equations with the same structure, which we may call *family of differential equations*. Lie Group application to family of differential equations have significant importance in generating maps between the members of the family. When preserving the structure of the family of differential equations, if an appropriate transformation exist, the transformation group of the family generate maps between equivalent but different members. In this study, we give the general perspective to the transformation groups of family of differential equations. An algorithm to determine the structure of the admissible members is expressed.

Keywords: Lie Group Application; Admissible Transformations, Equivalence Transformations; Exact Solution

1. Introduction

In classical physics, almost all field equations, representing the behavior of certain materials differ by some parameters which express the physical properties of the medium. In mathematics, such differential equations, involving some free functions of their variables represent a set of equations with the same structure. We call these equations as *family of differential equations*. Each member of the family is a differential equation with various properties such as, homogeneous, nonhomogeneous, constant or variable coefficient, linear or nonlinear.

Lie Groups have many applications to differential equations such as determining solutions, group classification, constructing their invariant quantities etc. Lie Group application to family of differential equations have significant importance not only in the meaning of the ones mentioned above, but also in the meaning of generating maps between the members of the family. An application to Lie groups; Equivalence Group of family of differential equations is a tool of generating maps between the members of the same family of equations. The idea of using Lie's classical invariant group theory (Lie 1884, 1888, 1897) for the family of differential equations is based on Ovsiannikov (1982) and has been developed and used by many researchers to various systems of differential equations representing great variety of physical problems.

In this study, after giving basic definitions with their examples, we shall give the general perspective to the transformation groups of family of differential equations. An alternative algorithm to determine the group will be developed and an example will be discussed to make the problem clear.

Definition 1 (Family of Equations). A set of differential equations

 $\mathcal{F}(x_i, u^{\alpha}, u^{\alpha}_{x_1, x_2, \dots x_p}, \phi_{k(q)}(x_i, u^{\alpha}, u^{\alpha}_{x_1, x_2, \dots x_p})) = 0$

is called a family of differential equations where x_i , i = 1, 2...n, are independent, $u^{\alpha}, \alpha = 1, 2, ...N$, dependent variables, ϕ_k are the smooth functions of their variables. $\phi_{k(q)}$ denotes the smooth functions ϕ_k and the partial derivatives w.r.t to both x_i 's u^{α} 's and $u^{\alpha}_{x_1,x_2,..x_p}$'s.

Example: $f(x, t, u, u_x)_x + g(x, t, u, u_x) = u_t$, $f(x, t, u, u_x, u_t)_x + g(x, t, u, u_x, u_t) = u_{tt}$ are one dimensional general family of diffusion equations and wave equations, respectively. It is clear that for different choice of the free functions f and g, we may have completely different type equations of the families, like constant coefficient, variable coefficient or more interestingly, linear and nonlinear equations.

2. Equivalence Transformations

Definition 2 (Equivalence Groups). For a given differential equation of the family the equivalence group \mathcal{E} is the group of smooth transformations of independent, dependent variables, their derivatives and smooth functions preserving the structure of the differential equation but transforms it into another equation.

Example: Equivalence Transformations: $\bar{x} = \bar{x}(x, t, u), \ \bar{t} = \bar{t}(x, t, u), \ \bar{u} = \bar{u}(x, t, u), \ \bar{u}_{\bar{x}} = \bar{u}_{\bar{x}}(x, t, u, u_x, u_t), \ \bar{u}_{\bar{t}} = \bar{u}_{\bar{t}}(x, t, u, u_x, u_t), \ \bar{f} = \bar{f}(x, t, u, u_x, u_t, f, g), \ \bar{g} = \bar{g}(x, t, u, u_x, u_t, f, g) \text{ generate maps between}$

$$f(x,t,u,u_x)_x + g(x,t,u,u_x) = u_t \longleftrightarrow \overline{f}(\overline{x},\overline{t},\overline{u},\overline{u}_{\overline{x}})_{\overline{x}} + \overline{g}(\overline{x},\overline{t},\overline{u},\overline{u}_{\overline{x}}) = \overline{u}_{\overline{t}}$$

To determine the equivalence transformations we shall use a geometric method developed by Harrison and Estabrook (1971) applied to Balance equations by Edelen (1980) for symmetry groups and finally applied to equivalence groups for first order Balance Equations by Özer (2004), for second order by Şuhubi (2000).

Definition 3 (Balance Equations).

$$\frac{\partial \Sigma^{i}(\mathbf{x}, u, \nabla u)}{\partial x^{i}} + \Sigma(\mathbf{x}, u, \nabla u) = 0, \qquad i = 1, 2, \dots n$$
(1)

is called a single Balance Equation of order two, where \mathbf{x} are independent variables, u is the dependent variable and ∇ is the gradient operator.

Almost all quasilinear field equations can be written as a Balance equation.

2.1. Method

To determine the equivalence groups for the Balance equation given in (1) we extend the (n + 1) dimensional manifold $M = \{x^i, u\}$ by adding the arbitrary functions $\{v_i = u_{,i}, \Sigma^i, \Sigma\}$ and to consider the functional dependencies:

$$s_j^i = \frac{\partial \Sigma^i}{\partial x^j}, \quad \sigma^i = \frac{\partial \Sigma^i}{\partial u}, \quad s^{ij} = \frac{\partial \Sigma^i}{\partial v_j}, \quad t_i = \frac{\partial \Sigma}{\partial x^i}, \quad \tau = \frac{\partial \Sigma}{\partial u}, \quad t^i = \frac{\partial \Sigma}{\partial v_i}$$

$$(2)$$

to the manifold with the coordinate cover $K = \{x^i, u, v_i, \Sigma^i, \Sigma, s^i_j, \sigma^i, s^{ij}, t_i, \tau, t^i\}$. A general vector field in the tangent space of the extended manifold covered by K is represented as follows:

$$V = X^{i} \frac{\partial}{\partial x^{i}} + U \frac{\partial}{\partial u} + V_{i} \frac{\partial}{\partial v_{i}} + S^{i} \frac{\partial}{\partial \Sigma^{i}} + \mathcal{H} \frac{\partial}{\partial \Sigma} + S^{i}_{j} \frac{\partial}{\partial s^{i}_{j}} + \mathcal{S}^{i} \frac{\partial}{\partial \sigma^{i}} + S^{ij} \frac{\partial}{\partial s^{ij}} + T_{i} \frac{\partial}{\partial t_{i}} + \mathcal{T} \frac{\partial}{\partial \tau} + T^{i} \frac{\partial}{\partial t^{i}}.$$
(3)

The explicit solutions of the infinitesimal generators, namely the coefficients of the vector field (3) on the tangent space of the extended manifold are obtained by Şuhubi (2000) as:

$$X^{i} = -\phi^{i}(x^{j}, u), \quad U = U(x^{j}, u), \quad V_{i} = D_{i}U + (D_{i}\phi^{j})v_{j},$$

$$S^{i} = (w + \frac{\partial\phi^{j}}{\partial u}v_{j})\Sigma^{i} - (D_{j}\phi^{i})\Sigma^{j} + \alpha^{ij}v_{j} + \beta^{i}, \quad \mathcal{H} = (w + \frac{\partial\phi^{i}}{\partial u}v_{i})\Sigma - D_{i}S^{i}. \quad (4)$$

where $D_i = \frac{\partial}{\partial x^i} + v_i \frac{\partial}{\partial u}$, $\alpha^{ij} = -\alpha^{ji}(x^k, u)$, $w = w(x^k, u)$ and the coefficients for the additional variables (2) as:

$$S_{j}^{i} = \frac{\partial F^{i}}{\partial x^{j}} + \frac{\partial F^{i}}{\partial \Sigma^{k}} s_{j}^{k} + \frac{\partial F^{i}}{\partial \Sigma} t_{j}, \quad S^{i} = \frac{\partial F^{i}}{\partial u} + \frac{\partial F^{i}}{\partial \Sigma^{k}} \sigma^{k} + \frac{\partial F^{i}}{\partial \Sigma} \tau, \quad S^{ij} = \frac{\partial F^{i}}{\partial v^{j}} + \frac{\partial F^{i}}{\partial \Sigma^{k}} s^{kj} + \frac{\partial F^{i}}{\partial \Sigma} t^{j},$$

$$T_{i} = \frac{\partial G}{\partial x^{i}} + \frac{\partial G}{\partial \Sigma^{j}} s_{i}^{j} + \frac{\partial G}{\partial \Sigma} t_{i}, \quad T = \frac{\partial G}{\partial u} + \frac{\partial G}{\partial \Sigma^{j}} \sigma^{j} + \frac{\partial G}{\partial \Sigma} \tau, \quad T^{i} = \frac{\partial G}{\partial v^{i}} + \frac{\partial G}{\partial \Sigma^{j}} s^{ji} + \frac{\partial G}{\partial \Sigma} t^{i}.$$

$$(5)$$

where $F^i = -s^i_j X^j - \sigma^i U - s^{ij} V_j + S^i$, $G = -t_i X^i - \tau U - t^i V_i + \mathcal{H}$.

2.2. Admissible Transformations between members of the family of equations

As we defined before, equivalence transformations map two equations belonging in the same family between each other; for a balance equation, the map can precisely be shown as:

$$\frac{\partial \Sigma^{i}(\mathbf{x}, u, \nabla u)}{\partial x^{i}} + \Sigma(\mathbf{x}, u, \nabla u) = 0 \longleftrightarrow \frac{\partial \overline{\Sigma}^{i}(\bar{\mathbf{x}}, \bar{u}, \bar{\nabla}\bar{u})}{\partial \bar{x}^{i}} + \Sigma(\bar{\mathbf{x}}, \bar{u}, \bar{\nabla}\bar{u}) = 0$$

The equivalence transformations for the given Balance equation are determined by solving the set of the following autonomous differential equations:

$$\frac{d\bar{x}^{i}}{d\epsilon} = -X^{i}(\bar{x}^{i}, \bar{u}), \quad \frac{d\bar{u}}{d\epsilon} = U(\bar{x}^{i}, \bar{u}), \quad \frac{d\bar{v}_{i}}{d\epsilon} = V_{i}(\bar{x}^{i}, \bar{u}, \bar{v}_{i}),
\frac{d\bar{\Sigma}^{i}}{d\epsilon} = S^{i}(\bar{x}^{i}, \bar{u}, \bar{v}_{i}, \bar{\Sigma}^{i}, \bar{\Sigma}), \quad \frac{d\bar{\Sigma}}{d\epsilon} = \mathcal{H}(\bar{x}^{i}, \bar{u}, \bar{v}_{i}, \bar{\Sigma}^{i}, \bar{\Sigma})$$
(6)

under the initial conditions: $\bar{x}(0) = x$, $\bar{y}(0) = y$, $\bar{t}(0) = t$, $\bar{u}(0) = u$, $\bar{f}(0) = f$, $\bar{g}(0) = g$ where the right sides of the ordinary differential equations are given in (4), here ϵ is the group parameter. The solution of (6) gives the complete set of admissible equivalence transformations between the members of the family without any restriction on the functional properties of the free functions Σ^i and Σ . Every particular functional dependence of those functions gives new determining equations so that their solution generates the admissible structure of possible maps. Studying such problems mainly interests in the answers of some questions:

- 1. Can any particular form of nonlinear equation in the family be mapped into a linear equation via point transformations? The same question for variable coefficient/constant coefficient, homogeneous/nonhomogeneous equations.
- 2. If there exist an admissible transformation between desired two types of equations, what are the corresponding exact transformations?
- 3. For some particular set of admissible transformations, what are the special forms for transformed equation?

To answer such questions we shall use an easy algorithm. Consider the family of differential equations in its most general form, find the infinitesimal generators for equivalence transformations. After determining the relations between the generators, step by step restrict the general form to the desired forms.

3. Application

In this section let us consider the diffusion equation which is widely studied by Özer in (2018) in the following form:

$$u_t - f(x, y, t, u, u_x, u_y)_x - g(x, y, t, u, u_x, u_y)_y = 0.$$
(7)

She examined the problem in the classical method, but here we shall use the method given above for the same problem and obtain the same results in a much more easier way. To write the given diffusion equation (7) as a Balance equation

$$x^{1} = x, \ x^{2} = y, \ x^{3} = t; \ v_{1} = u_{x}, \ v_{2} = u_{y}, \ v_{3} = u_{t}; \ \Sigma^{1} = f, \ \Sigma^{2} = g, \ \Sigma^{3} = 0, \ \Sigma = -v_{3}$$
(8)

must be taken. Because $\Sigma^3 = 0$, its corresponding coefficient in the vector field (3) must be set identically zero:

$$S^3 = 0. (9)$$

Moreover, the equality $\Sigma = -v_3$ generates another determining equation, via

$$\frac{\partial}{\partial v_3} = \frac{\partial}{\partial v_3} + \frac{\partial}{\partial \Sigma^3} \frac{\partial \Sigma^3}{\partial v_3} \Rightarrow \quad \mathcal{H} + V_3 = 0.$$
(10)

Solution of the determining equations (9) and (10) by using them from (4) yields

$$\begin{array}{rcl} X &=& X(x,y,t,u), \; Y = Y(x,y,t,u), \; T = T(t), \; U = U(x,y,t,u), \\ V_1 &=& U_x + (U_u + X_x)v_1 + X_uv_1^2 + Y_xv_2 + Y_uv_1v_2, \\ V_2 &=& U_y + (U_u + Y_y)v_2 + X_yv_1 + X_uv_1v_2 + Y_uv_2^2, \\ V_3 &=& U_t + (U_u + \dot{T})v_3 + X_tv_1 + X_uv_1v_3 + Y_tv_2 + Y_uv_2v_3, \\ S^1 &=& (U_u + \dot{T} - X_x + Y_uv_2)f - (X_y + Y_uv_2)g + \alpha^{12}v_2 + \beta^1, \\ S^2 &=& (U_u + \dot{T} - Y_y + X_uv_1)g - (Y_x + Y_uv_1)f - \alpha^{12}v_1 + \beta^2. \end{array}$$

Here we have the following relation to be satified between the vector field components

$$X_x + Y_y + h(t) = U_u + T.$$
 (11)

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The answer of the second question is hidden in the last equality. One can simply see that the nonlinear dependency on the dependent variable u which will possibly construct maps between linear and nonlinear members of (7) is impossible unless the transformation of at least one independent variables involve the dependent variable u.

Let us consider the diffusion equation with a restriction on the general form of (7) by

$$s^{12} = \frac{\partial f}{\partial u_y} = 0, \quad s^{21} = \frac{\partial f}{\partial u_x} = 0.$$
(12)

The additional determining equations which come from (12) are $S^{12} = S^{21} = 0$. Their solution, by using them the from (5), the infinitesimal generators for the transformations of local coordinates become X = X(x,t), Y = Y(y,t). That tells us by considering the above results with (11) together "the answer of the first question is NO". Basicly, as a result we may say the existence of admissible point transformations for linearization of such equations is only possible when either f depends on u_y or g depends on u_x . Thus we should continue to seek the restrictions for the following equations.

$$u_{t} - f(x, y, t, u, u_{x}, u_{y})_{x} - g(x, y, t, u, u_{x}, u_{y})_{y} = 0$$

$$u_{t} - f(x, y, t, u, u_{x})_{x} - g(x, y, t, u, u_{y})_{y} = 0$$

$$u_{t} - f(x, y, u, u_{x})_{x} - g(x, y, u, u_{y})_{y} = 0$$

$$u_{t} - f(x, u, u_{x})_{x} - g(y, u, u_{y})_{y} = 0$$

$$u_{t} - f(u, u_{x})_{x} - g(u, u_{y})_{y} = 0.$$

For every restriction, the structure of the infinitesimal generators gives a great classification of the members for the family of differential equations. Which types of differential equations are allowed to be mapped onto each other via point transformations.

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Space-Time Characteristics of Seismicity in Gümüşhane, Turkey: An Application of the Most Frequently Used Statistical Models

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The main purpose of this study is to make a detailed space-time analysis of earthquake activity in Gümüşhane, Turkey, at the beginning of 2018. For this purpose, we preferred the most frequently used statistical models for the evaluation of earthquake potential. In this context, a statistical assessment of the regional and temporal variations of the main seismicity parameters such as completeness magnitude Mc-value, seismotectonic b-value, standard normal deviate Z-test called seismic quiescence and GENAS algorithm to estimate all important rate changes of earthquake activity in different magnitude thresholds are achieved. For the detailed analyses, the region between the coordinates 39.5°N and 41.0°N in latitude and the co-ordinates 38.5°E and 40.5°E in longitude was selected as the study area. Earthquake catalog are compiled from the Boğazici University, Kandilli Observatory and Earthquake Research Institute (KOERI). This catalog includes about 47.27-years period from September 21, 1970 to December 27, 2017. It is homogeneous for duration magnitude, $M_{\rm d}$, and consists of 2802 shallow earthquakes having magnitudes greater than or equal to 1.0. Magnitude levels generally vary from 2.5 to 3.5 and, the earthquake magnitudes reach a maximum in $M_d=2.8$. The variation of Mc-value in time also shows a distribution between 2.5 and 3.0 after the year of 2000. So, average completeness magnitude for the region was taken as Mc=2.8. Temporal changes of b-value show that there is not any important decrease in recent years although some significant decreases in b-value before some strong earthquakes in the region between 1970 and 2018 are observed. Regional variations of b-value indicate that small b-values observed in and around Kelkit and Köse covering the north of Köse and the south of Kelkit may be significant in terms of the possible earthquake potential. The analysis of seismic quiescence Z-value shows that no anomalies of significant rate changes in the earthquake activity is detected in the study region at the beginning of 2018. To separate the magnitude bands where significant variations occur, the magnitude levels were separately evaluated by GENAS test. With this technique, important changes in the number of the larger and smaller earthquakes than a given magnitude versus time are described. The results show that a strong decrease is observed for both small and large earthquakes at the beginning of 2018. There is a remarkable compatibility between the results of seismic quiescence and the GENAS results. According to these results, the earthquake hazard is low and earthquake risk is minor in Gümüshane province of Turkey. Consequently, these types of statistical assessments of space-time characteristics may supply important clues for the intermediate term earthquake potential.

Keywords: Gümüşhane, Seismicity, Z-value, GENAS, Earthquake Potential

1. Introduction

Seismicity rate changes have been used in a large number of studies for many parts of the world in order to detect the precursory seismic quiescence occurred in and around focal areas several years before mainshocks. The quiescence hypothesis is firstly formulated by Wyss and Habermann (1988) and it postulates that some main shocks are preceded by seismic quiescence, which is a significant decrease of the mean seismicity rate. The duration of seismic quiescence before strong earthquakes which is expected to mapped case histories of seismic quiescence are needed to be 4.5 ± 3 years. Precursory seismic quiescence before some great earthquakes has been reported by many authors (e.g., Wiemer and Wyss, 1994; Öztürk, 2018). Main purpose of this study is to put forth the future earthquake potential in Gümüşhane at the beginning of 2018 by investigating whether there is a significant seismic quiescence as an observable precursor with Z-value technique, as well as *GENAS* modelling and seismotectonic *b*-value.

2. Data and Methods

The earthquake database is taken from Boğaziçi University, Kandilli Observatory and Earthquake Research Institute (KOERI). Main tectonics in Gümüşhane and vicinity (Figure 1a) are modified from different authors such as Şaroğlu et al. (1992) and Bozkurt (2001). Catalog is homogeneous for duration magnitude, M_d , and includes 2802 shallow earthquakes with magnitudes $1.0 \le M_d \le 6.5$ from 1970 to 2018 (Figure 1b).

We used the standard deviate Z-test, generating the LTA(t) (Log Term Average) function for the statistical evaluation of the confidence level (Wiemer and Wyss, 1994):

$$Z = (R_1 - R_2) / \sqrt{(S_1^2 / N_1) + (S_2^2 / N_2)}$$
(1)

where R_1 is the average earthquake activity rate in all period of catalog, R_2 is the mean activity rate in the considered time window, S_1^2 and S_2^2 are the standard deviations in these time intervals, and N_1 and N_2 the number of samples, *t* is the current time.

The frequency-magnitude relation of earthquakes was described by Gutenberg-Richter (1944) and gives a power-law distribution of earthquakes occurrences as follow:

 $\log_{10} N(M) = a - bM$

(2)

where N(M) is the expected number of earthquakes with magnitudes greater than or equal to M. *b*-value defines the slope of the frequency-magnitude distribution, and *a*-value is related to earthquake activity rate.

The *GENAS* algorithm estimates the cumulative numbers of different magnitude thresholds and describes the important variations in the number of earthquakes larger and smaller than a given magnitude versus time. We used the declustered earthquake catalogue for *GENAS* analysis.



Figure 1. (a) Active faults in and around Gümüşhane. Names of the faults: KLB-Kelkit Basin, BYB-Bayburt Basin, KÇFZ-Kelkit-Çoruh Fault Zone, KLFS-Kelkit Fault Segment, AÇFZ-Akdağ-Çayırlı Fault Zone, DYF-Dağyolu Fault, TAFZ-Tercan-Aşkale Fault Zone. Several significant centers are also given on the figure. (b) Earthquake epicenters between 1970 and 2018 with $M_d \ge 1.0$ as well as the declustered catalogue with $M_d \ge 2.8$. Stars represent strong main shocks with $M_d \ge 5.0$ with their occurrence times.

3. Results and Discussions

Some activities such as foreshocks, aftershocks and swarms generally masks temporal variations of the earthquake numbers and the related statistics. For this reason, it is necessary to remove the dependent events from the catalog. To make a quantitative evaluation of the precursory seismic quiescence, earthquake catalogue is declustered with the Reasenberg's (1985) algorithm. The cumulative number of earthquakes versus time for the original and

declustered catalogs are given in Figure 2. As seen in Figure 2, declustering process has removed dependent events from original catalogue and after this process, a more homogenous, reliable and robust earthquake catalogue has been obtained. Regional distribution of *b*-value and *Z*-value in and around Gümüşhane for the beginning of 2018 is shown in Figure 3a and 3b, respectively. Small *b*-values are observed in and around Kelkit and Köse covering the north of Köse and the south of Kelkit. Standard deviate *Z*-value shows no anomalies of important rate changes in the earthquake activity at the beginning of 2018. In addition to these statistical analyses, we used the *GENAS* technique in order to put forth all the significant seismicity rate changes in Gümüşhane (Figure 4). Results of the *GENAS* model show the important breaks in slope which begin from the end of data for all magnitude groups. Strong decrease is observed for both small and large earthquakes at the beginning of 2018. Consequently, a good correlation was observed between the results of seismic quiescence and the *GENAS*.



Figure 2. Cumulative number of earthquakes as a function of time for the original earthquake catalog with 2802 shallow events with $M_d \ge 1.0$, for the declustered catalog with 2336 events with $M_d \ge 1.0$ and for the declustered catalog with 1459 events with $M_d \ge 2.8$.



Figure 3. (a) Regional change of *b*-value with all the earthquakes with $M_d \ge 1.0$, (b) Regional variation of the *Z*-value using declustered catalogue with $M_d \ge 2.8$, in and around



Figure 4. Results of *GENAS* estimates for declustered earthquakes. Times of significant changes (at the 99% confidence level) are marked in blue for seismicity rate increases and in red for seismicity rate decreases as a function of different magnitude groups.

4. Conclusions

In this study, a space-time analysis of earthquake activity in Gümüşhane, Turkey, is achieved by applying the most frequently used statistical parameters such as seismic quiescence Zvalue, seismotectonic *b*-value and *GENAS*. Obtained results show that there is not a significant fluctuation in the seismic activity at the beginning of 2018, and the earthquake hazard is low and earthquake risk is minor in Gümüşhane province of Turkey.

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Size-Scaling Distributions of Earthquake Occurrences in Gümüşhane, Turkey: Magnitude Variations, Seismotectonic *b*-value and Fractal Dimension *Dc*-value

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In the scope of this study, a detailed statistical analysis of size-scaling distributions of earthquake occurrences in Gümüşhane, Turkey, at the beginning of 2018 was performed by evaluating the most frequently used size-scaling parameters such as completeness magnitude, Mc-value, described as the minimum magnitude of complete reporting, seismotectonic b-value, a power-law of size distribution of earthquakes, fractal dimension Dc-value, describing the size scaling attributes and clustering properties of earthquakes, annual probabilities and recurrence times of earthquakes as well as the magnitude distribution of earthquake activity. Statistical analyses were carried out in a rectangular area covered by the co-ordinates 39.5°N and 41.0°N in latitude and the coordinates 38.5°E and 40.5°E in longitude. Earthquake database is taken from Boğazici University, Kandilli Observatory and Earthquake Research Institute (KOERI). This catalog is homogeneous for duration magnitude, M_d , and includes 2802 shallow earthquakes having magnitude equal to and larger than 1.0 in about 47.27-years period between September 21, 1970 and December 27, 2017. The cumulative number of earthquakes against time show that any significant changes are not reported in seismicity from 1970 to 2003. However, the number of earthquakes gradually increases after 2003 and significant fluctuations in the earthquake activity are reported especially after the 2005s. Time-series analyses show that there are slight increases in the number of earthquakes in 2003 and 2017 and, there is a maximum increase in the number of events in 2012. The numbers of earthquakes show an exponential decay rate from the smaller to larger magnitudes and magnitude levels are change between 2.5 and 3.5 on average. Hence, the completeness magnitude for Gümüşhane region is taken as Mc=2.8. By using this Mc-value, b-value is calculated as 1.01 ± 0.02 with the maximum likelihood method. This result shows that magnitude-frequency distribution of earthquakes in Gümüshane is well represented with a *b*-value typically close to 1.0. By using 95% confidence interval and linear curve fitting technique, Dc-value is estimated as 1.57+0.03. For this distribution, the scale invariance in the cumulative statistics are selected between 5.11 and 89.01 km. This Dc-value indicates that seismic activity in Gümüşhane is more clustered at larger scales or in smaller areas. Analyses on annual probabilities of earthquake occurrences show that magnitude levels between 4.5 and 6.5 exhibits a value smaller than 1.0. Recurrence time of the earthquakes has a value of 30 years for $M_d=5.5$ and 100 years for M_d =6.0. These results reveal that Gümüşhane has not a noticeable earthquake potential for strong earthquake occurrences in the intermediate term.

Keywords: Gümüşhane, b-value, Dc-value, Annual Probability, Recurrence Time

1. Introduction

There are many statistical models for a comprehensive analysis of size-scaling distributions of earthquake occurrences. For this purpose, many researchers have used a lot of seismotectonic parameters to evaluate the earthquake potential for different regions (eg., Hirata, 1989; Polat et al., 2008; Öztürk, 2015). Some of these parameters are given as *b*-value, *Dc*-value, annual probability, recurrence time, moment and energy releases. The frequency-magnitude distribution is known as the *b*-value of Gutenberg-Richter relation (Gutenberg and Richter, 1944). The *b*-value reflects the relative numbers of both large and small earthquakes, and is related to the properties of the seismotectonic structures and stress distributions in time and space. Fractal dimension *Dc*-value defines the heterogeneity degree of seismicity in active fault system and some geological, mechanical or structural variations in heterogeneity (Mandelbort, 1982). In the scope of this study, these two seismotectonic parameters are analyzed as well as the completeness magnitude, annual probability and recurrence time in order to supply some useful outcomes for the evaluation of earthquake potential in Gümüşhane province of Turkey.

2. Data and Methods

The earthquake catalog for Gümüşhane and vicinity is compiled from Boğaziçi University, Kandilli Observatory and Earthquake Research Institute (KOERI). Tectonic structures in and around Gümüşhane are compiled from different authors such as Şaroğlu et al. (1992) and Bozkurt (2001) and given in Figure 1a. Catalog is homogeneous for duration magnitude, M_d and covers 2802 shallow earthquakes with magnitudes larger than or equal to M_d =1.0 from September 21, 1970 until December 27, 2017 (Figure 1b).

The relation between frequency and magnitude of earthquakes occurrences was given by Gutenberg-Richter (1944). This power-law distribution of earthquakes is given as follows:

$$\log_{10} N(M) = a - bM$$

(2)

where N(M) is the expected number of earthquakes with magnitudes greater than or equal to M. *b*-value defines the slope of the frequency-magnitude distribution, and *a*-value is related to earthquake activity rate. Completeness magnitude, *M*c-value, is a very important parameter

for high quality and reliable estimations, especially in the estimation of *b*-value. If *Mc*-value changes systematically as a function of time and space, temporal variations of *Mc*-value can cause potential wrong evaluation of seismicity behaviors.

The analysis of correlation dimension has been used as a powerful tool in order to quantify the self-similarity of a geometrical object. Correlation dimension Dc and the correlation sum C(r) was suggested by Grassberger and Procaccia (1983) as in the following:

$$Dc = \lim_{r \to 0} [\log C(r) / \log r]$$
(2)
$$C(r) = 2N_{R < r} / N(N-1)$$
(3)

where C(r) is the correlation function, r is the distance between two epicenters, and N is the number of earthquakes pairs separated by a distance R < r. If the epicenter distribution has a fractal structure, $C(r) \sim r^{Dc}$ is obtained.



Figure 1. (a) Tectonic structures of Gümüşhane and surrounding area. Names of the faults: KÇFZ-Kelkit-Çoruh Fault Zone, KLB-Kelkit Basin, KLFS-Kelkit Fault Segment, BYB-Bayburt Basin, DYF-Dağyolu Fault, AÇFZ-Akdağ-Çayırlı Fault Zone, TAFZ-Tercan-Aşkale

Fault Zone. Some significant centers are also shown on the figure. (b) Epicenters of 2802 shallow events with $M_d \ge 1.0$ between 1970 and 2018. Stars show the strong main events with

 $M_d \ge 5.0$. Dates of some strong events are also given on the figure.

3. Results and Discussions

Mc-value is a significant parameter for many statistical studies, and temporal changes in Mcvalue can affect the estimations of the seismotetonic parameters. For his reason, we aimed to use the maximum number of earthquakes for high-quality results. Figure 2 shows the temporal changes of Mc-value. Mc-value has great values before 2000 whereas it shows a decreasing trend after 2000. Average Mc-value for Gümüşhane from 1970 to 2018 is estimated as 2.8. Using this *Mc*-value, *b*-value is calculated as 1.01±0.02 (Figure 3a). Average *b*-value is given as 1.0 in literature and thus, frequency-magnitude distribution of earthquakes in and around Gümüşhane is well represented the Gutenberg-Richter law with a b-value typically close to 1. Dc-value is estimated as 1.57±0.03 with 95% confidence (Figure 3b). This log-log correlation function exhibits a clear linear range and scale invariance in the cumulative statistics between 5.11 and 89.01 km. The areas of increased complexity in active fault systems show higher Dc-value. The higher Dc-value is also quite sensitive to the heterogeneity in magnitude distribution. Annual probabilities of earthquake occurrences are given in Figure 4a. A value between 1 and 4 between magnitude levels 3.5 and 4.5, and a value of smaller than 1 between magnitude levels 4.5 and 6.5 are observed. Recurrence times of earthquake occurrences are plotted in Figure 4b. We observed quite smaller years (<1.0) for magnitude sizes from 3.5 to 4.0, and 1-10 years for magnitude sizes from 4.5 to 5.0. However, the values between 30 and 100 years are estimated for magnitude sizes between 5.5 and 6.0 while the values between 100 and 500 years are estimated for magnitude sizes between 6.0 and 6.5.



Figure 2. Completeness magnitude with time. Standard deviation (δMc) is also given.





Figure 3. (a) Gutenberg-Richter relation and frequency-magnitude distribution of earthquakes in Gümüşhane province. *Mc* and *a*-values are also given. (b) Correlation integral curve against distance. Black dots are the points in the scaling range. The slope of the blue line corresponds to the *Dc*-value and cyan lines represent the standard error.



Figure 4. (a) Annual probability and (b) Recurrence time of the earthquakes for different magnitude sizes in Gümüşhane province.

4. Conclusions

In the scope of this study, a statistical evaluation of earthquake activity for Gümüşhane province of Turkey at the beginning of 2018 is supplied by analyzing several seismotectonic parameters such as *b*-value, *Dc*-value, *Mc*-value, annual probability and recurrence time of earthquakes. *Mc*-value is estimated as 2.8. *b*-value is calculated as 1.01 ± 0.02 and is close to 1 and typical for earthquake catalogues. *Dc*-value is calculated as 1.57 ± 0.03 . seismicity is more clustered at larger scales (or in smaller areas) in Gümüşhane. The results of probability and recurrence time of the earthquakes suggested that Gümüşhane province of Turkey has not an important earthquake risk and hazard for strong earthquake occurrences at the beginning of 2018.

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Energy of wave and solutions of C-Holm equation under Lagrangian point of view

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Abstract

We deal with the Camassa-Holm equation $u_t - u_{xxt} + 2ku_x + 3uu_{xx} - uu_{xxx} = 0$ possesses a global continuous semigroup of weak conservative solutions for initial data $u|_{t=0} = \overline{u}$ in H^1 . The result is obtained by introducing a coordinate transformation into Lagrangian coordinates. To characterize conservative solutions it is necessary to include the energy density given by the positive Radon measure μ with $\mu_{ac} = (u^2 + u_x^2) dx$. The total energy is preserved by the solution.

Keywords: weak solutions; C-H equation; Radon measure; energy density; global solution; total energy.

1. Introduction

In this paper, we reformulate the Camassa – Holm equation using a different set of variables and obtain a semilinear system of ordinary differential equations, as Bressan and Constantin [1]. The Cauchy problem for the Camassa – Holm equation [4], [5].

$$u_t - u_{xxt} + 2ku_x + 3uu_{xx} - uu_{xxx} = 0, \qquad u|_{t=0} = \overline{u}$$
(1.1)

has received considerable attention the last decade. With k positive it models, see [2], [7], propagation of unidirectional gravitational waves in a shallow water approximation, with u representing the fluid velocity. The Camassa-Holm equation has a bi-Hamiltonian structure and is completely integrable. It has infinitely many conserved quantities. In particular, for smooth solutions the quantities are all time independent.

$$\int u \, dx, \qquad \int (u^2 + u_x^2) \, dx, \qquad \int (u^3 + u u_x^2) \, dx \tag{1.2}$$

In this article we consider the case k = 0 on the real line, that is

$$u_t - u_{xxt} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0$$
(1.3)

and henceforth we refer to (1.3) as the Camassa – Holm equation. The equation can be rewritten as the following system

$$u_t + uu_x + P_x = 0, (1.4a)$$

$$P - P_{xx} = u^2 + \frac{1}{2}u_x^2 \tag{1.4b}$$
More precisely, Constantin, Escher and Molinet [8, 10] showed some results. The Camassa – Holm equation possesses solutions, denoted (multi) peakons, of the form

$$u(t,x) = \sum_{i=1}^{n} p_i(t) e^{-|x-q_i(t)|},$$
(1.5)

where the $(p_i(t), q_i(t))$ satisfy the explicit system of ordinary differential equations

$$\dot{q}_i = \sum_{j=i}^n p_j e^{-|q_i - q_j|}, \qquad \dot{q}_i = \sum_{j=1}^n p_j sgn(q_i - q_j) e^{-|q_i - q_j|}$$

Higher peakons move faster than the smaller ones, and when a higher peakon overtakes a smaller, there is an exchange of mass, but no wave breaking takes place. However, if some of $p_i(0)$ have opposite sign, wave breaking may incur, see, e.g., [3, 6].

Bressan and Fonte [5, 11] presented another approach to the Camassa-Holm equation. The flow map $\bar{u} \rightarrow u(t)$ is, as we have seen, neither a continuous map on H^1 nor on L^2 .

2. Materials and Methods

We reformulate the equation using a different set of variables and obtain a semilinear system of ordinary differential equations, as Bressan and Constantin [1].

However, distinct variables from that simply corresponds to the transformation between Eulerian and Lagrangian coordinates.

Let u = u(t, x) denote the solution, and $y(t, \xi)$ the corresponding characteristics, thus $y_t(t, \xi) = u(t, y(t, \xi))$. Our new variables are $y(t, \xi)$,

$$U(t,\xi) = u(t,y(t,\xi)), \ H(t,\xi) = \int_{-\infty}^{y(t,\xi)} (u^2 + u_x^2) \, dx$$
(2.1)

where U corresponds to the Lagrangian velocity while H could be interpreted as the Lagrangian cumulative energy distribution. The characteristics $q(\xi; t)$ are defined as solutions of the equation

$$q_t(\xi; t) = m(q(\xi; t), t)$$

with the initial condition $q(\xi; t) = \xi$. Let we consider the momentum $m = u - u_{xx}$ of the system and introduce the variable p related direct to the momentum, as

$$p(\xi; t) = m(q(\xi; t), \frac{\partial_q}{\partial_{\xi}}(\xi; t)),$$

Then we can show that the following system

$$\begin{cases} y_t = U \\ U_t = -Q \\ H_t = U^3 - 2PU \end{cases}$$
(2.2)

is equivalent to the Camassa-Holm equation. Global existence of solutions of (2.2) is obtained starting from a contraction argument. As noted in [1], even if $H^1(R)$ is a natural space for the equation, there is no hope to obtain a group of solutions by only considering $H^1(R)$.

3. Results and Discussions for global solutions in Lagrangian coordinates

Assuming that u is smooth, it is not hard to check that

$$(u^{2} + u_{x}^{2})_{t} + (u(u^{2} + u_{x}^{2}))_{x} = (u^{3} - 2Pu)_{x}$$
(3.1)

Let us introduce the characteristics $y(t; \xi)$ defined as the solutions of

$$u_t(t;\xi) = u(t,y(t,\xi))$$
 (3.2)

for a given $y(0; \xi)$. Equation (3.1) gives us information about the evolution of the amount of energy contained between two characteristics. Indeed, given ξ_1 , ξ_2 in R, let $H(t) = \int_{y(t,\xi_1)}^{y(t,\xi_2)} (u^2 + u_x^2) dx$ be the energy contained between the two characteristic curves $y(t, \xi_1)$ and $y(t, \xi_2)$. Then, using (3.1) and (3.2), we obtain

$$\frac{dH}{dt} = \left((u^3 - 2Pu)^{\circ} y \right) \Big|_{\xi_1}^{\xi_2}$$
(3.3)

We now derive a system equivalent to (1.4). All the derivations in this section are formal and will be justified later. Let y still denote the characteristics. We introduce two other variables, the Lagrangian velocity and cumulative energy distribution, U and H, defined as $U(t,\xi) = u(t, y(t \xi))$ and $H(t) = \int_{-\infty}^{y(t,\xi)} (u^2 + u_x^2) dx$ (3.4)

From the definition of the characteristics, it follows that

$$U_t(t;\,\xi) = u_t(t,y) + y_t(t,\xi)u_x(t,y) = -P_x^{\circ}y(t,\xi)$$
(3.5)

This last term can be expressed uniquely in term of U, y, and H. From (1.4b), we obtain the following explicit expression for ,

$$P(\xi,t) = \frac{1}{2} \int_{R} e^{|x-z|} \left(u^{2}(t,z) + \frac{1}{2} u_{x}^{2}(t,z) \right) dz$$
(3.6)

Since $H_{\xi} = (u^2 + u_x^2)^{\circ} y y_{\xi}$

$$P_x^{\circ} y(t,\xi) = -\frac{1}{4} \int_R sgn(y(\xi) - y(\mu)) \exp(-|y(\xi) - y(\mu)|) \left(U^2 y_{\xi} + H_{\xi} \right) \mu \, d\mu \tag{3.7}$$

where the t variable has been dropped to simplify the notation. Later we will prove that y is an increasing function for any fixed time t. If, for the moment, we take this for granted, then $P_x \circ y$ is equivalent to Q where

$$Q(t,\xi) = -\frac{1}{4} \int_{R} sgn(\xi - \mu) \exp(-sgn(\xi - \mu)(y(\xi) - y(\mu))) \left(U^{2}y_{\xi} + H_{\xi} \right) \mu \, d\mu \quad (3.8)$$

$$P(t,\xi) = -\frac{1}{4} \int_{R} \exp(-sgn(\xi - \mu)(y(\xi) - y(\mu))) \left(U^{2}y_{\xi} + H_{\xi} \right) \mu \, d\mu$$
(3.9)

Thus $P_x \circ y$ and $P \circ y$ can be replaced by equivalent expressions given by (3.8) and (3.9) which only depend on our new variables U, H, and y. We introduce yet another variable, $\zeta(t; \xi)$, simply defined as $\zeta(t, \xi) = y(t, \xi) - \xi$. It will turn out that $\zeta \in L^{\infty}(R)$. We now derive a new system of equations, formally equivalent to the Camassa-Holm equation. Equations (3.5), (3.3) and (3.2) give us

$$\begin{cases} \zeta_t = U\\ U_t = -Q\\ H_t = U^3 - 2PU \end{cases}$$
(3.10)

As we will see, the system (3.10) of ordinary differential equations for (ζ, U, H) from [0; *T*] to *E* is well-posed, where *E* is Banach space to be defined in the next section. We have

$$Q_{\xi} = -\frac{1}{2}H_{\xi} - \left(\frac{1}{2}U^2 - P\right)y_{\xi} \quad \text{and} \quad P_{\xi} = Q \ y_{\xi}$$
(3.11)

Hence, differentiating (2.10) yields

$$\begin{cases} \zeta_{\xi t} = U_{\xi} \quad (or \; y_{\xi t} = U_{\xi}) \\ U_{\xi t} = \frac{1}{2} H_{\xi} - \left(\frac{1}{2} U^2 - P\right) y_{\xi} \\ H_{\xi t} = -2 Q U y_{\xi} + (3U^2 - 2P) U_{\xi} \end{cases}$$
(3.12)

The system (3.12) is semilinear with respect to the variables y_{ξ} , U_{ξ} and H_{ξ} .

4. Conclusions

Solutions of the Camassa-Holm blow up when characteristics arising from different points collide. It is important to notice that we do not get shocks as the Camassa-Holm preserves the H^1 norm and therefore solutions remain continuous.

However, it is not obvious how to continue the solution after collision time. It turns out that, when two characteristics collide, the energy contained between these two characteristics has a limit which can be computed from (3.3). 249

As we will see, knowing this energy enables us to prolong the characteristics and thereby the solution, after collisions.

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Water Wave Equation Arising On Logarithmic Quantum Mechanics

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Abstract

We introduce the model of equation that is closely related to the water wave equation. We study the global existence of week solution for this class of equation. Using the Gross logarithmic Sobolev inequality we establish the main theorem of existence of week solution for this class of equation arising from Logarithmic Quantum Mechanics. We can extend the results of [1, 2].

Keywords: logarithmic quantum mechanics; Gross-Sobolev inequality; logarithmic wave equation; global existence of solution; nonlinear effects.

1. Introduction

We deal with a mathematical analysis for the problem of water wave equation on Logarithmic quantum mechanics. The main difference between our work and [1,2] is: our problem is in k dimensional case on H_0^m and involves another nonlinear term $u \log |u|^k$; there is no restrictions on the coefficient of the logarithmic nonlinear term $u \log |u|^k$. Recently in [8] a numerical model is given. We mainly establish the global existence of weak solutions to the problem (1.2). Firstly we write the problem in a weak version. Secondly we construct approximate solutions by the Galerkin method. Finally we prove the convergence of the sequence of the approximate solutions. To get a priori estimates of the approximate solutions, we employ the Gross logarithmic Sobolev inequality and logarithmic Gronwall inequality.

In [1]- [2], Cazenave and Haraux established the existence a solution for the following equation

$$u_{tt} + A + u + u_t |u|^2 u = u ln |u|$$
, $x \in \Omega$, $t > 0$ (1.1)

for studying the dynamics of Q-ball in theoretical physics. In [2], Cazenave and Haraux established the existence and uniqueness of a solution for the Cauchy problem for the following equation in \mathbb{R}^n .

$$u_{tt} + A = u ln |u|^k , \qquad (1.2)$$

In the following section we state some lemmas. In the section 3 we give the proof of the theorem.

The logarithmic nonlinearity is of much interest in physics, since it appears naturally in cosmology and symmetric filed theories, quantum mechanics and nuclear physics [1, 14]. This type of problems have many applications in many branches of physics such as nuclear physics, optics and geophysics. It has been also introduced in the quantum field theory.

2. Materials and Methods

We denote by $||.||_p$ the $L^p(\Omega)$ norm, and $||\nabla||$ the Dirichlet norm in H_0^m . In particular, we denote $||.|| = ||.||_2$. We also use *C* to denote a universal positive constant that may have different values in different places. We denote by (\cdot, \cdot) the inner product in $L^2(\Omega)$. and by $\langle \cdot, \cdot \rangle$ the duality pairing between H_0^{-1} and H_0^{-m} . We also use C to denote a universal positive constant may take different values in different places. Let's we introduce the definition of weak solutions for the problem (1.2).

Def. A function u on [0, T] is called a weak solution u(x, t) of problem (1.2) on $\Omega \times [0, T]$, if $u \in C([0, T], H_0^m(\Omega)), u' \in C([0, T], L^2(\Omega)), u_0 = u^0(x)$ and $u'(0) = u^1(x)$ and u satisfies,

$$\langle u''(t), \emptyset \rangle + (\nabla u, \nabla \emptyset) + (u, \emptyset) + (u', \emptyset) - (u \log |u|^k, \emptyset) + (|u|^k u, \emptyset) = 0$$
 (2.1)

Lemma 2.1: (See [12]) Assume $v \in H_0^m(\Omega)$, and Ω is a bounded smooth domain in \mathbb{R}^n $(\Omega \subset \mathbb{R}^n)$. Then, for any a > 0, it holds that

$$\int_{\Omega} |v|^2 \log|v| \, dx \le \left(\frac{3}{4} \log \frac{4a}{e}\right) \|v\|_2^2 + \frac{a}{4} \|\nabla v\|_2^2 + \|v\|_2^2 \log\|v\|_2 \tag{2.2}$$

Lemma 2.2: (See [13]) Assume w(t) is nonnegative, $w(t) \in L^{\infty}(0, T)$, w(0) > 0 and

$$w(t) \le w(0) + a \int_{\Omega} w(s) \log [a + w(s)] ds \quad t \in (0, T),$$
 (2.3)

where a > 1 is a positive constant. Then we have

$$w(t) \le (a + w(0))^{e^{at}}, t \in (0, T),$$
 (2.4)

Lemma 2.3: (see [3, 5]) (Logarithmic Sobolev inequality) Let u be any function in $H_0^m(\Omega)$, and a > 0 be any number. Then

$$2\int_{\Omega} |u|^2 \ln \frac{|u|}{\|u\|} dx + n(1 + \log a) \|u\|^2 \le \frac{a^2}{\pi} \int_{\Omega} \left|\frac{1}{2}u\right|^2 dx$$
(2.5)

3. Results and Discussions for the main theorem and the proof

In this section we deal with main theorem of existence of global week solution. By using these lemmas and using the Gross logarithmic Sobolev inequality with the $c_{ombination}^{252}$

Galerkin method to construct approximate solutions, we can proof the main theorem. We carry out the proof of Theorem giving the solution u, where u is a weak solution of problem (1.2) on [0, T], where T is the maximal existence time of weak solution. The proof is based on Galerkin method. We use the Gross logarithmic Sobolev inequality and the logarithmic Gronwall inequality.

Theorem 3.1 Assume that $u^0(x) \in H_0^m(\Omega)$, and $u^1(x) \in L^2(\Omega)$. Then, the problem (1.2) admits global weak solution defined on [0, T] for any T > 0.

Proof: Let $\{w_j\}_{j=1}^{\infty}$ be the eigenfunctions of the operator $A = (\Delta)^m$ with zero Dirichlet boundary condition and $D(A) = H_0^m(\Omega) \cap H_0^1(\Omega)$. It is well-known that $\{w_j\}_{j=1}^{\infty}$ forms an orthonormal basis for $L^2(\Omega)$ as well as for $H_0^1(\Omega)$. Let P_k be the orthogonal projection of $L^2(\Omega)$ onto V_k = the linear span of $\{w_1, \ldots, w_k\}, k \ge 1$. Let $u_k = \sum_{j=1}^k g_{k_j}(t) w_j$ be an approximate solution to (1.2) in V_k . Then $u_k(t)$ verifies the following system of ODEs:

$$\langle \mathbf{u}_{k}^{''}(\mathbf{t}), w_{j} \rangle + (\nabla u_{k}(t), \nabla w_{j}) + (u_{k}, w_{j}) + (\mathbf{u}_{k}^{'}(\mathbf{t}), w_{j}) - (u_{k} \log |u_{k}|^{2}, w_{j}) + (|u_{k}|^{2}u_{k}, w_{j}) = 0$$

$$(3.1)$$

$$u_{k}(0) = P_{k} u^{0}(x), \quad u'_{k}(0) = P_{k} u^{1}(x)$$
(3.2)

for j = 1, ..., k. More specifically,

$$u_{k}(0) = \sum_{j=1}^{k} u_{k_{j}}(0) w_{j} , \quad u'_{k}(0) = \sum_{j=1}^{k} u'_{k_{j}}(0) w_{j}$$
$$u_{k_{i}}(0) = (u^{0}, w_{j}) , \qquad u'_{k_{i}}(0) = (u^{1}, w_{j}), \quad j = 1, \dots, k$$

where,

Now we try to get the a priori estimate for the approximate solutions
$$u_k(t)$$
 of the problem (1.2). Multiplying (3.1) by $g'_{k_j}(t)$ and summing with respect to j from 1 to k , we have

$$\frac{\mathrm{d}}{\mathrm{dt}} \left[\frac{1}{2} \| \mathbf{u}'_{k}(t) \|_{2}^{2} + \| u_{k}(t) \|_{2}^{2} - \int_{\Omega} |u_{k}(t)|^{2} \log |u_{k}(t)| \, dx + \frac{1}{4} \| u_{k}(t) \|_{4}^{4} \right] + \| \mathbf{u}'_{k}(t) \|_{2}^{2} = 0 \quad (3.3)$$

Integrating (3.3) over (0, t), $0 < t \leq T_k$, we get

$$\frac{1}{2} \| \mathbf{u}'_{k}(\mathbf{t}) \|_{2}^{2} + \frac{1}{2} \| \nabla u_{k}(\mathbf{t}) \|_{2}^{2} + \| u_{k}(\mathbf{t}) \|_{2}^{2} - \int_{\Omega} |u_{k}(\mathbf{t})|^{2} \log |u_{k}(\mathbf{t})| \, dx + \frac{1}{4} \| u_{k}(\mathbf{t}) \|_{4}^{4}$$

$$= \frac{1}{2} \| \mathbf{u}'_{k}(\mathbf{0}) \|_{2}^{2} + \frac{1}{2} \| \nabla u_{k}(\mathbf{0}) \|_{2}^{2} + \| u_{k}(\mathbf{0}) \|_{2}^{2} - \int_{\Omega} |u_{k}(\mathbf{0})|^{2} \log |u_{k}(\mathbf{0})| \, dx$$

$$+ \frac{1}{4} \| u_{k}(\mathbf{0}) \|_{4}^{4} + \int_{0}^{t} \| \mathbf{u}'_{k}(\mathbf{s}) \|_{2}^{2} \, ds \leq C_{0} - \int_{\Omega} |u_{k}(\mathbf{0})|^{2} \log |u_{k}(\underline{0})| \, dx \quad (3.4)$$

where $C_0 = C (\|u^0\|_{H^m(\Omega)}, \|u^1\|_{L^2(\Omega)})$ is a positive constant. We use the inequality

$$|t^2 \log t| \le C (1+t^3)$$
 for all $t > 0$ (3.5)

Now we use Lemma 2.1 introducing Gross-Sobolev. Then, we use now (3.5) again to estimate the logarithmic term

$$\int_{\Omega} |u_k \log u_k|^2 |^2 dx = 4 \int_{\Omega} |u_k|^2 (\log u_k|^2) dx \le C |\Omega| + C \int_{\Omega} |u_k|^2 dx \le C |\Omega| \left(||u_k(s)||_{H_0^m}^6 + 1 \right)$$
(3.6)

This implies that $u_k \log |u_k|^2_k$ is uniformly bounded in $L^2(0, T, L^2(\Omega))$. So, exists any function in $L^2(0, T, L^2(\Omega))$, such that $|u_k|^2 u_k$ converges in it. This is $u \log |u|^2$.

By Sobolev inequality

$$\int_{\Omega} ||u_k|^2 u_k|^2 dx = \int_{\Omega} |u_k|^6 \le C ||u_k(s)||_{H_0^m}^6 \le C$$
(3.7)

As above we explained there exist any function as $|u|^2 u$ in $L^2(0, T, L^2(\Omega))$ such that

$$|u_k|^2 u_k \to |u|^2 u$$
 in $L^2(0, T, L^2(\Omega))$ (3.8)

This clearly said that u satisfies the eq. (1.2) in the week sense. From (3.15) we have $u_k(0) \rightarrow u(0)$ weekly in $L^2(\Omega)$. Using eq.3.16 and by choosing $u_k(0) \rightarrow u^0$ strongly in $H_0^m(\Omega)$, we have

$$u(0) = u^0 \tag{3.9}$$

From (3.16), $\langle u''_k, w_j \rangle \rightarrow \langle u, w_j \rangle$ in $L^{\infty}(0,T)$. This implies that $\langle u'_k(0), w_j \rangle \rightarrow \langle u'(0), w_j \rangle$. Noting that $u'_k(0) \rightarrow u^1$ weekly in L^2 , than

$$u'(0) = u^{1} \tag{3.10}$$

From (3.22) and (3.23), the initial condition is satisfied. The theorem (3.1) is completed. The global existence of week solutions to the problem (1.2) is established.

4. Conclusions

This type of equation arising from many applications in many branches of physics such as nuclear physics, optics and geophysics [7, 9, 10].

The problem (1.1) is a relativistic version of logarithmic quantum mechanics introduced in [3, 4].

Based on the numerical simulations, we discuss how the Q-ball formation proceeds. Also we make a comment on possible deviation of the charge distributions from what was conjectured in the past. This type of problems have many applications in many branches of physics such as nuclear physics, optics and geophysics.

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Comparison of kernel functions in arrhytmia classification

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Abstract

Arrhytmia is one of the most common heart diseases in the world. Due to the complex nature of the electrocardiogram, the hand-operared diagnosis of arrhytmia is very tedious. In this article, a multi-class support vector machine based approach is proposed to solve the ECG multi-classification problem. To do so, several features besides RR interval are used. Various kernel functions in the multi-class support vector machine are tested for arrhytmia classification. Performance evaluation for the proposed method was tested over the MIT-BIH Arrhytmias Database.

Keywords: ECG, Arrhytmia classification, Multi-class support vector machine, Kernel functions

1. Introduction

Detection of arrhythmia classification is an important task in medicine. There are various methodologies for automatic detection of the cardiac arrhythmia classification which have been proposed in recent years. [1]-[7]. Khazaee et al. [1] used support vector machines (SVM) and genetic algorithms (GA) to detect arrhythmia, which is referred to as identification of premature ventricular contraction (V). But, their approach can only distinguish three types of arrhythmia, including Normal beat (N), V and others. It did not explicitly express the kernel function used in SVM The method present in [2] is based on Fisher Linear Discriminant. The RR interval duration and the distance between the occurrence of P and T waves are observed. Using these features Fisher's Linear Discriminant is applied. In [3] an SVM-based method for V arrhythmia detection is shown to be more efficient than Anfis. In [4] a new approach for feature selection and classification of cardiac arrhythmias based on particle swarm optimization-SVM (PSO-SVM) is proposed. In [5], a neuro-fuzzy approach for the ECGbased classification of heart rhythms is described. Here, the QRS complex signal is characterized by Hermite polynomials, whose coefficients feed the neuro-fuzzy classifier. In [6] arrhythmia Detection using Independent Component Analysis (ICA) and Wavelet transform to extract important features is proposed. In [7], a neural network classifier using wavelet and timing features is used for classifying beats of a large dataset. In machine

learning, multiclass classification is the problem of classifying instances into one of three or more classes. In this paper, we use a multi-class support vector machine (MSVM) to identify 3 heartbeat types; we distinguish normal heartbeat (N) from atrial premature beat (A) and premature ventricular contraction (V). Besides, we try to use different features for ECG classification and different kernel functions to classify them.

The organization of the paper is as follows. In Section 2, the basic working principle of SVM and the different kernel functions are introduced in detail. In Section 3 Kernel function of MSVM is applied to classification of arrhythmia and simulation results are discussed in detail. We express conclusions in the last section.

2. Methodology

The MSVM method is a machine learning method. It is based on the structural risk minimization principle and theoretical basis, by choosing the suitable subset function and the subset of the discriminant function [8,9]. This method can be used to identify N, A, and V. A selected part of the data is used to train the classifier, and the other part of the data is used as a test. The MIT-BIH database data is used for the validation of the method.

i) Multi-Class Support Vector Machine

In this section, we present a "one-against-one" multiclass support vector machine (MSVM) algorithm. The primary idea of MSVM is constructing separate hyperplanes between classes in feature space using support vectors. When we are given the module vector W_m (m = 1, 2, ..., i) and the number of shaded modules z_m ($z_m \in 1, 2, ..., i$), we can express the decision function for the training data by

$$g_{mn}(W) = x^{mn} \cdot \phi(W) + b_{mn} \tag{1}$$

We can solve the two-class classification problem by:

$$\min_{x^{mn}, b^{mn}, \gamma^{mn}} \frac{1}{2} (x^{mn})^T x^{mn} + C \sum_{t}^{j} (\gamma^{mn}) t$$

s.t. $(x^{mn})^T \phi(W_t) + b^{mn} \ge 1 - \gamma_t^{mn}, if : z_m = m$
 $(x^{mn})^T \phi(W_t) + b^{mn} \le 1 - \gamma_t^{mn}, if : z_m = m, \ \gamma^{mn} > 0$
(2)

Here C is the associated penalty for excessive deviation, γ^{mn} is the nonnegative variables, and ϕ is a function.

We can solve equation (2) by the Lagrange multipliers α^{mn} After we obtain the optimal solution $(\alpha^{mn})^*$, we can determine the optimal hyperplane parameters $(x^{mn})^*$ and $(b^{mn})^*$ can be determined, and we can write the indicator function as

$$sign\left[\sum_{t=1}^{j} z_t \left(\alpha^{mn}\right)_t^* \phi(W). \phi(W_t) + (b^{mn})^*\right]$$
(3)

We use the "one-against-one" approach to extend SVM to the multi-class situation. There are $c_2^k = r(r-1)/2$ classifiers used in training. Each classifier is then trained with two different classes. The strategy gives one vote to the n^{th} class, and the classes that receive the most votes serve as classification results.

ii) Kernel function

The performance of MSVM largely depends on the choice of the kernel function. As we change the kernel function, the training results will be different. Usually, there are four kinds of kernel functions commonly used for support vector machines. Their kernel functions are as follows:

Linear:	$K(w_m, w_n) = w_m. w_n$	(4)
Polynomial:	$K(w_m, w_n) = (w_m, w_n + 1)^d$	(5)
Radial Bases Function:	$K(w_m, w_n) = \exp(-\ w_m - w_n\ ^2 / 2\delta^2)$	(6)
Sigmoid:	$K(w_m, w_n) = \tanh(\epsilon w_m, w_n + \eta)$	(7)

3. Proposed Method

MIT-BIH database is recognized as one of the three standard ECG databases. It is provided by the Massachusetts Institute of Technology. In this paper, we use the data of the MIT-BIH arrhythmia database. In the MIT-BIH arrhythmia database, there are different types of heart disease data with 48 sets. A frequency of 360 HZ is used in ECG sampling. 258

MIT-BIH library contains all the period of the RR interval. Four local timing features can be extracted based on the RR interval, which can promote the ability of morphological characters' recognition. The effect of four feathers is most significant when distinguishing the similar heartbeats patterns. Four local timing features are an RR time interval ratio (IO), an RR time interval difference (IF), and two RR time intervals. The IO feature reflects the heartbeat rate deviation of two adjacent RR intervals and the IF feature reflects the deviation of non-adjacent RR intervals of heartbeat rate. They are defined as:

$$IF_i = K_{i+2} - K_{i+1} - (K_i - K_{i-1})$$
(8)

$$IO_i = (K_i - K_{i-1})/(K_{i+1} - K_i)$$
(9)

In the formulas above, K_i refers to the time at which the R-wave occurs.

Other than IO and IF, two other features which we will use are the preceding and the following RR time intervals for each kind of heartbeat. For a normal pulse, the value of IO approximately equals to 1, and IF equals to 0.

4. Results ans Discussions

To evaluate the workability of the proposed method, we used five records from the MIT-BIH database (108, 114, 200, 201, and 213). 50 % of the sample data are used for training the algorithm. The remained data are used for testing. The accuracy for classification of the algorithm is defined as the percentage of accurate classifications.

Table I shows the accuracy of the MSVM algorithm with various kernel functions in the detection of N, A, and V. In a Multiclass SVM, the associated penalty for excessive deviation C and the turning parameter δ involved in the RBF affects the classification accuracy. The values of C and δ used in our study are 65.21 and 31.02, respectively. These values were obtained experimentally. From the Table I, we can see that the MSVM with a linear kernel obtains the lowest accuracy. The quadratic polynomial kernel function improves the correctness of the linear kernel function by using higher order operation, and the overall accuracy reaches 95%. The RBF-MSVM shows better results in V detection. This result is consistent with results obtained in many SVM classification applications. According to our

experimental results, the RBF kernel function is better than the other kernel function types (linear and quadratic polynomial). The accuracy of RBF-MSVM is 96.30 %.

	Number	Number	Accuracy of test with different kernel			
	of beat used	of beat used	Linear	Quadratic	RBF	
	in the train	in the test		Polynomial		
Ν	4779	4780	95.61 %	95.82 %	96.21 %	
Α	49	50	98 %	98 %	100 %	
V	651	652	88.96 %	90.49 %	96.63 %	
Overall	5479	5482	94.84 %	95.20 %	96.30 %	

TABLE I. COMPARISON OF THE ACCURACY OF DIFFERENT MSVM

5. Conclusions

In this paper, we used four features of ECG obtained from the MIT-BIH database for an MSVM based classification algorithm. The method proposed distinguishes normal heartbeat from two types of arrhythmias. Besides, three M-SVMs with different kernel functions have been evaluated. Numerical results show that the RBF kernel function gives the highest accuracy, achieving the overall average accuracy of rate 96.30 %.

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Chebyshev Polynomial Coeffcient Bounds for an Unified Subclass of Bi-univalent Functions

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Abstract

The aim of this paper to discuss a newly constructed subclass of bi-univalent functions. Further, we establish Chebyshev polynomial bounds for the coefficients for the class $S_{\Sigma}^{\delta}(t)$.

Keywords: Coefficient bounds, Bi-univalent functions, Chebyshev polinomials

1. Introduction

Let *A* be the class of functions f of the form:

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots,$$
(1)

which are analytic in the open unit disc $U = \{z \in \Box | z | < 1\}$ normalized by f(0) = 0, f'(0) = 1.

Further, by S we shall denote the class of all functions in A which are univalent in U.

For two analytic functions, f and g, such that f(0) = g(0), we say that f is subordinate to g in U and write $f(z) \prec g(z), z \in U$, if there exists a Schwarz function w(z) with w(0) = 0 and $|w(z)| \le |z|, z \in U$ such that $f(z) = g(w(z)), z \in U$. Furthermore, if the function g is univalent in U, then we have the following equivalence;

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

The definition can be found in (Nehari 1952).

The Koebe-one quarter theorem (Duren 1983) ensures that the image of U under every univalent function $f \in A$ contains a disc of radius ¹/₄. Thus every univalent function f has an inverse f^{-1} satisfying $f^{-1}(f(z)) = z$, $z \in U$ and $f(f^{-1}(w)) = w$, $(|w| < r_0(f), r_0(f) \ge 1/4)$. Indeed, the inverse function f^{-1} is given by

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^3 + \cdots \stackrel{262}{\cdots}$$
(2)

A function $f \in A$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U. Let Σ denote the class of bi-univalent functions defined in U.

Many researchers have recently introduced and investigated several interesting subclasses of bi-univalent function class Σ and they have found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ (Altınkaya and Yalçın 2015, Brannan and Taha 1986, Srivastava et al. 2010). However, there are only a few works determining the general coefficient bounds $|a_n|$ for the analytic bi-univalent functions in the literature (Altınkaya and Yalçın 2015, Hamidi and Jahangiri 2014). The coefficient estimate problem for each of

$$|a_n|, n \in \Box - \{1, 2, 3\} (\Box = \{1, 2, 3, \ldots\})$$

is still an open problem.

One of the important tools in numerical analysis, from both theoretical and practical points of view, is Chebyshev polynomials. The majority of research papers dealing with specific orthogonal polynomials of Chebyshev family, contain mainly results of Chebyshev polynomials of first and second kinds $T_m(t)$ and $U_m(t)$ and their numerous uses in different applications, see for example, Doha (1994) and Mason (1967). In the case of a real variable t on (-1,1), they are defined by

$$T_m(t) = \cos m\theta, \ U_m(t) = \frac{\sin(m+1)\theta}{\sin \theta}$$

where the subscript *m* denotes the polynomial degree and where $t = \cos \theta$. If we choose

$$t = \cos \alpha, \ \alpha \in \left(-\frac{\pi}{3}, \frac{\pi}{3}\right),$$
 then
$$H(z,t) = \frac{1}{1 - 2tz + z^2} = 1 + \sum_{m=1}^{\infty} \frac{\sin(m+1)\alpha}{\sin \alpha} z^m.$$

Thus

$$H(z,t) = 1 + 2\cos\alpha z + (3\cos^2\alpha - \sin^2\alpha)z^2 + \cdots,$$

or equivalently

$$H(z,t) = 1 + U_1(t)z + U_2(t)z^2 + \dots (z \in U, t \in (-1,1)),$$
²⁶³

where $U_{m-1}(t) = \frac{\sin(m \arccos t)}{\sqrt{1-t^2}}$ ($m \in \Box$) are the Chebyshev polynomials of the second kind.

Also it is known that

$$U_m(t) = 2tU_{m-1}(t) - U_{m-2}(t)$$

and

$$U_1(t) = 2t, U_2(t) = 4t^2 - 1, U_3(t) = 8t^3 - 4t, \dots$$
(3)

Definition 1.1. A function $f \in \Sigma$ given by (1) is said to belong to the class

$$S_{\Sigma}^{\delta}(t) \left(0 < \delta \leq 1, t \in \left(\frac{1}{2}, 1\right], z, w \in U \right)$$

if the following subordinations are satisfied:

$$\frac{1}{2} \left(\frac{zf'(z)}{f(z)} + \left(\frac{zf'(z)}{f(z)} \right)^{\frac{1}{\delta}} \right) \prec H(z,t)$$

and

$$\frac{1}{2}\left(\frac{wg'(w)}{g(w)} + \left(\frac{wg'(w)}{g(w)}\right)^{\frac{1}{\delta}}\right) \prec H(w,t)$$

where the function g is the extension of f^{-1} to U.

2. Chebyshev polynomial coefficient bounds

In this section, we derive the resulting Chebyshev polynomial estimates for the initial coefficients $|a_2|$ and $|a_3|$ of functions $f \in S_{\Sigma}^{\delta}(t)$ given by the Taylor-Maclaurin series expansion (1).

Theorem 2.1. Let $f \in S_{\Sigma}^{\delta}(t)$. Then

$$|a_2| \le \frac{4\delta t \sqrt{2t}}{\sqrt{4(\delta^2 - \delta)t^2 + (\delta + 1)^2}}$$

and

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$$\left|a_{3}\right| \leq \frac{16\delta^{2}t^{2}}{\left(1+\delta\right)^{2}} + \frac{2\delta t}{1+\delta}.$$

Proof. Let $f \in S_{\Sigma}^{\delta}(t)$. In view of the definition of subordination, we can write

$$\frac{1}{2} \left(\frac{zf'(z)}{f(z)} + \left(\frac{zf'(z)}{f(z)} \right)^{\frac{1}{\delta}} \right) = 1 + U_1(t)\varphi(z) + U_2(t)\varphi^2(z),$$

(4)

$$\frac{1}{2} \left(\frac{wg'(w)}{g(w)} + \left(\frac{wg'(w)}{g(w)} \right)^{\frac{1}{\delta}} \right) = 1 + U_1(t)\psi(w) + U_2(t)\psi^2(w),$$

(5)

for some analytic functions φ , ψ such that $\varphi(0) = \psi(0) = 0$ and $|\varphi(z)| = |p_1 z + p_2 z^2 + \dots | < 1$, $|\psi(w)| = |q_1 w + q_2 w^2 + \dots | < 1$, Then,

$$|p_i| \le 1, \ |q_i| \le 1, \ \forall i \in \Box .$$
(6)

In the light of (4) and (5), we obtain

$$\frac{\delta+1}{2\delta}a_2 = U_1(t)p_1,$$

(7)

$$\frac{\delta+1}{2\delta}(2a_3-a_2^2)+\frac{(1-\delta)}{4\delta^2}a_2^2=U_1(t)p_2+U_2(t)p_1^2,$$

(8)

$$-\frac{\delta+1}{2\delta}a_2 = U_1(t)q_1,$$

(9)

$$\frac{(\delta+1)}{2\delta}(3a_2^2-2a_3)+\frac{(1-\delta)}{4\delta^2}a_2^2=U_1(t)q_2+U_2(t)q_1^2.$$

(10)

From (7) and (9), it follows that

 $p_1 = -q_1$

(11)

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and

$$\frac{(\delta+1)^2}{2\delta^2}a_2^2 = U_1^2(t)(p_1^2+q_1^2).$$

(12)

Now, by adding (8) and (10), we obtain

$$\left(\frac{\delta+1}{\delta}+\frac{1-\delta}{2\delta^2}\right)a_2^2 = U_1(t)(p_2+q_2)+U_2(t)(p_1^2+q_1^2).$$

(13)

Therefore, by using (11) in the equality (13), we obtain

$$\left(\frac{2\delta^2 + \delta + 1}{2\delta^2} - \frac{U_2(t)}{U_1^2(t)} \frac{(1+\delta)^2}{2\delta^2}\right) a_2^2 = U_1(t)(p_2 + q_2).$$

From (3) and (6), we immediately have

$$|a_2| \le \frac{4\delta t \sqrt{2t}}{\sqrt{4(\delta^2 - \delta)t^2 + (\delta + 1)^2}}$$

Additionaly, in order to calculate the bound on $|a_3|$, by subtracting (10) from (8), we obtain

$$\frac{2(\delta+1)}{\delta}a_3 - \frac{2(1+\delta)}{\delta}a_2^2 = U_1(t)(p_2 - q_2).$$

In view of (3), (6) and (12), we readily get the bound on $|a_3|$ as asserted in Theorem 2.1.

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Coeffcient estimates for two general subclasses of *m*-fold symmetric Bi-univalent functions

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Abstract

In the present paper, we introduce some new subclasses of \sum_m consisting of analytic and *m*-fold symmetric bi-univalent functions in the open unit disc U. Moreover, for functions belonging to the classes introduced here, we derive non-sharp estimates on the initial coefficients $|a_{m+1}|$ and $|a_{2m+1}|$.

Keywords: Coefficient estimates, m-fold symmetric bi-univalent functions, Starlike functions

1. Introduction

Let A be the class of functions f of the form:

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots,$$
 (1)

which are analytic in the open unit disc $U = \{z \in \Box | z | < 1\}$ normalized by f(0) = 0, f'(0) = 1.

Further, by S we shall denote the class of all functions in A which are univalent in U. The Koebe-one quarter theorem (Duren 1983) ensures that the image of U under every univalent function $f \in A$ contains a disc of radius ¹/₄. Thus every univalent function f has an inverse

$$f^{-1}$$
 satisfying $f^{-1}(f(z)) = z, z \in U$ and $f(f^{-1}(w)) = w, (|w| < r_0(f), r_0(f) \ge 1/4)$. Indeed,

the inverse function f^{-1} is given by

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^3 + \cdots$$
(2)

A function $f \in A$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U. Let Σ denote the class of bi-univalent functions defined in U. For a brief history and interesting examples in the class Σ , see Srivastava et al. (2010), (see also Altınkaya and Yalçın 2015, Brannan and Taha 1986, Hamidi and Jahangiri 2014).

For each function $f \in S$, the function

 $h(z) = \sqrt[m]{f(z^m)} \qquad (z \in U, m \in \Box)$

is univalent and maps the unit disc U into a region with *m*-fold symmetry. A function is said to be *m*-fold symmetric (see Pommerenke 1975) if it has the following normalized form:

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1}.$$
(3)

We indicate by S_m the class of *m*-fold symmetric univalent functions in U, which are normalized by the series expansion (3). In fact, the functions in the class S are *one*-fold symmetric. Analogous to the concept of *m*-fold symmetric univalent functions, we here introduced the concept of *m*-fold symmetric bi-univalent functions. Each function $f \in \Sigma$ generates a *m*-fold symmetric bi-univalent function for each integer $m \in \Box$. The normalized form of f is given as in (3) and the series expansion for f^{-1} , which has been recently proven by Srivastava et al. (2014), is given as follows:

$$g(w) = f^{-1}(w) = w - a_{m+1}w^{m+1} + ((m+1)a_{m+1}^2 - a_{2m+1})w^{2m+1} - \left[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1})\right]w^{3m+1}\cdots$$
(4)

We now define the following:

Definition 1.1. A function $f \in \Sigma_m$ given by (4) is said to belong to the class

$$S_{\Sigma_{-}}^{\mu}(\delta) \ (0 \le \mu < 1, 0 < \delta \le 1, z, w \in U)$$

if the following conditions are satisfied:

$$\operatorname{Re}\left[\frac{1}{2}\left(\frac{zf'(z)}{f(z)} + \left(\frac{zf'(z)}{f(z)}\right)^{\frac{1}{\delta}}\right)\right] > \mu$$

and

$$\operatorname{Re}\left[\frac{1}{2}\left(\frac{wg'(w)}{g(w)} + \left(\frac{wg'(w)}{g(w)}\right)^{\frac{1}{\delta}}\right)\right] > \mu$$
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where the function g is the extension of f^{-1} to U.

2. Coefficient Bounds

In this section, we derive the resulting estimates for the initial coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ of functions $f \in S^{\mu}_{\Sigma_m}(\delta)$ given by the Taylor-Maclaurin series expansion (3).

Theorem 2.1. Let $0 \le \mu < 1$. If $f \in A$ of the form (3) belongs to the class $S_{\Sigma_m}^{\mu}(\delta)$, then

$$\left|a_{m+1}\right| \leq \frac{2\delta}{m} \sqrt{\frac{2(1-\mu)}{2\delta^2 + \delta + 1}}$$

and

$$a_{2m+1} \Big| \le \frac{8(m+1)\delta^2(1-\mu)^2}{\left[m(1+\delta)\right]^2} + \frac{2\delta(1-\mu)}{m(1+\delta)}.$$

Proof. Let $f \in S^{\mu}_{\Sigma_m}(\delta)$. In view of Definition 1.1, we get

$$\frac{1}{2}\left(\frac{zf'(z)}{f(z)} + \left(\frac{zf'(z)}{f(z)}\right)^{\frac{1}{\delta}}\right) = \mu + (1-\mu)t(z)$$
(5)

and

$$\frac{1}{2} \left(\frac{wg'(w)}{g(w)} + \left(\frac{wg'(w)}{g(w)} \right)^{\frac{1}{\delta}} \right) = \mu + (1 - \mu)s(w).$$
(6)

Next, define the functions

$$t(z) = 1 + t_m z + t_{2m} z^2 + \dots$$
(7)

and

$$s(w) = 1 + s_m w + s_{2m} w^2 + \cdots$$
 (8)

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Then, t(z) and s(w) analytic in U with t(0) = 1 = s(0). Since the functions t(z) and s(w) have a positive real part in U, $|t_i| \le 2$ and $|s_i| \le 2$.

In the light of (5), (6) and (7), (8), we obtain

$$\frac{m(\delta+1)}{2\delta}a_2 = (1-\mu)t_m,\tag{9}$$

$$\frac{m(\delta+1)}{2\delta}(2a_{2m+1}-a_{m+1}^2) + \frac{m^2(1-\delta)}{4\delta^2}a_{m+1}^2 = (1-\mu)t_{2m},$$
(10)

$$-\frac{m(\delta+1)}{2\delta}a_2 = (1-\mu)s_m,\tag{11}$$

$$\frac{m(\delta+1)}{2\delta} \Big[(2m+1)a_{m+1}^2 - 2a_{2m+1} \Big] + \frac{m^2(1-\delta)}{4\delta^2} a_{m+1}^2 = (1-\mu)s_{2m}.$$
(12)

From (9) and (11), it follows that

$$t_m = -s_m \tag{13}$$

and

$$\frac{\left[m(\delta+1)\right]^2}{2\delta^2}a_{m+1}^2 = (1-\mu)^2(t_m^2 + s_m^2).$$
(14)

Now, by adding (10) and (12), we obtain

$$\left(\frac{m^2(\delta+1)}{\delta} + \frac{m^2(1-\delta)}{2\delta^2}\right)a_{m+1}^2 = (1-\mu)(t_{2m} + s_{2m}).$$

Therefore, we obtain

$$a_{m+1}^{2} = \frac{2\delta^{2}(t_{2m} + s_{2m})}{m^{2}(2\delta^{2} + \delta + 1)}$$

Applying $|t_i| \le 2$ and $|s_i| \le 2$ for the coefficients t_{2m} and s_{2m} , we immediately have

$$|a_{2m+1}| \leq \frac{8(m+1)\delta^2(1-\mu)^2}{[m(1+\delta)]^2} + \frac{2\delta(1-\mu)}{m(1+\delta)}.$$

Additionaly, in order to calculate the bound on $|a_{2m+1}|$, by subtracting (12) from (10), we obtain

$$\frac{2m(\delta+1)}{\delta}a_{2m+1} - \frac{m(m+1)(1+\delta)}{\delta}a_{m+1}^2 = (1-\mu)(t_{2m} - s_{2m}).$$

Applying $|t_i| \le 2$ and $|s_i| \le 2$ once again for the coefficients t_{2m} and s_{2m} , we readily get the bound on $|a_{2m+1}|$ as asserted in Theorem 2.1.

3. Conclusions

If we set $\delta = 1$ in Theorem 2.1, then the class $S_{\Sigma_m}^{\mu}(\delta)$ reduces to the classes $S_{\Sigma_m}^{\mu}$ and thus, we obtain the following corollary:

Corallary 3.1. Let $0 \le \mu < 1$. If $f \in A$ of the form (3) belongs to the class $S_{\Sigma_m}^{\mu}$, then

$$\left|a_{m+1}\right| \le \frac{\sqrt{2(1-\mu)}}{m}$$

and

$$|a_{2m+1}| \le 2 \frac{8(m+1)(1-\mu)^2}{m^2} + \frac{1-\mu}{m}.$$

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On Backlund Transformation of KdV Flow by Inextensibility

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Abstract

In this work, we obtain new characterizations about inextensible flow and KdV flow. Also, we present a new approach for computing the geometry properties of curves by integrable geometric curve flows. We use elliptic function expansion method in some new solutions by using the KdV flow. Finally, we obtain figures of this solutions.

Keywords: KdV flow, Bäcklund transformations, inextensible flows, elliptic function expansion method.

1. Introduction

In applied differential geometry, theory of curves in space is one of the significant study areas, [1-5]. In the theory of curves, helices, slant helices, and rectifying curves are the most fascinating curves. Flows of curves of a given curve are also widely studied, [7-13].

A particular nice feature of integrable systems is their relationship with invariant geometric flows of curves and surfaces in certain geometries. Those flows are invariant with respect to the symmetry groups of the geometries. A number of integrable equations have been shown to be related to motions of curves in Euclidean geometry, centro-equiaffine geometry, affine geometry, homogeneous manifolds and other geometries etc., and many interesting results have been obtained [14-18].

This study is organised as follows: Firstly, we present a new approach for computing the differential geometry properties of surfaces by using Bäcklund transformations of integrable geometric curve flows. We give some new solutions by using the extended Riccati mapping method. Finally, we obtain figures of this solutions.

2. Materials and Methods

Let $\gamma(s)$ be a smooth curve of constant torsion τ in R³, parametrized by arclength *s* . Let **T**, **N** and **B** be a Frenet frame, and k(s) the curvature of γ . For any constant C, suppose $\beta = \beta$ (s; k(s);C) is a solution of the differential equation

$$\frac{d\beta}{ds} = C\sin\beta - k$$

Then,

$$\widetilde{\gamma}(s) = \gamma(s) + \frac{2C}{C^2 + \tau^2} (\cos\beta \mathbf{T} + \sin\beta \mathbf{N})$$

is a curve of constant torsion τ , also parametrized by arclength s, [17].

Note that this transformation can be obtained by restricting the classical Bäcklund transformation for pseudospherical surfaces to the asymptotic lines of the surfaces with constant torsion.

We will restrict our attention to the geometric plane curve flows

$$\gamma_t = f\mathbf{T} + g\mathbf{N},$$

where f, g depend on the curvatures of the curves and their derivatives with respect to the arclength parameter, namely, these geometric flows are invariant with respect to the symmetry groups of the geometries, [17].

For a spacial curve $\gamma(s,t)$ in a given geometry, let $\tilde{\gamma}(s,t)$ be another curve related to through the following Bäcklund transformation

$$\widetilde{\gamma}(s,t) = \gamma(s,t) + \alpha(s,t)\mathbf{T} + \beta(s,t)\mathbf{N}.$$

3. Results and Discussions

Bäcklund Transformations for Plane Curve Flows

Consider the planar curve flow in the centro-equiaffine geometry, specified by

$$\gamma_t = h\mathbf{T} + f\mathbf{N},$$

where N and T are normal and tangent vectors of γ . One can compute the time evolution of N and T to get

$$\begin{pmatrix} \mathbf{T} \\ \mathbf{N} \end{pmatrix}_{t} = \begin{pmatrix} h_{s} - f & f_{s} + \phi h \\ -h & -f \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \end{pmatrix}.$$
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The Serret-Frenet formulae for curves in centro-equiaffine geometry reads

$$\mathbf{T}_{s} = \boldsymbol{\phi} \mathbf{N}, \mathbf{N}_{s} = -\mathbf{T}.$$

Assume that the flow is intrinsic, [17], a direct computation shows that the curvature satisfies

$$\phi_t = (D_s^2 + 4\phi + 2\phi_s \partial^{-1})f$$

Letting $\phi_s = f$ in above equation, we get the KdV equation

$$\phi_t = \phi_{sss} + 6\phi\phi_s \tag{1}$$

The corresponding KdV flow is

$$\gamma_t = \phi_s \mathbf{N} + 2\phi \mathbf{T}.$$

which was introduced firstly by Pinkall [16].

4. Application to Mathematica

We consider the following traveling wave transformation for Eq.(1)

$$\phi(s,t) = u(\gamma), \qquad \gamma = B(s - Qt), \tag{2}$$

where Q give the speed of the wave. Substituting Eq. (2) into Eq. (1), we obtain as follow,

$$-BQu'(\gamma) + 6Bu(\gamma)u'(\gamma) - B^{3}u'''(\gamma) = 0.$$
(3)

In this paper we will solve the Eq.(3) by using the cn elliptic function expansion method [6] as follows;

Assumed the solution of Eq. (3) is demonstrable as a finite series as follows:

$$\phi(s,t) = u(\gamma) = \sum_{j=0}^{N} \alpha_j c n^j [\gamma;n] + \sum_{j=1}^{N} \beta_j c n^{-j} [\gamma;n]$$
(4)

where cn [$\gamma;n$] is the Jacobi elliptic cn function with the parameter n $(0 < n < 1), \gamma = B(s - Qt)$ and $\alpha_0, \alpha_j, \beta_j$ for $j = \overline{1, N}$ are values to be definited.

Balancing $u^{'''}$ with u(u') in Eq. (3) gives

$$N+3 = N+N+1,5$$

$$\Rightarrow N=2.$$

Then the solution $u(\gamma)$ can write as follows,

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$$u(\gamma) = \alpha_0 + \alpha_1 cn[\gamma; n] + \alpha_2 cn^2[\gamma; n] + \beta_1 cn^{-1}[\gamma; n] + \beta_2 cn^{-2}[\gamma; n].$$
(5)

Substituting (6) into (3), collecting the coefficients of cn [γ ; n], and solving the obtaining system, several solution groups are obtained. One of them is as follows:

$$\begin{aligned} \alpha_0 &= \frac{1}{6} (B^2 (-4 + 8n^2) + v), \\ \alpha_1 &= 0, \\ \beta_1 &= 0, 7 \\ \alpha_2 &= -2B^2 n^2, \\ \beta_2 &= -2B^2 (-1 + n^2). \end{aligned}$$

From this result, Jacobi elliptic cn function solution is obtained as,

$$\phi(s,t) = \frac{1}{6} (B^2(-4+8n^2)+v) - 2B^2n^2cn^2[B(s-Qt);n] - 2B^2(-1+n^2)cn^{-2}[B(s-Qt);n] \quad (6)$$

When the parameter $n \rightarrow 1$, Eq. (8) is reduced to;

$$\phi(s,t) = \frac{1}{6}(4B^2 + v) - 2B^2 \sec h^2[B(s - Qt)]$$
⁽⁷⁾

which is the solitary wave solution of Eq.(1).

When the parameter $n \rightarrow 0$, Eq. (8) is reduced to;

$$\phi(s,t) = \frac{1}{6}(-4B^2 + v) + 2B^2 \csc^2[B(s - Qt)]$$
(8)

which is another solitary solution of Eq.(1).

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Numerical Solution of the Optimal Control Problem for Multilayered Materials

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Abstract

The problem of the determination of the transmission conditions and stresses on the common boundary between layers is very important nowadays in terms of engineering and applied mathematics. In this study the optimal control problem for the deformation of the laminate formed by different materials is investigated. A numerical algorithm for determining the properties of the thickness and hardness of the coating, taking into account the maximal of the deformation predicted by the influence of a certain force, is given. Physical and geometric interpretations of the obtained results are given with the help of a prepared computer program.

Keywords: Multilayered Material, Optimal Control Problem

1. Introduction

The contact problem related to the deformation of a rigid punch attracts the interest of mathematicians, mechanics and engineers for many years. The problem was considered by many authors: a historical review one can find in [7], [13], [17], [18] and [25]. Nowadays the problem has not lost its relevance (see the series of articles by Aleksandrov and his coauthors [1]-[6], by Borodich and his coauthors [8]-[14] and by Komvopoulos and his coauthors [17], [19]-[22], [26]). We would like to point out some works here. The paper [7] is devoted to the analysis of the infinitesimal deformations of a linear elastic anisotropic layer by using Stroh formalism method. The work [25] deals with the contact problem of a stiff spherical indenter with a composite plate by dint of the commercial software and the problem are simulated by a 2-D axisymmetric model. The results numerically obtained in [25] show independence of the indentation response of an orthotropic laminate from the material, the authors demonstrate dependence of the thickness of the multilayered material. In the paper [16] plane and axisymmetric contact problems for a three-layered elastic half-space are consided. A plane problem reduces to a singular integral equation with a Cauchy kernel in [16]. An analytical solution of this type of equations one can find in [23]. In turn, to obtain the solution of the integral equation the method of reduction to the corresponding conjugation problem can be used [24]. In addition, the solution of the integral equation can be obtained by numerical methods.

In the present paper, we give an analysis and numerical solution of the boundary value problem for the Lame system, modeling the contact problem for a multilayered material. By using the biquadratic basic functions, the transmission conditions are obtained on the boundaries of interlayer by the Finite Element Method and the interlayer stresses are analyzed.

2. Statement of the problem

The mathematical model of the contact problem related to the deformation of a rigid punch with a frictional pressure of a finite dimensional elastic material, which is a quadrilateral region, is expressed by the boundary value problem for the Lame equation as follows (see, for example [15]):

$$\begin{cases} \frac{\partial \sigma_{11}(u)}{\partial x} + \frac{\partial \sigma_{12}(u)}{\partial y} = F_1(x, y), \\ \frac{\partial \sigma_{12}(u)}{\partial x} + \frac{\partial \sigma_{22}(u)}{\partial y} = F_2(x, y), (x, y) \in \Omega \subset \mathbb{R}^2, \\ \begin{cases} u_2(x, 0) \leq -\alpha + \varphi(x), \ \sigma_{22}(x, 0) \leq 0, [u_2 + \alpha - \varphi] \sigma_{22}(x, 0) = 0, \\ \sigma_{12}(u) = 0, & (x, 0) \in \Gamma_0; \end{cases} \\ \end{cases}$$

$$\begin{cases} \sigma_{11}(u) = 0, \ \sigma_{12}(u) = 0, & (l_x, y) \in \Gamma_\sigma; \\ u_1(0, y) = 0, \ \sigma_{12}(u) = 0, & (0, y) \in \Gamma_1; \end{cases}$$

$$\sigma_{12}(u) = 0, \ u_2(x, -l_y) = 0, & (x, -l_y) \in \Gamma_u. \end{cases}$$
Here $\Omega = \{(x, y) : 0 < x < l_x, -l_y < y < 0\}, \ \partial \Omega = \Gamma_0 \cup \Gamma_\sigma \cup \Gamma_1 \cup \Gamma_u, \ \Gamma_0 = \{(x, 0) : 0 \leq x \leq l_x\}, \\ \Gamma_\sigma \{(l_x, y) : -l_y < y < 0\}, \ \Gamma_1 = \{(0, y) : -l_y < y < 0\}, \ \Gamma_1 = \{(x, -l_y) : 0 \leq x \leq l_x\} \text{ is the region occupied by the cross-section of the material under the influence of the punch and \ \partial \Omega = \Gamma_0 \cup \Gamma_\sigma \cup \Gamma_1 \cup \Gamma_u, \ \Gamma_0 = \{(x, 0) : 0 \leq x \leq l_x\}, \\ \Gamma_u = \{(x, -l_y) : 0 \leq x \leq l_x\} \text{ is the relevant part of the boundaries of the region. Since the condition at the \ \Gamma_0 \subset \partial \Omega \text{ boundary is given by inequality, the contact region of the punch$

 $\Gamma_c := \{(x, y) \in \Gamma_0 : u_2 = -\alpha + \varphi(x)\}$ is not certain, and therefore, even in the case of linear elasticity this problem is non-linear. In this study, the plate deformation problem of the layered material formed by different

materials with λ_k , μ_k are the Lame constants belonging to the Ω_k layer forming the Ω respectively, $u(x,y) = (u_1(x,y), u_2(x,y))$ is displacement function region, and $\sigma_{ii}(u) = \lambda_k div(u) + 2\mu_k \partial u_i / \partial x_i, \ \sigma_{ii}(u) = \mu_k (\partial u_i / \partial x_j + \partial u_i / \partial x_i), \ i, j \in \{1, 2\} \text{ are stress tensor,}$ are investigated. By using the biquadratic base functions, the transmission conditions were obtained at the $L_k := \{(x, y) \in \Omega : -l_x < x < l_x, 0 < l_k < l_y, l_0 = 0, l_{k_0} = l_y, k = \overline{1, k_0}\}$. $L_k = -\frac{1}{2} - \frac{1}{2}$ $\Omega_k \cap \Omega_{k+1}$ boundaries by the Finite Element Method and the interlayer stresses were analyzed.

 $\Gamma_0 U$

3. Results and Discussions

In order to carry out numerical experiments let us consider two examples for two layers materials: iron-copper and iron-steel. Let us refresh, that copper and steel more soft than iron. The upper layer in both examples is iron. The elasticity modules and Poisson's constants of these materials are $E_{Fe}=30000[kN/cm^2]$, $v_{Fe}=0.27$, $E_{Cu}=18100[kN/cm^2]$, $v_{Cu}=0.36$, $E_{St}=21000[kN/cm^2]$, $v_{St}=0.3$. In order to clarify the contact domain a_c we use the multigrid method. For the value of the indentation depth $\alpha=0.035$ cm and for the different thickness of layers we obtain the contact domain a_c and the values of force (P) effected the body.

	$E_{Fe} = 30000 \left[kN \right]$	cm^2], $v_{\text{Fe}} = 0.27$,	$E_{Fe} = 30000 \left[kN / cm^2 \right], v_{Fe} = 0.27,$		
	$E_{Cu} = 18100 [kN/$	cm^{2}], $v_{Cu} = 0.36$	$E_{st} = 21000 [kN/cm^2], v_{st} = 0.3$		
h _{Fe} [cm]	P.10 ²	a _c	P.10 ²	a _c	
0.1	3.4052	0.2720	3.5009	0.2649	
0.2	3.2611	0.2704	3.3830	0.2637	
0.3	3.2473	0.2689	3.3301	0.2624	
0.4	3.1462	0.2667	3.2901	0.2604	
0.5	3.1058	0.2642	3.2587	0.2581	

Table 1. The obtained values P and a_c corresponding to the different thickness h_{Fe} .



Figure 1. The graphics a_c(h) and P(h) for iron-copper and iron-steel body.

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A Note on Spacelike Surfaces via Inclined Curves as Geodesics

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Abstract

In this study, we study a spacelike surface in E_1^3 whose one of the principal curvatures is identically constant. We give some results about spacelike surfaces on which spacelike inclined curves lie as geodesic curves.

Key Words: Minkowski space, spacelike surfaces, inclined curves, geodesics.

1. Introduction

The study of special curves in surfaces is a well-examined topic in differential geometry. Using the relation between surfaces and curves lying in them, characterizations of surfaces have been given in some works. Planes, spheres and cylinders of revolution have been characterized in different manners [1, 7, 8, 9]

Tamura characterized surfaces in E^3 which contain helical geodesics under some additional conditions [8]. In [9], he generalized characterizations given by [8] to Riemannian space forms of non-negative curvatures. Görgülü and Hacısalihoğlu characterized surfaces by taking curves lying in them as inclined geodesic curves [1].

In this study, we try to see the results of [1] in Minkowski 3-space. We study spacelike surfaces and spacelike curves lying in them by using curvature properties.

2. Materials and Methods

Let M be a complete and smooth spacelike surface in Minkowski 3-space E_1^3 with the metric tensor $I = \langle .,. \rangle = dx^2 + dy^2 - dz^2$. Let $\chi(M)$ be the Lie algebra of all smooth tangent vector fields to M. Further, let ∇ and $\overline{\nabla}$ be the Levi-Civita connections of E_1^3 and M with the matric induced by the non-degnerate metric tensor $\langle .,. \rangle$, respectively. The second fundamental form II of M in E_1^3 is given by the Gauss formula $II(X,Y) = -\nabla_X Y + \overline{\nabla}_X Y$, for all

 $X, Y \in \chi(M)$ [2,4]. Let N be a unit normal vector field to M, then the shape operator S of M derived from N is a (1,1)-tensor field on M given by $\langle S(X), Y \rangle = \langle II(X,Y), N \rangle$, for all $X, Y \in \chi(M)$. It is also well-known that $D_X N = -S(X)$, for all $X \in \chi(M)$ [2,4]. Along with this study, we assume the shape operator S as a diagonalizable map. Also the mean curvature and Gauss curvature is defined as follows:

$$H = \frac{\varepsilon}{2} trace_{I}(S), \quad K = \varepsilon \det_{I}(S),$$

where $\varepsilon = \langle N, N \rangle$, and the subindex $_{I}$ denotes that these curvatures are computed due to the first fundamental form of the surface.

Definition 2.1. ([5]) *Spacelike angle*: Let x and y be spacelike vectors in E_1^3 that span a spacelike vector subspace; then we have $|\langle x, y \rangle| \le ||x|| ||y||$, and hence, there is a unique real number $\theta \ge 0$ such that $\langle x, y \rangle = ||x|| ||y|| \cos \theta$.

Let γ be a spacelike curve in E_1^3 and V_1 be the first Frenet vector field of γ . While $X \in \chi(\mathsf{E}_1^3)$

is a constant vector field, if

$$\langle V_1, X \rangle = \cos\varphi(\text{constant}),$$
 (1)

then γ is called an inclined curve (a general helix) in E_1^3 . φ is called slope angle and the space $Sp\{X\}$ is called slope axis [2]. Let

$$\gamma: I \subset E \to M \subset E_1^3$$
$$s \to \gamma(s)$$

be a spacelike inclined curve parametrized by the arc-length, then the Frenet formula for the frame field $\{V_1, V_2, V_3\}$ along the spacelike curve is given as

$$\begin{bmatrix} \nabla_{V_1} V_1 \\ \nabla_{V_1} V_2 \\ \nabla_{V_1} V_3 \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix},$$
(2)

where V_1, V_2 and V_3 denote the spacelike unit tangent, timelike principal normal and the spacelike binormal vector fields of the curve γ , respectively, and κ, τ are the curvature functions of γ [2].

Let *M* be a spacelike surface in E_1^3 and γ be a curve on *M*, and the tangent vector field of the curve γ is $\frac{d\gamma}{ds} = T$. The curve γ in \overline{M} is said to be geodesic provided [4] that

$$\overline{\nabla}_T T = 0. \tag{3}$$

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Theorem 2.2. ([3]) If E_1 , E_2 , E_3 is a principal frame field on $M \subset \mathsf{E}_1^3$, then

$$E_1[k_2] = (k_1 - k_2)\omega_{12}(E_2), E_2[k_1] = (k_1 - k_2)\omega_{12}(E_1).$$
(4)

Lemma 2.1. If two families of geodesics intersect at a constant angle everywhere on M, then M is a flat [6].

3. Results and Discussions

Characterizations of spacelike surfaces which contain spacelike inclined curves as geodesics

Theorem 3.1. Let *M* be a spacelike surface in E_1^3 and γ be a geodesic curve in *M*, then *M* has no umbilic points.

Proof. Let *M* be a spacelike surface in E_1^3 and

$$\gamma: I \subset E \to M \subset E_1^3$$
$$s \to \gamma(s)$$

be a geodesic curve in M, then

 $\overline{\nabla}_{V_1} V_1 = 0. \tag{5}$

Using (5), we obtain

$$\nabla_{V_1} V_1 = -II(V_1, V_1) \in \chi(M)^{\perp},$$
(6)

where $\chi(M)^{\perp}$ denotes the set of all smooth normal fields to M. Using (2) and (6), we have

$$II(V_1, V_1) = -\kappa V_2 \Longrightarrow \kappa = \langle II(V_1, V_1), V_2 \rangle.$$
⁽⁷⁾

This means that

 $V_{2} = N.$

By (1) and (2), we get

$$H(V_1, V_3) = -\nabla_{V_1} V_3 + \overline{\nabla}_{V_1} V_3 = -\tau V_2 \Longrightarrow \tau = \langle H(V_1, V_3), N \rangle.$$
(8)

The functions k_1 and k_2 are the principal curvatures in M, and E_1 and E_2 are the corresponding principal vector fields in M. Let θ be the angle between V_1 and E_1 . The unit tangent vector field V_1 is a spacelike vector field because of choosing the curve γ as spacelike one. We also choose arbitrarily the base vector E_1 as spacelike. After computations regarding to the angles by using the Definition 2.1, we have the following relation

$$\begin{cases} V_1 = \cos\theta E_1 + \sin\theta E_2\\ V_3 = -\sin\theta E_1 + \cos\theta E_2. \end{cases}$$
(9)

Using (2), we write

$$II(E_1, E_1) = k_1 N$$
, and $II(E_2, E_2) = k_2 N$

On the other hand, by means of the linear operator S and (15), we obtain

$$H(V_1, V_1) = \langle S(V_1), V_1 \rangle N = (-k_1 \cos^2 \theta - k_2 \sin^2 \theta) N,$$
(10)

and

$$II(V_1, V_3) = \langle S(V_1), V_3 \rangle N = (k_1 \sin \theta \cos \theta - k_2 \sin \theta \cos \theta) N.$$
⁽¹¹⁾

From (7), we find the curvature as follows:

$$c = k_1 \cos^2 \theta + k_2 \sin^2 \theta. \tag{12}$$

Similarly, from (8), we have the torsion as follows:

$$\mathbf{r} = (k_2 - k_1)\sin\theta\cos\theta. \tag{13}$$

From (12) and (13), we can express the harmonic curvature of the curve γ as

$$h = -\frac{\kappa}{\tau} = \frac{k_1 \cos^2 \theta + k_2 \sin^2 \theta}{(k_1 - k_2) \sin \theta \cos \theta},$$
(14)

then rearranging (14) gives us

$$h = -\frac{\kappa}{\tau} = \frac{\sqrt{H^2 + K}\cos 2\theta - H}{2\sqrt{H^2 + K}\sin 2\theta},$$
(15)

where $k_1 \neq k_2$, that is, *M* has no umbilic points.

Corollary 3.2. Let *M* be a spacelike flat surface in E_1^3 , then the spacelike geodesic curve γ makes a fixed angle with one of the parameter curves if and only if γ is an inclined geodesic curve in *M*.

Corollary 3.3. Let *M* be a spacelike minimal surface in E_1^3 , then the spacelike geodesic curve γ makes a fixed angle with one of the parameter curves if and only if γ is an inclined geodesic curve in *M*.

Theorem 3.4. Let M be a spacelike surface in E_1^3 whose one of the principal curvatures is identically constant. If M has a spacelike geodesic curve γ on itself which makes a fixed angle with one of the parameter curves, then M is either a plane, a sphere or a circular cylinder.

Proof. The orthogonal base $\{E_1, E_2\}$ for $\chi(M)$ can be chosen as follows:

$$\overline{\nabla}_{E_1}E_1 = \lambda E_2, \quad \overline{\nabla}_{E_2}E_2 = \mu E_1.$$

Using the connection equations

$$\overline{\nabla}_{\mathbf{v}} E_i = \sum \omega_{ij}(\mathbf{v}) E_j(\mathbf{p}),$$

we obtain

$$\overline{\nabla}_{E_1}E_2 = -\lambda E_1, \quad \overline{\nabla}_{E_2}E_1 = -\mu E_2.$$

Since γ is a geodesic in M, that is $\overline{\nabla}_{V_1}V_1 = 0$, we compute $\overline{\nabla}_{\cos\theta E_1} + \sin\theta E_2 \cos\theta E_1 + \sin\theta E_2 = 0.$ (16)

Evaluating (16), we find

$$(\sin^2\theta\mu - \sin\theta\cos\theta\lambda)E_1 + (\cos^2\theta\lambda - \sin\theta\cos\theta\mu)E_2 = 0.$$
(17)

Since $\{E_1, E_2\}$ is an orthogonal basis, we have the following equations:

$$\begin{cases} \sin^2 \theta \mu - \sin \theta \cos \theta \lambda = 0, \\ \cos^2 \theta \lambda - \sin \theta \cos \theta \mu = 0. \end{cases}$$
(18)

From the equations in (18), we obtain

$$\sin\theta\mu - \cos\theta\lambda = 0. \tag{19}$$

On the other hand, using the equations in (4) of Theorem 2.2, we have

$$E_1[k_2] = \mu(k_2 - k_1), E_2[k_1] = \lambda(k_1 - k_2).$$
⁽²⁰⁾

Taking $k_1 = \text{const.}$, by (20)₂, we arrive that

$$\lambda(k_1 - k_2) = 0, \tag{21}$$

the following cases occur:

Let $k_1 \neq k_2$, then (21) we have $\lambda = 0$, and from (19) we find that

$$\mu \sin \theta = 0 \tag{22}$$

which occurs two cases as follows:

<u>Case 1:</u> If $\mu = 0$, then we obtain

$$\overline{\nabla}_{E_1} E_1 = 0, \quad \overline{\nabla}_{E_2} E_2 = 0, \tag{23}$$

the equation (23) means that the parameter curves γ_1 and γ_2 are geodesics in M. These two

families of curves intersect at a constant angle $\frac{\pi}{2}$ since $\langle E_1, E_2 \rangle = 0$. From Corollary 3.3, *M* is spacelike flat surface. Hence *M* is a spacelike circular cylinder.

<u>**Case 2:**</u> If $\sin \theta = 0$, then $\theta = 0$, or $\theta = \pi$. As it is known that the angle θ is between V_1 and E_1 , so the vectors V_1 and E_1 become proportional since $\sin \theta = 0$. This means that M is also a spacelike flat surface from Corollary 3.3.

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On the Darboux Rotation Axis of Null Cartan Curves due to The Bishop Frame

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Abstract

In this study, the Darboux rotation axis of a null Cartan curve is obtained due to the Bishop frame in Minkowski 3-space. The axis is decomposed into two simultaneous rotations. By a simple mechanism, the axes of these simultaneous rotations are joined to each other. Also, some characterizations of null Cartan Darboux helices are given due to the Bishop frame. **Key Words:** Bishop frame, Null Cartan curve, Darboux rotation axis, Null Cartan Darboux Helix, Null slant helix.

1. Introduction

Null curves have a crucial place in physics [2]. Geometrical particle model is wholly based on the geometry of null curves in Minkowski spacetime. So the wave equations correspond to massive spinning particles of a spin by quantization [4]. Also null curves have been identified with geometrical particle models in Minkowski 3-space [9].

The study of pseudo null and null Cartan curves have been recently done via the Bishop frame by Grbovic and Nesovic [5]. As it is known, the construction of the Bishop frame dates back to the paper of R.L. Bishop in [1]. Bishop (parallel transport frame) frame is well defined and constructed in Euclidean and its ambient spaces [1, 3, 5].

This work consists of two sections. In the first section, the Darboux rotation axis of a null Cartan curve is obtained due to the Bishop frame. In the second section, some characterizations of null Cartan Darboux helices are given due to the Bishop frame.

2. Materials and Methods

The three dimensional Minkowski space E_1^3 is a real vector space E^3 endowed with the standard indefinite flat metric \langle , \rangle defined by

$$\langle , \rangle = -x_1 y_1 + x_2 y_2 + x_3 y_3, \tag{1}$$

where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ are any two vectors in E_1^3 . The pseudo-norm of an arbitrary vector $x \in \mathsf{E}_1^3$ is given by $||x|| = \sqrt{|\langle x, x \rangle|}$. Similarly, an arbitrary curve $\gamma = \gamma(s)$ in E_1^3 can locally be spacelike, timelike or null (lightlike) if its velocity vector γ' are, respectively, spacelike, timelike or null (lightlike), for every $s \in I \subset \mathsf{E}$. The curve $\gamma = \gamma(s)$ is called a unit speed curve if its velocity vector γ' is unit one i.e., $||\gamma'|| = 1$ [6].

A curve $\gamma: I \to \mathsf{E}_1^3$ is said to be a null curve provided that its tangent vector $\gamma' = T$ is a null vector. A null curve γ is called a null Cartan curve if it is given by the pseudo-arc parameter *s* defined by

$$s(t) = {}^{t}_{0} \sqrt{\left\| \gamma^{''}(u) \right\|} du.$$
⁽²⁾

There is only one Cartan frame $\{T, N, B\}$ along a non-geodesic null Cartan curve γ satisfying the Cartan equations

$$\begin{bmatrix} T'\\N'\\B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0\\-\tau & 0 & \kappa\\0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T\\N\\B \end{bmatrix},$$
(3)

where the curvature $\kappa(s) = 1$ and the torsion $\tau(s)$ is an arbitrary function in pseudo-arc parameter s [4].

The Bishop frame $\{T_1, N_1, N_2\}$ of a non-geodesic null Cartan curve in E_1^3 is positively oriented pseudo-orthonormal frame consisting of the tangential vector field T_1 , relatively parallel spacelike normal vector field N_1 and relatively parallel null transversal vector field N_2 [5].

Let γ be a null Cartan curve in E_1^3 parameterized by pseudo-arc *s* with the curvature $\kappa(s) = 1$ and the torsion $\tau(s)$. Then the Cartan equations of γ according to the Bishop frame are given as follows:

$$\begin{bmatrix} T_1' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} \kappa_2 & \kappa_1 & 0 \\ 0 & 0 & \kappa_1 \\ 0 & 0 & -\kappa_2 \end{bmatrix} \begin{bmatrix} T_1 \\ N_1 \\ N_2 \end{bmatrix},$$
(3)

where the first Bishop curvature $\kappa_1(s) = 1$ and the second Bishop curvature satisfies Riccati differential equation $\kappa_2'(s) = -\frac{1}{2}\kappa_2^{-2}(s) - \tau(s)$; the Bishop frame $\{T_1, N_1, N_2\}$ also satisfies the following relations:

and $T_1 \times N_1 = -T_1$, $N_1 \times N_2 = -N_2$, $N_2 \times T_1 = N_1$.

3. Results and Discussions

On Darboux Rotation Axis of Null Cartan Curves due to The Bishop Frame in E_1^3

Let γ be a null Cartan curve framed by the Bishop frame $\{T_1, N_1, N_2\}$ in Minkowski 3-space E_1^3 . Then via the Bishop frame, the angular momentum vector is found as

$$\partial = \kappa_2 N_1 + \kappa_1 N_2. \tag{5}$$

Therefore, the vectors N_1 and N_2 rotate with κ_1 and κ_2 angular speeds around the N_2 and N_1 , respectively. The norm of (5) is

$$\left\|\partial\right\| = \sqrt{\kappa_2^2} = \left|\kappa_2\right|$$

So, the Darboux vector is obtained as

$$\frac{\partial}{\left\|\partial\right\|} = \frac{\kappa_2 N_1 + \kappa_1 N_2}{\left|\kappa_2\right|} \,. \tag{6}$$

Then, the vector $(\frac{\partial}{\|\partial\|})'$ with the help of classical methods can be written as the linear combination of vectors T_1, N_1, N_2 , that is,

$$\left(\frac{\partial}{\left\|\partial\right\|}\right)' = -\frac{\varepsilon(\partial)\kappa_{2}'}{\kappa_{2}^{2}}N_{2},$$

where $\varepsilon(\partial) = sgn(\|\partial\|)$. This vector is also expressed as $(\frac{\partial}{\|\partial\|})' = wN_1 \times \frac{\partial}{\|\partial\|}$,

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where w is found as $w = \frac{\kappa_2'}{\kappa_2}$. Then, the vector $\frac{\partial}{\|\partial\|}$ rotates with angular speed w around vector N_1 . Further, the vector N_1 rotates with angular speed $\|\partial\|$ around $\frac{\partial}{\|\partial\|}$ due to the equation $N_1' = \partial \times N_1$. We obtain unit vector e from Darboux vector $\frac{\partial}{\|\partial\|}$ where $e = \frac{\partial}{\|\partial\|}$. So, we find a vector $e \times N_1$ which is orthogonal both e and N_1 .

4. On null Cartan Darboux helices due to the Bishop frame in E_1^3

Theorem. A null Cartan curve $\gamma: I \to \mathsf{E}_1^3$ framed by the Bishop frame $\{T_1, N_1, N_2\}$ is a Darboux helix if and only if the second Bishop curvature κ_2 is a non-zero constant or the axis is as in the form $U = -\langle T_1, U \rangle N_2$.

Proof. Suppose that γ is a null Cartan curve due to the Bishop frame $\{T_1, N_1, N_2\}$, a non-zero vector $U \in \mathsf{E}_1^3$ is given by $U(s) = \lambda_1 T_1 + \lambda_2 N_1 + \lambda_3 N_2$. Differentiating this equation with respect to s and using

The vector U is found as

$$U(s) = -e^{-\int \kappa_2 ds} T_1 + \int e^{-\int \kappa_2 ds} ds N_1 - \frac{1}{2} e^{-\int \kappa_2 ds} (\int e^{-\int \kappa_2 ds} ds)^2 N_2.$$
(7)

Furthermore, the equations (5) and (7) imply

$$\langle \partial, U \rangle = c_0 = constant.$$
 (8)

The null Cartan curve γ is a Darboux helix with an axis U according to the Definition of Darboux helix. Conversely, assume that γ is a null Cartan Darboux helix with an axis U. Then $\langle \partial, U \rangle = constant$. Differentiating (7) with respect to s and using (3), we find $\kappa'_2 \langle N_1, U \rangle = 0$.

It follows that $\kappa_{2}^{'} = 0$ or $\langle N_{1}, U \rangle = 0$. If

$$\langle N_1, U \rangle = 0,$$
(9)

differentiating (9) with respect to s and using (3) gives

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$$\left\langle N_2, U \right\rangle = 0. \tag{10}$$

With respect to the Bishop frame $\{T_1, N_1, N_2\}$, the axis U of γ can be decomposed as $U = -\langle N_2, U \rangle T_1 + \langle N_1, U \rangle N_1 - \langle T_1, U \rangle N_2$ (11)

Subtituting (9) and (10) into (11), we find $U = -\langle T_1, U \rangle N_2$.

Therefore $\kappa_{2} = 0$ and hence $\kappa_{2} = constant \neq 0$ or $U = -\langle T_{1}, U \rangle N_{2}$.

Corollary 1. Every null Cartan curve $\gamma: I \to \mathsf{E}_1^3$ with non-zero constant second Bishop curvature due to the Bishop frame is a Bishop slant helix with non-null axis.

Corollary 2. Every null Cartan slant helix $\gamma: I \to \mathsf{E}_1^3$ with non-zero constant second Bishop curvature is a Darboux helix due to the Bishop frame.

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A New Numerical Solutions for Fractional (1+1)-Dimensional Biswas-Milovic Equation

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Abstract

In this work, fractional (1+1)-dimensional Biswas-Milovic equation that defines the long-space optical communications solved by using the residual power series method (RPSM). The RPSM gets Maclaurin expansion of the solution. The solutions of present equation are computed in the shape of quickly convergent series with quickly calculable fundamentals by using mathematica software package. Explanation of the method is given graphical consequents and series solutions are made use of to represent our solution. The found consequents show that technique is a power and efficient method in conviction of solution for the fractional (1+1)-dimensional Biswas-Milovic equation.

Keywords: Residual power series method, (1+1)-dimensional Biswas-Milovic equation, Series solution.

1. Introduction

Fractional calculus and differential equations have become the focus of many recent studies, because of implementations in many fields [1-5]. Recently, there has been a significant analytical improvement in fractional differential equations and its applications. For some articles for fractional differential and fractional calculus equations, see [6-10].

In the present study, we apply RPSM to find powerful series solution for a fractional problem. The new method supplies the solution in the shape of a convergence series. An repeated algorithm is constituted for the designation of the infinite series solution [11-14].

In this work, we take up the Residual power series method to fractional Biswas-Milovic equation (FBME) for finding its numerical solutions. The FBME has as follows

$$iD_{t}^{\alpha}u^{m}(x,t) + \delta_{c}^{x}D_{0+}^{2\beta}u^{m}(x,t) + \eta |u(x,t)|^{2}u^{m}(x,t) = 0, \qquad (1.1)$$
$$x \in R, t > 0, 0 < \alpha, \beta \le 1,$$

where δ and η are two constants with $\delta \eta > 0$, *m* is a value with $m \ge 1$ and *D* is a Caputo sense derivative and $i = \sqrt{-1}$. For some articles for this equation, see [15-16]. We will find series solutions of Eq. (1.1) for m = 3, numerically by the RPSM. So far as we know, this is the first time which the state is conceived for Eq. (1.1).

2. Numerical Solutions of the FBME with RPSM algorithm

Firstly, we study the FBME for m = 3,

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$$iD_{t}^{\alpha}u^{3}(x,t) + \delta_{c}^{x}D_{0+}^{2\beta}u^{3}(x,t) + \eta |u(x,t)|^{2}u^{3}(x,t) = 0, \qquad (2.1)$$

by the initial condition $u(x,0) = e^{ix}$.

The RPSM propose the solution for Eqs. (2.1) with a fractional power series at point t = 0 [11]. Theorize that the solution selects the expansion shape,

$$u(x,t) = \sum_{n=0}^{\infty} f_n(x) \frac{t^{n\alpha}}{\Gamma(1+n\alpha)} \quad 0 < \alpha \le 1, x \in I, 0 \le t < R.$$
(2.2)

Next, we let $u_k(x,t)$ to refer k. truncated series of u(x,t),

$$u_{k}(x,t) = \sum_{n=0}^{k} f_{n}(x) \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}, \quad 0 < \alpha \le 1, x \in I, 0 \le t < R.$$
(2.3)

where $u_0(x,t) = f_0(x) = u(x,0) = f(x)$.

At first, to find the value of coefficients $f_n(x)$, $n = \overline{1, k}$ in series expanded of Eq.(2.3), we define residual function *Res*; for Eq.(1.1) by

$$Res(x,t) = i \frac{\partial^{\alpha} u^{3}(x,t)}{\partial t^{\alpha}} + \delta \frac{\partial^{2\beta} u^{3}(x,t)}{\partial x^{2\beta}} + \eta |u(x,t)|^{2} u^{3}(x,t)$$

and the k -th residual function, Res_k as follows:

$$Res_{k}(x,t) = i \frac{\partial^{\alpha} u_{k}^{3}(x,t)}{\partial t^{\alpha}} + \delta \frac{\partial^{2\beta} u_{k}^{3}(x,t)}{\partial x^{2\beta}} + \eta |u_{k}(x,t)|^{2} u_{k}^{3}(x,t), \ k = 1,2,3,...$$
(2.4)

As in [11-12], it is clear that Res(x,t) = 0 and $\lim_{k \to \infty} Res_k(x,t) = Res(x,t)$ for each $x \in I$ and $t \ge 0$.

Then, $D_t^{r\alpha} Res(x,t) = 0$, fractional derivative of a stationary in the Caputo's idea is zero and the fractional derivative $D_t^{r\alpha}$ of Res(x,t) and $Res_k(x,t)$ are pairing at t=0 with each $r=\overline{0,k}$. To give residual PS algorithm: Firstly, we replace the k-th truncated series of u(x,t) into Eq.(2.1). Secondly, we find the fractional derivative expression $D_t^{(k-1)\alpha}$ of both $Res_{u,k}(x,t)$, $k=\overline{1,\infty}$ and finally, we can solve found system

$$D_t^{(k-1)\alpha} \operatorname{Res}_{u,k}(x,0) = 0, 0 < \alpha \le 1, x \in I, k = \overline{1,\infty}.$$
(2.5)

to obtain the needed coefficients $f_n(x)$ for $n = \overline{1, k}$. in Eq. (2.5). Hence, to determine $f_1(x)$, we write k = 1 in Eq. (2.4),

$$Res_{1}(x,t) = i \frac{\partial^{\alpha} u_{1}^{3}(x,t)}{\partial t^{\alpha}} + \delta \frac{\partial^{2\beta} u_{1}^{3}(x,t)}{\partial x^{2\beta}} + \eta |u_{1}(x,t)|^{2} u_{1}^{3}(x,t),$$
(2.6)

where

$$u_1(x,t) = \frac{t^{\alpha}}{\Gamma(1+\alpha)} f_1(x) + f(x)$$

for $u(x,0) = f_0(x) = f(x) = u(x,0) = e^{ix}$.

Therefore,

$$Res_{1}(x,t) = if_{1}(x)(3e^{2ix} + 3e^{ix} \frac{t^{\alpha}}{\Gamma(1+\alpha)} f_{1}(x) + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} f_{1}(x)^{2}) + \eta(e^{ix} + \frac{t^{\alpha}}{\Gamma(1+\alpha)} f_{1}(x))^{3}(1+2\frac{t^{\alpha}}{\Gamma(1+\alpha)} \cos(x)f_{1}(x) + \frac{t^{2\alpha}}{\Gamma(1+\alpha)} f_{1}(x))^{3}(1+2\frac{t^{\alpha}}{\Gamma(1+\alpha)} \cos(x)f_{1}(x) + \frac{t^{2\alpha}}{\Gamma(1+\alpha)} f_{1}(x))^{3}(1+2\frac{t^{\alpha}}{\Gamma(1+\alpha)} \cos(x)f_{1}(x) + \frac{t^{2\alpha}}{\Gamma(1+\alpha)} f_{1}(x))^{3}(1+2\frac{t^{\alpha}}{\Gamma(1+\alpha)} f_{1}(x))^{3}(1+2\frac{t^{\alpha}}{\Gamma(1+\alpha)} \cos(x)f_{1}(x) + \frac{t^{\alpha}}{\Gamma(1+\alpha)} f_{1}(x))^{3}(1+2\frac{t^{\alpha}}{\Gamma(1+\alpha)} f_{1}(x))^{3}(1+2\frac{t^{\alpha}}{\Gamma(1+\alpha)} \cos(x)f_{1}(x) + \frac{t^{\alpha}}{\Gamma(1+\alpha)} f_{1}(x))^{3}(1+2\frac{t^{\alpha}}{\Gamma(1+\alpha)} \cos(x)f_{1}(x) + \frac{t^{\alpha}}{\Gamma(1+\alpha)} f_{1}(x))^{3}(1+2\frac{t^{\alpha}}{\Gamma(1+\alpha)} f_{1}(x))^{3}(1+2\frac{t^{\alpha}}{\Gamma(1+\alpha)} \cos(x)f_{1}(x) + \frac{t^{\alpha}}{\Gamma(1+\alpha)} f_{1}(x))^{3}(1+2\frac{t^{\alpha}}{\Gamma(1+\alpha)} f_{1}(x))^{3}(1+$$

From Eqs. (2.5) we deduce that $Res_1(x,0) = 0$ at t = 0 and thus,

$$f_1(x) = \frac{1}{3}ie^{ix}(\eta + \delta e^{i\pi\beta}).$$
 (2.7)

Therefore,

$$u_{1}(x,t) = \frac{1}{3}ie^{ix}(\eta + \delta e^{i\pi\beta})\frac{t^{\alpha}}{\Gamma(1+\alpha)} + e^{ix}.$$
 (2.8)

Likewise,

$$u_{2}(x,t) = e^{ix} + \frac{1}{3}ie^{ix}(\eta + \delta e^{i\pi\beta})\frac{t^{\alpha}}{\Gamma(1+\alpha)} - \frac{2}{9}e^{ix}(\eta + \delta e^{i\pi\beta})^{2}\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}.$$

$$u_{3}(x,t) = e^{ix} + \frac{1}{3}ie^{ix}(\eta + \delta e^{i\pi\beta})\frac{t^{\alpha}}{\Gamma(1+\alpha)} - \frac{2}{9}e^{ix}(\eta + \delta e^{i\pi\beta})^{2}\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} - \frac{1}{81}ie^{ix}(\eta + \delta e^{i\pi\beta})^{2}(23\eta + 14\delta e^{i\pi\beta}).$$
(2.9)
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To give a deficit overview of the content of our work, by the above recurrent connections, we can demonstrate some graphical consequents of equation (2.1)



Fig 1. The 3D graphic for the $u_3(x,t)$ approximate solution of the FBME (2.1) and its contour for $\alpha = \beta = 0.9, \delta = 1$ and $\eta = 2$. (a) Real division, (b) Contour of real division, (c) Imaginary division and, (d) Contour of imaginary division.



Fig 2. Solution of real part for the FBME when $\alpha = 0.5, 0.6, 0.7, 0.8, 0.9$ (t = 0.3 and $\beta = 0.5, \delta = 1$ and $\eta = 2$)

In figure 2, we plot the RPS approximate solution $u_3(x,t)$ for $\alpha = 0.5 - 0.9$ which are closing the exact solution as the number of α increase. These figure clear that the convergency of the approximate solutions to the exact solution related to the order of the solution and the exact error is being smaller as the order of the solution is increasing.

3. Conclusions

In this paper, we obtained approximate solutions that is given in the shape of power series of the FBME based on a fractional Caputo sense derivative and Residual power series method. It has been establish that the structure of this RPS method obsesses a very fast convergent series with easily calculable components using symbolic calculation software. The paper stressed our notion that the introduced process can be applied as an instead to get analytic solutions of different kinds of fractional linear and nonlinear partial differential equations practiced in mathematics, physics and engineering.

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Didactic conceptual sheets as complementary approach in flipped learning: Statistics and Biology

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Abstract

In the spirit of closing the gap between "classroom and distance learning", we propose a method to overcome some of the challenges inherent in teaching statistics to students enrolled in blended learning. Among the teaching techniques identified as essential, we have articulated most of our strategy to a "Flipped Learning" model. Building on existing research, we have developed a complementary approach that use the results of many researches in Flipped Learning, ICT and Didactics. This complementary approach, articulated on didactic conceptual sheets, has been used in teaching descriptive statistics for students in biology (Biostatistics).

In this article, we describe the activities developed for the implementation of didactic conceptual sheets in a "Flipped Learning" course. We have distinguished the effectiveness of this learning opportunity to help students improve their understanding of concepts related to statistics and biology. We have also identified concepts and misconceptions that need to be highlighted and clarified in a biostatistics course.

In light of our observations, we recommend a complementary training strategy (didactic concept sheets) that can be used in an interdisciplinary approach that articulates Mathematics-Biology in a flipped learning model.

Keyword(s): Interdisciplinarity (Math–Biology), Flipped learning, Didactic Conceptual Sheets, Misconceptions, Teacher practices

1. Introduction

Blended learning ("BL") is an education program that combines online digital media with traditional classroom methods. In "BL" approach student learns at least in part through delivery of content and instruction via digital and online media with some elemont of student

control over time, place, path, or pace. "BL" integration has been transforming higher education to provide more engaged learning experiences for students. "BL" is often combined with a Flipped Learning ("FL"). This "FL" approach is a pedagogical model in which the typical lecture and homework elements of a course are reversed. Students view short video lectures or other mulitmedia content asynchronously before the class session. Then in-class time is devoted to active learning such as discussions, project-based or problem-based assignments, or laboratory exercises. This teaching model allows instructors to guide student learning by answering student questions and helping them apply course concepts during class time. Activities that have been traditionally assigned as homework are now done in class with the instructor's support.

The present study was conducted at the Higher Institute of Education and Continuous Training (ISEFC). One of its programs is to provide continuous training face to face and/or non-face training. Driven by the demand to increase access to learning opportunities, educators were continually challenged to develop and integrate instructional delivery options, one of which was "BL" and more specifically "FL". This was a first-time experience for the faculty at the institution. In this context of learning and in the spirit of closing the gap between "classroom and distance learning", we propose a method to overcome some of the challenges inherent in teaching statistics to students enrolled in our "FL" model. Building on existing research, we have developed a complementary approach that use the results of many researches in "FL" (Garrison & Kanuka (2004), Rotellar & Cain, (2016)), ICT and Didactics. This complementary approach, articulated on biostatistics for biology students.

2. Materials and Methods

2.1. Student profile and targeted course

Our project was prepared and carried out from 2006-2018 in biostatistics course (descriptive statistics), in biology bachelor's degree (licence) at the ISEFC - Tunisia. (Table 1)

2.2. Learning model

Most important part of our "FL" model is a loop involving multiple processes and steps, which can be made effective with proper use of digital tools. Here's a list of tools and process that we use in our "FL" classrooms, categorized by 5 procedures in the loop: Content Source;

Virtual Learning Environment; Didactic Conceptual Sheets "DCS"; Involvement and participation of teacher and students; Communication tools, review and survey.

- Content sources:

* Supports and documents: To start the "FL", we used, and made available to students, our self-courses and selected resources such PDF, Excel Word and PowerPoint documents, website contents and pre-recorded lectures and videos.

* Disciplinary fields and key concepts: Teaching biostatistics is an example of integrating interdisciplinary learning activities (mathematics, statistics and biology). In this project, we have distinguished the effectiveness of this learning opportunity to help students improve their understanding of concepts related to statistics and biology. We have also identified concepts and misconceptions that need to be highlighted and clarified in a biostatistics course.

- Virtual Learning Environment: To create our lesson and share courses contents we used "Virtual Learning Environment" ("VLE") (usually: is Web-based; uses Web 2.0 (interactive / social Web / read and write Web) tools and simulates real-world educational modalities). According to this "VLE" we mainly used a Learning Management System (LMS) and a Wiki. We have privilegied: Moodle as the LMS to create an organized content and track the progress rate of the learners during all the flipped learning process. (Didaquest, 2018). Mediawiki as the Wiki interface to implement the "DCS" as complementary approach in "FL" and to create an expandable collection of interlinked web pages that allows any user to quickly and easily consult, create and edit content.

- Didactic conceptual sheets "DCS": We have combined the "Content sources" to "DCS" as a complementary approach to "FL", and implement it in our Wiki. The constituent elements of "DCS" are summarized in table 2 and further explained in Didaquest, 2018.

- Involvement and participation of teacher and students: When carrying out our flip learning model we respect 4 pillars and 11 indicators recommended (FLN, 2014). We combined this learning strategy to a didactics learning initiative that implements "DCS" to optimize the courses contents and to make learning and training activities more dynamic.

- Communication tools, review and survey: We have implemented in our VLE digital communications tools so as to come up with a compelling and more productive learning experience. Such tools empower the students allowing them to communicate, to engage, share their views, connect with the work of other students and edit a work collectively. These tools

also enabled us to follow the evolution of students' learning while carrying out several formatives evaluations and by collecting their assessments on the supports and tools used. The used Moodle and Wiki learning interface have several built-in evaluation tools such: counter which record usage, classifies visits by day, by visitor, most visited sections, duration, etc. We will present a brief summary of the results and specifically theses concerning the students' degree of satisfaction and the utility they showed in terms of the constituent elements of the didactical concept sheets (Table 3). In this activity students will rank components of "**DCS**".

Table 1: Student profile					Table 2 : Elements of a didactic concept sheet				
Year	Gender		Total		1. Translation / 2. Definitions: 2.1 Domain,				
	Woman	Man			Discipline, Thematic; 2.2 Written definition; 2.3				
2006 - 2007	13	7	20		Graphical definition / 3. Concepts or related				
2008 - 2009	15	8	23		concepts / 4. Examples, applications, uses / 5.				
2010 - 2011	12	8	20		Possible errors or confusions / 6. Possible				
2012 - 2013	13	11	24		questions / 7. Teaching Links and Programs: 7.1				
2014 - 2015	14	8	22		Ideas or Reflections related to his teaching; 7.2 Help				
2017 - 2018	17	6	23		and tips; 7.3 Education: Other links, sites or portals;				
Total	84	48	132		8. Bibliography				
Table 3: Rating and ranking scale questions									
Rating question: Student useful satisfaction					Ranking scale question using a scale of 1 to 12 for				
survey					student satisfaction survey.				
Please rate each of the following components on a					Please rank the following in order of importance from				
rating scale of 1-5, where 1 is 'not at all useful'					1 to 12 where 1 is most important to you and 12 is				
and 5 is 'very useful.'")					least important to you (justify your choices)				

3. Results and Discussions



Succinct overview and brief analysis of the results shows that all the students appreciate the use of "**DCS**". Their usefulness is more pronounced when the elements present practical aspects that help them in their learning, such as misconceptions, questions / answers, helps and tips. This study revealed many important elements, as enumerated in Didaquest (2018),

the most significant of which are: Highlighting major misunderstandings in biostatistics. And the importance of using examples from biology to clarify statistics contents. All the justifications given by students confirm these results (Didaquest, 2018). Moreover, and in reference to statistics' consultations provided by our Moodle and Wiki interfaces, the "DCS" are the most consulted elements, before, during and after the face-to-face learning sessions. In order of consultation, the questions / answers, then the conceptions and the graphical ones alone account for more than 63% of the consultations. It is clear that the "DCS", as well as helpful and appreciated, are also very used during the learning sessions related to the biostatistics contents. As a student pointed out, these didactic concept sheets are for her like a coach who accompanies and assists her during the learning phases.

4. Conclusions

The main advantage of the "DCS" is its large diversity of constitutif elements related to each concept, which can be exploited or rapident completed by the students through the wiki interface. Using the "DCS", we show that many students were attracted by popular misconceptions and questions related to biostatistics. In some cases, this appears to be due to failure to emphasize the relevance of knowledge from mathemathics instruction to biological contexts (Schwartz and al.,2016). The information gained from this study helped and will further be used to adapt the current biostatistics learning strategy. According to the results, and particularly the importance of their use, "DCS" seem to be a complementary tool in blended learning and particularly in flipped learning dedicated to biostatistics. Indeed, in addition to boosting and optimize content and learning sessions they seem to play a dynamic role as a coach that can meet the needs and expectations of students.

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Some New Quantum Codes over Nonbinary fields

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Abstract

In this paper, using CSS (Calderbank-Shor-Steane) construction method, we construct self-orthogonal codes over F_3 and we obtain new quantum codes with the following parameters: [[18,6,4]]₃, [[24, 0, 9]]₃, [[27, 15, 4]]₃, [[28, 20, 3]]₃, [[30, 22, 3]]₃, and [[32,24,3]]₃.

Keyword(s): quantum code, linear code, self-orthogonal code

1. Introduction

All notations about linear codes we take from [12, 16]. Let F_q^n be the *n*-dimensional vector space over the Galois field $F_q = GF(q)$. The Hamming weight of a vector $x \in F_q^n$, written wt(x), is the number of nonzero entries of x, and Hamming distance d(x,y) between two vectors $x, y \in F_q^n$ is defined to be the number of coordinates in which they differ. A *q*-ary linear [n,k,d] code C is a k-dimensional linear subspace of F_q^n with minimum Hamming distance d. A generator matrix of a code C is any $k \times n$ matrix whose rows are a basis of the code. A weight enumerator of a code C is the polynomial

$$C(z) = \sum_{i=0}^{n} A_i z^i$$

where A_i is the number of codewords of weight *i*.

The inner product is $(x,y) = x_1.y_1 + \ldots + x_n.y_n$, for two vectors $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in F_q^n . The *dual code* C^{\perp} of the code *C* (with respect to this inner product) is $C^{\perp} = \{v \in F_q^n | (u, v) = 0, \forall u \in C\}$. It is known that C^{\perp} is an [n, n-k] code. We say that a code *C* is *self-orthogonal* if the code C^{\perp} contains the codewords of *C*. If $C = C^{\perp}$ then the code *C* is *self-dual*.

The self-orthogonal q-ary linear codes are useful in order to construct quantum errorcorrecting codes (QECCs) [6]. In the last two decades several papers (see [2, 3, 7, 10, 13, 14]) were devoted to the construction of QECCs over different fields by using their connections to self-orthogonal q-ary codes. In this work, by constructing ternary self-orthogonal codes with dual distance at least 3, we improve some bounds on minimum distance of nonbinary QECCs. The paper is structured in the following way. Section 2 consists of definitions and general information about quantum computing and quantum codes. In Section 3 we describe the relationship between self-orthogonal codes over F_q and quantum codes and present the known method for constructing linear codes by their residual codes. In the last section, we summarize the obtained results.

2. Quantum information and Quantum codes

In [17] P.Shor gave a randomized algorithm for factorizing an integer in polynomial time on a quantum computer. Since factorization of large integers is the hard problem which underpins public-key encryption systems such as RSA, the importance of this result is obvious, and a large amount of research into the possibility of building a quantum computer is going on. The relationship between quantum information and classical information is a subject currently receiving much study. While there are many similarities, there are also substantial differences between the two. Classical information cannot travel faster than light, while quantum information appears to in some circumstances. Classical information can be duplicated, while quantum information cannot [6].

Otherwise, in a classical computer, each bit of information is stored by a transistor containing trillions of electrons. On a quantum computer, a single electron or nucleus in a magnetic field carries a bit of information. Interaction with the environment is much more serious, but decoherence puts a limit on the space and time resources available to a quantum computer.

Binary case: A quantum analogue of a bit of information is called a *qubit*. It is the state of a system in a 2-dimensional Hilbert space \mathbb{C}^2 . The two possible states of a qubit are labeled $|0\rangle$ and $|1\rangle$. To these two basis states orthonormal basis vectors can be associated $|0\rangle = e_0 = \binom{1}{0}$ and $|1\rangle = e_1 = \binom{0}{1}$. A qubit can be in a superposition of these two states $|\varphi\rangle = \alpha |0\rangle + \beta |1\rangle$, where $\alpha^2 + \beta^2 = 1$. An error, like any physical process, is a unitary transformation of the state space. The space of errors to a single qubit is 4-dimensional, and is spanned by the four unitary matrices (Pauli operators):

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad Y = iXZ$$

Nonbinary case: A *qudit* is a generalization of the qubit to a *q*-dimensional Hilbert space \mathbb{C}^q . For example, a qutrit (q = 3) is a three-state quantum system. The computation basis is then a set of three (orthogonal) states $|0\rangle$, $|1\rangle$ and $|2\rangle$ and an arbitrary qutrit is a linear combination of these three states $|\varphi\rangle = \alpha |0\rangle + \beta |1\rangle + \gamma |2\rangle$. A representation of an integer k in a qutrit system can be made by writing k in its ternary representation (for example, 73 = $2201_{(3)}$) and this can be encoded in a register of qutrits. The generalization of that to *q*-dimensional Hilbert space is easy. In this case the Pauli operators for a *q*-dimensional Hilbert space are defined by their action on the computational basis [11]:

$$X^{(q)}|j\rangle = |j+1\rangle, \ Z^{(q)}|j\rangle = |\omega^{j}\rangle$$

where $j \in F_q^n$ and ω is a primitive q-th root of unity. The matrix representations of X and Z for the qutrit are

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2\pi i/3} & 0 \\ 0 & 0 & e^{4\pi i/3} \end{pmatrix}$$

Definition 1 [10, 13]: Let $V = \bigotimes^n (\mathbb{C}^q) = \mathbb{C}^q \bigotimes \mathbb{C}^q \bigotimes \dots \bigotimes \mathbb{C}^q$, $\dim V = q^n$. A quantum code *C* is a *k*-dimensional subspace of *V*. If the dimension of the quantum code $\underset{\text{GD}}{\text{is } q^k}$, it will

be denoted by $C = [[n, k, d]]_q$, where *d* is the minimum distance. A code with minimum distance *d* is able to correct errors that affect no more than $\left|\frac{d-1}{2}\right|$ of the states.

3. The connection between self-orthogonal codes and Quantum codes

As it is too difficult to construct good quantum codes in general, we can consider simpler constructions. An interesting and simple construction of quantum codes was introduced in 1996 by Calderbank and Shor [5] and by Steane [18]. The CSS code construction provides a direct link to classical coding theory. The problem of finding QECCs is transformed into the problem of finding linear self-orthogonal codes under a certain inner product over the finite field with 2 elements. Later, this construction was generalized for quantum codes over different fields by Ketkar et. all [13].

Theorem 1 [5, 13]: Let C_1 and C_2 denote two classical linear codes with parameters $[n, k_1, d_1]_q$ and $[n, k_2, d_2]_q$ such that $C_2 \subseteq C_1$. Then there exists a $[[n, k_1 + k_2 - n, d]]_q$ quantum code with minimum distance $d = \min\{wt(c) \mid c \in C_1 \setminus C_2 \cup C_2 \setminus C_1\}$.

Corollary 1 [13]: If C is a classical linear $[n, k, d]_q$ code containing its dual, $C^{\perp} \subseteq C$, then there exists an $[[n, 2k - n, d]]_q$ quantum code.

In order to construct easier the proper self-orthogonal linear codes we use their *residual codes*.

Definition 2: Let G be a generator matrix of a linear $[n, k, d]_q$ code C. Then the residual code Res(C, c) of C with respect to a codeword c is the code generated by the restriction of G to the columns where c has a zero entry.

A lower bound on the minimum distance of the residual code is given by

Theorem 2[8]:Suppose C is an $[n, k, d]_q$ code and suppose $c \in C$ has weight w, where d > w(q - 1)/q. Then Res(C, c) is an $[n - w, k - 1, d']_q$ code with $d' \ge d - w + \left|\frac{w}{q}\right|$.

If w = d then $\operatorname{Res}(C, c)$ is an $[n - d, k - 1, d']_q$ code with $d' \ge \left\lfloor \frac{d}{q} \right\rfloor$. Inverting this operation we start from $[n - d, k - 1, d']_q$ residual code and search for an $[n, k, d]_q$ code. The other approach we use is to start from $[n - i, k, d']_q$ code (punctured code) and by extending it we search for an $[n, k, d]_q$ code.

Example 1: $[4,4,1]_3 \rightarrow [6,4,2]_3 \rightarrow [9,5,4]_3 \rightarrow [18,6,9]_3$

Example 2: $[2,2,1]_4 \rightarrow [3,2,2]_4 \rightarrow [8,3,5]_4 \rightarrow [28,4,20]_4$

4. Results

Using the method described in the previous section, when we search for an $[n, n-k, d]_q$ code (with dual distance at least given d^{\perp}) we construct residual codes for the code with these parameters. After that, we use the generator matrices of the obtained residual codes and we try to construct an $[n, n-k, d]_q$ code with dual distance at least d^{\perp} . Here we use the fact that the dual distance of the residual code also must be at least d^{\perp} [8, 16]. If the construction is successful we check the obtained code for self-orthogonality. As is shown in Example 1, when we search for $[18,6,9]_3$ code with dual distance 4, we construct $[9,5, \geq 3]_{d6}$ codes with $d^{\perp} \geq 4$ from their residual $[6,4, \geq 2]_3$ codes with $d^{\perp} \geq 4$ (extending the trivial $[4,4,1]_3$ code).

It is easier to construct linear codes with smaller dimensions because of the relatively small number of their codewords. In the cases for large dimension it is difficult to find all codewords in order to calculate their weights (i.e., to find the weight enumerator). In that case it is enough to check the number of the linearly independent columns of the generator matrix of the code. It is known [12, 16] that if for a given value d^{\perp} there are $d^{\perp} - 1$ linearly independent columns then the dual distance of the code is at least d^{\perp} . For these reasons, in our research we construct codes with smaller dimensions and check their dual distance.

Table 1 consists of the obtained results for self-orthogonal codes and the corresponding nonbinary quantum codes. The obtained codes are different from the parameters given in the papers [3, 9, 10, 15]. Also, by this method we construct some codes with the same parameters as in [3, 9] (for example [[20,10,4]]₃, [[20,12,3]]₃, [[24,16,3]]₃, [[26,18,3]]₃). All computer calculations were made by the program package Q-Extension [4]. In the Appendix 1 we present the generator matrices of the constructed self-orthogonal codes.

$[n,n-k,d]_q$	d⊥	$[[n,2k-n,d^{\perp}]]_q$	$[n,n-k,d]_q$	d⊥	$[[n, 2k-n, d^{\perp}]]_q$
[18,6,9]3	4	[[18,6,4]] ₃	[28,4,18] ₃	3	$[[28, 20, 3]]_3$
[24,12,9]3	9	$[[24, 0, 9]]_3$	[30,4,18] ₃	3	$[[30, 22, 3]]_3$
[27,6,15]3	4	$[[27, 15, 4]]_3$	[32,4,21]3	3	[[32,24,3]] ₃

5. Conclusions

In this paper, using CSS construction method, we construct ternary self-orthogonal codes and we prove the existence of the corresponding quantum codes. This research was partially supported by the project FSD-31-303-05/2018 of University of Veliko Tarnovo.

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Appendix 1

Here we present the generator matrices of the constructed self-orthogonal ternary codes.

$[27, 6, 15]_3$

A Result on the Behavior of Solutions of Third Order Linear Delay Differential Equations

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Abstract

This paper deals with the behaviour of solutions for scalar third order linear delay differential equations. By the use of two distinct real roots of the corresponding characteristic equation, a new result on the behavior of the solutions is obtained. **Keywords:** Delay differential equation, Characteristic equation, Roots, Asymptotic behavior

1. Introduction

The theory of delay differential equations is important both theoretical and practical interest. For the basic theory of delay differential equations, the reader is referred to the books by Bellman and Cooke [1], Driver [4], El'sgol'ts and Norkin [5], Hale and Verduyn Lunel [6], Kolmanovski and Myshkis [7] and Lakshmikantham, Wen and Zhang [8]. The very interesting asymptotic and stability results were given by Philos and Purnaras [9-10]. The techniques applied in [11,12] are originated in a combination of the methods used in [9,10]. Yeniçerioğlu [12] obtained some results on the qualitative behavior of the solutions of a second order linear autonomous delay differential equation with a single delay. The main idea in [12] is that of transforming the second order delay differential equation into a first order delay differential equation. The same idea will be used in this paper to obtain some general results. Recently, Cahlon and Schmidt et al. [2] have established the stability criteria for a third order delay differential equation. This equation is obtained the stability of third order delay differential equation using Pontryagin's theory for quasi-polynomials. However, we study the stability of the some problem using the method of characteristic roots.

Let us consider initial value problem for third order delay differential equation

$$y''' = py''(t) + \sum_{i \in I} p_i y''(t - \tau_i) + qy'(t) + \sum_{i \in I} q_i y'(t - \tau_i) + vy(t) + \sum_{i \in I} v_i y(t - \tau_i), \quad t \ge 0 \quad (1.1)$$

where I is an initial segment of natural numbers, p,q,v,p_i,q_i,v_i for $i \in I$ are real constants, and τ_i for $i \in I$ positive real numbers such that $\tau_{i_1} \neq \tau_{i_2}$ for $i_1, i_2 \in I$ with $i_1 \neq i_2$. Let's define $\tau = \max \tau_i$. (τ is a positive real number.)

In a previous paper [3], we considered Eq. (1.1) with $q_i = 0$ and $v_i = 0$, $i \in I$ which arose from a robotic model with damping and delay. There are no practical stability criteria of the zero solution of (1.1). Together with the delay differential equation (1.1), it is customary to specify an *initial condition* of the form

$$\mathbf{y}(t) = \boldsymbol{\varphi}(t) \qquad \text{for} \quad -\tau \le t \le 0, \tag{1.2}$$

where the initial function $\varphi(t)$ is a given twice continuously differentiable real-valued function on the initial interval $[-\tau, 0]$. Along with the delay differential equation (1.1), we associate the equation

$$\lambda^{3} = \lambda^{2} p + \lambda q + v + \sum_{i \in I} e^{-\lambda \tau_{i}} \left(\lambda^{2} p_{i} + \lambda q_{i} + v_{i} \right),$$
(1.3)
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which will be called the *characteristic equation of* (1.1). Eq. (1.3) is obtained from (1.1) by looking for solutions of the form $y(t)=e^{\lambda t}$ for $t \ge -\tau$. For a given real root λ_0 of the characteristic equation (1.3), we consider the (second order) delay differential equation

$$z''(t) = (p - 3\lambda_0)z'(t) + \sum_{i \in I} p_i e^{-\lambda_0 \tau_i} z'(t - \tau_i) + (q + 2p\lambda_0 - 3\lambda_0^2)z(t) + \sum_{i \in I} e^{-\lambda_0 \tau_i} (q_i + 2p_i\lambda_0)z(t - \tau_i) - \sum_{i \in I} e^{-\lambda \tau_i} (\lambda_0^2 p_i + \lambda_0 q_i + v_i) \int_{t - \tau_i}^t z(s) ds.$$
(1.4)

With the delay differential equation (1.4), we associate the equation

$$\delta^{2} = (p - 3\lambda_{0})\delta + q + 2p\lambda_{0} - 3\lambda_{0}^{2} + \sum_{i \in I} (p_{i} \delta + q_{i} + 2p_{i}\lambda_{0})e^{-(\lambda_{0} + \delta)\tau_{i}} - \delta^{-1} \sum_{i \in I} (p_{i}\lambda_{0}^{2} + q_{i}\lambda_{0} + v_{i})(1 - e^{-\delta\tau_{i}})e^{-\lambda_{0}\tau_{i}},$$
(1.5)

which is said to be the *characteristic equation of* (1.4). This equation is obtained from (1.4) by seeking solutions of the form $z(t)=e^{\delta t}$ for $t \ge -\tau$. For our convenience, we introduce some notations. For a given real root λ_0 of the characteristic equation (1.3), we set

$$\beta_{\lambda_0} = \sum_{i \in I} e^{-\lambda_0 \tau_i} \left\{ \tau_i \left(p_i \lambda_0^2 + q_i \lambda_0 + v_i - q_i - 2p_i \lambda_0 \right) \right\} - q - 2p\lambda_0 + 3\lambda_0^2$$
(1.6)

and, also, we define

$$L(\lambda_{0}; \phi) = \phi''(0) + (\lambda_{0} - p)\phi'(0) - \sum_{i \in I} p_{i}\phi'(-\tau_{i}) + (\lambda_{0}^{2} - p\lambda_{0} - q)\phi(0)$$

$$-\sum_{i \in I} (p_{i}\lambda_{0} + q_{i})\phi(-\tau_{i}) + \sum_{i \in I} e^{-\lambda_{0}\tau_{i}} (p_{i}\lambda_{0}^{2} + q_{i}\lambda_{0} + v_{i}) \int_{-\tau_{i}}^{0} e^{-\lambda_{0}s}\phi(s)ds ; \qquad (1.7)$$

in addition, provided that $\beta_{\lambda_0} \neq 0$, we define

$$\Phi_1(\lambda_0;\varphi)(t) = \varphi(t)e^{-\lambda_0 t} - \frac{L(\lambda_0;\varphi)}{\beta_{\lambda_0}} \qquad \text{for} \quad -\tau \le t \le 0.$$
(1.8)

The proof in the following theorem can be made in the same way as in the article [12].

For a given real roots λ_0 , δ_0 of the characteristic equations (1.3) and (1.5), respectively, we consider the (first order) delay differential equation

$$w'(t) = (p - 3\lambda_0 - 2\delta_0)w(t) + \sum_{i \in I} e^{-(\lambda_0 + \delta_0)\tau_i} p_i w(t - \tau_i) - \left(\sum_{i \in I} (p_i \delta_0 + q_i + 2p_i \lambda_0) e^{-(\lambda_0 + \delta_0)\tau_i} \right) \int_{t - \tau_i}^t w(s) ds + \sum_{i \in I} (p_i \lambda_0^2 + q_i \lambda_0 + v_i) e^{-\lambda_0 \tau_i} \int_{0}^{\tau_i} e^{-\delta_0 s} \left\{ \int_{t - s}^t w(u) du \right\} ds.$$
(1.9)

The characteristic equation of the delay differential equation (1.9) is

$$\gamma = p - 3\lambda_0 - 2\delta_0 + \sum_{i \in I} p_i e^{-(\lambda_0 + \delta_0 + \gamma)\tau_i} - \left(\sum_{i \in I} (p_i \delta_0 + q_i + 2p_i \lambda_0) e^{-(\lambda_0 + \delta_0)\tau_i} \right) \int_0^{\tau_i} e^{-\gamma s} ds$$
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$$+\sum_{i\in I} (p_i\lambda_0^2 + q_i\lambda_0 + v_i)e^{-\lambda_0\tau_i} \int_0^{\tau_i} e^{-\delta_0s} \left(\int_0^s e^{-\gamma u} du\right) ds$$
(1.10)

The last equation is obtained from (1.9) by seeking solutions of the form $w(t) = e^{\gamma t}$ for $t \ge -\tau$. For our convenience, we introduce some notations. For a given real root λ_0 of the characteristic equation (1.3) and a given real root δ_0 of the characteristic equation (1.5), we set

$$\eta_{\lambda_{0},\delta_{0}} = \sum_{i \in I} \left(p_{i}\delta_{0} + q_{i} + 2p_{i}\lambda_{0} \right) e^{-(\lambda_{0} + \delta_{0})\tau_{i}} \tau_{i} - p_{i} \right)$$
$$-p + 3\lambda_{0} + 2\delta_{0} - \sum_{i \in I} \left(p_{i}\lambda_{0}^{2} + q_{i}\lambda_{0} + v_{i} \right) e^{-\lambda_{0}\tau_{i}} \int_{0}^{\tau_{i}} s e^{-\delta_{0}s} ds$$
(1.11)

and let $\Phi_1(\lambda_0; \phi)$ be defined by (1.8). Also, we define

$$R(\lambda_{0},\delta_{0};\phi) = (\Phi_{1}(\lambda_{0};\phi)) (0) - \delta_{0}\Phi_{1}(\lambda_{0};\phi)(0) - (p - 3\lambda_{0} - 2\delta_{0})\Phi_{1}(\lambda_{0};\phi)(0) - e^{-\lambda_{0}\tau_{i}} \sum_{i \in I} p_{i}\Phi_{1}(\lambda_{0};\phi)(-\tau_{i}) + \left(\sum_{i \in I} (p_{i}\delta_{0} + q_{i} + 2p_{i}\lambda_{0})e^{-(\lambda_{0} + \delta_{0})\tau_{i}}\right) \int_{-\tau_{i}}^{0} e^{-\delta_{0}s}\Phi_{1}(\lambda_{0};\phi)(s)ds - \sum_{i \in I} (p_{i}\lambda_{0}^{2} + q_{i}\lambda_{0} + v_{i})e^{-\lambda_{0}\tau_{i}} \int_{0}^{\tau_{i}} e^{-\delta_{0}s} \left\{ \int_{-s}^{0} e^{-\delta_{0}u}\Phi_{1}(\lambda_{0};\phi)(u)du \right\} ds ,$$
(1.12)

where $(\Phi_1(\lambda_0; \phi))$ is derivative of $\Phi_1(\lambda_0; \phi)$; in addition, provided that $\eta_{\lambda_0, \delta_0} \neq 0$, we define

$$\Phi_{2}(\lambda_{0},\delta_{0};\phi)(t) = e^{-\delta_{0}t}\Phi_{1}(\lambda_{0};\phi)(t) - \frac{R(\lambda_{0},\delta_{0};\phi)}{\eta_{\lambda_{0},\delta_{0}}} \qquad \text{for } t \in [-\tau,0].$$
(1.13)

2. An Asymptotic Result

Theorem 2.1. Let λ_0 be real root of the characteristic equation (1.3), and let β_{λ_0} and $L(\lambda_0;\phi)$ be defined by (1.6) and (1.7), respectively. Furthermore, let δ_0 be real root of the characteristic equation (1.5), and let $\eta_{\lambda_0,\delta_0}$, $R(\lambda_0,\delta_0;\phi)$ and $\Phi_2(\lambda_0,\delta_0;\phi)$ be defined by (1.11), (1.12) and (1.13), respectively. Moreover, let γ_0 be a real root of the characteristic equation (1.10). Suppose that $\beta_{\lambda_0} \neq 0$ and $\eta_{\lambda_0,\delta_0} \neq 0$. (Note that, because of $\beta_{\lambda_0} \neq 0$, we always have $\delta_0 \neq 0$ and $\gamma_0 \neq -\delta_0$. Furthermore, because of $\eta_{\lambda_0,\delta_0} \neq 0$, we always have $\gamma_0 \neq 0$.) Set

$$\xi_{\lambda_{0},\delta_{0},\gamma_{0}} = \sum_{i\in I} p_{i}\tau_{i}e^{-(\lambda_{0}+\delta_{0}+\gamma_{0})\tau_{i}} - \left(\sum_{i\in I} (p_{i}\delta_{0}+q_{i}+2p_{i}\lambda_{0})e^{-(\lambda_{0}+\delta_{0})\tau_{i}}\right)_{0}^{\tau_{i}}e^{-\gamma_{0}s}s\,ds$$
$$+\sum_{i\in I} (p_{i}\lambda_{0}^{2}+q_{i}\lambda_{0}+v_{i})e^{-\lambda_{0}\tau_{i}}\int_{0}^{\tau_{i}}e^{-\delta_{0}s}\left\{\int_{0}^{s}e^{-\gamma_{0}u}u\,du\right\}ds$$
(2.1)
o, define 311

and, also, define

$$K(\lambda_{0}, \delta_{0}, \gamma_{0}; \phi) = \Phi_{2}(\lambda_{0}, \delta_{0}; \phi)(0) + \sum_{i \in I} p_{i} e^{-(\lambda_{0} + \delta_{0} + \gamma_{0})\tau_{i}} \int_{-\tau_{i}}^{0} e^{-\gamma_{0}s} \Phi_{2}(\lambda_{0}, \delta_{0}; \phi)(s) ds$$
$$-\left(\sum_{i \in I} (p_{i}\delta_{0} + q_{i} + 2p_{i}\lambda_{0})e^{-(\lambda_{0} + \delta_{0})\tau_{i}}\right)\int_{0}^{\tau_{i}} e^{-\gamma_{0}s} \left\{\int_{-s}^{0} e^{-\gamma_{0}u} \Phi_{2}(\lambda_{0}, \delta_{0}; \phi)(u) du\right\} ds$$
$$+\sum_{i \in I} (p_{i}\lambda_{0}^{2} + q_{i}\lambda_{0} + v_{i})e^{-\lambda_{0}\tau_{i}} \int_{0}^{\tau_{i}} e^{-\delta_{0}s} \left\{\int_{0}^{s} e^{-\gamma_{0}u} \left\{\int_{-u}^{0} e^{-\gamma_{0}\omega} \Phi_{2}(\lambda_{0}, \delta_{0}; \phi)(\omega) d\omega\right\} du\right\} ds. \quad (2.2)$$

Assume that

$$\mu_{\lambda_{0},\delta_{0},\gamma_{0}} = \sum_{i \in I} \left| p_{i} \right| e^{-(\lambda_{0}+\delta_{0}+\gamma_{0})\tau_{i}} \tau_{i} + \left(\sum_{i \in I} \left| p_{i}\delta_{0} + q_{i} + 2p_{i}\lambda_{0} \right| e^{-(\lambda_{0}+\delta_{0})\tau_{i}} \right) \int_{0}^{\tau_{i}} e^{-\gamma_{0}s} s \, ds$$

$$+ \sum_{i \in I} \left| p_{i}\lambda_{0}^{2} + q_{i}\lambda_{0} + v_{i} \right| e^{-\lambda_{0}\tau_{i}} \int_{0}^{\tau_{i}} e^{-\delta_{0}s} \left\{ \int_{0}^{s} e^{-\gamma_{0}u} u \, du \right\} ds < 1.$$

$$(2.3)$$

(This assumption guarantees that $1 + \xi_{\lambda_0, \delta_0, \gamma_0} > 0$.) Then the solution y of the IVP (1.1) and (1.2) satisfies

$$\lim_{t \to \infty} \left\{ e^{-\gamma_0 t} \left[e^{-(\lambda_0 + \delta_0)t} y(t) - \frac{L(\lambda_0; \phi)}{\beta_{\lambda_0}} e^{-\delta_0 t} - \frac{R(\lambda_0, \delta_0; \phi)}{\eta_{\lambda_0, \delta_0}} \right] \right\} = \frac{K(\lambda_0, \delta_0, \gamma_0; \phi)}{1 + \xi_{\lambda_0, \delta_0, \gamma_0}}.$$
(2.4)

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The Dirac equation solution of the generalized symmetric Woods-Saxon potential energy in one spatial dimension

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Abstract

We apply the two-component approach to the one-dimensional Dirac equation with a generalization of the Woods-Saxon potential energy that was proposed by including the surface effects. We find that the wave functions for the two-component spinor in terms of the Confluent Heun function.

Keywords: Dirac equation, Generalized Woods-Saxon potential energy, Confluent Heun equation.

1. Introduction

P. Dirac expressed a relativistic first order wave equation 90 years ago to abolish the flaws of the Klein-Gordon equation [Dirac 1928]. Dirac equation, with or without taking account the electromagnetic interaction, describes all spin half particles i.e. electrons, quarks. It was validated by its high accuracy in accounting the fine details of the hydrogen spectrum and it is still in use of scientist in their effort to describe the Nature.

R.D. Woods and D.S. Saxon in 1954 used a new potential energy to calculate the differential cross section of the protons that are scattered elastically by medium or heavy nuclei [Woods and Saxon 1954]. The high accurate results let the WSP to gain reputation. In 1983 G.R. Satchler proposed a generalization to the WSP, namely GWSP, where the surface interactions are considered [Satchler 1983]. In one dimension this potential has the form of

$$V(x) = \Theta(-x) \left[\frac{-V_0}{1 + e^{-\alpha(x+L)}} + \frac{W e^{-\alpha(x+L)}}{(1 + e^{-\alpha(x+L)})^2} \right] + \Theta(x) \left[\frac{-V_0}{1 + e^{\alpha(x-L)}} + \frac{W e^{\alpha(x-L)}}{(1 + e^{\alpha(x-L)})^2} \right]$$

The GWSP energy has been subjected to many articles. Among them, we examined the nonrelativistic analytical solution [Lütfüoğlu, Akdeniz, Bayrak 2016], and then we compared the WSP and GWSP in terms of thermodynamic functions [Lütfüoğlu CTP 2018]. Furthermore, we made a similar comparison in between the relativistic and non-relativistic solutions [Lütfüoğlu CJP 2018]. Very recently, we investigated the scattering and bound state solutions of the Klein Gordon equation in spin and pseudospin symmetry limits [Lutfuoglu, Lipovsky, Kriz 2018, Lütfüoğlu 2018].

2. Materials and Methods

We begin by the relativistic free-particle Dirac equation in Natural Units system ($\hbar = c = 1$)

$$\left(i\gamma^{\mu}\frac{\partial}{\partial x^{\mu}}-m\right)\Psi(x,t)=0\,,$$

An external potential energy can be coupled to the equation via the momentum and the mass quantities. Initially, the external potential that is under the investigation will only be coupled to the momentum quantity with a coupling constant, namely e, as known as minimal coupling.

$$\left[\gamma^{\mu}\left(i\frac{\partial}{\partial x^{\mu}}-eA_{\mu}\right)-m\right]\Psi(x,t)=0\,.$$

In (1 + 1) space-time dimension, we take the gamma matrices γ^0 and γ^1 to be the Pauli matrices σ^z and $i\sigma^x$, respectively. Furthermore, we choose the time component of the four-vector to be proportional to the potential energy, while the spatial component is zero. Note that the potential energy is time independent, therefore the wave function is chosen to be the multiplication of the terms that depend on time, e^{-iEt} , and space, $\psi(x)$. Then, the Dirac equation becomes

$$\left[\sigma^{z}\left(E-V(x)\right)-\sigma^{x}\frac{d}{dx}-m\right]\psi(x)=0.$$

We decompose the Dirac spinor, into two spinors, $u_1(x)$ and $u_2(x)$, given as

$$\psi(x) = \binom{u_1(x)}{u_2(x)}.$$

Then, we obtain two coupled first order differential equations

$$\frac{du_2(x)}{dx} = (E - V(x) - m)u_1(x),$$
$$\frac{du_1(x)}{dx} = -(E - V(x) + m)u_2(x).$$

We introduce new combination of the spinors as Flügge had done [Flügge 1974]

$$\phi(x) = u_1(x) + i u_2(x),$$

$$\chi(x) = u_1(x) - i u_2(x).$$

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After substituting the new-introduced spinors, we re-arrange the coupled first order differential equations and derive uncoupled second order differential equation system as follows:

$$\frac{d^2\phi(x)}{dx^2} + \left[\left(E - V(x) \right)^2 - m^2 + i \frac{d}{dx} V(x) \right] \phi(x) = 0,$$

$$\frac{d^2\chi(x)}{dx^2} + \left[\left(E - V(x) \right)^2 - m^2 - i \frac{d}{dx} V(x) \right] \chi(x) = 0.$$

3. Results and Discussions

We use the GWSP energy to solve the equations. Here, we present only the solution of one spinor, the other spinor's solution can be established similarly. Since the used potential is symmetric in the negative and positive region, we discuss only one solution to avoid a repetition. In order to have a dimensionless differential equation, we make the transformation

$$z = \left[1 + e^{-\alpha(x+L)}\right]^{-1},$$

and we express the GWSP energy and its derivative

$$V(x) \xrightarrow{\text{yields}} -V_0 z - W z(z-1),$$

$$\frac{dV(x)}{dx} \xrightarrow{\text{yields}} \alpha \ z \ (z-1)[(V_0 - W) + 2 \ W \ z].$$

After straightforward algebra, we obtain the second order differential equation as

$$\begin{split} \left[\frac{d^2}{dz^2} + \left(\frac{1}{z} + \frac{1}{z-1} \right) \frac{d}{dz} - \frac{\epsilon^2}{z^2(z-1)^2} + \frac{C^2}{(z-1)^2} + G^2 + \frac{B^2}{z(z-1)^2} + \frac{D^2}{z(z-1)} \right. \\ \left. + \frac{F^2}{z-1} \right] \phi_L(z) = 0, \end{split}$$

Note that, we abbreviate some terms with new definitions as given:

$$\epsilon^{2} = \frac{m^{2} - E^{2}}{\alpha^{2}},$$

$$B^{2} = \frac{2 E V_{0}}{\alpha^{2}},$$

$$C^{2} = \frac{V_{0}^{2}}{\alpha^{2}},$$

$$D^{2} = \frac{2 E W + i \alpha (V_{0} - W)}{\alpha^{2}},$$
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$$F^{2} = \frac{2 W(V_{0} + i \alpha)}{\alpha^{2}} ,$$
$$G^{2} = \frac{W^{2}}{\alpha^{2}} .$$

We consider the general solution with an ansatz

$$\phi_L(z) = z^{\mu}(z-1)^{\nu} e^{i G z} f(z),$$

where

$$\mu = \frac{i k}{\alpha},$$
$$\nu = \frac{i \kappa}{\alpha},$$

and we define the wave numbers as

$$\label{eq:k} \begin{split} k &= \sqrt{E^2 - m^2} \; , \\ \kappa &= \sqrt{(E+V_0)^2 - m^2} \end{split}$$

We obtain a second order differential equation to the f(z) function as given

$$\begin{aligned} \frac{d^2 f}{dz^2} + \left[2iG + \frac{1+2v}{z-1} + \frac{1+2\mu}{z}\right] \frac{df}{dz} \\ + \left[\frac{iG(2\mu+1) - 2\mu v - \mu - v + \zeta}{z} + \frac{iG(2v+1) + 2\mu v + \mu + v - \delta}{z-1}\right] f \\ = 0. \end{aligned}$$

Here

$$\begin{split} \zeta &= -2\epsilon^2 + B^2 - D^2 \,, \\ \delta &= -2\epsilon^2 + B^2 - D^2 - F^2 \,. \end{split}$$

This equation is known as the confluent HEUN equation [Heun 1888, Ronveaux 1995]

$$\frac{d^2w}{dy^2} + \left(\alpha + \frac{\gamma+1}{y-1} + \frac{\beta+1}{y}\right)\frac{dw}{dy} + \left(\frac{\nu}{y-1} + \frac{\mu}{y}\right)w = 0,$$

and its solutions are given in the form of $HeunC(\alpha, \beta, \gamma, \delta, \eta, y)$, and the coefficients should be obtained via the relations [Slavyanov and Lay 2000, Hortaçsu 2018]

$$\delta = \mu + \upsilon - \alpha \left(\frac{\beta + \gamma + 2}{2}\right),$$

$$\eta = \frac{\alpha(\beta + 1)}{2} - \mu - \left(\frac{\beta + \gamma + \gamma\beta}{2}\right).$$
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4. Conclusions

In this work, we show that the Dirac equation under the minimal coupling of Generalized Woods Saxon potential in (1+1) dimension owns a set of solutions in confluent Heun functions.

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Degenerate Genocchi Numbers Arising From Ordinary Differential Equations

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Abstract

Carlitz was the first to extend the classical Bernoulli polynomials, Euler polynomials and numbers, introducing them as q-Bernoulli and q-Euler numbers and polynomials. He introduced degenerate Bernoulli polynomials. Dolgy *et al.* defined and investigated the modified degenerate Bernoulli polynomials. Kim *et al.* and Kwon *et al.* proved some identities and recurrence relations. Kim *et al.* gave some explicit relation degenerate Bernoulli polynomials are some explicit relation degenerate Bernoulli polynomials. Kim *et al.* gave a symmetric identity for the degenerate Bernoulli polynomials.

In this work, we prove some identities between these polynomials. Further, we prove some relations between the degenerate Bernoulli polynomials and the degenerate second kind Stirling numbers of first kind.

Keywords: Bernoulli Polynomials and Numbers, Euler Polynomials and Numbers, Degenerate Bernoulli polynomials

1. Introduction

The classical Genocchi polynomials are defined in ([10], [11]) as

$$\sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!} = \frac{2t}{e^t + 1} e^{xt}, \quad |t| < \pi.$$
(1.1)

When x = 0, $G_n(0) = G_n$ are called the Genocchi numbers.

The classical Bernoulli polynomials and Bernoulli numbers are defined as the following equations in ([1]-[11]) respectively

$$\sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} = \frac{t}{e^{t} - 1} e^{xt} , \quad |t| < 2\pi$$
(1.2)

and

$$\sum_{n=0}^{\infty} B_n \frac{t^n}{n!} = \frac{t}{e^t - 1} , \qquad |t| < 2\pi .$$
 (1.3)

For $\lambda \in C$, Carlitz [1] defined degenerate Bernoulli polynomials the following generating function

$$\sum_{n=0}^{\infty} B_n(x|\lambda) \frac{t^n}{n!} = \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} (1+\lambda t)^{\frac{x}{\lambda}}, \quad |t| < 2\pi.$$
(1.4)

When x = 0, $B_n(\lambda) = B_n(0|\lambda)$ are called the degenerate Bernoulli numbers. Thus, by (1.4), we get

$$B_n(x|\lambda) = \sum_l^n \binom{n}{l} (x|\lambda)_{n-l} B_l$$
³¹⁹

where $(x|\lambda)_n = x(x-\lambda) \dots (x-\lambda(n-1))$.

Degenerate Genocchi polynomials and the higher order degenerate Genocchi numbers are defined by D. Lim [12] as following generating functions respectively;

$$\sum_{n=0}^{\infty} G_n(x|\lambda) \frac{t^n}{n!} = \frac{2t}{(1+\lambda t)^{\frac{1}{\lambda}} + 1} (1+\lambda t)^{\frac{1}{\lambda}}$$
(1.5)

and

$$\sum_{n=0}^{\infty} G_n(0|\lambda) \frac{t^n}{n!} = \frac{2t}{(1+\lambda t)^{\frac{1}{\lambda}} + 1}, r \in N.$$
(1.6)

Thus, by (1.5), we get

 $\lim_{\lambda \to 0} G_n(x|\lambda) = G_n(x), \lim_{\lambda \to 0} G_n(0|\lambda) = G_n, n \ge 0.$

In this note, we write as $G_n(x|\lambda) = G_n(x,\lambda)$.

2. Some Identities For The Degenerate Genocchi Numbers

In this section, we give the solution of the linear ordinary differential equations $F^{(k-1)}(t)$ and some recurrence relation for the degenerate Genocchi numbers.

Let

$$F = F(t) = \frac{1}{(1+\lambda t)^{\frac{1}{\lambda}} + 1}.$$
(2.1)

Then by derivative of (2.1), we get

$$F^{(1)} = \frac{d}{dt}F(t) = \frac{1}{\lambda}\log(1+\lambda)(F^{(2)} - F)$$
(2.2)

by using (2.2) and taking the derivative of (2.2), we have

$$2! F^{(3)} = \left(\frac{\lambda}{\log(1+\lambda)}\right)^2 F^{(2)} + 3F^{(1)}\left(\frac{\lambda}{\log(1+\lambda)}\right) + 2F.$$
(2.3)

Continuing this process, we can deduce

$$(N-1)! F^{(N)} = \sum_{k=1}^{N} a_k (N) F^{(k-1)} \left(\frac{\lambda}{\log(1+\lambda)}\right)^{k-1}$$
(2.4)

where $N \in N$, $F^{(k)} = \left(\frac{d}{dt}\right)^k F(t)$. Now we take derivative of (2.4) as follows

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$$N! F^{(N)} F^{(1)} = N! \left(F^{(N+1)} - F^{(N)} \right) \left(\frac{\log(1+\lambda)}{\lambda} \right).$$
(2.5)

From (2.4) and (2.5)

$$N! F^{(N+1)} = N \sum_{k=1}^{N} a_k (N) F^{(k-1)} \left(\frac{\lambda}{\log(1+\lambda)}\right)^{k-1} + \sum_{k=2}^{N+1} a_{k-1} (N) F^{(k-1)} \left(\frac{\lambda}{\log(1+\lambda)}\right)^{k-1}.$$
 (2.6)

Replacing N by N+1 in (2.4), we have

$$N! F^{(N+1)} = \sum_{k=1}^{N+1} a_k (N+1) F^{(k-1)} \left(\frac{\lambda}{\log(1+\lambda)}\right)^{k-1}.$$
(2.7)

Comparing the coefficient on the both sides of (2.6) and (2.7), we obtain

$$Na_1(N) = a_1(N+1), a_N(N) = a_{N+1}(N+1)$$
(2.8)

and

$$a_k(N+1) = Na_k(N) + a_{k-1}(N), 2 \le k \le N.$$
(2.9)

From (2.8), we easily get

$$a_1(N+1) = Na_1(N) = \dots = N! a_1.$$
(2.10)

From (2.4), we write as

$$F = \sum_{k=1}^{1} a_k (1)F = a_1(F).$$
(2.11)

Thus by (2.10) and (2.11), we get

$$a_1(1) = 1, a_1(N+1) = N!.$$
 (2.12)

For k=2 in (2.9)

$$a_2(N+1) = \sum_{k=0}^{N-1} (N)_k a_1(N-k)$$
(2.13)

where $(x)_N = x(x-1) \dots (x-N+1), N \in N, (x)_0 = 1.$

Let us take k=3 in (2.9). Then we have

$$a_3(N+1) = Na_3(N) + a_2(N) = \sum_{k=0}^{N-2} (N)_k a_2(N-k).$$
(2.14)

Continuing this process, we can deduce that

$$a_j(N+1) = \sum_{k=0}^{N-j+1} (N)_k a_{j-1}(N-k), 2 \le j \le N+1.$$
(2.15)

From (2.13),

$$a_2(N+1) = \sum_{k_1=0}^{N-1} (N)_{k_1} (N-k_1)!$$
(2.16)

$$a_{3}(N+1) = \sum_{k_{1}=0}^{N-2} (N)_{k_{2}} a_{2}(N-k_{2})$$

= $\sum_{k_{2}=0}^{N-2} \sum_{k_{1}=0}^{N-k_{2}-2} (N)_{k_{2}} (N-k_{2}-1)(N-k_{2}-k_{1}-2)!.$
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Continuing this process

$$a_{j}(N+1) = \sum_{k_{j-1}=0}^{N-j+1} \sum_{k_{j-2}=0}^{N-k_{j-1}-j+1} \dots \sum_{k_{2}=0}^{N-k_{j-1}-k_{j-2}-\dots-k_{2}-j+1} (N)_{k_{j}-1} (N-k_{j-1}-1)_{k_{j-2}} \times (N-k_{j-1}-k_{j-2}-2)_{k_{j-3}} \dots (N-k_{1}-k_{2}-\dots-k_{j-1}-j+1)!$$

$$(2.17)$$

where $2 \le j \le N + 1$. From (2.17) and (2.4), we obtain the following theorem (For details an above operation, see [5], [7], [8]).

Theorem. For $N \in N$, the ordinary differential equations

$$(N-1)! F^{N} = \sum_{k=0}^{N-j+1} a_{k} (N) F^{(k-1)} \left(\frac{\lambda}{\log(1+\lambda)}\right)^{k-1}$$

have a solution $F = F(t) = \frac{1}{(1+\lambda t)^{1/\lambda}+1}$.

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Playing with Continued Radicals, Fractions and Iterated Exponents

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Abstract

In mathematics, the study of Continued Radicals and Continued Fractions is very popular. In this paper we will present some notes on the calculation of the convergent continued radicals, fractions and exponents with integer elements and integer values by using the splitting of the integers under radicals in certain ways which are very bizarre yet interesting.

Continued Radicals

Let's start with some famous convergent continued radicals. For example, in real analysis or number theory courses, we see problems like this:

Problem 1: What is the value of *A* when

$$A = \sqrt{2\sqrt{2\sqrt{2}\dots}}.$$
 (1)

The following methods illustrate a few different ways to show that the above continued or nested radicals is convergent and its value is **2**.

<u>Method 1:</u> If A is the value of (1), then the same A is the value of the internal continued radicals from the second radical to all the way up to the end of the expression, i.e. we have

$$A = \sqrt{2\sqrt{2\sqrt{2}\dots}} = \sqrt{2(\sqrt{2\sqrt{2}\dots})} = \sqrt{2A}.$$

So, $A^2 = 2A$ and this simple equation has two solutions A = 2 and A = 0 and of course we reject A = 0 and A = 2 is the only acceptable value for (1).

<u>Method 2:</u> We may define a sequence to analyze the convergence of (1). Let's consider the sequence x_n defined by $x_1 = \sqrt{2}$ and $x_{n+1} = \sqrt{2x_n}$; n = 1, 2, 3, ... By using mathematical induction, it is easy to show that, this sequence is an increasing sequence and it is bounded above by 2 (prove it) and so the sequence is convergent and its limit, *l*, satisfies the equation $l = \sqrt{2l}$ and, similar to the above solution, we have l = 2. So, A = 2.

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<u>Method 3</u>: We can use the graphical method to investigate or find the fixed point of function $y = f(x) = \sqrt{2x}$ and observe that for any initial point the fixed point is (2, 2). That means 2 is the limit or the value of that A.

<u>Method 4</u>: Another method to figure out the value of (1) is by using series. We know that $\sqrt{2} = 2^{\frac{1}{2}}$ and $\sqrt{2\sqrt{2}} = (2\sqrt{2})^{\frac{1}{2}} = (2^{\frac{1}{2}})(2^{\frac{1}{4}}) = 2^{\frac{1}{2}+\frac{1}{4}}$ and in a similar manner, we have

$$A = \sqrt{2\sqrt{2\sqrt{2}}} = 2^{\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots}$$

The above geometric series with ratio $\frac{1}{2}$ is convergent and its sum is equal to one. So, A = 2. To accept that the above geometric series is convergent to 1, we may use the following simple high school level mathematics.

$$\mathbf{S} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = \frac{1}{2} \left(\mathbf{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \right) = \frac{1}{2} (\mathbf{1} + \mathbf{S}).$$

This simple equation implies that S = 1. Even we can use the following observation to show that S = 1, (see [3]). Consider a square with side equal to one and area equal to one. Then divide the square in half and continue dividing and adding the areas of the triangles.

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1$$

Proof Without Words



<u>Method 5:</u> After four different methods to calculate the value of the expression (1), here is my simple and interesting splitting method that I haven't seen used in any other place to solve the problem (1). The value of the continued fractions (1) is equal to 2 because:

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$$2 = \sqrt{4} = \sqrt{2 \times 2} = \sqrt{2 \times \sqrt{4}} = \sqrt{2 \times \sqrt{2 \times 2}} = \sqrt{2 \times \sqrt{2 \times \sqrt{4}}} = \sqrt{2 \times \sqrt{2 \times 2 \times 2}} = \cdots$$

So,
$$2 = \sqrt{2\sqrt{2\sqrt{2}\dots}}$$

This method is a very simple (it is at the high school level mathematics) and a very interesting way to show that (1) is convergent and its value is A = 2.

Now we consider the second famous convergent continued radicals:

$$B=\sqrt{2+\sqrt{2+\cdots}}.$$

Here again by using methods similar to 1 and 2, above, we can easily show that B = 2. But, by using our splitting technique we have: --

$$2 = \sqrt{4} = \sqrt{2 + 2} = \sqrt{2 + \sqrt{4}} = \sqrt{2 + \sqrt{2 + 2}} = \sqrt{2 + \sqrt{2 + \sqrt{4}}} = \sqrt{2 + \sqrt{2 + \sqrt{2 + 2}}} = \cdots$$
So,

$$2 = \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}.$$
 (2)

After these two examples, and finding the values of the continued radicals by splitting the numbers under the radicals, we will continue our presentation and we will work with more continued radicals.

$$3 = \sqrt{9} = \sqrt{6+3} = \sqrt{6+\sqrt{9}} = \sqrt{6+\sqrt{6+3}} = \sqrt{6+\sqrt{6+\sqrt{9}}} = \sqrt{6+\sqrt{6+\sqrt{6+\cdots}}}$$

So,
$$3 = \sqrt{6+\sqrt{6+\sqrt{6+\cdots}}}.$$
(3)

$$3 = \sqrt{6 + \sqrt{6 + \sqrt{6 + \cdots}}}.$$
 (3)

Another example of the splitting technique is

$$4 = \sqrt{16} = \sqrt{12 + 4} = \sqrt{12 + \sqrt{16}} = \sqrt{12 + \sqrt{12 + 4}} = \sqrt{12 + \sqrt{12 + \sqrt{16}}} = \cdots$$

So,

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$$4 = \sqrt{12 + \sqrt{12 + \sqrt{12 + \cdots}}}.$$
 (4)

The above three examples are three particular cases of the next problem.

Problem 2: Find all positive integers m such that the following continued radicals (5) are convergent with integer values C:

$$C = \sqrt{m + \sqrt{m + \sqrt{m + \cdots}}}.$$
 (5)

It is easy to show that (prove it), m must be the product of two consecutive positive integers, i.e. m = k(k + 1) for k = 1, 2, 3, ..., and C = k + 1, (see [1]). In other words, if m = k(k + 1) for k = 1, 2, 3, ..., then C = k + 1. By using the splitting technique we can arrange this time like this:

$$k + 1 = \sqrt{(k+1)^2} = \sqrt{k^2 + 2k + 1} = \sqrt{k(k+1) + (k+1)} = \sqrt{k(k+1) + \sqrt{(k+1)^2}} = \sqrt{k(k+1) + \sqrt{k(k+1) + (k+1)}} = \sqrt{k(k+1) + \sqrt{k(k+1) + (k+1)^2}} = \sqrt{k(k+1) + \sqrt{k(k+1) + \sqrt{k(k+1) + \cdots}}} = \sqrt{m + \sqrt{m + \sqrt{m + \cdots}}}.$$

For example, for $m = 2 = 1 \times 2$, $m = 6 = 2 \times 3$, or $m = 12 = 3 \times 4$ we have the above results (2), (3), and (4) and for $m = 20 = 4 \times 5$ we have:

$$5 = \sqrt{25} = \sqrt{20 + 5} = \sqrt{20 + \sqrt{25}} = \sqrt{20 + \sqrt{20 + 5}} = \sqrt{20 + \sqrt{20 + \sqrt{20 + \cdots}}}$$

So,

$$5 = \sqrt{20 + \sqrt{20 + \sqrt{20 + \cdots}}}.$$
 (6)

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We can even split 25 in different ways to get different nested radicals with the same value 5,

$$5 = \sqrt{25} = \sqrt{5 + 20} = \sqrt{5 + 4 \times 5} = \sqrt{5 + 4\sqrt{25}} = \sqrt{5 + 4\sqrt{5 + 4\sqrt{25}}} = \cdots$$

Hence

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$$5 = \sqrt{5 + 4\sqrt{5 + 4\sqrt{5 + 4\sqrt{5 + \cdots}}}}.$$

Or,

$$5 = \sqrt{25} = \sqrt{10 + 15} = \sqrt{10 + 3 \times 5} = \sqrt{10 + 3\sqrt{25}} = \cdots$$

Hence

$$5 = \sqrt{10 + 3\sqrt{10 + 3\sqrt{10 + 3\sqrt{10 + \cdots}}}}$$

How about the next two tricks?

$$3 = \sqrt{9} = \sqrt{3 + 2 \times 3} = \sqrt{3 + 2\sqrt{9}} = \sqrt{3 + 2\sqrt{3} + 2 \times 3} = \sqrt{3 + 2\sqrt{3}$$

We may try this splitting technique for more complicated iterated radicals too. For example, the solution of the following famous continued radicals belongs to our pioneer and genius mathematician, Ramanujan.



Ramanujan Continued Radicals

The solution of the following continued radicals problem is from the great mathematician Srinivasa Ramanujan. Over 100 years ago, he presented the solution of this problem with its following solution in a simple yet very interesting splitting technique in the Journal of Indian Mathematical Society; see [5, 6, and 7]. The problem was, compute the value of D when

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$$D = \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + 5\sqrt{1 + \cdots}}}}}.$$

To figure out the value of D we can calculate the first few terms of this continued radical. We can easily notice that these values are increasing and are close to three. By using a computer, I noticed that the approximate value of D is almost equal to 3. Actually, the real value of D is 3. Here is a very neat and simple proof on calculation of D from Ramanujan by using splitting technique:

$$3 = \sqrt{9} = \sqrt{1+8} = \sqrt{1+2\times4} = \sqrt{1+2\sqrt{16}} = \sqrt{1+2\sqrt{1+15}} = \sqrt{1+2\sqrt{1+3\times5}} = \sqrt{1+2\sqrt{1+3\sqrt{25}}} = \sqrt{1+2\sqrt{1+3\sqrt{1+24}}} = \sqrt{1+2\sqrt{1+3\sqrt{1+4\times6}}} = \sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{36}}}} = \sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+55}}}} = \sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+55}}}} = \sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+5\sqrt{49}}}}} = \cdots$$

This goes on and on and we never get to a dead end point. Because for any natural number n we have

$$n+1 = \sqrt{(n+1)^2} = \sqrt{1+n^2+2n} = \sqrt{1+n(n+2)} = \sqrt{1+n\sqrt{(n+2)^2}}$$
$$= \sqrt{1+n\sqrt{n^2+4n+4}} = \sqrt{1+n\sqrt{1+(n^2+4n+3)}} = \sqrt{1+n\sqrt{1+(n+1)(n+3)^2}} = \sqrt{1+n\sqrt{1+(n+1)(n+3)^2}} = \cdots$$

Hence $\boldsymbol{D} = \boldsymbol{3}$ and we have

$$3 = \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{1 + 5\sqrt{1 + \cdots}}}}}.$$

The second continued radicals in the same page of the same journal was to find the value of E when, (see [5])

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$$E = \sqrt{6+2}\sqrt{7+3}\sqrt{8+4}\sqrt{9+5}\sqrt{10+6}\sqrt{11+7}\sqrt{12+\cdots}}$$

Here again by using the similar to the above trick we can easily find that E = 4:

$$4 = \sqrt{16} = \sqrt{6 + 10} = \sqrt{6 + 2 \times 5} = \sqrt{6 + 2\sqrt{25}} = \sqrt{6 + 2\sqrt{7 + 18}} = \sqrt{6 + 2\sqrt{7 + 3\sqrt{36}}} = \sqrt{6 + 2\sqrt{7 + 3\sqrt{8 + 28}}} = \sqrt{6 + 2\sqrt{7 + 3\sqrt{8 + 28}}} = \sqrt{6 + 2\sqrt{7 + 3\sqrt{8 + 28}}} = \sqrt{6 + 2\sqrt{7 + 3\sqrt{8 + 4\sqrt{9 + 5\sqrt{64}}}} = \cdots$$
Hence

Hence

$$4 = \sqrt{6 + 2}\sqrt{7 + 3}\sqrt{8 + 4}\sqrt{9 + 5}\sqrt{10 + 6}\sqrt{11 + 7}\sqrt{12 + \cdots}.$$

Here is my final magic show. Similarly, by using our splitting technique, we can find few more continued radicals (think and figure out the tricks):

$$4 = \sqrt{1 + 3}\sqrt{1 + 4}\sqrt{1 + 5}\sqrt{1 + 6}\sqrt{1 + \cdots}$$
$$5 = \sqrt{1 + 4}\sqrt{1 + 5}\sqrt{1 + 6}\sqrt{1 + 7}\sqrt{1 + \cdots}$$
$$6 = \sqrt{6 + 5}\sqrt{6 + 5}\sqrt{6 + 5}\sqrt{6 + 5}\sqrt{6 + \cdots}$$

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$$6 = \sqrt{12 + 4\sqrt{12 + 4\sqrt{12 + 4\sqrt{12 + \cdots}}}}$$
$$6 = \sqrt{18 + 3\sqrt{18 + 3\sqrt{18 + 3\sqrt{18 + \cdots}}}}$$

I am sure that many interested readers will find and create many other convergent continued radicals with their values by using this interesting splitting technique.

One more comment. It is not true that any continued radicals with integers have integer value. For example the value of the following simple continued radicals is the Golden Ratio φ (prove it):

$$\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}}=\frac{1+\sqrt{5}}{2}=\varphi.$$

Interestingly, the value of the following simple continued fractions is also the Golden Ratio φ :

$$1 + \frac{1}{1$$

Over all, for any positive integer number n, we have, see [5 and 6],

$$\sqrt{n + \sqrt{n + \sqrt{n + \sqrt{n + \cdots}}}} = \frac{1 + \sqrt{1 + 4n}}{2}.$$
 (7)

And for any natural number $n \ge 2$, we have

$$\sqrt{n-\sqrt{n-\sqrt{n-\sqrt{n-\cdots}}}} = \frac{-1+\sqrt{1+4n}}{2}.$$
(8)

Actually, the expression (7) is true for any positive real number a. To show that for any positive real number a > 0, the continued radicals

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$$F = \sqrt{a + \sqrt{a + \sqrt{a + \cdots}}},$$
 (9)

is convergent, Wenjiang Tu in [4] proved that the following sequence x_n is convergent and its limit is equal to $F = \frac{1+\sqrt{1+4a}}{2}$.

$$x_1 = \sqrt{a}$$
, and $x_{n+1} = \sqrt{a + x_n}$; $n = 1, 2, 3, ...$

His proof has two parts. Obviously for any n = 1, 2, 3, ..., we have $x_n > \sqrt{a} > 0$ and $x_{n+1}^2 = a + x_n$. By using mathematical induction he showed that the sequence x_n increases. Easily, we notice that $x_2 = \sqrt{a + \sqrt{a}} > \sqrt{a} = x_1$, and if $x_n > x_{n-1}$ then

$$\begin{aligned} x_{n+1} - x_n &= \sqrt{a + x_n} - x_n = \frac{\left(\sqrt{a + x_n} - x_n\right)\left(\sqrt{a + x_n} + x_n\right)}{\left(\sqrt{a + x_n} + x_n\right)} \\ &= \frac{a + x_n - x_n^2}{\sqrt{a + x_n} + x_n} = \frac{a + x_n - a - x_{n-1}}{\sqrt{a + x_n} + x_n} = \frac{x_n - x_{n-1}}{\sqrt{a + x_n} + x_n} > 0. \end{aligned}$$

So, $x_{n+1} > x_n$ and this implies that the sequence x_n increases. Then he proved that x_n is bounded from above. By using $x_{n+1}^2 = a + x_n$,

$$x_{n+1} = \frac{a+x_n}{x_{n+1}} = \frac{a}{x_{n+1}} + \frac{x_n}{x_{n+1}} < \frac{a}{x_{n+1}} + 1 < \frac{a}{\sqrt{a}} + 1 = \sqrt{a} + 1$$

This shows that $\sqrt{a} + 1$ is an upper bound for the sequence. So x_n is a convergent sequence and its limit is equal to $F = \frac{1+\sqrt{1+4a}}{2}$, (see [4]).

It is obvious that $\sqrt{a} + 1 > \frac{1+\sqrt{1+4a}}{2}$, so $\sqrt{a} + 1$ is not the least upper bound. Here we try to find the least upper bound of x_n which is sharper than $\sqrt{a} + 1$. By using mathematical induction we prove that the sequence x_n is bounded by $\frac{1+\sqrt{1+4a}}{2}$. It is not difficult to show that $x_1 = \sqrt{a} < \frac{1+\sqrt{1+4a}}{2}$. Suppose $x_n < \frac{1+\sqrt{1+4a}}{2}$, then we show that $x_{n+1} < \frac{1+\sqrt{1+4a}}{2}$. We have $x_{n+1}^2 = a + x_n < a + \frac{1+\sqrt{1+4a}}{2} = \frac{2a+1+\sqrt{1+4a}}{2} = \frac{4a+2+2\sqrt{1+4a}}{4} = (\frac{1+\sqrt{1+4a}}{2})^2$.

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So, $x_{n+1} < \frac{1+\sqrt{1+4a}}{2}$, and the sequence is bounded from above and we have a convergent sequence with limit equal to $F = \frac{1+\sqrt{1+4a}}{2}$.

Now, the nice problem to discuss is finding all natural numbers n such that the odd number 1 + 4n, under radical, is a perfect square number, say k^2 with some odd integer k, which implies that the limit of the sequence $F = \frac{1+\sqrt{1+4n}}{2}$ is a positive integer. For example for n = 2, we have $F = \frac{1+\sqrt{1+4(2)}}{2} = 2$ or for n = 110, we have F = 11. Of course there are many natural numbers *n* such that $1 + 4n = k^2$ or $n = \frac{k^2 - 1}{4} = (\frac{k - 1}{2})(\frac{k + 1}{2})$, which is the product of two consecutive natural numbers and we have mentioned it before. In other words with any odd natural number k the continued radicals (7) is a convergent nested radicals with positive integer element $n = \frac{k^2 - 1}{2}$ and with integer value $F = \frac{1 + \sqrt{1 + 4n}}{2} = \frac{k + 1}{2}$.

Here again by using splitting method for n = 110, we have $F = \frac{1+\sqrt{1+4(110)}}{2} = \frac{1+21}{2} = 11$, and

$$11 = \sqrt{11^2} = \sqrt{121} = \sqrt{110 + 11} = \sqrt{110 + \sqrt{121}} = \sqrt{110 + \sqrt{110 + \dots}}$$

or (see [4])

0

$$\mathbf{11} = \sqrt{\mathbf{110} + \sqrt{\mathbf{110} + \sqrt{\mathbf{110} + \cdots}}}.$$

We can change the square roots to cube roots too. For example, see [5]

$$2 = \sqrt[3]{6 + \sqrt[3]{6 + \sqrt[3]{6 + \sqrt[3]{6 + \cdots}}}}$$

To show that the value of the above nested radicals is 2 we can use our splitting technique and approach like this:

$$2 = \sqrt[3]{8} = \sqrt[3]{6+2} = \sqrt[3]{6+\sqrt[3]{8}} = \dots = \sqrt[3]{6+\sqrt[3]{6+\sqrt[3]{6+\sqrt[3]{6+\sqrt[3]{6+\dots}}}}}$$

Similarly we have

$$3 = \sqrt[3]{27} = \sqrt[3]{24+3} = \sqrt[3]{24+\sqrt[3]{27}} = \dots = \sqrt[3]{24+\sqrt[3]{24+\sqrt[3]{24+\sqrt[3]{24+\frac{3}{\sqrt{24+\frac{3}{\sqrt{24+\frac{3}{24+\frac{3}{\sqrt{24+\frac{3}{24+\frac{3}{\sqrt{24+\frac{3}{24+\frac{3}{\sqrt{24+\frac{3}{24+\frac{3}{24+\frac{3}{\sqrt{24+\frac{3}{24+\frac{3}{\sqrt{24+\frac{3}{24+\frac{3}{\sqrt{24+\frac{3}{24+\frac{3}{\sqrt{24+\frac{3}{24+\frac{3}{\sqrt{24+\frac{3}{24+\frac{3}{\sqrt{24+\frac{3}{24+\frac{3}{\sqrt{24+\frac{3}{24+\frac{3}{\sqrt{24+\frac{3}{24+\frac{3}{\sqrt{24+\frac{3}{24+\frac{3}{\sqrt{24+\frac{3}{24+\frac{3}{\sqrt{24+\frac{3}{24}$$

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Continued Fractions

We can use the above splitting method and present some continued fractions. Here we present two of them and the interesting readers can create more examples:

Example 1:

$$1 = \frac{2}{2} = \frac{2}{1+1} = \frac{2}{1+\frac{2}{2}} = \frac{2}{1+\frac{2}{1+\frac{2}{1+1}}} = \frac{2}{1+\frac{2}{1+\frac{2}{1+\frac{2}{1+\cdots}}}} = \cdots$$
$$1 = \frac{2}{1+\frac{2}{1+\frac{2}{1+\frac{2}{1+\cdots}}}} =$$

 $A = \frac{2}{1+A}$ or A = 1.

Example 2:

$$1 = \frac{2}{2} = \frac{2}{1+1} = \frac{2}{1+\frac{3}{3}} = \frac{2}{1+\frac{3}{2+1}} = \frac{2}{1+\frac{3}{2+\frac{4}{4}}} = \frac{2}{1+\frac{3}{2+\frac{4}{3+1}}} = \frac{2}{1+\frac{3}{2+\frac{4}{3+\cdots}}}$$
$$1 = \frac{2}{1+\frac{3}{2+\frac{4}{3+\frac{5}{5+\frac{5}{$$

Iterated Exponents

Before we start to discuss about the calculation of iterated exponents, it is better to mention a very common mistake in calculating iterated exponents. We all know that the correct ways to calculate the followings expressions are computing them from left to right.

20 - 8 - 2 = 12 - 2 = 10 or $24 \div 6 \div 2 = 4 \div 2 = 2$.

But if we compute them from right to left, then we will get wrong results:

$$20 - 8 - 2 = 20 - 6 = 14$$
 or $24 \div 6 \div 2 = 24 \div 3 = 8$.

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But to calculate expressions 2^3^2 or $2^3^1^2$ we must compute them from right to left. In other words we must follow this convention rule $a^b^c = a^{b^c} = a^{(b^c)} = a^b^c$. For example $2^3^2 = 2^3^2 = 2^9 = 256$ not $2^3^2 = 8^2 = 64$. Unfortunately, this is a common mistake and even some famous calculators and internet math programs follow the left to right rule to calculate iterated exponents, see the following pictures from different devices.



Now we go back to our playing with exponent issue. From $2 = \sqrt{2}^2$ and $3 = \sqrt[3]{3}^3$ and repeating these on and on, we can easily get the following two famous iterated exponents, (for more on continued exponents or leaning towers of powers see for example [2] and the references within):



Interestingly, if we go one more step further we will get a big contradiction! Because, similar to the above process, from $4 = (\sqrt[4]{4})^4$ and repeating this we obtain

$$4 = \sqrt[4]{4}^{4\sqrt{4}}$$

But we know that $\sqrt[4]{4} = \sqrt{2}$ and so the above tower of powers is not equal to 4 but its value is equal to 2. Overall, for any positive integer $n \neq 4$ we have (see [2]),

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$$n = \sqrt[n]{n} \sqrt{n}^{n} \sqrt{n}^{\cdot}$$

References

- [1] Hossein Behforooz: *Problem* # 41.11, Problems and Solutions Section, Mathematical Spectrum, Vol. 41, No. 3, (2008/2009), page 139.
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- [3] Roger B. Nelsen, Proofs Without Words, MAA Publications, page 118.
- [4] Wenjiang Tu, *Comments on Calendar Problem # 29, October 2017,* NCTM, Mathematics Teacher, Vol. 111, No. 7 (May 2018), page 488.
- [5] https://en.wikipedia.org/wiki/Nested_radical
- [6] <u>https://www.youtube.com/watch?v=leFep9yt3JY</u>
- [7] https://www.youtube.com/watch?v=r5BGIi84arY

Öğretmen Adaylarının Bloom Taksonomisini Kullanım Yeterliliklerinin İncelenmesi

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Özet

Bu çalışmanın amacı öğretmen adaylarının Bloom taksonomisinin basamaklarına uygun, soru hazırlama yeterliliklerinin incelenmesidir. Araştırmada nitel araştırma desenlerinden biri olan durum çalışması benimsenmiştir. Çalışmaya İlköğretim Matematik ve Fen Bilimleri bölümü öğretmen adaylarından toplam 67 kişi katılmıştır. Uygulamada, öğretmen adaylarından 5., 6., 7. ve 8. sınıf matematik ve fen bilimleri müfredatı doğrultusunda konu sınırlandırması olmaksızın Bloom taksonomisinin her bir basamağına uygun soru hazırlamaları istenmiştir. Hazırlanan sorular Bloom taksonomisine uygunluk yönünden alanında uzman kişiler tarafından incelenmiş ve her bir sorunun basamaklara uygunluğu doğru ya da yanlış oluşuna göre frekansı alınarak tablolaştırılmıştır. Elde edilen sonuçlara göre öğretmen adaylarının alt bilişsel basamaklara uygun soru hazırlama yeterliliklerinin üst bilişsel basamaklara göre daha fazla olduğu belirlenmiştir.

Anahtar Kelimeler: Bloom Taksonomisi, Öğretmen Adayı, Matematik, Fen

1. Giriş

Bilişsel alan, bilgilerle ve bilgilerden doğan zihinsel yeteneklerle ilgili zihinsel öğrenmelerin yoğunlukta olduğu, zihinsel yetilerin geliştirildiği ve zihinsel faaliyetlerin ortaya çıkarılmaya çalışıldığı bir alan olup aynı zamanda öğretimin tasarlanmasına hizmet etmektedir (Yeşilyurt & Eser, 2010). Bilişsel süreçler; tanıma, yorumlama, akıl yürütme, ilişki bulma, genelleme yapma ve sonuç çıkarma gibi bazı bilişsel yeterliliklerden oluşmaktadır (Aydemir & Çiftçi, 2008). Bu anlamda eğitim hedefleri arasında bilgiyi tanıma ve hatırlama, bilgiyi kullanarak işlemler yapma, kavramlar, genellemeler ve kuramlar geliştirme sürecinde bilişsel yeterlilikler önemli bir yer tutmaktadır (Mutlu, Uşak & Aydoğdu, 2003). Bu yüzden, bilişsel süreçlerin çeşitliliğinin oluşturulması ve ölçme-değerlendirme faaliyetleri sonunda öğrencilerin gerçek başarı seviyelerinin belirlenmesi için öğretmenlerin sınavlarda tüm bilişsel düzeyde sorulara yer vermesi gerekmektedir (Baysen, 2006; Köğce & Baki, 2009).

Bilişsel alan üzerine yoğunlaşan bazı eğitim bilimciler bilişsel alanı farklı kategorilerde incelemişlerdir (Filiz, 2004). Bunlardan bir tanesi de bilim dünyasında en çok kabul edilen ve Bloom tarafından geliştirilen Bloom Taksonomisidir (Ralph, 1999). Bloom ve arkadaşları 1956 yılında yayınlanan ve orijinal adı "Taxonomy of Educational Objectives: Cognitiveand Affective Domains" olan eserde ilgili taksonomiyi "Bilgi, kavrama, uygulama, analiz, sentez ve değerlendirme" şeklinde hiyerarşik bir üslupla ele almışlardır (Bloom, Engelhart, Furst, Hill, & Krathwohl, 1956). Basitten karmaşığa doğru sıralanan bu taksonomide üst basamaklar alt basamaklardan daha fazla zihinsel bilgi ve beceri gerektirmektedir.

Bilişsel basamaklar içerisinde yer alan *Bilgi*, herhangi bir nesne veya olguyla ilgili özelliklerin görünce tanınması, sorunca söylenmesi veya herhangi bir yorum getirilmeden ezberden tekrar edilmesidir. *Kavrama*, bilgi seviyesinde kazanılan davranışların özümsenmesi, kendine mal edilmesi ve anlamının yakalanmasıdır. *Uygulama*, bilgi ve kavrama seviyesinde elde edilen bilgilerin yeni bir problemin çözümünde kullanılmasıdır. *Analiz*, herhangi bir bilgi bütününün öncelik-sonralık veya sebep-sonuç ilişkisi içerisinde öğelerine ayrılarak ortaya konulmasıdır. *Sentez*, bilgi, kavrama, uygulama ve analiz seviyelerinde elde edinlen bilgilerin belli ilişki ve kurallara göre birleştirilerek yeni ve orijinal bilgilerin/ürünlerin üretilmesidir. *Değerlendirme*, üretilen bilgilerin sebep ve sonuçlarla birlikte yorumlanması, savunulması ve yargılanmasıdır.

Öğrenmeyi ölçme ve başarıyı belirlemede sorular önemli birer araçtır. Başarıyı etkili bir şekilde ölçmek için ise, öğretim sürecinde nitelikli ve amaca uygun sorular kullanmak gereklidir. Etkili sorular kullanmanın yolu da, bilişsel seviyenin bütün basamaklarının dikkate alınmasıyla gerçekleşmektedir. Taksonomi basamaklarına göre hazırlanan sorular, öğretmene eğitimde kolaylık sağlamanın yanında öğrencinin üst düzey becerilerini kuvvetlendirmekte aynı zamanda soruların yalnızca bir ya da iki basamakla sınırlanmasını önlemektedir. Ayrıca İlköğretim Kurumları Yönetmenliği genel esaslarında; öğrencinin başarısını belirlemek amacıyla hazırlanan ölçme araçlarında sadece bilginin ölçülmesinin değil, kavrama, kendini ifade edebilme, yorumlayabilme, uygulama, analiz, sentez ve değerlendirme düzeyinde edindikleri davranışların da ölçülmesinin gerekli olduğu belirtilmektedir. Bu doğrultuda, mevcut çalışma Matematik ve Fen Bilimleri öğretmen adaylarının Bloom taksonomisinin basamaklarına uygun, soru hazırlama yeterliliklerinin incelenmesini amaçlamaktadır.

2. Yöntem

Araştırma Ölçme ve Değerlendirme dersi kapsamında durum çalışması niteliğinde gerçekleştirilmiştir. Çalışmaya Muş Alparslan Üniversitesi İlköğretim Matematik ve Fen Bilimleri bölümü öğretmen adaylarından toplam 67 kişi katılmıştır. Uygulamada, öğretmen adaylarından 5., 6., 7. ve 8. sınıf matematik ve fen bilimleri müfredatı doğrultusunda konu sınırlandırması olmaksızın Bloom taksonomisinin her bir basamağına uygun soru hazırlamaları istenmiştir. Hazırlanan toplam 402 soru Bloom taksonomisine uygunluk yönünden alanında uzman kişiler tarafından incelenmiştir. Basamakların tanımları, anahtar sözcükleri ve bu basamaklara uygun farklı türde soru örnekleri, öğretmen adaylarının hazırladıkları soruların Bloom taksonomisine uygunlukları yönünden güvenilir bir biçimde çözümlenmesine yardımcı olmuştur. Uzmanlar ilgili basamağı yansıtan soruları D (Doğru), yansıtmayanları ise Y (Yanlış) şeklinde kodlamışlardır. Daha sonra uzmanların kodladıkları sorular karşılaştırılmış ve uyum yüzdesi %90 olarak belirlenmiştir. Farklı kodlamaların ortaya çıktığı sorular üzerinde ortak görüş elde edilene kadar tartışmış ve kodlamalara son hali verilmiştir. Son olarak, öğretmen adaylarının her bir basamak için oluşturdukları doğru ve yanlış soruların frekans ve yüzdeleri hesaplanarak tablolaştırılmış ve yorumlanmıştır.

3. Bulgular ve Yorum

Öğretmen adaylarının oluşturdukları soruların bilişsel alanın basamaklarına uygunluk yönünden frekans ve yüzdelik dağılımları Tablo 1'de sunulmuştur.

Bilişsel Alan	D/Y	Matematik		Fen Bilimleri		Toplam	
Basamakları		n=37		n=30		n=67	
		n	%	n	%	n	%
Bilgi	Doğru	35	94,5	27	90	62	92,5
	Yanlış	2	5,4	3	10	5	7,4
Kavrama	Doğru	27	72,9	26	86,6	53	79,1
	Yanlış	10	27,0	4	13,3	14	20,9
Uygulama	Doğru	20	54,0	19	63,3	39	58,2
	Yanlış	17	45,9	11	36,6	28	41,8
Analiz	Doğru	9	24,3	12	40	21	31,3
	Yanlış	28	75,6	18	60	46	68,6
Sentez	Doğru	14	37,8	20	66,6	34	50,7
	Yanlış	23	62,1	10	33,3	33	49,2
Değerlendirme	Doğru	13	35,1	16	53,3	29	43,2
	Yanlış	24	64,8	14	46,6	338	56,7

Tablo 1. Öğretmen adaylarının oluşturdukları soruların bilişsel alan basamaklarına göre analizi

Tablo 1 incelendiğinde, matematik öğretmen adaylarının yaklaşık olarak %95'i bilgi, %73'ü kavrama, %54'ü uygulama, %24'ü analiz, %38'i sentez ve %35'i değerlendirme basamaklarında uygun sorular oluşturabilirken, fen bilgisi öğretmen adaylarının ise %90'ı bilgi, %87'si kavrama, %63'ü uygulama, %40'ı analiz, %67'si sentez ve %53'ü değerlendirme basamaklarında uygun şekilde sorular oluşturabilmişlerdir. Öğretmen adaylarının tamamına yakınının bilgi basamağında soru oluşturmada sorun yaşamadığı buna karşın büyük çoğunluğunun analiz basamağında soru oluşturmada sıkıntı yaşadığı tespit edilmiştir. Branşlara göre, fen bilimleri öğretmen adaylarının genel olarak kategorilere uygun doğru soru yazma oranının ilköğretim matematik öğretmen adaylarına göre daha fazla olduğu belirlenmiştir. Bu farklılığın ortaya çıkmasında, matematik ve fen derslerinin yapısal olarak farklılık göstermesi ve fen konularının doğal yaşamla ilişkilendirilmesinin kolay olmasının temel etken olabileceği düşünülmektedir.

Genel olarak öğretmen adaylarının basamaklara uygun sorular oluşturabilme sayılarının bilgi, kavrama, uygulama, sentez, değerlendirme ve analiz basamaklarına göre azalan bir sıra ile devam ettiği görülmektedir. Her iki programda da bilgi, kavrama ve uygulama basamaklarına yönelik uygun soru oluşturabilme yüzdeleri yüksek iken; analiz, sentez ve değerlendirme basamaklarında bu oranın düşük olduğu gözlenmiştir. Buradan hareketle öğretmen adaylarının genellikle üst bilişsel soru (analiz, sentez, değerlendirme) hazırlama yeterliliklerinin alt bilişsel soru (bilgi, kavrama, uygulama) hazırlama yeterliliklerine göre daha zayıf olduğu söylenebilir. Öğretmen adaylarının yaşadıkları bu güçlük onların basamaklar hakkında genel kavramları iyi özümseyememelerinden kaynaklanabileceği düşünülmektedir.

4. Sonuç ve Öneriler

Sonuçlara göre genel olarak öğretmen adaylarının azalan sırayla bilişsel alanın bilgi, kavrama, uygulama, sentez, değerlendirme ve analiz basamaklarına yönelik soru hazırlama yeterliliklerinin olduğu tespit edilmiştir. Öğretmen adayları en zor analiz, en kolay ise bilgi basamağına yönelik sorular oluşturabilmişlerdir. Dolayısıyla öğretmen adaylarının tamamına yakınının bilgi basamağında soru oluşturmada sorun yaşamadığı buna karşın büyük çoğunluğunun analiz basamağına uygun soru oluşturamadığı tespit edilmiştir. 340

Branşlara göre bakıldığında ise, fen bilimleri öğretmen adaylarının genel olarak kategorilere uygun soru yazma oranının ilköğretim matematik öğretmen adaylarına göre daha fazla olduğu görülmüştür. Tüm branş öğretmenlerinin öğrenci başarısını belirlemede aynı seviyedeki öğrenmeler yerine bilişsel süreçlerin çeşitliliğini artırmak için değişik seviyedeki öğrenmeleri belirleyebilecek türden sorular sorabilmesinin önemli olduğu düşünüldüğünde ileriki çalışmalarda nedenleri belirlenerek öğretmen adaylarının konu ile ilgili eksikliklerinin giderilmesine yönelik daha fazla çalışmanın yapılması önerilmektedir.

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